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Close to Regular Multipartite Tournaments

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Preface

In the last forty years Graph Theory has undergone an extensive and rapid development. One branch of Graph Theory, which is important for solving discrete organization and optimization problems, is the theory of directed graphs, or digraphs. The best studied class of directed graphs are the tournaments. Already in 1934 Rédei [21] proved that every tournament contains a Hamiltonian path. Other classical results can be found in the works of Camion [7], Moon [20], Harary and Moser [17] or Alspach [1].

Tournaments can be generalized to the class of semicomplete multipartite digraphs or to multipartite tournaments. A c-partite tournament is an orientation of a complete mutlipartite graph and a semicomplete multipartite digraph is obtained by replacing each edge of a complete multipartite graph by an arc or by a pair of mutually opposite arcs. These domains have only recently received attention in fundamental research. A very profound work, whose results will often be used throughout this thesis, is the Ph. D. thesis of Yeo [49]. Further results and surveys on the subject are the Habilitation thesis of Guo [9], the Ph. D. thesis of Tewes [23] and the articles [15] of Gutin and [31] of Volkmann.

In this thesis we will mainly examine the existence of directed cycles and directed paths (or short cycles and paths, respectively) with certain properties in multipartite tournaments. The example of an extended transitive tournament demonstrates that there are multipartite tournaments without any cycle and only with short paths. Since extended transitive tournaments are not strongly connected one approach is to analyze only strongly connected multipartite tournaments as done in [11] by Guo, Pinkernell and Volkmann. In this thesis the statements on the existence of cycles and paths depend on how much a multipartite tournament differs from being regular. Hence, we use a parameter introduced by Yeo [51], the global irregularity $i_g(D)$ of a digraph D, which is defined to be the difference between the maximum and the minimum occuring vertex-degree in D (out- or indegree). If $i_g(D) = 0$, then D is regular, and if $i_g(D) \leq 1$, then D is called almost regular.

This thesis is divided into three parts and eight chapters. In the first part, we study the existence of certain cycles in close-to-regular multipartite tournaments. The short second part, only consisting of Chapter 5, presents an analysis and an improvement of a result of Yeo [49] on the connectivity of close-to-regular multipartite tournaments. These results are useful for the third part of this thesis, in which we will mainly search for long paths in multipartite tournaments.

Chapter 1 contains an introduction to the terminology and notation used throughout this text. Furthermore, we present some results on the possible degrees of vertices in multipartite tournaments of a given global irregularity $i_g(D)$.

In Chapter 2, at the beginning of Part I, we take a look at the existence of cycles in almost regular multipartite tournaments whose length does not exceed the number of partite sets and which contain a given arc. Extending results of Alspach [1], Guo [9] and Volkmann [30, 32], we find an optimal integer c of partite sets of an almost regular c-partite tournament to ensure the existence of cycles of all lengths p with $p \in \{4, 5, \dots, c\}$ through a given arc. In detail, we distinguish the cases that there are at least two vertices in each partite set and that there is only one vertex in at least one partite set.

Chapter 3 also deals with cycles containing a given arc. But in contrast to Chapter 2, here the length of the cycles does not matter. Instead the number of partite sets the cycles include is considered. Inspired by a result of Goddard and Oellermann [8], Guo and Kwak [10] proved that every arc of a regular c-partite tournament D with $c \geq 4$ is contained in a cycle with vertices from exactly m partite sets for all $4 \leq m \leq c$. We extend this theorem to almost regular multipartite tournaments, and we show that the bounds we give are optimal in some sense.

In Chapter 4 we try to combine the themes of the last two chapters by searching for cycles of a given length and a given number of partite sets. The problem is to find sufficient conditions for a multipartite tournament to contain a cycle consisting of a given number of vertices from each partite set. At first, solving a problem of Volkmann [29], we show that every almost regular multipartite tournament with at least 5 partite sets contains a strongly connected subtournament of maximal order and thus, according to a well-known result of Moon [20], a cycle consisting of exactly one vertex from each partite set. Furthermore, we find cycles with exactly zero or one vertex from each partite set of a multipartite tournament with a given fixed global irregularity $i_g(D) \geq 2$. The number of partite sets that contribute exactly one vertex to the cycle depends on the global irregularity and on the number of partite sets. In the last section of the fourth chapter, we look for long cycles. Since, according to a result of Yeo [48], every regular multipartite tournament D is Hamiltonian, the next question is that of which multipartite tournaments contain a cycle with all but one vertex from each partite set. We prove that a regular c-partite tournament D with $c \geq 3$ and at least two vertices in each partite set contains a cycle with all but one vertex from each partite set with the exception of when $c = 4$ and there are two vertices in every partite set of D.

The second part of this thesis only consists of Chapter 5, which deals with connectivity in multipartite tournaments. In particular we study a bound for the (vertex-) connectivity developed by Yeo [51]. Since this bound is very useful for many applications, an interesting problem is to characterize the multipartite tournaments that realize this bound. Analyzing the proof of Yeo, we first find necessery conditions for a multipartite tournament to realize Yeo's bound. We obtain the structure of these multipartite tournaments. Second, using the conditions and certain classes of examples, we characterize all almost regular multipartite tournaments, which realize the bound.

In Chapter 6, at the beginning of Part III, we return to the problem of Chapter 4, but in a weaker form applied to paths. We look for sufficient conditions for a path to consist of a given number of vertices from each partite set. To get solutions for this problem, in the first section we improve and analyze a result of Yeo [51] (respectivly, of Gutin and Yeo [16]) on the path covering number of semicomplete multipartite digraphs. In the second section we consider "short" paths. The main result is that every regular multipartite tournament with at least two partite sets and two vertices in each partite set contains a path with exactly two vertices from each partite set. The third and last section of Chapter 6 deals with long paths. We show that almost all regular c-partite tournaments with $c \geq 4$ contain a path with all but s vertices from each partite set for a given integer $s \geq 1$.

The case $s = 0$, and thus the existence of Hamiltonian paths in multipartite tournaments, is the subject of Chapter 7. There the problem is to find, for each fixed irregularity $i_g(D) =: i$, an optimal value $g(i)$ such that every c-partite tournament, with the irregularity i and $c \geq g(i)$, contains a Hamiltonian path. Distinguishing the cases $i \leq 2$ and $i \geq 3$ we solve this problem completely.

In the last chapter of this thesis we also consider Hamiltonian paths. In contrast to Chapter 7 we consider Hamiltonian paths through a given arc. Analogous to Chapter 7, for a given irregularity i we can ask for the minimal number $h(i)$ of partite sets that ensures the property that for each arc of a multipartite tournament D with irregularity i there exists a Hamiltonian path containing this arc. Applying a result of Volkmann and Yeo [46] about Hamiltonian paths starting with a given arc in the first section, we show that almost all c-partite tournaments of a given irregularity i with $c \geq 4$ have the desired property. The main result of the second section is that $h(1) = 5$. In the last section we find a sufficient condition for an arc of an almost regular 3-partite tournament D to be contained in a Hamiltonian path of D.

Aachen, July 2004 Stefan Winzen

Contents

Chapter 1

Introduction

In this first section we present most of the terminology and the basic notation used throughout this thesis. As we assume a basic knowledge of graph theory and digraphs, we refer the reader that is unfamiliar with it to consult the books of Volkmann [28] or Bang-Jensen and Gutin [2]. Some special definitions that are only relevant in certain chapters will be defined in place where they are needed. If not stated otherwise, all *graphs* or *digraphs* of this thesis are simple and finite.

1.1 Terminology and notations

Definition 1.1 *[Digraphs]* A *digraph* D is an orientation of a *graph*. Every digraph D consists of vertices and arcs. The *vertex set* and *arc set* of a digraph D are denoted by $V(D)$ and $E(D)$, respectively. If $V(D)$ is finite, then we call the digraph D a *finite digraph*, and we define the *order* $n(D)$ of D by $n(D) = |V(D)|$ and the *size* $m(D)$ of D by $m(D) = |E(D)|$.

If xy is an arc of a digraph D, then we write $x \to y$ and say x *dominates* y. Furthermore, if we say x and y are *adjacent*, then we mean that there is an arc between these two vertices. We call a digraph *simple*, if firstly there are no two different parallel arcs and secondly there is no vertex, which is dominated by itself.

Definition 1.2 *[Neighborhood and degree]* If D is a digraph, then the *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the set of vertices dominating x. Therefore, if there is the arc $xy \in E(D)$, then y is an *outer neighbor* of x and x is an *inner neighbor* of y.

If X and Y are two disjoint vertex sets or subdigraphs of D such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$. Furthermore, $X \rightsquigarrow Y$ denotes the fact that there is no arc leading from Y to X. By $d(X, Y)$ we denote the number of arcs from the set X to the set Y, i.e., $d(X, Y) = |\{xy \in E(D) \mid x \in X, y \in Y\}|$.

The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are called the *outdegree* and *indegree* of x, respectively. Furthermore, the numbers $\delta_D^+ = \delta^+ = \min\{d^+(x) \mid x \in V(D)\}$ and $\delta_D^- = \delta^- = \min\{d^-(x) \mid x \in V(D)\}$

are the *minimum outdegree* and *minimum indegree*, respectively. Analogously, we define the numbers $\Delta^+_D = \Delta^+ = \max\{d^+(x) \mid x \in V(D)\}$ and $\Delta^-_D = \Delta^- = \max\{d^-(x) \mid x \in V(D)\}$, which are the *maximum outdegree* and *maximum indegree*, respectively.

Definition 1.3 *[Cycles and paths]* Let D be a digraph. A *directed path* or short a *path* of *length* l with $l \in \mathbb{N}$ is a sequence of $l+1$ pairwise disjoint vertices $v_0, v_1, \dots, v_l \in V(D)$ such that $v_i v_{i+1} \in E(D)$ for all $0 \le i < l$. We use the notation

$$P = v_0 v_1 \dots v_l.$$

A *directed cycle* or short a *cycle* of *length* l with $l \in \mathbb{N}$ is a sequence of l pairwise disjoint vertices $v_1, v_2, \dots, v_l \in V(D)$ such that $v_i v_{i+1} \in E(D)$ for all $1 \le i < l$ and $v_l v_1 \in E(D)$. If a cycle C is of the length l, then we say that C is an l-cycle. We use the notation

$$C = v_1 v_2 \dots v_l v_1.$$

A cycle or path of a digraph D is *Hamiltonian*, if it includes all the vertices of D. If D contains a Hamiltonian cycle, then we also say that D is *Hamiltonian*. A digraph D is called *pancyclic*, if it contains cycles of length n for all $n \in \{3, 4, \dots, |V(D)|\}$, and *even pancyclic*, if it contains cycles of all even lengths. If $x \in V(C)$ ($x \in V(P)$, respectively) for a cycle C (a path P, respectively), then we denote the successor of x in the given cycle (path) by x^+ and the predecessor by x^-. A digraph D is *cycle complementary*, if there exist two vertex-disjoint cycles C and C' such that $V(D) = V(C) \cup V(C')$. The *path covering number* of a digraph D ($pc(D)$) is the minimum number of paths in D that are pairwise vertex disjoint and cover the vertices of D.

Definition 1.4 *[Subdigraphs]* Let D be a digraph. A digraph H is called a *subdigraph* of D, if $V(H) \subseteq V(D)$ and $E(H) \subseteq E(D)$, and we write $H \subseteq D$. A *factor* is a subdigraph H of D with $V(H) = V(D)$. A factor is called a *cycle-factor*, if it consists of a set of vertex disjoint cycles, and it is a *k-path-cycle*, if it consists of a set of vertex disjoint paths and cycles, where k stands for the number of paths in the set. If X is an arbitrary vertex set of D, then we define $D[X]$ as the subdigraph induced by X. For any vertex set $X \subseteq V(D)$ and any vertex $x \in V(D)$ we define $D - X = D[V(D) - X]$ and $G - x = G - \{x\}$, respectively.

Definition 1.5 *[Converse]* If we replace in a digraph D every arc xy by yx, then we call the resulting digraph the *converse* of D, denoted by D^{-1}.

Definition 1.6 *[Connectivity]* A digraph D is *strongly connected* or *strong*, if, for each pair of vertices u and v, there is a path in D from u to v. A digraph D with at least $k+1$ vertices is *k-connected*, if for any set A of at most $k-1$ vertices the subdigraph $D-A$ obtained by deleting A is strong. The *connectivity* of D, denoted by $\kappa(D)$, is then defined to be the largest value of k such that D is k-connected. If $\kappa(D) = 1$, then the vertex x with the property that $D - x$ is not strong is called a *cut-vertex* of D, and if $\kappa(D) > 1$, then the vertex set S with the property that $|S| = \kappa(D)$ and $D - S$ is not strong is called a *separating set*. A strong subdigraph H of D is called a *component*, if there is no strong subdigraph $H' \subseteq D$ such that $H \subset H'$.

Definition 1.7 *[Irregularity]* There are several measures of how much a digraph differs from being regular. In [31] Volkmann defines the irregularity of a digraph D as $I(D) = \max |d^+(x) - d^-(y)|$ over all vertices x and y of D (including $x = y$). Other measures were given by Yeo [51] in 1999. He defines the *local irregularity* as

$$i_l(D) = \max_{x \in V(D)} |d^+(x) - d^-(x)|$$

and the *global irregularity*

$$i_g(D) = \max_{x \in V(D)} \{d^+(x), d^-(x)\} - \min_{y \in V(D)} \{d^+(y), d^-(y)\}.$$

Clearly, $i_l(D) \leq I(D) \leq i_g(D)$. If $i_g(D) = 0$, then D is *regular*; if $i_g(D) \leq 1$, then D is *almost regular*.

Definition 1.8 *[Multipartite tournaments and semicomplete multipartite digraphs]* A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. A *tournament* is a c-partite tournament with exactly c vertices. A *semicomplete multipartite digraph* is obtained by replacing each edge of a complete multipartite graph by an arc or by a pair of two mutually opposite arcs with the same end vertices. If $V_1, V_2, \ldots, V_c$ are the partite sets of a c-partite tournament or semicomplete c-partite digraph D and the vertex x of D belongs to the partite set V_i, then we define $V(x) = V_i$.

Definition 1.9 *[Independence and size of partite sets]* Let D be a digraph. A set $I \subseteq V(D)$ is called *independent*, if the subdigraph induced by I containes no arc. We call an independent set I *maximum*, if there is no independent set $I' \subseteq V(D)$ with $|I'| > |I|$. The cardinality of a maximum independent set is called the *independence number* denoted by $\alpha(D)$. Now, let $V_1, V_2, \ldots, V_c$ be the partite sets of a multipartite tournament D such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$. In this case, it follows that $\alpha(D) = |V_c|$. Analogously, we define $\gamma(D) = |V_1|$. If $|V_i| = n_i$ for $i = 1, 2, \ldots, c$, then we speak of the *partition-sequence* $(n_i) = n_1, n_2, \ldots, n_c$.

1.2 Degrees in multipartite tournaments

The possible vertex-degrees in a multipartite tournament D depend on the global irregularity of D and the cardinality of the partite sets of D. The following important result of Tewes, Volkmann and Yeo [24] shows the connection between these two sizes.

Lemma 1.10 (Tewes, Volkmann, Yeo [24]) *Let D be a c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$. Then $||V_i| - |V_j|| \leq 2i_g(D)$ for $1 \leq i, j \leq c$.*

In [24], there are also first bounds for vertex-degrees in close to regular multipartite tournaments.

Lemma 1.11 (Tewes, Volkmann, Yeo [24]) *Let D be a multipartite tournament. Then for every vertex x of D we have*

$$\frac{|V(D)| - \alpha(D) - i_g(D)}{2} \leq \frac{|V(D)| - \alpha(D) - i_l(D)}{2} \leq d^+(x), d^-(x) \quad \textit{and}$$

$$d^+(x), d^-(x) \leq \frac{|V(D)| - \gamma(D) + i_l(D)}{2} \leq \frac{|V(D)| - \gamma(D) + i_g(D)}{2}.$$

If we know the cardinality of the partite set $V(x)$, which contains the vertex $x \in V(D)$, then Lemma 1.11 can be improved as we can see in the following result, which can be found in [37] for the case that $i_g(D) \leq 1$ and in [44] for the general case that $i_g(D) \leq l$.

Lemma 1.12 *Let D be a multipartite tournament. If $x \in V(D)$ such that $|V(x)| = p$, then*

$$\frac{|V(D)| - p - i_g(D)}{2} \leq \frac{|V(D)| - p - i_l(D)}{2} \leq d^+(x), d^-(x) \quad \textit{and}$$

$$d^+(x), d^-(x) \leq \frac{|V(D)| - p + i_l(D)}{2} \leq \frac{|V(D)| - p + i_g(D)}{2}.$$

Proof. Let $i_l(D) \leq l$ and suppose that $d^+(x) \leq \frac{|V(D)|-p-l-1}{2}$. Because of $d^+(x) + d^-(x) = |V(D)| - |V(x)| = |V(D)| - p$, we conclude that $d^-(x) = |V(D)| - p - d^+(x) \geq \frac{|V(D)|-p+l+1}{2}$, which leads to $d^-(x) - d^+(x) \geq l + 1$, a contradiction to $i_l(D) \leq l$.

The rest of the proof follows analogously. □

This result is not always optimal as the following to lemmas show.

Lemma 1.13 (Volkmann, Winzen [44]) *If D is a multipartite tournament with $i_g(D) \leq l$ and $\gamma(D) = r$, then we have*

$$\frac{|V(D)| - \gamma(D) - 2l}{2} \leq d^+(x), d^-(x)$$

for all $x \in V(D)$. If furthermore $|V(x)| = r + 2l$, then it follows that

$$d^+(x), d^-(x) = \frac{|V(D)| - r - 2l}{2}.$$

Proof. Let $x \in V(D)$ be arbitrary. If $|V(x)| \leq r + l$, then the first assertion holds by Lemma 1.12. Hence, let $|V(x_1)| \geq r + l + 1$. Suppose that $d^+(x_1) \leq \frac{|V(D)|-r-2l-1}{2}$. Because of $i_g(D) \leq l$, we conclude that $d^+(y), d^-(y) \leq \frac{|V(D)|-r-1}{2}$ for all $y \in V(D)$. If we take a vertex $x_2 \in V(D)$ with $|V(x_2)| = r$, then we arrive at the contradiction

$$|V(D)| = d^+(x_2) + d^-(x_2) + r \leq |V(D)| - r - 1 + r = |V(D)| - 1.$$

Hence, it has to be $d^+(x) \geq \frac{|V(D)|-\gamma(D)-2l}{2}$ for all vertices $x \in V(D)$. Since the proof for $d^-(x)$ follows the same lines, the first assertion of this lemma is completed.

Let $x \in V(D)$ with $|V(x)| = r + 2l$. Suppose that $d^+(x) \geq \frac{|V(D)|-r-2l+1}{2}$. The fact that $|V(D)| = d^+(x)+d^-(x)+r+2l$ yields that $d^-(x) \leq \frac{|V(D)|-r-2l-1}{2}$, a contradiction to the first assertion of this lemma. This completes the proof of the lemma. □

Lemma 1.14 *Let D be a multipartite tournament with $i_g(D) \leq l$ and $\gamma(D) = r$. Then we have*

$$d^+(x), d^-(x) \leq \frac{|V(D)| + 2l - \alpha(D)}{2},$$

for all $x \in V(D)$. If especially $\alpha(D) = r + 2l$ and $x \in V(D)$ such that $|V(x)| = r$, then it follows that

$$d^+(x), d^-(x) = \frac{|V(D)| - r}{2}.$$

Proof. Let $x \in V(D)$ be arbitrary. If $|V(x)| \geq r + l$, then the first assertion holds by Lemma 1.12 and Lemma 1.10. Hence, let $|V(x)| \leq r + l - 1$. Suppose that $d^+(x) \geq \frac{|V(D)|+2l-\alpha(D)+1}{2}$. Because of $i_g(D) \leq l$, we conclude that $d^+(y), d^-(y) \geq \frac{|V(D)|-\alpha(D)+1}{2}$ for all $y \in V(D)$. If we take a vertex $x_1 \in V(D)$ with $|V(x_1)| = \alpha(D)$, then we arrive at the contradiction

$$|V(D)| = d^+(x_1) + d^-(x_1) + \alpha(D) \geq |V(D)| - \alpha(D) + 1 + \alpha(D) = |V(D)| + 1.$$

Hence, it has to be $d^+(x) \leq \frac{|V(D)|+2l-\alpha(D)}{2}$ for all vertices $x \in V(D)$. Since the proof for $d^-(x)$ follows the same lines , the first assertion of this lemma is completed.

Now, let $x \in V(D)$ with $|V(x)| = r$ and let $\alpha(D) = r + 2l$. Suppose that $d^+(x) \leq \frac{|V(D)|-r-1}{2}$. The fact that $|V(D)| = d^+(x) + d^-(x) + r$ yields that $d^-(x) \geq \frac{|V(D)|-r+1}{2}$, a contradiction to the first assertion of this lemma. This completes the proof of this lemma. □

For vertices that are contained in large partite sets the following upper bound presents an improvement of Lemma 1.12.

Lemma 1.15 (Winzen [47]) *Let D be a c-partite tournament with $i_g(D) \leq l$ and $\gamma(D) = r$. If $x \in V(D)$ such that $|V(x)| = r + 2l - k$ $(0 \leq k \leq 2l)$, then it follows that*

$$d^+(x), d^-(x) \leq \frac{|V(D)| - r - 2l + 2k}{2} = \frac{|V(D)| - |V(x)| + k}{2}.$$

Proof. Suppose that $d^+(x) \geq \frac{|V(D)|-r-2l+2k+1}{2}$. Then we conclude that $d^-(x) \leq |V(D)| - r - 2l + k - \frac{|V(D)-r-2l+2k+1}{2} = \frac{|V(D)|-r-2l-1}{2}$. Let $y \in V(D)$ such that

$|V(y)| = r$. Because of $i_g(D) \leq l$, it follows that $d^+(y), d^-(y) \leq d^-(x) + l \leq \frac{|V(D)|-r-1}{2}$, and we arrive at the contradiction

$$|V(D)| = d^+(y) + d^-(y) + r \leq |V(D)| - 1.$$

Hence, the assertion for $d^+(x)$ holds. The assertion for $d^-(x)$ follows analogously. This completes the proof of the lemma. $\square$

In the meantime, we will treat regular or almost regular tournaments. So, the lemmas above yield the following remarks, which can be found in [37, 38] and [43].

Remark 1.16 *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament. Then Lemma 1.10 implies that $r = |V_1| = |V_2| = \ldots = |V_c|$ and*

$$d^+(x), d^-(x) = \frac{(c-1)r}{2}$$

for all $x \in V(D)$. That means especially that c is odd, if r is odd.

Remark 1.17 *Let D be an almost regular c-partite tournament with $\gamma(D) = \alpha(D) = r$. In this case, Lemma 1.11 yields for all $x \in V(D)$ that*

$$\frac{(c-1)r-1}{2} \leq d^+(x), d^-(x) \leq \frac{(c-1)r+1}{2}.$$

Hence, if r is even or if c is odd, then we see that $d^+(x) = d^-(x) = \frac{(c-1)r}{2}$ and that D is regular.

Remark 1.18 *Let D be an almost regular c-partite tournament with $\alpha(D) = r+2$ and $\gamma(D) = r$. Then $|V(D)| - r$ is even. So the bounds in Lemma 1.12 can be improved by*

$$d^+(x), d^-(x) = \frac{|V(D)|-r-2}{2} \quad \text{if} \quad |V(x)| = r+2$$

or

$$d^+(x), d^-(x) = \frac{|V(D)|-r}{2} \quad \text{if} \quad |V(x)| = r.$$

Consequently, for the case that $\alpha(D) = r+2$, instead of Lemma 1.11, we can use the following result:

$$\frac{|V(D)|-r-2}{2} \leq d^+(x), d^-(x) \leq \frac{|V(D)|-r}{2}.$$

Now let us summarize some results of Lemma 1.12 and Remark 1.18.

Corollary 1.19 *If D is an almost regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $r = |V_1| \leq |V_2| \leq \ldots \leq |V_c| \leq r+2$, then for every vertex x of D we have*

$$\frac{|V(D)|-r-2}{2} \leq d^+(x), d^-(x).$$

Part I

Cycles in multipartite tournaments

Chapter 2

Cycles of a given length through an arc

In this chapter we study almost regular multipartite tournaments of a given (short) length through a given arc. It is very easy to see that every arc of a regular tournament belongs to a 3-cycle. The next example shows that this is not valid for regular multipartite tournaments in general.

Example 2.1 (Volkmann [32]) *Let C, C', and C'' be three induced cycles of length 4 such that $C \to C' \to C'' \to C$. The resulting 6-partite tournament D_1 is 5-regular, but no arc of the three cycles C, C', C'' is contained in a 3-cycle.*

Let H, H_1, and H_2 be three copies of D_1 such that $H \to H_1 \to H_2 \to H$. The resulting 18-partite tournament is 17-regular, but no arc of the cycles corresponding to the cycles C, C', and C'' is contained in a 3-cycle.

If we continue this process, we arrive at regular c-partite tournaments with arbitrary large c, which contain arcs that do not belong to any 3-cycle.

In 1998, Guo [9] proved the following generalization of Alspach's classical result [1] that every regular tournament is arc pancyclic.

Theorem 2.2 (Guo [9]) *Let D be a regular c-partite tournament with $c \geq 3$. If every arc of D is contained in a 3-cycle, then every arc of D is contained in an n-cycle for each $n \in \{4, 5, \ldots, c\}$.*

Now, the aim was to carry this result forward to almost regular multipartite tournaments. To reach this, Volkmann [30], [32] started with the following theorems.

Theorem 2.3 (Volkmann [32]) *Let D be an almost regular c-partite tournament.*

If $c \geq 8$, then every arc of D is contained in a 4-cycle.

If $c = 7$ and there are at least two vertices in every partite set, then every arc of D is contained in a 4-cycle.

Theorem 2.4 (Volkmann [30]) *Let D be an almost regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| = |V_2| = \ldots = |V_c| = r \geq 2$. If $c \geq 6$, then every arc of D is contained in an n-cycle for each $n \in \{4, 5, \ldots, c\}$.*

The main theorem of this chapter is the following extension and supplement of the Theorems 2.3 and 2.4

Theorem 2.5 (Volkmann, Winzen [42]) *Let D be an almost regular c-partite tournament and $e \in E(D)$ is an arbitrary arc of D. Then the following holds.*

a) If $c \geq 8$, then e is contained in an n-cycle for each $n \in \{4, 5, \dots, c\}$.

b) If $c = 7$ and there are at least two vertices in every partite set, then e is contained in an n-cycle for each $n \in \{4, 5, \dots, c\}$.

This result is also a supplement to the following theorem of Jakobsen.

Theorem 2.6 (Jakobsen [19]) *If T is an almost regular tournament of order $n \geq 8$, then every arc of T is contained in an m-cycle for each $m \in \{4, 5, \dots, n\}$.*

The following two examples, which can also be found in [32], show that the condition $c = 7$ in Theorem 2.5 b) and the condition $c \geq 8$ in Theorem 2.5 a) are best possible.

Example 2.7 *Let $V_1 = \{u\} \cup V_1'$ with $|V_1'| = 2$, $V_2 = \{v\} \cup V_2'$ with $|V_2'| = 2$, $V_3 = V_3' \cup V_3''$ with $|V_3'| = |V_3''| = 2$, and V_4, V_5, V_6 with $|V_4| = |V_5| = |V_6| = 2$ and $V_4 = \{x, y\}$ be the partite sets of a 6-partite tournament such that $u \rightarrow v \rightarrow V_1' \rightarrow (V_4 \cup V_5 \cup V_6) \rightarrow V_2' \rightarrow u \rightarrow (V_4 \cup V_5 \cup V_6) \rightarrow v$, $V_2' \rightarrow V_3 \rightarrow u$, $v \rightarrow V_3 \rightarrow V_1'$, $V_2' \rightarrow V_1'$, $V_4 \rightarrow V_5 \rightarrow V_6 \rightarrow V_4$, and $V_3' \rightarrow (V_6 \cup \{x\}) \rightarrow V_3'' \rightarrow (V_5 \cup \{y\}) \rightarrow V_3'$ (see Figure 2.1). The resulting 6-partite tournament is almost regular with at least two vertices in every partite set; however, the arc uv is not contained in a 4-cycle.*

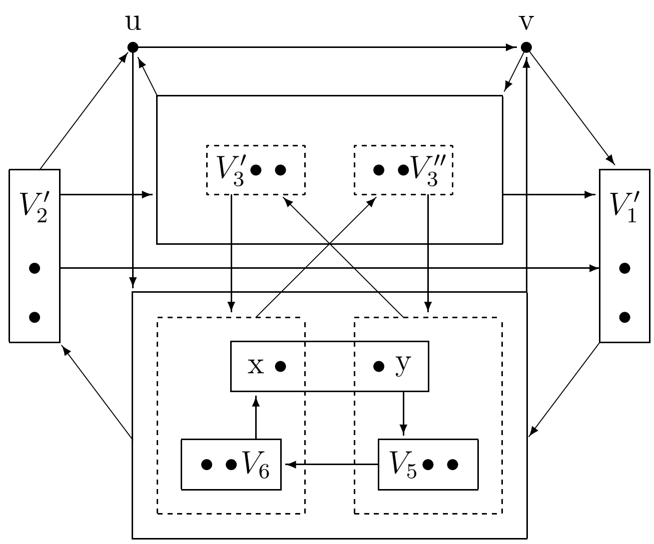

Figure 2.1: An almost regular 6-partite tournament with the property that the arc uv is not contained in a 4-cycle

Example 2.8 *Let* $V_1 = \{u, u_2\}$, $V_2 = \{v, v_2\}$, $V_3 = \{w_1, w_2, w_3\}$, $V_4 = \{x\}$, $V_5 = \{y\}$, $V_6 = \{z\}$, *and* $V_7 = \{a\}$ *be the partite sets of a* 7*-partite tournament such that* $u \to v \to u_2 \to \{a, x, y, z\} \to v_2 \to u \to \{a, x, y, z\} \to v \to V_3 \to u$, $v_2 \to u_2$, $v_2 \to V_3 \to u_2$, $w_1 \to a \to x \to y \to z \to a \to y \to w_1 \to z \to x \to w_1$, $w_2 \to z \to w_3 \to a \to w_2 \to x \to w_3 \to y \to w_2$ *(see Figure 2.2). The resulting* 7*-partite tournament is almost regular; however, the arc* uv *is not contained in a* 4*-cycle. Consequently, the condition* $c \geq 8$ *in Theorem 2.5 is necessary.*

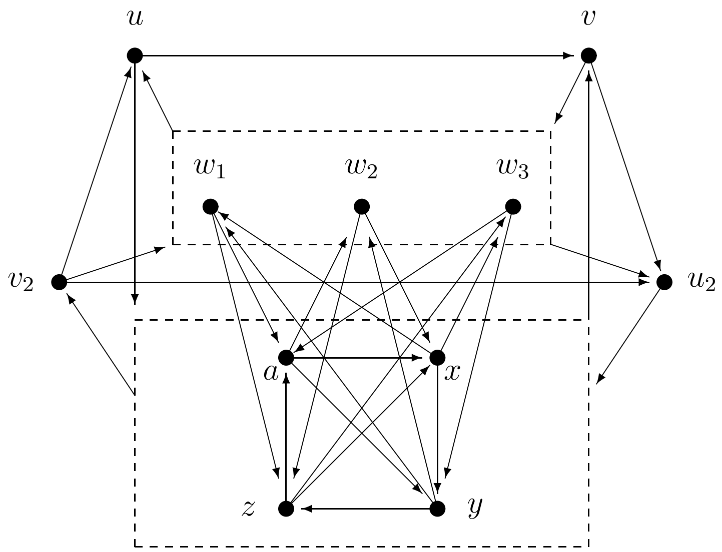

Figure 2.2: An almost regular 7-partite tournament with the property that the arc uv is not contained in a 4-cycle

Since the proof of Theorem 2.5 is very long and complicated, we will split it into two parts. In the first section, we will study the case that $c \geq 7$ and that there are at least two vertices in each partite set, whereas in the second section we will treat the case that $c \geq 8$ and $\gamma(D) = 1$.

2.1 The case $\gamma(D) \geq 2$

In this section we treat the case that there are at least two vertices in each partite set. The next well-known theoreom of Turán [27] (see also [28], p. 212) will be helpful in the main theorems of this and of the following section. To understand the terminology of this theorem, we first present two short definitions.

Definition 2.9 *[Underlying graph, Clique]*

a) Let D be a simple digraph. The unique graph G we obtain, if we replace each arc of D by an edge, is the *underlying graph* of D.

b) A complete subgraph H of a graph G is called a *clique*.

Theorem 2.10 (Turán 1941) *Let D be a digraph without 2-cycles. If the underlying graph D has no clique of order $p+1$, then*

$$|E(D)| \leq \frac{p-1}{2p}|V(D)|^2.$$

Let D be an almost regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $r = |V_1| \leq |V_2| \leq \ldots \leq |V_c|$. According to Lemma 1.10 and Theorem 2.4 it remains to consider the case that $r+1 \leq |V_c| \leq r+2$.

Theorem 2.11 (Volkmann, Winzen [37]) *Let D be an almost regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $2 \leq r = |V_1| \leq |V_2| \leq \ldots \leq |V_c| \leq r+2$ and $|V_c| \geq r+1$. If $c \geq 7$, then every arc of D is contained in an n-cycle for each $n \in \{4, 5, \ldots, c\}$.*

Proof. We prove the theorem by induction on n. For $n = 4$ the result follows from Theorem 2.3. Now let e be an arc of D and assume that e is contained in an n-cycle $C = a_n a_1 a_2 \ldots a_{n-1} a_n$ with $e = a_n a_1$ and $4 \leq n < c$. Suppose that $e = a_n a_1$ is not contained in any $(n+1)$-cycle.

Obviously, $|V(D)| = cr + k$ with $1 \leq k \leq c-1$, if $|V_c| = r+1$ and $2 \leq k \leq 2c-2$, if $|V_c| = r+2$. Firstly, we observe that $N^+(v) - V(C) \neq \emptyset$ for each $v \in V(C) = \{a_1, a_2, \ldots, a_n\}$, because otherwise Corollary 1.19, the fact that $r \geq 2$ and $k \geq 1$ yield the contradiction

$$n = |V(C)| \geq d^+(v) + 2 \geq \frac{cr + k - r - 2}{2} + 2 = \frac{(c-1)r + k + 2}{2} > c.$$

Analogously, one can show that $N^-(v) - V(C) \neq \emptyset$ for each $v \in V(C)$.

Next let S be the set of vertices that belong to partite sets not represented on C and define

$$X = \{x \in S \,|\, C \to x\}, \qquad Y = \{y \in S \,|\, y \to C\}.$$

Assume that $X \neq \emptyset$ and let $x \in X$. If there is a vertex $w \in N^-(a_n) - V(C)$ such that $x \to w$, then $a_n a_1 a_2 \ldots a_{n-2} x w a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. If $(N^-(a_n) - V(C)) \to x$, then $|N^-(x)| \geq |N^-(a_n) - V(C)| + |V(C)| \geq |N^-(a_n)| + 2$, a contradiction to the hypothesis that $i_g(D) \leq 1$. If there exists a vertex $b \in (N^-(a_n) - V(C))$ such that $V(b) = V(x)$, then b is adjacent with all vertices of C. In the case that $N^-(b) \cap V(C) \neq \emptyset$, let $l = \max_{1 \leq i \leq n-1}\{i \mid a_i \to b\}$. Then $a_n a_1 \ldots a_l b a_{l+1} \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. It remains to consider the case that $N^-(b) \cap V(C) = \emptyset$. If there is a vertex $u \in (N^-(b) - V(C)) = N^-(b)$ such that $x \to u$, then $a_n a_1 a_2 \ldots a_{n-3} x u b a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. Otherwise, $N^-(b) \to x$, and we arrive at the contradiction $d^-(x) \geq d^-(b) + |V(C)|$. Altogether, we have seen that $X \neq \emptyset$ is not possible, and analogously we find that $Y \neq \emptyset$ is impossible. Consequently, from now on we shall assume that $X = Y = \emptyset$.

By the definition of S, every vertex of $V(C)$ is adjacent to every vertex of S, and from our assumption $n < c$, we deduce that $S \neq \emptyset$. Now we distinguish different cases.

Case 1. There exists a vertex $v \in S$ with $v \to a_n$. Since $Y = \emptyset$, there is a vertex $a_i \in V(C)$ such that $a_i \to v$. If $l = \max_{1 \leq i \leq n-1}\{i \mid a_i \to v\}$, then $a_n a_1 \ldots a_l v a_{l+1} \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. This implies $a_n \to S$.

Case 2. There exists a vertex $v \in S$ with $a_1 \to v$. Since $X = \emptyset$, there is a vertex $a_i \in V(C)$ such that $v \to a_i$. If $l = \min_{2 \leq i \leq n-1}\{i \mid v \to a_i\}$, then $a_n a_1 \ldots a_{l-1} v a_l \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. This implies $S \to a_1$.

If $C = a_n a_1 a_2 \ldots a_n$ and $v \in S$, then the following three sets play an important role in our investigations

$$H = N^+(a_1) - V(C), \qquad F = N^-(a_n) - V(C), \qquad Q = N^-(v) - V(C).$$

Case 3. There exists a vertex $v \in S$ such that $v \to a_{n-1}$. If there is a vertex $a_i \in V(C)$ with $2 \leq i \leq n-2$ such that $a_i \to v$, then we obtain as above an $(n+1)$-cycle through $a_n a_1$, a contradiction. Thus, we investigate now the case that $v \to \{a_1, a_2, \ldots, a_{n-1}\}$. Because of $S \to a_1$, we note that every vertex of $N^+(a_1)$ is adjacent to v. If there is a vertex $x \in H$ such that $x \to v$, then $a_n a_1 x v a_3 a_4 \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. Therefore we assume now that $v \to H$. This leads to $d^+(v) \geq d^+(a_1) + 1$, and thus, because of $i_g(D) \leq 1$, it follows that $N^+(v) = N^+(a_1) \cup \{a_1\}$ and $a_1 \to \{a_2, a_3, \ldots, a_{n-1}\}$.

It is a simple matter to verify that $H \cap Q = \emptyset$, $S \cap H = \emptyset$ and $R = V(D) - (H \cup Q \cup V(v) \cup V(C)) = \emptyset$.

If there is an arc $x a_2$ with $x \in H$, then $a_n a_1 x a_2 a_3 \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction.

Subcase 3.1. Firstly, let H consist of vertices of only one partite set V_z. At least one vertex of V_z belongs to $V(C)$, that means $|H| \leq r+1$, if $|V_z| = r+2$, $|H| \leq r$, if $|V_z| = r+1$ and $|H| \leq r-1$, if $|V_z| = r$.

Because of Corollary 1.19 and $n \leq c-1$, we have

$$\frac{cr + k - r - 2}{2} - (c-3) \leq d^+(a_1) - (n-2) = |H|. \tag{2.1}$$

If $|V_z| = r$, then because of $|H| \leq r-1$, (2.1) yields $(c-3)r + k + 6 \leq 2c$. Since $r \geq 2$ and $k \geq 1$, this leads to the contradiction $2c + 1 \leq 2c$.

If $n = 4$, then we observe that $n \leq c-3$, and this implies

$$\frac{cr + k - r - 2}{2} - (c-5) \leq d^+(a_1) - (n-2) = |H| \leq r+1.$$

This leads again to $(c-3)r + k + 6 \leq 2c$, a contradiction. Consequently, it remains to treat the cases with $|V_z| \geq r+1$ and $n \geq 5$.

Subcase 3.1.1. Assume that $|V_c| = r+1$ and $|V_z| = r+1$. If $|V(a_1)| = r+1$ (and therefore $k \geq 2$), then (2.1) leads to $r = 2$, $|H| = r = 2$ and $k = 2$.

If $|V(a_1)| = r$, then together with Lemma 1.12 and $n \leq c-1$, we arrive at

$$\frac{cr + k - r - 1}{2} - (c-3) \leq d^+(a_1) - (n-2) = |H| \leq r,$$

and hence $(c-3)r+k+5 \leq 2c$. This leads to no contradiction, only if $r=2$, $|H|=r=2$ and $k=1$.

Consequently, it remains to consider the case that $|H|=r=2$ and $k=1$ or $k=2$ and $|V(a_1)|=r+1$. Therefore, we observe that $|V(v)|=r$.

Since $n \geq 5$, we have $Q \leadsto H$, because otherwise, if there are vertices $q \in Q$ and $h \in H$ such that $h \rightarrow q$, then $a_n a_1 h q v a_4 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Thus, for every vertex $h \in H$, we conclude that $d^+(h) \leq r-1+n-2=n-1$. Since $d^+(v)=d^+(a_1)+1=r+n-1=n+1$, this is a contradiction to $i_g(D) \leq 1$.

Subcase 3.1.2. Now let $|V_c|=r+2$. If $|V_z|=r+1$, then, because of $|H| \leq r$, (2.1) leads to $(c-3)r+k+4 \leq 2c$. Since in this case $k \geq 3$ and $r \geq 2$, this yields the contradiction $2c+1 \leq 2c$.

Finally, let $|V_z|=r+2$. Then (2.1) leads to the contradiction $c \leq 5$, if $r \geq 3$, and to the contradiction $1 \leq 0$, if $r=2$ and $k \geq 5$. Therefore, let $r=2$ and $k \in \{2,3,4\}$. Since $cr+k-r$ is even, the case $k=3$ is not possible.

Furthermore, we have a contradiction in (2.1), if $|H| \leq r$. Therefore, let $|H|=r+1$. Since $d^+(v)=d^+(a_1)+1$, we conclude that $|V(v)| \leq r+1$. Because of $n \geq 5$, analogously as in Subcase 3.1.1, we see that $(Q \cup \{a_1,a_2,v\}) \leadsto H$, and thus $d^+(h) \leq r+n-2=n$, if $h \in H$. On the other hand, we have seen that $d^+(v)=d^+(a_1)+1=r+1+n-1=n+2$, a contradiction to $i_g(D) \leq 1$.

Subcase 3.2. Let $n \geq 5$ and let H consist of vertices of more than one partite set. Then there is at least one arc $pq \in E(D[H])$. Let L be the set of all vertices in H with an inner neighbor in H, and $M=H-L$. Then we note that $L \neq \emptyset$. M consists of vertices of at most one partite set and $M \leadsto L$. If we take a vertex $q \in L$ with an inner neighbor $p \in H$, then it cannot be that $qa_3 \in E(D)$, because otherwise $a_n a_1 p q a_3 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Therefore let $a_3 \leadsto L$. If there is an arc xy with $x \in H$ and $y \in Q$, then $a_n a_1 x y v a_4 a_5 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Altogether, we have seen that $(Q \cup M \cup \{a_1,a_2,a_3\}) \leadsto L$.

First, let $|V(v)|=r+2$. Then, because of $d^+(v) \geq d^+(a_1)+1$, Remark 1.18 yields the contradiction

$$\frac{cr+k-r-2}{2}+1 \leq d^+(a_1)+1 \leq d^+(v)=\frac{cr+k-r-2}{2}.$$

Now let $|V(v)| \leq r+1$. Since $|R|=0$, for every vertex $q \in L$, we conclude that $d(q,V(D)-L) \leq n+r-3$, and thus, it follows with Corollary 1.19 that $d^+_{D[L]}(q)=d^+(q)-d(q,V(D)-L) \geq \frac{cr+k-r-2}{2}-r-n+3$. This implies

$$\begin{aligned} \frac{|L|(|L|-1)}{2} &\geq |E(D[L])| = \\ \sum_{q \in L} d^+_{D[L]}(q) &\geq |L|\left\{\frac{cr+k-r-2}{2}-r-n+3\right\}. \end{aligned} \tag{2.2}$$

The conditions $d^+(v) \geq d^+(a_1)+1$, $a_1 \rightarrow \{a_2,a_3,\ldots,a_{n-1}\}$, and Lemma 1.11 (respectively, Remark 1.18, if $|V_c|=r+2$) yield $|L|=|H|-|M|=d^+(a_1)-n+2-|M| \leq d^+(v)-1-n+2-|M| \leq \frac{cr+k-r+1}{2}-n+1-|M|$ (respectively,

$|L| \leq \frac{cr+k-r}{2} - n + 1 - |M|$, if $|V_c| = r+2$). Combining this with inequality (2.2), we obtain

$$\frac{cr+k-r+1}{2} - n - |M| \geq |L| - 1 \geq 2\left\{\frac{cr+k-r-2}{2} - r - n + 3\right\},$$

if $|V_c| = r+1$ and

$$\frac{cr+k-r}{2} - n - |M| \geq |L| - 1 \geq 2\left\{\frac{cr+k-r-2}{2} - r - n + 3\right\},$$

if $|V_c| = r+2$. This leads to $2n \geq (c-5)r + k + 7 + 2|M|$ (respectively, $2n \geq (c-5)r + k + 8 + 2|M|$, if $|V_c| = r+2$). Because of $k \geq 1$, $r \geq 2$ and $n \leq c-1$, this is a contradiction, if $|M| \geq 1$ (a contradiction, if $|V_c| = r+2$).

Consequently, it remains to consider the case that $|M| = 0$. This means that every vertex in $H = L$ has an inner neighbor in H. Therefore, $|L| = |H| \geq 3$, and every vertex in H is the last point of a path of length 2. If $a_4 \rightsquigarrow H$, then, because of $d(q, V(D) - L) \leq r + n - 4$, we obtain a contradiction as above. Thus, let $q_3a_4 \in E(D)$ with $q_3 \in H$, and let q_3 be the last point of the path $q_1q_2q_3$ in H, then $a_na_1q_1q_2q_3a_4 \ldots a_n$ is an $(n+1)$-cycle through a_na_1, a contradiction.

Subcase 3.3. Finally, let $n = 4$ and let H consist of vertices of more than one partite set. Let us define the set G by $G = N^+(a_3) - V(C)$. If there is a vertex $w \in F \cap G$, then $a_4a_1a_2a_3wa_4$ is a 5-cycle through a_4a_1, a contradiction. If there is an arc xy with $x \in G$ and $y \in F$, then $a_4a_1a_3xya_4$ is a 5-cycle, a contradiction. Consequently, it remains to consider the case that $F \cap G = \emptyset$ and $F \rightsquigarrow (G \cup \{a_3, a_4\})$.

According to Corollary 1.19, we have

$$|G| = |N^+(a_3)| - 1 \geq \frac{cr+k-r-2}{2} - 1 = \frac{cr+k-r-4}{2},$$

and thus, it follows for every vertex $x \in F$ that

$$\begin{aligned} d(V(D)-F, x) &\leq cr + k - |F| - |G| - 2 \\ &\leq \frac{cr+k+r+4}{2} - |F| - 2 = \frac{cr+k+r}{2} - |F|. \end{aligned}$$

This leads to

$$d^-_{D[F]}(x) \geq \frac{cr+k-r-2}{2} - \frac{cr+k+r}{2} + |F| = |F| - r - 1$$

for every $x \in F$. Hence, we conclude on the one hand that

$$|E(D[F])| = \sum_{x \in F} d^-_{D[F]}(x) \geq |F|(|F| - r - 1).$$

On the other hand, since $S \cap F = \emptyset$, the subdigraph $D[F]$ is 3-partite, and thus, Theorem 2.10 yields

$$|E(D[F])| \leq \frac{1}{3}|F|^2.$$

The last two inequalities imply $r \geq \frac{2}{3}|F| - 1$. Since $|F| = |N^-(a_4) - V(C)| \geq d^-(a_4) - 2$, we deduce from Corollary 1.19 that

$$\begin{aligned} r &\geq \frac{2|F|}{3} - 1 \geq \frac{cr + k - r - 6}{3} - 1 = \frac{cr + k - r - 9}{3} \\ \Leftrightarrow 3r &\geq (c-1)r + k - 9. \end{aligned} \tag{2.3}$$

Subcase 3.3.1. Let $|V_c| = r + 1$. Then, (2.3) leads to no contradiction, only if $c = 8$, $r = 2$ and $k = 1$ or if $c = 7$, $r = 2$ and $k \leq 3$.

Firstly, let $c = 8$, $r = 2$ and $k = 1$. Then we note that $|H| \leq 4$, and thus, it follows that

$$9 \leq |S| + 1 \leq d^+(a_4) \leq d^+(a_1) + 1 = |H| + 3 \leq 7,$$

a contradiction.

Therefore, it remains to consider the case that $c = 7$, $r = 2$ and $k \leq 3$. If $D[V(C)]$ is no tournament (that means that $V(a_2) = V(a_4)$), then we have $|S| \geq 4r = 8$ and $|H| \leq 3$, and therefore we arrive at the contradiction

$$9 \leq |S| + 1 \leq d^+(a_4) \leq d^+(a_1) + 1 = |H| + 3 \leq 6.$$

Consequently, we investigate the case that $D[V(C)]$ is a tournament. Then we see that

$$7 \leq |S| + 1 \leq d^+(a_4) \leq d^+(a_1) + 1 = |H| + 3,$$

and this yields $|H| \geq 4$. If $|H| = 4$, then we have equality in the last inequality chain, which implies $H \rightsquigarrow a_4$ and $a_2 \rightarrow a_4$. Let $x \in N^+(h) - V(C)$ with $h \in H$ such that $x \rightarrow a_2$, then $a_4 a_1 h x a_2 a_4$ is a 5-cycle, a contradiction. Consequently, $a_2 \rightsquigarrow N^+(h) - V(C)$ for every vertex $h \in H$. If every element of H has an outer neighbor in H, then there exists a 3-cycle or a 4-cycle in H. Now, we take a vertex $h_3 \in H - V(a_4)$ such that h_3 is contained in a cycle $h_3 h_1 h_2 h_3$ or $h_4 h_1 h_2 h_3 h_4$ in H. This leads to the 5-cycle $a_4 a_1 h_1 h_2 h_3 a_4$, a contradiction. Hence, there exists a vertex $h_0 \in H$ such that $N^+_{D[H]}(h_0) = \emptyset$. Since $a_2 \rightsquigarrow H$, $a_2 \rightarrow \{a_3, a_4\}$ and $N^+(h_0) \cap V(C) \subseteq \{a_3, a_4\}$, it follows that

$$\begin{aligned} d^+(a_2) &\geq |H| + 2 + |N^+(h_0) - V(C)| - |V(a_2) - \{a_2\}| \\ &\geq 4 + |N^+(h_0) - V(C)| \geq d^+(h_0) + 2, \end{aligned}$$

a contradiction to $i_g(D) \leq 1$.

Therefore, let $5 \leq |H| \leq 6$. Then H contains vertices of exactly three partite sets and $k \geq 2$. In the case that $|H| = 5$ (respectively, $|H| = 6$), the vertex a_4 has at most one (respectively two, if $|H| = 6$) further outer neighbors except S and a_1. If $a_2 \rightarrow a_4$, then $H_1 = H - N^+(a_4)$ consists of at least four elements and $H_1 \rightsquigarrow a_4$. Then, analogously to the case $|H| = 4$, we arrive at a contradiction.

Consequently, let $a_4 \rightarrow a_2$. Then, because of $|F| = |N^-(a_4) - V(C)| \geq d^-(a_4) - 1$, we get instead of (2.3) the better bound $r \geq \frac{cr+k-r-7}{3}$. Since $c = 7$, this yields $7 \geq 3r + k$, a contradiction to $k \geq 2$.

Subcase 3.3.2. Now let $|V_c| = r+2$. Then (2.3) leads to no contradiction, only if $c = 7$, $r = 2$ and $2 \leq k \leq 3$. Since, with respect to Remark 1.18, $k = 3$ is impossible, it remains to treat the cases when $|V(a_3)| = r$ or $|V(a_4)| = r$.

If $|V(a_3)| = r$, then we obtain with Remark 1.18 that

$$|G| = |N^+(a_3)| - 1 = \frac{cr+k-r}{2} - 1 = \frac{cr+k-r-2}{2}.$$

Following the same lines as above, we arrive at the inequality $(c-4)r + k \leq 6$ which leads to the contradiction $c \leq 6$.

If $|V(a_4)| = r$, then, according to Remark 1.18, we obtain the estimation

$$|F| = |N^-(a_4) - V(C)| \geq d^-(a_4) - 2 \geq \frac{cr+k-r}{2} - 2 = \frac{cr+k-r-4}{2}.$$

In this case, following the same way as above, we get the inequality $(c-4)r + k \leq 7$, which leads to the contradiction $c \leq 13/2$.

Summarizing the investigations of Case 3, we see that it remains to consider the case that $a_{n-1} \to S$.

Case 4. There exists a vertex $v \in S$ such that $a_2 \to v$. If we consider the converse of D, then, analogously to Case 3, it remains to treat the case that $S \to a_2$.

Summarizing the investigations in the Cases 1 - 4, we can assume in the following, usually without saying so, that

$$\{a_{n-1}, a_n\} \to S \to \{a_1, a_2\} \rightsquigarrow H. \tag{2.4}$$

Case 5. Let $n = 4$. Because of (2.4), we have $a_4 \to S$ and thus $S \cup \{a_1\} \subseteq N^+(a_4)$. If $D[V(C)]$ is 3-partite or 2-partite, then, in the case that $|V_c| = r+1$, we see that

$$1 + (c-3)r \leq |S| + 1 \leq d^+(a_4) \leq d^+(a_1) + 1 \leq |H| + 3 \leq 2r + 3,$$

and in the case that $|V_c| = r+2$, we obtain

$$\begin{array}{rcl} 1 + (c-3)r & \leq & |S| + 1 \leq d^+(a_4) \leq d^+(a_1) + 1 \leq |H| + 2 \leq 2r + 4 \\ \text{if} & & V(a_1) = V(a_3) \quad \text{and} \\ 1 + (c-3)r & \leq & |S| + 1 \leq d^+(a_4) \leq d^+(a_1) + 1 \leq |H| + 3 \leq 2r + 4 \\ \text{if} & & V(a_2) = V(a_4). \end{array}$$

All these cases yield a contradiction to $c \geq 7$. Consequently, it remains to consider the case that $D[V(C)]$ is a tournament.

Firstly, let $a_2 \to a_4$. If $a_1 \to a_3$ and $v \in S$, then $a_4a_1a_3va_2a_4$ is a 5-cycle, a contradiction. Let now $a_3 \to a_1$. If there are vertices $v \in S$ and $x \in H$ such that $x \to v$, then $a_4a_1xva_2a_4$ is a 5-cycle, a contradiction. Otherwise, we have $S \to H$. If we choose $v, w \in S$ such that $v \to w$, then $N^+(a_1) = H \cup \{a_2\}$ and $N^+(v) \supseteq H \cup \{a_1, a_2, w\}$, a contradiction to $i_g(D) \leq 1$.

Now assume that $a_4 \to a_2$. Firstly, let $a_1 \to a_3$. If there are vertices $v \in S$ and $x \in F = N^-(a_4) - V(C)$ such that $v \to x$, then $a_4a_1a_3vxa_4$ is a 5-cycle, a contradiction. Otherwise, we have $F \to S$. If we choose $v, w \in S$ such that $v \to w$, then we see that $N^-(a_4) = F \cup \{a_3\}$ and $N^-(w) \supseteq F \cup \{a_3, a_4, v\}$, a contradiction to $i_g(D) \leq 1$. In the remaining case that $a_3 \to a_1$, it follows from Corollary 1.19 that

$$\begin{aligned} cr + k &= |V(D)| \geq |H| + |F| + |S| + |V(C)| - |H \cap F| \\ &\geq \frac{cr + k - r - 2}{2} - 1 + \frac{cr + k - r - 2}{2} - 1 \\ &\quad +(c-4)r + 4 - |H \cap F| \\ &= 2cr + k - 5r - |H \cap F|. \end{aligned}$$

Consequently, $|H \cap F| \geq (c-5)r \geq 2r$ and thus, $H \cap F$ consists of at least two partite sets. If we choose $u_2, u_3 \in H \cap F$ such that $u_2 \to u_3$, then $C' = a_4a_1u_2u_3a_4$ is also a 4-cycle through a_4a_1. Since $u_2 \to a_4$, we arrive, analogously to above, at a contradiction.

Altogether, we have shown in the meantime that every arc of D belongs to a 5-cycle.

Case 6. Let $n \geq 5$ and assume that there exists a vertex $v \in S$ such that $v \to a_{n-2}$. If there is a vertex $a_i \in V(C)$ with $3 \leq i \leq n-3$ such that $a_i \to v$, then we obtain, as in Case 1, an $(n+1)$-cycle through a_na_1, a contradiction. Thus, we investigate now the case that $v \to \{a_1, a_2, \ldots, a_{n-2}\}$. If there is a vertex $h \in H$ such that $h \to v$, then $a_na_1hva_3a_4\ldots a_n$ is an $(n+1)$-cycle through a_na_1, a contradiction. Therefore, we assume now that $v \to H$. This leads to $d^+(v) \geq d^+(a_1)$, and thus, because of $i_g(D) \leq 1$, it follows that $a_1 \to \{a_2, a_3, \ldots, a_{n-1}\}$ or $a_1 \to \{a_2, a_3, \ldots, a_{n-1}\} - \{a_j\}$ for some $j \in \{3, 4, \ldots, n-1\}$ and $a_j \to a_1$ or $V(a_1) = V(a_j)$.

Subcase 6.1. Assume that $a_1 \to \{a_2, a_3, \ldots, a_{n-1}\}$. If there is a vertex $h \in H$ such that $h \to a_n$, then $a_na_1a_3a_4\ldots a_{n-1}vha_n$ is an $(n+1)$-cycle, a contradiction. Therefore, we may assume now that $a_n \to (H - V(a_n))$. If $a_{i-1} \to a_n$ for $3 \leq i \leq n-1$, then $a_na_1a_ia_{i+1}\ldots a_{n-1}va_2a_3\ldots a_{i-1}a_n$ is an $(n+1)$-cycle, a contradiction. Hence, it remains to treat the case that $a_n \to a_{i-1}$ or $a_{i-1} \in V(a_n)$ for $2 \leq i \leq n-1$. Let $\{a_1, a_2, \ldots, a_{n-2}\} = A \cup B$ such that $a_n \to A$ and $B \subseteq V(a_n)$. Then $N^+(a_1) = H \cup \{a_2, a_3, \ldots, a_{n-1}\}$ and $N^+(a_n) \supseteq A \cup S \cup (H - (V(a_n) - (B \cup \{a_n\})))$. This leads to

$$d^+(a_n) \geq |A| + |S| + |H| - (r + 1 - (|B| + 1)) = d^+(a_1) + |S| - r,$$

if $|V_c| = r+1$ (and $d^+(a_n) \geq d^+(a_1) + |S| - (r+1)$, if $|V_c| = r+2$). To get no contradiction, S has to consist of only one partite set, that means $n = c-1$, $D[V(C)]$ is a tournament, $B = \emptyset$ and $a_n \to \{a_1, a_2, \ldots, a_{n-2}\}$ (respectively, $n = c-1$, $D[V(C)]$ is a tournament or $n = c-2$, $r = 2$, $|S| = 2r = 4$, $|V(a_n)| = r+2 = 4$, $d^+(a_n) = d^+(a_1)+1$). Now define $R = V(D) - (H \cup F \cup S \cup V(C))$. Since $H \cap F = \emptyset$, we obtain by Corollary 1.19

$$|R| \leq cr + k - \left\{ \frac{cr + k - r - 2}{2} - (n-2) + \frac{cr + k - r - 2}{2} - 1 + |S| + n \right\}.$$

This yields $|R| \leq 1$, if $|S| = r$, $|R| = 0$, if $|S| = r+1$, and $|R| \leq -1$, if $|S| = 2r$ or $|S| = r+2$. Thus, it follows that $n = c-1$ and $|S| \leq r+1$ in all cases. Furthermore, we see that $|S| + |R| \leq r+1$.

If there is an arc $h \to y$ with $h \in H$ and $y \in F$, then we observe that $a_n a_1 a_4 \ldots a_{n-1} v h y a_n$ is an $(n+1)$-cycle, a contradiction. Hence let $(F \cup \{a_1, a_2, a_n, v\}) \rightsquigarrow H$. Now let L be the set of vertices in H having an inner neighbor in H, and let $M = H - L$. In the case that $L \neq \emptyset$ and $b \in L$, it cannot be that $ba_3 \in E(D)$, because otherwise $a_n a_1 a b a_3 a_4 \ldots a_n$ is an $(n+1)$-cycle, if $a \in H$ is an inner neighbor of b, a contradiction. Furthermore, we denote that $M \rightsquigarrow L$ and that M consists of vertices of at most one partite set.

Hence, for every vertex $b \in L$, we conclude that $d(b, V(D) - L) \leq n - 4 + |S| - 1 + |R| \leq r + n - 4 = r + c - 5$. Now it follows from Corollary 1.19 that

$$d^+_{D[L]}(b) = d^+(b) - d(b, V(D) - L) \geq \frac{cr + k - r - 2}{2} - r - c + 5.$$

This implies

$$\begin{aligned} \frac{|L|(|L|-1)}{2} &\geq |E(D[L])| = \sum_{b \in L} d^+_{D[L]}(b) \\ &\geq |L| \left\{ \frac{cr + k - r - 2}{2} - r - c + 5 \right\}. \end{aligned}$$

Furthermore, because of Lemma 1.11, we observe that $|L| = |H| - |M| = d^+(a_1) - (n-2) - |M| \leq \frac{cr+k-r+1}{2} - |M| - c + 3$. Combining these results, we arrive at

$$\frac{cr + k - r + 1}{2} - |M| - c + 2 \geq |L| - 1 \geq 2 \left\{ \frac{cr + k - r - 2}{2} - r - c + 5 \right\}.$$

The last inequality is equivalent to $(c-5)r \leq -k - 2|M| + 2c - 11 \leq -2|M| + 2c - 12$. Since $r \geq 2$, this leads to the contradiction $|M| \leq -1$.

Consequently, it remains to consider the case that $L = \emptyset$, that means that H consists of vertices of only one partite set. This partite set has to be $V(a_n)$, because otherwise, we observe that $N^+(a_n) \supseteq \{a_1, \ldots, a_{n-2}\} \cup H \cup S$ and $N^+(a_1) = H \cup \{a_2, \ldots, a_{n-1}\}$, a contradiction to $i_g(D) \leq 1$. This implies that $a_2 \to H$ and $\{a_3, \ldots, a_{n-1}\} \to H$, because otherwise, let $i = \min_{3 \leq l \leq n-1}\{l \mid h \to a_l\}$ with $h \in H$, then $a_n a_1 \ldots a_{i-1} h a_i \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Therefore, we have $(\{a_1, a_2, \ldots, a_{n-1}, a_n, v\} \cup F) \rightsquigarrow H$. Then we conclude for every vertex $h \in H$ that $\frac{cr+k-r-2}{2} \leq d^+(h) = d(h, V(D) - H) \leq |S| - 1 + |R| \leq r$, a contradiction to $c \geq 7$.

Subcase 6.2. Assume that there exists exactly one $j \in \{3, 4, \ldots, n-1\}$ such that $a_1 \to (\{a_2, a_3, \ldots, a_{n-1}\} - \{a_j\})$ and $a_j \to a_1$ or $V(a_j) = V(a_1)$ and that $n \geq 6$. This condition implies $d^+(v) \geq d^+(a_1) + 1$ and thus, because of $i_g(D) \leq 1$, $d^+(v) = d^+(a_1) + 1$. Furthermore, we note that $H \cap Q = \emptyset$ and $R = V(D) - (H \cup Q \cup V(v) \cup V(C)) = \emptyset$.

If there are vertices $x \in H$ and $y \in Q$ such that $x \to y$, then, because of $n \geq 6$, $a_n a_1 x y v a_4 a_5 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Hence, we assume that $(Q \cup \{a_1, a_2, v\}) \rightsquigarrow H$. Let L be the set of vertices q in H, which have an inner neighbor p in H. Furthermore, let $M = H - L$ and $|L| \neq 0$. Then we have $(Q \cup M \cup \{a_1, a_2, a_3, v\}) \rightsquigarrow L$.

Firstly, let $|V(v)| = r+2$. Then Remark 1.18 yields the contradiction

$$\frac{cr+k-r-2}{2} + 1 \le d^+(a_1) + 1 = d^+(v) = \frac{cr+k-r-2}{2}.$$

Secondly, let $|V(v)| = r+1$. Then, for every vertex $q \in L$, we conclude that $d(q, V(D) - L) \le |V(v)| + |V(C)| - 4 = r + n - 3$, and thus, it follows from Lemma 1.12 and Corollary 1.19 that

$$\begin{aligned} d^+_{D[L]}(q) &= d^+(q) - d(q, V(D) - L) \\ &\ge \frac{cr+k-r-2}{2} - r - n + 3, \quad \text{if} \quad k \ge 2 \\ \text{and} \quad d^+_{D[L]}(q) &\ge \frac{cr+k-r-1}{2} - r - n + 3, \quad \text{if} \quad k = 1. \end{aligned}$$

This implies

$$\begin{aligned} \frac{|L|(|L|-1)}{2} \ge |E(D[L])| &= \sum_{q \in L} d^+_{D[L]}(q) \\ &\ge |L| \left\{ \frac{cr+k-r-2}{2} - r - n + 3 \right\} \\ \text{and} \quad \frac{|L|(|L|-1)}{2} &\ge |L| \left\{ \frac{cr+k-r-1}{2} - r - n + 3 \right\}, \end{aligned}$$

respectively. The conditions $d^+(v) = d^+(a_1) + 1$, $a_1 \to (\{a_2, a_3, \ldots, a_{n-1}\} - \{a_j\})$ and Lemma 1.12 yield $|L| = |H| - |M| = d^+(a_1) - n + 3 - |M| = d^+(v) - n + 2 - |M| \le \frac{cr+k-r}{2} - |M| - n + 2$. Combining these results, we arrive at the inequalities

$$\begin{aligned} \frac{cr+k-r}{2} - |M| - n + 1 \ge |L| - 1 &\ge 2\left\{ \frac{cr+k-r-2}{2} - r - n + 3 \right\} \\ \text{and} \quad \frac{cr+k-r}{2} - |M| - n + 1 &\ge 2\left\{ \frac{cr+k-r-1}{2} - r - n + 3 \right\}, \end{aligned}$$

respectively. A transformation leads to $2n \ge (c-5)r + k + 2|M| + 6$ and $2n \ge (c-5)r + k + 2|M| + 8$, respectively. Since $n \le c-1$, $k \ge 2$ (respectively, $k = 1$) and $r \ge 2$, this yields a contradiction, if $|M| \ge 1$.

Thirdly, let $|V(v)| = r$. Then, for every vertex $q \in L$, we conclude ($|R| = 0$) that $d(q, V(D) - L) \le r + n - 4$, and analogously to above, we get the contradiction $|M| \le -1$.

The case that $|M| = 0$ yields a contradiction, analogously as in Subcase 3.2.

Consequently it remains to consider the possibility that $|L| = 0$, which means that H consists of vertices of only one partite set V_z. Firstly, let $|V_z| = r+2$ and $|V(a_1)| \ge r+1$ (this means $k \ge 3$). Because of $|N^+(a_1) \cap V(C)| = n-3, n \le c-1$ and Corollary 1.19, this leads to

$$\frac{cr+k-r-2}{2} - (c-4) \le d^+(a_1) - (n-3) = |H| \le r+1,$$

which is equivalent to $2c \geq (c-3)r + k + 4$, a contradiction, because of $r \geq 2$ and $k \geq 3$. Now let $|V_z| = r+2$ and $|V(a_1)| = r$. Then Remark 1.18 yields

$$\frac{cr + k - r}{2} - (c-4) \leq d^+(a_1) - (n-3) = |H| \leq r + 1,$$

hence $2c \geq (c-3)r + k + 6$, a contradiction. Finally, let $|V_z| \leq r+1$; then we arrive at

$$\frac{cr + k - r - 2}{2} - (c-4) \leq d^+(a_1) - (n-3) = |H| \leq r,$$

hence $2c \geq (c-3)r + k + 6$, a contradiction.

