A packing integer program arising in two-layer network design

Christian Raack[*] Arie M.C.A Koster[⋆]

[*]Zuse Institute Berlin
Takustr. 7, D-14195 Berlin

[⋆]Centre for Discrete Mathematics and its Applications (DIMAP)
Warwick Business School, University of Warwick, Coventry CV4 7AL, UK

Abstract

In this paper we study a certain cardinality constrained packing integer program which is motivated by the problem of dimensioning a cut in a two-layer network. We prove $\mathcal{NP}$-hardness and consider the facial structure of the corresponding polytope. We provide a complete description for the smallest nontrivial case and develop two general classes of facet-defining inequalities. This approach extends the notion of the well known cutset inequalities to two network layers.

Keywords: *packing integer programming, two-layer network design, cutset inequalities*

1 Introduction

Let A be a 0-1 matrix with $m \geq 2$ rows, $n \geq m$ columns, and the first m columns forming an identity matrix. We denote by $M := \{1, \dots, m\}$ and $N := \{1, \dots, n\}$ the row and column indices of A. The *length* ℓ_j of column $j \in N$ is defined as the sum of its entries, i. e., $\ell_j = \sum_{j=1}^{m} a_{ij}$. We set $\bar{\ell} := \max_{j \in N}(\ell_j)$. Depending on whether ℓ_j is odd or even we speak of odd and even columns of A. The index set for all odd columns is denoted by $O \subseteq N$. Obviously $M \subseteq O$. For any vector v and a subset of its indices S, let $v(S) := \sum_{j \in S} v_j$ throughout.

Let d be a n-dimensional 0-1 vector with $d_j = 1$ if and only if $j \in O$. Considering $b_0 \in \mathbb{Z}_+$ and a right hand side vector $b \in \mathbb{Z}_+^m$ we study the polytope

$$P := \operatorname{conv}\{x \in \mathbb{Z}_+^n : dx \geq b_0,\ Ax \leq b\}.$$

By aggregating variables we may assume that all columns of A differ. A valid inequality for P is called *nontrivial* if it is not a nonnegativity constraint and if it is not the *cardinality constraint* $dx \geq b_0$ or one of the *packing constraints* in the system $Ax \leq b$. The columns of A can be seen as incidence vectors for subsets of the base set M. Since the identity matrix is contained in A all singleton subsets are part of the problem. An integer point in P can be seen as a set *packing* where each element $i \in M$ is covered at most b_i times and the number of subsets with odd cardinality is at least b_0. The canonic packing x^0 (satisfying all packing constraints) is given by $x_j^0 := b_j$ for all $j \in M$ and $x_j^0 := 0$ for all $j \in N \backslash M$.

Our study of P is motivated by design problems for layered telecommunication networks [1, 7, 11]. In such stacked networks two (or more) layers are coupled in such a way that every upper layer link is represented by paths (between the corresponding end-nodes) in the underlying lower layer. In the following we provide a mixed integer programming formulation for a a two-layer network design problem and show that optimizing over the polytope P corresponds to the design problem for a cut (or a two-node two-layer network).

Two-layer network design Consider a first *physical layer* represented by a graph $G = (V, E)$. and a second complete *virtual layer* $H = (V, V \times V)$ defined by the same set of locations V and all possible *virtual links*. Every virtual link can be realised by (different) paths in the physical layer. Both graphs are simple and undirected. In general one may also consider subsets of V and $V \times V$ in the virtual graph. In practice, the graph G might represent a fiber topology of an optical transport network. In this case, a *virtual link* of H reflects the possibility to connect the corresponding end-nodes by a *light-path* in G using *wavelength division multiplexing* (WDM) technology [13]. Here we consider the physical graph to be fixed (not being subject to dimensioning). A realisation of a virtual link as a path in the physical layer will be called a light-path in the following.

Given a traffic matrix of user demands with respect to V, the task is to select light-paths and to equip them with capacities such that the user demands can be routed in the virtual layer. A demand can be routed using several virtual paths (paths in H) consisting of multiple virtual links. Flow can be fractional. Every edge of G provides only a fixed number of channels. Every light-path capacity module consumes one channel on every edge along the path in G.

