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Linear Fractional Order Differential Equations and their Solution

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Abstract

This project deals with the solutions to the equation

$$D^{\nu}x(t) = -\rho^{\nu}x(t) + f(t) \tag{0.0.0}$$

where $x:[0,\infty)\to\mathbf{R}$, $f:[0,\infty)\to\mathbf{R}$, $f\in L^1([0,\infty))$ and piecewise continuous, $t\geq 0$, and $\rho,\nu\in\mathbf{R}^+$.

Equations of this form have already been analysed for $\nu\in\mathbf{Z}$ and $\nu\in(0,2)$. The case $\nu\in(2,3)$ has recently been analysed numerically. This project gives a treatment of (0.0.0) for $\nu>2$.

Chapters 1,2,3, and 4 provided background material. Chapter 5 describes our new results on the behaviour of solutions to (0.0.0).

Declaration

This work is original and has not been submitted previously in support of any qualification or course.

Signed ________________________

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Contents

Introduction

The fractional calculus provides a generalisation of the notions of differentiation and integration of classical calculus. It arises out of attempts to solve integro-differential equations. An example of such a problem is the tautochrone problem that was studied by Abel, Oldham and Spannier (1974).

The purpose of this project is to investigate the solution of the linear fractional order differential equation:

$$D^{\nu}x(t) = -\rho^{\nu}x(t) + f(t) \tag{0.1.1}$$

where $x:[0,\infty)\to\mathbf{R}$, $f:[0,\infty)\to\mathbf{R}$, $f \in L^1([0,\infty))$ and piecewise continuous, and $\rho,\nu \in \mathbf{R}^+$; principally where $\nu \in (2,3)$.

There are two possible formulations of the initial condition, which are identical in the case $\nu \in \mathbf{Z}$. The mathematically simplest takes the limit as $t \to 0^+$ of fractional integrals as the starting values. The other requires only the same set of initial values as would be required for the ordinary differential equation of order $n \in \mathbf{N}$ where $n-1<\nu\leq n$.

These solutions are derived by the use of the Laplace transformation and are expressed in terms of generalized Mittag-Leffler functions.

Section 1 - Transcendental Function Theory

In tackling the fractional calculus it is convenient to survey the properties of two transcendental functions. These are the gamma function and the Mittag-Leffler function.

The gamma function, which includes as a special case the factorial numbers, provides a useful way of expressing the coefficients of terms that appear in summations, such as those which occur in the expansion of functions like $\frac{1}{\sqrt{1-x}}$. Theorems about the gamma function then allow us to prove important properties of the fractional differential operator.

The generalised Mittag-Leffler function includes as special cases the trigonometric and hyperbolic functions which occur when we solve the equation $D^{\nu}x(t) = -\rho^{\nu}x(t) + f(t)$ for integer values of ν.

Section 2 - Fractional Calculus

The formulation of the fractional calculus, and the supporting proofs, given here is principally due to Oldham and Spannier (1974). The fractional derivative can be approached either as a derivative or as an integral operator. Consequently two different definitions of the fractional differential operator are given. The definition of Grünwald-Letnikov generalises the notion of a derivative, whilst that of Riemann-Liouville generalises the notion of an integral by means of the Abel integral and ordinary differentiation. These two definitions have been shown to be equivalent and Oldham and Spannier's proof of this fact is given. A theorem Samko et al (1993) giving conditions for a linear fractional O.D.E. to possess a unique and continuous solution is given.

Section 3 - Laplace Transform Theory

We state a theorem on the Laplace transform of a convolution integral and a similar result for the Laplace transform of an integer order derivative. The Laplace transforms of the Mittag-Leffler function and the generalised Mittag-Leffler function, originally derived by Agarwal and Humbert, Erdélyi (1953), are also given.

Section 4 - Linear Fractional Differential Equations of order $\nu\in(0,2]$

This section summarizes the known results for the cases $\nu\in(0,1]$ and $\nu\in(1,2]$, as given by Gorenflow and Rutman (1994). The two cases $\nu\in(0,1]$ and $\nu\in(1,2]$ are dealt with separately because of qualitative differences in the behaviour of their respective solutions. The solutions to the equations are obtained by use of the Laplace transform technique. We find a description of the solution in terms of generalized Mittag-Leffler functions. An alternative proof, to that found in Gorenflow and Rutman, for the linear fractional o.d.e with incorporated conditions for $\nu\in(1,2]$ is given.

Section 5 - Linear Fractional Differential Equations where $\nu\in(2,\infty)$

This section investigates the third order case. Firstly by considering the 3rd order case and then by considering the case $\nu\in(2,3]$. The solution is again found to be expressible in terms of Mittag-Leffler functions and because for $\nu > 2$ these function have similar behaviour the analysis for the case $\nu\in(2,3]$ can be extended to cover all cases where $\nu\in(2,\infty)$.

Unstarred theorems are quoted from other sources, * theorems are from other sources but with a new proof, ** theorems are new.

Transcendental Function Theory

These functions are necessary for our discussion of the fractional calculus and the solution of (0.1.1) for two reasons. Firstly because the Grünwald-Letnikov definition of a fractional derivative is by means of a sum whose coefficients are the ratios of gamma functions, and secondly the solution to the differential equation $D^{\alpha} f = -\rho^{\alpha} f$, can be expressed in terms of the generalised Mittag-Leffler function.

1.1 The Gamma Function

The gamma function denoted $\Gamma(z)$ can be defined in several equivalent ways. See Erdélyi (1953), Whittaker and Watson (1927).

Infinite Product Definition

For $z \in \mathbf{C}$ $\qquad \Gamma(z) \equiv \lim_{N\to\infty}\left[\dfrac{N!\,N^{z}}{z(z+1)(z+2)\cdots(z+N)}\right]$ (1.1.1)

From this we see that as $z \to -n,\, n \in \mathbf{N},\, |\Gamma(z)| \to \infty$ or equivalently

$\dfrac{1}{|\Gamma(z)|} \to 0$.

Integral Transform Definition

$$\Gamma(z) \equiv \int_0^\infty t^{z-1}\exp(-t)dt \ , \qquad \text{where } z > 0 . \tag{1.1.2}$$

If we integrate this integral representation by parts we obtain the recurrence relation:

$$\Gamma(z) = \left[\frac{t^{z}}{z}\exp(-t)\right]_0^\infty + \frac{1}{z}\int_0^\infty t^{z}\exp(-t)dt$$

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

Hence $\Gamma(z+1) = z\Gamma(z)$ (1.1.3)

And therefore $\Gamma(n+1) = n!, \quad n \in \mathbf{N}$
(1.1.4)

Stirling's Theorem

For large $x > 0$

$$\Gamma(x) = x^{x-\frac{1}{2}} e^{-x}\sqrt{2\pi}\,\{1 + o(1)\}$$
(1.1.5)

Whittaker and Watson (1927)

Ratios of Gamma Functions

A relationship between the difference of two ratios of gamma functions, which we will need later on, is given by

$$\frac{\Gamma(j-q)}{\Gamma(j+1)} - \frac{\Gamma(j-q-1)}{\Gamma(j)} = \frac{\Gamma(j-q) - j\Gamma(j-q-1)}{\Gamma(j+1)}$$

$$= \frac{\Gamma(j-q-1)\{(j-q)-j\}}{\Gamma(j+1)}$$

$$= \frac{-q\Gamma(j-q-1)}{\Gamma(j+1)}$$

$$= \frac{\Gamma(-q)\Gamma(j-q-1)}{\Gamma(-q-1)\Gamma(j+1)}$$

(1.1.6)

The asymptotic expansion of the ratio of two gamma functions is given by

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\left[1 + \frac{(\alpha-\beta)(\alpha+\beta-1)}{2z} + O\left(\frac{1}{z^2}\right)\right] \quad (1.1.7)$$

Erdélyi (1953).

1.2 Mittag-Leffler Functions

A more complete bibliography and listing of the properties of the Mittag-Leffler function and the generalised Mittag-Leffler function is given in Erdélyi (1955).

The Mittag-Leffler function includes as special cases some of the trigonometric and hyperbolic functions. It can be represented by either a power series or a contour integral:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} = \frac{1}{2\pi i}\int_C \frac{t^{\alpha-1}e^t}{t^\alpha - z}\,dt \quad (1.2.1)$$

where the path of integration C is a loop which starts and ends at $-\infty$, and encircles the circular disc $|t| \le |z|^{1/\alpha}$ in the positive sense: $-\pi \le \arg t \le \pi$ on C.

Familiar examples of the Mittag-Leffler function are $E_1(z) = \exp z$ and $E_2(-z^2) = \cos z$, (ibid.).

For $\alpha > 0$, $E_\alpha(z)$ is an entire function of order $\frac{1}{\alpha}$, (ibid.).

The Laplace transform of $E_\alpha(z)$ may occur, in applications, when we attempt to calculate the inverse Laplace transform of functions such as $\frac{s^\alpha}{s^\alpha - 1}$, $\alpha > 0$, a proof of this is given in Chapter 3.

For $\alpha \in [0,1]$, $x \geq 0$, $E_\alpha(-x)$ is completely monotonic

i.e. $\quad (-1)^n \frac{d^n E_\alpha(-x)}{dx^n} \geq 0, n \in \mathbf{N}$,

the proof is based on the integral representation, (ibid.).

For $\alpha \in (0,2)$, $|\arg z| \leq \frac{\alpha\pi}{2}$ and $z \to \infty$ we have

$$E_\alpha(z) = \frac{1}{\alpha} \exp z^{1/\alpha} + O\left(|z|^{-1}\right) \qquad (1.2.2)$$

(ibid.).

Zeros of the Mittag-Leffler Function

There are three cases to be considered for the distribution of the zeros of the Mittag-Leffler function. These three cases will be shown to correspond to the behaviour of the solutions of the target differential equation $D^\alpha f = -\rho^\alpha f$. The three cases are:

(i) If $\alpha \in (0,1]$ the function has no real zeros.

(ii) If $\alpha \in (1,2)$ the function has an odd number of negative zeros.