Subcase 6.3. Assume that $n = 5$ and there is exactly one $j \in \{3, 4\}$ such that $a_1 \to (\{a_2, a_3, a_4\} - \{a_j\})$ and $a_j \to a_1$ or $V(a_j) = V(a_1)$.

Subcase 6.3.1. Let $a_1 \to \{a_2, a_3\}$ and $a_4 \to a_1$ or $V(a_4) = V(a_1)$. If there is a vertex $h \in H$ such that $h \to a_5$, then $a_5a_1a_3a_4vha_5$ is a 6-cycle, a contradiction. Therefore, we may assume that $a_5 \to (H - V(a_5))$. If $a_2 \to a_5$, then $a_5a_1a_3a_4va_2a_5$ is a 6-cycle, a contradiction. Hence, it remains to treat the case that $a_5 \to a_2$ or $V(a_5) = V(a_2)$. Let $\{a_1, a_2\} = A \cup B$ such that $a_5 \to A$ and $B \subseteq V(a_5)$. Then $N^+(a_1) = H \cup \{a_2, a_3\}$ and $N^+(a_5) \supseteq A \cup S \cup (H - (V(a_5) - (B \cup \{a_5\})))$. This leads to

$$d^+(a_5) \geq |A| + |S| + |H| - (r + 1 - (|B| + 1)) = d^+(a_1) + |S| - r,$$

if $|V(a_5)| = r + 1$ and

$$d^+(a_5) \geq |A| + |S| + |H| - (r + 2 - (|B| + 1)) = d^+(a_1) + |S| - (r+1), \quad (2.5)$$

if $|V(a_5)| = r + 2$. Since $i_g(D) \leq 1$, the set S consists of one ($n = c - 1$, if $|V(a_5)| = r + 1$) or of at most two ($n = c - 2$, if $|V(a_5)| = r + 2$) partite sets. Firstly, let $n = c - 1$. Then, since $n = 5$, this leads to a contradiction to $c \geq 7$. In the remaining case that $n = c - 2$ and $|V(a_5)| = r + 2$, we have $|V_c| = r+2$, $r = 2$ and $|S| = 2r = 4$. In this case, because of (2.5) and Remark 1.18, we arrive at the contradiction

$$\frac{cr + k - r - 2}{2} + 1 \leq d^+(a_1) + 1 = d^+(a_5) = \frac{cr + k - r - 2}{2}.$$

Subcase 6.3.2. Let $n = 5$ and assume that $a_1 \to \{a_2, a_4\}$ and $a_3 \to a_1$ or $V(a_3) = V(a_1)$. Analogously to Subcase 6.2, H consists of at least two partite sets. Hence, there exist vertices $x, y \in H$ such that $x \to y$. If $y \to a_5$, then $a_5a_1a_4vxya_5$ is a 6-cycle, a contradiction. Now let $W = H - V(a_5)$ and $U = \{x \in W | d^-_{D[H]}(x) = 0\}$. It follows that U is a subset of one partite set, which means $|U| \leq r$ (respectively, $|U| \leq r + 1$, if $|V_c| = r + 2$), and $a_5 \to (W - U)$. If $a_3 \to a_5$, then $a_5a_1a_4va_2a_3a_5$ is a 6-cycle, a contradiction. Hence, it remains to consider the case that $a_5 \to a_3$ or $V(a_5) = V(a_3)$. Let $\{a_1, a_3\} = A \cup B$ such that $a_5 \to A$ and $B \subseteq V(a_5)$. Then $N^+(a_1) = H \cup \{a_2, a_4\}$ and $N^+(a_5) \supseteq A \cup S \cup (H - ((V(a_5) - (B \cup \{a_5\})) \cup U))$ and therefore

$$d^+(a_5) \geq |A| + |S| + |H| - (r + 1 - (|B| + 1)) - |U| \geq d^+(a_1) + |S| - 2r,$$

if $|V_c| \le r+1$ and

$$d^+(a_5) \ge |A|+|S|+|H|-(r+2-(|B|+1))-|U| \ge d^+(a_1)+|S|-2(r+1),$$

if $|V_c| = r+2$. Because of $i_g(D) \le 1$, this yields a contradiction, if S consists of more than two (respectively, three, if $|V_c| = r+2$) partite sets. Let $|V_c| = r+2$ and let S consist of three partite sets; then we get a contradiction, if $r \ge 4$. If $r = 3$ and $|V(a_5)| = r+2$, then, because of Remark 1.18, we arrive at the contradiction

$$\frac{cr+k-r-2}{2}+1 \le d^+(a_1)+1 = d^+(a_5) = \frac{cr+k-r-2}{2}.$$

If $r = 3$ and $|V(a_5)| \le r+1$, then we have the contradiction

$$d^+(a_5) \ge |A|+|S|+|H|-(r+1-(|B|+1))-|U| \ge d^+(a_1)+r-1 = d^+(a_1)+2.$$

Consequently, it remains to treat the cases $n = c-2$, $|B| = 0, D[V(C)]$ is a tournament or $|V_c| = r+2$, $n = c-3$ and $r = 2$. If we define $U' = (N^+(a_1) \cap N^-(a_5)) - V(C)$, then $U' \subseteq U$ and U' consists of vertices of only one partite set V_y. Now let $J = N^-(a_5) - (U' \cup V(C))$ and $G = N^+(a_1) - (V_y \cup \{a_2, a_4\})$. In this case, we note that $G \ne \emptyset$, because otherwise $H = N^+(a_1) - \{a_2, a_4\} \subseteq V_y$, hence, it follows from Corollary 1.19

$$\frac{cr+k-r-2}{2}-2 \le d^+(a_1)-2 = |H| \le r+1,$$

a contradiction to $c \ge 7$. Therefore, assume that $G \ne \emptyset$. If there are vertices $x \in G$ and $y \in J \cup U'$ such that $x \to y$, then $a_5a_1a_4vxya_5$ is a 6-cycle, a contradiction.

Suppose next that there exist vertices $b \in G$ and $w \in S$ such that $b \to w$. If $w \to a_3$, then $a_5a_1bwa_3a_4a_5$ is a 6-cycle, a contradiction. So, we can assume that $a_3 \to w$. If there is a vertex $x \in (N^-(a_5) - V(C))$ such that $w \to x$, then $a_5a_1a_2a_3wxa_5$ is a 6-cycle, a contradiction. Thus, we can assume that $(N^-(a_5) - V(C)) \to w$. Altogether, we see that $N^-(a_5) \subseteq (N^-(a_5) - V(C)) \cup \{a_2, a_4\}$ and $N^-(w) \supseteq (N^-(a_5) - V(C)) \cup \{a_3, a_4, a_5, b\}$ and this yields the contradiction $d^-(w) \ge d^-(a_5)+2$. Consequently, it remains to treat the case that $S \to G$. If we define $R = V(D) - (H \cup J \cup S \cup V(C))$, then, because of

$$\begin{aligned} |J| &\ge |N^-(a_5)| - |U'| - 2 \ge \frac{cr+k-r-2}{2} - |U'| - 2 \\ &= \begin{cases} \frac{6r+k-2}{2} - |U'| - 2 & \text{, if } \quad n = c-2 = 5 \\ \frac{7r+k-2}{2} - |U'| - 2 & \text{, if } \quad n = c-3 = 5 \end{cases}, \end{aligned}$$

we obtain $|R| \le$

$$\begin{cases} 7r+k-\left\{\frac{6r+k-2}{2} - |U'| - 2 + \frac{6r+k-2}{2} - 2 + 2r + 5\right\} & \text{, if } \quad n = c-2 \\ 16+k-\left\{\frac{12+k}{2} - |U'| - 2 + \frac{12+k}{2} - 2 + 6 + 5\right\} & \text{, if } \quad n = c-3 \end{cases}$$

$$= \begin{cases} |U'| - r + 1 & \text{, if } \quad n = c-2 \\ |U'| - 3 & \text{, if } \quad n = c-3 \end{cases}.$$

Thus, we also see that $U' \neq \emptyset$. Let there be a vertex $y \in G$ such that $y \to a_3$. Because of $U' \subseteq U$ and $V_y \subseteq V(D) - G$, there exists a vertex $x \in U'$ such that $x \to y$. This leads to the 6-cycle $a_5a_1xya_3a_4a_5$, a contradiction. Hence, it remains that $(S \cup J \cup U' \cup \{a_1, a_2, a_3, a_5\}) \rightsquigarrow G$.

Firstly, let us observe the case that $n = c-2$. Then, for every vertex $x \in G$, we get $d(x, V(D)-G) \leq |R|+1+|V_y \cap H|-|U'| \leq 2-r+|V_y|-|V_y \cap V(C)| \leq 1-r+|V_y| \leq 3$ and thus, it follows that

$$d^+_{D[G]}(x) = d^+(x) - d(x, V(D) - G) \geq \frac{6r + k - 2}{2} - 3 = \frac{6r + k - 8}{2}.$$

This implies

$$\frac{|G|(|G| - 1)}{2} \geq |E(D[G])| = \sum_{x \in G} d^+_{D[G]}(x) \geq |G| \frac{6r + k - 8}{2}.$$

In view of Lemma 1.11, we have $|G| = d^+(a_1) - |V_y \cap H| - 2 \leq d^+(a_1) - 2 \leq \frac{6r+k-3}{2}$. Altogether, this leads to $\frac{6r+k-5}{2} \geq |G| - 1 \geq 6r + k - 8$, and thus, we obtain the inequality $6r + k \leq 11$, a contradiction.

Now let $n = c - 3$. Then, for every vertex $x \in G$, we conclude that $d(x, V(D)-G) \leq |R|+1+|V_y \cap H|-|U'| \leq -2+|V_y|-|V_y \cap V(C)| \leq -3+|V_y| \leq 1$ and thus, it follows that $d^+(x) \leq |G| = d^+(a_1) - |V_y \cap H| - 2 \leq d^+(a_1) - 2$, a contradiction to $i_g(D) \leq 1$.

Summarizing the investigations of Case 6, we see that it remains to treat the case when $a_{n-2} \to S$.

Case 7. Let $n = 5$. If we consider the cycle $C^{-1} = a_1a_5a_4a_3a_2a_1 = b_5b_1b_2b_3b_4b_5$ in the converse D^{-1} of D, then $\{b_4, b_5\} \to S \to \{b_1, b_2, b_3\}$. Since this is exactly the situation of Case 6, there exists in D^{-1} a 6-cycle, containing the arc $b_5b_1 = a_1a_5$, and hence there exists in D a 6-cycle through a_5a_1.

Case 8. Let $n \geq 6$. Assume that there exists a vertex $v \in S$ such that $a_3 \to v$. If we consider the converse of D, then in view of Case 6, it remains to consider the case that $S \to a_3$.

Case 9. Let $c > n \geq 6$. If there exist vertices $y \in S$ and $x \in H$ such that $x \to y$, then $a_na_1xya_3a_4 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Consequently, we assume now that $S \to H$. Let $y \in S$. If there exists a vertex $x \in H$ such that $x \to a_n$, then $a_na_1a_2 \ldots a_{n-2}yxa_n$ is an $(n+1)$-cycle, a contradiction. Hence, it remains to treat the case that $(S \cup \{a_1, a_2, a_n\}) \rightsquigarrow H$.

If $a_1 \to a_i$ and $a_{i-1} \to a_n$ for $i \in \{3, 4, \ldots, n-1\}$, then the $(n+1)$-cycle $a_na_1a_i \ldots a_{n-1}ya_2 \ldots a_{i-1}a_n$ yields a contradiction. Thus, if $a_1 \to a_i$ for some $i \in \{3, 4, \ldots, n-1\}$, then we may assume that $a_n \to a_{i-1}$ or $V(a_{i-1}) = V(a_n)$. Let $N = \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}$ be exactly the subset of $V(C) - \{a_2\}$ with the property that $a_1 \to N$. Then we define $A \cup B = \{a_{i_1-1}, a_{i_2-1}, \ldots, a_{i_k-1}\}$ such that $a_n \to A$ and $B \subseteq V(a_n)$. This definition and the fact that $a_n \to (H - V(a_n))$ lead to $N^+(a_1) = \{a_2\} \cup N \cup H$ and $N^+(a_n) \supseteq \{a_1\} \cup A \cup S \cup (H - (V(a_n) - (B \cup \{a_n\})))$. This implies

$$\begin{aligned} d^+(a_n) &\geq |A| + |S| + 1 + |H| - (r + 1 - (|B| + 1)) \\ &= |A| + |B| + |H| + |S| - r + 1 \\ &= d^+(a_1) + |S| - r, \end{aligned} \tag{2.6}$$

if $|V(a_n)| \leq r + 1$ and

$$d^+(a_n) \geq d^+(a_1) + |S| - (r + 1), \tag{2.7}$$

if $|V(a_n)| = r + 2$. If $|V(a_n)| = r + 2$ and S consists of two partite sets, then by (2.7), we conclude that $r = 2$ and $|S| = 2r = 4$, and thus, Remark 1.18 leads to the contradiction

$$\frac{cr + k - r - 2}{2} + 1 \leq d^+(a_1) + 1 \leq d^+(a_n) = \frac{cr + k - r - 2}{2}.$$

Hence, because of the bounds (2.6) and (2.7), we conclude that the case $n = c - 1$, $|B| = 0$ and $D[V(C)]$ is a tournament, remains to be considered.

Subcase 9.1. There exists a vertex $v \in S$ such that $v \rightarrow a_{n-3}$. If there is a vertex $a_i \in V(C)$ with $4 \leq i \leq n - 4$ such that $a_i \rightarrow v$, then we obtain, as in Case 1, an $(n + 1)$-cycle through $a_n a_1$, a contradiction. Thus, we investigate now the case that $v \rightarrow \{a_1, a_2, \ldots, a_{n-3}\}$. If $R_1 = V(D) - (H \cup Q \cup V(v) \cup V(C))$, then because of $|H| = |N^+(a_1) - V(C)| \geq d^+(a_1) - (n-2)$ and $|Q| = |N^-(v) - V(C)| = d^-(v) - 3$, we see with respect to Lemma 1.12 and Corollary 1.19 that

$$\begin{aligned} |R_1| &\leq cr + k \\ &\quad - \left\{ \frac{cr + k - r - 2}{2} - (n - 2) + \frac{cr + k - r - 1}{2} - 3 + r + n \right\} = \frac{5}{2}, \end{aligned}$$

if $|V(v)| = r$,

$$\begin{aligned} |R_1| &\leq cr + k \\ &\quad - \left\{ \frac{cr + k - r - 2}{2} - (n - 2) + \frac{cr + k - r - 2}{2} - 3 + r + 1 + n \right\} = 2, \end{aligned}$$

if $|V(v)| = r + 1$, and

$$\begin{aligned} |R_1| &\leq cr + k \\ &\quad - \left\{ \frac{cr + k - r - 2}{2} - (n - 2) + \frac{cr + k - r - 2}{2} - 3 + r + 2 + n \right\} = 1, \end{aligned}$$

if $|V(v)| = r + 2$. Altogether, we see that $|R_1| \leq 2$, if $|V(v)| \leq r + 1$ and $|R_1| \leq 1$, if $|V(v)| = r + 2$.

Subcase 9.1.1. Firstly, let H consist of vertices of only one partite set. Because of $|B| = 0$, according to (2.6) (respectively, (2.7)), this partite set has to be $V(a_n)$. Since $H \subseteq V(a_n) - \{a_n\}$, we have $a_2 \rightarrow H$ and thus $\{a_3, a_4, \ldots, a_{n-1}\} \rightarrow H$. If there are vertices $h \in H$ and $y \in F$ such that $h \rightarrow y$, then $a_n a_1 a_2 \ldots a_{n-2} h y a_n$ is an $(n + 1)$-cycle, a contradiction. Hence, $F \rightarrow H$. Consequently, $(N^-(a_n) \cup S) \rightarrow H$. Therefore, for $x \in H$, it follows that $d^-(x) \geq d^-(a_n) + |S| \geq d^-(a_n) + 2$, a contradiction to $i_g(D) \leq 1$.

Subcase 9.1.2. Now we assume that H consists of vertices of more than one partite set. Let L be the set of vertices in H, which have an inner neighbor in H and $M = H - L$. If there are vertices $q \in L$ and $p \in H$ such that $p \to q \to a_3$, then $a_n a_1 p q a_3 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Consequently, $a_3 \leadsto L$.

Firstly, let $n \geq 7$. Then, we have $Q \leadsto L$, because otherwise, if there are vertices $x \in Q$ and $q \in L$ such that $q \to x$, then $a_n a_1 q x v a_4 a_5 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Altogether, we observe that $(Q \cup V(v) \cup M \cup \{a_1, a_2, a_3, a_n\}) \leadsto L$. Because of $|R_1| \leq 2$, for every vertex $q \in L$, it follows that $d(q, V(D) - L) \leq n - 2 = c - 3$ and thus, Corollary 1.19 leads to

$$d^+_{D[L]}(q) = d^+(q) - d(q, V(D) - L) \geq \frac{cr + k - r - 2}{2} - c + 3.$$

This implies

$$\frac{|L|(|L| - 1)}{2} \geq |E(D[L])| = \sum_{q \in L} d^+_{D[L]}(q) \geq |L| \left\{ \frac{cr + k - r - 2}{2} - c + 3 \right\}.$$

Since $d^+(v) \geq |H| + (n - 3) = |H| + (c - 4)$, we conclude together with Lemma 1.11 that $|L| \leq d^+(v) - (n - 3) - |M| = d^+(v) - c + 4 - |M| \leq \frac{cr+k-r+1}{2} - c + 4 - |M|$. Combining these results, we arrive at

$$\frac{cr + k - r + 1}{2} - c + 3 - |M| \geq |L| - 1 \geq 2 \left\{ \frac{cr + k - r - 2}{2} - c + 3 \right\}.$$

This results in $(c - 1)r + k + 2|M| + 1 \leq 2c$, a contradiction, if $|M| \geq 1$.

The case $|M| = 0$ leads to a contradiction, analogously to Subcase 3.2.

It remains to treat the case that $n = 6$ and $c = n + 1 = 7$. We remember that $\{a_4, a_5, a_6\} \to S \to \{a_1, a_2, a_3\}$. We note that $H \cap F = \emptyset$, since $F \to a_6 \leadsto H$. If there are vertices $f \in F$ and $w \in S$ such that $w \to f$, then $a_6 a_1 a_2 a_3 a_4 w f a_6$ is a 7-cycle, a contradiction. Therefore, we have $F \to S$. Because of $H \cap F = \emptyset$, we see that $F \leadsto a_1$. Let $R_2 = V(D) - (H \cup F \cup S \cup V(C))$. Since $|B| = 0$ and $a_6 \to a_{i-1}$, if $a_1 \to a_i$ for $2 \leq i \leq n - 1$, we observe that $|N^+(a_1) \cap V(C)| + |N^-(a_6) \cap V(C)| \leq l + 5 - l = 5$, if $|N^+(a_1) \cap V(C)| = l$. Hence, Corollary 1.19 yields

$$|R_2| \leq cr + k - \left\{ \frac{cr + k - r - 2}{2} + \frac{cr + k - r - 2}{2} - 5 + |S| + n \right\} \leq 1.$$

From the fact that $v \to H$ and $N^+(v) \cap V(C) = \{a_1, a_2, a_3\}$, we deduce that $|N^+(a_1) \cap V(C)| \geq 2$. If $\{a_3\} \subseteq N^+(a_1)$ or $\{a_4\} \subseteq N^+(a_1)$, then $F \leadsto H$, because otherwise, if there are vertices $h \in H$ and $f \in F$ such that $h \to f$, then either $a_6 a_1 a_3 a_4 v h f a_6$ or $a_6 a_1 a_4 a_5 v h f a_6$ is a 7-cycle, a contradiction. Let L be the set of vertices in H, which have an inner neighbor in H and let $M = H - L$. Then it follows that $(M \cup F \cup S \cup \{a_1, a_2, a_3, a_6\}) \leadsto L$, and thus, since $|R_2| \leq 1$, for every vertex $q \in L$, we observe that $d(q, V(D) - L) \leq 3 = n - 3 = c - 4$ and, analogously as above, we get a contradiction. Consequently, let $N^+(a_1) \cap V(C) = \{a_2, a_5\}$, and thus $a_6 \to a_4$ and $d^+(a_1) = d^+(v) - 1$.

Assume that F consists of vertices of only one partite set V_b. In this case, we observe that $N^-(a_6) \subseteq F \cup (N^-(a_6) \cap V(C))$. Since $|N^+(a_6) \cap V(C)| \geq$

$|N^+(a_1) \cap V(C)| = 2$, it follows that $|N^-(a_6) \cap V(C)| \leq 3$ and thus $\frac{6r+k-2}{2} \leq d^-(a_6) \leq r+3$, if $|V_c| = r+1$. This yields the contradiction $4r+k \leq 8$. Hence, let us investigate the case that $|V_c| = r+2$. If $|V_b| = r+2$ and $|V(a_6)| \geq r+1$ (that means $k \geq 3$), then we arrive at the contradiction $\frac{6r+k-2}{2} \leq d^-(a_6) \leq r+4$. On the other hand, if $|V_b| \leq r+1$ or $|V(a_6)| = r$, we see that $\frac{6r+k-2}{2} \leq d^-(a_6) \leq r+3$ or $\frac{6r+k}{2} \leq d^-(a_6) \leq r+4$, in both cases a contradiction.

Consequently, it remains to consider the case that F consists of more than one partite set. Hence, there exists an arc $f_1 f_2 \in E(D[F])$, and the set F_1 of vertices in F having an outer neighbor in F is non-empty. Let $F_2 = F - F_1$. If there are vertices $f_1 \in F_1$, $h \in H$ and $f_2 \in F$ such that $h \to f_1 \to f_2$, then $a_6 a_1 a_5 v h f_1 f_2 a_6$ is a 7-cycle, a contradiction. Therefore, we may assume that $F_1 \leadsto H$. Furthermore, we see that $F_1 \leadsto a_4$, because otherwise $a_6 a_1 a_2 a_3 a_4 f_1 f_2 a_6$ is a 7-cycle, a contradiction. Because of $H \cap F = \emptyset$, we conclude that $F \leadsto a_1$. It is also easy to see that $F \leadsto a_5$ and $F \to S$, since otherwise we are able to construct a 7-cycle, a contradiction. Summarizing, we see that $F_1 \leadsto (H \cup S \cup F_2 \cup \{a_1, a_4, a_5, a_6\})$. Hence, since $|R_2| \leq 1$, for every vertex $f_1 \in F_1$, we conclude that $d(V(D) - F_1, f_1) \leq 3$, and thus, it follows from Corollary 1.19 that

$$d^-_{D[F_1]}(f_1) = d^-(f_1) - d(V(D) - F_1, f_1) \geq \frac{6r+k-2}{2} - 3.$$

This implies

$$\frac{|F_1|(|F_1|-1)}{2} \geq |E(D[F_1])| = \sum_{f_1 \in F_1} d^-_{D[F_1]}(f_1) \geq |F_1| \left\{ \frac{6r+k-2}{2} - 3 \right\}.$$

We see that $d^-(a_6) \geq |F|+2$, because otherwise, we arrive at the contradiction $d^+(a_6) \geq 4 + |H| - |V(a_6) - \{a_6\}| + |S| \geq d^+(a_1) + 2 + |S| - r \geq d^+(a_1) + 2$, if $|V(a_6)| \leq r+1$. If $|V(a_6)| = r+2$, then we obtain $d^+(a_6) \geq d^+(a_1) + 1$, a contradiction to Remark 1.18. Thus, it follows that $|F_1| \leq d^-(a_6) - 2 - |F_2| \leq \frac{6r+k+1}{2} - 2 - |F_2|$. Combining these results, we obtain

$$\frac{6r+k+1}{2} - 3 - |F_2| \geq |F_1| - 1 \geq 2 \left\{ \frac{6r+k-2}{2} - 3 \right\},$$

which can be transformed to $6r + k + 2|F_2| \leq 11$, a contradiction.

Subcase 9.2. Finally, we assume that $a_{n-3} \to S$. Then we see that $n = c - 1 \geq 7$. Let $R = V(D) - (H \cup F \cup S \cup V(C))$. If there is a vertex $w \in H \cap F$, then $a_n a_1 a_2 \ldots a_{n-2} v w a_n$ is an $(n+1)$-cycle, a contradiction. Consequently, let $H \cap F = \emptyset$. We have seen above that $|H| = d^+(a_1) - |N| - 1$ and $|N^+(a_n) \cap V(C)| \geq |N| + 1$. Hence $|N^-(a_n) \cap V(C)| \leq n - |N| - 2$, and thus $|F| = |N^-(a_n) - V(C)| \geq d^-(a_n) - (n - 2 - |N|)$. It follows from Corollary 1.19 that

$$\begin{aligned} |R| \leq\ & cr + k \\ & - \left\{ \frac{cr+k-r-2}{2} - |N| - 1 + \frac{cr+k-r-2}{2} - n + 2 + |N| + |S| + n \right\}, \end{aligned}$$

and thus $|R| \leq 1$, if $|S| = r$; $|R| = 0$, if $|S| = r+1$; and $|R| \leq -1$, if $|S| = r+2$. If there is an arc xy with $x \in H$ and $y \in F$, then $a_n a_1 a_2 \ldots a_{n-3} v x y a_n$ is an $(n+1)$-cycle, a contradiction. If there is an arc uy with $u \in S$ and $y \in F$, then $a_n a_1 a_2 \ldots a_{n-2} u y a_n$ is an $(n+1)$-cycle, a contradiction. Furthermore, if there is an arc $x a_{n-1}$ with $x \in H$, then $a_n a_1 a_2 \ldots a_{n-3} v x a_{n-1} a_n$ is an $(n+1)$-cycle, a contradiction. Consequently, it remains to treat the case that $(F \cup S \cup \{a_1, a_2, a_{n-1}, a_n\}) \leadsto H$ and $F \leadsto (\{a_1, a_{n-1}, a_n\} \cup S \cup H)$.

Subcase 9.2.1. Firstly, we investigate the case that $r = 2$. As seen above, for every vertex $h \in H$, we conclude that $d(h, V(D) - H) \leq n - 3 = c - 4$ and thus $d^+_{D[H]}(h) \geq \frac{cr+k-r-2}{2} - c + 4 = \frac{k+4}{2} \geq \frac{5}{2}$ and therefore $d^+_{D[H]}(h) \geq 3$. Hence, H contains at least 7 vertices. Furthermore, there is at least one vertex h_1 in H such that $d^+_{D[H]}(h_1) \leq \frac{|H|-1}{2}$. Since $N^+(a_1) = H \cup N \cup \{a_2\}$ and $i_g(D) \leq 1$, we conclude that $d^+(h_1) \geq |H| + |N|$. In addition, $(F \cup S \cup \{a_1, a_2, a_{n-1}, a_n\}) \leadsto H$, and thus $N^+(h_1) \subseteq V(C) \cup R \cup H$, which leads to

$$|N^+(h_1) \cap V(C)| + |R| + \frac{|H| - 1}{2} \geq d^+(h_1) \geq |H| + |N|.$$

This implies

$$|N^+(h_1) \cap V(C)| \geq \frac{|H| + 1}{2} + |N| - |R| \geq |N| + 3.$$

Let $a_i \in N^+(h_1) \cap V(C)$ $(3 \leq i \leq n-2)$. If $a_{i-1} \to a_n$, then we observe that $a_n a_1 h_1 a_i \ldots a_{n-2} v a_2 \ldots a_{i-1} a_n$ is an $(n+1)$-cycle, a contradiction. Therefore, in $V(C)$, a_n has at least $|N| + 3$ further outer neighbors except a_1. According to (2.6) and (2.7), this yields

$$d^+(a_n) \geq |N| + 4 + |H| + |S| - (r+1) = d^+(a_1) + 2 + |S| - r \geq d^+(a_1) + 2,$$

a contradiction to $i_g(D) \leq 1$.

Subcase 9.2.2. Assume that $|N| \geq \frac{c-6}{2}$ and $r \geq 3$. Since $|R| \leq 1$, for every vertex $h \in H$, we conclude that $d(h, V(D) - H) \leq n - 3 = c - 4$ and thus, it follows from Corollary 1.19 that

$$d^+_{D[H]}(h) = d^+(h) - d(h, V(D) - H) \geq \frac{cr + k - r - 2}{2} - c + 4.$$

This implies

$$\begin{aligned} \frac{|H|(|H|-1)}{2} &\geq |E(D[H])| = \sum_{h \in H} d^+_{D[H]}(h) \\ &\geq |H| \left\{ \frac{cr + k - r - 2}{2} - c + 4 \right\}. \end{aligned}$$

Since $|H| = d^+(a_1) - |N| - 1 \leq \frac{cr+k-r+1}{2} - |N| - 1 \leq \frac{cr+k-r+1}{2} - \frac{c}{2} + 2 = \frac{cr+k-r-c+5}{2}$, we obtain

$$\frac{cr + k - r - c + 3}{2} \geq |H| - 1 \geq cr + k - r - 2 - 2c + 8.$$

This inequality is equivalent to $(c-1)r+k \leq 3c-9$, a contradiction to $r \geq 3$.

Subcase 9.2.3. Now assume that $|N| \leq \frac{c-7}{2}$ and $r \geq 3$. Since $|R| \leq 1$, for every vertex $y \in F$, we conclude that $d(V(D)-F, y) \leq n-2 = c-3$ and thus, it follows from Corollary 1.19 that

$$d^-_{D[F]}(y) = d^-(y) - d(V(D)-F, y) \geq \frac{(c-1)r+k+4}{2} - c.$$

This implies

$$\frac{|F|(|F|-1)}{2} \geq |E(D[F])| = \sum_{y \in F} d^-_{D[F]}(y) \geq |F| \left\{ \frac{(c-1)r+k+4}{2} - c \right\}.$$

Since $i_g(D) \leq 1$, we conclude from (2.6) and (2.7) that $|N^+(a_n) \cap V(C)| \leq |N| + 3$, and thus $|N^-(a_n) \cap V(C)| \geq n - |N| - 4$. Hence, it follows that $|F| = |N^-(a_n) - V(C)| \leq d^-(a_n) - (n-|N|-4) \leq \frac{cr+k-r+1}{2} - (c-1) + 4 + \frac{c-7}{2} = \frac{(c-1)r+k+4-c}{2}$. Combining these results, we observe that

$$\frac{(c-1)r+k+2-c}{2} \geq |F| - 1 \geq (c-1)r+k+4-2c.$$

A transformation of this inequality leads to $3c \geq (c-1)r+k+6 \geq (c-1)r+7$, a contradiction to $r \geq 3$. This completes the proof of the theorem. □

2.2 The case $\gamma(D) = 1$

Now, it remains to consider the case that there is only one vertex in at least one partite set. Again, let D be an almost regular c partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $1 = |V_1| \leq |V_2| \leq \ldots \leq |V_c|$. According to Theorem 2.6 and Lemma 1.10 we may assume that $2 \leq |V_c| \leq 3$.

Theorem 2.12 (Volkmann, Winzen [42]) *Let D be an almost regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $1 = |V_1| \leq |V_2| \leq \ldots \leq |V_c| \leq 3$ and $|V_c| \geq 2$. If $c \geq 8$, then every arc of D is contained in an n-cycle for each $n \in \{4, \ldots, c\}$.*

Proof. We prove the theorem by induction on n. For $n = 4$, the result follows from Theorem 2.3. Now let e be an arc of D and assume that e is contained in an n-cycle $C = a_n a_1 a_2 \ldots a_n$ with $e = a_n a_1$ and $4 \leq n \leq c-1$. Suppose that $e = a_n a_1$ is not contained in any $(n+1)$-cycle.

Obviously, $|V(D)| = c+k$ with $1 \leq k \leq c-1$, if $|V_c| = 2$ and $2 \leq k \leq 2c-2$, if $|V_c| = 3$. Firstly, we observe that, if $n = 4$ and $|V_c| = 2$ or $n \leq 5$ and $|V_c| = 3$, then $N^+(v) - V(C) \neq \emptyset$ for each $v \in V(C)$, because otherwise Corollary 1.19, the fact that $k \geq 1$ (respectively, $k \geq 2$) and $c \geq 8$ yield the contradiction

$$4 = |V(C)| \geq d^+(v) + 2 \geq \frac{c+k-3}{2} + 2 \geq 5$$

or

$$5 \geq |V(C)| \geq d^+(v) + 2 \geq \frac{c+k-3}{2} + 2 > 5.$$

Analogously, one can show that $N^-(v) - V(C) \neq \emptyset$ for each $v \in V(C)$, in these cases.

Next, let S be the set of vertices that belong to partite sets not represented on C and define

$$X = \{x \in S \mid C \to x\}, \qquad Y = \{y \in S \mid y \to C\}.$$

Assume that $X \neq \emptyset$ and let $x \in X$. It follows that $N^-(v) - V(C), N^+(v) - V(C) \neq \emptyset$ for each $v \in V(C)$, because otherwise, we have $d^-(v), d^+(v) \leq n - 2$ and $d^-(x) \geq n$, a contradiction to $i_g(D) \leq 1$. If there is a vertex $w \in N^-(a_n) - V(C)$ such that $x \to w$, then $a_n a_1 a_2 \ldots a_{n-2} x w a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. If $(N^-(a_n) - V(C)) \to x$, then $|N^-(x)| \geq |N^-(a_n) - V(C)| + |V(C)| \geq |N^-(a_n)| + 2$, a contradiction to the hypothesis that $i_g(D) \leq 1$. If there exists a vertex $b \in (N^-(a_n) - V(C))$ such that $V(b) = V(x)$, then b is adjacent with all vertices of C. In the case that $N^-(b) \cap V(C) \neq \emptyset$, let $l = \max_{1 \leq i \leq n-1}\{i \mid a_i \to b\}$. Then $a_n a_1 \ldots a_l b a_{l+1} \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. It remains to consider the case that $N^-(b) \cap V(C) = \emptyset$. If there is a vertex $u \in (N^-(b) - V(C)) = N^-(b)$ such that $x \to u$, then $a_n a_1 a_2 \ldots a_{n-3} x u b a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. Otherwise, $N^-(b) \to x$, and we arrive at the contradiction $d^-(x) \geq d^-(b) + |V(C)|$. Altogether, we have seen that $X \neq \emptyset$ is not possible, and analogously we find that $Y \neq \emptyset$ is impossible. Consequently, from now on we shall assume that $X = Y = \emptyset$.

By the definition of S, every vertex of $V(C)$ is adjacent to every vertex of S, and since $n \leq c - 1$, we deduce that $S \neq \emptyset$. Now we distinguish different cases.

Case 1. There exists a vertex $v \in S$ with $v \to a_n$. Since $Y = \emptyset$, there is a vertex $a_i \in V(C)$ such that $a_i \to v$. If $l = \max_{1 \leq i \leq n-1}\{i \mid a_i \to v\}$, then $a_n a_1 \ldots a_l v a_{l+1} \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. This implies $a_n \to S$.

Case 2. There exists a vertex $v \in S$ with $a_1 \to v$. Since $X = \emptyset$, there is a vertex $a_i \in V(C)$ such that $v \to a_i$. If $l = \min_{2 \leq i \leq n-1}\{i \mid v \to a_i\}$, then $a_n a_1 \ldots a_{l-1} v a_l \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. This implies $S \to a_1$.

Case 3. There exists a vertex $v \in S$ such that $v \to a_{n-1}$. If there is a vertex $a_i \in V(C)$ with $2 \leq i \leq n-2$ such that $a_i \to v$, then we obtain as above an $(n+1)$-cycle through $a_n a_1$, a contradiction. Thus, we investigate now the case that $v \to \{a_1, a_2, \ldots, a_{n-1}\}$. Because of $S \to a_1$, we note that every vertex of $N^+(a_1)$ is adjacent to v. If there is a vertex $x \in (N^+(a_1) - V(C))$ such that $x \to v$, then $a_n a_1 x v a_3 a_4 \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. Therefore we assume now that $v \to (N^+(a_1) - V(C))$. This leads to $d^+(v) \geq d^+(a_1) + 1$, and thus, because of $i_g(D) \leq 1$, it follows that $N^+(v) = N^+(a_1) \cup \{a_1\}$ and $a_1 \to \{a_2, a_3, \ldots, a_{n-1}\}$.

If we define $H = N^+(a_1) - V(C)$ and $Q = N^-(v) - \{a_n\}$, then $H \cap Q = \emptyset$, $S \cap H = \emptyset$, and $R = V(D) - (H \cup Q \cup V(v) \cup V(C)) = \emptyset$.

If there is an arc $x a_2$ with $x \in H$, then $a_n a_1 x a_2 a_3 \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. Thus, we assume in the following that $a_2 \rightsquigarrow H$.

Subcase 3.1. Let $n = 4$. At first, let $|V_c| = 2$. If C consists of at most 3 partite sets, then it has to be $|S| \geq 5$ and thus, it follows that $d^+(a_4) \geq 6$. On the other hand, we see that $d^-(a_4) \leq |V(D)| - |S| - |V(a_4)| - |\{a_1\}| \leq 3$, a contradiction to $i_g(D) \leq 1$. Therefore, $D[V(C)]$ has to be a tournament.

Now, let $|V_c| = 3$. If $V(C)$ is 2-partite, then we observe that $d^+(a_4) \geq |S| + 1 \geq 7$ and $d^-(a_4) \leq |V(a_3) - \{a_1\}| \leq 2$, a contradiction to $i_g(D) \leq 1$. So, let C contain vertices of only 3 partite sets. If $|S| \geq 6$, then we see that $d^+(a_4) \geq 7$ and $d^-(a_4) \leq 5$, a contradiction. Consequently, it remains to investigate the case that $|S| = 5$, $c = 8$, $2 \leq k \leq 6$ and $10 \leq |V(D)| \leq 14$. Since $d^+(a_4) \geq 6$, it follows that $12 \leq |V(D)| \leq 14$. In view of Remark 1.18, it remains to treat the case that $|V(D)| = 13$. If $|V(a_4)| = 3$, then $d^+(a_4) = d^-(a_4) = 5$, a contradiction to $d^+(a_4) \geq 6$. If $|V(a_1)| = 3$, then $d^+(a_1) = d^-(a_1) = 5$, a contradiction to $d^-(a_1) \geq 6$. This implies $|V(a_1)|, |V(a_4)| \leq 2$ and thus $|V(D)| \leq 12$, a contradiction.

Consequently, if $n = 4$, then it is sufficient to investigate the case that $D[V(C)]$ is a tournament. We remind that we have shown above that $H \neq \emptyset$.

Subcase 3.1.1. Suppose that $|H| = 1$. This implies $d^+(v) = d^+(a_1) + 1 = 4$. On the other hand, we see that $d^+(a_4) \geq |S| + 1 \geq 5$, a contradiction to $i_g(D) \leq 1$.

Subcase 3.1.2. Let $|H| \geq 2$.

Subcase 3.1.2.1. Assume that $|H| = 2$ and $E(D[H]) = \emptyset$, which means that $|V_c| = |V(h)| = 3$ for $h \in H$. Then, it follows that $d^+(v) = d^+(a_1) + 1 = 5$, which yields

$$4 = d^+(a_1) \leq d^-(v) = |Q| + 1 \leq d^+(v) = 5,$$

and hence $3 \leq |Q| \leq 4$. Because of $d^+(a_4) \geq |S| + 1 \geq 5$, it remains to consider the case that $|S| = 4$, $d^+(a_4) = 5$, $c = 8$ and $a_2 \to a_4$. Since $|S| = 4$ and $S = V(v) \cup (Q \cap S)$, we see that we have to investigate the case $|Q - S| \leq 1$. If $H \subseteq V(a_4)$, then $d^-(a_4) \leq |\{a_2, a_3\}| + |Q - S| \leq 3$, a contradiction to $i_g(D) \leq 1$. Consequently, it has to be $H \to a_4$ and therefore also $H \leadsto a_3$, since otherwise $a_4a_1a_2a_3ha_4$ is a 5-cycle, if $h \in H$, a contradiction. Since $|V(v)| = 1$, at least three vertices of Q have to belong to $N^+(a_3)$, because otherwise, we arrive at the contradiction $d^+(a_3) \leq 3$. If there are vertices $q \in N^+(a_3) \cap Q$ and $h \in H$ such that $q \to h$, then $a_4a_1a_3qha_4$ is a 5-cycle, a contradiction. It remains to consider the case that $H \to (N^+(a_3) \cap Q)$. If $q \in Q \cap N^+(a_3)$ such that $q \to a_2$, then $a_4a_1hqa_2a_4$ is a 5-cycle, a contradiction. Let $q_1 \in N^+(a_3) \cap Q \cap S \neq \emptyset$ be a vertex such that $|N^-(q_1) \cap Q \cap S| \geq 1$. Then we arrive at $d^-(q_1) \geq |H| + 1 + |\{a_2, a_3, a_4\}| = 6$, a contradiction to $i_g(D) \leq 1$.

Subcase 3.1.2.2. Suppose now that $|H| \geq 2$ and $E(D[H]) \neq \emptyset$. Hence, there is an arc $p \to q$ in $E(D[H])$. If $q \to a_3$, then $a_4a_1pqa_3a_4$ is a 5-cycle, a contradiction. Hence, let $a_3 \leadsto q$. If $x \in N^+(q) - V(h)$, then $a_4 \leadsto x$, because otherwise, $a_4a_1pqxa_4$ is a 5-cycle, a contradiction.

Firstly, let $a_4 \to a_2$. Then, we have

$$N^+(a_4) \supseteq (N^+(q) - (V(C) \cup (V(a_4) - \{a_4\}))) \cup (N^-(q) \cap S) \cup \{a_1, a_2\}$$

and

$$N^+(q) \subseteq (N^+(q) - V(C)) \cup \{a_4\}.$$

If there is a vertex $x \in Q \cap S$ such that $x \to q$, then $|N^-(q) \cap S| \geq 2$ and we deduce that

$$d^+(a_4) \geq \begin{cases} d^+(q) + 1, & \text{if} \quad |V(a_4)| = 3 \\ d^+(q) + 2, & \text{if} \quad |V(a_4)| \leq 2 \end{cases},$$

in both cases a contradiction either to Remark 1.18 or to $i_g(D) \leq 1$. Therefore, let $q \to Q \cap S$. If $a_4 \to q$ or $q \in V(a_4)$, then similarly, we arrive at a contradiction. Hence, let $q \to a_4$. Furthermore, $p \in V(a_2)$, since otherwise, $a_4a_1a_2pqa_4$ is a 5-cycle, a contradiction. If there is a vertex $x \in Q \cap S$ such that $x \to a_3$, then $a_4a_1qxa_3a_4$ is a 5-cycle, a contradiction. Hence $Q \cap S \subseteq N^+(a_3)$. If there are vertices $x \in N^+(a_3) - \{a_4\}$ and $y \in N^-(a_4) - \{a_3\}$ such that $x \to y$, then $a_4a_1a_3xya_4$ is a 5-cycle, a contradiction. Consequently, we conclude that $N^-(a_4) - \{a_3\} \leadsto N^+(a_3) - \{a_4\}$. Let $v_1 \to v_2$ be an arc in $E(D[Q \cap S])$. Then, we observe that $d^+(v_2) \leq d^+(a_4) - 2 + |V(a_4) - \{a_4\}|$, and thus $|V(a_4)| \geq 2$. If $(V(a_4) - \{a_4\}) \to v_2$, then we see that $d^+(a_4) \geq d^+(v_2) + 2$, a contradiction. If $|V(a_4)| = 3$ and $|N^+(v_2) \cap (V(a_4) - \{a_4\})| = 1$, then it follows that $d^+(a_4) \geq d^+(v_2) + 1$, a contradiction to Remark 1.18. Hence, let $v_2 \to (V(a_4) - \{a_4\})$. Analogously, we conclude that there is no vertex $w \in Q \cap S$ such that $|N^-(w) \cap Q \cap S| \geq 2$. Let x_1, x_2, x_3 be three vertices of $Q \cap S$ belonging to three different partite sets, then they have to form a 3-cycle and $\{x_1, x_2, x_3\} \to (V(a_4) - \{a_4\})$. Furthermore, we see that $a_3 \to (V(a_4) - \{a_4\})$, because otherwise, if $d \in V(a_4) - \{a_4\}$ such that $d \to a_3$, then

$$\begin{aligned} N^+(a_4) &\supseteq (N^+(a_3) - (V(C) \cup (V(a_4) - \{a_4, d\}))) \cup \{v, a_1, a_2\} \quad \text{and} \\ N^+(a_3) &\subseteq (N^+(a_3) - V(C)) \cup \{a_4\}. \end{aligned}$$

If $|V(a_4)| = 3$, then this implies $d^+(a_4) \geq d^+(a_3) + 1$, a contradiction to Remark 1.18. If $|V(a_4)| = 2$, then $d^+(a_4) \geq d^+(a_3) + 2$, also a contradiction. Let $f \in V(a_4) - \{a_4\}$. Since $N^-(a_4) \leadsto N^+(a_3)$ and $f \in N^+(a_3)$, f has outer neighbors only in $N^+(a_4) - \{x_1, x_2, x_3\}$, a contradiction to $i_g(D) \leq 1$.

Secondly, let $a_2 \to a_4$. As above, we observe that $a_4 \leadsto (N^+(q) - V(C))$. If $V(q) \neq V(a_3)$, then because of $N^+(a_3) \cap N^-(a_4) = \emptyset$ we have $a_4 \leadsto q$ and thus

$$\begin{aligned} N^+(a_4) &\supseteq (N^+(q) \cup \{q\} - (V(C) \cup (V(a_4) - \{a_4\}))) \cup \{v, a_1\} \quad \text{and} \\ N^+(q) &= N^+(q) - V(C). \end{aligned}$$

This implies

$$d^+(a_4) \geq \begin{cases} d^+(q) + 1, & \text{if} \quad |V(a_4)| = 3 \\ d^+(q) + 2, & \text{if} \quad |V(a_4)| \leq 2 \end{cases}.$$

The first case is a contradiction to Remark 1.18, and the second case is a contradiction to $i_g(D) \leq 1$. Analogously, we arrive at a contradiction, if $V(q) = V(a_3)$ and $a_4 \to q$.

Let $A \subseteq H$ be the set of vertices having an inner neighbor in H. Then, it remains to treat the case that $V(q) = V(a_3)$ for all $q \in A$, $A \to a_4$ ($|A| \leq 2$) and $2 \leq |H| \leq 4$. If $B = H - A$, then we conclude that $B \subseteq V(a_2)$, because otherwise, if $p \in B - V(a_2)$ and $q \in A$, then $a_4a_1a_2pqa_4$ is a 5-cycle, a contradiction.

If $|H| = 2$, then $d^+(v) = d^+(a_1)+1 = 5$. Since $a_4 \to (V(v)\cup(Q\cap S)\cup\{a_1\})$ and thus $d^+(a_4) \geq 5$, this implies that $d^+(a_4) = 5$, $|V(v)| = 1$, $|Q \cap S| = 3$ and $H \to a_4$. If there is a vertex $v_1 \in Q \cap S$ such that $v_1 \to a_3$, then, as for the vertex v, it follows that $v_1 \to H \cup \{a_2\}$. Hence, we deduce that $d^+(v_1) \geq |H| + |\{v, a_1, a_2, a_3\}| = 6$, a contradiction. Thus, let $a_3 \to Q \cap S$. If there is a vertex $v_1 \in Q \cap S$ such that $v_1 \to x$ with $x \in \{p, q\}$, then $a_4a_1a_3v_1xa_4$ is a 5-cycle through e, a contradiction. If there is a vertex $v_1 \in Q \cap S$ such that $v_1 \to a_2$, then $a_4a_1a_3v_1a_2a_4$ is a 5-cycle, also a contradiction. Let $v_1, v_2 \in Q \cap S$ such that $v_1 \to v_2$. Summarizing our results, we observe that $d^-(v_2) \geq |H| + |\{a_2, a_3, a_4, v_1\}| = 6$, a contradiction.

Let $|H| = 4$, $H = \{p_1, p_2, q_1, q_2\}$ such that $p_i \to q_j$ with $i, j \in \{1, 2\}$. Then $d^+(v) = d^+(a_1) + 1 = 7$, $|V(a_2)| = |V(a_3)| = 3$ and because of Remark 1.18 $d^+(a_2) = d^-(q_1) = 6$. Since $d^-(v) = |Q| + 1 \geq 6$, we arrive at $|Q| \geq 5$. Furthermore, we see that $N^-(q_1) \supseteq \{p_1, p_2, v, a_1, a_2\}$. This implies $|N^-(q_1) \cap Q| \leq 1$, which means that $|N^+(q_1) \cap Q| \geq |Q| - 1 \geq 4$. If there exists a vertex $w \in N^+(q_1) \cap Q$ such that $w \to a_2$, then $a_4a_1q_1wa_2a_4$ is a 5-cycle, a contradiction. Therefore, we have

$$d^+(a_2) \geq |N^+(q_1) \cap Q| + |\{a_3, a_4, q_1, q_2\}| \geq 8,$$

a contradiction.

Assume now that $|H| = 3$, $H = \{p_1, p_2, q\}$ such that $p_i \to q$ for $i = 1, 2$. Then $d^+(v) = d^+(a_1) + 1 = 6$, $|V(a_2)| = 3$ and $d^+(a_2) = 5$. Since $d^-(v) = |Q| + 1 \geq 5$, we arrive at $|Q| \geq 4$. Furthermore, we see that $N^-(q) \supseteq \{p_1, p_2, v, a_1, a_2\}$. Since $d^-(q) = 5$, if $|V(q)| = 3$, and $d^-(q) \leq 6$, if $|V(q)| = 2$, we conclude that $|N^+(q) \cap Q| \geq |Q| - 1 \geq 3$. As above, we see that $a_2 \to N^+(q) \cap Q$. Therefore, we have

$$d^+(a_2) \geq |N^+(q) \cap Q| + |\{a_3, a_4, q\}| \geq 6,$$

a contradiction.

Consequently, it remains to treat the case that $|H| = 3$ and $H = \{p, q_1, q_2\}$ such that $p \to q_i$ for $i = 1, 2$. Then $d^+(v) = d^+(a_1) + 1 = 6$ and because of Lemma 1.12 and Remark 1.18 we observe that $|V(v)| \leq 2$ and $|V(D)| = 13$. Suppose that there is a vertex $x \in \{q_1, q_2\}$ such that $a_4 \to x$. This implies that $N^-(x) \supseteq \{a_1, a_2, a_4, p, v\}$. Since $|V(x)| = 3$, Remark 1.18 yields that $d^-(x) = 5$ and $x \to Q$. If $|V(v)| \geq 2$, then we conclude that $|S| \geq 5$ and thus $d^+(a_4) \geq 7$, a contradiction. Hence, let $|V(v)| = 1$ and therefore $|Q| = |V(D)| - |V(C)| - |H| - |V(v)| = 5$. If there is a vertex $y \in Q$ such that $y \to a_2$, then $a_4a_1xya_2a_4$ is a 5-cycle containing the arc e, a contradiction. Summarizing our results, we observe that $a_2 \to (Q \cup \{a_3, a_4, q_1, q_2\})$ and thus $d^+(a_2) \geq 9$, a contradiction. Hence, let $\{q_1, q_2\} \to a_4$. If $a_4 \to p$, then we define the cycle $C' = b_4b_1b_2b_3b_4 := a_4a_1pq_1a_4$. We observe that $v \to (\{b_1, b_2, b_3\} \cup (N^+(b_1) - V(C)))$, $|N^+(b_1) - V(C)| = 3$, $b_1 \to b_3$ and $b_4 \to b_2$ and as above we find a 5-cycle containing the arc $b_4b_1 = a_4a_1$, a contradiction. Hence, let $p \to a_4$. Let us take three vertices of $Q \cap S$ belonging to three different partite sets. Then, since $a_4 \rightsquigarrow N^+(a_3) - V(C)$, at least two of them have to be outer neighbors of a_3, because otherwise, there are vertices $v_1, v_2 \in Q \cap S$ such that

$a_4 \to \{v_1, v_2\} \to a_3$, and thus, it follows that

$$\begin{aligned} N^+(a_4) &\supseteq (N^+(a_3) - (V(C) \cup (V(a_4) - \{a_4\}))) \cup \{v, v_1, v_2, a_1\} \quad \text{and} \\ N^+(a_3) &= (N^+(a_3) - V(C)) \cup \{a_4\}. \end{aligned}$$

This implies that

$$d^+(a_4) \geq \begin{cases} d^+(a_3) + 1, & \text{if} \quad |V(a_4)| = 3 \\ d^+(a_3) + 2, & \text{if} \quad |V(a_4)| \leq 2 \end{cases},$$

in both cases a contradiction.

Consequently, let $N^+(a_3) \cap Q \cap S \supseteq \{x, y\}$ such that $x \to y$. If $y \to a_2$, then $a_4a_1a_3ya_2a_4$ is a 5-cycle, a contradiction. Hence, we have $a_2 \to y$. If $y \to u$ with $u \in \{p, q_1, q_2\}$, then $a_4a_1a_3yua_4$ is a 5-cycle, a contradiction. Hence, let $\{p, q_1, q_2\} \to y$. Altogether, we have that $N^-(y) \supseteq \{p, q_1, q_2, x, a_2, a_3, a_4\}$, a contradiction to $i_g(D) \leq 1$.

Subcase 3.2. Let $n \geq 5$. If there are vertices $x \in H$ and $y \in Q$ such that $x \to y$, then $a_na_1xyva_4 \dots a_n$ is an $(n+1)$-cycle, a contradiction. Hence, let $Q \rightsquigarrow H$.

Subcase 3.2.1. Assume that $|H| \geq 2$. At first, let there be an arc $p \to q$ in $E(D[H])$. If $q \to a_3$, then $a_na_1pqa_3 \dots a_n$ is an $(n+1)$-cycle through the arc a_na_1, a contradiction. Altogether, we observe that $d^-(q) \geq |\{p, v, a_1, a_2, a_3\}| + |Q| - |V(q) - \{q\}| \geq |Q| + 3 = d^-(v) + 2$, a contradiction to $i_g(D) \leq 1$.

Consequently it remains to consider the case that $E(D[H]) = \emptyset$, which means that $|H| = 2$ and thus $d^+(v) = d^+(a_1) + 1 = n + 1$. According to Lemma 1.12 and Remark 1.18, we have $|V(v)| \leq 2$. If $h \in H$, then we see that $d^+(h) \leq |V(v) - \{v\}| + |\{a_3, \dots, a_n\}| \leq n - 1$, a contradiction to $i_g(D) \leq 1$.

Subcase 3.2.2. Suppose that $|H| = 1$ and $h \in H$. In this case, we observe that $d^+(v) = d^+(a_1) + 1 = n$. According to Lemma 1.12 and Remark 1.18, we have $|V(v)| \leq 2$. Since $d^+(h) \leq |V(v) - \{v\}| + |\{a_3, \dots, a_n\}| \leq n - 1$, it follows that $d^+(h) = n - 1$, $h \in V(a_2)$ and $|V(v)| = 2$. Let $q \in Q - V(h) \neq \emptyset$. Because of $H \cap Q = \emptyset$, we conclude that $Q \rightsquigarrow a_1$. If $a_2 \to q$, then $a_na_1a_2qha_4a_5 \dots a_n$ is an $(n+1)$-cycle, a contradiction. If $a_i \to q$ with $3 \leq i \leq n - 1$, then $a_na_1a_3 \dots a_iqha_{i+1} \dots a_n$ is an $(n+1)$-cycle, also a contradiction. This implies that $Q \cap S \to \{v, h, a_1, a_2, \dots, a_{n-1}\}$, which means that $d^+(p) \geq n + 1$, if $p \in Q \cap S$, a contradiction. Hence, we have $Q \cap S = \emptyset$ and thus $S = V(v)$, $n = c - 1$ and $D[V(C)]$ is a tournament. Let x be a vertex with $V(x) = \{x\}$. Obviously, we have $x \in V(C)$. If $x = a_i$ with $i \in \{3, \dots, n-1\}$, then it follows that $d^-(a_i) \geq |Q - V(h)| + |\{a_{i-1}, a_1, v, h\}| = |Q| + 3 = d^-(v) + 2$, a contradiction to $i_g(D) \leq 1$. If $|V(a_1)| = 1$, then we conclude that $d^-(a_1) \geq |Q| + |V(v)| + |\{a_n\}| = d^-(v) + 2$, a contradiction. Because of $h \in V(a_2)$, we observe that $|V(a_n)| = 1$ and at least $n - 1$ of the n vertices of $V(C)$ belong to partite sets with at least two vertices. If $|V_c| = 3$, then we have $|Q| \geq |V(a_1) \cup V(a_2) \cup \dots V(a_{n-1})| - |\{a_1, a_2, \dots, a_{n-1}\}| - |H| \geq n - 1$ and $d^-(v) \geq n$. Together with Remark 1.18, this implies the contradiction

$$2n + 1 = |V(D)| = d^+(v) + d^-(v) + 2 \geq 2n + 2.$$

Hence, let $|V_c| = 2$. But now, for every $q \in Q$ we have that $q \notin V(h)$. Let there be a vertex $q \in Q$ such that $d^+_{D[Q]}(q) \geq 1$, then we see that $d^+(q) \geq$

$d^+_{D[Q]}(q) + |\{v, h, a_1, \ldots, a_{n-1}\}| - |V(q) - \{q\}| \geq n + 1$, a contradiction to $i_g(D) \leq 1$.

Subcase 3.2.3. Assume that $|H| = 0$. This yields $d^+(v) = d^+(a_1)+1 = n-1$. Because of $i_g(D) \leq 1$, it follows that $n - 1 \geq d^-(v) = |Q| + 1 \geq n - 2$, which means that $n - 3 \leq |Q| \leq n - 2$. As above we see that $Q \rightsquigarrow a_1$. If there is a vertex $q \in Q$ such that $a_2 \rightarrow q$, then $a_n a_1 a_2 q v a_4 \ldots a_n$ is an $(n + 1)$-cycle containing the arc e, a contradiction. If there are vertices $q \in Q$ and $a_i \in V(C)$ with $a_i \rightarrow q$ for $3 \leq i \leq n - 2$, then $a_n a_1 a_3 \ldots a_i q v a_{i+1} \ldots a_n$ is an $(n + 1)$-cycle, also a contradiction. Summarizing our results, we observe that $Q \rightsquigarrow \{a_1, a_2, \ldots, a_{n-2}, v\}$. Let L_1 be the set of vertices of $Q \cap S$ having an outer neighbor in Q. If $L_1 \neq \emptyset$ and $q_1 \in L_1$, then it follows that $d^+(q_1) \geq n$, a contradiction to $i_g(D) \leq 1$. Hence, let $L_1 = \emptyset$. Let L_2 be the set of vertices of Q having an outer neighbor in Q. Since $|Q| \geq n - 3$ and $|H| \neq 0$, if $|V_c| = 3$ and $n = 5$ (cf. the beginning of the proof of this theorem), we conclude that either $L_2 \neq \emptyset$ or $Q - S = \emptyset$ and $Q \cap S$ consists of vertices of only one partite set. At first let $Q - S = \emptyset$ and let $Q \cap S = S - V(v)$ be one partite set. If $q \in Q \cap S$, then we conclude that $d^+(q) \geq n - 1$, and thus $d^+(q) = n - 1$ and $|Q| = |V(q)| \leq 2$. Since S consists of only two partite sets, we see that $n = c - 2 \geq 6$ and thus $|Q| \geq n - 3 \geq 3$, a contradiction. Hence, let $L_2 \neq \emptyset$. If $q_2 \in Q_2$ and $q_2 \rightarrow q_1$ with $q_1 \in Q$, then we arrive at

$$\begin{aligned} d^+(q_2) &\geq |\{a_1, a_2, \ldots, a_{n-2}, v, q_1\}| - |V(q_2) - \{q_2\}| \\ &\geq \begin{cases} n-2, & \text{if} \quad |V(q_2)| = 3 \\ n-1, & \text{if} \quad |V(q_2)| = 2 \end{cases} . \end{aligned} \tag{2.8}$$

To get no contradiction to $i_g(D) \leq 1$ or to Remark 1.18, it follows that we have equality in (2.8), $d^+_{D[Q]}(q_2) = 1$ and $|V(q_2) \cap Q| = 1$ for all $q_2 \in L_2$, since otherwise, if there is a vertex $q_3 \in Q - \{q_1, q_2\}$ such that $q_2 \rightsquigarrow q_3$, then we observe that $N^+(q_2) \supseteq (\{a_1, a_2, \ldots, a_{n-2}, v, q_1, q_3\} - (V(q_2) - \{q_2\})$ and the right-hand side of (2.8) enlarges by one, a contradiction. If S consists of vertices of at least three partite sets, then, because of $R = \emptyset$ and thus $S - V(v) \subseteq Q$, we conclude that $Q \cap S$ contains vertices of at least two partite sets, a contradiction to $L_1 = \emptyset$. Consequently, it remains to treat the case that S consists of vertices of at most two partite sets.

Firstly, let S consist of vertices of one partite set. This yields $n = c - 1$, $Q \cap S = \emptyset$ and Q is a tournament with $|Q| \leq 3$. But now, we see that $n - 3 \leq |Q| \leq 3$, which means that $n = c - 1 \leq 6$, a contradiction to $c \geq 8$.

Secondly, let S consist of vertices of two partite sets. This implies that $n \geq c - 2$. To get no contradiction in (2.8), we deduce that $|Q \cap S| = 1$ and $q_2 \rightarrow Q \cap S$ for all $q_2 \in L_2$. Since $|V(q_2) \cap Q| = 1$, it follows that $|Q| \leq 2$, and thus $n - 3 \leq |Q| \leq 2$, which means that $c - 2 \leq n \leq 5$, a contradiction to $c \geq 8$.

Summarizing the investigations of Case 3, we see that there remains to consider the case that $a_{n-1} \rightarrow S$.

Case 4. There exists a vertex $v \in S$ such that $a_2 \rightarrow v$. If we consider the converse of D, then, analogously to Case 3, it remains to treat the case that $S \rightarrow a_2$.

If $C = a_n a_1 a_2 \ldots a_n$ and $v \in S$, then the following three sets play an important role in our investigations

$$H = N^+(a_1) - V(C), \qquad F = N^-(a_n) - V(C), \qquad Q = N^-(v) - V(C).$$

Summarizing the investigations in the Cases 1 - 4, we can assume in the following, usually without saying so, that

$$\{a_{n-1}, a_n\} \to S \to \{a_1, a_2\} \rightsquigarrow H \tag{2.9}$$

Case 5. Let $n = 4$. Because of (2.9), we see that $\{a_3, a_4\} \to S \to \{a_1, a_2\}$. Hence, we conclude that $N^+(a_4) \supseteq S \cup \{a_1\}$. Analogously as in Subcase 3.1, we observe that $D[V(C)]$ is a tournament.

Subcase 5.1. Let $a_1 \to a_3$. If $a_2 \to a_4$ and $v \in S$, then $a_4 a_1 a_3 v a_2 a_4$ is a 5-cycle, a contradiction. Consequently, let $a_4 \to a_2$. If there are vertices $v \in S$ and $x \in F$ such that $v \to x$, then $a_4 a_1 a_3 v x a_4$ is a 5-cycle, a contradiction. Hence, let $F \to S$. If we take vertices $v, w \in S$ such that $v \to w$, then we have $N^-(a_4) = F \cup \{a_3\}$ and $N^-(w) \supseteq F \cup \{a_3, a_4, v\}$, a contradiction to $i_g(D) \leq 1$.

Subcase 5.2. Let $a_3 \to a_1$ and assume that $a_2 \to a_4$. If there are vertices $v \in S$ and $x \in H$ such that $x \to v$, then $a_4 a_1 x v a_2 a_4$ is a 5-cycle, a contradiction. Otherwise, we have $S \to H$. If we take two vertices $v, w \in S$ such that $v \to w$, then we observe that $N^+(a_1) = H \cup \{a_2\}$ and $N^+(v) \supseteq \{a_1, a_2, w\} \cup H$, a contradiction to $i_g(D) \leq 1$.

Finally, let $a_4 \to a_2$. Because of Corollary 1.19, it follows that

$$\begin{aligned} c + k = |V(D)| &\geq |H| + |F| + |S| + |V(C)| - |H \cap F| \\ &\geq \frac{c+k-3}{2} - 1 + \frac{c+k-3}{2} - 1 + 4 + 4 - |H \cap F| \\ &= c + k + 3 - |H \cap F|, \end{aligned}$$

which leads to $|H \cap F| \geq 3$. Thus, $H \cap F$ contains vertices of at least two partite sets. Now, we take two vertices $u_2, u_3 \in H \cap F$ such that $u_2 \to u_3$. Then, $C' = a_4 a_1 u_2 u_3 a_4$ is a cycle through $a_4 a_1$ such that $a_1 \to u_3$ and $u_2 \to a_4$. Analogously to Subcase 5.1 with $a_2 \to a_4$, this yields a contradiction.

Therefore, we have seen that every arc of D is contained in a 5-cycle. From now on, let us suppose that $n \geq 5$.

Case 6. Let $n \geq 5$ and assume that there exists a vertex $v \in S$ such that $v \to a_{n-2}$. If there is a vertex $a_i \in V(C)$ with $3 \leq i \leq n-3$ such that $a_i \to v$, then we obtain, as in Case 1, an $(n+1)$-cycle through $a_n a_1$, a contradiction. Thus, we investigate now the case that $v \to \{a_1, a_2, \ldots, a_{n-2}\}$. If there is a vertex $h \in H$ such that $h \to v$, then $a_n a_1 h v a_3 a_4 \ldots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. Therefore, we assume now that $v \to H$. This leads to $d^+(v) \geq d^+(a_1)$, and thus, because of $i_g(D) \leq 1$, it follows that $a_1 \to \{a_2, a_3, \ldots, a_{n-1}\}$ or $a_1 \to \{a_2, a_3, \ldots, a_{n-1}\} - \{a_j\}$ for some $j \in \{3, 4, \ldots, n-1\}$ and $a_j \to a_1$ or $V(a_1) = V(a_j)$.

Subcase 6.1. Assume that $a_1 \to \{a_2, a_3, \ldots, a_{n-1}\}$. If there is a vertex $h \in H$ such that $h \to a_n$, then $a_n a_1 a_3 a_4 \ldots a_{n-1} v h a_n$ is an $(n+1)$-cycle, a contradiction. Therefore, we may assume now that $a_n \to (H - V(a_n))$.

If $a_{i-1} \to a_n$ for $3 \le i \le n-1$, then $a_n a_1 a_i a_{i+1} \dots a_{n-1} v a_2 a_3 \dots a_{i-1} a_n$ is an $(n+1)$-cycle, a contradiction. Hence, it remains to treat the case that $a_n \to a_{i-1}$ or $a_{i-1} \in V(a_n)$ for $2 \le i \le n-1$. If there is a vertex $x \in H \cap F$, then $a_n a_1 a_3 \dots a_{n-1} v x a_n$ is an $(n+1)$-cycle, a contradiction. Let $R = V(D) - (H \cup F \cup S \cup V(C))$. Since $a_n \rightsquigarrow \{a_1, \dots, a_{n-2}\}$, Corollary 1.19 leads to

$$|R| \le c + k - \left\{ \frac{c+k-3}{2} - (n-2) + \frac{c+k-3}{2} - 1 + 1 + n \right\} = 1,$$

if $|S| = 1$, $|R| \le 0$, if $|S| = 2$ and the contradiction $|R| \le -1$, if $|S| \ge 3$. Hence, it follows that $|S| \le 2$, and thus $n \ge 6$. If there are vertices $h \in H$ and $y \in F$ such that $h \to y$, then $a_n a_1 a_4 \dots a_{n-1} v h y a_n$ is an $(n+1)$-cycle containing the arc e, a contradiction. Consequently, let $F \rightsquigarrow H$.

Subcase 6.1.1. Suppose that $|H| \ge 2$. This implies that there are vertices $h_1, h_2 \in H$ such that $h_1 \rightsquigarrow h_2$. On the one hand, we have $d^+(v) \ge n - 2 + |H|$ and on the other hand, since $|S| + |R| \le 2$, we conclude that $d^+(h_2) \le |H - \{h_1, h_2\}| + |\{a_3, \dots, a_{n-1}\}| + |S - \{v\}| + |R| \le |H| - 2 + n - 3 + 1 = |H| + n - 4$. Combining these results we arrive at $d^+(v) - d^+(h_2) \ge 2$, a contradiction to $i_g(D) \le 1$.