The model we consider here is close to the formulation proposed by Raghavan and Stanojević [12], also see [1]. It has the advantage of a very compact description of the virtual layer flow. This is achieved by aggregating all flow variables for light-paths with the same end-nodes to a single variable. For every virtual link $\{v, w\} \in V \times V$ a set $P_{\{v,w\}}$ of admissible light-paths in the physical graph G is considered. Let P be the union of all these paths. Each path $p \in P$ can be equipped with multiples of a base *channel* capacity C at a certain cost. Every physical link $e \in E$ supports a total of B_e channels. We consider a set of commodities K modeling the given traffic forecast. With every commodity $k \in K$ and every node $v \in V$, a demand value D_v^k is associated such that $\sum_{v \in V} D_v^k = 0$.

We introduce the following variables. For every virtual link $\{v, w\}$ the variables f_{vw}^k and f_{wv}^k describe the flow between v and w in both directions w. r. t. commodity $k \in K$. The integer variable x_p counts the number of channel capacities for path p. The problem of minimizing the cost of a feasible capacity assignment satisfying the given traffic demands and the capacity restrictions on both layers can now be formulated as the problem of minimizing a linear function over the following set of constraints:

$$\sum_{w \in V \setminus \{v\}} (f_{vw}^k - f_{wv}^k) = D_v^k \qquad \forall v \in V, k \in K \tag{1}$$

$$\sum_{p \in P_{\{v,w\}}} C x_p - \sum_{k \in K} (f_{vw}^k + f_{wv}^k) \geq 0 \qquad \forall \{v, w\} \in V \times V \tag{2}$$

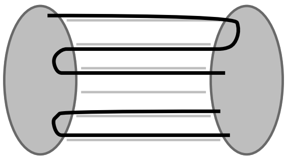
$$\sum_{p \in P \colon e \in p} x_p \leq B_e \qquad \forall e \in E \tag{3}$$

$$f_{vw}^k, f_{wv}^k \in \mathbb{R}_+, \; x_p \in \mathbb{Z}_+ \tag{4}$$

The flow conservation equations (1) ensure a feasible routing of the traffic. The virtual link capacity constraint (2) says that the flow between v and w must not exceed the total capacity installed on all corresponding paths. The physical link capacity constraint (3) restricts the number of light-path channels for every physical link e. An extension of the formulation above considering the design problem of virtual as well as physical links and nodes is used in [1].

Two-Layer cuts Consider a cut in the physical graph and all crossing light-paths, (i. e., all paths in P using at least one of the physical cut links, see Figure 1. Only if such a path uses an odd number of physical cut links, i. e., its end-nodes are in different shores of the cut, it can contribute to the transport of traffic across the cut. We assume that these odd paths have to be equipped with at least b_0 many capacity modules to allow for a feasible realization of the traffic across the cut. The cardinality constraint $dx \geq b_0$ reflects this requirement and can be seen as the (capacity forcing) cutset inequality [2, 3, 8] for the virtual cut. The value b_0 depends on the cut demand D and the size of the channel capacity C and can be computed as $b_0 = \lceil \frac{D}{C} \rceil$. The packing constraints $Ax \leq b$ are simply the physical channel limitations (3) for all cut links. The rows of A correspond to all physical cut links and the columns of A correspond to all light-paths crossing the cut. Since in practice typically all single-hop channels (light-paths using exactly one physical link) are part of the problem, the matrix A contains the identity matrix.

In this context, P is a two-layer network design polytope for two network nodes or a (two-layer) cutset polytope. Every cut in G defines a polytope of type P. Hence, facets of P extend the notion of cutset inequalities to two layers. Single-layer network design polyhedra, cutset polyhedra, and cutset inequalities have been studied for instance in [2, 3, 8].



$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Figure 1: Physical cut and crossing light-paths. Physical cut links correspond to rows. Paths correspond to columns of A. All singleton paths are part of the problem.