(iii) If $\alpha \geq 2$ the Mittag-Leffler function has infinitely many zeros on the negative real line and zeros nowhere else.

The integral $\int_0^\infty e^{-t} E_\alpha(t^\alpha z) dt = \frac{1}{1-z}$, $z \in \mathbf{C}$ was evaluated by Mittag-Leffler who showed that the region of convergence of the integral contains the unit circle and is bounded by the line $\operatorname{Re} z^{1/\alpha} = 1$. (ibid.)

Asymptotic Behaviour of Mittag-Leffler Functions

Mittag-Leffler showed that for $0<\alpha<2$ if $2\pi-\alpha\frac{\pi}{2}>\arg(z)>\alpha\frac{\pi}{2}$ then $\lim_{|z|\to\infty} E_{\alpha}(z)=0$. In particular $\lim_{t\to-\infty} E_{\alpha}(t)=0$.

Additionally we find (ibid.) the following result on the asymptotic behaviour of the Mittag-Leffler function :

For $|\arg(-z)|<\left(1-\frac{\alpha}{2}\right)\pi, z\to\infty, N\in\mathbf{N}$,

$$E_{\alpha}(z)=-\sum_{n=1}^{N-1}\frac{z^{-n}}{\Gamma(1-\alpha n)}+O\left(|z|^{-N}\right) \tag{1.2.3}$$

When $\alpha>2$ Mittag-Leffler showed that on the negative real axis

$$E_{\alpha}(-t)=\sum_{n=0}^{m-1}\frac{2}{\alpha}\exp\left(t^{\frac{1}{\alpha}}\cos\frac{2n+1}{\alpha}\pi\right)\cos\left(t^{\frac{1}{\alpha}}\sin\frac{2n+1}{\alpha}\pi\right) \tag{1.2.4}$$

where $t>0$ and $\frac{\alpha}{2}<2m<\frac{3\alpha}{2}$, Mittag-Leffler (1905).

1.3 Generalized Mittag-Leffler Function

The generalized Mittag-Leffler function is defined by the addition of a second parameter, Erdélyi (1955).

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+\beta)}=\frac{1}{2\pi i}\int_{C}\frac{t^{\alpha-\beta}e^{t}}{t^{\alpha}-z}dt \text{ where } \alpha,\beta>0 \tag{1.3.1}$$

where the path of integration C is a loop which starts and ends at $-\infty$, and encircles the circular disc $|t|\leq|z|^{1/\alpha}$ in the positive sense: $-\pi\leq\arg t\leq\pi$ on C.

Familiar examples of the generalised Mittag-Leffler function are $zE_{2,2}(z^2)=\sinh z$, and $zE_{2,2}(-z^2)=\sin z$ (ibid.).

Theorem 1.3

From Gorenflow and Rutman (1994) we have the equivalence

$$\alpha E_{\alpha}'\left(-(\rho t)^{\alpha}\right)=t^{\alpha-1}E_{\alpha,\alpha}\left(-(\rho t)^{\alpha}\right) \tag{1.3.2}$$

<u>Proof</u>

$$\frac{d}{dt}E_{\alpha}\left(-(\rho t)^{\alpha}\right)=-\alpha\rho^{\alpha}t^{\alpha-1}E_{\alpha}'\left(-(\rho t)^{\alpha}\right) \tag{1.3.3}$$

$$\frac{d}{dt}E_{\alpha}\left(-(\rho t)^{\alpha}\right)=\frac{d}{dt}\sum_{k=0}^{\infty}\frac{(-\rho t)^{\alpha k}}{\Gamma(\alpha k+1)}=-\rho^{\alpha}\sum_{k=1}^{\infty}\frac{(-\rho t)^{\alpha k-1}}{\Gamma(\alpha(k-1)+\alpha)}$$

$$=-\rho^{\alpha}t^{\alpha-1}\sum_{k=0}^{\infty}\frac{\left(-(\rho t)^{\alpha k}\right)}{\Gamma(\alpha k+\alpha)}$$

$$=-\rho^{\alpha}t^{\alpha-1}E_{\alpha,\alpha}\left(-(\rho t)^{\alpha}\right) \qquad (1.3.4)$$

Equating 1.3.3 with 1.3.4 gives 1.3.2.

When we use the Laplace transform to solve fractional linear O.D.E's the generalized Mittag-Leffler function occurs in the form:

$$E_{\nu,\nu-j}\left(-(\rho t)^{\nu}\right)=\sum_{k=0}^{\infty}\frac{-(\rho t)^{\nu k}}{\Gamma(\nu k+(\nu-j))}=\sum_{k=0}^{\infty}\frac{-(\rho t)^{\nu k}}{\Gamma(\nu(k+1)-j)} \qquad (1.3.5)$$

where $0\leq j\leq[\nu]$.

However there is an the integral relationship between $E_{\nu,\nu-j}$ and $E_{\nu,\nu-j-1}$ given by

$$\int_0^t t^{\nu-j}E_{\nu,\nu-j}\left(-(\rho t)^{\nu}\right)dt=\int_0^t\sum_{k=0}^{\infty}\frac{(-1)^k\rho^{\nu k}t^{\nu(k+1)-j}}{\Gamma(\nu(k+1)-j)}dt$$

$$=\left[\sum_0^{\infty}\frac{(-1)^k\rho^{\nu k}t^{\nu(k+1)-(j-1)}}{\Gamma(\nu(k+1)-(j-1))}\right]_0^t$$

$$=\left[t^{\nu-(j-1)}\sum_0^{\infty}\frac{(-1)^k(\rho t)^{\nu k}}{\Gamma(\nu(k+1)-(j-1))}\right]_0^t$$

$$=t^{\nu-(j-1)}E_{\nu,\nu-(j-1)}\left(-(\rho t)^{\nu}\right) \qquad (1.3.6)$$

or alternatively, Erdélyi (1955)

Theorem 1.4

$$\left(\frac{d}{dz}\right)^{m}\left[z^{\beta-1}E_{\alpha,\beta}\left(z^{\alpha}\right)\right]=z^{\beta-m-1}E_{\alpha,\beta-m}\left(z^{\alpha}\right) \tag{1.3.7}$$

where $m \in \mathbf{N}$.

Therefore since in the cases we shall be studying $\alpha-\beta \in \mathbf{Z}$ our solutions will, potentially, be expressible in terms of Mittag-Leffler functions and their derivatives.

2 The Fractional Calculus

2.1 The Grünwald-Letnikov Fractional Derivative of Order q

The Grünwald-Letnikov definition of the fractional derivative of a function generalises the notion of a backwards difference quotient of integer order. In the case $q = 1$, if the limit exists, the Grünwald-Letnikov fractional derivative is the left derivative of the function.

The Grünwald-Letnikov fractional derivative of order q of the function f is given by

$$ {}_aD_x^q f(x) = \lim_{N\to\infty}\left\{ \frac{\left[\frac{x-a}{N}\right]^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(x - j\left[\frac{x-a}{N}\right]\right) \right\} \qquad (2.1.1)$$

where $q \in \mathbf{C}$. Oldham & Spanier (1973)

If $q = -1$ then we have a Riemann sum which is the first integral.

If $q = 1$ then we have $\lim_{N\to\infty}\left\{ \frac{\left[f(x) - f\left(x - \left[\frac{x-a}{N}\right]\right)\right]}{\left[\frac{x-a}{N}\right]} \right\}$ which is the left derivative of f at x.

The fractional derivative of the unit function

Although the integer derivative of a constant function is zero this is not true for fractional derivatives of arbitrary order. Taking $a = 0$ and applying the above definition we have

$$D^q 1 = \lim_{N\to\infty}\left\{ \left[\frac{N}{x}\right]^q \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} \right\}$$

Using $\sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} = \frac{\Gamma(N-q)}{\Gamma(1-q)\Gamma(N)}$ and $\lim_{N\to\infty} N^q \frac{\Gamma(N-q)}{\Gamma(N)} = 1$ from ibid. gives

$$D^q 1 = \frac{x^{-q}}{\Gamma(1-q)}. \qquad (2.1.2)$$

For a function of constant value C we have

$$D^q C = \lim_{N\to\infty}\left\{\left[\frac{N}{x}\right]^q \sum_{j=0}^{N-1}\frac{C\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}\right\}$$
$$= C\lim_{N\to\infty}\left\{\left[\frac{N}{x}\right]^q \sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}\right\}$$
$$= CD^q 1$$

In particular if $C=0$ the $D^q 0 = 0D^q 1 = 0$, $x \neq 0$.

Example 2.1

From this result we see that $DD^q 1 = \dfrac{-qx^{-q-1}}{\Gamma(1-q)}$ whilst $D^q D1 = 0$

The following theorem gives a result on the addition of the D operators exponents.

Theorem 2.1

Let $n \in \mathbf{N}, q \in \mathbf{R}$ then if $\dfrac{d^n\left({}_aD_x^q f\right)}{dx^n}$ and ${}_aD_x^{n+q} f$ exist then

$$\frac{d^n\left({}_aD_x^q f\right)}{dx^n} = {}_aD_x^{n+q} f \qquad (2.1.3)$$

ibid.

Proof

Since q is arbitrary it suffices to prove the hypothesis in the cases $n=1$.