Subcase 6.1.2. Let $|H| = 1$ and $h \in H$. In this case, we have

$$d^-(h) \ge |F| + |\{v, a_n, a_1, a_2\}| - |V(h) - \{h\}| \ge \left\{ \begin{array}{ll} |F| + 2, & \text{if } \ |V(h)| = 3 \\ |F| + 3, & \text{if } \ |V(h)| = 2 \end{array} \right. ,$$

whereas $d^-(a_n) \le |F| + |\{a_{n-1}\}| = |F| + 1$, which means that $d^-(h) - d^-(a_n) \ge 1$, if $|V(h)| = 3$ and $d^-(h) - d^-(a_n) \ge 2$, if $|V(h)| = 2$, in both cases a contradiction.

Subcase 6.1.3. Assume that $H = \emptyset$. This implies that $d^+(a_1) = n - 2$ and $d^+(v) \ge n - 2$. If there are vertices $w \in S$ and $f \in F$ such that $w \to f$, then $a_n a_1 a_3 \dots a_{n-1} w f a_n$ is an $(n+1)$-cycle, a contradiction. Hence, we have $F \to S$. Since $n - 3 \le d^-(a_n) \le |F| + 1$, we conclude that $|F| \ge n - 4 \ge 2$, and thus $F \ne \emptyset$. Furthermore, we observe that

$$n - 1 \ge d^-(v) \ge |F| + 2 \quad \Rightarrow \quad |F| \le n - 3. \tag{2.10}$$

Since $H = \emptyset$, we see that $F \rightsquigarrow a_1$. If there is a vertex $f \in F$ such that $a_{n-1} \to f$, then $a_n a_1 \dots a_{n-1} f a_n$ is an $(n+1)$-cycle containing the arc e, a contradiction. If there is a vertex $f \in F$ such that $a_i \to f$ with $3 \le i \le n-3$, then $a_n a_1 a_3 \dots a_i f v a_{i+1} \dots a_n$ is an $(n+1)$-cycle, also a contradiction. Summarizing our results we observe that $F \rightsquigarrow (S \cup \{a_1, a_3, a_4, \dots, a_{n-3}, a_{n-1}, a_n\})$. Let $f \in F$ with $d^-_{D[F]}(f) \le \frac{|F|-1}{2}$. This yields

$$d^-(f) \le d^-_{D[F]}(f) + |\{a_2, a_{n-2}\}| + |R| \le \frac{|F| - 1}{2} + 2 + |R|. \tag{2.11}$$

Subcase 6.1.3.1. Suppose that $d^-(f) = n - 3$. In this case, the bound in (2.10) can be improved by $|F| + 2 \le d^-(v) \le n - 2$, which means that $|F| \le n - 4$ and thus $|F| = n - 4$. Combining this with (2.11) we arrive at $n - 3 \le \frac{n-5}{2} + 2 + |R| \le \frac{n+1}{2} \Rightarrow n \le 7$.

Firstly let $n = 6$. Because of $|S| \leq 2$, it follows that $n \geq c-2$, and thus $c = 8$, $|S| = 2$ and $|R| = 0$. But now, with (2.11) yields $n - 3 \leq \frac{n-5}{2} + 2 = \frac{n-1}{2} \Rightarrow n \leq 5$, a contradiction.

Secondly let $n = 7$. If $|R| = 0$, then we arrive at a contradiction as above. Hence, let $|R| = 1$. Since $d^-(f) = n-3$ we conclude that $d^+(v) = n-2$ and $d^-(v) \geq |F| + 2 = n-2$ and thus $d^-(v) = n-2$. If $x \in R$, then x is adjacent to v, a contradiction to $d^-(v) = d^+(v) = n-2$.

Subcase 6.1.3.2. Assume that $d^-(f) \geq n-2$. Combining (2.10) and (2.11) we see that

$$n - 2 \leq \frac{n-4}{2} + 2 + |R| \leq \frac{n+2}{2} \Rightarrow n \leq 6.$$

This implies that $n = 6$ and the inequalities in the last inequality-chain have to be equalities, which especially means that $|R| = 1$ and thus $|S| = 1$. This yields the contradiction $6 = n = c - 1 \geq 7$.

Subcase 6.2. Assume that $n = 5$ and there is exactly one $j \in \{3, 4\}$ such that $a_1 \to (\{a_2, a_3, a_4\} - \{a_j\})$ and $a_j \to a_1$ or $V(a_j) = V(a_1)$. In this case, we observe that $d^+(v) \geq d^+(a_1) + 1$.

Subcase 6.2.1. Let $a_1 \to \{a_2, a_3\}$ and $a_4 \to a_1$ or $V(a_4) = V(a_1)$. If there is a vertex $h \in H$ such that $h \to a_5$, then $a_5a_1a_3a_4vha_5$ is a 6-cycle, a contradiction. Therefore, we may assume that $a_5 \to (H - V(a_5))$. If $a_2 \to a_5$, then $a_5a_1a_3a_4va_2a_5$ is a 6-cycle, a contradiction. Hence, it remains to consider the case that $a_5 \to a_2$ or $V(a_5) = V(a_2)$. Let $\{a_1, a_2\} = A \cup B$ such that $a_5 \to A$ and $B \subseteq V(a_5)$. Then $N^+(a_1) = H \cup \{a_2, a_3\}$ and $N^+(a_5) \supseteq A \cup S \cup (H - (V(a_5) - (B \cup \{a_5\})))$. This leads to

$$d^+(a_5) \geq |A| + |S| + |H| - (3 - (|B| + 1)) = d^+(a_1) + |S| - 2.$$

This implies $|S| \leq 3$ and thus $c = 8$ and $|S| = 3$. Then we see that $d^+(a_5) \geq d^+(a_1) + 1$ such that we have equality in the last inequality chain. Especially, we observe that $|V(a_5)| = 3$, a contradiction to Lemma 1.12 and Remark 1.18.

Subcase 6.2.2. Let $a_1 \to \{a_2, a_4\}$ and $a_3 \to a_1$ or $V(a_3) = V(a_1)$. Since $N^+(v) = H \cup \{a_1, a_2, a_3\}$, we observe that $R = V(D) - (H \cup Q \cup V(v) \cup V(C)) = \emptyset$. If $a_3 \to a_5$, then $a_5a_1a_4va_2a_3a_5$ is a 6-cycle, a contradiction. If there exists a vertex $h \in H$ such that $h \to a_5$ and if $q \in Q \cap S \neq \emptyset$, then $a_5a_1a_4qvha_5$ is a 6-cycle, a contradiction. Let $A \cup B = \{a_1, a_3\}$ such that $a_5 \to A$ and $B \subseteq V(a_5)$, then it follows that $N^+(a_1) = H \cup \{a_2, a_4\}$ and $N^+(a_5) \supseteq S \cup A \cup (H - (V(a_5) - (B \cup \{a_5\})))$, and thus, we have

$$d^+(a_5) \geq |A| + |H| + |S| - (3 - (|B| + 1)) = d^+(a_1) + |S| - 2.$$

This implies $|S| \leq 3$ and thus $c = 8$ and $|S| = 3$. Then we see that $d^+(a_5) \geq d^+(a_1) + 1$ such that we have equality in the last inequality chain. Especially, we observe that $|V(a_5)| = 3$, because of Lemma 1.12 and Remark 1.18 a contradiction.

Subcase 6.3. Suppose that $n \geq 6$ and there is exactly one $j \in \{3, \ldots, n-1\}$ such that $a_1 \to (\{a_2, a_3, \ldots, a_{n-1}\} - \{a_j\})$ and $a_j \to a_1$ or $V(a_1) = V(a_j)$. In this case, we observe that $d^+(v) \geq d^+(a_1)+1$ and thus $d^+(v) = d^+(a_1)+1$. Since $Q \to v \to H$, it follows that $Q \cap H = \emptyset$. If $R = V(D) - (H \cup Q \cup V(v) \cup V(C))$, then obviously $R = \emptyset$. If there are vertices $x \in H$ and $y \in Q$ such that $x \to y$, then $a_n a_1 x y v a_4 \ldots a_n$ is an $(n+1)$-cycle through e, a contradiction. Summarizing our results, we see that

$$(Q \cup \{a_1, a_2, v\}) \rightsquigarrow H.$$

Subcase 6.3.1. Let $|H| \geq 2$. If there are vertices $h_1, h_2 \in H$ such that $h_1 \to h_2$, then it follows that $a_3 \rightsquigarrow h_2$, since otherwise $a_n a_1 h_1 h_2 a_3 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Hence we have

$$\begin{aligned} d^-(h_2) &\geq |Q| + |\{v, h_1, a_1, a_2, a_3\}| - |V(h_2) - \{h_2\}| \\ &\geq \left\{ \begin{array}{ll} |Q| + 3 = d^-(v) + 1, & \text{if} \quad |V(h_2)| = 3 \\ |Q| + 4 = d^-(v) + 2, & \text{if} \quad |V(h_2)| = 2 \end{array} \right. , \end{aligned}$$

in both cases a contradiction, either to $i_g(D) \leq 1$ or to Remark 1.18.

Consequently it remains to consider the case that $E(D[H]) = \emptyset$, which means that $H = \{h_1, h_2\}$ such that $h_1 \in V(h_2)$. If there are vertices $a_i \in V(C)$ with $i \in \{3, 4, \ldots, n\}$ and $h \in H$ such that $a_i \rightsquigarrow h$, then analogously as above we arrive at a contradiction. Hence let $H \to \{a_3, a_4, \ldots, a_n\}$. This yields that $a_n a_1 h_1 a_4 \ldots a_{n-1} v h_2 a_n$ is an $(n+1)$-cycle containing the arc e, a contradiction.

Subcase 6.3.2. Assume that $|H| = 1$ and $h \in H$. If there is a vertex $a_i \in N^+(h)$ with $3 \leq i \leq n$, then we conclude that $(Q - V(h)) \rightsquigarrow a_{i-2}$, since otherwise, if $q \in Q - V(h)$ such that $a_{i-2} \to q$, then $a_n a_1 \ldots a_{i-2} q h a_i \ldots a_n$ is an $(n+1)$-cycle, a contradiction. If $N^+(h) \cap V(C) = \{a_{i_1}, a_{i_2}, \ldots, a_{i_g}\}$, then we define $M = \{a_{i_1-2}, a_{i_2-2}, \ldots, a_{i_g-2}\}$. Furthermore we observe that $d^+(v) = n - 1 = d^+(a_1) + 1$. According to Remark 1.18, we have $|V(v)| \leq 2$. Because of $|Q| = d^-(v) - 2 \geq n - 4 \geq 2$, we see that there are vertices $q_1, q_2 \in Q$ such that $q_1 \rightsquigarrow q_2$.

Firstly, let $q_1 \notin V(h)$. This implies that

$$|N^+(h)| \leq |M| + |V(v) - \{v\}| \leq |M| + 1 \tag{2.12}$$

and

$$\begin{aligned} |N^+(q_1)| &\geq |M| + |\{q_2, v, h\}| - |V(q_1) - \{q_1\}| \\ &\geq \left\{ \begin{array}{ll} d^+(h), & \text{if} \quad |V(q_1)| = 3 \\ d^+(h) + 1, & \text{if} \quad |V(q_1)| = 2 \end{array} \right. . \end{aligned} \tag{2.13}$$

To get no contradiction, all inequalities in the inequality-chain of (2.12) and (2.13) have to be equalities, which especially means that $|V(v)| = 2$. If $a_3 \notin N^+(h)$, then, noticing that $q_1 \rightsquigarrow a_1$, we conclude that $a_1 \notin M$ and thus $N^+(q_1) \supseteq ((M \cup \{q_2, v, h, a_1\}) - (V(q_1) - \{q_1\}))$. Then similarly to (2.13), we arrive at a contradiction. Therefore, let $h \to a_3$. If $V(h) \neq V(a_2)$, then $a_n a_1 a_2 h a_3 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Consequently, let $V(h) = V(a_2)$. Let $v' \in V(v) - \{v\}$. Because of (2.12) and (2.13), it follows that $h \to v'$ and thus $a_3 \to v'$ since otherwise $a_n a_1 h v' a_3 \ldots a_n$ is an $(n+1)$-cycle through e, a contradiction. This implies $\{a_3, \ldots, a_n, h\} \to v'$ and thus $d^-(v') \geq n - 1$. Since $i_g(D) \leq 1$ we conclude that $d^-(v') = n - 1$ and $v' \to Q$. If $n \geq 7$, then $a_n a_1 h v' q v a_5 \ldots a_n$ is an $(n+1)$-cycle for any $q \in Q$, a contradiction. Hence, let $n = 6$, and thus $|S| \geq 3$ and $Q \cap S \neq \emptyset$. If there are vertices $s_1 \in Q \cap S$ and $\hat{q}_2 \in Q$ such that $s_1 \to \hat{q}_2$, then similarly as in (2.13), we arrive at the contradiction $d^+(s_1) \geq d^+(h) + 2$. This implies $Q \cap S$ consists of vertices of only one partite set, and thus we conclude that $c = 8$ and $D[V(C)]$ is a tournament. If there is a vertex a_i with $2 \leq i \leq 4$ such that $a_i \to a_6$, then $a_6 a_1 h v' q v a_i a_6$ is a 7-cycle for every $q \in Q$, a contradiction. This yields $d^+(a_6) \geq |\{a_1, a_2, a_3, a_4\}| + |S| \geq 7 = d^+(a_1) + 3$, a contradiction to $i_g(D) \leq 1$.

Secondly, let $q_1 \in V(h)$. If $|Q| \geq 3$, then there are vertices $q_1', q_2' \in Q$ such that $q_1' \rightsquigarrow q_2'$ and $q_1' \notin V(h)$ and as above this leads to a contradiction. Hence, let $|Q| = 2$ and thus, because of $|Q| \geq n - 4 \geq 2$, let $n = 6$. Since $c \geq 8$, we conclude that $Q \cap S \neq \emptyset$, $\{q_2\} = Q \cap S$, which implies that $c = 8$ and $D[V(C)]$ is a tournament. Furthermore we observe that

$$d^+(q_2) \geq |N^+(h) \cap V(C)| + |\{v, h\}| \geq d^+(h) + 1.$$

To get no contradiction to $i_g(D) \leq 1$, the equalities in the last inequality-chain and in (2.12) have to be equalities, which means that $|V(v)| = 2$, $h \to (V(v) - \{v\})$, and because of $q_2 \to \{a_1, a_2\}$, similarly as above it follows that $h \to \{a_3, a_4\}$, and thus $V(h) = \{h, a_2, q_1\}$. Let $v' \in V(v) - \{v\}$. If $v' \to a_3$, then $a_6 a_1 h v' a_3 a_4 a_5 a_6$ is a 7-cycle, a contradiction. Consequently, we have $a_3 \to v'$ and analogously as in Case 2, we arrive at $\{a_3, a_4, a_5, a_6, h\} \to v'$. Since $d^+(a_1) = 4$, this implies that $d^-(v') = 5$ and $v' \to Q$. If $h \to a_6$, then either $a_6 a_1 a_3 a_4 a_5 q_2 h a_6$ or $a_6 a_1 a_4 a_5 q_2 v h a_6$ is a 7-cycle, a contradiction. It follows that $a_6 \to \{h, a_1, v, v', q_2\}$ and thus $a_3 \to a_6$. But now $a_6 a_1 h v' q_2 a_2 a_3 a_6$ is a 7-cycle, a contradiction.

Subcase 6.3.3. Suppose that $|H| = 0$. If $a_1 \to a_i$ for some $i \in \{3, \ldots, n-1\}$ and $a_{i-1} \to a_n$, then $a_n a_1 a_i a_{i+1} \ldots a_{n-1} v a_2 a_3 \ldots a_{i-1} a_n$ is an $(n+1)$-cycle, a contradiction. Let $N^+(a_1) = \{a_{i_1}, \ldots, a_{i_{n-3}}\}$ and $A \cup B = \{a_{i_1-1}, \ldots, a_{i_{n-3}-1}\}$ such that $a_n \to A$ and $B \subseteq V(a_n)$. Then $|B| \leq 2$, $|S| \geq |B|+1$, $N^+(a_n) \supseteq A \cup S$ and thus

$$d^+(a_n) \geq |A| + |S| = d^+(a_1) - |B| + |S| \geq d^+(a_1) + 1, \qquad (2.14)$$

which means that $|S| = 1$, if $|B| = 0$, $|S| = 2$, if $|B| = 1$, and $|S| = 3$, if $|B| = 2$. According to Remark 1.18, the combination $|B| = 2$ and $d^+(a_n) \geq d^+(a_1) + 1$ is impossible. Hence let $|B| \leq 1$ and $|S| = |B| + 1 \leq 2$.

Since $|H| = 0$, we conclude that $d^+(a_1) = n - 3$, $d^+(v) = n - 2$ and $1 \leq n - 5 \leq |Q| = d^-(v) - 2 \leq n - 4$.

Firstly, let $|Q| = 1$. In this case we have $d^+(v) = n - 2 \geq 4$ and $d^-(v) = 3$, which implies that $n = 6 \leq c - 2$. Hence, we see that $|S| \geq 2$ and (2.14) yields $|S| = 2$, $Q = S - \{v\}$ and $D[V(C)]$ is a tournament, which means that $|B| = 0$, a contradiction to $|S| = 2$.

Secondly, let $|Q| = 2$ and $|V_c| = 3$. Then $d^+(v) = n - 2$ and $d^-(v) = 4$ and thus $n = 6$ or $n = 7$. If $n = 6 \leq c - 2$, then we conclude that $|S| \geq 2$ and (2.14) yields that $|S| = 2$ and $D[V(C)]$ is a tournament, which means that $|B| = 0$, a contradiction to $|S| = 2$. Consequently, let $n = 7$. In this case we have $d^+(v) = 5$ and $d^-(v) = 4$ and Remark 1.18 yields that $|V(v)| = 2$. Since

$|S| \leq 2$ and $c \geq 8$ we obtain that $|S| = 2$, $c = 8 = n+1$ and $D[V(C)]$ is a tournament and thus $|B| = 0$, also a contradiction to $|S| = 2$.

Thirdly, let $|Q| \geq 3$ or $|Q| = 2$ and $|V_c| = 2$. This implies that there are vertices $q_1, q_2 \in Q$ such that $q_1 \to q_2$. Because of (2.14), we have $N^+(a_n) \cap (Q - S) = \emptyset$. Let $q \in Q$ be arbitrary. Since $H = \emptyset$ we conclude that $q \rightsquigarrow a_1$. If $a_2 \to q$, then $a_n a_1 a_2 q v a_4 \ldots a_n$ is an $(n+1)$-cycle containing the arc e, a contradiction.

Assume that $a_1 \to a_3$. If $a_i \to q$ for an index i with $3 \leq i \leq n-3$, then $a_n a_1 a_3 \ldots a_i q v a_{i+1} \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Altogether, we see that $q_1 \rightsquigarrow \{v, a_1, \ldots, a_{n-3}, a_n, q_2\}$, if $q_1 \in Q - S$ and $q_1 \to \{v, a_1, \ldots, a_{n-3}, q_2\}$, if $q_1 \in Q \cap S$. It follows that

$$d^+(q_1) \geq \begin{cases} n-1 = d^+(a_1) + 2, & \text{if} \quad |V(q_1)| = 2 \\ n-2 = d^+(a_1) + 1, & \text{if} \quad |V(q_1)| = 3 \end{cases},$$

if $q_1 \in Q - S$ and $d^+(q_1) \geq n-1$, if $q_1 \in Q \cap S$, in all cases a contradiction either to $i_g(D) \leq 1$ or to Remark 1.18.

Consequently, it remains to consider the case that $a_3 \to a_1$ or $V(a_3) = V(a_1)$ and $a_1 \to \{a_2, a_4, \ldots, a_{n-1}\}$. If $n = 6$, then we deduce that $|S| = 2$ and $|B| = 0$, a contradiction to (2.14). Consequently, let $n \geq 7$. If $a_i \to q_1$ for $i \in \{4, \ldots, n-3\}$, then $a_n a_1 a_4 \ldots a_i q_1 q_2 v a_{i+1} \ldots a_n$ is an $(n+1)$-cycle containing the arc $a_n a_1$, a contradiction. At first let $q_1 \in Q \cap S$. This implies that $q_1 \to \{v, a_1, a_2, a_3, \ldots, a_{n-3}, q_2\}$ and thus $d^+(q_1) \geq n-1 = d^+(a_1) + 2$, a contradiction to $i_g(D) \leq 1$. Hence, we have $q_1 \in Q - S$ and $q_1 \rightsquigarrow \{v, a_1, a_2, a_4, \ldots, a_{n-3}, a_n, q_2\}$, which means that

$$d^+(q_1) \geq \begin{cases} n-2, & \text{if} \quad |V(q_1)| = 2 \\ n-3, & \text{if} \quad |V(q_1)| = 3 \end{cases}.$$

To get no contradiction to $i_g(D) \leq 1$, it has to be equality. This implies that $V(q_1) \neq V(a_{n-2})$ and $V(q_1) \neq V(a_{n-1})$ and $V(q_1) \neq V(a_3)$ and thus $\{a_3, a_{n-2}, a_{n-1}\} \to q_1$. The inequality-chain (2.14) yields that $|V(a_n)| \leq 2$. If $V(q_1) \neq V(a_n)$, then $a_n a_1 \ldots a_{n-1} q_1 a_n$ is an $(n+1)$-cycle, a contradiction. Consequently, let $V(q_1) = V(a_n)$ and thus $a_4 \notin V(q_1)$. This implies that $a_n a_1 a_2 a_3 q_1 a_4 \ldots a_n$ is an $(n+1)$-cycle through e, a contradiction.

Summarizing the investigations of Case 6, we see that there remains to treat the case that $a_{n-2} \to S$.

Case 7. Let $n = 5$. If we consider the cycle $C^{-1} = a_1 a_5 a_4 a_3 a_2 a_1 = b_5 b_1 b_2 b_3 b_4 b_5$ in the converse D^{-1} of D, then $\{b_4, b_5\} \to S \to \{b_1, b_2, b_3\}$. Since this is exactly the situation of Case 6, there exists in D^{-1} a 6-cycle, containing the arc $b_5 b_1 = a_1 a_5$, and hence there exists in D a 6-cycle through $a_5 a_1$.

Case 8. Let $n \geq 6$. Assume that there exists a vertex $v \in S$ such that $a_3 \to v$. If we consider the converse of D, then in view of Case 6, it remains to consider the case that $S \to a_3$.

Case 9. Let $c > n \geq 6$. If there are vertices $v \in S$ and $x \in H$ such that $x \to v$, then $a_n a_1 x v a_3 a_4 \ldots a_n$ is an $(n+1)$-cycle through e, a contradiction. Consequently, we have $S \to H$. If there is a vertex $x \in H$ such that $x \to a_n$, then $a_n a_1 a_2 \ldots a_{n-2} v x a_n$ is an $(n+1)$-cycle, also a contradiction. Summarizing our results, we see that $(S \cup \{a_1, a_2, a_n\}) \rightsquigarrow H$. If $a_1 \to a_i$ with $3 \le i \le n-1$ and $a_{i-1} \to a_n$, then $a_n a_1 a_i \ldots a_{n-1} v a_2 \ldots a_{i-1} a_n$ is an $(n+1)$-cycle containing the arc e, a contradiction. Let $N = \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}$ be exactly the subset of $V(C) - \{a_2\}$ such that $a_1 \to N$. Then we define $A \cup B = \{a_{i_1-1}, a_{i_2-1}, \ldots, a_{i_k-1}\}$ such that $a_n \to A$ and $B \subseteq V(a_n)$. Obviously $|B| \le 2$. Since $a_n \to (H - V(a_n))$, we deduce that $N^+(a_1) = \{a_2\} \cup N \cup H$ and $N^+(a_n) \supseteq \{a_1\} \cup A \cup S \cup (H - (V(a_n) - (B \cup \{a_n\})))$, and thus

$$\begin{aligned} d^+(a_n) &\ge \\ &\left\{ \begin{array}{ll} |A| + |S| + 1 + |H| - (3 - (|B| + 1)), & \text{if} \quad |V(a_n)| = 3 \\ |A| + |S| + 1 + |H| - (2 - (|B| + 1)), & \text{if} \quad |V(a_n)| \le 2 \end{array} \right. \\ &= \left\{ \begin{array}{ll} d^+(a_1) + |S| - 2, & \text{if} \quad |V(a_n)| = 3 \\ d^+(a_1) + |S| - 1, & \text{if} \quad |V(a_n)| \le 2 \end{array} \right. . \end{aligned} \tag{2.15}$$

This implies that $|S| = 1$ or $|S| = 2$ and thus $|B| \le 1$. Let $R_2 = V(D) - (H \cup F \cup S \cup V(C))$. Since $F \to a_n \rightsquigarrow H$, it follows that $H \cap F = \emptyset$. If there are vertices $\tilde{v} \in S$ and $f \in F$ such that $\tilde{v} \to f$, then $a_n a_1 \ldots a_{n-2} \tilde{v} f a_n$ is an $(n+1)$-cycle, a contradiction. Hence, let $F \to S$. Because of $F \cap H = \emptyset$, we observe that $F \rightsquigarrow a_1$. If there is a vertex $f \in F$ such that $a_{n-1} \to f$, then $a_n a_1 \ldots a_{n-1} f a_n$ is an $(n+1)$-cycle through e, a contradiction. Let $f \in F$ be arbitrary. If there is an index $i \in \{3, 4, \ldots, n-2\}$ such that $a_1 \to a_i$ and $a_{i-1} \to f$, then $a_n a_1 a_i \ldots a_{n-2} v a_2 \ldots a_{i-1} f a_n$ is an $(n+1)$-cycle containing the arc e, a contradiction. If $a_1 \to a_{n-1}$ and $a_{n-2} \to f$, then $a_n a_1 a_{n-1} v a_3 \ldots a_{n-2} f a_n$ is an $(n+1)$-cycle, also a contradiction. Summarizing our results, we observe that

$$F \rightsquigarrow (S \cup A \cup B \cup \{a_1, a_n, a_{n-1}\}). \tag{2.16}$$

Subcase 9.1. Assume that there is a vertex $v \in S$ such that $v \to a_{n-3}$. As in Case 1, we see that $v \to \{a_1, a_2, \ldots, a_{n-3}\}$.

Subcase 9.1.1. Let $H = \emptyset$. If there is a vertex $f \in F$, then (2.16) implies

$$\begin{aligned} d^+(f) &\ge |N| + |\{a_1, a_n, a_{n-1}\}| + |S| - |V(f) - \{f\}| \\ &\ge \left\{ \begin{array}{ll} |N| + 1 + |S| = d^+(a_1) + |S|, & \text{if} \quad |V(f)| = 3 \\ |N| + 2 + |S| = d^+(a_1) + 1 + |S|, & \text{if} \quad |V(f)| = 2 \end{array} \right. \\ &\ge \left\{ \begin{array}{ll} d^+(a_1) + 1, & \text{if} \quad |V(f)| = 3 \\ d^+(a_1) + 2, & \text{if} \quad |V(f)| = 2 \end{array} \right. , \end{aligned}$$

in both cases a contradiction either to Remark 1.18 or to $i_g(D) \le 1$. Hence, it remains to consider the case that $F = \emptyset$. According to (2.15), we have

$$d^+(a_n) \ge |A| + |S| + 1 \ge |A| + |S| + |B| = d^+(a_1) - 1 + |S|,$$

which means that there remain to treat the two following cases:

i) $|S| = 2$, $d^+(a_n) = d^+(a_1) + 1$, $|B| = 1$, $n = c - 1$, $|V(v)| = 1$ and $|V(a_n)| \le 2$. If $|V_c| = 3$, then we have $|V(a_1)| \ge 2$.

ii) $|S| = 1$ and thus $|B| = 0$, $n = c - 1$, $D[V(C)]$ is a tournament, $d^+(a_n) = d^+(a_1) + 1$ and $|V(a_n)| \le 2$. If $|V_c| = 3$, then we have $|V(a_1)| \ge 2$.

Let $a_1' \in V(a_1) - \{a_1\}$. If $a_1' \in V(C)$, then, because of $n = c-1$, we conclude that $|S| \geq 2$ and $|B| = 0$ or $|B| \geq 1$ and $|S| \geq 3$, in both cases a contradiction to i) and ii). Since $F = \emptyset$, it follows that $a_n \to a_1'$, and similarly as in i) and ii) we deduce that $d^+(a_n) \geq d^+(a_1) + 2$, a contradiction to $i_g(D) \leq 1$.

Hence, let $V(a_1) = \{a_1\}$ and thus $|V_c| = 2$. We observe that

$$\begin{aligned} |V(D)| &= d^+(a_n) + d^-(a_n) + |V(a_n)| = d^+(a_1) + 1 + d^-(a_n) + |V(a_n)| \\ &\geq d^+(a_1) + d^-(a_1) + |V(a_n)| = d^+(a_1) + d^-(a_1) + 1 + |V(a_n)| - 1 \\ &= |V(D)| + |V(a_n)| - 1. \end{aligned}$$

It follows that $|V(a_n)| = 1$ and thus $|B| = 0$, which means that it remains to treat the Case ii). If $R_2 \neq \emptyset$ and $x \in R_2$, then, because of $|V(a_n)| = 1$ we have $x \notin V(a_n)$. If $x \to a_n$, then $x \in F$, a contradiction to $F = \emptyset$. If $a_n \to x$, then as in ii) we conclude that $d^+(a_n) \geq d^+(a_1) + 2$, a contradiction to $i_g(D) \leq 1$. Consequently, it remains to investigate the case that $R_2 = \emptyset$. Since the Case ii) yields that $D[V(C)]$ is a tournament and $|S| = 1$, we conclude that $k = 0$, a contradiction to the hypothesis of this theorem.

Subcase 9.1.2. Suppose that H consists of vertices of only one partite set, which means that $|H| \leq 2$.

Subcase 9.1.2.1. Let $H \subseteq V(a_n) - \{a_n\}$.

Firstly, let $|B| = 0$. This yields that $\{a_2, a_3, \ldots, a_{n-1}\} \to H$, since otherwise for $l = \min\{2 \leq i \leq n-1 \mid \exists\, h \in H \text{ with } h \to a_i\}$, we have the $(n+1)$-cycle $a_n a_1 \ldots a_{l-1} h a_l \ldots a_n$, a contradiction. If there are vertices $h \in H$ and $f \in F$ such that $h \to f$, then $a_n a_1 a_2 \ldots a_{n-2} h f a_n$ is an $(n+1)$-cycle containing the arc e, a contradiction. Summarizing our results, we observe that $(S \cup F \cup \{a_1, a_2, \ldots a_{n-1}\}) \leadsto H$. If $h \in H$, then, because of $|N| \geq 1$, we have $d^-(a_n) \leq |F| + n - 3$ and thus

$$d^-(h) \geq \begin{cases} |S| + d^-(a_n) \geq d^-(a_n) + 1, & \text{if} \quad |V(h)| = 3 \\ |S| + d^-(a_n) + 1 \geq d^-(a_n) + 2, & \text{if} \quad |V(h)| = 2 \end{cases},$$

in both cases a contradiction either to Remark 1.18 or to $i_g(D) \leq 1$.

Secondly, let $|B| = 1$ and thus $|H| = 1$, $|V(a_n)| = 3$, $|S| = 2$ and $n = c - 1$. To get no contradiction using (2.15), we have $(Q - S) \to a_n$. If $n = 6 \leq c - 2$, then it follows that $|S| = 2$ and $D[V(C)]$ is a tournament, a contradiction to $|B| = 1$. Hence let $n \geq 7$. If there are vertices $q \in Q$ and $h \in H$ such that $h \to q$, then $a_n a_1 h q v a_4 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. This yields $Q \to H$. If $B \neq \{a_2\}$, then it follows that $d^-(h) \geq |Q| + |S| + |\{a_1, a_2\}| = |Q| + 4 \geq d^-(v) + 1$, a contradiction to Remark 1.18, since $|V(h)| = 3$. Consequently, it remains to consider the case that $B = \{a_2\}$, which means that $V(h) = \{a_n, a_2, h\}$, if $h \in H$, and $a_1 \to a_3$. Analogously we see that $h \to \{a_3, a_4, \ldots, a_{n-1}\}$. But now $a_n a_1 a_3 \ldots a_{n-2} v h a_{n-1} a_n$ is an $(n+1)$-cycle containing the arc e, a contradiction.

Subcase 9.1.2.2. Assume that $H \cap V(a_n) = \emptyset$. It follows that

$$d^+(a_n) \geq |A| + |S| + 1 + |H| \geq |A| + |B| + |S| + |H| = d^+(a_1) - 1 + |S|,$$

and there remain to treat the same two Cases i) and ii) as in Subcase 9.1.1.

Firstly, let $F = \emptyset$. If $|V_c| = 3$, then we arrive at a contradiction following the same lines as in Subcase 9.1.1. Hence let $|V_c| = 2$. Similarly as in Subcase 9.1.1 we conclude that it is sufficient to treat the Case ii) with $|V(a_n)| = 1$, $|B| = 0$ and $|R_2| = 0$ and thus $N^-(v) = \{a_{n-2}, a_{n-1}, a_n\}$. The fact that $4 = d^-(v) + 1 \geq d^+(v) \geq |\{a_1, a_2, \dots, a_{n-3}, h\}| = n - 2$ yields $n \leq 6 \leq c - 2$ and thus $|S| \geq 2$, a contradiction to the Case ii).

Secondly, let $F \neq \emptyset$. If there is a vertex $f \in F$ such that $d^-_{D[F]}(f) \geq 3$, then there is a vertex $\tilde{f} \in F$ with $d^+_{D[F]}(\tilde{f}) \geq 2$ and (2.16) implies that

$$\begin{aligned} d^+(\tilde{f}) &\geq |N| + |\{a_1, a_n, a_{n-1}\}| + 2 + |S| - |V(\tilde{f}) - \{\tilde{f}\}| \\ &\geq \begin{cases} |N| + 4 \geq d^+(a_1) + 1, & \text{if} \quad |V(\tilde{f})| = 3 \\ |N| + 5 \geq d^+(a_1) + 2, & \text{if} \quad |V(\tilde{f})| = 2 \end{cases}, \end{aligned}$$

in both cases a contradiction either to Remark 1.18 or to $i_g(D) \leq 1$. Hence, let $d^-_{D[F]}(f) \leq 2$ for all $f \in F$.

Suppose that there is a vertex $a_1' \in V(a_1) - \{a_1\}$. If $a_1' \in V(C)$, then the fact that $n \leq c - 1$ leads to $|S| \geq 2$ and $|B| = 0$ or $|S| \geq 3$ and $|B| \geq 1$, in both cases a contradiction to the Cases i) and ii). If $a_n \to a_1'$, then as in i) and ii) we see that $d^+(a_n) \geq d^+(a_1) + 2$, a contradiction. Hence, let $a_1' \to a_n$ and thus $a_1' \in F$. It follows that $a_1' \to \{a_2, a_3, \dots, a_n\}$ and since $F \to S$, we observe that $d^+(a_1') \geq n - 1 + |S|$. If there is a vertex $x \in R_2 - V(a_n)$, then $x \notin (F \cup V(C) \cup H)$ and thus $a_n \to x \to a_1$ and we arrive at the contradiction $d^+(a_n) \geq d^+(a_1) + 2$. Consequently, let $R_2 \subseteq V(a_n) - \{a_n\}$, and $|V(a_n)| \leq 2$ implies that $|R_2| \leq 1$. Altogether, it follows that

$$6 \geq |H| + |R_2| + d^-_{D[F]}(a_1') + 1 \geq d^-(a_1') + 1 \geq d^+(a_1') \geq n - 1 + |S|,$$

which means that either $|S| = 1$ and $n \leq 6$ or $|S| = 2$ and $n \leq 5$, in both cases a contradiction.

Consequently, it remains to consider the case that $V(a_1) = \{a_1\}$ and thus, because of i) and ii), $|V_c| = 2$. Let $f \in F$ be an arbitrary vertex. If $|S| = 2$ (Case i)), then (2.16) implies that

$$d^+(f) \geq |N| + |\{a_1, a_n, a_{n-1}\}| + |S| - |V(f) - \{f\}| \geq |N| + 4 = d^+(a_1) + 2,$$

a contradiction to $i_g(D) \leq 1$. Hence, let $|S| = 1$ (Case ii)). To get no contradiction as in the case $|S| = 2$, we deduce that $|F| = 1$ and $d^+(f) = d^+(a_1) + 1$. This leads to

$$\begin{aligned} |V(D)| &\geq d^+(f) + d^-(f) + 2 = d^+(a_1) + d^-(f) + 3 \geq d^+(a_1) + d^-(a_1) + 2 \\ &= |V(D)| + 1, \end{aligned}$$

a contradiction.

Subcase 9.1.3. Assume that H contains vertices of at least two partite sets, which means that there exist two vertices $p, q \in H$ such that $p \to q$. If $q \to a_3$, then $a_n a_1 p q a_3 \dots a_n$ is an $(n+1)$-cycle containing the arc $a_n a_1$, a contradiction. Hence, let $a_3 \rightsquigarrow q$.

Subcase 9.1.3.1. Suppose that $n \geq 7$. If there are vertices $x \in Q$ and $h \in H$ such that $h \to x$, then $a_n a_1 h x v a_4 \dots a_n$ is an $(n+1)$-cycle, a contradiction. Consequently, let $Q \rightsquigarrow H$. Let $q \in H$ with $d^-_{D[H]}(q) \geq \max\{1, \left\lceil \frac{|H|-2}{2} \right\rceil\}$. It follows that

$$\begin{aligned} d^-(q) &\geq |Q| + |S| + d^-_{D[H]}(q) + |\{a_1, a_2, a_3\}| - |V(q) - \{q\}| \\ &\geq \left\{ \begin{array}{ll} |Q| + |S| + 1 + d^-_{D[H]}(q), & \text{if} \quad |V(q)| = 3 \\ |Q| + |S| + 2 + d^-_{D[H]}(q), & \text{if} \quad |V(q)| = 2 \end{array} \right. \end{aligned}$$

and $d^-(v) \leq |Q| + 3$. Summarizing these results, we arrive at

$$d^-(q) - d^-(v) \geq \left\{ \begin{array}{ll} |S| - 2 + d^-_{D[H]}(q), & \text{if} \quad |V(q)| = 3 \\ |S| - 1 + d^-_{D[H]}(q), & \text{if} \quad |V(q)| = 2 \end{array} \right. . \tag{2.17}$$

If $|H| \geq 5$, then (2.17) yields

$$d^-(q) - d^-(v) \geq \left\{ \begin{array}{ll} 1, & \text{if} \quad |V(q)| = 3 \\ 2, & \text{if} \quad |V(q)| = 2 \end{array} \right. ,$$

in both cases a contradiction either to Remark 1.18 or to $i_g(D) \leq 1$. Hence, let $|H| \leq 4$.

Firstly, let $|H| = 4$. If H consists of vertices of 3 or 4 partite sets, then there is a vertex $\tilde{q} \in H$ such that $d^-_{D[H]}(\tilde{q}) \geq 2$ and (2.17) yields a contradiction, if we replace q by $\tilde{q}$. If H consists of vertices of only two partite sets, then it follows that $D[H]$ is a 4-cycle $h_1 h_2 h_3 h_4 h_1$ without any chord since otherwise (2.17) leads to a contradiction. This implies that

$$d^-(h_1) \geq |Q| + |S| + 1 + |\{a_1, a_2, a_3\}| - 1 = |Q| + |S| + 3 \quad \text{and} \quad |V(h_1)| = 3$$

and $d^-(v) \leq |Q| + 3$. Combining these results we arrive at $d^-(h_1) - d^-(v) \geq |S| \geq 1$ and $|V(h_1)| = 3$, a contradiction to Remark 1.18.

Secondly, let $|H| = 3$. If H contains vertices of 3 partite sets, then, to get no contradiction with (2.17), we deduce that $D[H]$ is a 3-cycle $h_1 h_2 h_3 h_1$. If without loss of generality $h_1 \notin V(a_4)$, then we observe that $a_4 \to h_1$, since otherwise $a_n a_1 h_2 h_3 h_1 a_4 \dots a_n$ is an $(n+1)$-cycle, a contradiction. But together with (2.17), this leads to a contradiction to $i_g(D) \leq 1$ or to Remark 1.18. If H contains vertices of only 2 partite sets, then either there is a vertex $q \in H$ with $d^-_{D[H]}(q) \geq 2$ or there are two vertices $h_1, h_2 \in H$ such that $h_1 \in V(h_2)$ and $d^-_{D[H]}(h_1) \geq 1$. Using (2.17), we arrive at a contradiction in both cases.

Finally, let $|H| = 2$ with the vertices $p, q \in H$ such that $p \to q$. This implies

$$d^-(q) \geq \left\{ \begin{array}{ll} |Q| + |S| + 2, & \text{if} \quad |V(q)| = 3 \\ |Q| + |S| + 3, & \text{if} \quad |V(q)| = 2 \end{array} \right. ,$$

and thus

$$d^-(q) - d^-(v) \geq \left\{ \begin{array}{ll} |S| - 1, & \text{if} \quad |V(q)| = 3 \\ |S|, & \text{if} \quad |V(q)| = 2 \end{array} \right. .$$

This leads to $|S| = 1$, $n = c - 1$, $D[V(C)]$ is a tournament, $|B| = 0$ and $Q \cap S = \emptyset$. If $q \notin V(a_3)$, then it follows that $a_3 \to q$, and thus $a_4 \rightsquigarrow q$, and as above this yields a contradiction either to $i_g(D) \leq 1$ or to Remark 1.18. Hence, let $q \in V(a_3)$ and $q \to a_4$. If $p \notin V(a_2)$, then $a_n a_1 a_2 p q a_4 \dots a_n$ is an $(n+1)$-cycle containing the arc e, a contradiction. Consequently, we deduce that $p \in V(a_2)$ and $V(a_n) \cap H = \emptyset$. Analogously as in (2.15), reminding that $|B| = 0$, we arrive at

$$d^+(a_n) \geq |A| + |S| + 1 + |H| + |B| = d^+(a_1) + 1, \tag{2.18}$$

which implies that $d^+(a_n) = d^+(a_1) + 1$ and $|V(a_n)| \leq 2$. Since $F \to S$, it follows that $F \subseteq Q$ and thus $F \leadsto H$. If $f \in F$, then with (2.16), we conclude that

$$\begin{aligned} d^+(f) &\geq |N| + |\{a_1, a_n, a_{n-1}\}| + |S| + |H| - |V(f) - \{f\}| \\ &\geq \begin{cases} |N| + 2 + |H| = d^+(a_1) + 1, & \text{if} \quad |V(f)| = 3 \\ |N| + 3 + |H| = d^+(a_1) + 2, & \text{if} \quad |V(f)| = 2 \end{cases}, \end{aligned}$$

in both cases a contradiction. Consequently, it remains to consider the case that $F = \emptyset$. Since $|S| = 1$, this implies that $a_n \leadsto Q \leadsto a_1$ and, because of (2.18), we have $Q \subseteq V(a_n) - \{a_n\}$, which means that $|Q| \leq 1$, and thus $d^-(v) \leq 4$. Summarizing our results, we arrive at

$$5 \geq d^+(v) \geq |\{p, q, a_1, \dots, a_{n-3}\}| = n - 1 \;\Rightarrow\; n \leq 6,$$

a contradiction to the assumption of this subcase.

Subcase 9.1.3.2. Suppose that $n = 6 \leq c - 2$. In this case, we observe that $|S| \geq 2$. To get no contradiction to (2.15), it follows that $|S| = 2$, $|V(v)| = 1$, $|B| = 0$, $D[V(C)]$ is a tournament and $V(a_n) - \{a_n\} \subseteq H$. Since $F \to a_6 \leadsto H$, it follows that $H \cap F = \emptyset$. Since $|B| = 0$ and $a_6 \to a_{i-1}$, if $a_1 \to a_i$ with $2 \leq i \leq n-1$ we conclude that $|N^+(a_1) \cap V(C)| + |N^-(a_6) \cap V(C)| \leq l + 5 - l = 5$, if $|N^+(a_1) \cap V(C)| = l$, and thus

$$|R_2| \leq c + k - \left\{ \frac{c+k-3}{2} + \frac{c+k-3}{2} - 5 + |S| + n \right\} = 0.$$

Summarizing the results of the Cases 1-8, we observe that $\{a_4, a_5, a_6\} \to S \to \{a_1, a_2, a_3\}$. Without loss of generality let $S = \{v, w\}$ such that $v \to w$. Since $v \to (H \cup \{w, a_1, a_2, a_3\})$ and $a_1 \to (H \cup (N^+(a_1) \cap V(C)))$, the fact that $i_g(D) \leq 1$ implies that $|N^+(a_1) \cap V(C)| \geq 3$ and thus $|N^-(a_6) \cap V(C)| \leq 2$ and $a_1 \to a_3$ or $a_1 \to a_4$. If there are vertices $h \in H$ and $f \in F$ such that $h \to f$, then $a_6 a_1 a_3 a_4 v h f a_6$ or $a_6 a_1 a_4 a_5 v h f a_6$ is a 7-cycle, a contradiction. Hence, let $F \leadsto H$. Let $p, q \in H$ such that $p \to q$. Then we see that

$$d^-(q) \geq |F| + |S| + |\{p, a_1, a_2, a_3\}| - |V(q) - \{q\}| \geq |F| + |S| + 2 = |F| + 4,$$

whereas $d^-(a_6) \leq |F| + 2$. This implies that $d^-(q) - d^-(a_6) \geq 2$, a contradiction to $i_g(D) \leq 1$.

Subcase 9.2. Assume that $a_{n-3} \to S$. Since $S \to a_3$, we conclude that $n \geq 7$. Let $v \in S$. If there is a vertex $w \in H \cap F$, then $a_n a_1 a_2 \dots a_{n-2} v w a_n$ is an $(n+1)$-cycle containing the arc e, a contradiction. Hence, let $H \cap F = \emptyset$. If there are vertices $x \in H$ and $y \in F$ such that $x \to y$, then $a_n a_1 a_2 \ldots a_{n-3} v x y a_n$ is an $(n+1)$-cycle through e, a contradiction. Consequently, let $F \rightsquigarrow H$. If $f \in F$, then together with (2.16), we arrive at

$$\begin{aligned} d^+(f) &\geq |N| + |\{a_1, a_n, a_{n-1}\}| + |S| + |H| - |V(f) - \{f\}| \\ &\geq \begin{cases} |N| + |H| + 2 = d^+(a_1) + 1, & \text{if} \quad |V(f)| = 3 \\ |N| + |H| + 3 = d^+(a_1) + 2, & \text{if} \quad |V(f)| = 2 \end{cases}, \end{aligned}$$

in both cases a contradiction either to Remark 1.18 or to $i_g(D) \leq 1$. Consequently it remains to treat the case that $F = \emptyset$. If there is a vertex $x \in H$ such that $x \to a_{n-1}$, then $a_n a_1 a_2 \ldots a_{n-3} v x a_{n-1} a_n$ is an $(n+1)$-cycle, a contradiction. Hence, let $a_{n-1} \rightsquigarrow H$. Let $h \in H$. If $a_i \to a_n$ and $h \to a_{i+1}$ for some $i \in \{3, 4, \ldots, n-2\}$, then $a_n a_1 h a_{i+1} \ldots a_{n-1} v a_3 \ldots a_i a_n$ is an $(n+1)$-cycle containing the arc e, a contradiction. If $a_2 \to a_n$ and $h \to a_3$, then $a_n a_1 h a_3 \ldots a_{n-2} v a_2 a_n$ is an $(n+1)$-cycle, also a contradiction. Let $N^-(a_n) \cap V(C) = N^-(a_n) = \{a_{j_1}, a_{j_2}, \ldots, a_{j_l}\}$ and $\tilde{N} = \{a_{j_1+1}, a_{j_2+1}, \ldots, a_{j_l+1}\}$. Summarzing our results, we observe that $(S \cup \{a_1, a_2\} \cup \tilde{N}) \rightsquigarrow H$ and thus

$$\begin{aligned} d^-(h) &\geq |\tilde{N}| + |S| + 2 - |V(h) - \{h\}| \\ &\geq \begin{cases} |\tilde{N}| + |S| \geq d^-(a_n) + 1, & \text{if} \quad |V(h)| = 3 \\ |\tilde{N}| + |S| + 1 \geq d^-(a_n) + 2, & \text{if} \quad |V(h)| = 2 \end{cases}, \end{aligned}$$

in both cases a contradiction either to Remark 1.18 or to $i_g(D) \leq 1$. Hence, let $H = \emptyset$. This leads to a contradiction analogously as in Subcase 9.1.1.

This completes the proof of this theorem. □

Combining the Theorems 2.11 and 2.12 with the Theorems 2.4 and 2.6, we arrive at Theorem 2.5, the main result of this chapter.

Chapter 3

Cycles through exactly m partite sets

In contrast to Chapter 2, the length of the cycles are not important here, but the number of partite sets, which are contained in the cycle. In 1991, Goddard and Oellermann [8] proved the following generalization of Moon's [20] theorem that every strong tournament is vertex pancyclic.

Theorem 3.1 (Goddard, Oellermann [8]) *Every vertex of a strongly connected c-partite tournament D belongs to a cycle that contains vertices from exactly m partite sets for each* $m \in \{3, 4, \ldots, c\}$.

Inspired by this theorem, in 1998 Guo and Kwak [10] (see also Guo [9]) studied cycles containing a given arc and vertices from exactly $m \leq c$ partite sets in regular c-partite tournaments. In a first step they proved the following theorem.

Theorem 3.2 (Guo, Kwak [10]) *Let D be a regular c-partite tournament with* $c \geq 3$*. Then the following holds:*

i) Every arc of D is in a cycle, which contains vertices from exactly 3 *or exactly* 4 *partite sets.*

ii) If $c \leq 5$ *or the cardinality common to the partite sets of D is odd, then every arc of D is in a cycle, which contains vertices from exactly* 3 *partite sets.*

Using this theorem as basis of induction, they showed that the following three theorems are valid.

Theorem 3.3 (Guo, Kwak [10]) *Let D be a regular c-partite tournament with* $3 \leq c \leq 5$*. Then every arc of D is in a cycle that contains vertices from exactly m partite sets for all m with* $3 \leq m \leq c$.

Theorem 3.4 (Guo, Kwak [10]) *Let D be a regular c-partite tournament with* $c \geq 4$*. Then every arc of D is in a cycle that contains vertices from exactly m partite sets for all m with* $4 \leq m \leq c$.

Theorem 3.5 (Guo, Kwak [10]) *Let D be a regular c-partite tournament with $c \geq 3$. If the cardinality common to all partite sets of D is odd, then every arc of D is in a cycle that contains vertices from exactly m partite sets for all m with $3 \leq m \leq c$.*

Note that Theorem 3.5 implies Alspach's [1] theorem that every regular tournament is arc pancyclic, since every partite set of a tournament has the cardinality exactly 1.

The aim is now to carry these results of Guo and Kwak over to almost regular multipartite tournaments. In the first section, we will extend Theorem 3.2 by showing that every arc of an almost regular c-partite tournament is in a cycle containing vertices from exactly 3 or exactly 4 partite sets, if $c \geq 4$ or if $c \geq 3$ and there are at least two vertices in each partite set. Examples will show that there are multipartite tournaments with an arbitrary large number of partite sets that have arcs, which are not in cycles through exactly 3 partite sets. A further example will demonstrate that the condition $c \geq 4$ is important, if there is only one vertex in at least one partite set. Using these results as basis of induction, in the second section, we will implement the induction step, which leads to the main result of this chapter.

Theorem 3.6 (Volkmann, Winzen [38]) *Let D be an almost regular c-partite tournament with $c \geq 4$. If there are at least two vertices in each partite set, then every arc of D is in a cycle that contains vertices from exactly m partite sets for all m with $4 \leq m \leq c$.*

An example will show that the condition that there are at least two vertices in each partite set is necessary, at least for $c = 4$.

In the last section of this chapter, we will pose some open problems on multipartite tournaments containing vertices of a given number of partite sets.

3.1 The basis of induction

Let $V_1, V_2, \ldots, V_c$ be the partite sets of an almost regular c-partite tournament D such that $r = |V_1| \leq |V_2| \leq \ldots \leq |V_c|$. If ab is an arbitrary arc of D such that $a \in V_i$ and $b \in V_j$ with $1 \leq i, j \leq c$, then the following partition of $V(D)$ is useful in the proofs of the next theorems.

$$
\begin{aligned}
&A_1 = N^-(b) \cap V_i, \quad A_2 = N^+(b) \cap V_i, \\
&B_1 = N^+(a) \cap V_j, \quad B_2 = N^-(a) \cap V_j, \\
&X = N^-(a) \cap \left(\bigcup_{l=1}^{c} V_l - (V_i \cup V_j) \right), \\
&Y = N^+(a) \cap N^-(b) \cap \left(\bigcup_{l=1}^{c} V_l - (V_i \cup V_j) \right) \\
&Z = N^+(a) \cap N^+(b) \cap \left(\bigcup_{l=1}^{c} V_l - (V_i \cup V_j) \right).
\end{aligned}
$$

Note that some of the defined sets (clearly except A_1 and B_1) might be empty.

Suppose that $X = \emptyset$. Then it follows that $N^-(a) = B_2$ and hence $N^+(a) = V(D) - (B_2 \cup V_i)$. If we set $d^+(a) = d^-(a) + \Delta_a$ with $\Delta_a \in \{-1, 0, 1\}$ and $\sum_{k \neq i,j} |V_k| = (c-2)r + h$ with $0 \leq h \leq 2(c-2)$, then we observe that $\Delta_a = |V(D) - (B_2 \cup V_i)| - |B_2| = |V_j| + (c-2)r + h - 2|B_2|$. As $|B_2| = |V_j| - |B_1|$ we obtain

$$|V_j| + \Delta_a = 2|B_1| + (c-2)r + h. \tag{3.1}$$

Theorem 3.7 (Volkmann, Winzen [38]) *Let D be an almost regular multipartite tournament with the partite sets $V_1, V_2, \ldots, V_c$. If $c \geq 4$, then every arc of D is in a cycle containing vertices from exactly 3 or exactly 4 partite sets. If $|V_1| = |V_2| = \ldots = |V_c| = r$, then this result also holds for $c = 3$*

Proof. According to Lemma 1.10 we can distinguish the three cases that $1 \leq r = |V_1| \leq |V_2| \leq \ldots \leq |V_c| = r + m$ with $m = 0, 1, 2$. Thus, we see that $|V(D)| = cr + k$ with $k = 0$, if $m = 0$, $1 \leq k \leq c - 1$, if $m = 1$, and $2 \leq k \leq 2c - 2$, if $m = 2$. If $m = 0$ and $c = 3$, then, according to Remark 1.17, D is regular, and Theorem 3.2 of Guo and Kwak yields the desired result. So, if $m = 0$, we can investigate the case that $c \geq 4$.

Let ab be an arbitrary arc of D such that $a \in V_i$ and $b \in V_j$ with $1 \leq i, j \leq c$, and let $A_1, A_2, B_1, B_2, X, Y, Z, \Delta_a$ and h be defined as in the beginning of this section.

Suppose that ab is not in a cycle, which contains vertices from exactly 3 partite sets. In particular, ab is not in a 3-cycle. Under this assumption, we firstly study the domination relationships among the partition sets of $V(D)$ listed above.

Firstly, we observe that

$$X \to b, \text{ i.e., } N^-(a) \cap N^+(b) \cap \left(\bigcup_{l=1}^{c} V_l - (V_i \cup V_j) \right) = \emptyset,$$

since otherwise, if there is a vertex $x \in X$ such that $b \to x$, then $abxa$ is a 3-cycle, a contradiction.

Now, we suppose that $X = \emptyset$. Since $c \geq 4$, (3.1) yields that $r + 3 \geq |V_j| + \Delta_a \geq 2 + 2r$, from which we obtain $r = 1$, $|B_1| = 1$, $h = 0$, $\Delta_a = 1$ and $|V_j| = 3$. By Remark 1.18, the fact that $\Delta_a \neq 0$ implies that $|V(a)| = |V_i| = r + 1 = 2$. Furthermore, we observe that $d^-(a) = |B_2| = |V_j| - |B_1| = 2$. Since $h = 0$, it remains to consider the partition-sequence $1, 1, 2, 3$. If $Z = \emptyset$, then we conclude that $|Y| = |V(D) - (V_i \cup V_j)| = 2$, and thus, it follows that $d^-(b) \geq 3$, because of Remark 1.18 and $|V_j| = 3 = r + 2$ a contradiction. Hence, we observe that there is a vertex $z \in Z$ and $|V(z)| = 1$. Remark 1.18 yields that $d^+(z) = d^-(z) = 3$. Since $\{a, b\} \to z$, there is a vertex $b_2 \in B_2$ such that $z \to b_2$ and $abzb_2a$ is a cycle with vertices from exactly 3 partite sets, a contradiction.

These considerations lead to $X \neq \emptyset$. Analogously, we see that the case $Z = \emptyset$ is impossible.

If there is an arc $a_2 \to x$ (respectively, $z \to b_2$) from A_2 to X (respectively, Z to B_2), then aba_2xa (respectively, $abzb_2a$) is a cycle containing vertices from exactly 3 partite sets, a contradiction. Hence,

$$X \to A_2 \quad \text{and} \quad B_2 \to Z. \tag{3.2}$$

If there is an arc $z \to a_2$ (respectively, $b_2 \to x$) from Z to A_2 (respectively, B_2 to X), then we also have $B_2 \to a_2$ (respectively, $b_2 \to A_2$), because otherwise, if there is a vertex $b_2 \in B_2$ (respectively, $a_2 \in A_2$) such that $a_2 \to b_2$, then $abza_2b_2a$ (respectively, aba_2b_2xa) is a cycle through exactly 3 partite sets, a contradiction. But this yields

$$d^-(a_2) \geq |X| + |B_2| + |\{b, z\}| = d^-(a) + 2$$

$$(\text{respectively,} \quad d^+(b_2) \geq |Z| + |A_2| + |\{a, x\}| = d^+(b) + 2),$$

a contradiction to $i_g(D) \leq 1$. Hence,

$$A_2 \to Z \quad \text{and} \quad X \to B_2. \tag{3.3}$$

Suppose now that the arc ab also does not belong to any cycle with vertices of exactly 4 partite sets. As a first consequence we observe that $X \rightsquigarrow Z$, since otherwise, if there are vertices $z \in Z$ and $x \in X$ such that $z \to x$, then $abzxa$ is a cycle with vertices from exactly 4 partite sets, a contradiction.

Assume that there exist vertices $b_1 \in B_1 - \{b\}$ and $x \in X$ such that $b_1 \to x$. If there is a vertex $a_2 \in A_2$ such that $a_2 \to b_1$, then aba_2b_1xa is a cycle through exactly 3 partite sets, a contradiction. If there is a vertex $z \in Z$ such that $z \to b_1$, then $abzb_1xa$ is a cycle containing vertices from exactly 3 or exactly 4 partite sets, a contradiction. Altogether, we see that $b_1 \to Z \cup A_2 \cup \{x\}$, which implies $d^+(b_1) \geq d^+(b) + 1$. Because of $i_g(D) \leq 1$, we conclude that $d^+(b_1) = d^+(b) + 1$ and $A_1 - \{a\} \to b_1$. If there are vertices $z \in Z$ and $a_1 \in A_1 - \{a\}$ such that $z \to a_1$, then $abza_1b_1xa$ is a cycle with vertices from exactly 3 or exactly 4 partite sets, a contradiction. Together with (3.2) and (3.3), for every vertex $z \in Z$, this yields

$$\begin{aligned} d^-(z) &\geq |X| + |V_i| + |B_2| + |\{b_1, b\}| - |V(z) - \{z\}| \\ &\geq \begin{cases} d^-(a) + 2, & \text{if} \quad |V(z)| \leq r + 1 \\ d^-(a) + 1, & \text{if} \quad |V(z)| = r + 2 \end{cases}, \end{aligned}$$

in both cases a contradiction either to $i_g(D) \leq 1$ or to Remark 1.18. Hence, we see that $X \to B_1$.

Now, assume that there are vertices $a_1 \in A_1 - \{a\}$ and $z \in Z$ such that $z \to a_1$. If there is a vertex $x \in X$ such that $a_1 \to x$, then $abza_1xa$ is a cycle containing vertices from exactly 3 or exactly 4 partite sets, a contradiction. Together with (3.2) and (3.3), for every vertex $x \in X$, this yields

$$\begin{aligned} d^+(x) &\geq |V_j| + |A_2| + |Z| + |\{a, a_1\}| - |V(x) - \{x\}| \\ &\geq \begin{cases} d^+(b) + 2, & \text{if} \quad |V(x)| \leq r + 1 \\ d^+(b) + 1, & \text{if} \quad |V(x)| = r + 2 \end{cases}, \end{aligned}$$

in both cases a contradiction either to $i_g(D) \leq 1$ or to Remark 1.18. Summarizing our results, we see that

$$X \rightsquigarrow Z \cup V_j \cup A_2 \cup \{a\} \quad \text{and} \quad V_i \cup X \cup B_2 \cup \{b\} \rightsquigarrow Z. \tag{3.4}$$

This leads to the following lower bounds for all $x \in X$ (respecively, all $z \in Z$)

$$\begin{aligned} d^+(x) &\geq |V_j| + |Z| + |A_2| + |\{a\}| - |V(x) - \{x\}| \\ &\geq \begin{cases} d^+(b) + 2, & \text{if } |V(x)| = r \\ d^+(b) + 1, & \text{if } |V(x)| = r + 1 \\ d^+(b) \quad , & \text{if } |V(x)| = r + 2 \end{cases}, \\ d^-(z) &\geq |V_i| + |B_2| + |X| + |\{b\}| - |V(z) - \{z\}| \\ &\geq \begin{cases} d^-(a) + 2, & \text{if } |V(z)| = r \\ d^-(a) + 1, & \text{if } |V(z)| = r + 1 \\ d^-(a) \quad , & \text{if } |V(z)| = r + 2 \end{cases}. \end{aligned}$$

To get no contradiction, it has to be $|V(x)|, |V(z)| \geq r + 1$ for all $x \in X$ and $z \in Z$. Furthermore, we conclude that the lower bounds of $d^+(x)$ and $d^-(z)$ must not increase by one, which means $|V_i| = |V_j| = r$, $V(x) - \{x\} \subseteq Z$ for all $x \in X$ and $V(z) - \{z\} \subseteq X$ for all $z \in Z$. If $r \geq 2$, then, because of $|V(x)| \geq r + 1$ and $V(x) - \{x\} \subseteq Z$ for all $x \in X$, there are at least two vertices $z_1, z_2 \in Z$ with $V(z_1) = V(z_2)$, a contradiction to $V(z) - \{z\} \subseteq X$ for all $z \in Z$. Hence, we examine the case that $r = 1$. This implies $V_i = \{a\}$, $V_j = \{b\}$ and $B_2 = A_2 = B_1 - \{b\} = A_1 - \{a\} = \emptyset$. Furthermore, we conclude that $d^+(b) = |Z|$ and $d^+(a) = |Z| + |Y| + |\{b\}|$, which yields $|Y| = 0$, $d^+(a) = d^+(b) + 1$ and, since $|V_j| = |V_i| = r$, Remark 1.18 yields $|V_c| = r + 1$. Because of $V(D) - (V_i \cup V_j) \subseteq X \cup Z$ and $c \geq 4$, there are at least two partite sets V_{x_1} and V_{x_2} in $V(D) - (V_i \cup V_j)$ such that $V_{x_1} = \{x_1, z_1\}$ and $V_{x_2} = \{x_2, z_2\}$. Furthermore, the fact that $V(x) - \{x\} \subseteq Z$ for all $x \in X$ and $V(z) - \{z\} \subseteq X$ for all $z \in Z$ implies that one vertex of V_{x_1} (respectively, V_{x_2}) is in X and the other one in Z. So, without loss of generality, let $x_1, x_2 \in X$ and $z_1, z_2 \in Z$ and $x_1 \to x_2$. But now we observe that

$$d^+(x_1) \geq |V_j| + |Z| - |V(x_1) - \{x_1\}| + |A_2| + |\{a, x_2\}| = d^+(b) + 2,$$

a contradiction to $i_g(D) \leq 1$. This completes the proof of the theorem. □

The following example shows that the supplement that every arc is in a cycle, which consists of vertices of exactly three *or four* partite sets is essential, since not every arc of an almost regular multipartite tournament is in a cycle containing vertices from exactly three partite sets.

Example 3.8 (Volkmann, Winzen [38]) *Let $V_1 = \{a, x_2, x_3\}$ and $V_2 = \{b, y_2, y_3\}$ be the two partite sets of a digraph D such that $a \to b \to x_2 \to y_2 \to x_3 \to y_3 \to a$, $b \to x_3$, $y_2 \to a$ and $y_3 \to x_2$. Furthermore, let D' and D'' be copies of D such that $D \to D' \to D'' \to D$. The resulting 6-partite tournament H (see also Figure 3.1) is almost regular, but the arc ab is not in any cycle containing vertices from exactly three partite sets.*

Let G, G', G'' be three copies of H such that $G \to G' \to G'' \to G$. The resulting 18-partite tournament is almost regular, but no copy of the arc ab is in a cycle containing vertices from exactly three partite sets.

If we continue this process, we arrive at almost regular c-partite tournaments with arbitrary large c, which contain arcs that do not belong to any cycle through exactly three partite sets.

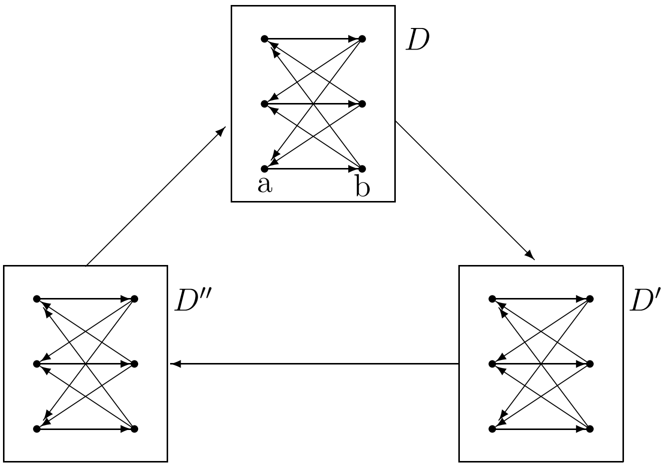

Figure 3.1: An almost regular 6-partite tournament with the property that the arc ab is in no cycle through exactly 3 partite sets

In the case that the maximal difference of the cardinality of the partite sets is exactly 2, Theorem 3.7 also holds, if the multipartite tournament consists of only three partite sets.