Basic observations In this paper, we study the complexity of optimizing over P as well as the polyhedral structure of P. For this, we introduce the following additional notation. Given a column index $j \in N$, the set $M[j] := \{i \in M \,:\, a_{ij} = 1\}$ contains all row indices with a nonzero entry in column j of A. Similarly, for a row index $i \in M$, the set $N[i] := \{j \in N \,:\, a_{ij} = 1\}$ corresponds to all columns with a nonzero entry in row i of A. For $j \in N$ we write $b_j := b(M[j])$. Note that b_j is well defined since it coincides with the right hand side of the packing constraint for $j \in M$. We denote by $e^j \in \{0,1\}^n$ the j-th unit vector for $j \in N$.

By a simple reduction from the decision version of MAXIMUM SET PACKING [5] it can be seen that already deciding whether P is nonempty or not is $\mathcal{NP}$ -complete if we allow for arbitrary $\{0,1\}$-matrices A. The situation however changes if A contains the identity matrix as claimed above. In this case the dimension of P only depends on the size of $b(M)$ compared to the size of b_0.

Lemma 1.1. *P is nonempty if and only if $b(M) \geq b_0$.*

Proof. Since $dx = \sum_{j \in O} x_j \leq \sum_{j \in N} \ell_j x_j$ and by aggregating all packing constraints $\sum_{j \in N} \ell_j x_j \leq b(M)$. we conclude that P is empty if $b(M) < b_0$. On the other hand, if $b(M) \geq b_0$, then $x^0 \in P$. □

Lemma 1.2. *P is full-dimensional if and only if $b_i \geq 1$ for all $i \in M$ and $b(M) \geq b_0 + \max(1,\ 2\left\lfloor \bar{\ell}\,/\,2 \right\rfloor)$.*

Proof. Let $\bar{j} = \operatorname{argmax}\{\ell_j \,:\, j \in N\}$.

Necessity: If $b_i = 0$ for some $i \in M$, then $x_i = 0$ for all feasible packings x and thus P is not full-dimensional. Assume that $b(M) \leq b_0$. Thus P is either empty (Lemma 1.1) or $b(M) = b_0$. If the latter is true, the only feasible vector is given by x^0 which gives a dimension of 0 and hence a contradiction. We may assume that $b(M) \geq b_0 + 1$. Since P is full-dimensional there exists a feasible assignment with $x_{\bar{j}} \geq 1$. For this assignment it holds that $dx = \sum_{j \in O} x_j \geq b_0$ if column $\bar{j}$ is even and $\sum_{j \in O \setminus \{\bar{j}\}} x_j \geq b_0 - x_{\bar{j}}$ if column $\bar{j}$ is odd. Summing up the packing constraints shows $b(M) \geq b_0 + 2\left\lfloor \bar{\ell}\,/\,2 \right\rfloor$.

Sufficiency: We construct $n+1$ affinely independent points in P. The first vector is given by x^0 which is feasible because $dx^0 = b(M) \geq b_0 + 1$. Since the cardinality constraint is not tight and $b_i \geq 1$ for all $i \in M$, every nonzero entry of x^0 can be reduced individually. More precisely, for $k \in M$ we consider the vector $x^k := x^0 - e^k$. Additionally, for columns $k \in N \backslash M$, we define the vectors $x^k := x^0 + e^k - \sum_{j \in M[k]} e^j$. It holds that $dx^k = b(M) - \ell_k$ or $dx^k = b(M) - \ell_k + 1$ depending on whether ℓ_k is even or odd. From $\ell_k \leq \bar{\ell}$ we get that $dx^k \geq b_0$ in both cases. The $n+1$ constructed vectors are clearly affinely independent. □

Lemma 1.2 implies that if P is not full dimensional it is either empty, contains a single point or there exists $j \in N$ such that $x_j = 0$ for all $x \in P$. It follows that by consecutively deleting variables that are fixed to zero and by excluding the trivial cases we may assume that P is full dimensional w. l. o. g. throughout the rest of this article. Due to length restrictions we have to omit most of the proofs.