Let $\delta_N x = \dfrac{[x-a]}{N}$

Then we can rewrite the definition as

$${}_aD_x^q f(x) = \lim_{N\to\infty}\left\{\frac{[\delta_N x]^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)} f(x - j\delta_N x)\right\}$$

and partitioning the interval $[a, x-\delta_N x]$ into $N-1$ equally spaced subintervals we have

$${}_aD_x^q f(x-\delta_N x) = \lim_{N\to\infty}\left\{\frac{[\delta_N x]^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-2}\frac{\Gamma(j-q)}{\Gamma(j+1)} f(x-\delta_N x - j\delta_N x)\right\}$$
$$= \lim_{N\to\infty}\left\{\frac{[\delta_N x]^{-q}}{\Gamma(-q)}\sum_{j=1}^{N-1}\frac{\Gamma(j-q-1)}{\Gamma(j)} f(x - j\delta_N x)\right\}$$

Therefore because the two series are uniformly convergent, in some neighbourhood of x, we can differentiate them term by term. Which gives:

$$\frac{d\left({}_aD_x^q f(x)\right)}{dx} = \lim_{N\to\infty}\left\{[\delta_N x]^{-1}\left[{}_aD_x^q f(x) - {}_aD_x^q f(x-\delta_N x)\right]\right\}$$

$$= \lim_{N\to\infty}\left\{\frac{[\delta_N x]^{-q-1}}{\Gamma(-q)}\left[\Gamma(-q)f(x)+\sum_{j=1}^{N-1}\left\{\frac{\Gamma(j-q)}{\Gamma(j+1)}-\frac{\Gamma(j-q-1)}{\Gamma(j)}\right\}f(x-j\delta_N x)\right]\right\}$$

But by 1.1.6

$$\frac{\Gamma(j-q)}{\Gamma(j+1)}-\frac{\Gamma(j-q-1)}{\Gamma(j)}=\frac{\Gamma(-q)\Gamma(j-q-1)}{\Gamma(-q-1)\Gamma(j+1)}$$

Therefore

$$\frac{d\left({}_aD_x^q f(x)\right)}{dx}=\lim_{N\to\infty}\left\{\frac{[\delta_N x]^{-q-1}}{\Gamma(-q-1)}\left[\sum_{j=1}^{N-1}\frac{\Gamma(j-q-1)}{\Gamma(j+1)}f(x-j\delta_N x)\right]\right\}$$
$$={}_aD_x^{q+1}f(x)$$

2.2 The Riemann-Liouville Fractional Derivative

The fractional integral of order q is given by the Abel integral

$${}_aJ_x{}^q(f)=\frac{1}{\Gamma(q)}\int_a^x (x-y)^{q-1}f(y)dy \qquad (2.2.1)$$

where $\operatorname{Re} q>0$.

An alternative approach to the definition of the fractional derivative is by means of the fractional integral.

Riemann-Liouville Fractional Derivative

The Riemann-Liouville definition of the fractional derivative is given by

$${}_aD_x{}^q f={}_aD_x{}^n\,{}_aJ_x{}^{n-q}f, n-1<q\le n,\ n\in\mathbf{N}. \qquad (2.2.2)$$

Theorem 2.2

Let $\lim_{x\to a}(x-a)^{1+q}f(x)=0$, $f\in L^1[a,\infty)$, $q<0$ then

$${}_aD_x^q f(x)={}_aJ_x{}^{-q}(f) \qquad (2.2.3)$$

and for $q\ge 0$: $\quad {}_aD_x^q f(x)={}_aD_x^n\,{}_aJ_x{}^{n-q}(f)$

where ${}_aD_x^q$ denotes the Grünwald-Letnikov version of the fractional derivative and $n\in\mathbf{N}$ such that $q-n\le 0$. Oldham and Spanier (1974)

Proof

First assume $q<-2$.

Let $0<\delta=\dfrac{x-a}{N}$, $f(x_j)=f(x-\delta j)$, and suppose $\varepsilon>0$ then $\exists N_1,N_2\in\mathbf{N}$ such that

$$-\frac{\varepsilon}{2}<{}_aD_x^q f(x)-\left\{\frac{\delta^{-q}}{\Gamma(-q)}\sum_{j=0}^{N_1-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}f(x_j)\right\}<\frac{\varepsilon}{2} \tag{2.2.4}$$

$$-\frac{\varepsilon}{2}<\sum_{j=0}^{N_2-1}\frac{f(x_j)\delta}{\Gamma(-q)[j\delta]^{q+1}}-{}_aJ_x^{-q}f(x)<\frac{\varepsilon}{2} \tag{2.2.5}$$

Let $N=N_1+N_2$

$$\left|{}_aD_x^q f(x)-{}_aJ_x^{-q}f(x)\right|<\left|\varepsilon+\frac{\delta^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}f(x_j)-\sum_{j=0}^{N-1}\frac{f(x_j)\delta}{\Gamma(-q)[j\delta]^{q+1}}\right| \tag{2.2.6}$$

therefore

$$\begin{aligned}\frac{\delta^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}f(x_j)-\sum_{j=0}^{N-1}\frac{f(x_j)\delta}{\Gamma(-q)[j\delta]^{q+1}}\\ =\frac{\delta^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}f(x_j)\left[\frac{\Gamma(j-q)}{\Gamma(j+1)}-\frac{1}{j^{1+q}}\right]\end{aligned} \tag{2.2.7}$$

From formula (1.1.7) for the asymptotic expansion of the ratio of two gamma functions we know that

$$\frac{\Gamma(j-q)}{\Gamma(j+1)}=j^{-1-q}\left[1+\frac{q(q+1)}{2j}+O(j^{-2})\right]\ \text{as}\ j\to\infty$$

therefore for N and J sufficiently large and with $J<N-1$

$$\frac{\delta^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}f(x_j)\left[\frac{\Gamma(j-q)}{\Gamma(j+1)}-j^{-1-q}\right]=\frac{[x-a]^{-q}}{\Gamma(-q)}\left\{\sum_{j=0}^{J-1}f(x_j)N^q\left[\frac{\Gamma(j-q)}{\Gamma(j+1)}-j^{-1-q}\right]+\frac{1}{N}\sum_{j=J}^{N-1}f(x_j)\left[\frac{j}{N}\right]^{-2-q}\left[\frac{q(q+1)}{2N}+\frac{O(j^{-1})}{N}\right]\right\} \tag{2.2.8}$$

so

$$\begin{aligned}\lim_{N\to\infty}\sum_{j=0}^{J}f(x_j)N^q\left[\frac{\Gamma(j-q)}{\Gamma(j+1)}-j^{-1-q}\right]&=\left\{\sum_{j=0}^{J}f(x_j)\left[\frac{\Gamma(j-q)}{\Gamma(j+1)}-j^{-1-q}\right]\right\}\lim_{N\to\infty}N^q\\&=0\end{aligned}$$

and since $q\le -2$

$$\left|\lim_{N\to\infty}\frac{1}{N}\sum_{j=J}^{N-1} f\left(x_j\right)\left[\frac{j}{N}\right]^{-2-q}\left[\frac{q(q+1)}{2N}+\frac{O\left(j^{-1}\right)}{N}\right]\right| \le \left|\lim_{N\to\infty}\frac{1}{N}\sum_{j=J}^{N-1} f\left(x_j\right).1.C.\frac{1}{N}\right|$$

$$=\left|\left(\lim_{N\to\infty}\frac{1}{N}\sum_{j=J}^{N-1} f\left(x_j\right)\right)\left(C.\lim_{N\to\infty}\frac{1}{N}\right)\right| \quad (2.2.9)$$

$$\le\left|\left(\int_a^x f(x)dx\right).C.\lim_{N\to\infty}\frac{1}{N}\right| = 0$$

For $q>-2$ choose $n\in\mathbf{N},\left(q-n\right)\le -2$. Then applying theorem 2.1 gives for the Grünwald-Letnikov derivative

$$_aD_x^q f = {}_aD_x^n\,{}_aD_x^{q-n} f = {}_aD_x^n\,{}_aJ_x^{n-q} f \qquad (2.2.10)$$

The Fractional Derivative of x^p

The Riemann-Liouville enables us to calculate a formula for the fraction derivative of a power of x which is the generalisation of the case from elementary calculus. We first consider the case when $q<0$.

$$D^q x^p = J^{-q}x^p = \frac{1}{\Gamma(-q)}\int_0^x \frac{y^p}{(x-y)^{q+1}}dy \text{ substituting } y=ux \text{ gives}$$

$$=\frac{x^{p-q}}{\Gamma(-q)}\int_0^x u^p(1-u)^{-q-1}du$$

$$=\frac{\Gamma(p+1)x^{p-q}}{\Gamma(p-q+1)} \qquad (2.2.11)$$

For $q>0$ we apply the previous theorem on the addition of operator exponents thus

$$D^q x^p = D^n D^{q-n} x^p \qquad \text{where } n\in\mathbf{N}, q-n\le 0$$

$$=D^n\frac{\Gamma(p+1)x^{p+n-q}}{\Gamma(p+n-q+1)}$$

$$=\frac{\Gamma(p+1)x^{p-q}}{\Gamma(p-q+1)}$$

2.3 Conditions under which $D^N D^q = D^q D^N$ holds

Whilst the condition $D^N D^q = D^q D^N$ is not true in general, as was shown in example 2.1, there exists a special case where identity occurs.

Theorem 2.3

Let N be a positive integer. Let f be a function such that $D^{N-1}f$ is continuous on $(0,\infty)$ and $D^N f$ is piecewise continuous on $(0,\infty)$, and let $q>0$. Then

$$D^{-q}f(t)=D^{-q-N}\left(D^N f(t)\right)-\sum_{k=0}^{N-1}\frac{t^{q+k}}{\Gamma(q+k+1)}D^{(k)}f(0) \tag{2.3.1}$$

Miller and Ross (1993)

Proof

The proof is by repeated iteration.

If $N=1$ we need to show that $D^{-q-1}\left[Df(t)\right]=D^{-q}f(t)-\frac{f(0)}{\Gamma(q)}t^q$

(2.3.2)

Let $\varepsilon>0,\eta>0$ be given. Then $(t-\xi)^{q-1}$ and $f(\xi)$ are differentiable on $[\eta,t-\varepsilon]$. Thus integrating by parts we have

$$\begin{aligned}\int_\eta^{t-\varepsilon}(t-\xi)^q\left[Df(\xi)\right]d\xi&=\left[(t-\xi)^q f(\xi)\right]_\eta^{t-\varepsilon}+q\int_\eta^{t-\varepsilon}(t-\xi)^{q-1}f(\xi)d\xi\\&=\varepsilon^q f(t-\varepsilon)-(t-\eta)^q f(\eta)+q\int_\eta^{t-\varepsilon}(t-\xi)^{q-1}f(\xi)d\xi\end{aligned}\tag{2.3.3}$$

Taking limits as ε and η goes to zero independently and dividing by $\Gamma((q+1)$ gives the result.