Theorem 3.9 (Volkmann, Winzen [38]) *Let D be an almost regular 3-partite tournament with the partite sets V_1, V_2, V_3 such that $1 \leq r = |V_1| \leq |V_2| \leq |V_3| = r+2$. Then every arc of D is in a cycle containing vertices of all partite sets.*

Proof. Let ab be an arbitrary arc of D. Suppose that ab is not in any cycle, containing vertices of all partite sets. Obviously, we have $|V(D)| = 3r + k$ with $2 \leq k \leq 4$. Let $a \in V_i$ and $b \in V_j$ with $1 \leq i, j \leq 3$. If we define $A_1, A_2, B_1, B_2, X, Y, Z, h$ and Δ_a as in the beginning of this section, then, following the same lines as in Theorem 3.7, we observe that

$$X \to A_2 \cup B_2 \cup \{a, b\} \to Z. \tag{3.5}$$

Suppose that $X = \emptyset$. Let $V_l = V(D) - (V_i \cup V_j)$. Since, $c = 3$, from (3.1) we get $|V_j| + \Delta_a = 2|B_1| + r + h$. This equality implies $B_1 = \{b\}$, $B_2 = V_j - \{b\}$ and $0 \leq h \leq 1$. If $h = 1$, then it follows that $\Delta_a = 1$, $|V_j| = r + 2$ and $|V_l| = r + 1$. By Remark 1.18 we have $|V(a)| = |V_i| = r + 1$. This is a contradiction since there is no partite set with r vertices. Hence, let $h = 0$ and thus $|V_l| = r$ and $0 \leq \Delta_a \leq 1$. First, we assume that $\Delta_a = 0$ and thus, according to (3.1), $|V_j| = r + 2$. If there is a vertex $z \in Z$, then (3.5) implies that $d^-(z) \geq |V_j| + 1 = r + 3$ and $d^+(z) \leq |V_i| - 1 \leq r + 1$, a contradiction. Consequently, we can consider the case that $Y = V_l$. If $|V_i| = r$, then we arrive at the contradiction $r + 1 = |Y| + 1 \leq d^-(b) \leq d^+(b) + 1 \leq |V_i| = r$. Since the partition-sequence $r, r+1, r+2$ is impossible, it remains to treat the case that $|V_i| = r + 2$. To get no contradiction to $i_g(D) \leq 1$, it follows that $A_2 = V_i - \{a\}$. If there are vertices $a_2 \in A_2$ and $y \in Y$ such that $a_2 \to y$, then we conclude that $B_2 \to y$, since otherwise, if there is a vertex $b_2 \in B_2$ such that $y \to b_2$, then aba_2yb_2a is a cycle through all 3 partite sets. But now we arrive at the contradiction $d^-(y) \geq r + 3$ and $d^+(y) \leq r + 1$. Hence, let $Y \to A_2$, which implies that $A_2 \to B_2 \to Y$. If $a_2, a_2' \in A_2$, $b_2, b_2' \in B_2$ and $y \in Y$, then $aba_2b_2ya_2'b_2'a$ is a cycle through all partite sets, a contradiction. Second, let $\Delta_a = 1$. Since $|V_c| = r + 2$, Remark 1.18 yields $|V_i| = r + 1$, and thus $|V_j| = r + 2$, a contradiction to $i_g(D) \leq 1$.

Analogously, we see that the case $Z = \emptyset$ is impossible. Consequently, it remains to consider the case that $X, Z \neq \emptyset$. Now, analogously to Theorem 3.7, we get the relationships (3.4) and the conditions $|V_i| = |V_j| = r = 1$ and $|V_c| = r + 1$, a contradiction. $\square$

Nevertheless Theorem 3.7 cannot be improved in the sense that in all almost regular c-partite tournaments with $c \geq 3$, every arc is in a cycle containing vertices from exactly three or exactly four partite sets. This can be seen in the following simple example, which shows a 3-partite tournament with an arc ab that is not contained in any cycle through all partite sets.

Example 3.10 (Volkmann, Winzen [38]) *Let $V_1 = \{a, x_2\}$, $V_2 = \{b, y_2\}$ and $V_3 = \{z\}$ be the three partite sets of the multipartite tournament D such that $a \to b \to x_2 \to y_2 \to z \to x_2$ and $y_2 \to a \to z \to b$ (see Figure 3.2). Then the arc ab is not contained in any cycle with vertices of exactly three (and clearly also not four) partite sets.*

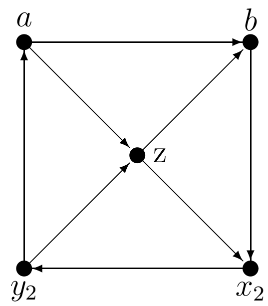

Figure 3.2: An almost regular 3-partite tournament with the property that the arc ab is in no cycle through exactly 3 partite sets

In the last example, there is one partite set containing only one vertex. If we add the condition that there are at least two vertices in every partite set, then we can improve Theorem 3.7.

Theorem 3.11 (Volkmann, Winzen [38]) *Let D be an almost regular multipartite tournament with the partite sets $V_1, V_2, \ldots, V_c$. If $c \geq 3$ and there are at least two vertices in each partite set, then every arc of D is in a cycle containing vertices from exactly 3 or exactly 4 partite sets.*

Proof. If $c \geq 4$ or $|V_1| = |V_2| = |V_3|$, then the assertion holds with Theorem 3.7. If $r = |V_1| \leq |V_2| \leq |V_3| = r+2$, then the assertion follows from Theorem 3.9. Therefore, it remains to consider the case that $c = 3$ and $2 \leq r = |V_1| \leq |V_2| \leq |V_3| = r+1$.

Let ab be an arbitrary arc of D. Suppose that ab is not in any cycle, containing vertices of all partite sets. Obviously, we have $|V(D)| = 3r + k$ with $1 \leq k \leq 2$. Let $a \in V_i$ and $b \in V_j$ with $1 \leq i, j \leq 3$. If we define $A_1, A_2, B_1, B_2, X, Y, Z, h$ and Δ_a as in the beginning of this section, then, following the same lines as in Theorem 3.7, we observe that

$$X \to A_2 \cup B_2 \cup \{a, b\} \to Z. \tag{3.6}$$

Suppose that $X = \emptyset$. Let $V_l = V(D) - (V_i \cup V_j)$. With $c = 3$ and the fact that $|V_j| \leq r+1$, (3.1) implies $B_1 = \{b\}$, $h = 0$, $\Delta_a = 1$, $|V_l| = r$, $|V_j| = r+1$ and $|B_2| = r$. If there is a vertex $z \in Z$, then (3.6) yields that $d^-(z) \geq |V_j| + 1 = r+2$ and $d^+(z) \leq |V_i| - 1 \leq r$, a contradiction. Hence, let $Y = V_l$. If $|V_i| = r$, then we arrive at the contradiction $r+1 = |V_l| + 1 \leq d^-(b) \leq d^+(b) + 1 \leq |A_2| + 1 \leq r$. Hence, let us suppose that $|V_i| = r+1$. To get no contradiction to $i_g(D) \leq 1$, it follows that $|A_2| = r$. If there are vertices $a_2 \in A_2$ and $y \in Y$ such that $a_2 \to y$, then we deduce that $B_2 \to y$, since otherwise, if there is a vertex $b_2 \in B_2$ such that $y \to b_2$, then aba_2yb_2a is a cycle with vertices from all partite sets, a contradiction. But this yields the contradiction $d^-(y) \geq r+2$ and $d^+(y) \leq r$. Consequently, it follows that $Y \to A_2$, and thus $A_2 \to B_2 \to Y$. If $a_2, a_2' \in A_2$, $b_2, b_2' \in B_2$ and $y \in Y$, then $aba_2b_2ya_2'b_2'a$ is a cycle through all 3 partite sets, a contradiction.

Analogously, we observe that the case $Z = \emptyset$ is impossible. Consequently, it remains to treat the case that $X, Z \neq \emptyset$. Now, analogously to Theorem 3.7, we get the relationships (3.4) and the condition $|V_i| = |V_j| = r = 1$, a contradiction to $r \geq 2$. This completes the proof of the theorem. $\square$

3.2 The induction-step

We take Theorem 3.11 as basis of induction to show Theorem 3.6. Next, we will present the induction-step.

Theorem 3.12 (Volkmann, Winzen [38]) *Let D be an almost regular c-partite tournament with $c \geq 4$ and at least two vertices in each partite set. If an arc of D is in a cycle that contains vertices from exactly m partite sets for some m with $3 \leq m < c$, then it is also in a cycle that contains vertices from exactly $m+1$ partite sets.*

Proof. Let $V_1, V_2, \ldots, V_c$ be the partite sets of D such that $2 \leq r = |V_1| \leq |V_2| \leq \ldots \leq |V_c| = r + o$ with $o = 0$, $o = 1$ or $o = 2$. Obviously, we have $|V(D)| = cr+k$ with $k = 0$, if $o = 0$, $1 \leq k \leq c-1$, if $o = 1$, and $2 \leq k \leq 2c-2$, if $o = 2$. Let v_1v_2 be an arc that is in a cycle, say $C = v_1v_2 \ldots v_tv_1$, which contains vertices from exactly m partite sets for some $3 \leq m < c$. Suppose that v_1v_2 is not part of a cycle containing vertices from exactly $m+1$ partite sets. Assume without loss of generality that $v_1 \in V_i$ and $v_2 \in V_j$ for some $1 \leq i, j \leq c$. If $I = \{i_{m+1}, \ldots, i_c\}$ is the maximal set of indices such that $V(C) \cap V_l = \emptyset$ for all $l \in I$, then we define the sets X and Y by

$$X = N^-(v_1) \cap \left(\bigcup_{l \in I} V_l \right), \quad Y = N^+(v_1) \cap \left(\bigcup_{l \in I} V_l \right).$$

It is clear that $X \cup Y = \bigcup_{l \in I} V_l$ and every vertex of $X \cup Y$ is adjacent with all vertices in C.

Firstly, let us suppose that $X \neq \emptyset$. If there is a vertex $x \in X$ such that $v_t \to x$, then $v_1 v_2 \ldots v_t x v_1$ is a cycle through exactly $m+1$ partite sets, a contradiction. If such a vertex does not exist, then $X \to v_t$. Since $X \to \{v_1, v_t\}$, we observe that, if some $v_i \in V(C)$ dominates a vertex $x \in X$, then let $n = \max\{l \mid v_l \to x\}$ and $v_1 v_2 \ldots v_n x v_{n+1} \ldots v_t v_1$ is a cycle through exactly $m+1$ partite sets. Now, we assume that $X \to V(C)$.

Now, let $H = N^+(v_2) - V(C)$. If there is an arc $h \to x$ with $h \in H$ and $x \in X$, then let firstly be $h \in V_l$ with $l \notin I$. In this case $v_1 v_2 h x v_3 \ldots v_t v_1$ is a cycle through exactly $m+1$ partite sets, a contradiction. Consequently, let $h \in V_l$ with $l \in I$. If $m = 3$, then $v_1 v_2 h x v_1$ is a cycle through exactly 4 partite sets, a contradiction. Otherwise, if $m \geq 4$, then let p be the index such that $\{v_p, v_{p+1}, \ldots, v_t, v_1\} - V(v_2)$ consists of vertices from exactly $m-2$ partite sets. In this case, $v_1 v_2 h x v_p \ldots v_t v_1$ is a cycle containing vertices of exactly $m+1$ partite sets, a contradiction. For all $x \in X$, this leads to

$$d^+(x) \geq |H - (V(x) - \{x\})| + |V(C)|$$

whereas

$$d^+(v_2) \leq |H| + |V(C))| - 2.$$

If $H \cap V(x) = \emptyset$, then we arrive at a contradiction to $i_g(D) \leq 1$. Hence, let $y \in H \cap V(x)$. Since $H \cap X = \emptyset$, we conclude that $y \in Y$. Now let $z \in N^-(x)$ and assume that $y \to z$. If $z \in V_l$ with $l \notin I$, then $v_1 v_2 y z x v_3 \ldots v_t v_1$ is a cycle through exactly $m+1$ partite sets, a contradiction. Thus, let $z \in V_l$ with $l \in I$. If $m = 3$, then $v_1 v_2 y z x v_1$ is a cycle through exactly 4 partite sets, and if $m \geq 4$, then we choose the index p as above and $v_1 v_2 y z x v_p \ldots v_t v_1$ is a cycle through exactly $m+1$ partite sets, in both cases a contradiction. Hence, let $N^-(x) \to y$. If $y \to v_i$ for some $3 \leq i \leq t$, then let $n = \min\{q \mid 2 \leq q \leq i-1, v_q \to y\}$. Now, $v_1 v_2 \ldots v_n y v_{n+1} \ldots v_t v_1$ is a cycle through exactly $m+1$ partite sets, a contradiction. Altogether, we see that $\{v_1, v_2, \ldots, v_t\} \cup N^-(x) \to y$, and thus it follows that

$$d^-(y) \geq d^-(x) + t \geq d^-(x) + 3,$$

a contradiction to $i_g(D) \leq 1$.

Consequently, there remains to consider the case that $X = \emptyset$. This implies that $v_1 \to Y$ and $Y = \bigcup_{l \in I} V_l$. Now, we distinguish different cases.

Case 1. Let there be a vertex $y \in Y$ such that $v_2 \to y$. Then we have $V(C) \to y$, since otherwise, if we choose let $n = \min\{z \mid y \to v_z\}$, then $v_1 v_2 \ldots v_{n-1} y v_n \ldots v_t v_1$ is a cycle through exactly $m+1$ partite sets, a contradiction. If $v_1 \rightsquigarrow N^+(y)$, then it follows that $d^-(y) = |V(C)| + |N^-(y) - V(C)|$

and $d^-(v_1) \le |V(C)| - 2 + |N^-(y) - V(C)|$, a contradiction to $i_g(D) \le 1$. Therefore, there is a 3-cycle v_1yzv_1. Obviously, the case $z \in Y \cup V(C)$ is impossible, and thus $v_1v_2 \dots v_tyzv_1$ is a cycle through exactly $m+1$ partite sets, a contradiction.

Altogether we see that there remains the case $Y \to v_2$.

Case 2. Suppose that there exists a vertex $y \in Y$ such that $v_3 \to y$. As in Case 1 we observe that in this case $V(C) - \{v_2\} \to y$. In the following we will denote the sets F and H by $F = N^-(y) - V(C)$ and $H = N^+(y) - V(C)$, respectively. If there is a 3-cycle v_1yzv_1, then, analogously as in Case 1, we arrive at a contradiction. Hence, let $v_1 \leadsto N^+(y)$. It follows that $d^-(y) = |V(C)|-1+|F|$ and $d^-(v_1) \le |V(C)|-2+|F|$. Because of $i_g(D) \le 1$, this leads to $N^-(v_1) = (V(C)-\{v_1,v_2\}) \cup F$, $d^-(y) = d^-(v_1)+1$, $V(v_1)-\{v_1\} \subseteq N^+(y)$ and $Y - V(y) \subseteq N^+(y)$. Since $r \ge 2$, we conclude that $V(v_1) - \{v_1\} \ne \emptyset$. Let $H' = H - Y$. Then we have $\{v_4, v_5, \dots, v_t\} \leadsto H'$, because otherwise, if there are vertices $h' \in H'$ and v_l such that $h' \to v_l$ for some $4 \le l \le t$, then $v_1v_2 \dots v_{l-1}yh'v_l \dots v_tv_1$ is a cycle containing vertices from exactly $m+1$ partite sets, a contradiction. Furthermore, if there are vertices $f \in F$ and $h' \in H'$ such that $h' \to f$, then $v_1v_2 \dots v_tyh'fv_1$ is a cycle through exactly $m+1$ partite sets, a contradiction. Summarizing our results, we see that $(F \cup \{y, v_1, v_4, v_5, \dots, v_t\}) \leadsto H'$.

Subcase 2.1. Assume that there are vertices $h' \in H'$ and $y' \in V(y) - \{y\}$ such that $h' \to y'$. It follows that $F \to y'$, since otherwise, if there is a vertex $f \in F$ such that $y' \to f$, then $v_1v_2 \dots v_tyh'y'fv_1$ is a cycle through exactly $m+1$ partite sets, a contradiction. If there exists a vertex $v_l \in V(C)$ with $4 \le l \le t$ such that $y' \to v_l$, then $v_1v_2 \dots v_{l-1}yh'y'v_l \dots v_1$ is a cycle containing vertices from exactly $m+1$ partite sets, a contradiction. Hence, let $(\{v_1, v_4, \dots, v_t, h'\} \cup F) \to y'$. We arrive at

$$d^-(y') \ge |F| + |V(C)| - 1 = d^-(y) = d^-(v_1) + 1.$$

To get no contradiction to $i_g(D) \le 1$, it follows that $y' \to (H - \{h'\}) \cup \{v_3\}$. If there is a vertex v_l ($4 \le l \le t$) such that $v_2 \to v_l$, then we observe that $v_1v_2v_l \dots v_tyh'y'v_3 \dots v_{l-1}v_1$ is a cycle through exactly $m+1$ partite sets, a contradiction. If there is a vertex $f \in F$ such that $v_2 \to f$, then $v_1v_2fyh'y'v_3 \dots v_tv_1$ is a cycle containing vertices from exactly $m+1$ partite sets, a contradiction. If $v_2 \to h'$, then $v_1v_2h'y'v_3 \dots v_tv_1$ is a cycle through exactly $m+1$ sets, also a contradiction. Hence, we have $(F \cup \{h', v_1, v_4, \dots, v_t\} \cup Y) \leadsto v_2$, and thus

$$d^+(v_2) \le |H| - 1 - |Y - V(y)| - |V(v_2) \cap H| + |\{v_3\}| \le |H|,$$

whereas $d^+(y) = |H| + 1$. This implies that $v_2 \to H - \{h'\}$ and $H'' := H' - \{h'\} = H - \{h'\}$. If there exist vertices $h'' \in H''$ and $y'' \in Y - \{y\}$ such that $h'' \to y''$, then analogously as above, we observe that $h'' \to v_2$, a contradiction. Hence, let $Y = V(y) \to H''$. According to Corollary 1.19, we have $d^+(y) \ge 3$, and thus $|H| \ge 2$, which means that $H'' \ne \emptyset$. Consequently, there is a vertex $h'' \in H''$ such that $d^+_{D[H'']}(h'') \le \frac{|H|-2}{2}$. Summarizing our results, we arrive at

$$|H| \le d^+(h'') \le \frac{|H|-2}{2} + 2.$$

Since $|H| \geq 2$, this yields $|H| = 2$ and $h'' \to h'$. Now, $v_1v_2h''h'y'v_3 \dots v_tv_1$ is a cycle through all $m+1$ partite sets, a contradiction.

Subcase 2.2. Suppose that $V(y) \to H'$. Since $V(v_1) - \{v_1\} \subseteq H'$, the observations above yield that $(\{v_4, v_5, \dots, v_t\} \cup F) \to (V(v_1) - \{v_1\})(\subseteq H')$. This implies that

$$\begin{aligned} d^-(v_1') &\geq |F| + |V(C)| - 3 + |V(y)| \geq |F| + |V(C)| - 1 \\ &= d^-(v_1) + 1 \end{aligned}$$

for all vertices $v_1' \in V(v_1) - \{v_1\}$. To get no contradiction to $i_g(D) \leq 1$, it follows that $|V(y)| = 2$ and $(V(v_1) - \{v_1\}) \to \{v_2, v_3\} \cup (H - V(v_1))$. Analogously as in Subcase 2.1, replacing the path $yh'y'v_3$ by $yv_1'v_3$, we see that $(F \cup \{v_4, v_5, \dots, v_t\}) \rightsquigarrow v_2$. Hence, we arrive at

$$d^+(v_2) \leq |H| - |Y - V(y)| - |V(v_2) \cap H| - |V(v_1) \cap H| + 1 \leq |H| - r + 2 \leq |H|,$$

whereas $d^+(y) = |H| + 1$. This implies that $v_2 \to H - V(v_1) =: H''$, $|H \cap V(v_1)| = 1$ and $Y - V(y) = \emptyset$, which means $H' = H$. Following the same lines as in Subcase 2.1, replacing there h' by v_1', we arrive at $H'' = \{h''\}$ such that $h'' \to v_1'$, a contradiction to $(V(v_1) - \{v_1\}) \to (H - V(v_1))$.

Summarizing the investigations of Case 2, we see that $Y \to v_3$. Observing the converse D^{-1} of D, we conclude that $v_t \to Y$ and therefore $t \geq 4$.

Case 3. Finally, let $\{v_t, v_1\} \to Y \to \{v_2, v_3\}$. Let us define the sets U and W by $W = N^+(v_2) - V(C)$ and $U = N^-(v_1) - V(C)$, respectively. It is not difficult to show that, if there is an arc leading from W to Y (respectively, from Y to U), or if $Y \to W$ (respectively, $U \to Y$) and there is an arc from W to v_1 (respectively, from v_2 to U), then the multipartite tournament contains a cycle through v_1v_2 and exactly $m+1$ partite sets, a contradiction. Hence, we may assume that $Y \cup \{v_1, v_2\} \rightsquigarrow W$ and $U \rightsquigarrow Y \cup \{v_1, v_2\}$ and $U \cap W = \emptyset$.

If there exists a vertex $v_l \in V(C)$ such that $v_2 \to v_l$ and $v_{l-1} \to v_1$, then obviously $l \geq 4$ and $v_1v_2v_l \dots v_tyv_3 \dots v_{l-1}v_1$ is a cycle through exactly $m+1$ partite sets for some $y \in Y$, a contradiction. Therefore, from now on, we investigate the case that $v_1 \to v_{l-1}$ or $V(v_1) = V(v_{l-1})$, if $v_2 \to v_l$.

If there are vertices $u \in U$ and $v_l \in V(C)$ with $l \geq 4$ such that $v_2 \to v_l$ and $v_{l-1} \to u$, then $v_1v_2v_l \dots v_tyv_3 \dots v_{l-1}uv_1$ is a cycle through exactly $m+1$ partite sets, a contradiction. Hence, we may assume that $u \to v_{l-1}$ or $V(u) = V(v_{l-1})$, if $v_2 \to v_l$. Analogously, we see that $v_{l+1} \to w$ or $V(w) = V(v_{l+1})$, if $w \in W$ and $v_l \to v_1$ with $l < t$.

If there is an arc $w \to u$ from W to U, then $v_1v_2wuyv_3 \dots v_tv_1$ is a cycle containing vertices from exactly $m+1$ partite sets, a contradiction. Therefore, we have $U \rightsquigarrow W$.

If $y \in Y$ is an arbitrary vertex, then these results yield the following three lower bounds

$$\begin{aligned} |N^+(v_1)| &\geq |Y| + |W| + |N^+(v_2) \cap V(C)| - |V(v_1) - \{v_1\}| \\ &\geq |V(y)| + |N^+(v_2)| - |V(v_1) - \{v_1\}| \\ &\geq \begin{cases} |N^+(v_2)| & \text{if } |V(v_1)| \leq r+1 \\ |N^+(v_2)| - 1 & \text{if } |V(v_1)| = r+2 \end{cases} \end{aligned} \tag{3.7}$$

$$\begin{aligned}|N^+(u)| &\geq |Y| + |W| + |N^+(v_2) \cap V(C)| - 1 \\ &\quad +|\{v_1, v_2\}| - |V(u) - \{u\}| \\ &\geq |V(y)| + |N^+(v_2)| + 1 - |V(u) - \{u\}| \\ &\geq \left\{ \begin{array}{ll} |N^+(v_2)| + 1 & \text{if } \ |V(u)| \leq r+1 \\ |N^+(v_2)| & \text{if } \ |V(u)| = r+2 \end{array} \right. ,\end{aligned} \tag{3.8}$$

for every $u \in U$ and

$$\begin{aligned}|N^-(w)| &\geq |Y| + |U| + |N^-(v_1) \cap V(C)| - 1 \\ &\quad +|\{v_1, v_2\}| - |V(w) - \{w\}| \\ &\geq |V(y)| + |N^-(v_1)| + 1 - |V(w) - \{w\}| \\ &\geq \left\{ \begin{array}{ll} |N^-(v_1)| + 1 & \text{if } \ |V(w)| \leq r+1 \\ |N^-(v_1)| & \text{if } \ |V(w)| = r+2 \end{array} \right. ,\end{aligned} \tag{3.9}$$

for every $w \in W$. If the right-hand side of (3.7) increases by at least two or the right-hand side of (3.8) or (3.9) increases by at least one, then we arrive at a contradiction either to $i_g(D) \leq 1$ or to Remark 1.18. This leads to $|V(u)|$, $|V(w)| \geq r+1$ for $u \in U$ and $w \in W$. Another consequence is that $|Y| = r$, if $U \cup W \neq \emptyset$, and $|Y| \leq r+1$, if $U \cup W = \emptyset$. Anyway, Y consists of exactly one partite set. Furthermore, the bounds (3.7)-(3.9) yield $|U|, |W| \leq 1$, since otherwise, the right-hand side of (3.8) or (3.9) increases by one, a contradiction. Let $U \neq \emptyset$ and $u \in U$. Because of $v_1 \to v_2$, we conclude that $v_t \to u$, since otherwise the right-hand side of (3.8) increases by one, a contradiction. If we observe the cycle $C' = b_1 b_2 \ldots b_{t+1} b_1 := v_1 v_2 \ldots v_t u v_1$ such that $b_1 = v_1$, then we see that C' fulfills $\{b_{t+1}, b_1\} \to Y \to \{b_2, b_3\}$. Hence, we can replace C by C', which means that, without of generality, we may suppose that $U = \emptyset$. Analogously, it remains to treat the case that $W = \emptyset$.

Let $y \in Y$. If we define $U' = N^-(y) - V(C)$ and $W' = N^+(y) - V(C)$, then we conclude that $V(D) = V(y) \cup V(C) \cup U' \cup W'$. Let $w' \in W'$. If $w' \to v_1$, then it follows that $w' \in U$, and thus we have $w' \in N^-(y) - V(C)$, a contradiction to the definition of W'. Since $W = \emptyset$, this yields $v_1 \rightsquigarrow w' \rightsquigarrow v_2$ and the right-hand side of (3.7) increases by one. Analogously, we observe that $v_1 \rightsquigarrow u' \rightsquigarrow v_2$ for each $u' \in U'$. To get no contradiction in (3.7), it has to be $|U' \cup W'| \leq 1$.

Subcase 3.1. Suppose that $m = 3$, and thus $c = 4$. Let $V_b = V(D) - (Y \cup V(v_1) \cup V(v_2))$. We observe that $N^-(v_1) \cap V_b \neq \emptyset$, since otherwise, we arrive at

$$\begin{aligned}\frac{3r+k-2}{2} &\leq d^-(v_1) \leq |V(v_2) - \{v_2\}| \leq r, \\ \text{if} &\quad |V(v_2)| \leq r+1, \\ \frac{3r+k-2}{2} &\leq d^-(v_1) \leq |V(v_2) - \{v_2\}| = r+1, \\ \text{if} &\quad |V(v_2)| = r+2,\, |V(v_1)| \geq r+1 \quad \text{and thus} \quad k \geq 3 \quad \text{and} \\ \frac{3r+k}{2} &= d^-(v_1) \leq |V(v_2) - \{v_2\}| = r+1,\end{aligned}$$

$$\text{if} \qquad |V(v_2)| = r+2 \quad \text{and} \quad |V(v_1)| = r,$$

in all cases a contradiction. If $N^+(v_2) \cap (V(C) - \{v_3\}) = \emptyset$, then Corollary 1.19 yields $\frac{3r+k-2}{2} \leq d^+(v_2) \leq 2$, a contradiction.

Suppose that there exists an index $q \geq 4$ as small as possible such that $v_2 \to v_q$ and that there is an index $l < q$ with $v_l \to v_1$. This index l let be chosen as large as possible. Now, let us observe the cycle $C' = v_1 v_2 v_q \ldots v_t y v_3 \ldots v_l v_1$. If C' does not contain vertices from all the 4 partite sets, then we conclude that $V_b \subseteq V(D) - V(C') \subseteq [\{v_{l+1}, \ldots, v_{q-1}\} \cup U' \cup W']$. Since $v_1 \rightsquigarrow U' \cup W' \cup \{v_{l+1}, \ldots, v_{q-1}\}$, we arrive at $N^-(v_1) \cap V_b = \emptyset$, a contradiction.

Altogether, we see that an index q chosen as above does not exist. Let y_1 be the largest index such that $v_2 \to v_{y_1}$. This implies that $v_1 \rightsquigarrow \{v_2, v_3, \ldots, v_{y_1-1}\}$. If $v_{y_1} \to v_1$, then we have the 3-cycle $v_1 v_2 v_{y_1} v_1$, a contradiction to $t \geq 4$. Hence, we deduce that $N^-(v_1) \subseteq \{v_{y_1+1}, v_{y_1+2}, \ldots, v_t\}$. If there is no arc leading from v_3 to $\{v_{y_1+1}, v_{y_1+2}, \ldots, v_t\}$, then we arrive at

$$\begin{aligned} d^-(v_3) &\geq d^-(v_1) + |Y| + |\{v_1, v_2\}| - |V(v_3) - \{v_3\}| \\ &\geq \begin{cases} d^+(v_1) + 2, & \text{if} \quad |V(v_3)| \leq r+1 \\ d^+(v_1) + 1, & \text{if} \quad |V(v_3)| = r+2 \end{cases}, \end{aligned}$$

in both cases a contradiction. Therefore, let $y_2 > y_1$ be the largest index such that $v_3 \to v_{y_2}$. Firstly, let $v_l \to y$ for some $y \in Y$ and $4 \leq l \leq y_2 - 1$ (notice that, because of $y_1 \geq 4$, it has to be $y_2 \geq 5$). This yields $v_a \to y$ for all $l \leq a \leq t$, since otherwise, we can find a cycle through all 4 partite sets, a contradiction. Let x_1 be the smallest index in $\{4, 5, \ldots, y_1\}$ such that $v_2 \to v_{x_1}$. Now, let us observe the cycle $C' := v_1 v_2 v_{x_1} \ldots v_{y_2-1} y v_3 v_{y_2} \ldots v_t v_1$. If C' does not contain vertices from all 4 partite sets, then we conclude that $V_b \subseteq \{v_4, v_5, \ldots, v_{x_1-1}\} \cup U' \cup W'$, and thus $N^-(v_1) \cap V_b = \emptyset$, a contradiction. Hence, we arrive at $Y \to \{v_2, v_3, v_4, v_5, \ldots, v_{y_2-1}\}$, and thus $d^+(y) \geq d^+(v_2) + 1$ for all $y \in Y$, $y_2 = y_1 + 1$ and $v_2 \to \{v_3, \ldots, v_{y_1}\}$, which means $\{v_3, \ldots, v_{y_1}\} \cap V(v_2) = \emptyset$. Let x_2 be the first index such that $v_{x_2} \to v_1$ ($x_2 \geq y_2$). If $\{v_{x_2+1}, \ldots, v_t\} \rightsquigarrow v_4$, then we conclude that

$$\begin{aligned} d^-(v_4) &\geq d^-(v_1) - 1 + |Y| + |\{v_1, v_2, v_3\}| - |V(v_4) - \{v_4\}| \\ &\geq \begin{cases} d^-(v_1) + 2, & \text{if} \quad |V(v_4)| \leq r+1 \\ d^-(v_1) + 1, & \text{if} \quad |V(v_4)| = r+2 \end{cases}, \end{aligned}$$

in both cases a contradiction. Therefore, let $v_4 \to v_{y_3}$ with $y_3 > y_2$. If we notice that either $v_3 \in V_b$ or $v_4 \in V_b$, then we observe that $v_1 v_2 v_3 v_{y_2} y v_4 v_{y_3} \ldots v_t v_1$ is a cycle through all 4 partite sets, a contradiction.

Subcase 3.2. Let $m \geq 4$ and thus $c \geq 5$. Using Corollary 1.19, we arrive at $d^+(v_2) \geq \frac{(c-1)r+k-2}{2} \geq \frac{7}{2}$, which means $d^+(v_2) \geq 4$ and v_2 has at least four outer neighbors in $V(C)$.

Suppose that there is an index $q \geq 4$ as small as possible such that there is an index $l < q$ with $v_l \to v_1$. This index l let be chosen as large as possible. If the cycle $C' = v_1 v_2 v_q \ldots v_t y v_3 \ldots v_l v_1$ does not contain vertices from all $m+1$ partite sets, then the remaining partite sets have to be in $\{v_{l+1}, \ldots, v_{q-1}\} \cup U' \cup W'$. Furthermore, the choice of the indices l and q implies $v_1 \rightsquigarrow \{v_{l+1}, \ldots, v_{q-1}\} \rightsquigarrow v_2$. If the partite sets, which do not appear in C'

are only part of $\{v_{l+1}, \ldots, v_{q-1}\}$, then there are at least two vertices v_{x_1} and v_{x_2} such that $v_1 \rightsquigarrow \{v_{x_1}, v_{x_2}\}$ and $\{v_{x_1+1}, v_{x_2+1}\} \rightsquigarrow v_2$, which leads to a contradiction to (3.7). Let $w' \in W'$ be part of a partite set that does not appear in C'. Hence, we have $U' = \emptyset$, $l+1 = q-1$ and $v_{l+1} \in V(w')$, since otherwise, the right-hand side of (3.7) increases by at least two, a contradiction. Therefore, there are vertices from exactly m partite sets in C'. Now, we see that $r = 2$ and $|V(w')| = r = 2$. This and the fact that $v_1 \rightarrow v_2$ yield $q \geq 5$. If $w' \rightarrow v_3$, then $v_1 v_2 v_q \ldots v_t y w' v_3 \ldots v_l v_1$ is a cycle with vertices from exactly $m+1$ partite sets, a contradiction. If $q \geq 6$ and $w' \rightarrow v_b$ with $4 \leq b \leq l$, then we observe inductively that $v_1 v_2 v_q \ldots v_t y v_3 \ldots v_{b-1} w' v_b \ldots v_l v_1$ is a cycle with vertices from m+1 partite sets, a contradiction. Hence, let $\{v_3, \ldots, v_l\} \rightarrow w'$. If there is a vertex $y' \in V(y) - \{y\}$ such that $w' \rightarrow y'$, then $v_1 v_2 v_q \ldots v_t y w' y' v_3 \ldots v_l v_1$ is a cycle with vertices from exactly $m+1$ partite sets, a contradiction. If there is a vertex v_b in $V(C)$ with $4 \leq b \leq t$ such that $v_b \rightarrow y$ and $w' \rightarrow v_{b+1}$ $(t+1 \equiv 1)$, then $v_1 v_2 \ldots v_b y w' v_{b+1} \ldots v_1$ is a cycle containing vertices from exactly $m+1$ partite sets, a contradiction.

Firstly, let $v_l \rightarrow y$. This implies $\{v_l, v_{l+1}, \ldots, v_t, v_1\} \rightarrow y$, and thus $N^+(y) \subseteq \{w', v_2, \ldots, v_{l-1}\}$, which means $d^+(y) \leq l-1$. Because of Corollary 1.19, on the other hand, we have $d^+(y) \geq \frac{(c-1)r+k-1}{2} \geq \frac{7}{2}$, which implies $l \geq 5$. Altogether, it follows that

$$d^-(w') \geq d^-(y) - 2 + |Y| + l - 2 \geq d^-(y) + 3,$$

a contradiction to $i_g(D) \leq 1$. Otherwise, if $y \rightarrow v_l$, then it follows that

$$d^-(w') \geq d^-(y) - 1 + |Y| + 1 \geq d^-(y) + 2,$$

again a contradiction to $i_g(D) \leq 1$.

Altogether, we see that an index q chosen as above does not exist. Let z' be the largest index such that $v_2 \rightarrow v_{z'}$ (notice that $z' \geq 6$). This implies that $v_1 \rightsquigarrow \{v_2, v_3, \ldots, v_{z'-1}\}$, and thus $N^-(v_1) \subseteq \{v_{z'}, v_{z'+1}, \ldots, v_t\}$. If there is a vertex $y \in Y$ such that $v_{z'-1} \rightarrow y$, then it follows that $\{v_{z'-1}, \ldots, v_t, v_1\} \rightarrow y$, and thus, we have $d^-(y) \geq d^-(v_1)+2$, a contradiction to $i_g(D) \leq 1$. Therefore, we may assume that $Y \rightarrow \{v_2, v_3, \ldots, v_{z'-1}\}$. Let z'' be the smallest index such that $v_{z''} \rightarrow v_1$.

Firstly, let $v_2 \rightsquigarrow v_{z'-2}$. Then there is an arc from $v_{z'-2}$ to $\{v_{z''+1}, \ldots, v_t\}$, since otherwise, we observe that

$$\begin{aligned} d^-(v_{z'-2}) &\geq d^-(v_1) - 1 + |\{v_{z'-3}, v_1, v_2\}| + |Y| - |V(v_{z'-2}) - \{v_{z'-2}\}| \\ &\geq \left\{ \begin{array}{ll} d^-(v_1) + 2 & \text{if} \quad |V(v_{z'-2})| \leq r+1 \\ d^-(v_1) + 1 & \text{if} \quad |V(v_{z'-2})| = r+2 \end{array} \right. . \end{aligned}$$

Both cases yield a contradiction, either to $i_g(D) \leq 1$ or to Remark 1.18. Consequently, let $v_{z'-2} \rightarrow v_{y_1}$ with $y_1 \in \{z''+1, \ldots, t\}$. Let $y \in Y$ and let $y_2 < y_1$ be the largest index such that $v_{y_2} \rightarrow v_1$. Let us suppose that the cycle $C' := v_1 v_2 \ldots v_{z'-2} v_{y_1} \ldots v_t y v_{z'-1} \ldots v_{y_2} v_1$ does not contain vertices of exactly $m+1$ partite sets. Then there is a partite set V_b such that $V_b \subseteq \{v_{y_2+1}, v_{y_2+2}, \ldots, v_{y_1-1}\} \cup U' \cup W'$. Since $v_1 \rightsquigarrow \{v_{y_2+1}, v_{y_2+2}, \ldots, v_{y_1-1}\} \cup U' \cup W'$

and $\{v_{y_2+2}, v_{y_2+3}, \ldots, v_{y_1}\} \cup U' \cup W' \rightsquigarrow v_2$, (3.7) implies that $|V_b| \leq 1$, a contradiction to $r \geq 2$.

Secondly, let $v_{z'-2} \to v_2$. Since $v_1 \rightsquigarrow v_{z'-3}$, this yields that the right-hand side of (3.7) increases by 1. To get no contradiction, it follows that $v_2 \rightsquigarrow v_{z'-1}$ and $\{v_{z'}, v_{z'+1}, \ldots, v_t\} \to v_1$, which means that $z' = z''$. This implies that there is an arc from $v_{z'-1}$ to $\{v_{z'+1}, v_{z'+2}, \ldots, v_t\}$, since otherwise, we observe that

$$\begin{aligned} d^-(v_{z'-1}) &\geq d^-(v_1) - 1 + |\{v_{z'-2}, v_1, v_2\}| + |Y| - |V(v_{z'-1}) - \{v_{z'-1}\}| \\ &\geq \begin{cases} d^-(v_1) + 2 & \text{if} \quad |V(v_{z'-1})| \leq r+1 \\ d^-(v_1) + 1 & \text{if} \quad |V(v_{z'-1})| = r+2 \end{cases}. \end{aligned}$$

Both cases yield a contradiction, either to $i_g(D) \leq 1$ or to Remark 1.18. Consequently, let $v_{z'-1} \to v_{z_1}$ with $z_1 \in \{z'+1, z'+2, \ldots, t\}$. If there is a vertex $y \in Y$ such that $v_{z'} \to y$, then we conclude that $\{v_{z'}, v_{z'+1}, \ldots, v_t, v_1\} \to y$ and $v_1 v_2 v_{z'} \ldots v_{z_1-1} y v_3 \ldots v_{z'-1} v_{z_1} \ldots v_t v_1$ is a cycle with vertices from exactly $m+1$ partite sets, a contradiction. Hence, let $Y \to v_{z'}$. For an arbitrary vertex $y \in Y$, it follows that $v_1 v_2 \ldots v_{z'-1} v_{z_1} \ldots v_t y v_{z'} \ldots v_{z_1-1} v_1$ is a cycle through $m+1$ partite sets, a contradiction.

This completes the proof of the theorem. □

Combining the results of the Theorems 3.11 and 3.12, we arrive at Theorem 3.6.

The next example shows that the condition that there are at least two vertices in each partite set is necessary, at least for $c = 4$.

Example 3.13 *Let $V_1 = \{a\}$, $V_2 = \{b, b_2\}$, $V_3 = \{c\}$, and $V_4 = \{y\}$ be the partite sets of a 4-partite tournament such that $a \to b \to c \to b_2 \to y \to c \to a \to y \to b$ and $b_2 \to a$ (see Figure 3.3). The resulting 4-partite tournament is almost regular, however, the arc ab is on a cycle with vertices from exactly 3 partite sets, but not from all 4 partite sets.*

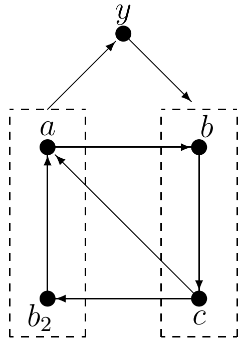

Figure 3.3: An almost regular 4-partite tournament with the property that the arc ab is in no cycle through exactly 4 partite sets

3.3 Open problems

The results in the last two sections lead us to the following problems.

Problem 3.14 (Volkmann, Winzen [38]) *Let D be a c-partite tournament with $i_g(D) \leq i$ and at least r vertices in each partite set. For all i, find the smallest values $g(i)$ and $f(i, g(i))$ with the property that every arc of D is contained in a cycle through m partite sets for all $m \in \{4, 5, \ldots c\}$, if $r \geq g(i)$ and $c \geq f(i, g(i))$.*

According to the Theorems 3.4 and 3.6, we have $g(0) = 1$, $f(0, 1) = 4$, $g(1) = 2$ and $f(1, 2) = 4$.

Problem 3.15 (Volkmann, Winzen [38]) *Let D be a c-partite tournament with $i_g(D) \leq i$ and r vertices in each partite set. For all i, c and r find optimal values $g_1(i, c, r)$ and $g_2(i, c, r)$ such that every arc of D is contained in a cycle through exactly m partite sets for all $g_1(i, c, r) \leq m \leq g_2(i, c, r)$.*

Chapter 4

Combining Chapters 2 and 3

There is an extensive literature on cycles in multipartite tournaments, see e.g., Bang-Jensen and Gutin [2], Guo [9], Gutin [15], Volkmann [31] and Yeo [49]. Many results are about the existence of cycles of a given length as Theorem 2.5, the main result of Chapter 2, and the following result of Bondy [6].

Theorem 4.1 (Bondy [6]) *Each strongly connected c-partite tournament T contains a cycle of order m for each $m \in \{3, 4, \dots, c\}$.*

Other articles treat the existence of cycles containing vertices of a given number of partite sets. Good examples are Theorem 3.1 of Goddard and Oellermann [8] and Theorem 3.6, the main theorem of Chapter 3. An interesting question is now to find sufficient conditions for a multipartite tournament such that we are able to combine these two categories of results, which means to solve the following problem.

Problem 4.2 (Volkmann, Winzen [43]) *Which conditions have to be fulfilled such that a c-partite tournament with the partite sets $V_1, V_2, \dots, V_c$ contains a cycle with exactly r_i vertices of V_i for all $1 \le i \le c$ and given integers $0 \le r_i \le |V_i|$?*

In the first section, we will search for strong subtournaments in almost regular multipartite tournaments. The following theorem of Moon [20] shows that every strongly connected tournament is Hamiltonian, which implies the connection between strong subtournaments and Problem 4.2.

Theorem 4.3 (Moon [20]) *Every vertex of a strongly connected tournament T is contained in a cycle of order m for all $3 \le m \le |V(T)|$.*

Solving a Problem posed by L. Volkmann [29], in the second section, we will study the existence of strong subtournaments in multipartite tournaments of higher irregularity. Obviously, the order of strong subtournaments that can be guaranteed will be smaller the higher the irregularity of the multipartite tournament is.

In the third section, we will search for long cycles. In 1997, A. Yeo [48] gave a solution of Problem 4.2 for regular c-partite tournaments in the case that $r_i = |V_i|$ for all $1 \le i \le c$.

Theorem 4.4 (Yeo [48]) *Every regular multipartite tournament D is Hamiltonian.*

Here, we will study the next step, which means that we will solve Problem 4.2 for regular c-partite tournaments in the case that $r_i = |V_i| - 1$ for all $1 \le i \le c$.

4.1 Strong subtournaments when $i_g(D) \le 1$

In 1999, L. Volkmann [29] proved the following theorem about strong subtournaments in multipartite tournaments.

Theorem 4.5 (Volkmann [29]) *Let D be an almost regular c-partite tournament with $c \ge 4$. Then D contains a strongly connected subtournament of order p for every $p \in \{3, 4, \ldots, c-1\}$.*

The next example of Volkmann [29] shows that Theorem 4.5 is best possible for $c = 4$, even for regular multipartite tournaments.

Example 4.6 (Volkmann [29]) *Let $V_i = V_i' \cup V_i''$ with $|V_i'| = |V_i''| = t$ for $i = 1, 2, 3, 4$ be the partite sets of a 4-partite tournament such that $V_1' \to V_2' \to V_3' \to V_1'$, $V_1'' \to V_2'' \to V_3'' \to V_1''$,*

$$(V_1' \cup V_2' \cup V_3') \to V_4' \to (V_1'' \cup V_2'' \cup V_3'') \to V_4'' \to (V_1' \cup V_2' \cup V_3'),$$

$$V_1' \to V_3'' \to V_2' \to V_1'' \to V_3' \to V_2'' \to V_1'.$$

Now it is a simple matter to check that the resulting 4-partite tournament is $3t$-regular without a strongly connected subtournament of order 4.

However, for $c \ge 5$, Volkmann [29] presented the following conjecture.

Conjecture 4.7 (Volkmann [29]) *If D is an almost regular c-partite tournament with $c \ge 5$, then D contains a strongly connected subtournament of order c.*

In the following, we will settle Conjecture 4.7 in affirmative. To reach this, we need some results about tournaments.

Definition 4.8 *[Tournament Q_n]* By Q_n we define the tournament of order n consisting of a path $x_1 x_2 \ldots x_n$ and all arcs $x_i x_j$ where $i > j + 1$.

Lemma 4.9 (Thomassen [26]) *A strong tournament T on n vertices has three vertices y_1, y_2, y_3 such that $T - y_i$ is strong for $i = 1, 2, 3$, unless T is isomorphic to Q_n.*

Lemma 4.10 (Yeo [49]) *If X is a non-empty vertex set of a digraph D, then*

$$i_l(D) \geq \frac{|d(X, V(D)-X) - d(V(D)-X, X)|}{|X|},$$

which means, if D is almost regular, then it follows that

$$|X| \geq |d(X, V(D)-X) - d(V(D)-X, X)|.$$

Lemma 4.11 (Volkmann [29]) *Let T be a strongly connected tournament of order $|V(T)| \geq 4$. Then there exists a vertex $u \in V(T)$ of maximum outdegree such that for all $x \in V(T)-\{u\}$, the subtournament $T-x$ has a Hamiltonian path with the initial vertex u.*

Now we are able to prove the main result of this section.

Theorem 4.12 (Volkmann, Winzen [39]) *Let D be an almost regular c-partite tournament with $c \geq 5$. Then D contains a strongly connected subtournament of order c.*

Proof. Let $V_1, V_2, \ldots, V_c$ be the partite sets of D with $r = |V_1| \leq |V_2| \leq \ldots \leq |V_c| \leq r+2$. Obviously, $|V(D)| = cr + k$ with $k = 0$, if $|V_c| = r$, $1 \leq k \leq c-1$, if $|V_c| = r+1$, and $2 \leq k \leq 2c-2$, if $|V_c| = r+2$. According to Theorem 4.5, there exists a strongly connected subtournament $T_{c-1} = D[\{v_1, v_2, \ldots, v_{c-1}\}]$ of order $c-1$ in D. Let V_p be the partite set without any vertex in T_{c-1}. Assume now that D does not contain any strong subtournament of order c.

If there is a vertex $z \in V_p$ with an inner and an outer neighbor in T_{c-1}, then $D[\{z, v_1, v_2, \ldots, v_{c-1}\}]$ is a strong subtournament of order c, a contradiction. Hence, let $V' \subseteq V_p$ and $V'' = V_p - V'$ such that $V(T_{c-1}) \rightarrow V'$, if $V' \neq \emptyset$, and $V'' \rightarrow V(T_{c-1})$, if $V'' \neq \emptyset$. Let U be the set of vertices of $V(D)-(V_p \cup V(T_{c-1}))$ being dominated by at least one vertex of V', and let W be the set of vertices of $V(D)-(V_p \cup V(T_{c-1}))$ that are not dominated by any vertex of V', that means $W \rightarrow V'$. These definitions lead us to the following claim.

Claim 1. If there is a vertex $v \in V(T_{c-1})$ that is the initial vertex of a Hamiltonian path in $T_{c-1} - \{y\}$ for all $y \in T_{c-1} - \{v\}$, then $v \rightsquigarrow U$. Otherwise, if there is a vertex $u \in U \cap V(v_j)$ with $v_j \in V(T_{c-1})$ such that $u \rightarrow v$, then let $v' \in V'$ with $v' \rightarrow u$. In this case $D[(V(T_{c-1}) - \{v_j\}) \cup \{v', u\}]$ is a strongly connected tournament, a contradiction.

Now, we distinguish different cases.

Case 1. Let $|V'| \leq 1$ or $|V''| \leq 1$. Without loss of generality, we suppose that $|V''| \leq 1$. Then, let $V := V'$. The definitions of the sets U and W yield $d(V, V(D)-V) \leq |V||U|$ and $d(V(D)-V, V) \geq |V|(|V(D)-(U \cup V)|-1)$. According to Lemma 4.10, we have $|V| \geq d(V(D)-V, V) - d(V, V(D)-V) \geq |V|(|V(D)| - 2|U| - |V| - 1)$, which is equivalent with

$$|U| \geq \frac{|V(D)| - |V| - 2}{2}. \tag{4.1}$$

Because of $c-1 \geq 4$ and Lemma 4.11 there is a vertex $v \in V(T_{c-1})$ such that for all vertices $y \in V(T_{c-1}) - \{v\}$, the subtournament $T_{c-1} - \{y\}$ contains a Hamiltonian path with the initial vertex v. With Claim 1, we obtain that $v \rightsquigarrow U$. According to Lemma 4.11, the vertex v has maximum outdegree in T_{c-1}, and thus $d^+_{D[V(T_{c-1})]}(v) \geq 2$. Because of (4.1), it follows that

$$\begin{aligned}
d^+(v) &\geq |V| + |U - (V(v) - \{v\})| + d^+_{D[V(T_{c-1})]}(v) \\
&\geq |V| + \frac{|V(D)| - |V| - 2}{2} - |V(v)| + 3 \\
&= \frac{|V(D)| + |V| - 2}{2} - |V(v)| + 3 \\
&\geq \begin{cases} \frac{|V(D)|+r-3}{2} - (r+2) + 3, & \text{if} \quad |V(v)| = r+2 \\ \frac{|V(D)|+r-3}{2} - (r+1) + 3, & \text{if} \quad |V(v)| = r+1 \\ \frac{|V(D)|+r-3}{2} - r + 3 \qquad , & \text{if} \quad |V(v)| = r \end{cases} \\
&= \begin{cases} \frac{|V(D)|-r-1}{2}, & \text{if} \quad |V(v)| = r+2 \\ \frac{|V(D)|-r+1}{2}, & \text{if} \quad |V(v)| = r+1 \\ \frac{|V(D)|-r+3}{2}, & \text{if} \quad |V(v)| = r \end{cases},
\end{aligned}$$

in all cases a contradiction, either to Remark 1.18 or to Lemma 1.12.

Case 2. Let $|V'| \geq 2$ and $|V''| \geq 2$. This implies that $r \geq 2$, since otherwise we arrive at $|V_p| \geq 4 = r+3$, a contradiction to Lemma 1.10.

If H is the set of partite sets containing a vertex $v \in V(T_{c-1})$ such that the subdigraph $T_{c-1} - \{v\}$ is still strongly connected, then let us define the sets $\hat{W}$ and $\hat{U}$ by $\hat{U} = U \cap H$ and analogously $\hat{W} = W \cap H$.

Now, it follows that $V(T_{c-1}) \rightsquigarrow \hat{U}$, since otherwise let $v_i \in V(T_{c-1})$, $\hat{u} \in \hat{U} \cap V(v_j)$ with $v_j \in V(T_{c-1})$ and $v' \in V'$ such that $v' \to \hat{u} \to v_i$, then $D[(V(T_{c-1}) - \{v_j\}) \cup \{\hat{u}, v'\}]$ is a strongly connected tournament, a contradiction. This implies $V'' \to \hat{U}$, because otherwise let $\hat{u} \in \hat{U} \cap V(v_j)$ with $v_j \in V(T_{c-1})$ and $v'' \in V''$ such that $\hat{u} \to v''$, then $D[(V(T_{c-1}) - \{v_j\}) \cup \{\hat{u}, v''\}]$ is a strongly connected tournament of order c, a contradiction. If there are vertices $\hat{u} \in \hat{U} \cap V(v_j)$ with $v_j \in V(T_{c-1})$ and $v \in N^-(v'') \cap V(v_i)$ with $v_i \in V(T_{c-1})$ such that $\hat{u} \to v$, then $D[(V(T_{c-1}) - \{v_i, v_j\}) \cup \{\hat{u}, v'', v\}]$ is a strong tournament, a contradiction. Consequently, let us assume that $N^-(v'') \rightsquigarrow \hat{U}$ for all $v'' \in V''$. Altogether, we see that

$$(V(T_{c-1}) \cup V'' \cup N^-(v'')) \rightsquigarrow \hat{U}, \tag{4.2}$$

if $v'' \in V''$. Let $\hat{w} \in \hat{W} \cap N^-(v'')$ with $v'' \in V''$. Assume that there exists a vertex $v_i \in V(T_{c-1})$ such that $v_i \to \hat{w}$. If $\hat{w} \in V(v_j)$ with $v_j \in V(T_{c-1})$, then $D[(V(T_{c-1}) - \{v_j\}) \cup \{\hat{w}, v''\}]$ is a strongly connected tournament of order c, a contradiction. Therefore, let $\hat{w} \rightsquigarrow V(T_{c-1})$. This yields $\hat{w} \rightsquigarrow U$, since otherwise let $u \in U$ and $v' \in V'$ such that $v' \to u \to \hat{w}$. If $u \in V(v_i)$ and $\hat{w} \in V(v_j)$ with $v_i, v_j \in V(T_{c-1})$, then $D[(V(T_{c-1}) - \{v_i, v_j\}) \cup \{u, \hat{w}, v'\}]$ is a strongly connected tournament, a contradiction. Altogether, this leads to

$$N^-(v'') \cap \hat{W} \rightsquigarrow (V(T_{c-1}) \cup U \cup V'). \tag{4.3}$$

Subcase 2.1. Suppose that $\hat{U} \neq \emptyset$. Because of (4.2) this yields $N^-(v'') \subseteq W \cup (U - \hat{U})$. If there is a vertex $v'' \in V''$ such that $v'' \to \hat{W}$ or if $\hat{W} = \emptyset$, then $N^-(v'') \cap V(\hat{u}) = \emptyset$ for all $\hat{u} \in \hat{U}$, which yields that $N^-(v'') \to \hat{U}$, and thus, according to (4.2), we have

$$\begin{aligned} d^-(\hat{u}) &\geq d^-(v'') + |V(T_{c-1}) - V(\hat{u})| + |N^-(\hat{u}) \cap V'| + |N^-(\hat{u}) \cap V''| \\ &\geq d^-(v'') + c - 2 + 3, \end{aligned}$$

if $\hat{u} \in \hat{U}$, a contradiction to $i_g(D) \leq 1$. Hence, let $\hat{w} \in \hat{W} \cap N^-(v'')$ with $v'' \in V''$. If we define $\hat{c}$ by $\hat{c} = |V(\hat{w}) \cap U|$, then, because of (4.3), for all $v' \in V'$ we see that

$$\begin{aligned} d^+(v') + 1 &\geq d^+(\hat{w}) \geq |U| - \hat{c} + |V'| + |N^+(\hat{w}) \cap V''| + |V(T_{c-1}) - V(\hat{w})| \\ &\geq d^+(v') - \hat{c} + |V'| + 1 + c - 2, \end{aligned}$$

and thus, we conclude that $\hat{c} \geq |V'| + c - 2$. Let $\hat{u} \in V(\hat{w}) \cap \hat{U}$. If $\tilde{c} = |V(\hat{u}) \cap \hat{W}|$, then for $v'' \in V''$ it follows that

$$\begin{aligned} d^-(v'') + 1 &\geq d^-(\hat{u}) \\ &\geq d^-(v'') + |V(T_{c-1}) - V(\hat{u})| + |V''| + |N^-(\hat{u}) \cap V'| - \tilde{c} \\ &\geq d^-(v'') + c - 2 + |V''| + 1 - \tilde{c}. \end{aligned}$$

This implies $\tilde{c} \geq |V''| + c - 2$, and we arrive at the contradiction

$$|V(\hat{u})| \geq \tilde{c} + \hat{c} + 1 \geq |V_p| + 2c - 3 \geq |V_p| + 7.$$

Subcase 2.2. Let $\hat{U} = \emptyset$. This leads us to a further claim.

Claim 2. If $v \in V(T_{c-1})$ is the initial vertex of a Hamiltonian path in $T_{c-1} - \{y\}$ for all $y \in T_{c-1} - \{v\}$, then v is a cut-vertex of D, since otherwise $V(v) - \{v\} \subseteq \hat{W}$. Using Claim 1 this yields $v \to U$ and thus $d^+(v) \geq |U| + |V'| \geq d^+(v') + 2$ for all $v' \in V'$, a contradiction to $i_g(D) \leq 1$.

If T_{c-1} is 2-strong, then it follows that $U = \emptyset$, and thus $d^+(v') = 0$ with $v' \in V'$, a contradiction.

Consequently, there remains the case that T_{c-1} contains a cut-vertex x, which means $T_{c-1} - \{x\}$ is not strongly connected and consists of strongly connected components $T'_1, T'_2, \ldots, T'_l$ such that $V(T'_i) \to V(T'_j)$, if $1 \leq i < j \leq l$. Furthermore, there exist vertices $v'_1 \in V(T'_1)$ and $v'_l \in V(T'_l)$ such that $v'_l \to x \to v'_1$.

If there are two vertices $\hat{w} \in \hat{W}$ and $v'' \in V''$ such that $\hat{w} \to v''$, then, analogously as in the last subcase, we see that $|V(\hat{w}) \cap U| \geq |V'| + c - 2 > 0$, a contradiction to $\hat{U} = \emptyset$. Consequently, let $V'' \to \hat{W}$.

Let $z_1 \to z_2 \to \ldots \to z_m$ be a Hamiltonian path in $T_{c-1} - \{x\}$ such that $z_m \to x \to z_1$.

Firstly, let $x \to z_2$. In this case, $T_{c-1} - \{z_1\}$ is strongly connected. Hence, $V(z_1)$ belongs to H. Subcase 2.1 implies that $V(z_1) \cap U = \emptyset$, and since $V'' \to \hat{W}$, we see that $V'' \to V(z_1)$. If z_1 is the initial vertex of a Hamiltonian path in $T_{c-1} - \{y\}$ for all $y \in V(T_{c-1}) - \{z_1\}$, then, according to Claim 1, we arrive at $z_1 \rightsquigarrow U$, a contradiction to Claim 2.

Secondly, we investigate the case that $z_{m-1} \to x$. Then $T_{c-1} - \{z_m\}$ is strongly connected. Hence, $V(z_m) \subseteq H$, $V(z_m) \cap U = \emptyset$ and $V'' \to V(z_m)$. If z_m is the terminal vertex of a Hamiltonian path in $T_{c-1} - \{y\}$ for all $y \in V(T_{c-1}) - \{z_m\}$, then we observe that $N^-(v'') \rightsquigarrow z_m$, since otherwise, if there are vertices $v'' \in V''$ and $v \in V(v_j)$ with $v_j \in V(T_{c-1})$ such that $z_m \to v \to v''$, then $D[(V(T_{c-1}) - \{v_j\}) \cup \{v, v''\}]$ is a strong tournament of order c, a contradiction. Because of $V'' \to \hat{W}$ we even have $N^-(v'') \to z_m$, which means

$$d^-(z_m) \geq d^-(v'') + 1 + |V''| > d^-(v'') + 1,$$

if $v'' \in V''$, a contradiction.

Subcase 2.2.1. Let $V(T_1') = \{v_1', \hat{v}_1, \tilde{v}_1\}$ such that $x \to v_1' \to \tilde{v}_1 \to \hat{v}_1 \to v_1'$.

Firstly, we investigate the case that $x \to \{v_1', \hat{v}_1, \tilde{v}_1\}$. In this case we observe that $V(v_1') \cup V(\hat{v}_1) \cup V(\tilde{v}_1) \subseteq H$. If y is an arbitrary vertex in $V(T_{c-1}) - \{x\}$, then it is straightforward to see that x is the initial vertex of a Hamiltonian path in $T_{c-1} - \{y\}$. Claim 1 yields that $x \rightsquigarrow U$. If $\tilde{c} = |V(x) \cap U|$ and $v' \in V'$, then we arrive at

$$\begin{array}{rcl}
d^+(v') + 1 & \geq & d^+(x) = |U| - \tilde{c} + |N^+(x) \cap W| + d^+_{D[V(T_{c-1})]}(x) + |V'| \\
& \geq & d^+(v') - r + d^+_{D[V(T_{c-1})]}(x) + |N^+(x) \cap W| + 2, \\
\text{if} & & \tilde{c} \leq r \quad \text{and by Remark 1.18} \\
d^+(v') & \geq & d^+(x) = |U| - \tilde{c} + |N^+(x) \cap W| + d^+_{D[V(T_{c-1})]}(x) + |V'| \\
& \geq & d^+(v') - (r+1) + d^+_{D[V(T_{c-1})]}(x) + |N^+(x) \cap W| + 2, \\
\text{if} & & \tilde{c} = r + 1.
\end{array} \tag{4.4}$$

In both cases, we obtain

$$|N^+(x) \cap W| \leq r - 1 - d^+_{D[V(T_{c-1})]}(x). \tag{4.5}$$

If $l = 2$ and $T_2' = \{v_2'\}$, then $c = 6$ and for $v' \in V'$ we have

$$3r + 2 \leq |V(T_{c-1})| + |\hat{W}| \leq d^-(v') \leq d^+(v') + 1 \leq |U| + 1 \leq 2r + 3,$$

a contradiction to $r \geq 2$. Otherwise we observe that there is a vertex $a \in V(T_2')$ such that $T_{c-1} - \{a\}$ is strong.

Hence, we may assume now that $|\hat{W}| \geq 4r - 4$. By (4.5), we conclude that for all $\hat{w} \in \hat{W}$ except at most $r - 1 - d^+_{D[V(T_{c-1})]}(x)$ vertices we have $\hat{w} \to x$.

Using (4.5), we will show next that there exists at least one vertex $\tilde{w} \in \hat{W} \cap N^-(x)$ such that

$$\begin{array}{rcl}
|N^+(\tilde{w}) \cap W| & \geq & \dfrac{3r - 3 - (r - 1 - d^+_{D[V(T_{c-1})]}(x))}{2} \\
& = & r + \dfrac{d^+_{D[V(T_{c-1})]}(x) - 2}{2}.
\end{array} \tag{4.6}$$

Let F be the set $\hat{W}$ without the at most $r-1-d^+_{D[V(T_{c-1})]}(x)$ vertices, which are dominated by x. For every vertex $f \in F$ we conclude that

$$d^+_{D[F]}(f) + d^-_{D[F]}(f) \geq 3r - 3 - (r - 1 - d^+_{D[V(T_{c-1})]}(x)).$$

This implies

$$\begin{aligned} 2\sum_{f\in F} d^+_{D[F]}(f) &= \sum_{f\in F}(d^+_{D[F]}(f) + d^-_{D[F]}(f)) \\ &\geq |F|(3r - 3 - (r - 1 - d^+_{D[V(T_{c-1})]}(x))), \end{aligned}$$

which immediately yields (4.6).

If we remove one of the vertices that are no cut-vertices of T_{c-1} and a cut-vertex $\tilde{x} \neq x$ (obviously $\tilde{x} \in V(T_l'))$ from T_{c-1}, then it is easy to see that x is still an initial vertex of a Hamiltonian path in the remaining tournament. This implies that $\tilde{w} \to U - V(x)$, since otherwise let $v' \in V'$, $u \in (U - V(x)) \cap V(v_i)$ and $\tilde{w} \in V(v_j)$ with $v_i, v_j \in V(T_{c-1})$ such that $v' \to u \to \tilde{w}$. In this case $D[(V(T_{c-1}) - \{v_i, v_j\}) \cup \{v', u, \tilde{w}\}]$ is a strongly connected tournament, a contradiction. Therefore and because of (4.6), we arrive at

$$\begin{aligned} |U| - \tilde{c} + |N^+(x) \cap W| + d^+_{D[V(T_{c-1})]}(x) + |V'| &\overset{(4.4)}{=} d^+(x) \\ \geq\ d^+(\tilde{w}) - 1 \overset{(4.6)}{\geq} |U| - \tilde{c} + |V'| + r + &\frac{d^+_{D[V(T_{c-1})]}(x) - 4}{2}, \end{aligned}$$

which leads to

$$r \leq |N^+(x) \cap W| + \frac{d^+_{D[V(T_{c-1})]}(x)}{2} + 2 \overset{(4.5)}{\leq} r + 1 - \frac{d^+_{D[V(T_{c-1})]}(x)}{2},$$

a contradiction to $d^+_{D[V(T_{c-1})]}(x) \geq 3$.

Hence, it remains the case that $\hat{v}_1 \to x$ or $\tilde{v}_1 \to x$. At first, we investigate the case that $x \to \{v_1', \tilde{v}_1\}$ (that means $\hat{v}_1 \to x$). Now, it is straightforward to see that $\hat{v}_1$ is the initial vertex of a Hamiltonian path in $T_{c-1} - \{y\}$, if y is an arbitrary vertex in $V(T_{c-1}) - \{\hat{v}_1\}$. According to Claim 2, this contradicts the fact that $T_{c-1} - \{\hat{v}_1\}$ is strongly connected.

Consequently, we have to investigate the case that $\tilde{v}_1 \to x$. It is easy to check that in this case $\tilde{v}_1$ is the initial vertex of a Hamiltonian path in $T_{c-1} - \{y\}$ for an arbitrary $y \in V(T_{c-1}) - \{\tilde{v}_1\}$. Analogously as in Claim 2, we observe that $V(\tilde{v}_1) \cap U \neq \emptyset$, since otherwise we arrive at $\tilde{v}_1 \to U$, a contradiction to $i_g(D) \leq 1$. If $V(\hat{v}_1) \cap W \cap N^-(\tilde{v}_1) = \emptyset$, then for all $v' \in V'$ we arrive at the contradiction

$$\begin{aligned} d^+(\tilde{v}_1) &\geq |U| - |V(\tilde{v}_1) \cap U| + |V'| + |V(\hat{v}_1) \cap W| + d^+_{D[V(T_{c-1})]}(\tilde{v}_1) \\ &\geq d^+(v') - (r+1) + 2 + r - 1 + 3 = d^+(v') + 3. \end{aligned}$$

Hence, let $\hat{w} \in V(\hat{v}_1) \cap W \cap N^-(\tilde{v}_1)$. If there are vertices $v' \in V'$ and $u \in (U - (V(\tilde{v}_1) \cup V(x))) \cap V(v_j)$ with $v_j \in V(T_{c-1})$ such that $v' \to u \to \hat{w}$, then $\tilde{v}_1$ is still the initial vertex of a Hamiltonian path in $V(T_{c-1}) - \{v_j, \hat{v}_1\}$ and

$D[(V(T_{c-1}) - \{v_j, \hat{v}_1\}) \cup \{u, v', \hat{w}\}]$ is a strong tournament, a contradiction. Hence, we conclude that $\hat{w} \to U - (V(\tilde{v}_1) \cup V(x))$. If there is a vertex $\tilde{x} \in V(x) \cap U$ such that $\tilde{x} \to \hat{w}$, then $v'_1 \to \hat{w}$, since otherwise let $v' \in V'$ such that $v' \to \tilde{x}$. Now, $D[(V(T_{c-1}) - \{x, \hat{v}_1\}) \cup \{\tilde{x}, v', \hat{w}\}]$ is a strong tournament of order c, a contradiction. If there is a vertex $\tilde{v} \in V(\tilde{v}_1) \cap U$ such that $\tilde{v} \to \hat{w}$, then we also conclude that $v'_1 \to \hat{w}$, since otherwise let $v' \in V'$ with $v' \to \tilde{v}$. Then $D[(V(T_{c-1}) - \{\tilde{v}_1, \hat{v}_1\}) \cup \{\tilde{v}, v', \hat{w}\}]$ is a strongly connected tournament, a contradiction. If $\hat{w} \to U$, then it follows that $d^+(\hat{w}) \geq d^+(v') + |V'| \geq d^+(v') + 2$, if $v' \in V'$, a contradiction. Consequently, there exists a vertex $v \in (V(\tilde{v}_1) \cup V(x)) \cap U$ such that $v \to \hat{w}$. As seen above, this yields $v'_1 \to \hat{w}$. Now, let us examine the tournament $\tilde{T}_{c-1} := (T_{c-1} - \{\hat{v}_1\}) \cup \{\hat{w}\}$. Since there is no arc leading from $\tilde{T}_{c-1} - \{x, v'_1\}$ to v'_1, the vertex x is a cut-vertex of $\tilde{T}_{c-1}$, that means that $\tilde{T}_{c-1} - \{x\}$ consists of the strong components $\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_o$ with $V(\hat{T}_i) \to V(\hat{T}_j)$, if $i < j$. Now, it is easy to see that $v'_1 \in V(\hat{T}_1)$. Furthermore, we observe that $\hat{w} \in V(\hat{T}_2)$. This can be seen by taking the last vertex z_i of a Hamiltonian path in $T_{c-1} - \{x\}$ that dominates $\hat{w}$. If such a vertex does not exist, then clearly $V(\hat{T}_2) = \{\hat{w}\}$. If otherwise $z_i \in V(T'_s)$, then $V(\hat{T}_2) = \{\hat{w}, \tilde{v}_1\} \cup V(T'_2) \cup \ldots \cup V(T'_s)$. Now, it is straightforward to verify that v'_1 is the initial vertex of a Hamiltonian path in $\tilde{T}_{c-1} - \{y\}$ for an arbitrary vertex $y \in V(\tilde{T}_{c-1}) - \{v'_1\}$. According to the results in the beginning of Subcase 2.2, this yields $\hat{w} \to x$, which implies that $\tilde{T}_{c-1} - \{\tilde{v}_1\}$ is strongly connected. Following the same lines as in Subcase 2.1, we arrive at $V(\tilde{v}_1) - \{\tilde{v}_1\} \subseteq \hat{W}$, which means $V(\tilde{v}_1) \cap U = \emptyset$, a contradiction.