2 Complexity

Given weights $w \in \mathbb{Z}^n$, we consider the problem of optimizing a linear function over P:

$$\min\{wx \,:\, x \in P\} \tag{P}$$

We first observe that if all columns of A have at most two entries ($\bar{\ell} \leq 2$) the problem (P) can be solved efficiently. If $\bar{\ell} = 1$, then A is the identity matrix and $\binom{-d}{A}$ is totally unimodular. Hence P is already completely described by the cardinality, packing, and nonnegativity constraints. Now consider the case that $\bar{\ell} = 2$, which implies that for every column of the constraint matrix $\binom{-d}{A}$ the sum of the absolute values of its entries is 2. By Edmonds and Johnson [4], the corresponding optimization problem can be seen as a generalized b-matching problem or a matching problem on bidirected graphs [14, chapter 36]. A complete description of P is obtained by adding all $\{0, 1/2\}$-Chvátal-Gomory cuts (all *blossom* inequalities) [4, 6, 14]. Also in this cases the problem (P) can be solved in strongly polynomial time. Notice that the case $\bar{\ell} \leq 2$ is of particular practical interest since for a single-node cut in a two-layer network it holds that a light-path visits the cut at most twice.

In the following we show that optimizing over P is strongly $\mathcal{NP}$ -hard in general. For the maximization version of (P) there is a straightforward reduction from MAXIMUM SET PACKING [5].

Proposition 2.1. *The optimization problem* (P) *is strongly $\mathcal{NP}$ -hard.*

The corresponding reduction uses nonpositive weights only. But it turns out that also the minimization version (nonnegative weights) of (P) is $\mathcal{NP}$ -hard (in contrast to the minimization version of standard SET PACKING). Notice that network design typically means minimizing the cost of certain resources. Here we prove an even stronger result for 0-1 weights by reduction from MAXIMUM INDEPENDENCE SET [5].

Theorem 2.2. *The optimization problem* (P) *with $w_j \in \{0, 1\}$ for all $j \in N$ is strongly $\mathcal{NP}$ -hard.*

Proof. The problem (P) is clearly in $\mathcal{NP}$. We reduce MAXIMUM INDEPENDENT SET to (P). Let $G = (V, E)$ be a connected graph with $|E| \geq |V|$ (MAXIMUM INDEPENDENT SET is in $\mathcal{P}$ for trees, see [10]) and let $K \in \mathbb{Z}_+$. We have to decide whether there is a subset $S \subset V$ with $|S| \geq K$ which is independent, that is, for every edge $\{v, w\} \in E$ it holds that $|S \cap \{v, w\}| \leq 1$. The set of incident edges to $v \in V$ is denoted by $\delta(v)$. Let $U \subseteq V$ be the set of nodes in G with even node degree. We define the matrix A as follows. Set $m := |E| + |U|$ and identify the first $|E|$ rows of A with edges of G and all other rows with nodes in U. The number of columns is defined by $n := m + |V|$. The first m columns form an identity matrix again. Every column $j > m$ represents a node $j \in V$ with $M[j] := \delta(j) \cup \{j\}$ if $j \in U$ and $M[j] := \delta(j)$ if $j \in V \backslash U$. This way all columns of A have odd length. Set $b_i := 1$ for all $i \in E \cup U$ and $b_0 := K$. The weights are defined such that $w_j := 1$ for $j \leq m$ and $w_j := 0$ otherwise.

In the following we show that using this reduction there exists an independent set in G of size at least K if and only if there exists an integer solution $x \in P$ with weight $wx \leq 0$. Let first $x \in \mathbb{Z}_+$ be a vector in P. Such a solution exists since $b(M) \geq |E| \geq |V| \geq K = b_0$, see Lemma 1.2. We define $S := \{j \in N \,:\, x_j = 1\}$. It follows that $S \subseteq V$ if $wx \leq 0$. From the cardinality constraint we get that $|S| = dx \geq b_0 = K$ because all columns of A are odd. From $Ax \leq b$ it follows that $|S \cap \{v, w\}| = \sum_{j \in S} x_j \leq 1$ for all edges $\{v, w\} \in E$. Hence S is an independent set of size at least K. Now let S be an independent set of size at least K. We construct an integer solution in P by setting $x_j := 1$ for all $j \in S$ and $x_j := 0$ otherwise. It holds that $x \in P$ because $dx \geq K = b_0$ and $\sum_{j \in S} x_j = |S \cap \{v, w\}| \leq 1$. $\square$

3 Polyhedral Studies

In this section we study the facial structure of P. We start by considering trivial facets and properties of nontrivial facets. Next, we provide a complete description of P for the case $m = 3$. Based on this description, we develop two classes of general facet-defining inequalities for P. Recall that we assume P to be full-dimensional.