If $N=2$ we replace q by $q+1$ and f by Df in (2.3.2) which gives

$$\begin{aligned}D^{-q-2}\left[D^2f(t)\right]&=D^{-q-1}\left[Df\right]-\frac{Df(0)}{\Gamma(q+2)}t^{q+1}\\&=D^{-q}f(t)-\frac{f(0)}{\Gamma(q+1)}t^q-\frac{Df(0)}{\Gamma(q+2)}t^{q+1}\end{aligned}\tag{2.3.4}$$

2.4 Linear Fractional O.D.E's.

The following definition is given by Samko et al (1993):

The Cauchy-type problem is

$$D^\nu x(t)=f(t,x),n-1<\nu\le n,n=1,2,\cdots\tag{2.4.1}$$

with initial conditions

$$D^{\nu-k}x(t)\Big|_{x=+0}=b_k,k=1,2,\cdots,n$$

Let $R_n=\left\{(t,x)\in D{:}0<t\le h,\left|t^{n-\nu}x(t)-\frac{b_n}{\Gamma(\nu-n+1)}\right|\le a\right\}$

with $a>\sum_{k=0}^{n-1}\frac{h^{n-k}b_k}{\Gamma(\nu-n+1)}$ where a,h,b_0 are certain constants and $R_n\subset D\subseteq\mathbf{R}^n$.

Theorem 2.4

Let $f(t,x)$ be a real valued function continuous in D and Lipschitzian with respect to x:

$$\left|f(t,x_1)-f(t,x_2)\right|\leq A\left|x_1-x_2\right|$$

and let the condition $\left|\sup_{(t,x)\in D} f(t,x)\right| = b_0 < \infty$ hold.

Then there exists a unique continuous solution of the Cauchy-type problem (2.4.1) for $n=1,2,\cdots$ in the domain $R_1 \subset D$, Samko et al (1993).

2.5 Incorporated Initial Conditions

This is a generalisation, of the initial value problem, proposed by Mainardi (1994) and Nonnenmacher and Glöckle (1991). Consider the linear fractional O.D.E. given by

$$D^{\nu}\left(u(t)-\sum_{i=0}^{n-1}\frac{u_0^{(i)}t^i}{i!}\right) = -\rho^{\nu}u + f \tag{2.5.1}$$

where $0<n-1<\nu\leq n, n\in\mathbf{N}$, $u^{(i)}(0)=u_0^{(i)}, i<n$.

If u satisfies the hypotheses of theorem 2.3 we can rewrite equation (2.5.1) as $\quad DD^{\nu-n}D^{n-1}\left(u(t)-\sum_{i=0}^{n-1}\frac{u_0^{(i)}t^i}{i!}\right) = -\rho^{\nu}u + f$

which after differentiating out becomes

$$DJ^{n-\nu}\left(u^{(n-1)}(t)-u_0^{(n-1)}\right) = -\rho^{\nu}u + f \tag{2.5.2}$$

which is a convolution integral. Chapter 4 shows how this integral equation can be solved by the application of Laplace transform theory.

3 Laplace Transform

3.1 Laplace Transform and Convolution Integrals

The Abel integral is a convolution integral. There is a powerful result connecting the Laplace transform of a convolution integral of two functions with the product of the Laplace transforms of those two functions.

Convolution Theorem for Fourier Transforms

Definition

If f is Riemann integrable on $[a, b]$ for every $b \geq a$, and if the limit

$\lim_{b \to +\infty} \int_a^b f(x)dx$ exists then f is said to be improper Riemann-integrable on $[a,+\infty)$. Apostol (1977)

Theorem

If $f, g \in L(\mathbf{R})$ and at least one of them is continuous and bounded on $\mathbf{R}$ and $h = f * g$ then for every real u we have

$$\int_{-\infty}^{\infty} h(x)e^{-ixu}dx = \left(\int_{-\infty}^{\infty} f(t)e^{-ixu}dx\right)\left(\int_{-\infty}^{\infty} f(t)e^{-ixu}dx\right) \qquad (3.1.1)$$

Furthermore the integral on the left exists both as a Lebesgue integral and an improper Riemann integral. Apostol (1977)

From this theorem we can deduce its analogue for the Laplace transform. Namely

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g) \qquad (3.1.2)$$

Applying this theorem to the Laplace transform of a fractional integral for $\mathrm{Re}(q) > 0$ yields:

$$\mathcal{L}\left(J^q f\right) = \mathcal{L}\left(\frac{1}{\Gamma(q)}\int_a^x [x-y]^{q-1} f(y)dy\right) = \frac{1}{\Gamma(q)}\mathcal{L}\left(\left[x^{q-1}\right]\right)\mathcal{L}(f)$$

Thus $\mathcal{L}\left(J^q f\right) = \dfrac{1}{\Gamma(q)}\dfrac{\Gamma(q)}{s^q}\bar{f} = \dfrac{\bar{f}}{s^q}$ (3.1.3)

Where $\mathcal{L}(f)=\bar{f}$.

3.2 The Laplace Transform of Derivatives

The Laplace Transform of an nth derivative is given by the formula Doetsch (1961)

$$\mathcal{L}\left(D^n f\right)=s^n \bar{f}(s)-\sum_{k=0}^{n-1} D^k f(+0) s^{n-k-1} \tag{3.2.1}$$

3.3 Laplace Transform of Mittag-Leffler Functions

Theorem 3.1

The Mittag-Leffler function $E_{\alpha}\left(-(\rho t)^{\alpha}\right)$ is the inverse Laplace transform of the function $\dfrac{s^{\alpha-`1}}{s^{\alpha}+\rho^{\alpha}}$.

Proof

$$\frac{s^{\alpha-1}}{s^{\alpha}+\rho^{\alpha}}=\frac{1}{s}\left(1+\left(\frac{\rho}{s}\right)^{\alpha}\right)^{-1}=\left(\frac{1}{s}-\frac{\rho^{\alpha}}{s^{\alpha+1}}+\frac{\rho^{2\alpha}}{s^{2\alpha+1}}-\frac{\rho^{3\alpha}}{s^{3\alpha+1}}+-\cdots\right) \tag{3.3.1}$$

Inverting (3.3.1) with respect to s gives

$$1-\frac{(\rho t)^{\alpha}}{\Gamma(\nu+1)}+\frac{(\rho t)^{2\alpha}}{\Gamma(2\nu+1)}-\frac{(\rho t)^{3\alpha}}{\Gamma(3\nu+1)}+-\cdots=E_{\nu}\left(-(\rho t)^{\alpha}\right)$$

Theorem 3.3.2

$t^{\beta-1}E_{\alpha,\beta}\left(-(\rho t)^{\nu}\right)$ is the inverse Laplace transform of $\dfrac{s^{\alpha-\beta}}{s^{\alpha}+\rho^{\alpha}}$.

Proof

$$\frac{s^{\alpha-\beta}}{s^{\alpha}+\rho^{\alpha}}=\frac{1}{s^{\beta}}\left(1-\frac{\rho^{\alpha}}{s^{\alpha}}+\frac{\rho^{2\alpha}}{s^{2\alpha}}-\frac{\rho^{3\alpha}}{s^{3\alpha}}+-\cdots\right)$$

$$=\frac{1}{s^{\beta}}-\frac{\rho^{\alpha}}{s^{\alpha+\beta}}+\frac{\rho^{2\alpha}}{s^{2\alpha+\beta}}-\frac{\rho^{3\alpha}}{s^{3\alpha+\beta}}+-\cdots \tag{3.3.2}$$

Inverting (3.3.2) with respect to s gives

$$\frac{t^{\beta-1}}{\Gamma(\beta)}-\frac{t^{\beta-1}(\rho t)^{\alpha}}{\Gamma(\alpha+\beta)}+\frac{t^{\beta-1}(\rho t)^{2\alpha}}{\Gamma(2\alpha+\beta)}-\frac{t^{\beta-1}(\rho t)^{3\alpha}}{\Gamma(3\alpha+\beta)}+-\cdots=\sum_{k=0}^{\infty}\frac{t^{\beta-1}(-\rho t)^{\alpha}}{\Gamma(\alpha k+\beta)}$$

$$=t^{\beta-1}E_{\alpha,\beta}\left(-(\rho t)^{\alpha}\right)$$ as required.

3.4 An Alternative Representation

An alternative notation for the case $t^{\nu-1}E_{\nu,\nu}(z)$, where $\nu>0$, has been introduced by Gorenflo & Rutman (Gorenflo & Rutman 1994).

Definition 3.4.1

$$w(t) = \nu t^{\nu-1} E'_{\nu}(z) = t^{\nu-1} E_{\nu,\nu}(z) \qquad (3.4.1)$$

where $z = -(\rho t)^{\nu}$.

If in addition $\nu > 1$, then $sW(s) = \frac{s}{s^{\nu} + \rho^{\nu}}$ is the Laplace transform of $w'(t)$.

Proof

Since $(\nu > 1) \wedge (t = 0) \to w(0) = 0$ applying formula 3.2.1 gives

$$L(w'(t)) = sL(w(t)) - w(0) = sW(s).$$

Additionally we need w' and w''. We can calculate these derivatives from formula 1.2.6.

Hence $w'(t) = t^{\nu-2} E_{\nu,\nu-1}\left(-(\rho t)^{\nu}\right)$ (3.4.2)

Hence $w''(t) = t^{\nu-3} E_{\nu,\nu-2}\left(-(\rho t)^{\nu}\right)$ (3.4.3)

These are a specific case of the more general formula given by Podlubny (1995)

$$_0 D_t^{\gamma}\left(t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}\left(\lambda t^{\alpha}\right)\right) = t^{\alpha k+\beta-\gamma-1} E_{\alpha,\beta-\gamma}^{(k)}\left(\lambda t^{\alpha}\right) \qquad (3.4.5)$$

4 Linear Fractional O.D.E. of Order $\nu\in(0,2]$

In this section I give a summary of the known results for the equation

$$D^{\nu}x=-\rho^{\nu}x \tag{4.0.1}$$

in the case $\nu\in(0,2]$, the development is as in Gorenflow and Rutman (1994), with a new proof being given for the case of incorporated initial conditions.