Subcase 2.2.2. Let $|T'_l| = 3$. If we consider the converse D^{-1} of D, then, in view of Subcase 2.2.1, it remains the case that $|T'_l| \neq 3$.

Subcase 2.2.3. Let $T'_1 = \{v'_1\}$ and $T'_l = \{v'_l\}$. In this case, the only possible cut-vertices of T_{c-1} are x, v'_1 and v'_l, which means $\hat{W} \supseteq V(D) - (V(x) \cup V(v'_1) \cup V(v'_l) \cup V(T_{c-1}) \cup V_p)$, and thus $|\hat{W}| \geq (c-4)(r-1)$.

Obviously, we have $5 \leq c \leq 7$, since otherwise, if $c \geq 8$, then it follows that $|\hat{W}| \geq 4r - 4$ and $|U| \leq 3r + 3$, and thus for $v' \in V'$

$$4r + 3 \leq |V(T_{c-1})| + |\hat{W}| \leq d^-(v') \leq d^+(v') + 1 \leq |U| + 1 \leq 3r + 4,$$

a contradiction to $r \geq 2$.

Firstly, let $c = 7$. Let $z_1 z_2 \ldots z_5$ be a Hamiltonian path in $T_{c-1} - \{x\}$ such that $z_1 = v'_1$ and $z_5 = v'_l$. If $x \to z_2$, then $T_{c-1} - \{v'_1\}$ is strongly connected and it follows that

$$4r + 2 \leq d^-(v') \leq d^+(v') + 1 \leq 2r + 3,$$

if $v' \in V'$, a contradiction. Hence, let $z_2 \to x$. Analogously, we see that $x \to z_4$. This implies that $l = 5$ and $V(T'_i) = \{z_i\}$ with $i = 1, 2, \ldots, 5$ and $z_2 \to x \to z_4$. Without loss of generality let $z_3 \to x$. Now, z_1 is a cut-vertex of T_{c-1} such that $T_{c-1} - \{z_1\}$ consists of the strong components $\hat{T}_1, \hat{T}_2, \hat{T}_3$ with $V(\hat{T}_1) \to (V(\hat{T}_2) \cup V(\hat{T}_3))$, $V(\hat{T}_2) \to V(\hat{T}_3)$ and $V(\hat{T}_3) = \{z_4, z_5, x\}$, a contradiction to Subcase 2.2.2.

Secondly, let $c = 6$. In this case, we have $l = 4$ and $V(T'_i) = \{z_i\}$ for $1 \leq i \leq 4$. Furthermore, we observe that $z_2 \to x \to z_3$, since otherwise, because of $|U| \leq 2r+2$ and $|W| \geq 3r-3$, it follows that

$$3r+2 \leq d^-(v') \leq d^+(v')+1 \leq 2r+3,$$

a contradiction to $r \geq 2$. Now, v'_4 is a cut-vertex of T_{c-1} such that $T_{c-1} - \{z_4\}$ consists of the strong components $\hat{T}_1, \hat{T}_2$ with $V(\hat{T}_1) \rightarrow V(\hat{T}_2)$ and $V(\hat{T}_1) = \{x, z_1, z_2\}$, a contradiction to Subcase 2.2.1.

Thirdly, let $c = 5$ and $V(T'_i) = \{z_i\}$ for $i = 1, 2, 3$. Without loss of generality, we may suppose that $x \rightarrow z_2$. In this case, $T_{c-1} - \{z_1\}$ is strongly connected and we arrive at $|U| + |W - \hat{W}| \leq 2r+2$ and $|\hat{W}| \geq 2r-2$. Because of $N^+(v'')$, $N^-(v') \supseteq V(T_{c-1}) \cup \hat{W}$, $N^+(v') \subseteq U$ and $N^-(v'') \subseteq U \cup (W - \hat{W})$ for all $v' \in V'$ and $v'' \in V''$, this yields that

$$\begin{aligned} 2r+2 \leq d^-(v') &\leq d^+(v')+1 \leq 2r+3 \quad \text{and} \\ 2r+2 \leq d^+(v'') &\leq d^-(v'')+1 \leq 2r+3 \end{aligned}$$

if $v' \in V'$ and $v'' \in V''$. If $W \neq \hat{W}$ or if there exist vertices $u \in U$ and $v' \in V'$ such that $u \rightarrow v'$, then we observe that

$$2r+3 \leq |\hat{W}| + 1 + |V(T_{c-1})| \leq d^-(v') \leq d^+(v')+1 \leq |U| \leq 2r+2,$$

a contradiction. Hence, we conclude that $N^+(v') = U$ for all $v' \in V'$ and $W - \hat{W} = \emptyset$. Furthermore, we observe that

$$U = (V(z_3) \cup V(x)) - \{z_3, x\} \quad \text{and} \quad W = \hat{W} = (V(z_1) \cup V(z_2)) - \{z_1, z_2\}.$$

Analogously, we see that $N^-(v'') = U$ for all $v'' \in V''$. Now, it is straightforward to see that there remain the cases that either $|U| = 2r+2$ and $|W| = 2r-2$ ($d^+(v') = d^-(v') = 2r+2$) or $|U| = 2r+1$ and $|W| = 2r-2$ ($d^-(v') = 2r+2 = d^+(v')+1$) or $|U| = 2r+2$ and $|W| = 2r-1$ ($d^-(v') = 2r+3 = d^+(v')+1$).

Clearly, x is the initial vertex and z_2 is the terminal vertex of a Hamiltonian path in $T_{c-1} - \{z_3\}$ and z_1 is the initial vertex and z_3 is the terminal vertex of a Hamiltonian path in $T_{c-1} - \{x\}$. This implies $x \rightarrow V(z_3) \cap U \rightarrow z_2$ and $z_1 \rightarrow V(x) \cap U \rightarrow z_3$, and since $U = (V(x) \cup V(z_3)) - \{z_3, x\}$, we conclude that $N^+(z_3) \cap U = N^-(x) \cap U = \emptyset$. If there is a vertex $y_1 \in N^-(x) \cap V(z_1) \cap W$, then it follows that $y_1 \rightarrow V(z_3) \cap U$ since otherwise let $y_3 \in V(z_3) \cap U$ and $v' \in V'$ such that $v' \rightarrow y_3 \rightarrow y_1$. In this case $v' \rightarrow y_3 \rightarrow y_1 \rightarrow x \rightarrow z_2 \rightarrow v'$, a contradiction. Analogously, we conclude that $y_2 \rightarrow (V(z_3) \cap U)$, if $y_2 \in N^-(x) \cap V(z_2) \cap W$.

Using $U \rightarrow V''$, analogously as above, we arrive at $V(x) \cap U \rightarrow \hat{v}_2$, if $\hat{v}_2 \in N^+(z_3) \cap V(z_2) \cap W$, and $V(x) \cap U \rightarrow \hat{v}_1$, if $\hat{v}_1 \in N^+(z_3) \cap W \cap V(z_1)$.

Let $\hat{v}_3 \in V(z_3) \cap U$ and $\hat{x} \in V(x) \cap U$. If there is a vertex $\tilde{v}_2 \in N^-(x) \cap N^+(z_3) \cap V(z_2)$, then the cycle $z_1 \hat{x} \tilde{v}_2 \hat{v}_3 v'' z_1$ is c-cycle through all c partite sets, a contradiction. If there is a vertex $\tilde{v}_1 \in N^-(x) \cap N^+(z_3) \cap V(z_1)$, then $z_2 v' \hat{x} \tilde{v}_1 \hat{v}_3 z_2$ is a c-cycle through all c partite sets, a contradiction.

Hence, it remains the case that $N^-(x) \cap N^+(z_3) \cap W = \emptyset$. Because of $2r-2 \leq |W| \leq 2r-1$, we conclude that $|N^-(x) \cap W| \leq r-1$ or $|N^+(z_3) \cap W| \leq r-1$, and thus $|N^+(x) \cap W| \geq r-1$ or $|N^-(z_3) \cap W| \geq r-1$. Since

$x \to (V(z_3) \cap U) \cup V' \cup \{z_1, z_2\}$ and $V'' \cup \{z_1, z_2\} \cup (V(x) \cap U) \to z_3$, in the case that $|N^+(x) \cap W| \geq r-1$ we obtain

$$\begin{aligned} d^+(x) &\geq |V'| + |V(z_3) \cap U| + 2 + |N^+(x) \cap W| \\ &\geq \begin{cases} 2r+3, & \text{if} \quad |V(z_3) \cap U| = r \quad \text{(that means} \quad |U| = 2r+1) \\ 2r+4, & \text{if} \quad |V(z_3) \cap U| = r+1 \end{cases} \end{aligned}$$

and in the case that $|N^-(z_3) \cap W| \geq r-1$

$$\begin{aligned} d^-(z_3) &\geq |V''| + |V(x) \cap U| + 2 + |N^-(z_3) \cap W| \\ &\geq \begin{cases} 2r+3, & \text{if} \quad |V(x) \cap U| = r \quad \text{(that means} \quad |U| = 2r+1) \\ 2r+4, & \text{if} \quad |V(x) \cap U| = r+1 \end{cases} . \end{aligned}$$

Since, as seen above, for all $v' \in V'$ we have $d^+(v') = 2r+2$, if $|U| = 2r+2$, and $d^+(v') = 2r+1$, if $|U| = 2r+1$, in all cases we arrive at a contradiction to $i_g(D) \leq 1$.

Subcase 2.2.4. For all cut-vertices let $|T_1'| > 3$ or $|T_l'| > 3$.

Firstly, we want to show that

$$|\hat{W}| \geq 3r - 3. \tag{4.7}$$

To reach this, we start with the case that $l = 2$, $|T_1'| > 3$ and $T_2' = \{v_2'\}$. Let $z_1 z_2 \ldots z_t z_1$ be a Hamiltonian cycle in T_1' such that $x \to z_1$. If v_2' is the terminal vertex of a Hamiltonian path in $T_{c-1} - \{y\}$ for all $y \in T_{c-1} - \{v_2'\}$, then, as at the beginning of Subcase 2.2, we conclude that $x \to z_t$. Analogously, it follows that $x \to z_{t-1}$ and inductively that $x \to T_1'$. This yields that x and v_2' are the only cut-vertices of T_{c-1}. Since $c \geq 7$, it follows that $|U| \leq 2r+2$ and $|W| \geq 4r-4$. Therefore we obtain for $v' \in V'$ the contradiction

$$4r + 2 \leq d^-(v') \leq d^+(v') + 1 \leq 2r + 3.$$

Consequently, it remains the case that there exists a vertex $y \in T_{c-1} - \{v_2'\}$ such that in $T_{c-1} - \{y\}$, there exists no Hamiltonian path with v_2' as terminal vertex. This implies $(T_1' - \{v_1'\}) \to x$, since otherwise let $x \to z_s$ with $s \neq 1$. If we remove an arbitrary vertex z_q from T_1', then it can easily be seen that either $x \to z_{q+1}$ or there exists a vertex z_e in the Hamiltonian path of $T_1' - \{z_q\}$ such that $z_e \to x \to z_{e+1}$. In all cases, there is a Hamiltonian path in $T_{c-1} - \{z_q\}$ with the terminal vertex v_2', a contradiction. Now, we conclude that $T_{c-1} - \{v_2'\}$ is strongly connected, that means $V(v_2') \subseteq H$ and $V(v_2') - \{v_2'\} \subseteq \hat{W}$.

If $l \geq 3$ or $|T_2'| \geq 3$, then there is at least one vertex $v_j \in V(T_{c-1}) - (V(T_1') \cup \{x\})$ such that $T_{c-1} - \{v_j\}$ is strongly connected.

Summarizing our results, we see that there always exists a vertex $y \in (V(T_2') \cup V(T_3') \cup \ldots V(T_l'))$ such that $V(y) - \{y\} \subseteq W$. Now we will show that there are at least two further vertices with this property in $V(T_1')$. Let Q_n be the tournament consisting of the path $x_1 x_2 \ldots x_n$ and all arcs $x_i x_j$ where $i > j+1$. If $T_1' \neq Q_n$, then Lemma 4.9 yields that there are three vertices y_1, y_2, $y_3 \in V(T_1')$ such that $T_1' - \{y_i\}$ is strongly connected. Two of them surely are different from v_1'. In this case, we get (4.7). Hence, let $T_1' = Q_n$. Firstly let us suppose that $x \to x_{n-1}$. In this case, x_n surely is the initial vertex of a Hamiltonian path in $T_{c-1} - \{x\}$. Furthermore, $x_n x_1 \ldots x_{n-3} T'_2 \ldots T'_l x x_{n-1}$ is a Hamiltonian path in $T_{c-1} - \{x_{n-2}\}$ with initial vertex x_n. Now, it is easy to see that x_n is the initial vertex of a Hamiltonian path $z_1 z_2 \ldots z_{c-3}$ of $T_{c-1} - \{x, y\}$ for all $y \in T_{c-1} - \{x, x_n, x_{n-2}\}$. If $z_{c-3} \to x$, then clearly x_n is the initial vertex of the Hamiltonian path $z_1 z_2 \ldots z_{c-3} x$ in $T_{c-1} - \{y\}$. Otherwise, if $x \to z_{c-3}$, then let $q = \max\{i | z_i \to x\}$. If there is a vertex $z_j \neq v'_l$ such that $z_j \to x$, then the index q exists and $z_1 z_2 \ldots z_i x z_{i+1} \ldots z_{c-3}$ is a Hamiltonian path of $T_{c-1} - \{y\}$ with the initial vertex x_n. Altogether, we have seen, that x_n is the initial vertex of a Hamiltonian path in $T_{c-1} - \{y\}$ with $y \in V(T_{c-1}) - \{x_n\}$ arbitrary (except the case that $x \to T_{c-1} - \{x, v'_l\}$, but in this case we immediately arrive at (4.7)). Furthermore, $T_{c-1} - \{x_n\}$ is strongly connected, a contradiction to Claim 2. Hence, let $x_{n-1} \to x$. In this case similarly as above we obtain that x_{n-1} is the initial vertex of a Hamiltonian path in $T_{c-1} - \{y\}$, if y is an arbitrary vertex in $V(T_{c-1}) - \{x_{n-1}\}$. If $T_{c-1} - \{x_{n-1}\}$ is strongly connected, then, as above, we arrive at a contradiction. Therefore, let $x_n \to x$. Notice that $T_{c-1} - \{x_n\}$ is strongly connected and thus $V(x_n) - \{x_n\} \subseteq \hat{W}$. If $x \to x_i$ with $i \geq 2$, then $T_{c-1} - \{x_1\}$ is also strongly connected and we arrive at (4.7). Thus, it remains the case that $T'_1 - \{x_1\} \to x$ and $x \to x_1$. Now, it is straightforward to see that x_{n-3} is a cut-vertex of T_{c-1} such that $T_{c-1} - \{x_{n-3}\}$ consists of the strong components $\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_s$ with $V(\hat{T}_i) \to V(\hat{T}_j)$, if $i < j$, and $V(\hat{T}_1) = \{x_{n-2}, x_{n-1}, x_n\}$, a contradiction as in Subcase 2.2.1.

Altogether, we have shown that (4.7) is valid, if $|T'_1| > 3$. Caused by symmetry, we arrive at the same result, if $|T'_l| > 3$.

By Lemma 4.11, it follows that there exists a vertex v of maximum outdegree in T_{c-1} such that v is the initial vertex of a Hamiltonian path in $T_{c-1} - \{y\}$, if $y \in V(T_{c-1}) - \{v\}$ is an arbitrary cut-vertex of T_{c-1}. If we remove a further vertex $y' \in V(T_{c-1}) - \{y, v\}$, which is not a cut-vertex of T_{c-1}, then we will show that v is still an initial vertex of a Hamiltonian path in $T_{c-1} - \{y, y'\}$.

Let us consider the digraph $T_{c-1} - \{y\}$, for an arbitrary cut-vertex $y \neq v$ of T_{c-1}. According to Lemma 4.11, it follows that v is the initial vertex of a Hamiltonian path in $T_{c-1} - \{y\}$. If $\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_t$ are the strong components of $T_{c-1} - \{y\}$ such that $V(\tilde{T}_i) \to V(\tilde{T}_j)$, if $i < j$, then we conclude that $v \in V(\tilde{T}_1)$. Now, let $y' \in V(T_{c-1}) - \{v, y\}$ be a vertex such that $T_{c-1} - \{y'\}$ is strongly connected. If $y' \in V(\tilde{T}_i)$ for $i \geq 2$, then it is easy to see that v is still the initial vertex of a Hamiltonian path in $T_{c-1} - \{y, y'\}$. Thus, let $y' \in V(\tilde{T}_1)$. If y' is not a cut-vertex of $\tilde{T}_1$, then v is also an initial vertex of a Hamiltonian path in $T_{c-1} - \{y, y'\}$. Consequently, let y' be a cut-vertex of $\tilde{T}_1$ with $|\tilde{T}_1| \geq 3$ (because of Subcase 2.2.1, we may even suppose that $|\tilde{T}_1| \geq 4$) such that $\tilde{T}_1 - \{y'\}$ consists of the strong components $\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_q$ with $V(\hat{T}_i) \to V(\hat{T}_j)$, if $i < j$. In this case, there exist vertices $\hat{z}_1 \in V(\hat{T}_1)$ and $\hat{z}_q \in V(\hat{T}_q)$ such that $\hat{z}_q \to y' \to \hat{z}_1$. Since y' is not a cut-vertex of T_{c-1}, there exists a vertex $\hat{z}'_1 \in V(\hat{T}_1)$ such that $y \to \hat{z}'_1$. If $v \in V(\hat{T}_1)$, then it is obvious that v is the initial vertex of a Hamiltonian path in $T_{c-1} - \{y, y'\}$. If $v \in V(\hat{T}_i)$ with $i \neq q$, then it is easy to see that $T_{c-1} - \{v\}$ is strongly connected and we arrive at a contradiction to Claim 2. If $v \in V(\hat{T}_i)$ with $i \geq 4$

or if $v \in V(\hat{T}_3)$ and $|V(\hat{T}_i)| \geq 3$ for at least one i with $1 \leq i \leq 3$ or if $v \in V(\hat{T}_2)$ with $|V(\hat{T}_1)| \geq 4$ or $|V(\hat{T}_1)|, |V(\hat{T}_2)| \geq 3$, then we arrive at a contradiction to the fact that the vertex v is of maximum outdegree in T_{c-1}. If $\{v\} = V(\hat{T}_3)$ ($q = 3$) and $|V(\hat{T}_1)| = |V(\hat{T}_2)| = 1$ or if $\{v\} = V(\hat{T}_2)$ ($q = 2$) and $|V(\hat{T}_1)| = 3$ or if $v \in V(\hat{T}_2)$ with $|V(\hat{T}_2)| \geq 3$ and $|V(\hat{T}_1)| = 1$, then the condition that v is of maximum outdegree in T_{c-1} implies that $v \rightarrow \{y, y'\}$ and there are at least two vertices that dominate y'. One of these vertices is v and the other one is in $\{y\} \cup (V(\tilde{T}_1) - \{y', v, \hat{z}_1\})$. In all cases, it is straightforward to show that $T_{c-1} - \{v\}$ is strongly connected, a contradiction to Claim 2. Finally, it remains the case that $\{v\} = V(\hat{T}_2)$ and $|V(\hat{T}_1)| = 1$. But now, we observe that $|V(\tilde{T}_1)| = 3$ and Subcase 2.2.1 yields a contradiction.

Summarizing our results, we see that there exists a vertex v that is the initial vertex of a Hamiltonian path in $T_{c-1} - \{\tilde{y}\}$ and in $T_{c-1} - \{y, y'\}$ with $y, y', \tilde{y} \in V(T_{c-1}) - \{v\}$ arbitrary such that y is a cut-vertex of T_{c-1} and y' is not a cut-vertex of T_{c-1}.

If $\hat{W}$ consists of vertices of at least four partite sets, that means $|\hat{W}| \geq 4r - 4$, then analogously as in Subcase 2.2.1 with $x \rightarrow \{v_1', \tilde{v}_1, \hat{v}_1\}$, we arrive at a contradiction, since $d^+_{D[V(T_{c-1})]}(v) \geq 3$. Hence, let $\hat{W}$ consist of vertices of exactly three partite sets, that means

$$3r - 3 \leq |\hat{W}| \leq 3r + 3. \tag{4.8}$$

According to Claim 1, it follows that $v \rightsquigarrow U$. If $\tilde{c} = |U \cap V(v)|$ and $v' \in V'$, then, using Remark 1.18, we observe that

$$\begin{aligned} d^+(v') + 1 &\geq d^+(v) = |U| - \tilde{c} + |V'| + d^+_{D[V(T_{c-1})]}(v) + |N^+(v) \cap W| \\ &\geq d^+(v') - r + 2 + d^+_{D[V(T_{c-1})]}(v) + |N^+(v) \cap W|, \end{aligned}$$

if $|V(v)| \leq r + 1$ and

$$\begin{aligned} d^+(v') &\geq d^+(v) = |U| - \tilde{c} + |V'| + d^+_{D[V(T_{c-1})]}(v) + |N^+(v) \cap W| \\ &\geq d^+(v') - (r + 1) + 2 + d^+_{D[V(T_{c-1})]}(v) + |N^+(v) \cap W|, \end{aligned}$$

if $|V(v)| = r + 2$. Both cases lead to

$$|N^+(v) \cap W| \leq r - 1 - d^+_{D[V(T_{c-1})]}(v).$$

This implies that for all $\hat{w} \in \hat{W}$ except at most $r - 1 - d^+_{D[V(T_{c-1})]}(v)$ vertices, it has to be $\hat{w} \rightarrow v$.

If $\tilde{w} \in V(v_i) \cap \hat{W} \cap N^-(v)$ with $v_i \in V(T_{c-1})$ and if there are vertices $u \in (U - V(v)) \cap V(v_j)$ with $v_j \in V(T_{c-1})$ and $v' \in V'$ such that $v' \rightarrow u \rightarrow \tilde{w}$, then our considerations above imply that $D[(V(T_{c-1}) - \{v_i, v_j\}) \cup \{u, \tilde{w}, v'\}]$ is a strong tournament of order c, a contradiction. Thus, we have $(\hat{W} \cap N^-(v)) \rightarrow (U - V(v))$.

Using (4.8), we will show next that there is at least one vertex $\tilde{w} \in \hat{W} \cap N^-(v)$ such that

$$\begin{aligned} |N^+(\tilde{w}) \cap W| &\geq \frac{2r - 2 - (r - 1 - d^+_{D[V(T_{c-1})]}(v))}{2} \\ &= \frac{r}{2} + \frac{d^+_{D[V(T_{c-1})]}(v) - 1}{2}. \end{aligned} \tag{4.9}$$

Let F be the set $\hat{W}$ without the at most $r - 1 - d^+_{D[V(T_{c-1})]}(x)$ vertices, which are dominated by v. For every vertex $f \in F$ we conclude that

$$d^+_{D[F]}(f) + d^-_{D[F]}(f) \geq 2r - 2 - (r - 1 - d^+_{D[V(T_{c-1})]}(v)).$$

This implies

$$\begin{aligned} 2\sum_{f \in F} d^+_{D[F]}(f) &= \sum_{f \in F}(d^+_{D[F]}(f) + d^-_{D[F]}(f)) \\ &\geq |F|(2r - 2 - (r - 1 - d^+_{D[V(T_{c-1})]}(v))). \end{aligned}$$

This leads to (4.9). Firstly, let there be at least one vertex $f_0 \in F$ such that $d^+_{D[F]}(f_0) + d^-_{D[F]}(f_0) > 2r - 2 - (r - 1 - d^+_{D[V(T_{c-1})]}(v))$. Then we even have $2\sum_{f \in F} d^+_{D[F]}(f) > |F|(2r - 2 - (r - 1 - d^+_{D[V(T_{c-1})]}(v)))$, which immediately yields that there is at least one vertex $\tilde{w} \in \hat{W} \cap N^-(v)$ such that

$$|N^+(\tilde{w}) \cap W| > \frac{r}{2} + \frac{d^+_{D[V(T_{c-1})]}(v) - 1}{2}. \tag{4.10}$$

Because of (4.10), we obtain

$$|U| - \tilde{c} + |V'| + \frac{r}{2} + \frac{d^+_{D[V(T_{c-1})]}(v) - 1}{2} < d^+(\tilde{w}) \leq d^+(v') + 1 \leq |U| + 1,$$

which leads to

$$|V'| + \frac{r}{2} + \frac{d^+_{D[V(T_{c-1})]}(v) - 3}{2} < \tilde{c} \leq r + 1,$$

and this yields

$$|V'| \leq \frac{r + 4 - d^+_{D[V(T_{c-1})]}(v)}{2}. \tag{4.11}$$

Let w be a vertex of maximum indegree such that w is the terminal vertex of a Hamiltonian path in $T_{c-1} - \{y\}$, if y is an arbitrary vertex in $V(T_{c-1}) - \{w\}$. Considering the converse D^{-1} of D, we conclude that $|V''| \leq \frac{r+4-d^-_{D[V(T_{c-1})]}(w)}{2}$. Combining this with (4.11), we arrive at

$$d^+_{D[V(T_{c-1})]}(v) + d^-_{D[V(T_{c-1})]}(w) \leq 8, \tag{4.12}$$

since otherwise by (4.11) we obtain the contradiction

$$|V_p| = |V'| + |V''| \leq \frac{r + 4 - d^+_{D[V(T_{c-1})]}(v)}{2} + \frac{r + 4 - d^-_{D[V(T_{c-1})]}(w)}{2} < r.$$

Let $x \notin \{v, w\}$ be a cut-vertex of T_{c-1}. If $T_{c-1} - \{x\}$ consists of the strong components $T'_1, T'_2, \ldots, T'_l$ with $V(T'_i) \to V(T'_j)$ for $i < j$ such that $|T'_1| > 3$ or $|T'_l| > 3$ and $l \geq 3$ or $c \geq 9$, then it is easy to see that $v \in V(T'_1)$ and $w \in V(T'_l)$ with $d^+_{D[V(T_{c-1})]}(v) + d^-_{D[V(T_{c-1})]}(w) \geq 9$, a contradiction to (4.12).

Because of this, there remains the case $l = 2$ and $T'_2 = \{v'_2\}$ to discuss, if $|T'_1| > 3$. The case that $|T'_l| > 3$ follows analogously. Since, according to our assumptions, v'_2 is the terminal vertex of a Hamiltonian path in $T_{c-1} - \{y\}$, if $y \in V(T_{c-1}) - \{v'_2\}$ is an arbitrary vertex, as in the beginning of this subcase, we arrive at a contradiction.

Hence, according to the proof of (4.9), there remains to treat the case that (4.9) is fulfilled with equality for all $\tilde{w} \in \hat{W} \cap N^-(v)$. This is possible, only if $|\hat{W}| = 3r - 3$, $d^+_{D[V(T_{c-1})]}(v) = r - 1$, which means that $|N^+(v) \cap W| = 0$, and if $D[\hat{W}]$ is an $(r-1)$-regular 3-partite tournament. Let $\hat{w} \in \hat{W}$ be an arbitrary vertex. Then we obtain

$$r + 2 + |U| - \tilde{c} \leq |U| - \tilde{c} + |\{v\}| + |V'| + r - 1 \leq d^+(\hat{w}) \leq d^+(v') + 1 \leq |U| + 1.$$

This implies that $\tilde{c} = r + 1$, $|V'| = 2$, $T_{c-1} - \{v\} \rightsquigarrow \hat{W}$ and $d^+(\hat{w}) = d^+(v') + 1$ for all $\hat{w} \in \hat{W}$ and $v' \in V'$. According to Remark 1.18 it follows that $|V(v')| \geq r + 1$. Observing the converse D^{-1} of D we conclude that $|V''| = 2$, and thus $|V(v')| = |V'| + |V''| = 4 \geq r + 1 \Leftrightarrow r \leq 3$. Since in this subcase obviously $|T_{c-1}| \geq 6$, and thus $c \geq 7$, we arrive at the contradiction

$$\frac{5}{2} \leq \frac{c-2}{2} \leq d^+_{D[V(T_{c-1})]}(v) = r - 1 \leq 2.$$

This completes the proof of the theorem. □

Combining this result with Theorem 4.3 or Theorem 4.5 we arrive at the following corollary.

Corollary 4.13 (Volkmann, Winzen [39]) *Let D be an almost regular c-partite tournament with $c \geq 5$. Then D contains a strongly connected subtournament of order p for every $p \in \{3, 4, \ldots, c\}$.*

4.2 Strong subtournaments when $i_g(D) \geq 2$

In this section we want to deal with the following problem, which was posed by L. Volkmann [29] in 1999 (see also [31], Problem 2.32).

Problem 4.14 (Volkmann [29]) *Determine other sufficient conditions for (strongly connected) c-partite tournaments to contain strong subtournaments of order p for some $4 \leq p \leq c$.*

The complexity of the proof of Theorem 4.12 makes it clear that the statement of this theorem becomes false, if we enlarge $i_g(D)$ without changing the rest of the parameters. This also demonstrates the following example.

Example 4.15 (Winzen [47]) *Let the partite sets of a multipartite tournament D be defined as $V_1 = \{z_1\}$, $V_2 = \{z_2\}$, $V_3 = \{z_3, \hat{z}_3\}$, $V_4 = \{x, \hat{x}\}$ and $V_5 = \{v', v''\}$ such that $x \to z_1 \to z_2 \to z_3 \to x \to z_2$, $z_1 \to z_3$, $x \to \hat{z}_3 \to \{z_1, z_2\} \to \hat{x} \to z_3$, $\{x, z_1, z_2, z_3\} \to v' \to \{\hat{z}_3, \hat{x}\} \to v'' \to \{x, z_1, z_2, z_3\}$ and $\hat{z}_3 \to \hat{x}$ (see also Figure 4.1). The resulting 5-partite tournament D with $i_g(D) \leq 2$ does not contain a strong subtournament of order 5.*

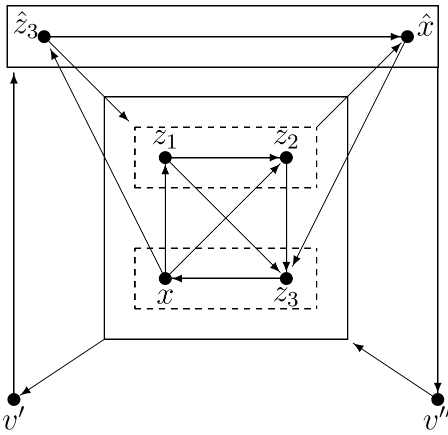

Figure 4.1: A 5-partite tournament with $i_g(D) = 2$ and without a strong subtournament of order 5.

It is very probably that the size of strong subtournaments decreases, if the global irregularity increases. In this section we will present a result that guarantees strong subtournaments of a size depending on the global irregularity $i_g(D)$. Besides the results of the previous section, we need the following two lemmas.

Theorem 4.16 (Yeo [49]) *If D is a multipartite tournament, then*

$$\kappa(D) \geq \frac{|V(D)| - \alpha(D) - 2i_l(D)}{3}.$$

Lemma 4.17 (Winzen [47]) *If D is a c-partite tournament with $r \geq 2$ vertices in each partite set, then there are vertices $x, y \in V(D)$ such that $d^-(x), d^+(y) \geq c - 1$.*

Proof. For every vertex $x \in V(D)$ we observe that

$$d^-(x) + d^+(x) = |V(D)| - |V(x)| \geq (c-1)r \geq 2(c-1).$$

Counting all outdegrees and indegrees in D we obtain that

$$2 \sum_{x \in V(D)} d^+(x) = 2 \sum_{x \in V(D)} d^-(x) = \sum_{x \in V(D)} (d^+(x) + d^-(x)) \geq |V(D)| 2(c-1),$$

which immediately implies the statement of this lemma. □

Following the same lines as in the proof of Theorem 4.5 we will show the following main theorem of this section.

Theorem 4.18 (Winzen [47]) *Let D be a c-partite tournament with at least 3 vertices in each partite set, $i_g(D) \leq l$, $c \geq l+2$ and $l \geq 2$. Then D contains a strongly connected subtournament of order p for every $p \in \{3, 4, \ldots, c-l+1\}$.*

Proof. Let $V_1, V_2, \ldots, V_c$ be the partite sets of D and let $r = \gamma(D)$. Because of Lemma 1.10 we obtain $3 \leq r \leq |V_i| \leq r + 2l$ for all $i \in \{1, 2, \ldots, c\}$, and thus we have $|V(D)| = cr + k$ with $0 \leq k \leq 2l(c-1)$. We proceed the proof by induction on the order p of strongly connected subtournaments. Theorem 4.16 yields that

$$\begin{aligned} \kappa(D) &\geq \frac{cr + k - \alpha(D) - 2i_l(D)}{3} \geq \frac{(c-1)r - 2i_g(D)}{3} \\ &\geq \frac{(l+1)r - 2l}{3} \geq \frac{3l + 3 - 2l}{3} = 1 + \frac{l}{3} > 1. \end{aligned}$$

This implies that D is strongly connected. Hence, according to Theorem 4.1, there exists a 3-cycle in D, which is a strong subtournament of order 3.

Now, let $c \geq l + 3$ and T_p be a strong subtournament of order p with $3 \leq p \leq c - l$. Suppose that D does not contain a strong subtournament of order $p+1$. Without loss of generality, we assume that $T_p = D[\{v_1, v_2, \ldots, v_p\}]$ with $v_i \in V_i$ for $i = 1, 2, \ldots, p$. If there is a vertex $z \in V_{p+1}, V_{p+2}, \ldots, V_c$ such that z has an inner and an outer neighbor in T_p, then $D[\{z, v_1, v_2, \ldots, v_p\}]$ is a strong subtournament of order $p + 1$, a contradiction. If such a vertex does not exist, then let $V_i' \subseteq V_i$ and $V_i'' = V_i - V_i'$ such that $V(T_p) \to V_i'$ when $V_i' \neq \emptyset$, and $V_i'' \to V(T_p)$ when $V_i'' \neq \emptyset$, for $i = p+1, p+2, \ldots, c$. In addition, we define $V' = V_{p+1}' \cup V_{p+2}' \cup \ldots \cup V_c'$ and $V'' = V_{p+1}'' \cup V_{p+2}'' \cup \ldots \cup V_c''$. Now we distinguish two cases.

Case 1. Let $V' \neq \emptyset$ and $V'' \neq \emptyset$. If there is an arc xy with $x \in V'$ and $y \in V''$, then $D[\{x, y, v_1, v_2, \ldots, v_p\}]$ is a strong subtournament of order $p + 2$. As a consequence of Theorem 4.3, we see immediately that there also exists a strong subtournament of order $p + 1$, a contradiction. Hence, we conclude that $V'' \rightsquigarrow V'$. Furthermore, let $R = V(D) - (V' \cup V'' \cup V(T_p))$ and $|V_i'| = t_i$ for $p + 1 \leq i \leq c$, and without loss of generality, we assume that $t_{p+1} \geq t_{p+2} \geq \ldots \geq t_c$.

Subcase 1.1. Let $V_c'' \neq \emptyset$. In this case, we choose the index s such that

$$\begin{cases} t_s \geq 2 \wedge t_{s+1} \leq 1, & \text{if} \quad t_{p+1} \geq 2 \wedge t_c \leq 1 \\ s = c - 1 \quad , & \text{if} \quad t_c \geq 2 \\ s = p + 1 \quad , & \text{if} \quad t_{p+1} \leq 1 \end{cases}.$$

Let $v \in V(D')$ with $D' = D[V_{p+1}' \cup V_{p+2}' \cup \ldots \cup V_s']$ such that v is of maximum indegree in D'. Furthermore let $w \in V(D'')$ with $D'' = D[V_{s+1}'' \cup V_{s+2}'' \cup \ldots \cup V_c'']$ a vertex of maximum outdegree in D''. Since each of the vertex-sets $V_{s+1}', V_{s+2}', \ldots, V_c'$ consists of at most one vertex (for the case that $s \neq c-1$), and because of $r \geq 3$, each of the vertex-sets $V_{s+1}'', V_{s+2}'', \ldots, V_c''$ (for $s \neq c-1$) has to consist of at least two vertices. Hence, according to the choice of the parameter s, Lemma 4.17 yields that $d_{D'}^-(v) \geq s - p - 1$ and $d_{D''}^+(w) \geq c - s - 1$ (even if $s = c - 1$). Let $v \in V_i$ and $w \in V_j$. If $|V_j| = r + b$ and $d^-(w) = \frac{|V(D)| - r - a}{2}$, then it follows that $d^+(w) = |V(D)| - r - b - d^-(w) = \frac{|V(D)| - r + a - 2b}{2}$

and because of $i_g(D) \leq l$ we arrive at $d^+(v) \geq \frac{|V(D)|-r+a-2(b+l)}{2}$. Summarizing our results we observe that

$$\begin{aligned} |N^+(v) \cap R| &\geq \\ & \max\left\{0, \frac{|V(D)|-r+a-2(b+l)}{2} - \sum_{\substack{m=p+1 \\ m \neq i}}^{c} (t_m) + s - p - 1\right\} \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} |N^-(w) \cap R| &\geq \\ & \max\left\{0, \frac{|V(D)|-r-a}{2} - \sum_{\substack{m=p+1 \\ m \neq j}}^{c} (r - t_m) - s_1 + c - s - 1\right\} \end{aligned} \tag{4.14}$$

with $0 \leq s_1 \leq \min\{k-b, 2l(c-p-1)\}$ such that $|(V' \cup V'') - V_j| = (c-p-1)r + s_1$. If $|R| = pr - p + s_2$, then we observe that $0 \leq s_2 \leq \min\{k-b, 2lp\}$ and $s_1 + s_2 \leq k - b$. Because of $t_i \geq t_j$, $s_1 + s_2 \leq k - b$, $p \geq 3$ and $c - p \geq l$, (4.13) and (4.14) imply that

$$\begin{aligned} |N^+(v) \cap R| + |N^-(w) \cap R| &\geq (c-1)r + k - b - l + t_i - t_j - s_1 \\ &\quad -(c-p-1)r + c - s - 1 + s - p - 1 \\ &\geq pr + k - b - s_1 - 2 \geq pr + s_2 - 2 \\ &\geq pr - p + s_2 + 1 = |R| + 1. \end{aligned}$$

Hence, there exists a vertex $x \in ((N^+(v) \cap R) \cap (N^-(w) \cap R))$. Without loss of generality, let $x \in V_1$. Since $V(T_p) \to v$ and $w \to V(T_p)$, and since v and w are in different partite sets, we conclude that $D[\{v, x, w, v_3, v_4, \ldots, v_p\}]$ is a strongly connected tournament of order $p + 1$, a contradiction.

Subcase 1.2. Let $V_c'' = \emptyset$. This implies $V_c' = V_c$ and $t_{p+1} \geq t_{p+2} \geq \ldots \geq t_c = |V_c| \geq r$. If $|V'| = (c-p)r + l_1$ and $|V''| = l_2$, then it follows that $1 \leq l_1 + l_2 \leq \min\{k, 2l(c-p)\}$. Let $w \in V''_{j_{max}}$ with $j_{max} \in \{p+1, p+2, \ldots, c\}$ such that $|V''_{j_{max}}| =: t''_{max}$ is maximal. According to Lemma 4.17, there is a vertex $v \in V' - V'_{j_{max}}$ such that $d^+_{D[V']}(v) \geq c - p - 2 \geq l - 2$. If $|V(v)| = r + b$ and $d^+(v) = \frac{|V(D)|-r-a}{2}$, then, analogously as in Subcase 1.1, we see that $d^-(w) \geq \frac{|V(D)|-r+a-2(b+l)}{2}$, and we conclude that

$$\begin{aligned} |N^+(v) \cap R| &\geq \\ & \max\left\{0, \frac{|V(D)|-r-a}{2} - (c-p-1)r - l_1 + b - t''_{max} + l - 2\right\} \end{aligned} \tag{4.15}$$

and

$$|N^-(w) \cap R| \geq \max\left\{0, \frac{|V(D)|-r+a-2(b+l)}{2} - l_2 + t''_{max}\right\}. \tag{4.16}$$

If $|R| = pr - p + s_2$, then it follows that $0 \leq s_2 \leq \min\{k, 2lp\}$ and $s_2 + l_1 + l_2 \leq k$. Analogously as in Subcase 1.1, we obtain by (4.15) and (4.16) that $|N^+(v) \cap R| + |N^-(w) \cap R| > |R|$. Hence, again there exists a vertex $x \in ((N^+(v) \cap R) \cap (N^-(w) \cap R))$. If, without loss of generality, $x \in V_1$, then, since v and w are in different partite sets, $D[\{v, x, w, v_3, v_4, \ldots, v_p\}]$ is a strong subtournament of order $p+1$, a contradiction.

Case 2. Let $V' = \emptyset$ or $V'' = \emptyset$. Without loss of generality, we discuss the case that $V'' = \emptyset$. This implies that $V_i' = V_i$ for $p+1 \leq i \leq c$, and we write V instead of V'. Let U contain all vertices of $V(D) - (V \cup V(T_p))$ that are dominated by at least one vertex of V, and let W be the set of vertices from $V(D) - (V \cup V(T_p))$, which are not dominated by any vertex from V. Thus, $W \rightarrow V$ and hence it follows that $d(V, V(D) - V) \leq |V||U|$ and $d(V(D) - V, V) \geq |V||V(D) - (U \cup V)|$. Now Lemma 4.10 yields that $l|V| \geq d(V(D) - V, V) - d(V, V(D) - V) \geq |V|(|V(D)| - |V| - 2|U|)$, and this implies that

$$|U| \geq \frac{|V(D)| - |V| - l}{2}. \tag{4.17}$$

We now consider the following two subcases.

Subcase 2.1. Let $p = 3$. Consider that V consists of $c - p \geq l$ partite sets.

Suppose firstly that there is a vertex $w \in W$ that dominates two vertices of $V(T_p)$. This implies $w \rightsquigarrow U$, since otherwise let $v' \in V$ and $u \in U$ such that $v' \rightarrow u \rightarrow w$. In this case v', u, w and the vertex of $V(T_p)$, which is dominated by w and is in another partite set than u induce a strong tournament of order 4, a contradiction. According to Lemma 4.17, there is a vertex $v' \in V$ such that $d^-_{D[V]}(v') \geq l-1$. Hence, we arrive at $d^-(v') \geq l-1+|W|+|V(T_p)| = |W|+l+2$ and $d^-(w) \leq |W - \{w\}| = |W| - 1$, a contradiction to $i_g(D) \leq l$. Since each vertex $w \in W$ has exactly two neighbors in $V(T_p)$, we conclude that $d(W, V(T_p)) \leq d(V(T_p), W)$.

Now let there be a vertex $u \in U$ that dominates two vertices in $V(T_p)$. This yields that u, a vertex of V, which dominates u and the two vertices of $V(T_p)$ not belonging to the same partite set as u induce a strongly connected subtournament of order 4, a contradiction. Analogously as for the set W, we conclude that $d(U, V(T_p)) \leq d(V(T_p), U)$.

Together with Lemma 4.10 we obtain

$$\begin{aligned} 3l = l|V(T_p)| &\geq d(V(T_p), V(D) - V(T_p)) - d(V(D) - V(T_p), V(T_p)) \\ &= d(V(T_p), V) + d(V(T_p), U) + d(V(T_p), W) \\ &\quad -d(V, V(T_p)) - d(U, V(T_p)) - d(W, V(T_p)) \\ &\geq d(V(T_p), V) = |V(T_p)||V| \geq 9l, \end{aligned}$$

a contradiction.

Subcase 2.2 Let $p \geq 4$. According to Lemma 4.11, there is a vertex $v \in V(T_p)$ such that for all $y \in V(T_p) - \{v\}$ the subtournament $T_p - \{y\}$ contains a Hamiltonian path with the initial vertex v. If there is a vertex $u \in U$ such that $u \rightarrow v$, then let $w \in V$ with $w \rightarrow u$. If $u \in V_t$, then w, u and v_j with $1 \leq j \leq p$ and $j \neq t$ induce a strongly connected subtournament of order $p+1$, a contradiction. If otherwise, there is no such vertex u, then clearly $v \rightsquigarrow U$. By Lemma 4.11, the vertex v is of maximum outdegree in T_p and thus

$d^+_{D[V(T_p)]}(v) \geq 2$. If $v \in V_i$ with $|V_i| = r + 2l - m$ $(0 \leq m \leq 2l)$, then, because of $|V| \geq lr$, $r \geq 3$, (4.17) and Lemma 1.15, we arrive at the contradiction

$$\begin{aligned}
\frac{|V(D)| - |V_i| + m}{2} &\geq d^+(v) \geq |V| + |U - (V_i - \{v\})| + d^+_{D[V(T_p)]}(v) \\
&\geq |V| + \frac{|V(D)| - |V| - l}{2} - |V_i| + 3 \\
&= \frac{|V(D)| - |V_i| + m + 1}{2} + \frac{|V| - |V_i| - m - l + 5}{2} \\
&\geq \frac{|V(D)| - |V_i| + m + 1}{2} + \frac{(l-1)r - 3l + 5}{2} \\
&> \frac{|V(D)| - |V_i| + m + 1}{2} + \frac{(l-1)(r-3)}{2} \\
&\geq \frac{|V(D)| - |V_i| + m + 1}{2}.
\end{aligned}$$

This completes the proof of the theorem. □

Neglecting a finite family of multipartite tournaments, Theorem 4.18 enlarges Theorem 4.5 to classes of multipartite tournaments with $i_g(D) \leq l$ for $l \geq 2$. If we omit the condition of Theorem 4.18 that there are at least three vertices in each partite set, then the proof becomes much more complicated. Nevertheless, we believe that also in this case the theorem remains valid, if D is strongly connected and the number of partite sets is sufficiently large.

Conjecture 4.19 (Winzen [47]) *If D is a strongly connected c-partite tournament with c sufficiently large and $i_g(D) \leq l$. Then D contains a strongly connected subtournament of order p for every $p \in \{3, 4, \ldots, c - l + 1\}$.*

If Conjecture 4.19 is valid, then the bounds for c and p as in Theorem 4.18 would be best possible as the following example demonstrates.

Example 4.20 *Let $V_1 = \{v_1\}$, $V_2 = \{v_2\}$, $V_3 = \{v_3\}$ and $V_4 = \{v_4, v_4'\}$ be the partite sets of the multipartite tournament D such that $v_1 \to v_2 \to v_3 \to v_4' \to v_2 \to v_4 \to v_1 \to v_3$, $v_1 \to v_4'$ and $v_4 \to v_3$ (see also Figure 4.2). Then we observe that D is a strongly connected c-partite tournament with $i_g(D) = l = 2$, $c = 4 = l + 2$ and without any strong subtournament of order $4 = c - l + 2$.*

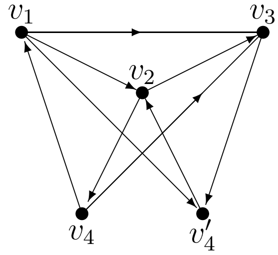

Figure 4.2: A 4-partite tournament with $i_g(D) = 2$ and without a strong subtournament of order 4.

Even if we enlarge the number c of partite sets of a multipartite tournament D, then there is not always a strong subtournament of order $c - i_g(D) + 2$, which can be seen in Example 4.15.

4.3 Long cycles

In this section, we will treat the case that D is a regular c-partite tournament. For this class of digraphs, we will give a solution of Problem 4.2 with $r_i = |V_i| - 1$ for all $1 \leq i \leq c$. According to Corollary 4.13, we may suppose that D has at least 3 vertices in each partite set, if $c \geq 5$. For the bipartite case let us define a special family of graphs.

Definition 4.21 *[Bipartite tournament $B(r_1, r_2, r_3, r_4)$]* Let R_1, R_2, R_3, R_4 be pairwise disjoint independent sets of vertices with $|R_i| = r_i$ for $1 \leq i \leq 4$. Then the bipartite tournament $B = B(r_1, r_2, r_3, r_4)$ is defined by $V(B) = R_1 \cup R_2 \cup R_3 \cup R_4$ such that $R_i \rightarrow R_{i+1}$ for $i = 1, 2, 3$ and $R_4 \rightarrow R_1$.

Since the vertices of a cycle in a bipartite tournament D alternate between the two partite sets of D, Beineke and Little [5] (for a stronger form, see also Zhang [53]) gave a solution to our problem, if $c = 2$.

Theorem 4.22 (Beineke, Little [5]) *A bipartite tournament is even pancyclic, if it is Hamiltonian and is not isomorphic to the bipartite tournament $B(r, r, r, r)$ with $r \geq 2$.*

If we remove one vertex of each partite set in the bipartite tournament $B(r, r, r, r)$, then obviously the remaining bipartite tournament is not Hamiltonian. The case that $c = 3$ is also solved, if we pay attention to the next result.

Theorem 4.23 (Volkmann [35]) *Let D be a regular 3-partite tournament with $|V(D)| \geq 6$. Then D contains two complementary cycles of length 3 and $|V(D)| - 3$, unless D is isomorphic to the digraph $D_{3,2}$ of Figure 4.3.*

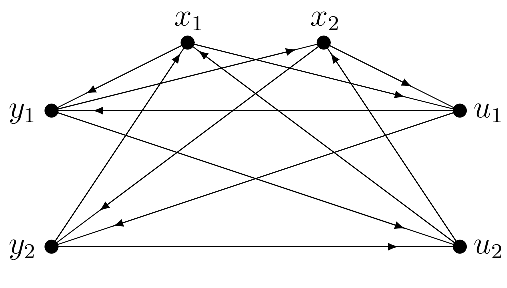

Figure 4.3: The 2-regular 3-partite tournament $D_{3,2}$

Since a 3-cycle contains vertices of exactly 3 partite sets and the digraph $D_{3,2}$ contains the cycle $x_2y_2u_2x_2$, we see that a regular 3-partite tournament with r vertices of each partite set always contains a cycle with exactly $r-1$ vertices of every partite set.

In the following, we will show that all regular c-partite tournaments with r vertices in every partite set contain a cycle with exactly $r-1$ vertices of each partite set, if $c \geq 5$ or $c = 4$ and $r \geq 3$. To reach this, we need the following results about Hamiltonicity. We start with the following well-known result of Rédei.

Theorem 4.24 (Rédei [21]) *Every tournament has a Hamiltonian path.*

Theorem 4.25 (Yeo [48]) *If D is a multipartite tournament with $\kappa(D) \geq \alpha(D)$, then D is Hamiltonian.*

Theorem 4.26 (Camion [7]) *A tournament is strongly connected, if and only if it is Hamiltonian.*

Theorem 4.27 (Yeo [48]) *Let D be a $(\lfloor q/2 \rfloor + 1)$-connected c-partite tournament such that $\alpha(D) \leq q$. If D has a cycle-factor, then D is Hamiltonian.*

Theorem 4.28 (Yeo [51]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament D such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$. If*

$$\begin{aligned} i_l(D) &\leq \min\left\{|V(D)| - 3|V_c| + 1, \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 2}{2}\right\} \quad \textit{or} \\ i_g(D) &\leq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 2}{2}, \end{aligned}$$

then D is Hamiltonian.

Lemma 4.29 (Yeo [49]; Gutin, Yeo [16]) *A digraph D has no cycle-factor (respectively, with $pc(D) > k \geq 1$) if and only if its vertex set $V(D)$ can be partitioned into four subsets Y, Z, R_1, and R_2 such that*

$$R_1 \rightsquigarrow Y \quad \textit{and} \quad (R_1 \cup Y) \rightsquigarrow R_2, \tag{4.18}$$

where Y is an independent set and $|Y| > |Z|$ (respectively, $|Y| > |Z| + k$).

First, let us investigate the case that we can choose the vertices the desired cycle consists of.

Theorem 4.30 (Volkmann, Winzen [43]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D with $c \geq 4$ and $|V_1| = |V_2| = \ldots = |V_c| = r \geq 2$. Furthermore, let X be an arbitrary subset of $V(D)$ consisting of m partite sets with exactly k vertices and $c - m$ partite sets with exactly $k-1$ vertices for $0 < m \leq c$ and $1 \leq k \leq r-1$. If*

$$r \geq \begin{cases} \left\lceil \frac{2k(c-1)-2}{c-3} \right\rceil + k & \textit{and} \quad m = c \\ \left\lceil \frac{2k(c-1)-1}{c-3} \right\rceil + k & \textit{and} \quad m = c-1 \\ \left\lceil \frac{(2k-3)c+3m-2k+3}{c-3} \right\rceil + k & \textit{and} \quad m \leq c-2 \end{cases},$$

then D contains a cycle C such that $V(C) = V(D) - X$.

Proof. Let $D' = D - X$ with the partite sets $V_1', V_2', \ldots, V_c'$ such that $|V_1'| \leq |V_2'| \leq \ldots \leq |V_c'| \leq |V_1'| + 1$. Since D is regular, it follows that

$$i_g(D') \leq \begin{cases} k(c-1), & \text{if} \quad c-1 \leq m \leq c \\ (k-1)(c-1)+m, & \text{if} \quad m \leq c-2 \end{cases}.$$

If

$$\begin{cases} k(c-1) \\ (k-1)(c-1)+m \end{cases} \leq \frac{|V(D')| - |V_{c-1}'| - 2|V_c'| + 2}{2} \begin{array}{l} \text{, if} \quad c-1 \leq m \leq c \\ \text{, if} \quad m \leq c-2 \end{array},$$

then Theorem 4.28 implies that D' is Hamiltonian, and hence the desired result. To show this, let us note that

$$\frac{|V(D')| - |V_{c-1}'| - 2|V_c'| + 2}{2} = \begin{cases} \frac{(c-3)(r-k)+2}{2}, & \text{if} \quad m = c \\ \frac{(c-3)(r-k)+1}{2}, & \text{if} \quad m = c-1 \\ \frac{(c-3)(r-k)+c-m-1}{2}, & \text{if} \quad m \leq c-2 \end{cases}.$$

If we distinguish the cases $m = c$, $m = c-1$ and $m \leq c-2$, then, noticing that $r \in \mathbb{N}$, equivalent transformations yield the bounds for r as in the assumptions of this theorem. This completes the proof of the theorem. $\square$

In the following, we will only treat the case that $m = c$ and $k = 1$. In this case Theorem 4.30 leads to the next corollary.

Corollary 4.31 (Volkmann, Winzen [43]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D such that $|V_1| = |V_2| = \ldots = |V_c| = r$. Furthermore, let $x_i \in V_i$ be arbitrary for all $1 \leq i \leq c$. If $c \geq 5$ and $r \geq 4$ or $c = 4$ and $r \geq 6$, then there exists a cycle C in D such that $V(C) = \bigcup_{i=1}^{c}(V_i - x_i)$.*

The following example shows that the condition of Corollary 4.31 that $r \geq 4$, if $c \geq 5$, is best possible.

Example 4.32 (Volkmann, Winzen [43]) *Let $p \in \mathbb{N}$ and let D be a regular $(2p+1)$-partite tournament with $r = 3$ vertices in each partite set. If D consists of three regular disjoint subtournaments H_1, H_2, H_3 of order $2p+1$ such that $H_1 \leadsto H_2 \leadsto H_3 \leadsto H_1$, then $D' = D - V(H_1)$ contains no Hamiltonian cycle.*

Nevertheless, if $r = 3$, and thus, according to Remark 1.16, $c = 2p+1$, then there exist vertices $x_1, x_2, \ldots, x_c$ with $x_i \in V_i$ such that D contains a cycle C with $V(C) = \bigcup_{i=1}^{c}(V_i - x_i)$, as the following theorem demonstrates.

Theorem 4.33 *Let $V_1, V_2, \ldots, V_{2p+1}$ be the partite sets of a regular $(2p+1)$-partite tournament with $p \geq 2$ such that $|V_1| = |V_2| = \ldots = |V_{2p+1}| = 3$. Then D contains a cycle with exactly 2 vertices of each partite set.*

Proof. Suppose that D does not contain a cycle with exactly 2 vertices of each partite set. Let T_1 be a subtournament of D with $|V(T_1)| = 2p+1$. Then we define $D' = D - V(T_1)$. Since D is regular, Remark 1.16 with $r = 3$ implies $d^+(x), d^-(x) = 3p$ and thus $d^+_{D'}(x), d^-_{D'}(x) \geq p$.

Firstly, let D' be 2-connected. Because of $\alpha(D') = 2$, Theorem 4.25 yields that D' is Hamiltonian, a contradiction.

Secondly, let D' be not strong. Then D' can be partitioned into the strong components $D_1, D_2, \ldots, D_t$ such that $D_i \rightsquigarrow D_j$ for $i < j$. The fact that $d^-_{D_1}(x) \geq p$ for all $x \in V(D_1)$ implies $|V(D_1)| \geq 2\delta^-_{D_1}+1 \geq 2p+1$. Analogously, we observe that $|V(D_t)| \geq 2p+1$. Since $|V(D_1)| + |V(D_2)| + \ldots + |V(D_t)| = 4p+2$, we deduce that $t = 2$ and $|D_1| = |D_2| = 2p+1$. This is possible, only if $D_2 \rightsquigarrow T_1 \rightsquigarrow D_1$ and D_1, D_2, T_1 are regular tournaments. Hence, D is the multipartite tournament of Example 4.32. If $a_1a_2 \ldots a_{2p+1}a_1$ is a Hamiltonian cycle of T_1, $v_1 \in V(D_1) \cap V(a_1)$ and $b_1b_2 \ldots b_{2p+1}b_1$ is a Hamiltonian cycle of D_2 such that $b_1 \in V(a_1)$, then $a_1a_2 \ldots a_{2p+1}v_1b_2b_3 \ldots b_{2p+1}a_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction.

Thirdly, let D' be exactly 1-connected. This yields that D' contains a cut-vertex u such that $D' - \{u\}$ consists of the strong components $D_1, D_2, \ldots, D_t$ with the property that $D_i \rightsquigarrow D_j$ for $i < j$. Furthermore, there are vertices $v_1 \in V(D_1)$ and $v_t \in V(D_t)$ such that $v_t \to u \to v_1$. Since $d^-_{D_1}(x) \geq p-1$ for all $x \in V(D_1)$, we conclude that $|V(D_1)| \geq 2\delta^-_{D_1}+1 \geq 2p-1$. Analogously, we see that $|V(D_t)| \geq 2p-1$. Without loss of generality, let $|V(D_1)| \leq |V(D_t)|$, since otherwise we use the converse D^{-1} of D. Now we distinguish the two possible cases $|V(D_1)| = 2p-1$ and $|V(D_1)| = 2p$.

Case 1. Suppose that $|V(D_1)| = 2p-1$. This is possible, only if D_1 is a $(p-1)$-regular tournament with $u \to V(D_1)$, $V(T_1) \rightsquigarrow V(D_1)$ and $2p-1 \leq |V(D_t)| \leq 2p+2$. Let $C = a_1a_2 \ldots a_{2p-1}a_1$ be a Hamiltonian cycle of D_1.

Subcase 1.1. Let $|V(D_t)| = 2p-1$. As above, we deduce that D_t is a regular tournament with a Hamiltonian cycle $\tilde{C} = b_1b_2 \ldots b_{2p-1}b_1$ such that $V(D_t) \rightsquigarrow V(T_1)$ and $V(D_t) \to u$. The fact that $|V(D_2)| + |V(D_3)| + \ldots + |V(D_{t-1})| = 3$ implies that $t = 3$ or $t = 5$.

Firstly, let $t = 3$. In this case, D_2 is a 3-cycle $c_1c_2c_3c_1$. Without loss of generality, we may suppose that $a_{2p-1} \notin V(c_1)$ and $b_1 \notin V(c_3)$. Now, $a_1a_2 \ldots a_{2p-1}c_1c_2c_3b_1b_2 \ldots b_{2p-1}ua_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction.

Secondly, let $t = 5$. This yields that $|D_2| = |D_3| = |D_4| = 1$ such that $D_2 = \{v_2\}$, $D_3 = \{v_3\}$ and $D_4 = \{v_4\}$. If $v_2 \notin V(v_3)$ and $v_3 \notin V(v_4)$, then the vertices of $V(C)$ and $V(\tilde{C})$ can be chosen such that $a_{2p-1} \notin V(v_2)$ and $b_1 \notin V(v_4)$. Now, $a_1a_2 \ldots a_{2p-1}v_2v_3v_4b_1b_2 \ldots b_{2p-1}ua_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction. If $v_2 \in V(v_3)$ and $v_2' \in V(T_1) \cap V(v_3)$, then, without loss of generality, the numbering of the cycles C and $\tilde{C}$ can be chosen such that $v_4 \notin V(b_2)$ and $a_{2p-1} \notin V(b_1)$. In this case, we see that $b_1ua_1v_3v_4b_2b_3 \ldots b_{2p-1}v_2'a_2a_3 \ldots a_{2p-1}b_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction. Analogously, we arrive at a contradiction, if $v_3 \in V(v_4)$.

Subcase 1.2. Assume that $|V(D_t)| = 2p$ and thus $t = 4$ and $|V(D_2)| = |V(D_3)| = 1$. Let $D_2 = \{v_2\}$ and $D_3 = \{v_3\}$.

Subcase 1.2.1. Suppose that D_4 is Hamiltonian with the Hamiltonian cycle $C' = b_1 b_2 \ldots b_{2p} b_1$.

Firstly, let $v_2 \notin V(v_3)$. Because of

$$2p^2 \leq \sum_{x \in V(D_4)} d^+_{D'}(x) \leq \frac{2p(2p-1)}{2} + d(D_4, u) = 2p^2 - p + d(D_4, u),$$

we deduce that $d(D_4, u) \geq p \geq 2$. Hence, there exists a vertex $b_i \in V(C')$ such that $b_i \to u$ and $b_i^+ \notin V(v_3)$. Now the vertices of C can be numerated such that $a_{2p-1} \notin V(v_2)$ and $a_1 a_2 \ldots a_{2p-1} v_2 v_3 b_i^+ \ldots b_i^- b_i u a_1$ is a Hamiltonian cycle of D', a contradiction.

Secondly, let $v_2 \in V(v_3)$. This implies that D_4 is a tournament. Let $v_2' \in V(T_1) \cap V(v_3)$. If $v_2' \to D_4$, then we observe that $d^+_{D_4}(y) \geq 3p - (|V(T_1)| - 2) - |\{u\}| = p$ for all $y \in V(D_4)$, and thus

$$2p^2 - p = |E(D_4)| \geq 2p^2,$$

a contradiction. Let $\{v_2', u\} = \{x, y\}$ and $x \to y$. If $y' \in N^-(x) \cap V(D_4)$, then let $y' b_2 b_3 \ldots b_{2p} y'$ be the Hamiltonian cycle of D_4. Summarizing our results, we see that $a_1 a_2 \ldots a_{2p-1} v_2 b_2 b_3 \ldots b_{2p} y' x y a_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction.

Subcase 1.2.2. Let D_4 be not Hamiltonian. Since D_4 is strongly connected, Theorem 4.26 implies that D_4 is no tournament. The fact that D_1 is a tournament and $u \to D_1$ yields that D_4 consists of vertices of exactly $2p-1$ partite sets, and thus $v_2 \notin V(v_3)$.

Let $x \in V(T_1)$ be arbitrary. Then we observe that

$$\begin{aligned} 6p^2 &= \sum_{y \in V(D_4)} d^+(y) \leq \sum_{y \in V(D_4)} d^+_{D_4}(y) + d(D_4, u) + d(D_4, T_1) \\ &\leq 2p^2 - p - 1 + 2p - |V(u) \cap V(D_4)| - |N^+(u) \cap V(D_4)| \\ &\quad +4p^2 - |N^+(x) \cap V(D_4)|, \end{aligned}$$

and it follows that

$$|V(u) \cap V(D_4)| + |N^+(x) \cap V(D_4)| + |N^+(u) \cap V(D_4)| \leq p - 1. \tag{4.19}$$

Theorem 3.1 implies that D_4 contains a cycle C' with vertices of all the $2p-1$ partite sets of D_4, and thus $L(C') = 2p - 1$. Let $\{v_4\} = V(D_4) - V(C')$ and $v_4' \in V(T_1) \cap V(v_4)$. If $C' = b_1 b_2 \ldots b_{2p-1} b_1$, then, according to (4.19), there are at least $|V(C')| - (p-1) = p \geq 2$ vertices $b_i, b_j \in V(C') - (V(u) \cup N^+(v_4') \cup N^+(u))$ such that $\{b_i, b_j\} \to u$ and $\{b_i, b_j\} \rightsquigarrow v_4'$. Let $b_j \to v_4'$. If $v_3 \notin V(b_i^+)$, then the vertices of C can be numerated such that $a_{2p-1} \notin V(b_j^+)$ and $a_{2p-2} \notin V(v_2)$, and we see that $a_1 a_2 \ldots a_{2p-2} v_2 v_3 b_i^+ \ldots b_j v_4' a_{2p-1} b_j^+ \ldots b_i^- b_i u a_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction. If $v_3 \in V(b_i^+)$ and thus $v_2 \notin V(b_i^+)$, $v_3 \notin V(b_j^+)$ and $V(C) \to v_3$, then the vertices of C can be numerated such that $a_{2p-2} \notin V(v_2)$. This implies that $a_1 a_2 \ldots a_{2p-2} v_2 b_i^+ \ldots b_j v_4' a_{2p-1} v_3 b_j^+ \ldots b_i^- b_i u a_1$ is a cycle with exactly two vertices from every partite set, also a contradiction.

Subcase 1.3. Assume that $|V(D_t)| = 2p+1$. This implies $t = 3$ and $|V(D_2)| = 1$. Let $V(D_2) = \{v_2\}$.