Lemma 3.1. *Row $i \in M$ of the system $Ax \leq b$ defines a facet of P.*

Lemma 3.2. *The cardinality constraint $dx \geq b_0$ defines a facet of P if and only if $b_0 > 0$.*

Lemma 3.3. *Let $j \in N$. The nonnegativity constraint $x_j \geq 0$ defines a facet of P if either $j \in N \backslash M$ or $b(M) - b_j \geq b_0 + \max(1,\ 2\lfloor \bar{\ell}/2 \rfloor)$.*

Lemma 3.4. *Let $\alpha x \leq \alpha_0$ be a nontrivial facet-defining inequality for P and let $j \in N$. If $j \in M$ then $\alpha_j \leq 0$. If $j \in N \backslash O$, then $\alpha_j \geq 0$. Moreover $\alpha_i \leq \alpha_j$ for all $i \in M[j]$ and $\sum_{i \in M[j]} \alpha_i \leq \alpha_j$.*

Proof. First assume that $j \in M$. Since $\alpha x \leq \alpha_0$ is not one of the packing constraints there is a feasible point x on the facet that is not tight in row j. Hence x_j can be increased without leaving P which gives $\alpha_j \leq 0$. Now let $j \in N \backslash M$. Since $\alpha x \leq \alpha_0$ is not a nonnegativity constraint there is a point x on the facet with $x_j \geq 1$. If ℓ_j is even entry x_j can be reduced maintaining feasibility. Hence $\alpha_j \geq 0$. Moreover, we can construct new feasible packings from x by reducing x_j and increasing x_i for (some or all) $i \in M[j]$. This shows $\alpha_i \leq \alpha_j$ for all $i \in M[j]$ and $\sum_{i \in M[j]} \alpha_i \leq \alpha_j$. □

Corollary 3.5. *If $\alpha x \leq \alpha_0$ is a facet-defining inequality for P with $\alpha_j \geq 0$ for all $j \in N$, then it is either one of the packing constraints or $\alpha_j = 0$ for all $j \in M$.*

Proof. If $\alpha x \leq \alpha_0$ is nontrivial then $\alpha_j \leq 0$ for all $j \in M$ by Lemma 3.4. □

Lemma 3.6. *If $\alpha x \leq \alpha_0$ is a facet-defining inequality for P with $\alpha_j \leq 0$ for all $j \in N$, then it is either a nonnegativity constraint or the cardinality constraint.*

Proof. From $\alpha x \leq \alpha_0$ being a facet follows $\alpha_0 \leq 0$. If it is not the cardinality constraint, then there is a point x^* on the facet with $\sum_{j \in O} x_j^* > b_0$. Let $j \in N$ with $x_j^* > 0$. We may reduce x_j^*. The resulting vector is feasible and has to satisfy $\alpha x \leq \alpha_0$, hence $\alpha_j = 0$. It follows that for all $j \in N$ either $x_j^* = 0$ or $\alpha_j = 0$. Hence $\alpha x^* = 0$ which implies $\alpha_0 = 0$ and thus $\alpha x \leq \alpha_0$ is a nonnegativity constraint. □

Complete description for $m = 3$. In the context of two-layer network design, m small is of particular interest since physical networks are sparse in practice, i. e., cuts typically have a small number of physical links. If $m = 2$, then $\bar{\ell} \leq 2$ for which a complete description (by blossom inequalities) is known as mentioned in Section 2. Here we aim to study the case $m = 3$ with equal right hand sides, reading as follows:

$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 \qquad\qquad\qquad &\geq b_0 && (5)\\
x_1 \qquad\quad + x_4 + x_5 \qquad + x_7 &\leq \beta && (6)\\
x_2 \qquad + x_4 + x_5 + x_6 \qquad &\leq \beta && (7)\\
x_3 + x_4 \qquad + x_6 + x_7 &\leq \beta && (8)
\end{aligned}$$