The behaviour of the solution of 4.0.1 for the case $\nu\in(0,1]$ is qualitatively different to the behaviour of its solution for the case $\nu\in(1,2]$. Therefore the two cases are considered separately.

The case $\nu\in(0,1]$ is dealt with in section 4.1. I first consider the solution to 4.0.1 in the limiting case of $\nu=1$. Secondly I solve 4.0.1 for the case $\nu\in(0,1]$ using the Laplace transform, this solution is of limited usefulness because it requires an initial condition that is not readily available. Thirdly I solve 4.0.1 in the case where the initial value of the function is specified.

The case $\nu\in(1,2]$ is dealt with in section 4.2. I first consider the solution to 4.0.1 in the limiting case of $\nu=2$. Secondly I solve 4.0.1 for the case $\nu\in(1,2]$ using the Laplace transform, this solution is of limited usefulness because it requires two initial condition that involve fraction operators. Thirdly I solve 4.0.1 in the case where the initial value of the function and its derivative is specified.

4.1 Linear Fractional O.D.E's of Order $\nu\in(0,1]$

4.1.1 Limiting Case

The limiting case is of order 1 and has the form

$$Dx(t)=-\rho x(t)+f(t), t\geq 0, \rho>0 \tag{4.1.1}$$

Taking the Laplace transform of this equation gives

$$s\overline{x}-x(0)=-\rho\overline{x}+F(s)$$

rearranging this for $\overline{x}$ gives

$$\overline{x}=\frac{x(0)}{s+\rho}+\frac{F(s)}{s+\rho}$$

Therefore the general solution is

$$x(t) = x(0)\exp(\rho t) + \frac{1}{\rho}\int_0^t f(t-\tau)\exp(\rho\tau)d\tau \qquad (4.1.2)$$

4.1.2 Homogeneous Linear Fractional O.D.E.

If $\nu \in (0,1), \rho > 0$ we have the homogenous linear fractional O.D.E.

$$D^\nu x = -\rho^\nu x \qquad (4.1.3)$$

Using the Riemann-Liouville definition of the fractional derivative we have:

$$D^\nu x = DJ^{1-\nu}x = D\frac{1}{\Gamma(1-\nu)}\int_0^t \frac{x(y)}{(t-y)^\nu}dy \qquad (4.1.4)$$

Taking the Laplace transform of both sides of (4.1.2) yields

$$\mathcal{L}\left(D^\nu x\right) = s\mathcal{L}\left(J^{1-\nu}x\right) - J^{1-\nu}x(u)\big|_{u=0} = -\rho^\nu \bar{x} \qquad (4.1.5)$$

$$\frac{\bar{x}}{s^{-\nu}} - J^{1-\nu}x(u)\big|_{u=0} = -\rho^\nu \bar{x} \to \bar{x} = \frac{J^{1-\nu}x(u)\big|_{u=0}}{s^\nu + \rho^\nu} \qquad (4.1.6)$$

And by Theorem 3.2 on the Laplace transform of the Generalized Mittag-Leffler function with $\alpha = \nu$ and $\beta = \nu$ we have

$$x(t) = J^{1-\nu}x(u)\big|_{u=0} t^{\nu-1}E_{\nu,\nu}\left(-(\rho t)^\nu\right)$$

$$x(t) = J^{1-\nu}x(u)\big|_{u=0} \nu t^{\nu-1}E'_{\nu}\left(-(\rho t)^\nu\right) \qquad (4.1.7)$$

$$x(t) = J^{1-\nu}x(u)\big|_{u=0} w(t) \qquad (4.1.8)$$

By Theorem 2.4 since $x(t)$ is continuous on $(0,\infty)$ we can conclude that this is the unique solution to 4.1.3.

4.1.3 Inhomogeneous Linear Fractional O.D.E. of Order ν∈(0,1]

In this case the equation has the form $D^\nu x = -\rho^\nu x + f(t)$ which, by the convolution property for Laplace transforms, introduces the additional term $\mathcal{L}^{-1}\left\{\frac{F(s)}{s^\nu + \rho^\nu}\right\} = \int_0^t w(\tau)f(t-\tau)dt$

(4.1.9)

into the solution so that it becomes (where $w(t)$ is the function defined in (3.4):

$$x(t) = J^{1-\nu} x(u)\big|_{u=0} w(t) + \int_0^t w(\tau) f(t-\tau) d\tau \tag{4.1.10}$$

which is uniquely determined up to $J^{1-\nu} x(u)\big|_{u=0}$.

4.1.4 Incorporated Initial Conditions

The above solution has the disadvantage of requiring the value of the derivatives of fractional integrals as $t \to 0^+$ as initial conditions. To avoid having to specify the value of a fractional derivative, as $t \to 0^+$, as an initial condition we can incorporate x_0 in to the original equation. Which gives an equation of the form

$$D^{\nu}\left(x(t)-x_0\right) = -\rho^{\nu}x(t) + f(t), \tag{4.1.11}$$

where $t \geq 0, \rho > 0, \nu \in (0,1], x(0) = x_0$.

The solution to (4.1.11) is

$$x(t) = x_0 E_{\nu}\left(-(\rho t)^{\nu}\right) + \int_0^t w(\tau) f(t-\tau) d\tau \tag{4.1.12}$$

Proof

$$D^{\nu}\left(x(t)-x_0\right) = DJ^{1-\nu}\left(x(t)-x_0\right) = -\rho^{\nu}x(t) + f(t)$$

Taking the Laplace transform of the second representation and noting that $\lim_{x \to 0} J^{1-\nu}x(0) = 0$ gives us

$$s\frac{1}{s^{1-\nu}}\left(\bar{x} - \frac{x_0}{s}\right) = \frac{1}{s^{-\nu}}\left(\bar{x} - \frac{x_0}{s}\right) = -\rho^{\nu}\bar{x} + \bar{f} \tag{4.1.13}$$

Rearranging 4.1.13 gives

$$\bar{x} = \frac{x_0 s^{\nu-1}}{\left(s^{\nu}+\rho^{\nu}\right)} + \frac{\bar{f}}{s^{\nu}+\rho^{\nu}} \tag{4.1.14}$$

Applying Theorem 3.2 to obtain the inverse Laplace transform of (4.1.14) we have

$$x(t) = x_0\left(1 - \frac{(\rho t)^{\nu}}{\Gamma(\nu+1)} + \frac{(\rho t)^{2\nu}}{\Gamma(2\nu+1)} - + \cdots\right) + \int_0^t w(\tau) f(t-\tau) d\tau$$

$$= x_0 E_{\nu}\left(-(\rho t)^{\nu}\right) + \int_0^t w(\tau) f(t-\tau) d\tau$$

4.1.5 Asymptotic Behaviour of $E_\nu(.)$

We have from the definition of the Mittag-Leffler function the power series

$$E_\nu\left(-(\rho t)^\nu\right) = 1 - \frac{(\rho t)^\nu}{\Gamma(1+\nu)} + O\left((\rho t)^{2\nu}\right) \text{ as } t \to 0.$$

(4.1.15)

From Erdélyi (1953) we have the following result on the asymptotic behaviour of the Mittag-Leffler function :

For $|\arg(-z)| < \left(1 - \frac{\alpha}{2}\right)\pi, z \to \infty, N \in \mathbf{N}$,

$$E_\alpha(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(1-\alpha n)} + O\left(|z|^{-N}\right)$$

In this case $\alpha = \nu \in (0,1]$ so the result applies to the negative real line

Rewriting formula 1.2.2 with the current notation and putting $z = -(\rho t)^\nu$ gives

$$E_\nu\left(-(\rho t)^\nu\right) \sim \frac{1}{\Gamma(1-\nu)}(\rho t)^{-\nu}, \; t \to \infty$$

(4.1.16)

From (4.1.15) we conclude that for small t, the smaller ν the faster the function decays as t increases. From (4.1.16) we conclude that for large t the function decays slower the smaller ν, the fastest decay occurring in the limiting case when $\nu = 1$ since the exponential function decays faster than any polynomial. It should be noted that $w'(0) = -\infty, 0 < \nu < 1$ and $w'(0) = -\rho$ when $\nu = 1$.

From the, complete monotonicity of the Mittag-Leffler function for $0 < \nu < 1$, we have $E_\nu(0) = 1$, and $E_\nu(-\infty) = 0$. In the homogeneous case the solution will have extreme value at $t = 0$ which will decay to zero as $t \to \infty$. This behaviour of the solution, in this case, has been referred to as fractional relaxation.