Subcase 1.3.1. Suppose that D_3 is Hamiltonian with the Hamiltonian cycle $C' = b_1b_2 \dots b_{2p+1}b_1$. Let $u' \in V(T_1) \cap V(u)$. If $|N^-(u) \cap V(D_3)| = 1$ and $|N^-(u') \cap V(D_3)| \leq 1$, then we conclude that $d(D_3, u) \leq 1$ and $|N^+(u') \cap V(D_3)| \geq 2p-1$, and thus

$$\begin{aligned}(2p+1)3p &= \sum_{y \in V(D_3)} d^+(y) = \sum_{y \in V(D_3)} d^+_{D_3}(y) + d(D_3, u) + d(D_3, T_1) \\ &\leq 2p^2 + p + 1 + (2p+1)2p - (2p-1) = 6p^2 + p + 2,\end{aligned}$$

a contradiction to $p \geq 2$. Hence, it follows that $|N^-(u) \cap V(D_3)| \geq 2$ or $|N^-(u') \cap V(D_3)| \geq 2$. If $|N^-(u) \cap V(D_3)| \geq 2$, then the vertices of C' can be numerated such that $b_{2p+1} \to u$ and $b_1 \notin V(v_2)$. Let $a_{2p-1} \notin V(v_2)$. Then $a_1a_2 \dots a_{2p-1}v_2b_1b_2 \dots b_{2p+1}ua_1$ is a cycle with exactly 2 vertices of every partite set, a contradiction. Analogously, the case that $|N^-(u') \cap V(D_3)| \geq 2$ leads to a contradiction.

Subcase 1.3.2. Let D_3 be not Hamiltonian. Since D_3 is strongly connected, Theorem 4.26 implies that D_3 is not a tournament. Since $\{u\} \cup V(D_1)$ consists of $2p$ partite sets, it follows that D_3 consists of vertices of exactly $2p$ partite sets and $v_2 \to D_3$. Analogously as in Subcase 1.2.2, we see that

$$|V(u) \cap V(D_3)| + |N^+(x) \cap V(D_3)| + |N^+(u) \cap V(D_3)| \leq 2p \tag{4.20}$$

for an arbitrary vertex $x \in V(T_1)$. Theorem 3.1 implies that D_3 contains a cycle C' with vertices of all the $2p$ partite sets of D_3.

Hence, let $L(C') = 2p$ such that $C' = b_1b_2 \dots b_{2p}b_1$. Let us define $\{v_3\} = V(D_3) - V(C')$ and $\{v_3'\} = V(T_1) \cap V(v_3)$.

Assume that $u \rightsquigarrow C'$. Since $N^-(u) \cap V(D_3) \neq \emptyset$, it follows that $v_3 \to u$. Furthermore (4.20) yields that $C' \rightsquigarrow v_3'$. If $\{\tilde{v}_3\} = V(C') \cap V(v_3)$, then let the vertices of C be numerated such that $a_{2p-1} \notin V(v_2)$. In this case $a_1v_3ua_2a_3 \dots a_{2p-1}v_2\tilde{v}_3^+ \dots \tilde{v}_3^-v_3'a_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction. Hence, let $N^-(u) \cap V(C') \neq \emptyset$ and $b_i \to u$.

Suppose now that $v_3' \rightsquigarrow C'$. This yields that $v_2 \to v_3'$, since otherwise we observe that

$$3p = d^+(v_3') \geq |V(D_1)| + |V(C')| - 1 + |\{v_2\}| = 4p - 1,$$

a contradiction to $p \geq 2$. If the vertices of C are numerated such that $a_{2p-2} \notin V(v_2)$ and $a_{2p-1} \notin V(b_i^+)$, then $a_1a_2 \dots a_{2p-2}v_2v_3'a_{2p-1}b_i^+ \dots b_i^-b_iua_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction. Consequently, let $N^-(v_3') \cap V(C') \neq \emptyset$.

Let $\{x, y\} = \{u, v_3'\}$ such that $x \to y$ and $b \in V(C')$ with $b \to x$. If $a_{2p-1} \notin V(v_2)$, then we see that $a_1a_2 \dots a_{2p-1}v_2b^+ \dots b^-bxya_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction.

Subcase 1.4. Assume that $|V(D_t)| = 2p+2$. This implies that $t = 2$.

Subcase 1.4.1. Suppose that D_2 is Hamiltonian with the Hamiltonian cycle $C' = b_1b_2 \dots b_{2p+2}b_1$. It is easy to see that the vertices of C and C' can be numerated such that $b_{2p+2} \to u$ and $a_{2p-1} \notin V(b_1)$. Now, we observe that $a_1a_2 \dots a_{2p-1}b_1b_2 \dots b_{2p+2}ua_1$ is a Hamiltonian cycle of D', a contradiction.

Subcase 1.4.2. Let D_2 be not Hamiltonian. Since D_1 is a tournament and $u \notin V(x)$ for all $x \in V(D_1)$, we conclude that D_2 contains vertices of exactly $2p+1$ partite sets. Theorem 3.1 implies that D_2 contains a cycle C' with vertices of all the $2p+1$ partite sets of D_2. If $L(C') = 2p+2$, then D_2 is Hamiltonian and Subcase 1.4.1 yields a contradiction.

Consequently, it remains to consider the case that $L(C') = 2p+1$ such that $C' = b_1b_2 \dots b_{2p+1}b_1$. Let us define $\{v_2\} = V(D_2) - V(C')$ and $\{v_2'\} = V(T_1) \cap V(v_2)$. If $N^-(u) \cap V(C') = \emptyset$, then we observe that

$$3p = d^+(u) \geq |V(D_1)| + |V(C')| - 1 = 2p - 1 + 2p = 4p - 1,$$

a contradiction to $p \geq 2$. Hence, there exists a vertex $b_i \in V(C')$ such that $b_i \to u$. Analogously, we see that there exists a vertex $b_j \in V(C')$ such that $b_j \to v_2'$.

Taking into acount that either $u \to v_2'$ or $v_2' \to u$ and that the vertices of C can be numerated such that $a_{2p-1} \notin V(b_i^+) \cup V(b_j^+)$, we observe that either $a_1a_2 \dots a_{2p-1}b_i^+ \dots b_i^- b_i u v_2' a_1$ or $a_1a_2 \dots a_{2p-1}b_j^+ \dots b_j^- b_j v_2' u a_1$ is a cycle of D with exactly 2 vertices of every partite set, a contradiction.

Case 2. Assume that $|V(D_1)| = 2p$. This implies that $|V(D_t)| = 2p$ and $t = 3$ or $|V(D_t)| = 2p+1$ and $t = 2$. Let D_1 consist of vertices of exactly k partite sets with $p \leq k \leq 2p$. It follows that

$$\begin{aligned}
6p^2 &= \sum_{y \in V(D_1)} d^-(y) = \sum_{y \in V(D_1)} d^-_{D_1}(y) + d(u, D_1) + d(T_1, D_1) \\
&\leq \frac{2(2p-k)(2p-2) + (2k-2p)(2p-1)}{2} + 2p - |V(u) \cap V(D_1)| \\
&\quad -|N^-(u) \cap V(D_1)| + 4p^2 - \sum_{x \in V(T_1)} |N^-(x) \cap V(D_1)| \\
&= 6p^2 + k - p - |V(u) \cap V(D_1)| - |N^-(u) \cap V(D_1)| \\
&\quad - \sum_{x \in V(T_1)} |N^-(x) \cap V(D_1)|,
\end{aligned}$$

and thus

$$|N^-(u) \cap V(D_1)| + \sum_{x \in V(T_1)} |N^-(x) \cap V(D_1)| \leq k - p - |V(u) \cap V(D_1)|. \quad (4.21)$$

Let $y_1 \in V(D_2) \cup \ldots \cup V(D_t)$ be an arbitrary vertex. We will show that there exists a Hamiltonian path in $D_1 \cup \{u, y_1\}$ with the initial vertex u and the terminal vertex y_1. Suppose that this is not true.

Firstly, assume that D_1 is Hamiltonian with the Hamiltonian cycle $C = a_1a_2 \dots a_{2p}a_1$. If $|N^+(u) \cap V(D_1)| \geq 2$, then, without loss of generality, let $u \to a_1$ and $a_{2p} \notin V(y_1)$. But now $ua_1a_2 \dots a_{2p}y_1$ is a Hamiltonian path in $D_1 \cup \{u, y_1\}$, a contradiction. Hence, let $|N^+(u) \cap V(D_1)| = 1$. Together with (4.21), this implies

$$\begin{aligned}
p - |V(u) \cap V(D_1)| &\geq k - p - |V(u) \cap V(D_1)| \\
&\geq |N^-(u) \cap V(D_1)| \geq 2p - 1 - |V(u) \cap V(D_1)|,
\end{aligned}$$

a contradiction to $p \geq 2$.

Secondly, let D_1 be not Hamiltonian, and thus, according to Theorem 4.26, $k \neq 2p$. Theorem 3.1 implies that D_1 contains a cycle with vertices of all the k partite sets. Let $C = a_1 a_2 \dots a_l a_1$ be a cycle, which fulfills this condition and which has the maximal cardinality of all these cycles that contain vertices of all the k partite sets of D_1. If $L(C) = 2p$, then D_1 is Hamiltonian and as above, we arrive at a contradiction. Hence, let $L(C) < 2p$ and $T_1' = D_1 - V(C)$. It is obvious that T_1' is a tournament and, according to Theorem 4.24, T_1' contains a Hamiltonian path $P = b_1 b_2 \dots b_{2p-l}$.

If $|N^+(u) \cap V(C)| \leq p - 1$, then it follows that $|N^-(u) \cap V(C)| \geq k - p + 1 - |V(u) \cap V(C)|$, a contradiction to (4.21). Hence, we conclude that

$$|N^+(u) \cap V(C)| \geq p \ \ (\geq 2). \tag{4.22}$$

Let $u \to a_i$. If $a_i^- \to b_1$, then, noticing that $P \to y_1$, $u a_i a_i^+ \dots a_i^- b_1 b_2 \dots b_{2p-l} y_1$ is a Hamiltonian path in $D_1 \cup \{u, y_1\}$, a contradiction. Consequently let $b_1 \rightsquigarrow a_i^-$. Suppose that $b_j \to b_1$ for some $j \geq 3$. Let $j_{max} = \max\{j \geq 3 \mid b_j \to b_1\}$.

At first let $a_i^- \in V(b_1)$. If $b_1 \to a_{i-2}$, then, because of the maximality of C, we deduce that $b_1 \rightsquigarrow C$, and thus

$$p - 1 \leq d^-_{D_1}(b_1) \leq |V(T_1')| - 2 \quad \Rightarrow \quad |V(T_1')| \geq p + 1,$$

a contradiction. Hence, let $a_{i-2} \to b_1$. Now, the maximality of C implies that $a_i^- \to \{b_2, b_3, \dots, b_{2p-l}\}$. If $j_{max} \neq 2p - l$, then $D_1 \cup \{u, y_1\}$ contains the Hamiltonian path $u a_i a_i^+ \dots a_i^- b_2 b_3 \dots b_{j_{max}} b_1 b_{j_{max}+1} \dots b_{2p-l} y_1$ and if $j_{max} = 2p - l$, then $u a_i a_i^+ \dots a_i^- b_2 b_3 \dots b_{2p-l} b_1 y_1$ is a Hamiltonian path of $D_1 \cup \{u, y_1\}$, in both cases a contradiction.

Consequently, it remains to consider the case that $b_1 \to a_i^-$. If $a_p \in V(b_1) \cap V(C)$, then the maximality of C implies that $b_1 \to \{a_{p+1}, a_{p+2}, \dots, a_i^-\}$, and thus $p \neq i$. If $b_1 \to a_{p-1}$, then analogously as above, we see that $b_1 \rightsquigarrow C$, a contradiction. Again the maximality of C yields that $a_p \to \{b_2, b_3, \dots b_{2p-l}\}$. If $j_{max} \neq 2p - l$, then $b_2 b_3 \dots b_{j_{max}} b_1 b_{jmax+1} \dots b_{2p-l}$ is a Hamiltonian path of T_1' and if $j_{max} = 2p - l$, then $b_2 b_3 \dots b_{2p-l} b_1$ is a Hamiltonian path of T_1'. Both Hamiltonian paths have the initial vertex b_2. Analogously as above, we see that $b_2 \to a_i^-$, $b_2 \to \{a_{q+1}, a_{q+2}, \dots, a_i^-\}$ and $a_q \to \{b_1, b_3, b_4, \dots, b_{2p-l}\}$, if $a_q \in V(b_2) \cap V(C)$ $(q \neq i)$. Without loss of generality, we may suppose that $i > q > p$ (modulo l). But now, the fact that $a_q \to b_1$ and $b_1 \to \{a_{p+1}, a_{p+2}, \dots, a_{i-1}\}$ yields a contradiction.

Summarizing our results, we see that $b_1 \to \{b_2, b_3, \dots, b_{2p-l}\}$. Now, suppose that $u \to b_1$. Let $a_w \in V(y_1) \cap V(C)$ (or $a_w \in V(C) - V(b_{2p-l})$ be arbitrary, if $V(C) \cap V(y_1) = \emptyset$). Then it follows that $a_w \to b_{2p-l}$, since otherwise $u b_1 b_2 \dots b_{2p-l} a_w a_w^+ \dots a_w^- y_1$ is a Hamiltonian path in $D_1 \cup \{u, y_1\}$, a contradiction. The maximality of C implies that $a_{w+1} \rightsquigarrow b_{2p-l}$. If $m \notin \{1, 2, \dots, l\} - \{w, w + 1\}$ and $b_{2p-l} \to a_m$, then $u b_1 b_2 \dots b_{2p-l} a_m a_m^+ \dots a_m^- y_1$ is a Hamiltonian path of $D_1 \cup \{u, y_1\}$, a contradiction. Altogether, we have $C \rightsquigarrow b_{2p-l}$. If $a_n \in V(b_{2p-l})$, then we conclude that $a_n \to b_{2p-l-1}$, since otherwise

$$u b_1 b_2 \dots b_{2p-l-1} a_n a_n^+ \dots a_n^- b_{2p-l} y_1 \tag{4.23}$$

is a Hamiltonian path in $D_1 \cup \{u, y_1\}$. The maximality of D_1 yields that $a_{n+1} \rightsquigarrow b_{2p-l-1}$. To get no contradiction as in (4.23), we deduce that $C \rightsquigarrow b_{2p-l-1}$. Successively, it follows that $C \rightsquigarrow \{b_1, b_2, \ldots, b_{2p-l}\}$, a contradiction to the strong connectivity of D_1.

Consequently, let $b_1 \to u$ and thus $d^-_{D_1}(b_1) \geq p$. Furthermore, using (4.22) and the results above, we conclude that

$$\begin{aligned} |N^+_{D_1}(b_1)| &\geq |N^+(u) \cap V(C)| - |V(b_1) \cap V(C)| + |V(T_1') - \{b_1\}| \\ &\geq |N^+(u) \cap V(C)| - 1 \geq p - 1. \end{aligned}$$

Altogether, we arrive at the contradiction

$$2p = |V(D_1)| = d^+_{D_1}(b_1) + d^-_{D_1}(b_1) + 2 \geq 2p + 1.$$

Hence, for an arbitrary vertex $y_1 \in V(D_1) \cup \ldots \cup V(D_t)$ there exists a Hamiltonian path of $D_1 \cup \{u, y_1\}$ with the initial vertex u and the terminal vertex y_1.

Subcase 2.1. Assume that $|V(D_t)| = 2p$, and thus $t = 3$ and $D_2 = \{v_2\}$. Observing the converse D^{-1} of D, we see that for an arbitrary vertex $y_2 \in V(D_1) \cup V(D_2)$, there exists a Hamiltonian path of $D_2 \cup \{u, y_2\}$ with the initial vertex y_2 and the terminal vertex u. Choosing $y_1 = y_2 = v_2$, we get a Hamiltonian cycle of D', a contradiction.

Subcase 2.2. Suppose that $|V(D_t)| = 2p + 1$, and thus $t = 2$. According to (4.21), we have

$$\sum_{x \in V(T_1)} |N^-(x) \cap V(D_1)| \leq k - p.$$

We conclude that there are at least $k - (k - p) = p \geq 2$ vertices in $V(T_1)$ belonging to partite sets represented in $V(D_1)$ such that they (weakly) dominate D_1. Hence, let $w_1 \in V(T_1)$ with $w_1 \rightsquigarrow D_1$ and $x_1 \in V(D_1) \cap V(w_1)$. Let $D'' = [D' \cup \{w_1\}] - \{x_1\}$. Assume that there is a vertex $x \in V(D'')$ such that $d^+_{D''}(x) \leq p - 1$ or $d^-_{D''}(x) \leq p - 1$. This yields the contradiction $3p = d^+_D(x), d^-_D(x) \leq p - 1 + |V(T_1)| - 1 = 3p - 1$. Hence, let $d^+_{D''}(x), d^-_{D''}(x) \geq p \geq 2$ for all $x \in V(D'')$, and thus

$$d(D_2, w_1) \geq p - 1 \geq 1. \tag{4.24}$$

If $D_1 - \{x_1\}$ is not strongly connected, then let $D_1', D_2', \ldots, D_{t'}'$ be the strong components of $D_1 - \{x_1\}$ such that $D_i' \rightsquigarrow D_j'$ for $i < j$. If $D_1' \rightsquigarrow u$, then it follows that $d^-_{D_1'}(y) \geq p - 1$ for all $y \in V(D_1')$, and thus $|V(D_1')| \geq 2p - 1$, a contradiction to $|V(D_1)| = 2p$. Consequently, we may assume that there is a vertex $y \in D_1'$ such that $u \to y$, if $D_1 - \{x_1\}$ is not strongly connected. If $D_1 - \{x_1\}$ is strongly connected and $D_1 - \{x_1\} \rightsquigarrow u$, then we see that

$$2p - 1 \leq |V(u) \cap V(D_1)| + |N^-(u) \cap V(D_1)| \leq k - p \Rightarrow 3p - 1 \leq k \leq 2p,$$

a contradiction to $p \geq 2$. Consequently, we observe that there is a vertex $y \in V(D_1) - \{x_1\}$ such that $u \to y$, if $D_1 - \{x_1\}$ is strong.

The results above guarantee that D'' is strong. If D'' is 2-connected, then Theorem 4.25 yields that D'' is Hamiltonian, a contradiction. Hence, D'' is exactly 1-connected. Obviously, the vertices u and w_1 are no cut-vertices of D''. Since $D_1 - \{x_1\} \rightsquigarrow D_2$, $x_1 \in V(w_1)$, $N^-(w_1) \cap V(D_2) \neq \emptyset$, $N^-(u) \cap V(D_2) \neq \emptyset$, $w_1 \rightsquigarrow D_1 - \{x_1\}$ and $d^+_{D''}(x), d^-_{D''}(x) \geq 2$ for all $x \in V(D'')$, in $D_1 - \{x_1\}$, there is also no cut-vertex of D''. Hence, let $x' \in V(D_2)$ be a cut-vertex of D''. Because of (4.24), $N^+(u) \cap V(D'_1) \neq \emptyset$, $w_1 \rightsquigarrow D_1$, $d^-_{D''}(u) \geq 2$ and $d^-_{D''}(w_1) \geq 2$, the vertex x' is no cut-vertex of D_2, only if $x' \to \{w_1, u\} \rightsquigarrow D_2 - \{x'\} =: \hat{D}$ and $u \to w_1$. Since $d^-_{D''}(w_1) \geq p$, this implies that $p = 2$, and thus $|\hat{D}| = 4$. Let $y \in \hat{D} - V(w_1)$ such that $d^+_{\hat{D}}(y) = 1$. Then we observe that $d^+(y) \leq 1 + |\{x'\}| + |V(T_1)| - 2 = 5$, a contradiction. Hence, x' is a cut-vertex of D_2. Let $D''_1, D''_2, \ldots, D''_{t''}$ be the strong components of $D_2 - \{x'\}$ such that $D''_i \rightsquigarrow D''_j$, if $i < j$.

Suppose that there is a vertex $y \in V(D''_{t''})$ with $y \to u$. Since $N^+(u) \cap V(D'_1) \neq \emptyset$, $w_1 \rightsquigarrow D_1$ and $d^-_{D''}(w_1) \geq 2$, we conclude that $D'' - \{x'\}$ is strongly connected, a contradiction. Consequently, let $u \rightsquigarrow D''_{t''}$. This yields that $d^+_{D''_{t''}}(x) \geq 3p - (|V(T_1)| - 1) - |\{x'\}| = p - 1$ for all $x \in V(D''_{t''})$, and thus $|V(D''_{t''})| \geq 2\delta^+_{D''_{t''}} + 1 \geq 2p - 1$. To get no contradiction, it follows that $t'' = 2$, $|V(D''_2)| = 2p - 1$ and $D''_2 \rightsquigarrow T_1 \cup \{x'\}$. Since $D_1 - \{x_1\} \rightsquigarrow D_2$, $d^-_{D''}(u) \geq 2$ and $w_1 \rightsquigarrow D_1 - \{x_1\}$, we deduce that $D'' - \{x'\}$ is strongly connected, a contradiction.

This completes the proof of the theorem. □

Combining Corollary 4.31 with Corollary 4.13 and Theorem 4.33, it can be seen that we have found a solution of Problem 4.2 for regular multipartite tournaments and $r_i = |V_i| - 1$ for all $1 \leq i \leq c$.

Corollary 4.34 (Volkmann, Winzen [43]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D with $c \geq 5$ such that $|V_1| = |V_2| = \ldots = |V_c| = r \geq 2$. Then D contains a cycle with exactly $r - 1$ vertices of each partite set.*

Now, we shall prove the main theorem of this section.

Theorem 4.35 (Volkmann, Winzen [43]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D with $|V_1| = |V_2| = \ldots = |V_c| = r \geq 2$. If $c \geq 5$ or $c = 4$ and $r \geq 4$ or $c = 3$ or $c = 2$ and D is not isomorphic to $B(s, s, s, s)$, then D contains a cycle with exactly $r - 1$ vertices from each partite set.*

Proof. If $c \geq 5$, then Corollary 4.34 yields the desired result. Since, according to Theorem 4.4, D is Hamiltonian the result for $c = 2$ follows directly from Theorem 4.22 and for the case $c = 3$ we use the Theorems 4.1 and 4.23. Hence, let $c = 4$. According to Remark 1.16, r has to be even. If $r \geq 6$, then Corollary 4.31 leads to the desired result.

Consequently, it remains to consider the case that $c = r = 4$. Suppose that D does not contain any cycle with exactly 3 vertices of every partite set. Let T_1 be a subtournament of D of order 4 and $D' = D - V(T_1)$. This implies that $\alpha(D') = 3$. With respect to Remark 1.16, we observe that $d^+(x), d^-(x) = 6$

for all $x \in V(D)$ and $d^+_{D'}(x), d^-_{D'}(x) \geq 3$ for all $x \in V(D')$. Now we distinguish different cases.

Case 1. Let $\kappa(D') \geq 3$. In this case, Theorem 4.25 yields that D' is Hamiltonian, a contradiction.

Case 2. Assume that $\kappa(D') = 0$. Let $D_1, D_2, \ldots, D_t$ be the strong components of D such that $D_i \leadsto D_j$ for $i < j$. Since $d^-_{D_1}(x) \geq 3$ for all $x \in V(D_1)$, we deduce that $|V(D_1)| \geq 7$. Analogously, we conclude that $|V(D_t)| \geq 7$. Hence, we arrive at the contradiction $12 = |V(D')| \geq |V(D_1)| + |V(D_t)| \geq 14$.

Case 3. Suppose that $\kappa(D') = 1$. Let u be a cut-vertex of D' such that $D' - u$ consists of the strong components $D_1, D_2, \ldots, D_t$ with $D_i \leadsto D_j$ for $i < j$. This implies that $d^-_{D_1}(x) \geq 2$ for all $x \in V(D_1)$ and thus, since $c = 4$, we conclude that $|V(D_1)| \geq 6$. Analogously, we observe that $|V(D_t)| \geq 6$, a contradiction to $|V(D')| = 12$.

Case 4. Assume that $\kappa(D') = 2$.

Firstly, let D' contain a cycle-factor. In this case, because of $\alpha(D') = 3$, Theorem 4.27 yields that D' is Hamiltonian, a contradiction.

Secondly, let D' contain no cycle-factor. Now, Lemma 4.29 implies that $V(D')$ can be partitioned into four subsets Y, Z, R_1 and R_2 such that $R_1 \leadsto Y$ and $(R_1 \cup Y) \leadsto R_2$, where Y is an independent set and $|Y| > |Z|$.

If $|Z| \leq 1$, then we deduce that $\kappa(D') \leq 1$, a contradiction to $\kappa(D') = 2$. If $|Z| \geq 3$, then Y has to be an independent set with $|Y| \geq 4$, a contradiction to $\alpha(D') = 3$. Hence, let $|Z| = 2$ and $|Y| = 3$, which means that Y is a partite set of D'. Without loss of generality, let $|R_1| \leq |R_2|$.

Assume that $|R_1| = 0$. This yields that $|R_2| = 7$ and thus $d^+_{D'}(y) \geq 7$ for all $y \in Y$, a contradiction to $d^+(x), d^-(x) = 6$ for all $x \in V(D)$.

Now, let $1 \leq |R_1| \leq 2$. In this case, we see that there is a vertex $x \in R_1$ with $d^-_{D[R_1]}(x) = 0$ and thus $d^-_{D'}(x) \leq |Z| = 2$, a contradiction.

Finally, let $|R_1| = 3$. Because of $d^-_{D'}(x) \geq 3$ for all $x \in R_1$, we conclude that $D[R_1]$ is a 3-cycle and $Z \rightarrow R_1$. Since $D' - Y$ and R_1 consist of vertices of 3 partite sets, this is impossible. This completes the proof of the theorem. □

For the case that $c = 4$ and $r = 2$, Theorem 4.35 is not true in general as Example 4.6 with $t = 1$ demonstrates (see also Figure 4.4).

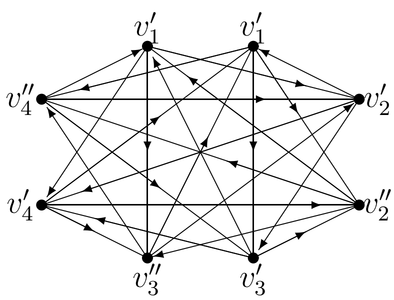

Figure 4.4: A regular 4-partite tournament without a strong subtournament of order 4

The results of Theorems 4.4 and the Corollaries 4.13 and 4.34 lead us to the following conjecture.

Conjecture 4.36 (Volkmann, Winzen [43]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D with $c \geq 5$ such that $|V_1| = |V_2| = \ldots = |V_c| = r \geq 2$. Then D contains a cycle with exactly m vertices of each partite set for every $m \in \{1, 2, \ldots, r\}$.*

Note that, according to Theorem 4.30, for a given m, Conjecture 4.36 is valid, if c and r are sufficiently large.

Part II

Connectivity

Chapter 5

An improvement of Yeo's result

Let D be a c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \le |V_2| \le \ldots \le |V_c|$. In 1998, Yeo [49] proved the useful bound

$$\kappa(D) \ge \left\lceil \frac{|V(D)| - \alpha(D) - 2i_l(D)}{3} \right\rceil \tag{5.1}$$

for each c-partite tournament D (see also Theorem 4.16).

In general, this bound cannot be improved as the following example demonstrates (see also [36]).

Example 5.1 (Volkmann [36]) *Let $q \ge 1$ be an integer, and let $c = 3q + 1$. We define the families $\mathcal{F}_q$ of c-partite tournaments with the partite sets $W_1, W_2, \ldots, W_q$ and*

$$W_{q+1} = A_{q+1} \cup B_{q+1}, W_{q+2} = A_{q+2} \cup B_{q+2}, \ldots, W_c = A_c \cup B_c$$

with $2|A_i| = 2|B_i| = |W_j| = 2t$ for $i = q+1, q+2, \ldots, c$ and $j = 1, 2, \ldots, q$ as follows. The partite sets $W_1, W_2, \ldots, W_q$ induce a $t(q-1)$-regular q-partite tournament H, the sets $A_{q+1}, A_{q+2}, \ldots, A_c$ induce a tq-regular $(2q+1)$-partite tournament A, and the sets $B_{q+1}, B_{q+2}, \ldots, B_c$ induce a tq-regular $(2q+1)$-partite tournament B. In addition, let $H \to A \rightsquigarrow B \to H$. Obviously, if $D \in \mathcal{F}_q$, then D is a $3qt$-regular c-partite tournament with the separating set $V(H)$ and thus $\kappa(D) = 2qt = q\alpha(D)$.

Since Yeo's result is often used to solve problems depending on the global irregularity, it would be interesting to solve the following general problem.

Problem 5.2 *For each integer $i \ge 0$ find all multipartite tournaments D with $i_g(D) = i$ and the property that*

$$\kappa(D) = \left\lceil \frac{|V(D)| - |V_c| - 2i}{3} \right\rceil.$$

In Section 5.1, we will analyze the proof of Theorem 4.16. With this method we will extend this result by working out – for each given integer $j \ge 0$ – the structure of those multipartite tournaments D with $i_l(D) = j$ the bound (5.1) is tight for. This structure implies a well known bound of Thomassen [25] on the connectivity of tournaments of given irregularity.

In Section 5.2, we will study Problem 5.2 for $i = 0$ and $i = 1$. For the case that D is a regular tournament, Volkmann [36] (for the case that $c = 4$ see also [34]) proved the following bound, which solves Problem 5.2 for $i = 0$.

Theorem 5.3 (Volkmann [36]) *If D is a regular c-partite tournament with $c \geq 2$, then*

$$\kappa(D) \geq \left\lceil \frac{|V(D)| - \alpha(D) + 1}{3} \right\rceil,$$

with exception of the case that D is a member of the families $\mathcal{F}_q$.

Using this structure of the multipartite tournaments, which realize (5.1), in the beginning of Section 5.2, we will present a shorter proof of Theorem 5.3. Furthermore, we will extend this result to almost regular multipartite tournaments, which means that we will present a solution of Problem 5.2 for $i = 1$.

5.1 An analysis of Yeo's result

The following four results (for Lemma 5.5 see also Lemma 4.10 and for Theorem 5.7 see also Theorem 4.16) were given in [49] and [51]. The informations about the cases of equality can implicitly be found in the proofs of the theorems.

Lemma 5.4 (Yeo [51]) *Let D be a semicomplete multipartite digraph with the partite sets $V_1, V_2, \ldots, V_c$. Let $X \subset Y \subseteq V(D)$ and let $y_i = |Y \cap V_i|$ and $x_i = |X \cap V_i|$ for all $i = 1, 2, \ldots, c$. Then*

$$\begin{aligned} & \frac{d(X, Y - X) + d(Y - X, X)}{|X|} + \frac{d(X, Y - X) + d(Y - X, X)}{|Y - X|} \\ \geq \; & |Y| - \max\{y_i \mid i = 1, 2, \ldots, c\}. \end{aligned}$$

Furthermore, if equality holds above, then $y_i - 2x_i = y_j - 2x_j$ and $y_j - x_j = y_i - x_i$ for all $1 \leq i, j \leq c$.

Lemma 5.5 (Yeo [51]) *If D is a semicomplete c-partite digraph, then the following holds.*

$$i_l(D) \geq \max_{\emptyset \neq X \subseteq V(D)} \left\{ \frac{|d(X, V(D) - X) - d(V(D) - X, X)|}{|X|} \right\}$$

In the case of equality, we observe that $d^+(x) = d^-(x) + i_l(D)$ for all $x \in X$, if $d(X, V(D) - X) \geq d(V(D) - X, X)$ and $d^-(x) = d^+(x) + i_l(D)$ for all $x \in X$, if $d(V(D) - X, X) \geq d(X, V(D) - X)$.

Theorem 5.6 (Yeo [49]) *Let D be a semicomplete multipartite digraph with the partite sets $V_1, V_2, \ldots, V_c$, and let S be a separating set in D. Let Q_1 and*

Q_2 be a partition of $V(D) - S$, such that $Q_1 \rightsquigarrow Q_2$, and let $v' = \max\{|V_i \cap (V(D) - S)| \mid i = 1, 2, \ldots, c\}$. Then the following holds.

$$i_l(D) \geq \frac{|V(D)| - 3|S| - v'}{2} \tag{5.2}$$

In the case of equality in (5.2) we have also equality in Lemma 5.4 with $X = Q_1$ and $Y = V(D) - S$. Furthermore, it follows that $|Q_1| = |Q_2|$, $S \rightarrow Q_1$ and $d(Q_1, V(D) - Q_1) \geq |Q_1||S|$, and we have equality in Lemma 5.5 with $X = Q_1$.

This immediately leads to Yeo's main result.

Theorem 5.7 (Yeo [49]) *If D is a semicomplete multipartite digraph, then (5.1) holds.*

Furthermore, if equality holds in (5.1), then we observe equality in (5.2) and there is a partite set V_i such that $|V_i| = \alpha(D)$ and $V_i \subseteq V(D) - S$.

These results yield the following corollary, which structures the multipartite tournaments that realize (5.1).

Corollary 5.8 (Volkmann, Winzen [44]) *Let D be a multipartite tournament with $\kappa(D) = \frac{|V(D)| - 2i_l(D) - \alpha(D)}{3}$ and let S be a separating set with $|S| = \kappa(D)$. Then the following holds.*

i) $\frac{|V(D)| - 2i_l(D) - \alpha(D)}{3} \in \mathbb{N}_0$.

ii) There is no partite set V_i of D such that $V_i \cap (V(D) - S) \neq \emptyset$ and $V_i \cap S \neq \emptyset$.

iii) For all partite sets V_i of D with $V_i \subseteq V(D) - S$ it follows that $|V_i| = \alpha(D)$.

iv) $V(D) - S$ can be partitioned in the sets Q_1 and Q_2 with $Q_1 \rightsquigarrow Q_2$ such that $|Q_1| = |Q_2|$, $Q_2 \rightarrow S \rightarrow Q_1$ and $D[Q_1]$ and $D[Q_2]$ are strong.

v) $d^+(q_1) = d^-(q_1) + i_l(D) = \frac{|V(D)| - \alpha(D) + i_l(D)}{2}$ for all $q_1 \in Q_1$ and $d^-(q_2) = d^+(q_2) + i_l(D) = \frac{|V(D)| - \alpha(D) + i_l(D)}{2}$ for all $q_2 \in Q_2$.

vi) $\alpha(D)$ is even.

vii) Every partite set V_i of D with $V_i \subseteq V(D) - S$ can be partitioned in two disjoined sets of vertices V_i' and V_i'' such that $|V_i'| = |V_i''|$, $V_i' \subseteq Q_1$ and $V_i'' \subseteq Q_2$.

viii) $D[Q_1]$ and $D[Q_2]$ are regular multipartite tournaments.

Proof. Since $\kappa(D)$ is a non-negative integer, i) follows immediately. Let Q_1 and Q_2 be a partition of $V(D) - S$ such that $Q_1 \rightsquigarrow Q_2$. According to Theorem 5.7, there is a partite set V_i of D such that $V_i \subseteq V(D) - S$ and

$|V_i| = \alpha(D)$. Now Lemma 5.4 with $x_i = |Q_1 \cap V_i|$ and $y_i = |V_i \cap (V(D) - S)|$ yields that

$$\begin{aligned} |V_i \cap (V(D) - S)| - 2|V_i \cap Q_1| &= |V_j \cap (V(D) - S)| - 2|V_j \cap Q_1| \quad \text{and} \\ |V_i \cap (V(D) - S)| - |V_i \cap Q_1| &= |V_j \cap (V(D) - S)| - |V_j \cap Q_1| \end{aligned}$$

for all indices j with $V_j \cap (V(D) - S) \neq \emptyset$. This is possible only if $|V_i \cap Q_1| = |V_j \cap Q_1|$ and $|V_i \cap (V(D) - S)| = |V_i| = \alpha(D) = |V_j \cap (V(D) - S)|$ for all these indices j. This implies ii) and iii).

According to Theorem 5.6, we have $|Q_1| = |Q_2|$. If $D - S$ does not consist of two strong components of the same cardinality, then we can choose a partition Q_1 and Q_2 of $V(D) - S$ such that $Q_1 \rightsquigarrow Q_2$ and $|Q_1| \neq |Q_2|$, a contradiction. Furthermore Theorem 5.6 leads to $S \to Q_1$. Observing the converse D^{-1} of D, we arrive at $Q_2 \to S$. Altogether we have shown iv).

Since, according to Theorem 5.6, $d(Q_1, V(D) - Q_1) \geq |Q_1||S| = d(V(D) - Q_1, Q_1)$, Lemma 5.5 yields $d^+(q_1) = d^-(q_1) + i_l(D)$ for all $q_1 \in Q_1$ and, caused by symmetry, $d^-(q_2) = d^+(q_2) + i_l(D)$ for all $q_2 \in Q_2$. Using Lemma 1.12 with $p = \alpha(D)$, we arrive at v).

As seen above, Lemma 5.4 implies $|V_i \cap Q_1| = |V_j \cap Q_1|$ for all indices i and j with $V_i, V_j \subseteq V(D) - S$. Because of $|Q_1| = |Q_2|$, this exactly means vii) and thus with iii) we deduce that vi) is valid.

According to vii), we have $d(x, Q_2) = d(y, Q_2)$ for all $x, y \in Q_1$. Because of v), $D[Q_1]$ has to be a regular multipartite tournament. Caused by symmetry, $D[Q_2]$ is also a regular multipartite tournament, which means that viii) is valid.

This completes the proof of this corollary. □

This result yields a simple method to check, whether the inequality (5.1) can be improved.

Corollary 5.9 (Volkmann, Winzen [44]) *Let D be a multipartite tournament. If $\alpha(D)$ is odd, then it follows that*

$$\kappa(D) \geq \left\lceil \frac{|V(D)| - 2i_l(D) - \alpha(D) + 1}{3} \right\rceil.$$

In the case of a tournament T we observe that $\alpha(T) = 1$ is odd and $i_g(T) = i_l(T) =: i(T)$. Hence, Corollary 5.9 implies the following result of Thomassen [25].

Theorem 5.10 (Thomassen [25]) *If T is a tournament with $i(T) \leq k$, then*

$$\kappa(D) \geq \left\lceil \frac{|V(T)| - 2k}{3} \right\rceil.$$

Another consequence of Corollary 5.9 is the following result.

Corollary 5.11 (Volkmann, Winzen [44]) *If D is a c-partite tournament with $c \geq 2$, $i_g(D) = 2k + 1$ for an integer $k \geq 0$ and $\alpha(D) = \gamma(D)$, then the following holds.*

$$\kappa(D) \geq \left\lceil \frac{|V(D)| - \alpha(D) - 2i_l(D) + 1}{3} \right\rceil = \left\lceil \frac{|V(D)| - \alpha(D) - 4k - 1}{3} \right\rceil$$

5.2 Almost regular multipartite tournaments

Using the results of the last section we are able to present a shorter proof of Theorem 5.3.

Theorem 5.12 (Volkmann [36]) *Let D be a regular c-partite tournament with $c \geq 2$. Then*

$$\kappa(D) \geq \left\lceil \frac{|V(D)| - \alpha(D) + 1}{3} \right\rceil,$$

with exception of the case that D is a member of the families $\mathcal{F}_q$.

Proof. If $V_1, V_2, \ldots, V_c$ are the partite sets of D, then $|V_1| = |V_2| = \ldots = |V_c| = r$, $\alpha(D) = r$, and $i_l(D) = 0$. Suppose that $\kappa(D) = \frac{|V(D)|-\alpha(D)}{3} = \frac{(c-1)r}{3}$. It follows that i)-viii) of Corollary 5.8 holds. Especially ii) yields that $|S| = sr$ for an integer s. On the other hand, we see that $|S| = \kappa(D) = \frac{c-1}{3}r$ and thus $s = \frac{c-1}{3} \in \mathbb{N}$, which means that $c = 3q + 1$ for an integer q and $|S| = qr = q\alpha(D)$. Since, according to iv), $Q_2 \to S \to Q_1$ with $|Q_1| = |Q_2|$ and D is regular, $D[S]$ has also to be regular. With Corollary 5.8 vii) and viii) we conclude that D belongs to the families $\mathcal{F}_q$. □

Now we will examine almost regular multipartite tournaments. Using Corollary 5.8, we arrive at the following result.

Corollary 5.13 (Volkmann, Winzen [44]) *Let D be a multipartite tournament such that $\kappa(D) = \frac{|V(D)|-2i_l(D)-\alpha(D)}{3}$ and $i_g(D) = i_l(D) \geq 1$. Then it follows that $\alpha(D) < \gamma(D) + 2i_g(D)$.*

Proof. According to iii) and v) in Corollary 5.8, we observe that

$$d^+(q_1) = d^-(q_1) + i_g(D) = \frac{|V(D)| - \alpha(D) + i_g(D)}{2}$$

and $|V(q_1)| = \alpha(D)$ for all $q_1 \in Q_1$. Assume that $\alpha(D) \geq \gamma(D) + 2i_g(D)$. Lemma 1.10 yields that $\alpha(D) = \gamma(D) + 2i_g(D)$. Now Lemma 1.13 leads to the contradiction

$$d^+(q_1) = \frac{|V(D)| - \gamma(D) - 2i_g(D)}{2} = \frac{|V(D)| - \alpha(D)}{2}.$$

□

The following examples will present the families of the multipartite tournaments with $i_g(D) = 1$, which realize (5.1).

Example 5.14 (Volkmann, Winzen [44]) *Let k, m, r, p, l, v, q, c and k_1 be integers, which fulfill one of the following properties:*

1) $r = 2p+1 \geq 1$, $k = 3m \geq 3$, $l = 2v$, $0 \leq v \leq \frac{m-1}{4p+2}$, $k_1 = m-1-2v(2p+1)$, $q = 2v + 2vp + m$ and $c = 3q + 1$.

2) $r = 4p+3 \geq 3$, $k = 3m \geq 3$, $0 \leq l \leq \frac{m-1}{4p+3}$, $k_1 = m-1-l(4p+3)$, $q = 2l+2lp+m$ *and* $c = 3q+1$.

3) $r = 12p+3 \geq 3$, $k = 3m$, $m \geq 8p+3$, $1 \leq l \leq \frac{4p+m}{12p+3}$, $k_1 = 4p+m-l(12p+3)$, $q = m-2p-1+l(6p+2)$ *and* $c = 3q+2$.

4) $r = 6p+3 \geq 3$, $k = 3m$, $m \geq 4p+3$, $l = 2v+1 \geq 1$, $0 \leq v \leq \frac{m-4p-3}{12p+6}$, $k_1 = 2p+m-(2v+1)(6p+3)$, $q = m-p-1+(2v+1)(3p+2)$ *and* $c = 3q+2$.

5) $r = 12p+11 \geq 11$, $k = 3m+1$, $m \geq 8p+8$, $1 \leq l \leq \frac{m+4p+3}{12p+11}$, $k_1 = 4p+3+m-l(12p+11)$, $q = m-2p-2+l(6p+6)$ *and* $c = 3q+2$.

6) $r = 6p+5 \geq 5$, $k = 3m+1$, $m \geq 4p+4$, $l = 2v+1$, $0 \leq v \leq \frac{m-4p-4}{12p+10}$, $k_1 = 2p+1+m-(2v+1)(6p+5)$, $q = m-p-1+(2v+1)(3p+3)$ *and* $c = 3q+2$.

7) $r = 12p+7 \geq 7$, $k = 3m+2$, $m \geq 8p+5$, $1 \leq l \leq \frac{m+4p+2}{12p+7}$, $k_1 = 4p+2+m-l(12p+7)$, $q = m-2p-1+l(6p+4)$ *and* $c = 3q+2$.

8) $r = 6p+1 \geq 1$, $k = 3m+2$, $m \geq 4p+1$, $l = 2v+1$, $0 \leq v \leq \frac{m-4p-1}{12p+2}$, $k_1 = 2p+m-(2v+1)(6p+1)$, $q = m-p+(2v+1)(3p+1)$ *and* $c = 3q+2$.

9) $r = 12p+3 \geq 3$, $k = 3m$, $m \geq 4p+2$, $0 \leq l \leq \frac{m-4p-2}{12p+3}$, $k_1 = m-4p-2-l(12p+3)$, $q = m+2p+1+l(6p+2)$ *and* $c = 3q$.

10) $r = 6p+3 \geq 3$, $k = 3m$, $m \geq 8p+5$, $l = 2v+1$, $0 \leq v \leq \frac{m-8p-5}{12p+6}$, $k_1 = m-2p-2-(2v+1)(6p+3)$, $q = m+p+1+(2v+1)(3p+2)$ *and* $c = 3q$.

11) $r = 6p+1 \geq 1$, $k = 3m+1$, $m \geq 8p+2$, $l = 2v+1$, $0 \leq v \leq \frac{m-8p-2}{12p+2}$, $k_1 = m-2p-1-(2v+1)(6p+1)$, $q = m+p+1+(2v+1)(3p+1)$ *and* $c = 3q$.

12) $r = 12p+7 \geq 7$, $k = 3m+1$, $m \geq 4p+3$, $0 \leq l \leq \frac{m-4p-3}{12p+7}$, $k_1 = m-4p-3-l(12p+7)$, $q = m+2p+2+l(6p+4)$ *and* $c = 3q$.

13) $r = 6p+5 \geq 5$, $k = 3m+2$, $m \geq 8p+7$, $l = 2v+1$, $0 \leq v \leq \frac{m-8p-7}{12p+10}$, $k_1 = m-2p-2-(2v+1)(6p+5)$, $q = m+2+p+(2v+1)(3p+3)$ *and* $c = 3q$.

14) $r = 12p+11 \geq 11$, $k = 3m+2$, $m \geq 4p+4$, $0 \leq l \leq \frac{m-4p-4}{12p+11}$, $k_1 = m-4p-4-l(12p+11)$, $q = m+3+2p+l(6p+6)$ *and* $c = 3q$.

If the properties in i) are valid $(i = 1, 2, \ldots, 14)$ *for the indices* k, m, r, p, l, v, q, c *and* k_1, *then we define the families* $\mathcal{G}_q^i$ *of* c*-partite tournaments with the partite sets*

$$W_1 = A_1 \cup B_1, W_2 = A_2 \cup B_2, \ldots, W_{k-k_1} = A_{k-k_1} \cup B_{k-k_1} \quad \text{and}$$

$W_{k-k_1+1}, W_{k-k_1+2}, \ldots, W_c$ with $2|A_i| = 2|B_i| = |W_j| = r+1$ for $i = 1, 2, \ldots, k-k_1$ and $j = k-k_1+1, k-k_1+2, \ldots, k$ and $|W_{k+1}| = |W_{k+2}| = \ldots = |W_c| = r$ as follows.

The partite sets $W_{k-k_1+1}, W_{k-k_1+2}, \ldots, W_c$ induce a $(q+l)$-partite tournament H such that $d_H^+(x) = d_H^-(x)$ for all $x \in W_{k+1} \cup W_{k+2} \cup \ldots \cup W_c$ and $|d_H^+(x) - d_H^-(x)| = 1$ for all $x \in W_{k-k_1+1} \cup W_{k-k_1+2} \cup \ldots \cup W_k$; the sets $A_1, A_2, \ldots, A_{k-k_1}$ induce a $\left[(c-q-l-1)\frac{r+1}{4}\right]$-regular $(c-q-l)$-partite tournament A; and analogously we see that the sets $B_1, B_2, \ldots, B_{k-k_1}$ induce a $\left[(c-q-l-1)\frac{r+1}{4}\right]$-regular $(c-q-l)$-partite tournament B. In addition, let $H \rightarrow A \rightsquigarrow B \rightarrow H$. If $D \in \mathcal{G}_q^i$ for $i = 1, 2, \ldots, 14$, then it is straightforward to show that D is a c-partite tournament with $i_g(D) = i_l(D) = 1$ containing the separating set $V(H)$ such that $|V(H)| = \kappa(D) = (c-k+k_1)r + k_1 = \frac{|V(D)| - \alpha(D) - 2}{3}$.

Example 5.15 (Volkmann, Winzen [44]) *Let k, m, r, p, l, v, q, c and k_1 be integers, which fulfill one of the following properties:*

1) $r = 2+4p \geq 2$, $k = 3m+1$, $m \geq 1+2p$, $1 \leq l \leq \frac{m}{1+2p}$, $k_1 = m - l(1+2p)$, $q = m + l(1+p)$ and $c = 3q+1$.

2) $r = 4+4p \geq 4$, $k = 3m+1$, $m \geq 4+4p$, $l = 2v+2$, $0 \leq v \leq \frac{m-4-4p}{4+4p}$, $k_1 = m - (2v+2)(2p+2)$, $q = m + (v+1)(2p+3)$ and $c = 3q+1$.

3) $r = 6+12p \geq 6$, $k = 3m+1$, $m \geq 5+10p$, $l = 2v$, $1 \leq v \leq \frac{1}{6}(1 + \frac{m}{1+2p})$, $k_1 = 1 + 2p + m - 6v - 12pv$, $q = 4v + 6pv + m - p - 1$ and $c = 3q+2$.

4) $r = 10+12p \geq 10$, $k = 3m+2$, $m \geq 10p+8$, $l = 2v$, $1 \leq v \leq \frac{m+2p+2}{12p+10}$, $k_1 = 2p + 2 + m - 12pv - 10v$, $q = 6v + 6pv + m - p - 1$ and $c = 3q+2$.

5) $r = 2+12p \geq 2$, $k = 3m$, $m \geq 10p+2$, $l = 2v$, $1 \leq v \leq \frac{m+2p}{12p+2}$, $k_1 = 2p + m - 2v - 12pv$, $q = 2v + 6pv + m - p - 1$ and $c = 3q+2$.

6) $r = 6p+2 \geq 2$, $k = 3m$, $m \geq 2p+1$, $l = 2v+1$, $0 \leq v \leq \frac{m-2p-1}{6p+2}$, $k_1 = m - 2p - 2v - 6pv - 1$, $q = 2v + 3pv + m + p$ and $c = 3q+2$.

7) $r = 6+6p \geq 6$, $k = 3m+1$, $m \geq 2p+2$, $l = 2v+1$, $0 \leq v \leq \frac{1}{6}(\frac{m}{p+1} - 2)$, $k_1 = m - 2p - 6pv - 6v - 2$, $q = 4v + 3vp + m + p + 1$ and $c = 3q+2$.

8) $r = 4+6p \geq 4$, $k = 3m+2$, $m \geq 2p+1$, $l = 2v+1$, $0 \leq v \leq \frac{m-2p-1}{4+6p}$, $k_1 = m - 2p - 1 - 6pv - 4v$, $q = 3v + 3pv + m + p + 1$ and $c = 3q+2$.

9) $r = 10+12p \geq 10$, $k = 3m$, $m \geq 2p+2$, $l = 2v$, $0 \leq v \leq \frac{m-2p-2}{10+12p}$, $k_1 = m - 2 - 2p - 10v - 12pv$, $q = 6v + 6pv + m + 1 + p$ and $c = 3q$.

10) $r = 6+12p \geq 6$, $k = 3m+1$, $m \geq 2p+1$, $l = 2v$, $0 \leq v \leq \frac{m-2p-1}{6+12p}$, $k_1 = m - 2p - 1 - 12pv - 6v$, $q = 4v + 6pv + m + p + 1$ and $c = 3q$.

11) $r = 2+12p \geq 2$, $k = 3m+2$, $m \geq 2p$, $l = 2v$, $0 \leq v \leq \frac{m-2p}{2+12p}$, $k_1 = m - 2p - 2v - 12pv$, $q = 2v + 6pv + m + 1 + p$ and $c = 3q$.

12) $r = 6p + 4 \geq 4$, $k = 3m$, $m \geq 3 + 4p$, $l = 2v + 1$, $0 \leq v \leq \frac{m-3-4p}{4+6p}$, $k_1 = m - 3 - 4p - 4v - 6pv$, $q = 3v + 2 + 3pv + m + 2p$ *and* $c = 3q$.

13) $r = 6 + 6p \geq 6$, $k = 3m + 1$, $m \geq 4 + 4p$, $l = 2v + 1$, $0 \leq v \leq \frac{m-4-4p}{6+6p}$, $k_1 = m - 4 - 4p - 6v - 6pv$, $q = 4v + 3pv + 2p + m + 3$ *and* $c = 3q$.

14) $r = 2 + 6p \geq 2$, $k = 3m + 2$, $m \geq 4p + 1$, $l = 2v + 1$, $0 \leq v \leq \frac{m-4p-1}{2+6p}$, $k_1 = m - 4p - 2v - 6pv - 1$, $q = 2v + 2 + 3pv + m + 2p$ *and* $c = 3q$.

If the properties in i) are valid $(i = 1, 2, \ldots, 14)$ *for the indices* k, m, r, p, l, v, q, c *and* k_1*, then we define the families* $\mathcal{H}_q^i$ *of* c*-partite tournaments with the partite sets*

$$W_1 = A_1 \cup B_1, W_2 = A_2 \cup B_2, \ldots, W_{k-k_1} = A_{k-k_1} \cup B_{k-k_1} \quad \text{and}$$

$W_{k-k_1+1}, W_{k-k_1+2}, \ldots, W_c$ *with* $2|A_i| = 2|B_i| = |W_j| = r+2$ *for* $i = 1, 2, \ldots, k-k_1$ *and* $j = k - k_1 + 1, k - k_1 + 2, \ldots, k$ *and* $|W_{k+1}| = |W_{k+2}| = \ldots = |W_c| = r$ *as follows.*

The partite sets $W_{k-k_1+1}, W_{k-k_1+2}, \ldots, W_c$ *induce a local regular* $(q + l)$*-partite tournament* H*; the sets* $A_1, A_2, \ldots, A_{k-k_1}$ *induce a* $\left[(c - q - l - 1)\frac{r+2}{4}\right]$*-regular* $(c - q - l)$*-partite tournament* A*; and the sets* $B_1, B_2, \ldots, B_{k-k_1}$ *induce a* $\left[(c - q - l - 1)\frac{r+2}{4}\right]$*-regular* $(c - q - l)$*-partite tournament* B*. In addition, let* $H \rightarrow A \rightsquigarrow B \rightarrow H$*. If* $D \in \mathcal{H}_q^i$ *for* $i = 1, 2, \ldots, 14$*, then it is left to the reader to show that* D *is a* c*-partite tournament with* $i_g(D) = 1$ *and* $i_l(D) = 0$ *containing the separating set* $V(H)$ *such that* $|V(H)| = \kappa(D) = (c - k + k_1)r + 2k_1 = \frac{|V(D)|-\alpha(D)}{3}$.

There are no other c-partite tournaments with $i_g(D) = 1$ and $\kappa(D) = \frac{|V(D)|-\alpha(D)-2i_l(D)}{3}$ as we can see in the following theorem.

Theorem 5.16 (Volkmann, Winzen [44]) *Let* D *be an almost regular* c*-partite tournament with* $c \geq 2$*. Then,*

$$\kappa(D) \geq \left\lceil \frac{|V(D)| - \alpha(D) - 2i_l(D) + 1}{3} \right\rceil,$$

with exception of the case that D *is a member of one of the families* $\mathcal{F}_q$*,* $\mathcal{G}_q^i$ *or* $\mathcal{H}_q^i$ *with* $i \in \{1, 2, \ldots, 14\}$*.*

Proof. If $i_g(D) = 0$, then the assertion follows from Theorem 5.12. Hence, we may assume that D is a c-partite tournament with $i_g(D) = 1$ and the partite sets $V_1, V_2, \ldots, V_c$ such that $r = |V_1| \leq |V_2| \leq \ldots \leq |V_c| = \alpha(D)$. According to Lemma 1.10, we have $r \leq \alpha(D) \leq r + 2$. Suppose that $\kappa(D) = \frac{|V(D)|-2i_l(D)-\alpha(D)}{3}$. Let S, Q_1 and Q_2 be defined as in Corollary 5.8 and observe that i)-viii) of this corollary holds. Now we distinguish different cases.

Case 1. Let $\alpha(D) = r$. In this case, Corollary 5.11 yields a contradiction.

Case 2. Assume that $\alpha(D) = r + 1$. This implies that $i_l(D) = i_g(D) = 1$ and according to Corollary 5.8 vi), r is odd. Hence, we may suppose that $r = 2p+1$ for an integer $p \geq 0$. Let $|V(D)| = cr+k$ with $0 < k < c$. Because of

Corollary 5.8 v), we deduce that the number of partite sets with the cardinality r has to be odd, which means that $c-k$ is odd. Again with Corollary 5.8 we see that S consists of all the $c-k$ partite sets of cardinality r and of $k_1 \geq 0$ partite sets of cardinality $r+1$. This yields that $|S| = (c-k+k_1)r + k_1$. Since $|Q_1| = |Q_2|$ and $Q_2 \to S \to Q_1$, it follows that $d^+_{D[S]}(x) = d^-_{D[S]}(x)$ for all vertices x belonging to a partite set of cardinality r and $|d^+_{D[S]}(x) - d^-_{D[S]}| = 1$ for all vertices $x \in S$ belonging to a partite set of cardinality $r+1$.

Subcase 2.1. Let $c = 3q+1$. If $D[S]$ is $(q+l)$-partite, then we arrive at

$$\begin{aligned} |S| &= (c-k+k_1)r + k_1 = (q+l)r + k_1 = \frac{cr+k-2-(r+1)}{3} \\ &= qr + \frac{k}{3} - 1 = (q+l)r + \frac{k}{3} - 1 - lr. \end{aligned}$$

This implies that $k = 3m$ for an integer $m \geq 1$, $c-k+k_1 = q+l$ and $k_1 = m-1-l(2p+1) \geq 0$. It follows that $l \leq \frac{m-1}{2p+1}$ and

$$\begin{aligned} & c-k+k_1 = 3q+1-3m+m-1-l(2p+1) = q+l \\ \Leftrightarrow\ & 2q = 2l+2lp+2m \Rightarrow q = l+lp+m. \end{aligned}$$

This leads to

$$\begin{aligned} & c = 3q+1 = 3l+3lp+3m+1 = k+1+3l+3lp < c+1+3l+3lp \\ \Rightarrow\ & 1+3l+3lp > 0 \Rightarrow l > -\frac{1}{3+3p} \Rightarrow l \geq 0. \end{aligned}$$

Since $c-k = 1+3l+3lp = 1+3l(p+1)$ has to be odd, it follows that l is even or p is odd.

If $l = 2v$, then we deduce that $0 \leq v \leq \frac{m-1}{4p+2}$, $m \geq 1$, $k_1 = m-1-2v(2p+1)$ and $q = 2v+2vp+m$. Corollary 5.8 yields that D belongs to the families $\mathcal{G}^1_q$.

If $p = 2s+1 \geq 1$, then it follows that $r = 4s+3 \geq 3$, $m \geq 1$, $0 \leq l \leq \frac{m-1}{4s+3}$ and $q = 2l+2ls+m$. Corollary 5.8 implies that D is an element of the families $\mathcal{G}^2_q$.

Subcase 2.2. Suppose that $c = 3q+2$ for an integer $q \geq 0$. If $D[S]$ is $(q+l)$-partite, then we observe that

$$\begin{aligned} |S| &= (c-k+k_1)r + k_1 = (q+l)r + k_1 = \frac{cr+k-r-3}{3} \\ &= qr + \frac{r+k}{3} - 1 = (q+l)r + \frac{r+k}{3} - 1 - lr, \end{aligned}$$

and thus $c-k+k_1 = q+l$ and $k_1 = \frac{r+k}{3} - 1 - lr$. This leads to

$$\begin{aligned} & 3q+2-k+\frac{r+k}{3} - 1 - lr = 3q+1+\frac{r-2k}{3} - lr = q+l \\ \Rightarrow\ & 2q = \frac{2k-r}{3} - 1 + l(r+1) \ \Rightarrow\ q = \frac{2k-r-3}{6} + l\frac{r+1}{2}. \end{aligned}$$

Since $r = 2p+1$, we have

$$q = \frac{2k-2p-4}{6} + l(p+1) = \frac{k-p-2}{3} + l(p+1). \tag{5.3}$$

Subcase 2.2.1. Let $k = 3m$ for an integer $m \geq 1$. With (5.3) we arrive at $q = m - \frac{p+2}{3} + l(p+1)$ and thus $p = 3s+1$, $r = 6s+3$ and $q = m-s-1+l(3s+2)$ for an $s \in \mathbb{N}_0$. Furthermore we see that

$$\begin{aligned} c &= 3q + 2 = 3m - 3s - 3 + 3l(3s+2) + 2 = k - 3s - 1 + 3l(3s+2) \\ &< c - 3s - 1 + 3l(3s+2) \end{aligned}$$

$$\Rightarrow -3s - 1 + 3l(3s+2) > 0 \Rightarrow l > \frac{3s+1}{9s+6} \Rightarrow l \geq 1$$

and

$$k_1 = \frac{r+k}{3} - 1 - lr = 2s + m - l(6s+3) \geq 0 \Rightarrow l \leq \frac{2s+m}{6s+3}.$$

Since $c - k = -3s - 1 + 3l(3s+2) = 3(3ls + 2l - s) - 1$ is odd, we conclude that $3ls - s = s(3l-1)$ is even and thus s is even or l is odd.

If $s = 2n$ with $n \in \mathbb{N}_0$, then we arrive at $r = 12n+3$, $q = m - 2n - 1 + l(6n+2)$, $k_1 = 4n + m - l(12n+3)$, $1 \leq l \leq \frac{4n+m}{12n+3}$ and thus $m \geq 8n+3$. According to Corollary 5.8, D is a member of the families $\mathcal{G}_q^3$.

If $l = 2v+1$ for an integer v, then it follows that $0 \leq v \leq \frac{m-4s-3}{12s+6}$, $m \geq 4s+3$, $q = m - s - 1 + (2v+1)(3s+2)$ and $k_1 = 2s + m - (2v+1)(6s+3)$. Again with Corollary 5.8 we deduce that D is an element of the families $\mathcal{G}_q^4$.

Subcase 2.2.2. Assume that $k = 3m+1$ with $m \in \mathbb{N}_0$. According to (5.3), we have $q = m - \frac{p+1}{3} + l(p+1)$ and thus $p = 3s+2$ for an integer $s \geq 0$, $r = 6s+5$ and $q = m - s - 1 + l(3s+3)$. Furthermore we conclude that

$$\begin{aligned} & c = 3q + 2 = 3m - 3s - 3 + 3l(3s+3) + 2 = k - 2 - 3s + 3l(3s+3) \\ \Rightarrow\ & -2 - 3s + 3l(3s+3) > 0 \Rightarrow l > \frac{3s+2}{3(3s+3)} \Rightarrow l \geq 1 \end{aligned}$$

and

$$k_1 = \frac{r+k}{3} - 1 - lr = 2s + 1 + m - l(6s+5) \geq 0 \Rightarrow l \leq \frac{m+2s+1}{6s+5}.$$

Since $c - k = -2 - 3s + 3l(3s+3) = -2 + 3(3ls + 3l - s)$ is odd, we observe that $3ls + 3l - s$ is odd. This is possible only if s is odd or l is odd.

If $s = 2n+1$ for an integer $n \geq 0$, then it follows that $r = 12n+11$, $1 \leq l \leq \frac{m+4n+3}{12n+11}$, $m \geq 8n+8$, $k_1 = 4n+3+m-l(12n+11)$ and $q = m-2n-2+l(6n+6)$. According to Corollary 5.8, we deduce that D belongs to the families $\mathcal{G}_q^5$.

If $l = 2v+1$ for an integer v, then it follows that $0 \leq v \leq \frac{m-4s-4}{12s+10}$, $m \geq 4s+4$, $k_1 = 2s+1+m-(2v+1)(6s+5)$ and $q = m-s-1+(2v+1)(3s+3)$. Hence, using Corollary 5.8 we conclude that D is a member of the families $\mathcal{G}_q^6$.

Subcase 2.2.3. Suppose that $k = 3m+2$ with $m \in \mathbb{N}_0$. According to (5.3), we observe that $q = m - \frac{p}{3} + l(p+1)$, and thus $p = 3s$, $r = 6s+1$ and $q = m - s + l(3s+1)$ with $s \geq 0$. Furthermore we see that

$$\begin{aligned} & c = 3q + 2 = 3m - 3s + 3l(3s+1) + 2 = k - 3s + 3l(3s+1) \\ \Rightarrow\ & -3s + 3l(3s+1) > 0 \Rightarrow l > \frac{3s}{9s+3} \Rightarrow l \geq 1 \end{aligned}$$

and

$$k_1 = \frac{r+k}{3} - 1 - lr = 2s + m - l(6s+1) \geq 0 \Rightarrow l \leq \frac{m+2s}{6s+1}.$$

Since $c - k = -3s + 3l(3s+1) = 3(3ls + l - s)$ is odd, we see that $3ls + l - s$ is odd. This is possible only if s is odd or l is odd.

If $s = 2n+1$ with $n \in \mathbb{N}_0$, then it follows that $r = 12n+7$, $1 \leq l \leq \frac{m+4n+2}{12n+7}$, $m \geq 8n+5$, $q = m - 2n - 1 + l(6n+4)$ and $k_1 = 4n + 2 + m - l(12n+7)$. Using Corollary 5.8 we deduce that D is an element of the families $\mathcal{G}_q^7$.

If $l = 2v + 1$ for an integer v, then we have $0 \leq v \leq \frac{m-4s-1}{12s+2}$, $m \geq 4s+1$, $q = m - s + (2v+1)(3s+1)$ and $k_1 = 2s + m - (2v+1)(6s+1)$. Again with Corollary 5.8 we conclude that D is a member of the families $\mathcal{G}_q^8$.

Subcase 2.3. Let $c = 3q$ for an integer $q \geq 1$. If S is $(q+l)$-partite, then it follows that

$$\begin{aligned} |S| &= (c - k + k_1)r + k_1 = (q+l)r + k_1 = \frac{cr + k - r - 3}{3} \\ &= qr + \frac{k-r}{3} - 1 = (q+l)r + \frac{k-r}{3} - 1 - lr, \end{aligned}$$

and thus $c - k + k_1 = q + l$ and $k_1 = \frac{k-r}{3} - 1 - lr$. This implies that

$$\begin{aligned} & 3q - k + \frac{k-r}{3} - 1 - lr = q + l \Rightarrow 2q = \frac{r+2k}{3} + 1 + l(r+1) \\ \Rightarrow\ & q = \frac{r+2k+3}{6} + l\frac{r+1}{2}. \end{aligned}$$

Because of $r = 2p+1$ this means that

$$q = \frac{2k+2p+4}{6} + l(p+1) = \frac{k+p+2}{3} + l(p+1). \tag{5.4}$$

Subcase 2.3.1. Assume that $k = 3m$ with $m \in \mathbb{N}$. According to (5.4), this leads to $q = m + \frac{p+2}{3} + l(p+1)$, and thus $p = 3s+1$, $r = 6s+3$ and $q = m + s + 1 + l(3s+2)$ for an integer $s \geq 0$. Furthermore we observe that

$$\begin{aligned} & c = 3q = 3m + 3s + 3 + 3l(3s+2) = k + 3s + 3 + 3l(3s+2) \\ \Rightarrow\ & 3s + 3 + 3l(3s+2) > 0 \Rightarrow l > \frac{-s-1}{3s+2} \Rightarrow l \geq 0 \end{aligned}$$

and

$$k_1 = \frac{k-r}{3} - 1 - lr = m - 2s - 2 - l(6s+3) \geq 0 \Rightarrow l \leq \frac{m-2s-2}{6s+3}.$$

Since $c - k = 3s + 3 + 3l(3s+2) = 3 + 3(s + 3ls + 2l)$ is odd, we deduce that $s + 3ls = s(1+3l)$ is even, which means that s is even or l is odd.

If $s = 2n$ with $s \in \mathbb{N}_0$, then it follows that $r = 12n+3$, $q = m + 2n + 1 + l(6n+2)$, $0 \leq l \leq \frac{m-4n-2}{12n+3}$, $m \geq 4n+2$ and $k_1 = m - 4n - 2 - l(12n+3)$. According to Corollary 5.8, we see that D is member of the families $\mathcal{G}_q^9$.

If $l = 2v+1$ for an integer v, then we arrive at $0 \leq v \leq \frac{m-8s-5}{12s+6}$, $m \geq 8s+5$, $q = m+s+1+(2v+1)(3s+2)$ and $k_1 = m-2s-2-(2v+1)(6s+3)$. Again with Corollary 5.8 we observe that D belongs to the families $\mathcal{G}_q^{10}$.