Notice that the columns in (6)-(8) correspond to all nonempty subsets of $M = \{1, 2, 3\}$. We consider the polytope $\mathcal{P}^3 := \operatorname{conv}\{x \in \mathbb{Z}_+^7 \ :\ x \text{ satisfies (5) - (8)}\}$. We assume that $\beta, b_0 \in \mathbb{Z}_+ \backslash \{0\}$ and that $\mathcal{P}^3$ is full dimensional, hence by Lemma 1.2 it holds that $3\beta \geq b_0 + 2$. It suffices to study $\mathcal{P}^3$ since all other instances having $m = 3$ can be obtained by fixing subsets of x_4, x_5, x_6 or x_7 to zero which gives nonempty faces of $\mathcal{P}^3$. Consequently, a complete description for $\mathcal{P}^3$ means a complete description for $m = 3$. Setting $p := \lfloor (3\beta - b_0)/2 \rfloor$ and $q := \lfloor (2\beta - b_0)/2 \rfloor$, the following inequalities are obviously valid for $\mathcal{P}^3$:

$$\begin{aligned}
x_4 + x_5 + x_6 + x_7 &\leq p && (9)\\
-x_1 \qquad\qquad\qquad + x_6 \qquad &\leq q && (10)\\
-x_3 \quad + x_5 \qquad\qquad &\leq q && (11)\\
-x_2 \qquad\qquad\qquad + x_7 &\leq q && (12)
\end{aligned}$$

These inequalities are obtained by aggregating subsets of (5)-(8) and applying a $\{0, \frac{1}{2}\}$-Chvátal-Gomory step. The subsets are {(5)-(8)}, {(5), (7), (8)}, {(5), (6), (7)}, and {(5), (6), (8)}, respectively. In the following we will make use of the following integral points several times:

$$x^1 = (s, s, s, p, 0, 0, 0), x^2 = (0, t, t, 0, 0, q, 0), x^3 = (0, 0, 0, \beta, 0, 0, 0), x^4 = (0, 0, 0, b_0 + 1, -s, -s, -s)$$

where $s := \lceil (b_0 - \beta) / 2 \rceil$ and $t := \lceil b_0 / 2 \rceil$. Notice that $x^1 \in \mathcal{P}^3$ whenever $\beta \leq b_0$, that x^2 is in $\mathcal{P}^3$ if $2\beta \geq b_0$, and that x^3, x^4 are valid if $\beta > b_0$.

Lemma 3.7. *Inequality* (9) *defines a facet of $\mathcal{P}^3$ if and only if $b_0 - \beta$ is odd.*

Proof. Necessity: If $b_0 - \beta$ is even or equivalently $3\beta - b_0$ is even, then (9) is the sum of (5)-(8).

Sufficiency: Setting $y := x^1$ if $\beta < b_0$ and $y := x^4$ if $\beta \geq b_0$ the following seven affinely independent points are on the face defined by (9):

$$y, \quad y - e^4 + e^5, \quad y - e^4 + e^6, \quad y - e^4 + e^7, \quad y - e^4 + e^3 + e^5, \quad y - e^4 + e^1 + e^6, \quad y - e^4 + e^2 + e^7$$

Notice that from the fact that $b_0 - \beta$ is odd follows that $p + s = \beta$, $p + 3s = b_0 + 1$, and $b_0 + 1 - 2s = \beta$. □

Lemma 3.8. *The inequalities* (10) - (12) *define facets of $\mathcal{P}^3$ if and only if b_0 is odd and $2\beta - b_0 \geq 1$.*

Proof. By symmetry, it suffices to prove the result for (10). *Necessity:* If b_0 is even or equivalently $2\beta - b_0$ is even, then (10) is the sum of (5), (7) and (8). If $2\beta - b_0 \leq -1$, then the sum of (5), (7) and (8) dominates (10). *Sufficiency:* The following affinely independent points are on the face defined by (10):

$$x^2, \quad x^2 - e^2, \quad x^2 - e^3, \quad x^2 - e^2 - e^3 + e^4, \quad x^2 - e^2 + e^5, \quad x^2 - e^3 + e^7, \quad x^2 - e^2 - e^3 + e^1 + e^6$$