4.2 Linear Fractional O.D.E's of Order $\nu \in (1,2]$

4.2.1 Limiting Case

The limiting case is of order 2 and has the form

$$D^2 x(t) = -\rho^2 x(t) + f(t), t \geq 0, \rho > 0 \qquad (4.2.1)$$

Taking the Laplace transform of this equation gives

$$s^2 \bar{x} - s x'(0) - x(0) = -\rho^2 \bar{x} + F(s)$$

$$\bar{x} = \frac{sx'(0)}{s^2+\rho^2} + \frac{x(0)}{s^2+\rho^2} + \frac{F(s)}{s^2+\rho^2}$$

Therefore the general solution is

$$x(t) = x'(0)\cos\rho t + \frac{x(0)}{\rho}\sin\rho t + \frac{1}{\rho}\int_0^t f(t-\tau)\sin\rho\tau d\tau \qquad (4.2.2)$$

4.2.2 Inhomogeneous Linear Fractional O.D.E. of Order $\nu \in (1,2]$

$$D^{\nu}x(t) = -\rho^{\nu}x(t) + f(t), t > 0, \rho > 0, 1 < \nu < 2 \qquad (4.2.3)$$

By Theorem 2.1 this is equivalent to

$$D^{2}J^{2-\nu}x(t) = -\rho^{\nu}x(t) + f(t). \qquad (4.2.4)$$

Taking the Laplace transform of (4.2.4) yields

$$s^{2}L\left\{J^{2-\nu}x(t)\right\} - sJ^{2-\nu}x(\tau)\big|_{\tau=0} - DJ^{2-\nu}x(\tau)\big|_{\tau=0} = -\rho^{\nu}\bar{x} + \bar{f}$$

$$s^{2}\frac{x(t)}{s^{2-\nu}} + \rho^{\nu}\bar{x} = sJ^{2-\nu}x(\tau)\big|_{\tau=0} + DJ^{2-\nu}x(\tau)\big|_{\tau=0} + \bar{f}$$

$$s^{\nu}\bar{x}(t) + \rho^{\nu}\bar{x} = sJ^{2-\nu}x(\tau)\big|_{\tau=0} + DJ^{2-\nu}x(\tau)\big|_{\tau=0} + \bar{f}$$

$$\bar{x} = \frac{sJ^{2-\nu}x(\tau)\big|_{\tau=0}}{s^{\nu}+\rho^{\nu}} + \frac{DJ^{2-\nu}x(\tau)\big|_{\tau=0}}{s^{\nu}+\rho^{\nu}} + \frac{\bar{f}}{s^{\nu}+\rho^{\nu}} \qquad (4.2.5)$$

Applying Theorem 3.2 to obtain the inverse Laplace transform of (4.2.5) yields

$$\begin{aligned} x(t) = {} & J^{2-\nu}x(\tau)\big|_{\tau=0}t^{\nu-2}E_{\nu,\nu-1}\left((-\rho t)^{\nu}\right) + DJ^{2-\nu}x(\tau)\big|_{\tau=0}t^{\nu-1}E_{\nu,\nu}\left((-\rho t)^{\nu}\right) \\ & + f(t) * \left(t^{\nu-1}E_{\nu,\nu}\left((-\rho t)^{\nu}\right)\right) \end{aligned} \qquad (4.2.6)$$

Equivalently from section 3.4 we can rewrite (4.2.6) as

$$x(t) = J^{\nu-2}x(\tau)\big|_{\tau=0}w'(t) + DJ^{2-\nu}x(\tau)\big|_{\tau=0}w(t) + (w*f)(t) \qquad (4.2.7)$$

In (4.2.7) $J^{\nu-2}x(\tau)\big|_{\tau=0}$ and $DJ^{\nu-2}x(\tau)\big|_{\tau=0}$ are to be understood as their respective limiting values as $\tau \to 0^{+}$.

We cannot write $J^{1-\nu}$ instead of $DJ^{2-\nu}$ since the operators J^{α} are only defined for $\alpha \geq 0$ but in this case since $\nu \in (1,2]$ we will have $1-\nu < 0$.
If $J^{\nu-2}x(\tau)\big|_{\tau=0} \neq 0$ then because $w'(t) \sim \dfrac{t^{\nu-2}}{\Gamma(\nu-1)}$ is unbounded in any neighbourhood of the origin so is $x(t)$.

4.2.3 Incorporated Initial Conditions - *

Incorporating the initial conditions in to the second order case we have

$$D^{\nu}\left(x(t)-x_0-x_0't\right)=-\rho^{\nu}x(t)+f(t) \tag{4.2.8}$$

where $t>0, \rho>0, \nu\in(0,1], x(0)=x_0, x'(0)=x_0'$..

Assuming the solution to (4.2.8) is once differentiable and Laplace transformable then by Theorems 2.1 and 2.3 we have

$$\begin{aligned} D^{\nu}\left(x(t)-x_0-x_0't\right) &= D^2D^{\nu-2}\left(x(t)-x_0-x_0't\right) \\ &= D^2D^{\nu-3}D(x(t)-x_0-x_0't) \\ &= DD^{\nu-2}D(x(t)-x_0-x_0't_0) \\ &= DJ^{2-\nu}(x'(t)-x_0') \end{aligned}$$

Hence $$DJ^{2-\nu}(x'(t)-x_0't) \quad =-\rho^{\nu}x(t)+f(t) \tag{4.2.9}$$

Taking the Laplace transform of (4.2.9) yields

$$\begin{aligned} \frac{s}{s^{2-\nu}}\mathcal{L}(x')-\frac{x_0'}{s^{2-\nu}} &= \frac{s}{s^{2-\nu}}\left[s\bar{x}-x_0'\right]-\frac{x_0'}{s^{2-\nu}} \\ &= \frac{\bar{x}}{s^{-\nu}}-\frac{x_0'}{s^{2-\nu}}-\frac{x_0}{s^{1-\nu}} \qquad (4.2.10) \\ &= -\rho^{\nu}\bar{x}+\bar{f} \end{aligned}$$

Rearranging gives

$$\bar{x}\left(s^{\nu}+\rho^{\nu}\right)=\frac{x_0}{s^{1-\nu}}+\frac{x_0'}{s^{2-\nu}}+\bar{f}$$

$$\bar{x}=\frac{x_0}{s^{1-\nu}\left(s^{\nu}+\rho^{\nu}\right)}+\frac{x_0'}{s^{2-\nu}\left(s^{\nu}+\rho^{\nu}\right)}+\frac{\bar{f}}{\left(s^{\nu}+\rho^{\nu}\right)} \tag{4.2.11}$$

Applying Theorem 3.2 the inverse Laplace transform of this is

$$x(t)=x_0E_{\nu,1}\left(-(\rho t)^{\nu}\right)+x_0't^{\nu-1}E_{\nu,2}\left(-(\rho t)^{\nu}\right)+\left(t^{\nu-1}E_{\nu,\nu}\left(-(\rho t)^{\nu}\right)\right)*f(t) \tag{4.2.12}$$

Or equivalently

$$x(t)=x_0w_0(t)\_+x_0'w_1(t)+\int_0^t w(\tau)f(t-\tau)d\tau \tag{4.2.13}$$

4.2.4 Asymptotic Behaviour

In the case $\nu=2$, $E_2\left(-(\rho t)^2\right)=\cos\rho t$ and $E_{2,2}\left(-(\rho t)^2\right)=\dfrac{\sin\rho t}{\rho t}$.

So for the limiting case of the homogeneous equation the solution is an oscillation of constant amplitude.

For $1 < \nu < 2$ $E_{\nu}(-\infty) = 0$. If we refer to Mittag-Leffler's paper of 1904 the addition of the second parameter does not alter the argument by which he concludes the value of the limit, additionally $E_{\nu,2}(t) \to 0, t \to -\infty$ faster than $t^{\nu-1} \to \infty$. See Mittag-Leffler 1904 pg. 122 (29) and 124 (36). Therefore in the homogeneous case the magnitude of the solution decays to zero as time increases. From 1.2(ii) we know that $E_{\nu,1}\left(-(\rho t)^{\nu}\right)$ will have an odd number of zeros, the number of zeros being larger the larger ν, when $\nu = 2$ the function has an infinite number of zeros. From Theorem 1.3 we have $t^{\nu-1} E_{\nu,2}\left(-(\rho t)^{\nu}\right) = \nu E_{\nu}'\left(-(\rho t)^{\nu}\right)$ therefore this component will have an even number of zeros. Hence the term fractional oscillation equation.

5 Linear Fractional O.D.E's Order $\nu \in (2,\infty)$

This section applies the methods that we developed in chapter 4 for solving linear fractional O.D.E's to the case where the order of differentiation is greater than 2. This chapter is divided in to two sections. In section 5.1 the three cases for $\nu \in (2,3]$ (limiting integer equation of order 3, fractional integral initial conditions, and incorporated initial conditions) are studied in the same order as they were for the cases $\nu \in (0,1]$ and $\nu \in (1,2]$. The solutions are again found to be expressible in terms of generalised Mittag-Leffler functions. In section 5.2 the results obtained in section 5.1 are extended to cover all cases where $\nu > 2$.

5.1 Linear Fractional O.D.E's of Order $\nu \in (2,3]$

5.1.1 Third Order O.D.E.

The limiting case of this example is

$$D^3 x(t) = -\rho^3 x(t) + f(t),\ t \geq 0,\ \rho > 0$$

(5.1.1.1)

with initial conditions $x(0) = x_0, x'(0) = x_0', x''(0) = x_0''$.
Taking the Laplace transform of (5.1.1.1) gives

$s^3\bar{x} - s^2 x_0 - s x_0' - x_0'' = -\rho^3 \bar{x} + \bar{f}$ rearranging this for $\bar{x}$ gives

$$\bar{x} = \frac{s^2 x_0}{s^3 + \rho^3} + \frac{s x_0'}{s^3 + \rho^3} + \frac{x_0''}{s^3 + \rho^3} + \frac{\bar{f}}{s^3 + \rho^3}$$

(5.1.1.2)

By Theorem 3.2 we can invert (5.1.1.2) and give the solution in terms of generalised Mittag-Leffler functions, which gives

$$x(t) = x_0 E_{3,1}\left(-(\rho t)^3\right) + x_0' t E_{3,2}\left(-(\rho t)^3\right) + x_0'' t^2 E_{3,3}\left(-(\rho t)^3\right) + \int_0^t \tau^2 E_{3,3}\left(-(\rho\tau)^3\right) f(t-\tau) d\tau$$

(5.1.1.3)

Alternatively we could have used the partial fraction decomposition of the factorisation $\frac{1}{s^3+\rho^3} = \frac{1}{(s-\rho)(s-\omega\rho)(s-\omega^2\rho)}$ where $\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\omega^2 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$ to obtain the solution of the homogeneous equation as

$$x(t) = A\exp(-\rho t) + B\exp\left(\frac{\rho t}{2}\right)\cos\left(\frac{\sqrt{3}\rho t}{2}\right) + C\exp\left(\frac{\rho t}{2}\right)\sin\left(\frac{\sqrt{3}\rho t}{2}\right)$$

(5.1.1.4)
where $A, B, C \in \mathbf{R}$ since $x(t)$ is a real valued function.

This only has a stable solution for $B = C = 0$ i.e. when $x(0) = -\rho x'(0) = \rho^2 x''(0)$. If $B = C = 0$ the solution behaves like the first order case otherwise we have an oscillatory solution of exponentially increasing amplitude. If $B \neq 0$ and $C \neq 0$ we have two exponentially growing sine waves of the same frequency but 90° out of phase with each other.