Subcase 2.3.2. Suppose that $k = 3m+1$ for an integer $m \geq 0$. With (5.4) this yields $q = m + \frac{p}{3} + 1 + l(p+1)$ and thus $p = 3s$, $r = 6s+1$ and $q = m+s+1+l(3s+1)$ with $s \in \mathbb{N}_0$. Furthermore, we conclude that

$$\begin{aligned} c = 3q = 3m+3s+3+3l(3s+1) = k+3s+2+3l(3s+1) \\ \Rightarrow \quad 3s+2+3l(3s+1) > 0 \Rightarrow l > -\frac{3s+2}{9s+3} \Rightarrow l \geq 0 \end{aligned}$$

and

$$k_1 = \frac{k-r}{3} - 1 - lr = m-2s-1-l(6s+1) \geq 0 \Rightarrow l \leq \frac{m-2s-1}{6s+1}.$$

Since $c-k = 3s+2+3l(3s+1) = 2+3(3ls+l+s)$ is odd, it follows that $3ls+l+s$ is odd. This is possible only if s is odd or l is odd.

If $l = 2v+1$ for an integer v, then we arrive at $0 \leq v \leq \frac{m-8s-2}{12s+2}$, $m \geq 8s+2$, $q = m+s+1+(2v+1)(3s+1)$ and $k_1 = m-2s-1-(2v+1)(6s+1)$. According to Corollary 5.8, D is an element of the families $\mathcal{G}_q^{11}$.

If $s = 2n+1$ for an integer $n \geq 0$, then it follows that $r = 12n+7$, $0 \leq l \leq \frac{m-4n-3}{12n+7}$, $m \geq 4n+3$, $q = m+2n+2+l(6n+4)$ and $k_1 = m-4n-3-l(12n+7)$. Again with Corollary 5.8 we see that D belongs to the families $\mathcal{G}_q^{12}$.

Subcase 2.3.3. Let $k = 3m+2$ with $m \in \mathbb{N}_0$. Using (5.4), we observe that $q = m+1+\frac{p+1}{3} + l(p+1)$, and thus $p = 3s+2$, $r = 6s+5$ and $q = m+2+s+l(3s+3)$. Furthermore we have

$$\begin{aligned} c = 3q = 3m+6+3s+3l(3s+3) = k+4+3s+3l(3s+3) \\ \Rightarrow \quad 4+3s+3l(3s+3) > 0 \Rightarrow l > -\frac{4+3s}{9s+9} \Rightarrow l \geq 0 \end{aligned}$$

and

$$k_1 = \frac{k-r}{3} - 1 - lr = m-2s-2-l(6s+5) \geq 0 \Rightarrow l \leq \frac{m-2s-2}{6s+5}.$$

The fact that $c-k = 4+3(3ls+3l+s)$ is odd implies that $3ls+3l+s = s(3l+1)+3l$ is odd. This is possible only if l is odd or if s is odd.

If $l = 2v+1$ for an integer v, then we deduce that $0 \leq v \leq \frac{m-8s-7}{12s+10}$, $m \geq 8s+7$, $q = m+2+s+(2v+1)(3s+3)$ and $k_1 = m-2s-2-(2v+1)(6s+5)$. According to Corollary 5.8, we have that D is a member of the families $\mathcal{G}_q^{13}$.

If $s = 2n+1$ for an integer $n \geq 0$, then we observe that $r = 12n+11$, $0 \leq l \leq \frac{m-4n-4}{12n+11}$, $m \geq 4n+4$, $q = m+3+2n+l(6n+6)$ and $k_1 = m-4n-4-l(12n+11)$. Using Corollary 5.8 it follows that D belongs to the families $\mathcal{G}_q^{14}$.

Case 3. Let $\alpha(D) = r+2$. According to Corollary 5.13 we have $i_l(D) < i_g(D)$ and hence $i_l(D) = 0$. Because of $Q_2 \to S \to Q_1$, $D[S]$ has to be local regular. Since $V_i \subseteq S$ for all partite sets V_i with $|V_i| \leq r+1$, this implies that D does not contain any partite set of order $r+1$. Hence, let $|V(D)| = cr+2k$ such that $0 < k < c$. Using Corollary 5.8 we observe that S contains all the

$c-k$ partite sets of order r. If S contains additionally k_1 partite sets of order $r+2$, then it follows that $|S| = (c-k+k_1)r + 2k_1$.

Subcase 3.1. Suppose that $c = 3q+1$ with $q \in \mathbb{N}$. If S is $(q+l)$-partite, then it follows that

$$\begin{aligned} |S| &= (c-k+k_1)r + 2k_1 = (q+l)r + 2k_1 = \frac{cr + 2k - (r+2)}{3} \\ &= qr + \frac{2k-2}{3} = (q+l)r + \frac{2k-2}{3} - lr. \end{aligned}$$

Since $|S| \in \mathbb{N}_0$, we observe that $k = 3m+1$ for an integer $m \geq 0$, and thus $c - k + k_1 = q + l$ and $k_1 = m - l\frac{r}{2}$. This yields that

$$\begin{aligned} & 3q + 1 - (3m+1) + m - l\frac{r}{2} = 3q - 2m - l\frac{r}{2} = q + l \\ \Rightarrow\ & 2q = 2m + l(1 + \frac{r}{2}) \Rightarrow q = m + l\frac{r+2}{4}. \end{aligned}$$

If $l \leq 0$, then we arrive at $q \leq m$ and thus $c \leq k$, a contradiction. Hence, let $l \geq 1$. Because of $q \in \mathbb{N}$, we conclude that $l\frac{r+2}{4} \in \mathbb{N}$. Since r is even this implies that $r = 2 + 4p$ or $r = 4 + 4p$ and $l = 2v + 2$ for integers $p, v \geq 0$.

If $r = 2 + 4p$, then we see that $q = m + l(1+p)$ and $k_1 = m - l(1+2p)$. The fact that $k_1 \geq 0$ yields that $1 \leq l \leq \frac{m}{1+2p}$ and thus $m \geq 1 + 2p$. Using Corollary 5.8, it is obvious that D belongs to the families $\mathcal{H}_q^1$.

If $r = 4 + 4p$ and $l = 2v + 2$, then it follows that $q = m + (v+1)(2p+3)$ and $k_1 = m - (2v+2)(2p+2)$. Because of $k_1 \geq 0$ we have $0 \leq v \leq \frac{m-4-4p}{4+4p}$ and thus $m \geq 4 + 4p$. Again with Corollary 5.8 we deduce that D is a member of the families $\mathcal{H}_q^2$.

Subcase 3.2. Assume that $c = 3q + 2$ for an integer $q \geq 0$. Let $r = 2 + 2p$ with $p \in \mathbb{N}_0$. If S is $(q+l)$-partite, then we observe that

$$\begin{aligned} |S| &= (c-k+k_1)r + 2k_1 = (q+l)r + 2k_1 = \frac{cr + 2k - r - 2}{3} \\ &= qr + \frac{r + 2k - 2}{3} = (q+l)r + \frac{r+2k-2}{3} - lr, \end{aligned}$$

and thus $c - k + k_1 = q + l$ and $k_1 = \frac{r+2k-2}{6} - l\frac{r}{2}$. This implies

$$\begin{aligned} & 3q + 2 - k + \frac{r+2k-2}{6} - l\frac{r}{2} = 3q + 2 + \frac{r - 4k - 2}{6} - l\frac{r}{2} = q + l \\ \Rightarrow\ & 2q = l(1 + \frac{r}{2}) - 2 + \frac{4k + 2 - r}{6} \\ \Rightarrow\ & q = \frac{r+2}{4}l + \frac{4k - r + 2}{12} - 1 = \frac{3lr + 6l + 4k - r + 2}{12} - 1. \end{aligned}$$

This leads to

$$\begin{aligned} & c = 3q + 2 = k + \frac{3lr + 6l - r + 2}{4} - 1 < c + \frac{3lr + 6l - r + 2}{4} - 1 \\ \Rightarrow\ & \frac{3lr + 6l - r + 2}{4} - 1 > 0 \Rightarrow 3lr + 6l - r + 2 > 4 \\ \Rightarrow\ & 3l(r+2) > r + 2 \Rightarrow l > \frac{1}{3} \Rightarrow l \geq 1, \end{aligned}$$

which means that $v \geq 1$, if $l = 2v$, and $v \geq 0$, if $l = 2v + 1$. Since $r = 2 + 2p$, we observe that

$$q = l - 1 + \frac{3lp + 2k - p}{6}. \tag{5.5}$$

Subcase 3.2.1. Let $l = 2v$ for an integer v. Then (5.5) leads to

$$q = 2v + vp + \frac{2k - p}{6} - 1. \tag{5.6}$$

Subcase 3.2.1.1. Assume that $k = 3m + 1$ with $m \in \mathbb{N}_0$. Now (5.6) yields

$$q = 2v + vp + m + \frac{2 - p}{6} - 1,$$

and thus $p = 6s + 2$, $r = 6 + 12s$ and $q = 4v + 6vs + m - s - 1$ for an integer $s \geq 0$. Furthermore, we conclude that

$$\begin{aligned} k_1 &= \frac{r + 2k - 2}{6} - l\frac{r}{2} = 1 + 2s + m - 6v - 12vs \geq 0 \\ \Rightarrow \quad 1 &\leq v \leq \frac{1 + 2s + m}{6 + 12s} = \frac{1}{6}\left(1 + \frac{m}{1 + 2s}\right), \end{aligned}$$

and thus $m \geq 5 + 10s$. Using Corollary 5.8 we see that D is an element of the families $\mathcal{H}_q^3$.

Subcase 3.2.1.2. Let $k = 3m + 2$ with $m \in \mathbb{N}_0$. Using (5.6) we arrive at

$$q = 2v + vp + m + \frac{4 - p}{6} - 1,$$

and thus $p = 6s + 4$, $r = 12s + 10$ and $q = 6v + 6vs + m - s - 1$ for an integer $s \geq 0$. Furthermore we observe that

$$k_1 = \frac{r + 2k - 2}{6} - l\frac{r}{2} = 2s + 2 + m - 12vs - 10v \geq 0 \;\Rightarrow\; 1 \leq v \leq \frac{2s + 2 + m}{10 + 12s},$$

which implies that $m \geq 10s + 8$. According to Corollary 5.8, D belongs to the families $\mathcal{H}_q^4$.

Subcase 3.2.1.3. Suppose that $k = 3m$ with $m \in \mathbb{N}$. Then (5.6) leads to

$$q = 2v + vp + m - \frac{p}{6} - 1,$$

and thus $p = 6s$, $r = 2 + 12s$ and $q = 2v + 6vs + m - s - 1$ for an integer $s \geq 0$. Furthermore we see that

$$k_1 = \frac{r + 2k - 2}{6} - l\frac{r}{2} = 2s + m - 2v - 12vs \geq 0 \;\Rightarrow\; 1 \leq v \leq \frac{2s + m}{12s + 2},$$

which yields that $m \geq 10s + 2$. Using Corollary 5.8 it follows that D is a member of the families $\mathcal{H}_q^5$.

Subcase 3.2.2. Assume that $l = 2v + 1$ for an integer v. In this case (5.5) yields that

$$q = 2v + vp + \frac{p + k}{3}. \tag{5.7}$$

Subcase 3.2.2.1. Let $k = 3m$ with $m \in \mathbb{N}$. Using (5.7) we deduce that

$$q = 2v + vp + m + \frac{p}{3},$$

which leads to $p = 3s$, $r = 2 + 6s$ and $q = 2v + 3vs + m + s$ for an integer $s \geq 0$. Furthermore it follows that

$$k_1 = \frac{r + 2k - 2}{6} - l\frac{r}{2} = m - 2s - 2v - 6vs - 1 \geq 0 \Rightarrow 0 \leq v \leq \frac{m - 2s - 1}{2 + 6s},$$

and thus $m \geq 2s + 1$. According to Corollary 5.8, D is an element of the families $\mathcal{H}_q^6$.

Subcase 3.2.2.2. Suppose that $k = 3m + 1$ with $m \in \mathbb{N}_0$. With (5.7) we arrive at

$$q = 2v + vp + m + \frac{p+1}{3},$$

and thus $p = 3s + 2$, $r = 6s + 6$ and $q = 4v + 3vs + m + s + 1$ for an integer $s \geq 0$. Furthermore we see that

$$k_1 = \frac{r + 2k - 2}{6} - l\frac{r}{2} = m - 2s - 6vs - 6v - 2 \geq 0 \Rightarrow 0 \leq v \leq \frac{1}{6}\left(\frac{m}{s+1} - 2\right),$$

which implies that $m \geq 2s + 2$. Hence, again with Corollary 5.8 we observe that D belongs to the families $\mathcal{H}_q^7$.

Subcase 3.2.2.3. Let $k = 3m + 2$ with $m \in \mathbb{N}_0$. According to (5.7), we have

$$q = 2v + vp + m + \frac{p+2}{3},$$

and thus $p = 3s + 1$, $r = 6s + 4$ and $q = 3v + 3vs + m + s + 1$ for an integer $s \geq 0$. Furthermore, we observe that

$$k_1 = \frac{r + 2k - 2}{6} - l\frac{r}{2} = m - 2s - 1 - 6vs - 4v \geq 0 \Rightarrow 0 \leq v \leq \frac{m - 2s - 1}{4 + 6s},$$

which means that $m \geq 2s + 1$. Using Corollary 5.8 we conclude that D is a member of the families $\mathcal{H}_q^8$.

Subcase 3.3. Assume that $c = 3q$ with $q \in \mathbb{N}$. Let $r = 2 + 2p$ for an integer $p \geq 0$. If S is $(q + l)$-partite, then we conclude that

$$\begin{aligned} |S| &= (c - k + k_1)r + 2k_1 = (q + l)r + 2k_1 = \frac{cr + 2k - (r + 2)}{3} \\ &= qr + \frac{2k - 2 - r}{3} = (q + l)r + \frac{2k - 2 - r}{3} - lr. \end{aligned}$$

This implies that $c - k + k_1 = q + l$ and $k_1 = \frac{2k-2-r}{6} - l\frac{r}{2}$, and thus

$$\begin{aligned} 3q - k + \frac{2k - 2 - r}{6} - l\frac{r}{2} = 3q - \frac{4k + 2 + r}{6} - l\frac{r}{2} = q + l \\ \Rightarrow \quad 2q = \frac{r + 2 + 4k}{6} + l\frac{r + 2}{2} \Rightarrow q = \frac{r + 2 + 4k + 3lr + 6l}{12}. \end{aligned}$$

This leads to

$$\begin{aligned} c = 3q = k + \frac{r+2+3lr+6l}{4} &< c + \frac{r+2+3lr+6l}{4} \\ \Rightarrow \quad \frac{r+2+3lr+6l}{4} &> 0 \Rightarrow l > -\frac{1}{3} \Rightarrow l \geq 0, \end{aligned}$$

which means that $v \geq 0$, if $l = 2v$ or $l = 2v+1$ for an integer v. Furthermore, since $r = 2+2p$, we deduce that

$$q = l + \frac{2+p+2k+3lp}{6}. \tag{5.8}$$

Subcase 3.3.1. Let $l = 2v$ for an integer v. Using (5.8) we see that

$$q = 2v + vp + \frac{2+p+2k}{6}. \tag{5.9}$$

Subcase 3.3.1.1. Assume that $k = 3m$ with $m \in \mathbb{N}$. According to (5.9), we have

$$q = 2v + vp + m + \frac{2+p}{6},$$

and thus $p = 6s+4$, $r = 10+12s$ and $q = 6v+6vs+m+1+s$ for an integer $s \geq 0$. Furthermore it follows that

$$k_1 = \frac{2k-2-r}{6} - l\frac{r}{2} = m-2-2s-10v-12vs \geq 0 \Rightarrow 0 \leq v \leq \frac{m-2-2s}{10+12s},$$

which yields that $m \geq 2s+2$. Using Corollary 5.8 we conclude that D belongs to the families $\mathcal{H}_q^9$.

Subcase 3.3.1.2. Suppose that $k = 3m+1$ with $m \in \mathbb{N}_0$. Using (5.9) we see that

$$q = 2v + vp + m + \frac{p+4}{6},$$

and thus $p = 6s+2$, $r = 12s+6$ and $q = 4v+6vs+m+s+1$ for an integer $s \geq 0$. Furthermore we observe that

$$k_1 = \frac{2k-2-r}{6} - l\frac{r}{2} = m-2s-1-12vs-6v \geq 0 \Rightarrow 0 \leq v \leq \frac{m-2s-1}{6+12s},$$

which means that $m \geq 2s+1$. According to Corollary 5.8, we deduce that D is an element of the families $\mathcal{H}_q^{10}$.

Subcase 3.3.1.3. Let $k = 3m+2$ with $m \in \mathbb{N}_0$. According to (5.9), we arrive at

$$q = 2v + vp + m + 1 + \frac{p}{6},$$

and thus $p = 6s$, $r = 2+12s$ and $q = 2v+6vs+m+1+s$ for an integer $s \geq 0$. Furthermore we conclude that

$$k_1 = \frac{2k-2-r}{6} - l\frac{r}{2} = m-2s-2v-12vs \geq 0 \Rightarrow 0 \leq v \leq \frac{m-2s}{2+12s},$$

which leads to $m \geq 2s$. Using Corollary 5.8 we observe that D is a member of the families $\mathcal{H}_q^{11}$.

Subcase 3.3.2. Assume that $l = 2v + 1$ for an integer v. According to (5.8), this yields

$$q = 2v + 1 + vp + \frac{1 + 2p + k}{3}. \tag{5.10}$$

Subcase 3.3.2.1. Suppose that $k = 3m$ with $m \in \mathbb{N}$. Using (5.10) we observe that

$$q = 2v + 1 + vp + m + \frac{1 + 2p}{3},$$

and thus $p = 3s + 1$, $r = 4 + 6s$ and $q = 3v + 2 + 3vs + m + 2s$. Furthermore we see that

$$k_1 = \frac{2k - 2 - r}{6} - l\frac{r}{2} = m - 3 - 4s - 4v - 6vs \geq 0 \Rightarrow 0 \leq v \leq \frac{m - 3 - 4s}{4 + 6s},$$

which leads to $m \geq 3 + 4s$. According to Corollary 5.8, D belongs to the families $\mathcal{H}_q^{12}$.

Subcase 3.3.2.2. Let $k = 3m + 1$ with $m \in \mathbb{N}_0$. Using (5.10) we have

$$q = 2v + 1 + vp + m + \frac{2p + 2}{3},$$

and thus $p = 3s + 2$, $r = 6s + 6$ and $q = 4v + 3vs + 2s + m + 3$ for an integer $s \geq 0$. Furthermore we see that

$$k_1 = \frac{2k - 2 - r}{6} - l\frac{r}{2} = m - 4 - 4s - 6v - 6vs \geq 0 \Rightarrow 0 \leq v \leq \frac{m - 4 - 4s}{6 + 6s},$$

which yields that $m \geq 4 + 4s$, According to Corollary 5.8, it follows that D is an element of the families $\mathcal{H}_q^{13}$.

Subcase 3.3.2.3. Assume that $k = 3m + 2$ with $m \in \mathbb{N}_0$. Using (5.10) we observe that

$$q = 2v + 2 + vp + m + \frac{2p}{3},$$

and thus $p = 3s$, $r = 6s + 2$ and $q = 2v + 2 + 3vs + m + 2s$. Furthermore it follows that

$$k_1 = \frac{2k - 2 - r}{6} - l\frac{r}{2} = m - 4s - 2v - 6vs - 1 \geq 0 \Rightarrow 0 \leq v \leq \frac{m - 4s - 1}{2 + 6s},$$

which means that $m \geq 4s + 1$. According to Corollary 5.8, we conclude that D belongs to the families $\mathcal{H}_q^{14}$. This completes the proof of the theorem. $\square$

Part III

Paths in multipartite tornaments

Chapter 6

Paths of a given vertex-structure

In Chapter 4, we searched for cycles in (almost) regular multipartite tournaments with a given number s of vertices from each partite set. Since the proofs of these theorems become the complex, the more s differs from 1 and $\alpha(D)$, it is an interesting question to solve the following weaker form of Problem 4.2.

Problem 6.1 (Volkmann, Winzen [45]) *Which conditions have to be fulfilled such that a c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ contains a path with exactly s_i vertices of V_i for all $1 \le i \le c$ and given integers $0 \le s_i \le |V_i|$?*

In this chapter we will consider Problem 6.1 for some given small (Section 6.2) and large (Section 6.3) values s_i.

According to the Theorems 4.4, 4.12 and 4.35, we have solutions for Problem 6.1 for the cases that D is an almost regular c-partite tournament and $s_i = 1$ for all $1 \le i \le c$, D is a regular c-partite tournament and $s_i = \alpha(D)$ for all $1 \le i \le c$ or $s_i = \alpha(D) - 1$ for all $1 \le i \le c$. Furthermore Theorem 4.24 solves Problem 6.1 for the case that $s_i = 1$ for all $1 \le i \le c$.

To get more solutions for special choices of s_i in Problem 6.1, we need some results about the existence of Hamiltonian paths. In 1988, Gutin [14] gave a characterization of semicomplete multipartite digraphs having a Hamiltonian path.

Theorem 6.2 (Gutin [14]) *A semicomplete multipartite digraph D has a Hamiltonian path if and only if it contains a 1-path-cycle factor.*

This result was used to prove another result of Gutin and Yeo [16].

Theorem 6.3 (Gutin, Yeo [16]) *Let D be a semicomplete multipartite digraph with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \le |V_2| \le \ldots \le |V_c|$. If there exists a positive integer k such that*

$$i_l(D) \le \min\left\{|V(D)| - 3|V_c| + 2k + 1, \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 2}{2}\right\},$$

then $pc(D) \le k$.

In Section 6.1, we will give a supplement to Theorem 6.3 and some related results, which will be usefull afterwards.

In Section 6.2, we will give a solution of Problem 6.1 with $s_i = 2$ for all $1 \leq i \leq c$ by showing that every regular c-partite tournament with at least $r \geq 2$ vertices in each partite set contains a path with exactly two vertices from each partite set. Furthermore, we will give some results for the case that D is a regular multipartite tournament and $2 \leq s_i \leq 3$ for all $1 \leq i \leq c$ in Problem 6.1.

In Section 6.3, we will treat the case that D is a regular c-partite tournament with exactly r vertices in each partite set and $s_i = \alpha(D) - s$ for $1 \leq i \leq c$ and a given integer $2 \leq s \leq r - 1$. We will show that almost all regular c-partite tournaments D with $c \geq 4$ contain a path with exactly $r - s$ vertices from each partite set, if $s \in \mathbb{N}$ is an arbitrary integer and r is the cardinality of every partite set of D. Especially we will prove that each regular c-partite tournament D with $c \geq 5$ and at least $r \geq 5s - 3$ vertices in each partite set or with $c = 4$ and at least $r \geq 7s - 5$ vertices of each partite set contains a path with exactly $r - s$ vertices of each partite set. Nevertheless, we conjecture that this result also holds for all $r \geq s + 1$.

6.1 Path covering number and irregularity

In this section, we will show the connection between the path covering number and the irregularity of a semicomplete multipartite digraph by presenting a slight improvement of a result of Yeo [51] and Gutin and Yeo [16]. Furthermore, we will analyze those semicomplete multipartite digraphs, which realize the developed result.

The following theorem is a useful supplement to Theorem 6.3. The proof is similar to the proof of Lemma 4.3 in [51] and Theorem 3.2 in [16].

Theorem 6.4 ((Stella,) Volkmann, Winzen [40, 22]) *Let D be a semicomplete c-partite digraph with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$. Assume that $pc(D) > k$ for an integer $k \geq 1$ (respectively, let $k = 0$, if D has no cycle-factor). According to Theorem 4.29, $V(D)$ can be partitioned into subsets Y, Z, R_1, R_2 satisfying (4.18) such that $|Z| + k + 1 \leq |Y| \leq |V_c| - t$ with an integer $t \geq 0$. Let V_i be the partite set with the property that $Y \subseteq V_i$. If $Y_1 = R_1 \cap V_i$, $Y_2 = R_2 \cap V_i$, $Q = V(D) - Z - V_i$, $Q_1 = Q \cap R_1$ and $Q_2 = Q \cap R_2$, then*

$$i_l(D) \geq |V(D)| - 3|V_c| + 2t + 2k + 2,$$

if $Q_1 = \emptyset$ or $Q_2 = \emptyset$,

$$i_g(D) \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + |Y_2|}{2},$$

if $Q_1 = \emptyset$,

$$i_g(D) \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + |Y_1|}{2},$$

if $Q_2 = \emptyset$, *and*

$$i_g(D) \geq i_l(D) \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + t}{2},$$

if $Q_1 \neq \emptyset$ *and* $Q_2 \neq \emptyset$.

Proof. Let $V(D)$ be partitioned into the subsets Y, Z, R_1, R_2 satisfying (4.18) such that $|Z| + k + 1 \leq |Y| \leq |V_c| - t$ for integers $k \geq 1$ and $t \geq 0$. If Y_1, Y_2, Q_1 and Q_2 are defined as above, then we observe that $|Z| \leq |Y| - 1 - k \leq |V_c| - 1 - k - t$, $Q_1 \to Y \to Q_2$, $(Q_1 \cup Y_1) \rightsquigarrow (Q_2 \cup Y_2)$ and $Y_1 \cup Y_2 \cup Y \subseteq V_i$. If $i = c$, then let $j = c - 1$ and if $i < c$, then let $j = c$. We now consider the following three cases.

Case 1. Let $Q_1 = \emptyset$. Then $Q_2 = Q$ and we obtain

$$\begin{aligned} & d(Y, V(D) - Y) - d(V(D) - Y, Y) \\ \geq \;& |Y||Q_2| - |Y||Z| \geq |Y|(|V(D)| - |V_i| - 2|Z|) \\ \geq \;& |Y|(|V(D)| - |V_c| - 2(|V_c| - 1 - k - t)) \\ = \;& |Y|(|V(D)| - 3|V_c| + 2 + 2k + 2t). \end{aligned}$$

According to Lemma 5.5, this implies that $i_l(D) \geq |V(D)| - 3|V_c| + 2 + 2k + 2t$, and hence, we have one part of the desired result. We will now show the second part.

Let $\delta^* = \min\{d^-(w) \mid w \in V_i\}$. Since $Y \subseteq V_i$ and thus $d^-(y) \leq |Z|$ for all $y \in Y$ we observe that $\delta^* \leq |Z| \leq |Y| - k - 1 \leq |V_i| - |Y_2| - 1 - k$. Let $\Delta^* = \max\{d^+(w), d^-(w) \mid w \in V(D) - V_i\}$ and note that $d^+(w) + d^-(w) \geq |V(D)| - |V_j|$ for all $w \in V(D) - V_i$. The fact that $\sum_{x \in Q_2}(d^-(x) - d^+(x)) \geq |Q_2|(|Y| - |Z| - |Y_2|) \geq |Q_2|(1 + k - |Y_2|)$ implies that there is a vertex $q \in Q_2$ such that $d^-(q) \geq d^+(q) + k - |Y_2| + 1$. This leads to $2d^-(q) - k + |Y_2| - 1 \geq d^+(q) + d^-(q) \geq |V(D)| - |V_j|$, and thus we conclude that $\Delta^* \geq \frac{|V(D)| - |V_j| + k - |Y_2| + 1}{2}$. This implies

$$\begin{aligned} i_g(D) \;&\geq\; \Delta^* - \delta^* \geq \frac{|V(D)| - |V_j| + k - |Y_2| + 1}{2} - |V_i| + |Y_2| + k + 1 \\ &=\; \frac{|V(D)| - |V_j| - 2|V_i| + 3k + 3 + |Y_2|}{2} \\ &\geq\; \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + |Y_2|}{2}, \end{aligned}$$

and the second part is proved.

Case 2. Let $Q_2 = \emptyset$. This is analogously to Case 1 (change the orientation of all the arcs in D).

Case 3. Let $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. Since $|V_i| + |V_j| \leq |V_{c-1}| + |V_c|$, we deduce that $|Q| - |V_j| \geq |V(D)| - |V_i| - |Z| - |V_j| \geq |V(D)| - |V_{c-1}| - |V_c| - (|V_c| - 1 - k - t)$. By Lemma 5.4 with $X = Q_1$ and $Y = Q_1 \cup Q_2 = Q$ and because of $Q \cap V_i = \emptyset$, it follows that

$$\begin{aligned} & \frac{d(Q_1, Q_2) + d(Q_2, Q_1)}{|Q_1|} + \frac{d(Q_1, Q_2) + d(Q_2, Q_1)}{|Q_2|} \\ = \;& \frac{d(Q_1, Q_2)}{|Q_1|} + \frac{d(Q_1, Q_2)}{|Q_2|} \geq |Q| - |V_j| \\ \geq \;& |V(D)| - |V_{c-1}| - 2|V_c| + 1 + k + t. \end{aligned}$$

Thus,

$$\begin{array}{llll}
\text{(i)} & \dfrac{d(Q_1,Q_2)}{|Q_1|} & \geq & \dfrac{|V(D)|-|V_{c-1}|-2|V_c|+1+k+t}{2}-|Y_2|+|Y_1| \text{ or} \\
\text{(ii)} & \dfrac{d(Q_1,Q_2)}{|Q_2|} & \geq & \dfrac{|V(D)|-|V_{c-1}|-2|V_c|+1+k+t}{2}+|Y_2|-|Y_1|.
\end{array}$$

Assume that (i) holds as the case when (ii) holds can be treated similarly. Because of $R_1 = Q_1 \cup Y_1$ and $R_2 = Q_2 \cup Y_2$, Lemma 5.5 yields

$$\begin{aligned}
i_g(D) \geq i_l(D) &\geq \frac{d(Q_1,V(D)-Q_1)-d(V(D)-Q_1,Q_1)}{|Q_1|} \\
&= \frac{d(Q_1,Q_2)}{|Q_1|}+\frac{d(Q_1,Y\cup Y_2)-d(Y\cup Y_2,Q_1)}{|Q_1|} \\
&+ \frac{d(Q_1,Z\cup Y_1)-d(Z\cup Y_1,Q_1)}{|Q_1|}-\frac{d(Q_2,Q_1)}{|Q_1|} \\
&\geq \left(\frac{|V(D)|-|V_{c-1}|-2|V_c|+1+k+t}{2}-|Y_2|+|Y_1|\right) \\
&+ (|Y|+|Y_2|)-(|Z|+|Y_1|) \\
&= \frac{|V(D)|-|V_{c-1}|-2|V_c|+1+k+t}{2}+|Y|-|Z| \\
&\geq \frac{|V(D)|-|V_{c-1}|-2|V_c|+3+3k+t}{2}.
\end{aligned}$$

This completes the proof of the theorem. □

Theorem 6.4 with $t = 0$ leads immediately to the following result on the path covering number.

Corollary 6.5 (Volkmann, Winzen [40]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a semicomplete multipartite digraph D such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$. If there exists a positive integer k such that $i_g(D) \leq \frac{|V(D)|-|V_{c-1}|-2|V_c|+3k+2}{2}$, then $pc(D) \leq k$.*

An analysis of the proof of the Theorem 6.4 yields the following result.

Corollary 6.6 (Stella, Volkmann, Winzen [22]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a semicomplete c-partite digraph D such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$. Let $pc(D) > k$ for an integer $k \geq 1$ and let $Y, Z, R_1, R_2, Q, Q_1, Q_2, V_i, Y_1$ and Y_2 be defined as in Theorem 6.4.*

If $Q_1 = \emptyset$ and $i_g(D) = \frac{|V(D)|-|V_{c-1}|-2|V_c|+3k+3+|Y_2|}{2}$, then the following holds.

i) $\min\{d^-(w) \mid w \in V_i\} = |Z| = |Y| - k - 1$.

ii) $|Y| = |V_i| - |Y_2|$, *which means that* $|Y_1| = 0$ *and* $|V_i \cap Z| = 0$.

iii) $Y \to Q_2 \to (Y_2 \cup Z)$.

iv) $d^-(q_2) = d^+(q_2) + k - |Y_2| + 1$ *for all* $q_2 \in Q_2$.

v) $\max\{d^+(w), d^-(w) \mid w \in V(D) - V_i\} = d^-(q)$ *for a vertex* $q \in Q_2$ *such that* $|V(q)| = |V_{c-1}|$

vi) $i_g(D) = \max\{d^-(q) \mid q \in Q_2\} - \min\{d^-(w) | w \in V_i\}$.

vii) $|V_i| = |V_c|$.

viii) $|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + |Y_2|$ *is even.*

Let $j = c - 1$, *if* $i = c$ *and* $j = c$, *if* $i < c$. *If* $Q_1 \neq \emptyset$ *and* $Q_2 \neq \emptyset$ *and* $i_g(D) = \frac{|V(D)|-|V_{c-1}|-2|V_c|+3k+3+t}{2}$, *then we conclude that*

a) $i_g(D) = i_l(D)$.

b) $\{|V_i|, |V_j|\} = \{|V_c|, |V_{c-1}|\}$.

c) $V_i \cap Z = \emptyset$, $|Z| = |Y| - 1 - k$, $|Y| = |V_c| - t$.

d) there is equality in Lemma 5.4 with $X = Q_1$ *and* $Y = Q = Q_1 \cup Q_2$, *which means that* $|V_m \cap Q_1| = |V_l \cap Q_1|$ *and* $|V_m \cap Q| = |V_l \cap Q|$ *for all* $1 \le l, m \le c$ *such that* $V_m \cap Q \neq \emptyset$ *and* $V_l \cap Q \neq \emptyset$.

e) $V_j \subseteq Q$.

f) $\frac{d(Q_1,Q_2)}{|Q_1|} = \frac{|V(D)|-|V_{c-1}|-2|V_c|+1+k+t}{2} - |Y_2| + |Y_1|$ *and*
$\frac{d(Q_1,Q_2)}{|Q_2|} = \frac{|V(D)|-|V_{c-1}|-2|V_c|+1+k+t}{2} + |Y_2| - |Y_1|$.

g) $d^+(q_1) = d^-(q_1) + i_g(D)$ *for all* $q_1 \in Q_1$ *and* $d^-(q_2) = d^+(q_2) + i_g(D)$ *for all* $q_2 \in Q_2$.

h) $Q_2 \to (Z \cup Y_2)$, $(Z \cup Y_1) \to Q_1$.

j) $|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + t$ *is even.*

If we especially observe the case $k = 1$, then we arrive at the following result.

Theorem 6.7 (Volkmann, Winzen [40]) *Let* $V_1, V_2, \ldots, V_c$ *be the partite sets of the semicomplete* c*-partite digraph* D *such that* $1 \le r = |V_1| \le |V_2| \le \ldots \le |V_c| \le r + p$ *for an integer* $p \ge 0$. *If* $c \ge \max\{2, 3 + \frac{2i_g(D)-5+p}{r}\}$, *then* D *contains a Hamiltonian path.*

Proof. Clearly, D contains a Hamiltonian path if and only if $pc(D) = 1$. Hence, according to Corollary 6.5 with $k = 1$, it is sufficient to show that $i_g(D) \le \frac{|V(D)|-|V_{c-1}|-2|V_c|+5}{2}$. Because of $c \ge 3 + \frac{2i_g(D)-5+p}{r}$, we conclude that $i_g(D) \le \frac{(c-3)r+5-p}{2}$, and together with $|V_1|, |V_2|, \ldots, |V_{c-2}| \ge r$, $|V_c| \le r+p$ and $c \ge 2$ this implies

$$\begin{aligned}\frac{|V(D)| - |V_{c-1}| - 2|V_c| + 5}{2} &= \frac{|V_1| + |V_2| + \ldots + |V_{c-2}| - |V_c| + 5}{2} \\ &\ge \frac{(c-3)r - p + 5}{2} \ge i_g(D),\end{aligned}$$

the desired result. □

If D is a multipartite tournament, then, according to Lemma 1.10, we can choose $p = 2i_g(D)$ in the previous theorem.

Corollary 6.8 (Volkmann, Winzen [40]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament D such that $1 \leq r = |V_1| \leq |V_2| \leq \ldots \leq |V_c|$. If $c \geq \max\{2, \frac{4i_g(D)-5}{r} + 3\}$, then D contains a Hamiltonian path.*

6.2 Two or three vertices of each partite set

To treat Problem 6.1 with $s_i = 2$ or $2 \leq s_i \leq 3$ for all $1 \leq i \leq c$ we still need the following two results.

Theorem 6.9 (Guo, Volkmann [13]) *Every partite set of a strongly connected c-partite tournament has at least one vertex that lies on cycles of each length m for $m \in \{3, 4, \ldots, c\}$.*

Theorem 6.10 (Yeo [50]) *Let D be a regular c-partite tournament with $c \geq 4$ and the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| = |V_2| = \ldots = |V_c| = r$. If w is an arbitrary vertex in D, then for all integers p with $3 \leq p \leq (c-2)r + 2$, there exists a p-cycle C in D such that $w \in V(C)$.*

Theorem 6.11 (Volkmann, Winzen [45]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D with $c \geq 2$ such that $|V_1| = |V_2| = \ldots = |V_c| = r \geq 2$. Then D contains a path with exactly two vertices from each partite set.*

Proof. If $r = 2$, then, according to Theorem 4.4, D is Hamiltonian and we are done. In the following let $r \geq 3$.

Case 1. Let $c \geq 5$. In view of Theorem 4.12, there exists a strongly connected subtournament T_1 of order c, which contains, by Theorem 4.26, a Hamiltonian cycle C. If T_2 is an arbitrary subtournament in $D - V(T_1)$ of order c, then, by Theorem 4.24, T_2 has a Hamiltonian path P'. Now $C \cup P'$ is a 1-path-cycle factor of $D[V(C) \cup V(P')]$. Applying Theorem 6.2, we see that the Hamiltonian path P of $D[V(C) \cup V(P')]$ has the desired properties.

Case 2. Let $c = 2$. In view of Theorem 4.4, D has a Hamiltonian path. Every part P of the Hamiltonian path with $|V(P)| = 4$ has the desired properties.

Case 3. Let $c = 3$. In view of Theorem 4.1, there exists a 3-cycle C_3. Analogously to Case 1, we arrive at the desired path.

Case 4. Let $c = 4$. This yields that $r = 2s \geq 4$ is even, and $d^+(x) = d^-(x) = 3s$ for each $x \in V(D)$. Let $V_1 = \{x_1, x_2, \ldots, x_r\}$, $V_2 = \{y_1, y_2, \ldots, y_r\}$, $V_3 = \{u_1, u_2, \ldots, u_r\}$, and $V_4 = \{v_1, v_2, \ldots, v_r\}$. With respect to Theorem 6.10, there exists a 5-cycle C_5 in D. If C_5 contains vertices from four partite sets, then we obtain the desired path as above.

If C_5 contains vertices from three partite sets, then let, without loss of generality, $C_5 = x_1y_1u_1x_2y_2x_1$. If there exists a path $v_iu_tv_j$ with $t \geq 2$ and

$i \neq j$, then Theorem 6.2 leads to the desired path. If not, then we can assume, without loss of generality, that there is a vertex $u_t \in V_3 - \{u_1\}$ such that $u_t \to V_4$.

Case 4.1. There exists one of the arcs x_1u_t or y_1u_t or x_2u_t or y_2u_t. In a first step, we will show that $(V(C_5) - \{w\}) \to V_4$, if $w \to u_t$ with $w \in \{x_1, y_1, x_2, y_2\}$.

Firstly, let $w = y_2$. If $v_i \to x_1$ or $v_i \to y_1$ or $v_i \to u_1$ or $v_i \to x_2$ for an i with $1 \leq i \leq r$, then we arrive at the cycle $u_tv_ix_1y_1u_1x_2y_2u_t$ or $u_tv_iy_1u_1x_2y_2u_t$ or $u_tv_iu_1x_2y_2u_t$ or $u_tv_ix_2y_2u_t$ through four partite sets and we are done. Hence we investigate now the case that $\{x_1, y_1, u_1, x_2\} \to V_4$.

If $w \in \{x_1, y_2\}$, then analogously as above we arrive at the desired result $(V(C_5) - \{w\}) \to V_4$. Hence, let $x_2u_t \in E(D)$. If $v_i \to y_2$ or $v_i \to x_1$ or $v_i \to y_1$ with $1 \leq i \leq r$, then we arrive at the cycle $u_tv_iy_2x_1y_1u_1x_2u_t$ or $u_tv_ix_1y_1u_1x_2u_t$ or $u_tv_iy_1u_1x_2u_t$ through four partite sets and we are done. If $v_i \to u_1$, then let us observe an arbitrary vertex v_j with $1 \leq j \leq r$ and $i \neq j$. If $v_j \to y_1$, then as above we arrive at a cycle of length 5 through 4 partite sets and we are done. If $y_1 \to v_j$, then $u_tv_iu_1x_2u_t$ and $y_2x_1y_1v_j$ is an 1-path-cycle factor and Theorem 6.2 yields the desired result.

So from now on, we may suppose that $(V(C_5) - \{w\}) \to V_4$, if $w \to u_t$ with $w \in \{x_1, y_1, x_2, y_2\}$. Without loss of generality, let $w = y_2$. Because of $d^+(u_t) = d^-(u_t) = 3s \geq 6$, there exist vertices x_p and v_j or y_p and v_j with $p \geq 3$ such that $v_j \to x_p \to u_t$ or $v_j \to y_p \to u_t$, say $v_j \to y_p \to u_t$. Applying Theorem 6.2 on the 1-path-cycle factor $x_1y_1u_1x_2v_k$ and $v_jy_pu_tv_j$ with $j \neq k$, we obtain the desired path of order 8.

Case 4.2. Let $u_t \to \{x_1, x_2, y_1, y_2\}$. If there is no arc leading from v_j with $1 \leq j \leq r$ to $N^-(u_t) - V(C_5) = N^-(u_t) \subseteq V_1 \cup V_2$, then we observe that $|N^-(v_j)| \geq |N^-(u_t)| + |\{u_t\}|$, a contradiction to the regularity of D. Hence, let $v_jx_pu_tv_j$ or $v_jy_pu_tv_j$ be a 3-cycle with $1 \leq j \leq r$ and $p \geq 3$. Since the other case follows similarly, we will treat the case that there exists the 3-cycle $C = v_jy_pu_tv_j$. Let $v_i \in V_4 - \{v_j\}$. If $y_2 \to v_i$, then we define the path $P' = y_1u_1x_2y_2v_i$, and if $v_i \to y_2$, then let $P' = v_iy_2x_1y_1u_2$. Applying Theorem 6.2, we see that the Hamiltonian path P of $D[V(C_5) \cup V(P')]$ has the desired properties. □

Theorem 6.12 (Volkmann, Winzen [45]) *Let V_1, V_2, V_3 be the partite sets of a regular 3-partite tournament D such that $|V_1| = |V_2| = |V_3| = r \geq 3$. Then D contains a path P with at exactly three vertices from each partite set.*

Proof. If $r = 3$, then, according to Theorem 4.4, D is Hamiltonian and we are done. Let now $r \geq 4$. According to Theorem 4.1, there exists a 3-cycle C_3 in D. Theorem 4.16 shows that

$$\kappa(D - V(C_3)) \geq \frac{3r - 3 - (r-1) - 4}{3} = \frac{2r - 6}{3} > 0.$$

Therefore, Theorem 4.1 yields a further 3-cycle C_3^* in $D - V(C_3)$. Let $P = a_1a_2a_3$ be a path in $D - (V(C_3) \cup V(C_3^*))$ with exactly one vertex from each partite set. Then $C_3 \cup C_3^* \cup P$ is a 1-path-cycle factor, and Theorem 6.2 leads to the desired result. □

Theorem 6.13 (Volkmann, Winzen [45]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D with $c \geq 2$ such that $|V_1| = |V_2| = \ldots = |V_c| = r \geq 3$. Then D contains a path P with at least two and at most three vertices from each partite set such that*

$$|V(P)| \geq 2c + \frac{c}{2} + \frac{r-c}{2(r-1)}.$$

Proof. Case 1. Let $2 \leq c \leq 3$. If $c = 2$ or $c = 3$, then the desired result follows from Theorem 4.4 or Theorem 6.12, respectively.

Case 2. Let $c = 4$. It follows that $r \geq 4$ is even. In view of Theorem 4.1, there exists a 3-cycle $C_3 = x_1x_2x_3x_1$ such that, without loss of generality, $x_i \in V_i$ for $1 \leq i \leq 3$.

Subcase 2.1. Let $r = 4$. According to Theorem 4.16, we deduce that $\kappa(D) \geq 4$ and thus we conclude that the subdigraph $D - V(C_3)$ is strongly connected. By Theorem 6.9, there exists a vertex $w \in V_4$, which is contained in a 3-cycle $C_3' = wy_1y_2w$ of $D - V(C_3)$. Let, without loss of generality, $y_i \in V_i$ for $i = 1, 2$. According to Theorem 4.24, there exist a path P' of length three in $D - (V(C_3) \cup V(C_3'))$ such that all vertices of P' belong to different partite sets, and $C_3 \cup C_3' \cup P'$ is a 1-path-cycle factor of $D[V(C_3) \cup V(C_3') \cup V(P')]$. Applying Theorem 6.2, we see that the Hamiltonian path P of $D[V(C_3) \cup V(C_3') \cup V(P')]$ has at least two and at most three vertices from each partite set such that

$$|V(P)| = 10 = 2c + \frac{c}{2} + \frac{r-c}{2(r-1)}.$$

Subcase 2.2. Let $r \geq 6$. According to Theorem 5.3, we deduce that $\kappa(D) \geq 7$ or D is a member of the family $\mathcal{F}_1$ of Example 5.1. If $D \in \mathcal{F}_1$, then it is easy to verify that there exists a path P with the desired properties. Otherwise, in view of Theorem 6.9, we can choose two vertex disjoint 3-cycles C_3' and C_3'' in $D - V(C_3)$ such that both of these cycles contain vertices v and w of the partite set V_4. If $C_3' = vy_1y_2v$ and $C_3'' = wu_1u_2w$ such that, for example, $y_i, u_i \in V_i$ for $i = 1, 2$, then we choose in $D - (V(C_3) \cup V(C_3') \cup V(C_3''))$ a path $P' = ab$ with $a \in V_3$ and $b \in V_4$. Now the 1-path-cycle factor $V(C_3) \cup V(C_3') \cup V(C_3'') \cup V(P')$ of $D[V(C_3) \cup V(C_3') \cup V(C_3'') \cup V(P')]$ yields togehter with Theorem 6.2 the desired path P with

$$|V(P)| = 11 \geq 2c + \frac{c}{2} + \frac{r-c}{2(r-1)}.$$

Case 3. Let $c \geq 5$. In view of Theorem 4.12, there exists a strongly connected subtournament T_c of order c, which contains, by Theorem 4.26, a Hamiltonian cycle C. Now let P^* be a longest path in $D' = D - V(T_c)$ with at least one and at most two vertices from each partite set in D' such that the first c vertices of P^* belong to different partite sets. Let

$$P^* = x_1x_2 \ldots x_cy_1y_2 \ldots y_t.$$

If $t \geq \frac{c}{2} + \frac{r-c}{2(r-1)}$, then $C \cup P^*$ is a 1-path-cycle factor of $D[V(C) \cup V(P^*)]$. Applying Theorem 6.2, we see that the Hamiltonian path P of $D[V(C) \cup V(P^*)]$ has the desired properties.

Suppose now that $t < \frac{c}{2} + \frac{r-c}{2(r-1)}$, and let $V_1', V_2', \ldots, V_{c-t}'$ be the partite sets in $D' - V(P^*)$ with the property that $V_i' \cap V(y_j) = \emptyset$ for all $1 \leq i \leq c-t$ and $1 \leq j \leq t$. It follows that $|V_i'| = r-2$ for $1 \leq i \leq c-t$. Furthermore, it is straightforward to verify that

$$(V_1' \cup V_2' \cup \ldots \cup V_{c-t}') \to \{y_1, y_2, \ldots, y_t\}.$$

Subcase 3.1. There exists an arc $y_t x_i$ for any i with $1 \leq i \leq c-t$. This leads to the cycle $C_1 = x_i x_{i+1} \ldots x_c y_1 y_2 \ldots y_t x_i$ in D' with $|V(C_1)| = c+t-i+1$. According to Theorem 4.24, there is a path $P' = a_1 a_2 \ldots a_{c-t}$ in $D'[V_1' \cup V_2' \cup \ldots \cup V_{c-t}']$ with exactly one vertex from each partite set. Now the path

$$W = a_1 a_2 \ldots a_{c-t} y_1 y_2 \ldots y_t x_i x_{i+1} \ldots x_c$$

is also a path with at least one and at most two vertices from each partite set in D' such that the first c vertices belong to different partite sets. Because of $|V(W)| = 2c+1-i \geq c+t+1$, we arrive at the contradiction to our assumption that P^* is the longest path with these properties.

Subcase 3.2. Assume that $\{x_1, x_2, \ldots, x_{c-t}\} \rightsquigarrow y_t$. Since D is regular, $t < \frac{c}{2} + \frac{r-c}{2(r-1)}$, and $(V_1' \cup V_2' \cup \ldots \cup V_{c-t}') \to y_t$, we finally obtain the contradiction

$$\begin{aligned}
\frac{r(c-1)}{2} &= d_D^-(y_t) \geq c-t+(c-t)(r-2) = c(r-1) - t(r-1) \\
&> c(r-1) - \Big(\frac{c}{2} + \frac{r-c}{2(r-1)}\Big)(r-1) \\
&= c(r-1) - \frac{c(r-1)}{2} - \frac{r-c}{2} = \frac{r(c-1)}{2}.
\end{aligned}$$

□

6.3 $\alpha(D) - s$ vertices of each partite set

In this section, we will look for paths in regular c-partite tournaments with exactly $\alpha(D) - s$ vertices from each partite set. For the case that $s = 1$, Theorem 4.35 yields a solution, if $c \geq 5$ or $c = 4$ and $r \geq 4$ or $c = 3$ or if $c = 2$ and D is not isomorphic to the bipartite digraph $B(t,t,t,t)$ of Definition 4.21. If $c = 4$, then the fact that D is regular implies that r is even. According to Theorem 4.24, it follows that a regular 4-partite tournament always contains a path with exactly $1 = 2-1$ vertices of each partite set. Since the bipartite tournament $B(t-1, t-1, t, t)$ also contains a Hamiltonian path, we arrive at the following corollary.

Corollary 6.14 (Volkmann, Winzen [45]) *Every regular c-partite tournament with at least $r \geq 2$ vertices in each partite set contains a path with exactly $r-1$ vertices of each partite set.*

So, from now on, we may suppose that $s \geq 2$.

Theorem 6.15 (Volkmann, Winzen [45]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D with $c \geq 4$ and $|V_1| = |V_2| = \ldots = |V_c| = r \geq 2$. Furthermore, let X be an arbitrary subset of $V(D)$ consisting of m partite sets with exactly s vertices and $c-m$ partite sets with exactly $s-1$ vertices for $0 < m \leq c$ and $1 \leq s \leq r-1$. If*

$$r \geq \begin{cases} 3s + \left\lceil \frac{4s-5}{c-3} \right\rceil & \text{and} \quad m = c \\ 3s + \left\lceil \frac{4s-4}{c-3} \right\rceil & \text{and} \quad m = c-1 \\ 3s - 2 + \left\lceil \frac{4s+2m-8}{c-3} \right\rceil & \text{and} \quad m \leq c-2 \end{cases},$$

then D contains a path P such that $V(P) = V(D) - X$.

Proof. Let $D' = D - X$ with the partite sets $V_1', V_2', \ldots, V_c'$ such that $|V_1'| \leq |V_2'| \leq \ldots \leq |V_c'| \leq |V_1'| + 1$. Since D is regular, it follows that

$$i_g(D') \leq \begin{cases} s(c-1), & \text{if} \quad c-1 \leq m \leq c \\ (s-1)(c-1) + m, & \text{if} \quad m \leq c-2 \end{cases}.$$

If D' contains a Hamiltonian path P, then this path P has the desired properties. Using Theorem 6.7 with $p = 0$, if $m = c$, and $p = 1$, if $m \leq c-1$, we see that it is sufficient to show that

$$3 + \frac{2i_g(D') - 5 + p}{r - s} \leq \left\{ \begin{array}{lll} 3 + \frac{2s(c-1)-5}{r-s}, & \text{if} & m = c \\ 3 + \frac{2s(c-1)-4}{r-s}, & \text{if} & m = c-1 \\ 3 + \frac{2(s-1)(c-1)+2m-4}{r-s}, & \text{if} & m \leq c-2 \end{array} \right\} \leq c.$$

If we distinguish the cases $m = c$, $m = c-1$ and $m \leq c-2$, then, noticing that $r \in \mathbb{N}$, equivalent transformations yield the bounds for r as in the assumptions of this theorem. This completes the proof of the theorem. □

This result immediately yields the following two corollaries.

Corollary 6.16 (Volkmann, Winzen [45]) *Every regular c-partite tournament with $c \geq 4$ and at least $r \geq 7s - 5$ in each partite set contains a path with exactly $r - s$ vertices from each partite set for a given integer $s \in \mathbb{N}$.*

Corollary 6.17 (Volkmann, Winzen [45]) *Almost all regular multipartite tournaments D with at least four partite sets contain a path with exactly $r - s$ vertices from each partite set for a given integer $s \in \mathbb{N}$, if r is the cardinality of each partite set of D.*

Theorem 6.18 (Volkmann, Winzen [45]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D with $c \geq 5$ such that $|V_1| = |V_2| = \ldots = |V_c| = r$. If $r \geq 5s - 3$ for an integer $s \geq 2$, then D contains a path with exactly $r - s$ vertices of each partite set.*

Proof. Let X be an arbitrary subset of $V(D)$ with exactly s vertices of each partite set ($2 \leq s \leq r-1$). Theorem 6.15 with $m = c$ guarantees the existence of a Hamiltonian path P of $D' := D - X$ that has the desired properties, if $r \geq 3s + \lceil \frac{4s-5}{c-3} \rceil$. Since $c \geq 5$, the proof is complete for the case that $r \geq 5s - 2$.

Hence, let $r = 5s - 3$ and let D' be defined as above. If D' contains a Hamiltonian path, then we arrive at the desired result. Hence, assume that D' does not contain any Hamiltonian path. If

$$c \geq 3 + \frac{2s(c-1)-5}{4s-3} \Leftrightarrow c \geq 5 + \left\lceil \frac{1}{2s-3} \right\rceil = 6,$$

then Theorem 6.7 leads to a contradiction. Hence, let $c = 5$ and thus $d_D^+(x) = d_D^-(x) = 2r = 10s - 6$ for all $x \in V(D)$. If $i_g(D') \leq s(c-1) - 1$, then because of

$$5 \geq 5 - \frac{1}{4s-3} = 3 + \frac{2s(c-1)-7}{4s-3} \geq 3 + \frac{2i_g(D')-5}{r-s}$$

Theorem 6.7 yields a contradiction. Thus we may suppose that $i_g(D') = s(c-1) = 4s$. Let $V_1', V_2', \ldots, V_c'$ be the partite sets of D' and let us define $Y, Z, R_1, R_2, Q, Q_1, Q_2, t, V_i', Y_1$ and Y_2 as in Theorem 6.4.

Firstly, we suppose that $|Q_1| = 0$. If $|Y_2| > 0$, then Theorem 6.4 with $k = 1$ implies that

$$\begin{aligned} i_g(D') &\geq \frac{|V(D')| - |V_{c-1}'| - 2|V_c'| + 6 + |Y_2|}{2} = \frac{(c-3)(r-s) + 6 + |Y_2|}{2} \\ &= r - s + 3 + \frac{|Y_2|}{2} = 4s + \frac{|Y_2|}{2} > 4s, \end{aligned}$$

a contradiction. Hence let $|Y_2| = 0$ and $i_g(D') = \frac{|V(D')|-|V_{c-1}'|-2|V_c'|+6}{2} = 4s$. Applying Corollary 6.6 we see that $|Y| = |V_i'| = r - s = 4s - 3$ (ii)) and $|Z| = 4s - 5$ (i)). This yields that $|Q_2| = |V(D')| - |Y| - |Z| = 5(r-s) - (4s-3) - (4s-5) = 12s - 7$. Since $Y \rightarrow Q_2$ (iii)), we arrive at $d^+(y) \geq 12s - 7$ for all $y \in Y \neq \emptyset$, because of $s \geq 2$ a contradiction to $d^+(y) = 10s - 6$. Analogously, we see that the case that $|Q_2| = 0$ is impossible.

Secondly, we assume that $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. If $t > 0$, then Theorem 6.4 with $k = 1$ implies that

$$i_g(D') \geq \frac{|V(D')| - |V_{c-1}'| - 2|V_c'| + 6 + t}{2} > 4s,$$

a contradiction. Hence let $t = 0$ and $i_g(D') = \frac{|V(D')|-|V_{c-1}'|-2|V_c'|+6}{2} = 4s$. Applying Corollary 6.6 we see that $|Y| = |V_c'| = 4s - 3 = |Z| + 2$ (c)) and thus $|Y_1| = |Y_2| = 0$. Now Corollary 6.6 f) implies that $|Q_1| = |Q_2|$, and we arrive at the contradiction

$$2|Q_1| = |Q_1| + |Q_2| = |Q| = |V(D')| - |Y| - |Z| = 12s - 7.$$

This completes the proof of this theorem. □

The results above lead us to the following conjecture.

Conjecture 6.19 (Volkmann, Winzen [45]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D with $c \geq 2$ such that $|V_1| = |V_2| = \ldots = |V_c| = r \geq 2$. Then D contains a path with exactly m vertices of each partite set for every $m \in \{1, 2, \ldots, r\}$.*

Notice that, according to the Theorems 4.4, 4.22, 4.24, 6.11 and Corollary 6.14, the conjecture holds for $c = 2$ or $m \in \{1, 2, r-1, r\}$.

Chapter 7

The existence of a Hamiltonian path

Hamiltonian cycles in multipartite tournaments are well studied (see e.g. [3, 4, 12, 18, 48, 51]). A good example for this is Yeo's [51] result that every regular multipartite tournament is Hamiltonian (Theorem 4.4).

On the other hand, it is not paid much attention on the existence of Hamiltonian paths in such digraphs. Apart from Theorem 6.2 of Gutin [14], there are not many results concerning this theme.

The aim of this chapter is to solve the following problem.

Problem 7.1 (Volkmann, Winzen [40]) *For all i find the smallest value, $g(i)$, with the property that all c-partite tournaments with $i_g \leq i$ and $c \geq g(i)$ have a Hamiltonian path.*

This means that we give a solution of Problem 6.1 with $s_j = \alpha(D)$ for all $1 \leq j \leq c$, if D is a c-partite tournament of a given global irregularity i.

In the first section of this chapter, we will examine the case that $i \leq 2$. If D is regular, then Theorem 4.4 clearly guarantees the existence of a Hamiltonian path, which is already shown by Zhang [52] in 1989. Using a sufficient condition for multipartite tournaments with an arbitrary large irregularity number to contain a Hamiltonian path, we will show that every almost regular c-partite tournament D contains a Hamiltonian path with the exception that $c = 2$ and one partite set consists of two vertices more than the other partite set. Furthermore, we will precisely examine the case that $i_g(D) = 2$. In this case D contains a Hamiltonian path, if $c \geq 5$. If $c = 4$, then there is only a finite family of graphs that do not contain any Hamiltonian path. Finally, we will show that almost all c-partite tournaments with $c \geq 4$ and $i_g \leq i$ for $i \geq 2$ contain a Hamiltonian path. Since there are infinitely many 2-partite and 3-partite tournaments with given irregularity $i_g(D) \geq 2$ that have no Hamiltonian path at all, the bound $c \geq 4$ is best possible.

In the last section, we will solve Problem 7.1 completely by proving that $g(i) = 4i - 4$, if $i \geq 3$.

7.1 The case $i_g(D) \le 2$

First, we want to treat the case of an almost regular c-partite tournament D. For the case $c = 2$ we can use the following theorem to examine the existence of a Hamiltonian path.

Theorem 7.2 (Volkmann [33]) *If D is an almost regular bipartite tournament with the partite sets X, Y such that $1 \le |X| \le |Y|$, then every arc of D is contained in a Hamiltonian path if and only if $|Y| \le |X| + 1$ and D is not isomorphic to $T_{3,3}$, where $T_{3,3}$ is the bipartite tournament presented in Figure 7.1.*

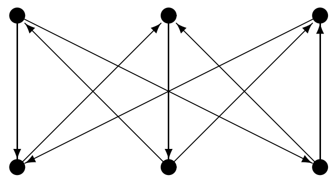

Figure 7.1: The almost regular bipartite tournament $T_{3,3}$

Using this result together with Corollary 6.8 we arrive at the following theorem.

Theorem 7.3 (Volkmann, Winzen [40]) *Let D be an almost regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \le |V_2| \le \ldots \le |V_c|$. Then D contains a Hamiltonian path if and only if $c \ge 3$ or $c = 2$ and $|V_2| \le |V_1| + 1$.*

Proof. Firstly, let $c = 2$. Suppose that $|V_2| = |V_1| + 2$. Then it is obvious that D does not contain any Hamiltonian path, because the vertices of this path would alternately be part of the partite sets V_1 and V_2. Hence let $|V_2| \le |V_1| + 1$. In this case Theorem 7.2 shows that D contains a Hamiltonian path, since $T_{3,3}$ is Hamiltonian.

Secondly, let $c \ge 3 \ge \max\{2, 3 - \frac{1}{r}\}$ for all $r \in \mathbb{N}$. In this case, Corollary 6.8 yields the desired result. □

The case that $i_g(D) = 2$ is more complicated as the following considerations demonstrate.

Theorem 7.4 (Volkmann, Winzen [40]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament D such that $1 \le r = |V_1| \le |V_2| \le \ldots \le |V_c|$. If $i_g(D) = 2$, $c \ge 4$ and D doesn't have one of the partition-sequences $1, 1, 2, 4$; $1, 2, 3, 5$; $1, 1, 3, 4$ and $2, 2, 4, 6$, then D contains a Hamiltonian path.*

Proof. Since $\max\{\frac{8-5}{r} + 3 \mid r \in \mathbb{N}\} = 6 > 3$, Corollary 6.8 yields the desired result, if $c \ge 6$.

Hence, let $c = 4$ or $c = 5$ and assume that D does not contain any Hamiltonian path. Let the sets $Y, Z, R_1, R_2, Q, Q_1, Q_2, Y_1, Y_2$ be defined as in the proof of Theorem 6.4. Since the case that $Q_2 = \emptyset$ follows analogously as the case that $Q_1 = \emptyset$, in the following we will always distinguish the two cases that $Q_1 = \emptyset$ or $Q_1, Q_2 \neq \emptyset$.

Case 1. Let $c = 5$. If $r \geq 2$, then because of $c > \max\{\frac{8-5}{r} + 3 \mid r \geq 2\} = \frac{9}{2} > 3$ and Corollary 6.8 D contains a Hamiltonian path and the proof is finished. If $|V_5| \leq 4$, then $\frac{|V_1|+|V_2|+|V_3|-|V_5|+5}{2} \geq 2 = i_g(D)$, a contradiction to Corollary 6.5. If $|V_5| = 5$ and $|V_1| + |V_2| + |V_3| \geq 4$, then we analogously arrive at a contradiction. Altogether, we see that there remain to consider the following five partition-sequences.

Subcase 1.1. Let $(n_i) = 1, 1, 1, 1, 5$. In this case, we see that $d^+(x) = d^-(x) = 4$, if $x \in V_1 \cup V_2 \cup V_3 \cup V_4$ and $d^+(x) = d^-(x) = 2$, if $x \in V_5$, which means $i_l(D) = 0$. Since $\frac{|V(D)|-|V_4|-2|V_5|+6}{2} = 2 > i_l(D)$, it remains to consider the case that $Q_1 = \emptyset$ in Theorem 6.4. If $Y = V_5$, then it follows that $|Z| \leq 3$ and thus $|Q_2| \geq 1$. This yields $d^-(x) \geq 5$ for all $x \in Q_2$, a contradiction to $d^-(x) \leq 4$ for all $x \in V(D)$. If $|Y| = 4$, then it follows that $|Z| \leq 2$ and thus $|Q_2| \geq 2$. This implies that there is an arc $pq \in E(D[Q_2])$. Since $d^-(q) \geq 5$, we arrive at a contradiction. If $|Y| \leq |V_5| - 2$, then $|V(D)| - 3|V_5| + 8 = 2 > i_l(D)$ contradicts Theorem 6.4 with $t = 2$.

Subcase 1.2. Let $(n_i) = 1, 1, 1, 2, 5$. Since $i_g(D) = 2$, this is impossible.

Subcase 1.3. Let $(n_i) = 1, 1, 1, 3, 5$. This yields $d^+(x) = d^-(x) = 5$, if $x \in V_1 \cup V_2 \cup V_3$, $d^+(x) = d^-(x) = 4$ or $\{d^+(x), d^-(x)\} = \{3, 5\}$, if $x \in V_4$ and $d^+(x) = d^-(x) = 3$, if $x \in V_5$.

Firstly, we assume that $Q_1 = \emptyset$. If $Y = V_5$, then we conclude that $|Z| \leq 3$ and thus $|Q_2| \geq 3$. Since $d^+(x) = d^-(x) = 3$ for all $x \in V_5$, it follows that $|Z| = |Q_2| = 3$ and $Z \to Y$. It is obvious that there are either in Q_2 or in Z two vertices of different partite sets. Hence, there is an arc $p \to q$ that is either in $E(D[Q_2])$ or in $E(D[Z])$. If $pq \in E(D[Q_2])$, then $d^-(q) \geq 6$, and if $pq \in E(D[Z])$, then $d^+(p) \geq 6$, in both cases a contradiction. If $|Y| = 4$, then we see that $|Z| \leq 2$ and thus $|Q_2| \geq 4$, a contradiction to $d^+(x) = 3$ for all $x \in V_5$. Hence, let $|Y| \leq |V_5| - 2$. But now, $|V(D)| - 3|V_5| + 8 = 4 > i_l(D)$ contradicts Theorem 6.4.

Consequently, it remains to consider the case that $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. Note that $\frac{|V(D)|-|V_4|-2|V_5|+6}{2} = 2 = i_g(D)$. According to Theorem 6.4 and Corollary 6.6 we have $t = 0$ and $|Y| = |Z|+2$. This implies $Y = V_5$ and $|Z| = |Y|-2 = 3$. Hence, we conclude that $|Q| = 3$ and, without loss of generality, let $|Q_1| = 1$ and $|Q_2| = 2$. If there is an arc leading from Q_1 to Q_2 and $q_1 \in Q_1$, then it follows that $d^+(q_1) \geq |Y| + 1 = 6$, a contradiction. Since $Q_1 \rightsquigarrow Q_2$, we obtain $Q = V_4$ and thus $Z = V_1 \cup V_2 \cup V_3$. Let $D' = D[V_1 \cup V_2 \cup V_3]$. Because of $Q_2 \to Z \to Q_1$, we have on the one hand that

$$\begin{aligned} 15 = \sum_{x \in V(D')} d^+(x) &= d(V(D'), Q_1) + \sum_{x \in V(D')} d^+_{D'}(x) + d(V(D'), Y) \\ &= 3 + 3 + d(V(D'), Y), \end{aligned}$$

which means $d(V(D'),Y)=9$. Since $Q_1\to Y\to Q_2$, we observe on the other hand that

$$15=\sum_{y\in Y} d^-(y)=d(Q_1,Y)+d(V(D'),Y)=5+d(V(D'),Y),$$

which means $d(V(D'),Y)=10$, a contradiction.
Subcase 1.4. Let $(n_i)=1,1,1,4,5$. Since $i_g(D)=2$, this is impossible.
Subcase 1.5. Let $(n_i)=1,1,1,5,5$. This yields that $d^+(x)=d^-(x)=6$, if $x\in V_1\cup V_2\cup V_3$ and $d^+(x)=d^-(x)=4$, if $x\in V_4\cup V_5$, which means $i_l(D)=0$. Since $|V(D)|-3|V_5|+3=1>i_l(D)$ and $\frac{|V(D)|-|V_4|-2|V_5|+5}{2}=\frac{3}{2}>i_l(D)$, this contradicts Theorem 6.3.