Notice that if b_0 is odd then $2t = b_0 + 1$ and $q + t = \beta$. □

Theorem 3.9. *The polytope $\mathcal{P}^3$ is completely described by the the inequalities* (5) – (12)*.*

General facets. It has been shown above that the Chvátal rank of P is 1 also in the case $m = \bar{\ell} = 3$. All facet-defining inequalities are $\{0, 1/2\}$-cuts. But not every combination of rows of the initial formulation gives rise to a facet-defining inequality. Only those $\{0, 1/2\}$-cuts that combine the rows of $Ax \leq b$ with the cardinality constraint are strong. This observation motivates the following two general classes of facet-defining inequalities. The first class of facets generalizes inequalities (10)–(12) and the second class is similar to (9). Both inequalities are rank 1 mixed integer rounding (MIR) inequalities [9].

Let $i_1, i_2 \in M$ be two arbitrary rows of A. We assume w. l. o. g. that $i_1 = 1$ and $i_2 = 2$. For $k \in \{0, 1, 2\}$ we set $N^k := \{j \in N \,:\, |M[j] \cap \{1, 2\}| = k\}$. Hence N^k corresponds to all columns that have k entries in the first two rows of A. We set $q := \lfloor (b_1 + b_2 - b_0) / 2 \rfloor$. Aggregating rows i_1, i_2 and the cardinality constraint, dividing by 2 and rounding down left and right hand sides gives

$$\sum_{j \in N^2 \setminus O} x_j - \sum_{j \in N^0 \cap O} x_j \leq q \tag{13}$$

Theorem 3.10. *Inequality* (13) *is valid and defines a facet of P if the following conditions hold:*

1. $b_1 + b_2 - b_0 > 0$ *is positive and odd,* $N^2 \setminus O \neq \emptyset$*, and* $|b_1 - b_2| \leq b_0 - 1$
2. $b_i \geq q + 2$ *for all* $i \in M[j], j \in N^2 \setminus O$

Condition 1 is necessary for (13) *to define a facet. A further necessary condition is $b_i \geq q + 1$ for all $i \in M[j], j \in N^2 \setminus O$.*

Another nontrivial facet-defining inequality is derived as follows. Let us assume there is a column $k \in N \setminus M$ with the property that k has at most one entry in common with any other column in A, i. e., it holds that $|M[k] \cap M[j]| \leq 1$ for every $j \in N, j \neq k$. Let $N_k^0, N_k^1 \subseteq N$ denote the columns of A that have no entry in common with column k and that have exactly one entry in common with column k, respectively. We denote by ℓ_k^0 the length of the longest odd column in N_k^0, thus $\ell_k^0 = \max\{\ell_j \,:\, j \in N_k^0 \cap O\}$ and by r the remainder of the division of $b_k - b_0$ by $s := 2 \lfloor \ell_k / 2 \rfloor$. Set $p := \lfloor (b_k - b_0) / s \rfloor$. Now we aggregate all rows corresponding to $M[k]$ and the cardinality constraint, and consider the $1/s$-MIR inequality

$$(s - r)\, x_k - \sum_{j \in N_k^0 \cap O} x_j \leq (s - r)\, p \tag{14}$$

Theorem 3.11. *Inequality* (14) *is valid and defines a facet of P if the following conditions hold:*

1. $r \geq 1$, $b_k > b_0$, *and* $b_i > p$ *for all* $i \in M[k]$
2. *Either* $M \backslash M[k] = \emptyset$ *or* $b(M) - b_k \geq s - r + \ell_k^0$.

Condition 1 is necessary for (14) *to define a facet.*

4 Concluding remarks

In this paper we have discussed the complexity and the polyhedral properties of a combinatorial structure appearing in the context of dimensioning cuts in two-layer networks. The corresponding problem has been described as a cardinality constrained packing integer program and has been proven to be strongly $\mathcal{NP}$ -hard. Based on the complete description of the smallest nontrivial instance two classes of facet defining inequalities have been identified. These inequalities generalize the well known cutset inequalities to two network layers. Future work involves the separation of these inequalities and evaluation of the practical value of these inequalities.

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