5.1.2 The Inhomogeneous Linear Fractional O.D.E. of Order ν∈(2,3] - **

This equation has the form

$$D^{\nu} x(t) = -\rho^{\nu} x(t) + f(t), t > 0, \rho > 0, 2 < \nu \leq 3 \qquad (5.1.2.1)$$

Equivalently by Theorem 2.1

$$D^3 J^{3-\nu} x(t) = -\rho^{\nu} x(t) + f(t) \qquad (5.1.2.2)$$

Taking the Laplace transform of (5.1.2.2) yields

$$s^3 \mathcal{L}\left\{J^{3-\nu} x(t)\right\} - s^2 J^{3-\nu} x(\tau)\big|_{\tau=0} - sDJ^{3-\nu} x(\tau)\big|_{\tau=0} - D^2 J^{3-\nu} x(\tau)\big|_{\tau=0} = -\rho^{\nu}\bar{x} + \bar{f}$$

$$\frac{s^3\bar{x}}{s^{3-\nu}} - s^2 J^{3-\nu} x(\tau)\big|_{\tau=0} - sDJ^{3-\nu} x(\tau)\big|_{\tau=0} - D^2 J^{3-\nu} x(\tau)\big|_{\tau=0} = -\rho^{\nu}\bar{x} + \bar{f}$$

$s^{\nu}\bar{x} + \rho^{\nu}\bar{x} = s^2 J^{3-\nu} x(\tau)\big|_{\tau=0} + sDJ^{3-\nu} x(\tau)\big|_{\tau=0} + D^2 J^{3-\nu} x(\tau)\big|_{\tau=0} + \bar{f}$ rearranging this for $\bar{x}$ gives

$$\bar{x} = \frac{s^2 J^{3-\nu} x(\tau)\big|_{\tau=0}}{s^{\nu} + \rho^{\nu}} + \frac{sDJ^{3-\nu} x(\tau)\big|_{\tau=0}}{s^{\nu} + \rho^{\nu}} + \frac{D^2 J^{3-\nu} x(\tau)\big|_{\tau=0}}{s^{\nu} + \rho^{\nu}} + \frac{\bar{f}}{s^{\nu} + \rho^{\nu}}$$

(5.1.2.3)

Therefore by Theorem 3.2 the inverse of (5.1.2.3) is

$$x(t) = J^{3-\nu} x(\tau)\big|_{\tau=0} t^{\nu-3} E_{\nu,\nu-2}\left(-(\rho t)^{\nu}\right) + DJ^{3-\nu} x(\tau)\big|_{\tau=0} t^{\nu-2} E_{\nu,\nu-1}\left(-(\rho t)^{\nu}\right)$$
$$+ D^2 J^{3-\nu} x(\tau)\big|_{\tau=0} t^{\nu-1} E_{\nu,\nu}\left(-(\rho t)^{\nu}\right) + \int_0^t t^{\nu-1} E_{\nu,\nu}\left(-(\rho t)^{\nu}\right) f(t-\tau)\, d\tau$$

(5.1.2.4)

5.1.3 Incorporated Initial Conditions - **

In this case we have

$$D^{\nu}\left(x(t)-x_0-x_0't-\frac{x_0''t^2}{2}\right)=-\rho^{\nu}x(t)+f(t)$$

(5.1.3.1)

Applying Theorems 2.1 and 2.3, as in section 2.5, to the left hand side of (5.1.3.1) we obtain

$$D^{\nu}\left(x(t)-x_0-x_0't-\frac{x_0''t^2}{2}\right)=D^3D^{\nu-3}\left(x(t)-x_0-x_0't-\frac{x_0''t^2}{2}\right) \quad \text{by Theorem 2.1}$$

$$=D^3D^{\nu-5}D^2\left(x(t)-x_0-x_0't-\frac{x_0''t^2}{2}\right) \quad \text{by Theorem 2.3}$$

$$=D^3D^{\nu-5}\left(x''(t)-x_0''\right)$$ differentiating out term by term

$$=D\ \ D^{\nu-3}\left(x''(t)-x_0''\right)$$ by Theorem 2.1

Therefore we can rewrite (5.1.3.1) as

$$DJ^{3-\nu}\left(x''(t)-x_0''\right)=-\rho^{\nu}x(t)+f(t)$$

(5.1.3.2)

Laplace transforming (5.1.3.2) gives

$$\frac{s}{s^{3-\nu}}\left[s^2\bar{x}-sx_0-x_0'\right]-\frac{x_0''}{s^{3-\nu}}=-\rho^{\nu}\bar{x}+\bar{f}$$

(5.1.3.3)

Rearranging this gives

$$\left(s^{\nu}+\rho^{\nu}\right)\bar{x}=\frac{x_0}{s^{1-\nu}}+\frac{x_0'}{s^{2-\nu}}+\frac{x_0''}{s^{3-\nu}}+\bar{f}$$

Therefore $\bar{x}=\dfrac{x_0}{s^{1-\nu}\left(s^{\nu}+\rho^{\nu}\right)}+\dfrac{x_0'}{s^{2-\nu}\left(s^{\nu}+\rho^{\nu}\right)}+\dfrac{x_0''}{s^{3-\nu}\left(s^{\nu}+\rho^{\nu}\right)}+\dfrac{\bar{f}}{\left(s^{\nu}+\rho^{\nu}\right)}$

(5.1.3.4)

By Theorem 3.2 the inverse Laplace transform of (5.1.3.4) is

$$x(t)=x_0E_{\nu,1}\left(-(\rho t)^{\nu}\right)+x_0'tE_{\nu,2}\left(-(\rho t)^{\nu}\right)+x_0't^2E_{\nu,3}\left(-(\rho t)^{\nu}\right)$$
$$+\left\{t^{\nu-1}E_{\nu,\nu}\left(-(\rho t)^{\nu}\right)\right\}*f(t)$$

(5.1.3.5)

By Theorems 1.3 and 1.4 we can rewrite this in terms of the Mittag-Leffler function for parameter ν.

$$x(t) = x_0 E_\nu\left(-(\rho t)^\nu\right) + x_0' J E_\nu\left(-(\rho t)^\nu\right) + x_0' J^2 E_\nu\left(-(\rho t)^\nu\right) + \left\{\nu E_\nu'\left(-(\rho t)^\nu\right)\right\} * f(t)$$

(5.1.3.6)

5.1.4 Asymptotic Behaviour of the Solution in the Homogeneous Case

From (1.2.4) we have for some m which fulfils the condition $\frac{\nu}{2} < 2m < \frac{3\nu}{2}$ the following identity for the Mittag-Leffler function.

$$E_\nu\left(-(\rho t)^\nu\right) = \sum_{n=0}^{m-1} \frac{2}{\nu} \exp\left(\rho t \cos\frac{2n+1}{\nu}\pi\right) \cos\left(\rho t \sin\frac{2n+1}{\nu}\pi\right)$$

At the lower bounding integer case of $\nu = 2$ we have a constant oscillation corresponding to the square roots of -1, at the upper bounding integer case of $\nu = 3$ we have a diverging oscillatory solution corresponding to the cube roots of -1.

This suggests the following conjecture for the behaviour of the solution to (5.1.3.1).
For $2 < \nu < 3$ $\cos\left(\rho t \sin\frac{1}{\nu}\pi\right)$ will generate two exponentially diverging sine waves of the same period $\dfrac{2\pi}{\rho \sin\frac{\pi}{\nu}}$ but of different phase angles.
At $\nu = 3$ the term associated with $\exp(-\rho t)$ will appear.

5.2 Linear Fraction O.D.E's of Order $\nu \in (2,\infty)$

The Mittag-Leffler function has three types of behaviour as $\nu \in (0,1]$, $\nu \in (1,2]$, or $\nu \in (2,\infty)$. Having investigated $\nu \in (2,3]$ it is natural to try to extend the result to deal with the whole interval $(2,\infty)$.

5.2.1 n^{th} Order O.D.E.

In the integer order case the homogeneous n^{th} order linear differential equation $\sum_{k=0}^{n} a_k D^{n-k} x(t) = 0$

(5.2.1.1)

with $n(>0) \in \mathbf{N}$ has the characteristic equation $\sum_{k=0}^{n} a_k \lambda^{n-k} = 0$.

The following theorem give the composition of the particular solutions of (5.2.1.1).

Theorem - 5.2.1

If the roots of the characteristic equation of the nth order linear differential equation (5.2.1.1) are λ_k with multiplicity m_k then the particular solutions of (5.2.1.1) have the form $e^{\lambda_k x}, xe^{\lambda_k x}, x^{m_k - 1} e^{\lambda_k x}$. Tikhonov, Vasil'eve, Sveshnikov (1980)

For the case we are considering the equation has the form

$$D^n x = -\rho^n x \quad \rho > 0$$
(5.2.1.2)

with initial conditions $x(0) = 0, \cdots, x^{(n-1)}(0) = x^{(n-1)}{}_0$.

This has characteristic equation $\lambda^n + \rho^n = 0$
(5.2.1.3)

Writing the n[th] roots of -1 as

$$\omega_{n,j} = \exp\left(\frac{(2j+1)i\pi}{n}\right) = \alpha_{n,j} + i\beta_{n,j}$$,where $j = 0,1,\cdots,n-1$, and $\alpha_{n,j}, \beta_{n,j} \in \mathbf{R}$.

Then the solutions of (5.2.1.2) are

$$x(t) = \sum_{j=0}^{n-1} A_j \exp\left(\rho t \omega_{n,j}\right)$$

(5.2.1.4)

Since $x(t)$ is a real function $\exists a_j, b_j \in \mathbf{R}$, $j = 0,1,\cdots,n-1$ such that

$$x(t) = \sum_{j=0}^{n-1} \exp\left(\rho\alpha_{n,j} t\right)\left(a_j \cos\left(\rho\beta_{n,j} t\right) + b_j \sin\left(\rho\beta_{n,j} t\right)\right)$$

(5.2.1.5)

So after the 2[nd] order case, since the equation $x^n + \rho^n$ will have some roots whose real part is positive and the solution of (5.1.2.1) will be dominated by a sum of diverging oscillations of period $\frac{\rho\beta_{n,j}\pi}{n}$ where $\left|\frac{\beta_{n,j}\pi}{n}\right| < \frac{\pi}{2}$; except when the initial conditions are such as to force the coefficients of these terms to be zero. This is because in the complex plane the n[th] roots of −1 form a regular polygon of n vertices which is symmetrical about the real axis.

Conjectured Behaviour of the Solution to (5.2.1.2) for Non-integral Orders of Differentiation.

Consider the equation $D^\nu x = -\rho^\nu x$, $n-1 < \nu \le n$. The behaviour for integer derivatives suggests that there will be components $\exp\left(\rho\alpha_{\nu,j} t\right)\cos\left(\rho\beta_{\nu,j} t\right)$ and $\exp\left(\rho\alpha_{\nu,j} t\right)\sin\left(\rho\beta_{\nu,j} t\right)$ in the solution that starting from the lower integer value solution $\exp\left(\rho\alpha_{n-1,j} t\right)\cos\left(\rho\beta_{n-1,j} t\right)$ and $\exp\left(\rho\alpha_{n-1,j} t\right)\sin\left(\rho\beta_{n-1,j} t\right)$ tend to their respective upper integer value solutions $\exp\left(\rho\alpha_{n,j} t\right)\cos\left(\rho\beta_{n,j} t\right)$ and $\exp\left(\rho\alpha_{n,j} t\right)\sin\left(\rho\beta_{n,j} t\right)$ as ν increases in the interval $[n-1, n]$.

5.2.2 Homogeneous Linear Fractional O.D.E. ν∈(2,∞) - **

In this case we have $D^{\nu}x = -\rho^{\nu}x,\, n-1 < \nu \le n,\, n \in \mathbf{N}$
(5.2.2.1)
with initial conditions $D^k J^{n-\nu} x(\tau)\big|_{\tau=0} = x_0^{(\nu+k-n)} \quad k = 0,1,\cdots,n-1$

Theorem 5.2.2
The general solution of (5.2.2.1) is
$$x(t) = \sum_{k=0}^{n-1} x_0^{(k,\nu)} t^{\nu+k-n} E_{\nu,\nu+k-(n-1)}\left(-(\rho t)^{\nu}\right) \quad (5.2.2.2)$$
Proof

$$\mathcal{L}\left(D^{\nu} x\right) = \mathcal{L}\left(D^{n} J^{n-\nu} x\right)$$
(5.2.2.3)

Taking the Laplace transform of the r.h.s. of (5.2.2.3) and using (3.2.1) yields
$$\begin{aligned}\mathcal{L}\left(D^n J^{n-\nu} x\right) &= s^n \mathcal{L}\left(J^{n-\nu} x\right) - \sum_{k=0}^{n-1} D^k J^{n-\nu} x(\tau)\big|_{\tau=0} s^{n-k-1} \\ &= s^n \mathcal{L}\left(J^{n-\nu} x\right) - \sum_{k=0}^{n-1} x_0^{(\nu+k-n)} s^{n-k-1} \\ &= s^n \frac{\bar{x}}{s^{n-\nu}} - \sum_{k=0}^{n-1} x_0^{(\nu+k-n)} s^{n-k-1} \\ &= -\rho^{\nu} \bar{x}\end{aligned}$$
(5.2.2.4)

Rearranging (5.2.2.4) for $\bar{x}$ gives
$$\bar{x} = \sum_{k=0}^{n-1} \frac{x_0^{(\nu+k-n)} s^{n-k-1}}{\left(s^{\nu} + \rho^{\nu}\right)}$$
(5.2.2.5)
Applying Theorem 3.2 to obtain the inverse Laplace transform of (5.2.2.5) we have
$$x(t) = \sum_{k=0}^{n-1} x_0^{(\nu+k-n)} t^{\nu+k-n} E_{\nu,\nu+k-(n-1)}\left(-(\rho t)^{\nu}\right)$$
(5.2.2.6)

In the inhomogeneous case the inverse Laplace transform of the function $\dfrac{\bar{f}}{s^{\nu}+\rho^{\nu}}$ will occur. Hence

$$x(t) = \sum_{k=0}^{n-1} x_0^{(k,\nu)} t^{\nu+k-n} E_{\nu,\nu+k-(n-1)}\left(-(\rho t)^{\nu}\right) \quad + \; f(t) * E_{\nu,\nu}\left(-(\rho t)^{\nu}\right)$$

5.2.3 Incorporated Initial Conditions - **

Theorem 5.2.3

The solution of 5.2.2.1 with incorporated initial conditions is

$$x(t) = \sum_{k=0}^{n-1} x_0^{(k)} J^k E_{\nu}\left(-(\rho t)^{\nu}\right)$$

Proof

Starting with equation (5.2.2.1) we have

$$D^{\nu}\left(x - \sum_{i=0}^{n-1} \frac{x^{(i)}{}_0 t^i}{i!}\right) = -\rho^{\nu} x, n-1 < \nu < n, n \in \mathbf{N}$$

(5.2.3.1)

with initial conditions $x^{(i)}(0) = x_0^{(i)}, i = 0,1,\cdots,n$.

Applying Theorems 2.1 and 2.3 as before ultimately gives

$$D^{\nu} x = DJ^{n-\nu} D^{n-1}\left(x - \sum_{i=0}^{n-1} \frac{x_0^{(i)} t^i}{i!}\right) = -\rho^{\nu} x$$

(5.2.3.2)

Which after differentiating out term by term becomes

$$D^{\nu} x = DJ^{n-\nu}\left(x^{(n-1)} - x_0^{(n-1)}\right) = -\rho^{\nu} x$$

(5.2.3.3)

Taking the Laplace transform of (5.2.3.3) gives

$$\frac{s}{s^{n-\nu}} L\left(x^{(n-1)}\right) - \frac{x_0^{(n-1)}}{s^{n-\nu}} = -\rho^{\nu} \overline{x}$$

(5.2.3.4)

$$\frac{1}{s^{n-\nu}}\left(s^n \overline{x} - \sum_{k=0}^{n-1} x_0^{(k)} s^{n-k}\right) = -\rho^{\nu} \overline{x}$$

(5.2.3.5)

Rearranging this for $\overline{x}$ gives

$$\overline{x} = \sum_{k=0}^{n-1} x_0^{(k)} \frac{s^{\nu-k}}{\left(s^{\nu} + \rho^{\nu}\right)}$$

(5.2.3.6)

By Theorem 3.2 the inverse Laplace transform of (5.2.3.6) is

$$x(t) = \sum_{k=0}^{n-1} x_0^{(k)} t^k E_{\nu,k+1}\left(-(\rho t)^{\nu}\right)$$

(5.2.3.7)

By Theorem 1.4 this can be rewritten in terms of Mittag-Leffler functions as

$$x(t) = \sum_{k=0}^{n-1} x_0^{(k)} J^k E_\nu\left(-(\rho t)^\nu\right)$$

5.2.4 Asymptotic Behaviour of the Solution of the Homogeneous Equation

From formulas 1.2.4 we know that $E_\nu\left(-(\rho t)^\nu\right)$ can be expressed as the sum of products of the exponential function with sine and cosine functions. The integral of the product of the exponential function with sine and cosine functions is itself a product of the exponential function with sine and cosine functions. Therefore for t large the solution will be dominated by a sum of exponentially diverging oscillations, whose frequencies are in harmonic series, and whose limiting values are those of the solutions to the limiting integer order equation

$$D^n x = -\rho^n x, n-1 < \nu \le n, n \in \mathbf{N} \qquad (5.2.3.8)$$

More specifically, and continuing the conjecture of 5.1.4, for $\nu \in (2m, 2m+1]$, where $m \in \mathbf{Z}$, as $\nu \to 2m+1$ those solutions of the equation $D^{2m}x = -\rho^{2m}x$ which lie in the upper half plane, will travel on a circle, of radius ρ, in the negative sense, towards the solutions of the equation $D^{2m+1}x = -\rho^{2m+1}x$. Those solutions of the equation $D^{2m}x = -\rho^{2m}x$ which lie in the lower half plane, will travel on a circle, of radius ρ, in the positive sense towards the solutions of the equation $D^{2m+1}x = -\rho^{2m+1}x$.

When ν reaches $2m+1$ the solution $\exp(-\rho t)$ will appear in the solution, and as ν becomes greater than $2m+1$ this will split in to a conjugate pair of solutions travelling towards those solutions of the case $\nu = 2m+2$ corresponding to $\exp\left(\frac{(2m+1)i\pi}{2m+2}\right)$ and $\exp\left(\frac{(2m+3)i\pi}{2m+2}\right)$.

Those solutions of the fractional O.D.E. tending to solutions of the O.D.E. corresponding to a root of -1 whose real part is less than or equal to zero will exponentially decay, whilst those solutions of the fractional O.D.E. tending to solutions of the O.D.E. corresponding to a root of -1 whose real part is greater than zero will exponentially diverge.

5.3 Conclusion

It has been shown in Theorems 5.2.2 and 5.2.3 that the solutions to the equation

$$D^\nu x = -\rho^\nu x, \; \nu, \rho \in \mathbf{R}^+ \qquad (5.3.1)$$

are expressible in terms of sums of generalised Mittag-Leffler functions or the integrals of Mittag-Leffler functions. The Mittag-Leffler function has three kinds of behaviour depending on the value of ν and therefore equation (5.3.1) has three different kinds of behaviour. For $\nu \in (0,1]$ we have a monotonically decaying solution. For $\nu \in (1,2)$ we get a damped oscillation with a finite number of zeros. For $\nu = 2$ we get a steady state oscillation and for $\nu > 2$ we get a sum of diverging oscillations and decaying oscillations.

These results are consistent with the numerical approximations given by Dr. L. Blank in her paper.

Clearly it necessary to determine whether the conjectured solutions are the true solutions. This could be done by following up Mittag-Leffler's original work and further investigating the Hankel integral by which he established formula 1.2.4

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