Case 2. Let $c=4$. If $r\geq 3$, then $4\geq\max\{2,\frac{3}{r}+3\}$ and Corollary 6.8 yields that D contains a Hamiltonian path, a contradiction. If $r=2$ and $|V_4|\leq 5$ or $r=2$, $|V_4|=6$ and $|V_1|+|V_2|\geq 5$, then $\frac{|V(D)|-|V_3|-2|V_4|+5}{2}\geq 2=i_g(D)$ leads to a contradiction to Corollary 6.5. If $r=1$ and $|V_4|\leq 3$ or $r=1$, $|V_4|=4$ and $|V_1|+|V_2|\geq 3$ or $r=1$, $|V_4|=5$ and $|V_1|+|V_2|\geq 4$, then $\frac{|V_1|+|V_2|-|V_4|+5}{2}\geq 2=i_g(D)$, a contradiction to Corollary 6.5.

Summarizing our results, we see that, according to the assertion of this theorem, there remain to treat 14 partition-sequences.
Subcase 2.1. Because of $i_g(D)=2$, the partition-sequences $1,1,1,5$; $1,1,3,5$; $1,1,5,5$; $1,2,2,5$; $1,2,4,5$; $2,2,3,6$ and $2,2,5,6$ are impossible.
Subcase 2.2. Let $(n_i)=1,1,2,5$. In this case, we obtain that $d^+(x)=d^-(x)=4$, if $x\in V_1\cup V_2$, $\{d^+(x),d^-(x)\}=\{3,4\}$, if $x\in V_3$, and $d^+(x)=d^-(x)=2$, if $x\in V_5$, which means $i_l(D)=1$. Since $\frac{|V_1|+|V_2|-|V_4|+6}{2}=\frac{3}{2}>i_l(D)$, it remains to consider the case that $Q_1=\emptyset$ in Theorem 6.4. If $Y=V_4$, then we conclude that $|Z|\leq 3$ and thus $|Q_2|\geq 1$, a contradiction to $d^-(x)\leq 4$ for all $x\in V(D)$. If $|Y|=4$, then it follows that $|Z|\leq 2$ and $|Q_2|\geq 2$. This implies that there is an arc $p\to q$ that is either in $E(D[Q_2])$ or in $E(D[Z])$. Since $d^-(x)=2$ for all $x\in Y$, we conclude that $|Z|=2$ and $Z\to Y$. If $pq\in E(D[Q_2])$, then $d^-(q)\geq 5$ and if $pq\in E(D[Z])$, then $d^+(p)\geq 5$, in both cases a contradiction. Hence, let $|Y|\leq 3=|V_4|-2$. Then because of $|V(D)|-3|V_4|+8=2>i_l(D)$ we arrive at a contradiction to Theorem 6.4.
Subcase 2.3. Let $(n_i)=1,1,4,5$. This yields that $d^+(x)=d^-(x)=5$, if $x\in V_1\cup V_2$, $\{d^+(x),d^-(x)\}=\{3,4\}$, if $x\in V_3$, and $d^+(x)=d^-(x)=3$, if $x\in V_5$, which means $i_l(D)=1$. Since $\frac{|V_1|+|V_2|-|V_4|+6}{2}=\frac{3}{2}>i_l(D)$, it remains to consider the case that $Q_1=\emptyset$ in Theorem 6.4. If $Y=V_4$, then it follows that $|Z|\leq 3$ and thus $|Q_2|\geq 3$. This implies that there is a vertex $q_2\in Q_2\cap V_3$, since $Y\to Q_2$ a contradiction to $d^-(x)\leq 4$ for all $x\in V_3$. Consequently, let $|Y|\leq 4$. In this case, the fact that $|V(D)|-3|V_4|+6=2>i_l(D)$ contradicts Theorem 6.4.
Subcase 2.4. Let $(n_i)=1,2,5,5$. This yields that $d^+(x)=d^-(x)=6$, if $x\in V_1$, $\{d^+(x),d^-(x)\}=\{5,6\}$, if $x\in V_2$, and $d^+(x)=d^-(x)=4$, if $x\in V_3\cup V_4$, which means $i_l(D)=1$. Since $\frac{|V_1|+|V_2|-|V_4|+5}{2}=\frac{3}{2}>i_l(D)$ and $|V(D)|-3|V_4|+3=1=i_l(D)$, we have a contradiction to Theorem 6.3.
Subcase 2.5. Let $(n_i)=1,1,1,4$. In this case, we observe that $d^+(x)=d^-(x)=3$, if $x\in V_1\cup V_2\cup V_3$, and $\{d^+(x),d^-(x)\}=\{1,2\}$, if $x\in V_4$, which means

$i_l(D) = 1$. Because of $\frac{|V_1|+|V_2|-|V_4|+6}{2} = 2 > i_l(D)$, it remains to consider the case that $Q_1 = \emptyset$ in Theorem 6.4. If $Y = V_4$, then we conclude that $|Z| \leq 2$ and thus $|Q_2| \geq 1$, a contradiction to $d^-(x) \leq 3$ for all $x \in V(D)$. If $|Y| = 3$, then $|Z| \leq 1$ and $|Q_2| \geq 2$. Hence, there exists an arc $p \rightarrow q$ with $p, q \in Q_2$. Since $d^-(q) \geq 4$, we arrive at a contradiction. Consequently, let $|Y| \leq 2$. In this case, $|V(D)| - 3|V_4| + 8 = 3 > i_l(D)$ contradicts Theorem 6.4.
Subcase 2.6. Let $(n_i) = 1, 1, 4, 4$. This yields $\{d^+(x), d^-(x)\} = \{4, 5\}$, if $x \in V_1 \cup V_2$ and $d^+(x) = d^-(x) = 3$, if $x \in V_3 \cup V_4$, which means $i_l(D) = 1$. Since $\frac{|V_1|+|V_2|-|V_4|+5}{2} = \frac{3}{2} > i_l(D)$ and $|V(D)| - 3|V_4| + 3 = 1 = i_l(D)$, we arrive at a contradiction to Theorem 6.3.
Subcase 2.7. Let $(n_i) = 2, 2, 2, 6$. In this case, we observe that $d^+(x) = d^-(x) = 5$, if $x \in V_1 \cup V_2 \cup V_3$, and $d^+(x) = d^-(x) = 3$, if $x \in V_4$, which means $i_l(D) = 0$. Because of $\frac{|V_1|+|V_2|-|V_4|+6}{2} = 2 > i_l(D)$, it remains to consider the case that $Q_1 = \emptyset$ in Theorem 6.4. If $Y = V_4$, then we conclude that $|Z| \leq 4$ and thus $|Q_2| \geq 2$. Since $Y \rightarrow Q_2$, we have a contradiction to $d^-(x) \leq 5$ for all $x \in V(D)$. If $|Y| = 5$, then it follows $|Z| \leq 3$ and $|Q_2| \geq 3$. This yields the existence of an arc $p \rightarrow q$ with $p, q \in Q_2$. Hence, $d^-(q) \geq 6$, a contradiction. Consequently, let $|Y| \leq 4$. But now $|V(D)| - 3|V_4| + 8 = 2 > i_l(D)$ contradicts Theorem 6.4.
Subcase 2.8. Let $(n_i) = 2, 2, 6, 6$. This implies that $d^+(x) = d^-(x) = 7$ for all $x \in V_1 \cup V_2$ and $d^+(x) = d^-(x) = 5$ for all $x \in V_3 \cup V_4$, which means $i_l(D) = 0$. Since $\frac{|V_1|+|V_2|-|V_4|+5}{2} = \frac{3}{2} > i_l(D)$ and $|V(D)| - 3|V_4| + 3 = 1 > i_l(D)$, this is a contradiction to Theorem 6.3. □

For the partition-sequences $1, 1, 2, 4$; $2, 2, 4, 6$; $1, 2, 3, 5$ and $1, 1, 3, 4$ there are multipartite tournaments that do not have any Hamiltonian path as the following four examples demonstrate.

Example 7.5 (Volkmann, Winzen [40]) *Let* $V_1 = \{u\}$, $V_2 = \{v\}$, $V_3 = \{x_1, x_2\}$ *and* $V_4 = \{y_1, y_2, y_3, y_4\}$ *be the partite sets of a 4-partite tournament* D *such that* $(V_1 \cup V_2) \rightarrow x_1 \rightarrow V_4 \rightarrow x_2 \rightarrow (V_1 \cup V_2)$ *and* $u \rightarrow v \rightarrow y_4 \rightarrow u \rightarrow y_1 \rightarrow v \rightarrow y_3 \rightarrow u \rightarrow y_2 \rightarrow v$ *(see Figure 7.2). Then* $i_g(D) = 2$ *and* D *has the partition-sequence* $1, 1, 2, 4$ *but no Hamiltonian path.*

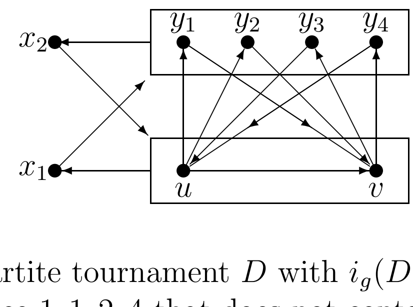

Figure 7.2: A 4-partite tournament D with $i_g(D) = 2$ and the partition-sequence $1, 1, 2, 4$ that does not contain a Hamiltonian path

Example 7.6 (Volkmann, Winzen [40]) *Let* D *be a multipartite tournament with the partite sets* $V_1 = \{u_1, u_2\}$, $V_2 = \{v_1, v_2\}$, $V_3 = \{x_1, x_2, x_3, x_4\}$

and $V_4 = \{y_1, y_2, y_3, y_4, y_5, y_6\}$. *If* $V_4 \to \{x_3, x_4\} \to (V_1 \cup V_2) \to \{x_1, x_2\} \to V_4$, $u_1 \to v_1 \to u_2 \to v_2 \to u_1$ *and* $\{y_1, y_2, y_3\} \to \{u_2, v_2\} \to \{y_4, y_5, y_6\} \to \{u_1, v_1\} \to \{y_1, y_2, y_3\}$ *(see Figure 7.3), then* D *is a* 4*-partite tournament with* $i_g(D) = 2$ *and the partition-sequence* $2, 2, 4, 6$ *that does not contain any Hamiltonian path.*

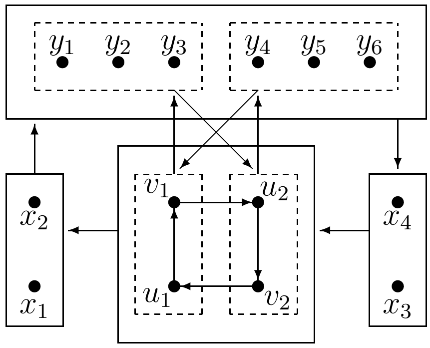

Figure 7.3: A 4-partite tournament D with $i_g(D) = 2$ and the partition-sequence $2, 2, 4, 6$ that does not contain a Hamiltonian path

Example 7.7 (Volkmann, Winzen [40]) *Let* D *be a* 4*-partite tournament with the partite sets* $V_1 = \{u\}$, $V_2 = \{v_1, v_2\}$, $V_3 = \{x_1, x_2, x_3\}$ *and* $V_4 = \{y_1, y_2, y_3, y_4, y_5\}$ *such that* $V_4 \to \{x_1, x_2\} \to (V_1 \cup V_2) \to x_3 \to V_4$, $V_2 \to V_1$, $\{v_2, u\} \to \{y_1, y_2\} \to v_1$, $\{v_1, u\} \to \{y_3, y_4\} \to v_2$ *and* $V_2 \to y_5 \to V_1$ *(see Figure 7.4). Then* D *is a* 4*-partite tournament with* $i_g(D) = 2$ *and the partition-sequence* $1, 2, 3, 5$ *that does not contain a Hamiltonian path.*

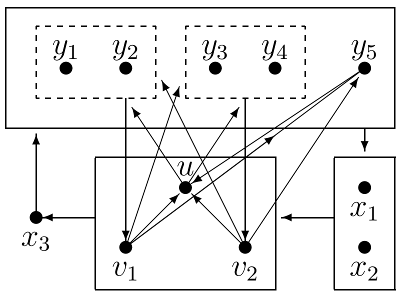

Figure 7.4: A 4-partite tournament D with $i_g(D) = 2$ and the partition-sequence $1, 2, 3, 5$ that does not contain a Hamiltonian path

Example 7.8 (Volkmann, Winzen [40]) *Let* $V_1 = \{u\}$, $V_2 = \{v\}$, $V_3 = \{x_1, x_2, x_3\}$ *and* $V_4 = \{y_1, y_2, y_3, y_4\}$ *be the partite sets of a* 4*-partite tournament* D *such that* $V_4 \to \{x_1, x_2\} \to (V_1 \cup V_2) \to x_3 \to V_4$, $u \to v \to \{y_3, y_4\} \to u$, $u \to \{y_1, y_2\}$ *and* $y_1 \to v \to y_2$ *(see Figure 7.5). Then* D *is a* 4*-partite tournament with* $i_g(D) = 2$ *and the partition-sequence* $1, 1, 3, 4$ *that does not contain any Hamiltonian path.*

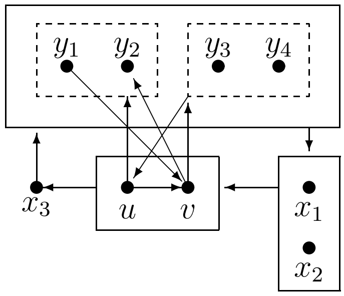

Figure 7.5: A 4-partite tournament D with $i_g(D) = 2$ and the partition-sequence $1, 1, 3, 4$ that does not contain a Hamiltonian path

In the case that $c = 2$ or $c = 3$ and $i_g(D) \geq 2$, there are infinitely many digraphs D that do not contain any Hamiltonian path as we can see in the following example.

Example 7.9 (Volkmann, Winzen [40]) *Let D be a bipartite tournament with $i_g(D) \geq 2$. If V_1, V_2 are the partite sets of D such that $|V_1| + 2 \leq |V_2| \leq |V_1| + 2i_g(D)$, then clearly, D does not contain a Hamiltonian path.*

Now, let D be a 3-partite tournament with the partite sets V_1, V_2, V_3. If $|V_1| = |V_2| = r$ and $|V_3| = r + i_g(D)$ with $i_g(D) \geq 2$ such that $V_1 \to V_2 \to V_3 \to V_1$, then D does not contain a Hamiltonian path.

All the presented examples show that Theorem 7.4 is best possible. Combining Corollary 6.8 with Example 7.9, we observe the following.

Corollary 7.10 (Volkmann, Winzen [40]) *Let $i \geq 2$ be an arbitrary integer. Then all, except a finite number, of c-partite tournaments with $i_g \leq i$ and $c \geq 4$ have a Hamiltonian path. Furthermore the bound $c \geq 4$ is best possible.*

Proof. If $r \geq 4i - 5$, then Corollary 6.8 yields that a c-partite tournament D with $c \geq 4$ and at least r vertices in each partite set contains a Hamiltonian path. Because of Lemma 1.10 there are only finitely many c-partite tournaments with $i_g \leq i$ and at most $4i - 6$ vertices in the smallest partite set. Thus, the first part of this corollary is proved.

Example 7.9 demonstrates that there are infinitely many 3-partite tournaments with $i_g \geq 2$ that do not contain any Hamiltonian path and the proof of this corollary is complete. □

7.2 The case $i_g(D) \geq 3$

Let $g(i)$ be defined as in Problem 7.1. According to Theorem 4.4 we have $g(0) = 2$. Moreover, in the previous section we have seen that $g(1) = 3$ (Theorem 7.3) and $g(2) = 5$ (Theorem 7.4). In this section, we will solve Problem 7.1 completely by proving that $g(i) = 4i - 4$, if $i \geq 3$.

The following two families of examples show that $g(i) \geq 4i - 4$, if $i \geq 3$.

Example 7.11 (Stella, Volkmann, Winzen [22]) *If $i \geq 3$ is an integer and $c = 4i - 5$, then we define the c-partite tournament D_i with the partite sets $V_j = \{v_j\}$ for $1 \leq j \leq 4i - 6$ and $V_{4i-5} = \{y_1, y_2, \ldots, y_{2i}\}$ as follows.*

The partite sets $V_1, V_2, \ldots, V_{2i-3}$ induce an $(i-2)$-regular tournament A and the partite sets $V_{2i-2}, V_{2i-1}, \ldots, V_{4i-6}$ induce an $(i-2)$-regular tournament B. In addition, let $A \to B \to (V_{4i-5} - \{y_{2i}\}) \to A \to y_{2i} \to B$ (see Figure 7.6 for D_3). It is straightforward to see that D_i is a $(4i-5)$-partite tournament with $i_g(D_i) = i$ that does not contain any Hamiltonian path.

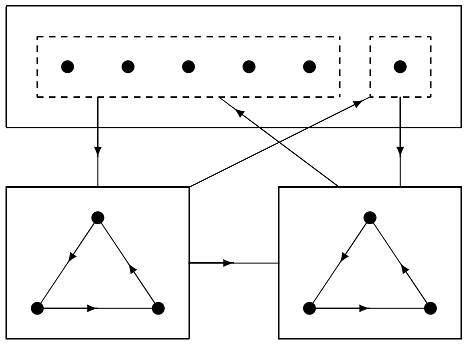

Figure 7.6: The 7-partite tournament D_3 without any Hamiltonian path

Example 7.12 (Stella, Volkmann, Winzen [22]) *If $i \geq 3$ is an integer and $c = 4i-5$, then we define the c-partite tournament G_i with the partite sets $V_j = \{v_j\}$ for $1 \leq j \leq 4i - 6$ and $V_{4i-5} = \{y_1, y_2, \ldots, y_{2i+1}\}$ as follows.*

The partite sets $V_1, V_2, \ldots, V_{2i-3}$ induce an $(i-2)$-regular tournament A and the partite sets $V_{2i-2}, V_{2i-1}, \ldots, V_{4i-6}$ induce an $(i-2)$-regular tournament B. In addition, let $A \to B \to (V_{4i-5} - \{y_{2i}, y_{2i+1}\}) \to A \to \{y_{2i}, y_{2i+1}\} \to B$. It is straightforward to see that G_i is a $(4i-5)$-partite tournament with $i_g(G_i) = i$ that does not contain any Hamiltonian path.

Note that for all multipartite tournaments G_i with $i \geq 3$ it follows that $i_l(G_i) = 0$. This demonstrates that the existence of a Hamiltonian path does not depend on the local irregularity. There are local regular c-partite tournaments with c arbitrary large that do not contain any Hamiltonian path.

If $i_g(D) \geq 3$, then Corollary 6.8 is not best possible as we can see in the following theorem.

Theorem 7.13 (Stella, Volkmann, Winzen [22]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament D with $i_g(D) \geq 3$ such that $1 \leq r = |V_1| \leq |V_2| \leq \ldots \leq |V_c|$. If $c \geq \frac{4i_g(D)-6}{r} + 3$, then D contains a Hamiltonian path.*

Proof. Suppose that D does not have a Hamiltonian path, which implies $pc(D) > 1$. According to Corollary 6.8, this leads to $c = \frac{4i_g(D)-6}{r} + 3$. Regarding

Theorem 6.7, we observe that $p = 2i_g(D)$ and D has the partition-sequence $r, r, \ldots, r, |V_{c-1}|, r + 2i_g(D)$. To get no contradiction, it follows that $d^+(x) = d^-(x) = \frac{(c-2)r+|V_{c-1}|}{2}$ for all $x \in V_c$ and $d^+(y) = d^-(y) = \frac{(c-2)r+|V_{c-1}|}{2} + i_g(D)$ for all $y \in V_1 \cup V_2 \cup \ldots \cup V_{c-2}$. In other words,

$$\big(|d^+(x) - d^-(x)| > 0 \quad \text{for a vertex} \quad x \in V(D)\big) \Rightarrow x \in V_{c-1}. \tag{7.1}$$

Note that because of $c = \frac{4i_g(D)-6}{r} + 3$ it follows that $i_g(D) = \frac{(c-3)r-2i_g(D)+6}{2} = \frac{|V(D)|-|V_{c-1}|-2|V_c|+6}{2}$.

Let the sets $Y, R_1, R_2, Z, Q, Q_1, Q_2, V_i, Y_1, Y_2$ and t be defined as in Theorem 6.4.

Case 1. Let $Q_1 = \emptyset$, and thus $Q = Q_2$. This yields that i)-viii) of Corollary 6.6 with $|Y_2| = 0$ and $k = 1$ are valid. In particular we deduce from ii), iv) and vii) that $|Y| = |V_i| = |V_c| = r + 2i_g(D)$ and $d^-(q_2) = d^+(q_2) + 2$ for all $q_2 \in Q_2 = Q$. Now, (7.1) leads to $Q \subseteq V_{c-1}$, and thus $V_1 \cup V_2 \cup \ldots \cup V_{c-2} \subseteq Z$. Using i), this yields

$$\begin{aligned} r + 2i_g(D) &= |Y| = |Z| + 2 \geq (c-2)r + 2 \\ &= \left(\frac{4i_g(D)-6}{r} + 1\right) r + 2 = 4i_g(D) - 4 + r, \end{aligned} \tag{7.2}$$

a contradiction to $i_g(D) \geq 3$.

Case 2. Assume that $Q_2 = \emptyset$. By symmetry, we arrive at a contradiction similarly to Case 1.

Case 3. Suppose that $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. Hence a)-j) of Corollary 6.6 with $t = 0$ and $k = 1$ hold. With c) we see that $|Y| = |V_c|$ and thus $|Y_1| = |Y_2| = 0$. Using g), it follows that $d^+(q_1) = d^-(q_1) + i_g(D)$ for all $q_1 \in Q_1$, and $d^-(q_2) = d^+(q_2) + i_g(D)$ for all $q_2 \in Q_2$. According to (7.1), we have $Q \subseteq V_{c-1}$ and thus $V_1 \cup V_2 \cup \ldots \cup V_{c-2} \subseteq Z$. Hence, we arrive at the contradiction (7.2). This completes the proof of this theorem. $\square$

Since $\max\left\{\frac{4i_g-6}{r} + 3 \mid r \in \mathbb{N}\right\} = 4i_g(D) - 3$, we conclude that $g(i) \leq 4i - 3$, if $i \geq 3$. In the next step, we even will show that $g(i) \leq 4i - 4$, and thus $g(i) = 4i - 4$, if $i \geq 3$.

Theorem 7.14 (Stella, Volkmann, Winzen [22]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament such that $1 \leq r = |V_1| \leq |V_2| \leq \ldots \leq |V_c|$. If $c = 4i_g(D) - 4$ and $i_g(D) \geq 3$, then D contains a Hamiltonian path.*

Proof. If $r \geq 2$, then Theorem 7.13 yields the desired result. Let $r = 1$, $c = 4i_g(D) - 4$ and suppose that D does not contain a Hamiltonian path, which means that $pc(D) > 1$. If $|V_c| \leq 2i_g(D) - 1$, then $\frac{|V(D)|-|V_{c-1}|-2|V_c|+5}{2} \geq \frac{c+4-2i_g(D)}{2} = i_g(D)$. Corollary 6.5 with $k = 1$ yields $pc(D) \leq 1$, a contradiction. If $|V_c| = 2i_g(D)$ and $|V_1| + |V_2| + \ldots + |V_{c-2}| \geq c - 1$, then, similarly as in the proof of Theorem 6.7 with $p = 2i_g(D) - 1$, we see that D contains a Hamiltonian path, a contradiction. Analogously, if $|V_c| = 2i_g(D) + 1$ and

$|V_1| + |V_2| + \ldots + |V_{c-2}| \geq c$, then we arrive at a contradiction. The following three cases remain to be considered.

Case 1. Assume that $|V_c| = 2i_g(D)$. As seen above this implies that D has the partition-sequence $1, 1, \ldots, 1, |V_{c-1}|, 2i_g(D)$.

If $|V_{c-1}| = 2p + 1$ with $p \in \mathbb{N}_0$, then we conclude that $d^+(x) = d^-(x) = 3i_g(D) + p - 3$, if $x \in V_1 \cup V_2 \cup \ldots \cup V_{c-2}$ and $\{d^+(y), d^-(y)\} = \{2i_g(D) + p - 3, 2i_g(D) + p - 2\}$ for all $y \in V_c$.

If $|V_{c-1}| = 2m$ with $m \in \mathbb{N}$, then it follows that $\{d^+(x), d^-(x)\} = \{3i_g(D) + m - 4, 3i_g(D) + m - 3\}$ for all $x \in V_1 \cup V_2 \cup \ldots \cup V_{c-2}$ and $d^+(y) = d^-(y) = 2i_g(D) - 3 + m$ for all $y \in V_c$.

In both cases we deduce that

$$\big(|d^+(x) - d^-(x)| > 1 \quad \text{for a vertex} \quad x \in V(D)\big) \Rightarrow x \in V_{c-1}. \tag{7.3}$$

Furthermore, we observe that $i_g(D) = \frac{4i_g(D)-6-2i_g(D)+6}{2} = \frac{|V(D)|-|V_{c-1}|-2|V_c|+6}{2}$. Let $Y, R_1, R_2, Z, Q, Q_1, Q_2, V_i, t, Y_1$ and Y_2 be defined as in Theorem 6.4.

Subcase 1.1. Let $Q_1 = \emptyset$, and thus $Q = Q_2$. This yields that i)-viii) of Corollary 6.6 with $|Y_2| = 0$ and $k = 1$ are valid. If we consider ii), iv) and vii), then we see that $|Y| = |V_i| = |V_c| = 2i_g(D)$ and $d^-(q_2) = d^+(q_2) + 2$ for all $q_2 \in Q_2 = Q$. Now (7.3) leads to $Q \subseteq V_{c-1}$, and thus $V_1 \cup V_2 \cup \ldots \cup V_{c-2} \subseteq Z$. Using i), this yields

$$2i_g(D) = |Y| = |Z| + 2 \geq (c-2) + 2 = 4i_g(D) - 4,$$

a contradiction to $i_g(D) \geq 3$.

Subcase 1.2. Assume that $Q_2 = \emptyset$. By symmetry, we arrive at a contradiction similar to Subcase 1.1.

Subcase 1.3. Suppose that $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. Hence a)-j) of Corollary 6.6 with $t = 0$ and $k = 1$ hold. With c), we see that $|Y| = |V_c|$, and thus $|Y_1| = |Y_2| = 0$. Using g), it follows that $d^+(q_1) = d^-(q_1) + i_g(D)$ for all $q_1 \in Q_1$ and $d^-(q_2) = d^+(q_2) + i_g(D)$ for all $q_2 \in Q_2$. According to (7.3), we have $Q \subseteq V_{c-1}$ and thus $V_1 \cup V_2 \cup \ldots \cup V_{c-2} \subseteq Z$. Hence, as in Subcase 1.1 we arrive at a contradiction.

Case 2. Let D have the partition-sequence $1, 1, \ldots, 1, 2, |V_{c-1}|, 2i_g(D) + 1$. If $|V_{c-1}| = 2m$ for an integer $m \geq 1$, then there is a vertex $x \in V_1$ such that $d^+(x) \geq 3i_g(D) + m - 2$ or $d^-(x) \geq 3i_g(D) + m - 2$ and a vertex $y \in V_c$ such that $d^+(y) \leq 2i_g(D) + m - 3$ or $d^-(y) \leq 2i_g(D) + m - 3$, a contradiction to the definition of $i_g(D)$. Hence, in the following we can assume that $|V_{c-1}| = 2p + 1$ for an integer $p \geq 1$. This implies that $d^+(x) = d^-(x) = 3i_g(D) - 2 + p$ for all $x \in V_1 \cup V_2 \cup \ldots \cup V_{c-3}$, $\{d^+(y), d^-(y)\} = \{3i_g(D) + p - 3, 3i_g(D) + p - 2\}$ for all $y \in V_{c-2}$ and $d^+(z) = d^-(z) = 2i_g(D) + p - 2$ for all $z \in V_c$. In other words this means that

$$\big(|d^+(x) - d^-(x)| > 1 \quad \text{for a vertex} \quad x \in V(D)\big) \Rightarrow x \in V_{c-1}. \tag{7.4}$$

Note that according to the given partition-sequence it follows that $i_g(D) = \frac{4i_g(D)+1-(2i_g(D)+1)}{2} = \frac{|V(D)|-|V_{c-1}|-2|V_c|+6}{2}$. Let $Y, R_1, R_2, Z, Q, Q_1, Q_2, V_i, Y_1, Y_2$ and t be defined as in Theorem 6.4.

Subcase 2.1. Let $Q_1 = \emptyset$, and thus $Q = Q_2$. This yields that i)-viii) of Corollary 6.6 with $|Y_2| = 0$ and $k = 1$ are valid. If we consider ii), iv) and vii), we see that $|Y| = |V_i| = |V_c| = 1 + 2i_g(D)$ and $d^-(q_2) = d^+(q_2) + 2$ for all $q_2 \in Q_2 = Q$. Now, (7.4) leads to $Q \subseteq V_{c-1}$, and thus $V_1 \cup V_2 \cup \ldots \cup V_{c-2} \subseteq Z$. Using i) this yields

$$2i_g(D) + 1 = |Y| = |Z| + 2 \geq c + 1 = 4i_g(D) - 3,$$

a contradiction to $i_g(D) \geq 3$.

Subcase 2.2. Assume that $Q_2 = \emptyset$. By symmetry, we arrive at the same contradiction as in Subcase 2.1.

Subcase 2.3. Suppose that $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. Hence a)-j) of Corollary 6.6 with $t = 0$ and $k = 1$ hold. With c) we see that $|Y| = |V_c|$ and thus $|Y_1| = |Y_2| = 0$. Using g), we see that $d^+(q_1) = d^-(q_1) + i_g(D)$ for all $q_1 \in Q_1$ and $d^-(q_2) = d^+(q_2) + i_g(D)$ for all $q_2 \in Q_2$. According to (7.4), we have $Q \subseteq V_{c-1}$ and thus $V_1 \cup V_2 \cup \ldots \cup V_{c-2} \subseteq Z$. Analogously as in Subcase 2.1 we obtain a contradiction.

Case 3. Let D have the partition-sequence $1, 1, \ldots, 1, |V_{c-1}|, 2i_g(D) + 1$. If $|V_{c-1}| = 2m+1$ for an integer $m \geq 0$, then we deduce that there are vertices $x \in V_1$ and $y \in V_c$ such that $d^+(x) \geq 3i_g(D)+m-2$ or $d^-(x) \geq 3i_g(D)+m-2$ and $d^+(y) \leq 2i_g(D)+m-3$ or $d^-(y) \leq 2i_g(D)+m-3$, a contradiction. Hence, we may assume that $|V_{c-1}| = 2p$ with $p \in \mathbb{N}$, $d^+(x) = d^-(x) = 3i_g(D) - 3 + p$ for all $x \in V_1 \cup V_2 \cup \ldots \cup V_{c-2}$ and $d^+(y) = d^-(y) = 2i_g(D) + p - 3$ for all $y \in V_c$. This leads to

$$\big(|d^+(x) - d^-(x)| > 0 \quad \text{for a vertex} \quad x \in V(D)\big) \Rightarrow x \in V_{c-1}. \tag{7.5}$$

Note that according to the given partition-sequence it follows that $i_g(D) = \frac{|V(D)|-|V_{c-1}|-2|V_c|+7}{2}$. Let the sets $Y, Z, R_1, R_2, Q, Q_1, Q_2, V_i, V_j, Y_1, Y_2$ and t be defined as in Theorem 6.4.

Subcase 3.1. Let $t \geq 1$.

Subcase 3.1.1. Suppose that $Q_1 = \emptyset$.

First, let $i \neq c$, and thus $j = c$. Since $|Y| \geq |Z| + 2$, we conclude that $i = c - 1$. The fact that $|V_c|$ is odd whereas $|V_{c-1}|$ is even implies that $|V_j| = |V_c| = |V_{c-1}| + s = |V_i| + s$ for an integer $s \geq 1$. As in Case 1 of the proof of Theorem 6.4 with $k = 1$, we see that

$$i_g(D) \geq \frac{|V(D)| - |V_j| - 2|V_i| + 6}{2} = \frac{|V(D)| - |V_{c-1}| - 2|V_c| + s + 6}{2}.$$

To present no contradiction it follows that $s = 1$, and thus $|V_i| = |V_{c-1}| = 2i_g(D)$. Furthermore, equality holds in the last inequality. This is possible, only if $|Y_2| = 0$ and ii) and iv) of Corollary 6.6 hold. Using ii), we see that $Y = V_{c-1}$, and iv) means that $d^-(q_2) = d^+(q_2) + 2$ for all $q_2 \in Q_2$. According to (7.5), this yields $Q \subseteq V_{c-1}$, a contradiction to $Y = V_{c-1}$.

Second, let $V_i = V_c$. If $|Y_2| \geq 2$, then Theorem 6.4 yields a contradiction. If $|Y_2| = 1$, then Theorem 6.4 and Corollary 6.6 iv) imply that $d^-(q_2) = d^+(q_2)+1$ for all $q_2 \in Q_2$, and thus $Q \subseteq V_{c-1}$ and $V_1 \cup V_2 \cup \ldots \cup V_{c-2} \subseteq Z$. This leads to the contradiction

$$2i_g(D) = |Y| \geq |Z| + 2 \geq c = 4i_g(D) - 4.$$

If $|Y_2| = 0$, then we conclude that $|Z| \leq |Y| - 2 \leq |V_i| - 3$. If δ^* and Δ^* are defined as in Case 1 of the proof of Theorem 6.4, then, as there, we observe that

$$i_g(D) \geq \Delta^* - \delta^* \geq \frac{|V(D)| - |V_j| + 2}{2} - |V_i| + 3 \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 8}{2},$$

a contradiction.

Subcase 3.1.2. Let $Q_2 = \emptyset$. If we reverse each arc of D, we arrive at a contradiction by using Subcase 3.1.1.

Subcase 3.1.3. Assume that $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. To get no contradiction it follows that $t = 1$ and a)-g) of Corollary 6.6 hold, and thus especially $d^+(q_1) = d^-(q_1) + i_g(D)$ for all $q_1 \in Q_1$ and $d^-(q_2) = d^+(q_2) + i_g(D)$ for all $q_2 \in Q_2$. As above, this yields a contradiction.

Subcase 3.2. Let $t = 0$ and thus $|Y| = |V_c|$ and $|Y_1| = |Y_2| = 0$.

Subcase 3.2.1. Suppose that $Q_1 = \emptyset$. If $d^-(q_2) \geq d^+(q_2) + 2$ for all $q_2 \in Q_2$, then analogously as above we arrive at the contradiction

$$2i_g(D) + 1 = |Y| \geq |Z| + 2 \geq c = 4i_g(D) - 4.$$

Hence, observing Case 1 of the proof of Theorem 6.4, we conclude that there is a vertex $q \in Q \cap V_{c-1}$ such that $d^-(q) \geq d^+(q) + 3$. Let δ^* and Δ^* be defined as in the proof of Theorem 6.4. Similarly as there, we deduce that $\Delta^* \geq \frac{|V(D)| - |V_j| + 3}{2}$ and thus $i_g(D) \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 7}{2}$. To present no contradiction, it must be the case that $|Z| = |Y| - 2 = 2i_g(D) - 1$ and

$$\begin{aligned} |Z| = 2i_g(D) - 1 &= \delta^* = \delta(G) = \min\{d^+(x), d^-(x) \mid x \in V(D)\} \\ &= 2i_g(D) + p - 3, \end{aligned}$$

and thus $p = 2$ and $|V_{c-1}| = 4$. It follows that D has the partition-sequence $1, 1, \ldots, 1, 4, 2i_g(D) + 1$ and $\{d^+(x), d^-(x)\} = \{3i_g(D) - 2, 3i_g(D) - 3\}$ or $\{d^+(x), d^-(x)\} = \{3i_g(D) - 1, 3i_g(D) - 4\}$ for all $x \in V_{c-1}$. Since $|Y| = 2i_g(D) + 1 = |Z| + 2$, it follows that $|Q_2| = |V(D)| - |Y| - |Z| = 2i_g(D) - 1$. The facts that $Y \to Q_2$ and $d^+(y) = 2i_g(D) - 1$ for all $y \in Y = V_c$ yield that $Z \to Y$. If $|V_{c-1} \cap Q_2| \leq 1$, then $D[Q_2]$ is a tournament and there is a vertex $q_2 \in Q_2$ such that $d^-_{D[Q_2]}(q_2) \geq i_g(D) - 1$. This leads to the contradiction

$$d^-(q_2) \geq |Y| + d^-_{D[Q_2]}(q_2) \geq 3i_g(D).$$

Analogously, we arrive at a contradiction, if $|V_{c-1} \cap Z| \leq 1$. Now the remaining case is that $|V_{c-1} \cap Q_2| = |V_{c-1} \cap Z| = 2$. As seen above, we have $d^-_{D[Q_2]}(q_2) \leq i_g(D) - 2$ for all $q_2 \in Q_2$, and thus we arrive at

$$|E(Q_2)| = \sum_{q_2 \in Q_2} d^-_{D[Q_2]}(q_2) \leq (2i_g(D) - 1)(i_g(D) - 2) = 2i_g^2(D) - 5i_g(D) + 2.$$

On the other hand we observe that

$$|E(Q_2)| = \frac{(2i_g(D) - 3)(2i_g(D) - 2) + 2(2i_g(D) - 3)}{2} = 2i_g^2(D) - 3i_g(D).$$

This yields the contradiction $i_g(D) \leq 1$.

Subcase 3.2.2. Assume that $Q_2 = \emptyset$. Caused by symmetry this leads to a contradiction analogously as in Subcase 3.2.1.

Subcase 3.2.3. Finally, let $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. If $i_g(D) \geq i_l(D) + 1$, then Theorem 6.4 yields that $i_g(D) \geq \frac{|V(D)|-|V_{c-1}|-2|V_c|+8}{2}$, a contradiction. Hence let $i_g(D) = i_l(D)$. This implies that there is a vertex $x \in V_{c-1}$ such that $\{d^+(x), d^-(x)\} = \{\frac{7i_g(D)-5}{2}, \frac{5i_g(D)-5}{2}\}$ and thus

$$\frac{7i_g(D)-5}{2} = 3i_g(D) - 3 + p \Rightarrow |V_{c-1}| = 2p = i_g(D) + 1.$$

Hence, we may assume that D has the partition-sequence $1, 1, \ldots, 1, i_g(D) + 1, 2i_g(D) + 1$ and $i_g(D)$ is odd. Observing the proof of Theorem 6.4, we recognize that the case $|Y| > |Z| + 2$ also yields a contradiction. Consequently it remains to consider the case that $|Z| = |Y| - 2 = 2i_g(D) - 1$. This implies that

$$|Q| = |V(D)| - |Y| - |Z| = 3i_g(D) - 4.$$

Since $i_g(D)$ is odd we may assume, without loss of generality, that $|Q_1| \geq |Q_2| + 1$ and thus $|Q_1| \geq \frac{3i_g(D)-3}{2}$. If there is a vertex $x \in Q_2 - V_{c-1}$, then we arrive at the contradiction

$$d^-(x) \geq |Q_1| + |Y| \geq 3i_g(D) - 1 + \frac{i_g(D)+1}{2}.$$

Hence, let $Q_2 \subseteq V_{c-1}$ and $|Q_1| = \frac{3i_g(D)-3}{2} + l$ for an integer $l \geq 0$, and thus $|Q_2| = \frac{3i_g(D)-5}{2} - l$. Now we conclude for an arbitrary vertex $x \in Q_2$ that

$$d^-(x) \geq |Y| + |Q_1 - V_{c-1}| \geq |Y| + |Q_1| - |V_{c-1}| + |Q_2| = 4i_g(D) - 4.$$

Since $4i_g(D) - 4 > 3i_g(D) - 3 + \frac{i_g(D)+1}{2} = d^+(y) + i_g(D)$ with $y \in Y$ arbitrary if and only if $i_g(D) \geq 4$, it remains to treat the case that $i_g(D) = 3$, D has the partition-sequence $1, 1, 1, 1, 1, 1, 4, 7$, $|Y| = 7 = |Z| + 2 = |Q| + 2$, $d^+(x) = d^-(x) = 8$ for all $x \in V_1 \cup V_2 \cup \ldots \cup V_6$ and $d^+(z) = d^-(z) = 5$ for all $z \in V_8$. Because of $|Q| = 5$, there is a vertex $v \in Q \cap (V_1 \cup V_2 \cup \ldots \cup V_6)$. Without loss of generality, let $v \in Q_1$. If $|Q_2| \geq 2$, then we see that $d^+(v) \geq 9$, a contradiction. Hence, let $|Q_1| = 4$ and $|Q_2| = 1$. It follows that $Q_2 \cup (Q_1 - \{v\}) = V_7$, $(Q_1 - \{v\}) \to v$, $Q_2 \to Z \to Q_1$ and $D[Z]$ is a tournament. Since

$$40 = \sum_{z \in Z} d^+(z) = |Z||Q_1| + d(Z, Y) + \sum_{z \in Z} d^+_{D[Z]}(z) = 30 + d(Z, Y),$$

we deduce that $d(Z, Y) = 10$. On the other hand, we observe that

$$35 = \sum_{y \in Y} d^-(y) = |Y||Q_1| + d(Z, Y) = 28 + d(Z, Y),$$

and thus $d(Z, Y) = 7$, a contradiction.

This completes the proof of this theorem. □

Combining the Examples 7.11 and 7.12 together with the Theorems 7.13 and 7.14, we arrive at the following main result of this chapter.

Corollary 7.15 (Stella, Volkmann, Winzen [22]) *If D is a multipartite tournament with c partite sets and*

$$3 \leq i_g(D) \leq \frac{c+4}{4},$$

then D contains a Hamiltonian path. Moreover, for $i_g(D) \geq 3$ the upper bound of $i_g(D)$ is optimal.

Chapter 8

Hamiltonian paths containing a given arc

After the examination of the existence of Hamiltonian paths in close to regular multipartite tournaments, in this chapter, we search for Hamiltonian paths containing a given arc. If D is regular, then the following result holds.

Theorem 8.1 (Volkmann, Yeo [46]) *Every arc of a regular c-partite tournament D is contained in a Hamiltonian path of D.*

Now, it is an interesting extension of Theorem 8.1 to solve the following problem, which is similar to Problem 7.1.

Problem 8.2 (Volkmann, Winzen [41]) *For all i find the smallest value, $h(i)$, with the property that each arc of all c-partite tournaments with $i_g \leq i$ and $c \geq h(i)$ is contained in a Hamiltonian path.*

In Section 8.1, we will show that $h(i) \leq 4i+4$ and that almost all c-partite tournaments of a given irregularity i with $c \geq 4$ have the property that each arc is contained in a Hamiltonian path. In the Sections 8.2 and 8.3, we will examine almost regular c-partite tournaments. In particular, we have shown that $h(1) = 5$. For the cases that $c = 2$ we have Theorem 7.2 and if all partite sets have the same size, then we can use the following theorem of Volkmann [33].

Theorem 8.3 (Volkmann [33]) *Let D be an almost regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| = |V_2| = \ldots = |V_c|$. Then each arc of D is contained in a Hamiltonian path if and only if D is not isomorphic to $T_{3,3}$ with $T_{3,3}$ as in Figure 7.1.*

In Section 8.2, we will prove that $h(1) \leq 5$. In the case that $c = 4$ and D does not have the partition-sequence $1, 1, 2, 3$ we still can prove that each arc is contained in a Hamiltonian path of D (which means that $h(1) = 5$), but the improvement may not be worth the additional effort. In the case that $c = 3$ there are infinite families of such digraphs with the property that not every arc is contained in a Hamiltonian path of D. Nevertheless, in the last section, we will present an interesting sufficient condition for an almost regular 3-partite tournament D with the property that a given arc is contained in a Hamiltonian path of D.

8.1 Hamiltonian paths starting with a given arc

Recently, Volkmann and Yeo [46] found the following sufficient condition for a c-partite tournament to contain a Hamiltonian path starting with a given arc.

Theorem 8.4 (Volkmann, Yeo [46]) *Let D be a c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$, and let e be an arc in D. If*

$$|V(D)| \geq 2i_g(D) + 2|V_c| + |V_{c-1}| + 1,$$

then there exists a Hamiltonian path in D, starting with the arc e.

This leads to the following result.

Corollary 8.5 (Volkmann, Winzen [41]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament D such that $r = |V_1| \leq |V_2| \leq \ldots \leq |V_c|$ and let e be an arc of D. If*

$$c \geq 3 + \frac{4i_g(D) + 1}{r},$$

then there is a Hamiltonian path with the initial arc e.

Proof. According to Theorem 8.4, D has a Hamiltonian path starting with e, if

$$|V_1| + |V_2| + \ldots + |V_{c-2}| \geq |V_c| + 2i_g(D) + 1. \tag{8.1}$$

Since Lemma 1.10 yields that $r \leq |V_i| \leq r + 2i_g(D)$ we arrive at (8.1), if

$$\begin{aligned} & (c-2)r - 2i_g(D) - 1 - (r + 2i_g(D)) \geq 0 \\ \Leftrightarrow \quad & c \geq \frac{4i_g(D) + 3r + 1}{r} = 3 + \frac{4i_g(D) + 1}{r}, \end{aligned}$$

which is the desired result. □

This immediately implies the following corollary.

Corollary 8.6 (Volkmann, Winzen [41]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament D such that $r = |V_1| \leq |V_2| \leq \ldots \leq |V_c|$ and let e be an arc of D. If $c \geq 4i_g(D) + 4$ or if $c \geq 3 + p$ and $r \geq \frac{4i_g(D)+1}{p}$ for an integer $p \in \mathbb{N}$, then there exists a Hamiltonian path in D starting with the arc e.*

Since every Hamiltonian path starting with a given arc e obviously contains the arc e, Corollary 8.6 yields that $h(i) \leq 4i_g(D) + 4$ with $h(i)$ defined as in Problem 8.2. Another consequence of Corollary 8.6 is the following interesting result.

Corollary 8.7 (Volkmann, Winzen [41]) *Almost all multipartite tournaments D with at least four partite sets and a given constant irregularity $i_g(D)$ have the property that every arc of D is the first arc of a Hamiltonian path of D.*

In the case $c = 3$ the assertion of Corollary 8.7 becomes false. As we will see in the last section, there are infinitely many almost regular 3-partite tournaments D with the property that not every arc is contained in a Hamiltonian path of D.

If we know more about the sizes of the partite sets, then, using Theorem 8.4, we arrive at the following corollary.

Corollary 8.8 (Volkmann, Winzen [41]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament D such that $r = |V_1| \leq |V_2| \leq \ldots \leq |V_c|$ and let e be an arbitrary arc of D. Furthermore, let p, q, b be integers with $2 \leq p \leq c-2$, $0 \leq q \leq 2i_g(D)$ and $0 \leq b \leq 2i_g(D) - q$ such that $|V_p| \geq q + r$ and $|V_c| \leq r + 2i_g(D) - b$.*

a) If $c \geq 3 + \frac{4i_g(D)+1-b+(p-2)q}{r+q}$, then there is a Hamiltonian path in D with the initial arc e.

b) If $c \geq 3 + s$ and $r \geq \frac{4i_g(D)+1-b+(p-s-2)q}{s}$ for an integer $s \in \mathbb{N}$, then there exists a Hamiltonian path, starting with the arc e.

Proof. a) Since $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$, the conditions of this corollary yield

$$\begin{aligned} & |V_1| + |V_2| + \ldots + |V_{c-2}| - |V_c| - 2i_g(D) - 1 \\ \geq\ & (p-1)r + (c-2-(p-1))(r+q) - (r + 2i_g(D) - b) - 2i_g(D) - 1 \\ =\ & -(p-1)q + c(r+q) - 3r - 2q - 4i_g(D) + b - 1. \end{aligned}$$

According to Theorem 8.4, it is sufficient to show that the last expression is ≥ 0. This is fulfilled, if

$$\begin{aligned} c \ \geq\ & \frac{3r + 4i_g(D) + 1 - b + 3q + (p-2)q}{r+q} \\ =\ & 3 + \frac{4i_g(D) + 1 - b + (p-2)q}{r+q}, \end{aligned}$$

the desired result.

b) follows immediately from a). □

8.2 Almost regular c-partite tournaments ($c \geq 4$)

In this section we search for almost regular c-partite tournaments with $c \geq 4$ that contain a Hamiltonian path through a given arc. To do this we need an analysis of multipartite tournaments having a cycle-factor but no Hamiltonian cycle. But firstly, we give a definition.

Definition 8.9 *[$C_1 \simeq\!> C_2$]* Let D be a digraph, with two disjoint cycles C_1 and C_2. We shall write $C_1 \simeq\!> C_2$ when the following is true. There is a vertex $x_1 \in V(C_1)$ such that $x_1 \rightsquigarrow V(C_2)$, and there is no vertex $y_1 \in V(C_1)$, such that $V(C_2) \rightsquigarrow y_1$. Furthermore, there is a vertex $x_2 \in V(C_2)$, such that $V(C_1) \rightsquigarrow x_2$, and there is no vertex $y_2 \in V(C_2)$, such that $y_2 \rightsquigarrow V(C_1)$.

Theorem 8.10 (Yeo [48]) *Let D be a multipartite tournament with a cycle-factor F of t cycles such that t is minimum. Then the cycles in F can be labeled in a unique way $C_1, C_2, \ldots, C_t$ such that $C_i \Rightarrow C_j$ for all i, j satisfying $1 \leq i < j \leq t$.*

The next theorem of Yeo is a slight reformulation of a result in his paper mentioned above.

Theorem 8.11 (Yeo [48]) *If D is a multipartite tournament having a cycle-factor but no Hamiltonian cycle, then for some minimal cycle-factor F with the unique numbering $C_1, C_2, \ldots, C_t$ as in Theorem 8.10 and for all pairs of indices i and j with $1 \leq i < j \leq t$, there exists a partite set $V^*(i,j)$ such that $\{x^+, y^-\} \subseteq V^*(i,j)$ when xy is an arc from C_j to C_i. Furthermore $y^- \to x$ and $y \to x^+$.*

Corollary 8.12 (Volkmann, Yeo [46]) *Let D and F be as in the last theorem. Then either $d(C_1, F - V(C_1)) \geq 2d(F - V(C_1), C_1)$ or $t = 1$.*

Lemma 8.13 (Yeo [51]) *If there is a cycle-factor in a semicomplete multipartite digraph D, but no Hamiltonian cycle and if $V_1, V_2, \ldots, V_c$ are the partite sets of D such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$, then $i_l(D) \geq \frac{|V(D)|-|V_{c-1}|-2|V_c|+3}{2}$.*

Using these very helpful results, we arrive at the following theorem.

Theorem 8.14 (Volkmann, Winzen [41]) *Let D be an almost regular c-partite tournament with at least r vertices in each partite set. If one of the following conditions is fulfilled then every arc of D is contained in a Hamiltonian path of D.*
(i) $c \geq 7$
(ii) $c = 6$ *and* $r \geq 2$
(iii) $c = 5$ *and* $r \geq 3$
(iv) $c = 4$ *and* $r \geq 4$

Proof. Let $V_1, V_2, \ldots, V_c$ be the partite sets of D. According to Lemma 1.10, we observe that $r = |V_1| \leq |V_2| \leq \ldots \leq |V_c| \leq r+2$. Because of the Theorems 8.1 and 8.3, it remains to consider the cases $i_g(D) = 1$ and $|V_c| \geq r+1$. Let $e = uv$ be an arbitrary arc of D.

According to Corollary 8.6, D has a Hamiltonian path starting with e, if $c \geq 8$, $c = 6, 7$ and $r \geq 2$, $c = 5$ and $r \geq 3$ or $c = 4$ and $r \geq 5$. Hence, there remain to consider the cases $c = 7$ and $r = 1$ and $c = 4$ and $r = 4$.

Suppose that e is not contained in any Hamiltonian path of D. If we assume that a digraph D' has no cycle-factor, then we define the sets Q_1, Q_2, V_i, Y, Z, R_1 and R_2 as in Theorem 6.4. In the following we will often use the subdigraph $D' = D - v$ with the partite sets $V'_1, V'_2, \ldots, V'_p$ such that $|V'_1| \leq |V'_2| \leq \ldots \leq |V'_p|$. It is obvious that $c - 1 \leq p \leq c$.

Case 1. Let $c = 4$ and $r = 4$. If $|V_2| \geq 5$ or $|V_4| \leq 5$, then Corollary 8.8 b) with $p = 2$, $q = 1$ and $b = 0$ or with $b = 1$ and $q = 0$ implies that D has a Hamiltonian path with the initial arc e, a contradiction. Since $i_g(D) = 1$, the partition-sequence $4, 4, 5, 6$ is impossible. Consequently, there remain to treat the partition-sequences $4, 4, 4, 6$ and $4, 4, 6, 6$.

Subcase 1.1. Let $(n_i) = 4, 4, 4, 6$. It follows that $d^+(x) = d^-(x) = 7$ for $x \in V_1 \cup V_2 \cup V_3$ and $d^+(x) = d^-(x) = 6$ for $x \in V_4$.

Subcase 1.1.1. Suppose that one of the vertices u and v is in V_4, say $v \in V_4$. We conclude that $D' = D - v$ has the partition-sequence $4, 4, 4, 5$ such that $\{d^+(x), d^-(x)\} = \{6, 7\}$ for $x \in V_1' \cup V_2' \cup V_3'$ and $d^+(x) = d^-(x) = 6$ for $x \in V_4'$. Since $\frac{|V(D')|-|V_3'|-2|V_4'|+2}{2} = \frac{5}{2} > 1 = i_g(D')$ Theorem 4.28 implies that D' is Hamiltonian, and thus D contains a Hamiltonian path with the terminal arc e, a contradiction.

Subcase 1.1.2. Assume that $\{u, v\} \subseteq V_1 \cup V_2 \cup V_3$. Again, we define $D' = D - v$. This implies that D' has the partition-sequence $3, 4, 4, 6$ and $i_l(D') = 1$.

Subcase 1.1.2.1. Let D' have a cycle factor. Since $\frac{|V(D')|-|V_3'|-2|V_4'|+3}{2} = 2 > i_l(D')$, Lemma 8.13 implies that D' is Hamiltonian, and thus D contains a Hamiltonian path with the terminal arc e, a contradiction.

Subcase 1.1.2.2. Let D' have no cycle-factor. Since $\frac{|V(D')|-|V_3'|-2|V_4'|+3}{2} = 2 > i_l(D')$, Theorem 6.4 with $k = 0$ yields $Q_1 = \emptyset$ or $Q_2 = \emptyset$. If $|Y| \leq 5$ or $|Z| \leq |Y| - 2$, then we arrive at a contradiction to Theorem 6.4 with $t \geq 1$ or $k \geq 1$. Hence, let $|Y| = 6$ and $|Z| = 5$, and thus $|Q| = 6$.

Firstly, let $Q_1 = \emptyset$. This implies $Q = Q_2$. Since $|Q| = 6$, it is a simple matter to verify that there is a vertex $q_2 \in Q_2$ such that $d^-_{D[Q_2]}(q_2) \geq 2$, and because of $Y \to Q_2$, we obtain the contradiction $d^-(q_2) \geq 8$.

Analogously, we arrive at a contradiction, if $Q_2 = \emptyset$.

Subcase 1.2. Let $(n_i) = 4, 4, 6, 6$. It follows that $d^+(x) = d^-(x) = 8$ for $x \in V_1 \cup V_2$ and $d^+(x) = d^-(x) = 7$ for $x \in V_3 \cup V_4$.

Subcase 1.2.1. Suppose that one of the vertices u and v is in $V_3 \cup V_4$, say $v \in V_3 \cup V_4$. This implies that $D' = D - v$ has the partition-sequence $4, 4, 5, 6$. Because of $\frac{|V(D')|-|V_3'|-2|V_4'|+2}{2} = 2 \geq i_g(D')$, Theorem 4.28 yields that D' is Hamiltonian and thus D contains a Hamiltonian path with the terminal arc e, a contradiction.

Subcase 1.2.2. Let $\{u, v\} \subseteq V_1 \cup V_2$. If we define $D' = D - v$, then D' has the partition-sequence $3, 4, 6, 6$ such that $d^+(x) = d^-(x) = 8$ for $x \in V_1'$, $\{d^+(x), d^-(x)\} = \{7, 8\}$ for $x \in V_2'$ and $\{d^+(x), d^-(x)\} = \{6, 7\}$ for $x \in V_3' \cup V_4'$. Since $\frac{|V(D')|-|V_3'|-2|V_4'|+2}{2} = \frac{3}{2} > 1 = i_l(D')$ and $|V(D')| - 3|V_4'| + 1 = 2 > i_l(D')$, Theorem 4.28 implies that D' has a Hamiltonian cycle, and thus D has a Hamiltonian path with the terminal arc $e = uv$, a contradiction.

Case 2. Let $c = 7$ and $r = 1$. If $|V_5| \geq 2$ or $|V_7| \leq 2$, then Corollary 8.8 b) implies that D has a Hamiltonian path with the initial arc e, a contradiction. Since $i_g(D) = 1$, the partition-sequence $1, 1, 1, 1, 1, 2, 3$ is impossible. Hence, there remain to consider the partition-sequences $1, 1, 1, 1, 1, 1, 3$ and $1, 1, 1, 1, 1, 3, 3$.

Subcase 2.1. Let $(n_i) = 1, 1, 1, 1, 1, 1, 3$. This implies $d^+(x) = d^-(x) = 4$ for $x \in V_1 \cup V_2 \cup \ldots \cup V_6$ and $d^+(x) = d^-(x) = 3$ for $x \in V_7$.

Subcase 2.1.1. Suppose that one of the vertices u and v is part of V_7, say $v \in V_7$. Then let $D' = D - v$. This implies $i_g(D') \leq 2$ and D' has the partition-sequence $1,1,1,1,1,1,2$. Since $\frac{|V(D')|-|V_6'|-2|V_7'|+2}{2} = \frac{5}{2} > i_g(D')$, Theorem 4.28 yields that D' contains a Hamiltonian cycle, and hence, D has a Hamiltonian path with e as the terminal arc, a contradiction.

Subcase 2.1.2. Assume that $\{u,v\} \subseteq V_1 \cup V_2 \cup \ldots \cup V_6$. It follows that $D' = D - v$ has the partition-sequence $1,1,1,1,1,3$. Furthermore, we see that $\{d^+(x), d^-(x)\} = \{3,4\}$ for $x \in V_1' \cup V_2' \cup \ldots \cup V_5'$ and $\{d^+(x), d^-(x)\} = \{2,3\}$ for $x \in V_6'$. This shows that $i_l(D') = 1$.

Subcase 2.1.2.1. Let D' have a cycle-factor. Since $\frac{|V(D')|-|V_5'|-2|V_6'|+3}{2} = 2 > i_l(D')$, Lemma 8.13 implies that D' is Hamiltonian, and hence D contains a Hamiltonian path with the terminal arc e, a contradiction.

Subcase 2.1.2.2. Let D' have no cycle-factor. Since $\frac{|V(D')|-|V_5'|-2|V_6'|+3}{2} = 2 > i_l(D')$, Theorem 6.4 with $k=0$ yields $Q_1 = \emptyset$ or $Q_2 = \emptyset$. If $|Y| \leq 2$ or $|Z| \leq |Y| - 2$, then $|V(D')| - 3|V_6'| + 4 = 3 > i_l(D')$ contradicts Theorem 6.4 with $t \geq 1$ or $k \geq 1$. Hence, let $|Y| = 3$ and $|Z| = 2$.

Firstly, let $Q_1 = \emptyset$, which means $Q = Q_2 = \{q_1, q_2, q_3\}$. According to (4.18) and noticing that $Q_2 \subseteq R_2$, it follows that $Y \to Q \to Z = \{z_1, z_2\} \to Y$, $q_1 \to q_2 \to q_3 \to q_1$. If, without loss of generality, $z_1 \to z_2$, then we deduce that $z_2 \to v \to z_1$ and $Q \to v \to Y = \{y_1, y_2, y_3\}$. If $z_2 = u$, then $y_1 q_1 z_2 v y_2 q_2 z_1 y_3 q_3$ is a Hamiltonian path through e, and if, without loss of generality, $q_1 = u$, then $y_1 q_1 v y_2 q_2 z_1 y_3 q_3 z_2$ is a Hamiltonian path containing $e = uv$, a contradiction in both cases.

Secondly, let $Q_2 = \emptyset$, which means $Q = Q_1 = \{q_1, q_2, q_3\}$. According to (4.18) and noticing that $Q_1 \subseteq R_1$, it follows that $Y \to Z = \{z_1, z_2\} \to Q \to Y$, $q_1 \to q_2 \to q_3 \to q_1$, $z_1 \to z_2 \to v \to z_1$ and $Y \to v \to Q$. Since $u, v \in V_1 \cup V_2 \cup \ldots \cup V_6$, we deduce that $u = z_2$. But now $q_3 y_1 z_2 v q_1 y_2 z_1 q_2 y_3$ is a Hamiltonian path containing e, a contradiction.

Subcase 2.2. Let $(n_i) = 1,1,1,1,1,3,3$. The fact that $i_g(D) = 1$ implies that $d^+(x) = d^-(x) = 5$ for $x \in V_1 \cup V_2 \cup \ldots \cup V_5$ and $d^+(x) = d^-(x) = 4$ for $x \in V_6 \cup V_7$.

Subcase 2.2.1. Suppose that one of the vertices u and v is in $V_6 \cup V_7$, say $v \in V_6 \cup V_7$. It follows that $D' = D - v$ has the partition-sequence $1,1,1,1,1,2,3$. Because of $\frac{|V(D')|-|V_6'|-2|V_7'|+2}{2} = 2 \geq i_g(D')$, Theorem 4.28 yields that D' has a Hamiltonian cycle and thus D contains a Hamiltonian path with the terminal arc $e = uv$, a contradiction.

Subcase 2.2.2. Let $\{u,v\} \subseteq V_1 \cup V_2 \cup \ldots \cup V_5$. We observe that $D' = D - v$ has the partition-sequence $1,1,1,1,3,3$ such that $\{d^+(x), d^-(x)\} = \{4,5\}$ for $x \in V_1' \cup V_2' \cup V_3' \cup V_4'$ and $\{d^+(x), d^-(x)\} = \{3,4\}$ for $x \in V_5' \cup V_6'$. Since $\frac{|V(D')|-|V_5'|-2|V_6'|+2}{2} = \frac{3}{2} > 1 = i_l(D')$ and $|V(D')| - 3|V_6'| + 1 = 2 > i_l(D')$, Theorem 4.28 yields that D' is Hamiltonian and thus D contains a Hamiltonian path with the terminal arc e, a contradiction.

This completes the proof of the theorem. □

Next we will prove that we can omit the assumption $r \geq 2$ in Condition (ii) of Theorem 8.14 and $r \geq 3$ in Condition (iii) of the same theorem.

Theorem 8.15 *If D is an almost regular c-partite tournament with $c \geq 5$, then every arc of D is contained in a Hamiltonian path.*

Proof. Let $V_1, V_2, \ldots, V_c$ be the partite sets of D. According to Lemma 1.10, we observe that $1 \leq r = |V_1| \leq |V_2| \leq \ldots \leq |V_c| \leq r+2$. Because of the Theorems 8.1, 8.3 and 8.14, it remains to consider the cases that $i_g(D) = 1$, $|V_c| \geq r+1$ and $5 \leq c \leq 6$. Let $e = uv$ be an arbitrary arc of D and suppose that e is not contained in any Hamiltonian path of D. If we assume that a digraph D' has no cycle-factor, then we define the sets Q_1, Q_2, V_i, Y, Z, R_1 and R_2 as in Theorem 6.4. Furthermore, let $Y_1 = R_1 \cap V_i$ and $Y_2 = R_2 \cap V_i$. In the following we will often use the subdigraph $D' = D - v$ with the partite sets $V'_1, V'_2, \ldots, V'_p$ such that $|V'_1| \leq |V'_2| \leq \ldots \leq |V'_p|$. It is obvious that $c - 1 \leq p \leq c$.

Case 1. Let $c = 6$. If $|V_3| \geq 2$ or $|V_4| \geq 3$ or $|V_4| = 2$ and $|V_6| = 2$, then Corollary 8.8 implies that D has a Hamiltonian path with the initial arc e, a contradiction. Since $i_g(D) = 1$, the partition sequences $1,1,1,1,1,3$; $1,1,1,1,3,3$ and $1,1,1,2,2,3$ are impossible. Consequently, there remain to consider the following four partition-sequences.

Case 1.1. Let $(n_i) = 1,1,1,2,3,3$. This yields that $d^+(x) = d^-(x) = 5$ for $x \in V_1 \cup V_2 \cup V_3$, $\{d^+(x), d^-(x)\} = \{4,5\}$ for $x \in V_4$ and $d^+(x) = d^-(x) = 4$ for $x \in V_5 \cup V_6$.

Subcase 1.1.1. Assume that one of the vertices u and v is in $V_5 \cup V_6$, say $v \in V_5 \cup V_6$. This yields $D' = D - v$ has the partition-sequence $1,1,1,2,2,3$. Because of $\frac{|V(D')| - |V'_5| - 2|V'_6| + 2}{2} = 2 = i_g(D')$, Theorem 4.28 implies that D' contains a Hamiltonian cycle, a contradiction.

Subcase 1.1.2. Suppose that one of the vertices u and v is in V_4, say $v \in V_4$. In this case we observe that $D' = D - v$ has the partition-sequence $1,1,1,1,3,3$. It follows that $\{d^+(x), d^-(x)\} = \{4,5\}$ for $x \in V'_1 \cup V'_2 \cup \ldots \cup V'_4$ and $\{d^+(x), d^-(x)\} = \{3,4\}$ for $x \in V'_5 \cup V'_6$. Since $\frac{|V(D')| - |V'_5| - 2|V'_6| + 2}{2} = \frac{3}{2} > 1 = i_l(D')$ and $|V(D')| - 3|V'_6| + 1 = 2 > i_l(D')$, Theorem 4.28 implies that D' is Hamiltonian, a contradiction.

Subcase 1.1.3. Let $\{u, v\} \subseteq V_1 \cup V_2 \cup V_3$. This yields that $D' = D - v$ has the partition-sequence $1,1,2,3,3$ such that $\{d^+(x), d^-(x)\} = \{4,5\}$ for $x \in V'_1 \cup V'_2$, $d^+(x) = d^-(x) = 4$ or $\{d^+(x), d^-(x)\} = \{3,5\}$ for $x \in V'_3$ and $\{d^+(x), d^-(x)\} = \{3,4\}$ for $x \in V'_4 \cup V'_5$.

Subcase 1.1.3.1. Assume that D' has no cycle-factor. Since $|V(D')| - 3|V'_6| + 2 = 3 > i_l(D')$, Theorem 6.4 with $k = 0$ leads to $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. If $|Y| \leq 2$ or $|Z| \leq |Y| - 2$, then Theorem 6.4 with $t \geq 1$ or $k \geq 1$ yields a contradiction. Hence, let $|Y| = 3$ and $|Z| = 2$.

If $|Q_1| = 2$ and $|Q_2| = 3$, then let $q_1, q'_1 \in Q_1$. Since $Q_1 \leadsto Y \cup Q_2$ and $d^+(q_1) \leq 5$, there exists a vertex $q_2 \in V(q_1) \cap Q_2$. If $q'_1 \in V(q_1)$, then it follows that $|V(q_1)| = 3$ and $d^+(q_1) \geq 5$, a contradiction. If $q'_1 \notin V(q_1)$, then because of $d^+(q_1) \leq 5$ (respectively, $d^+(q_1) \leq 4$, if $|V(q_1)| = 3$), we deduce that $q'_1 \to q_1$ and thus $d^+(q'_1) \geq 6$ or $d^+(q'_1) \geq 5$ and $q'_1 \in V'_4 \cup V'_5$, in both cases a contradiction.

The case that $|Q_1| = 3$ and $|Q_2| = 2$ follows analogously.

If $|Q_1| = 1$ and $|Q_2| = 4$, then either $d^+(q_1) \geq 6$ or $d^+(q_1) \geq 5$ and $|V(q_1)| = 3$, if $q_1 \in Q_1$, in both cases a contradiction. The case that $|Q_1| = 4$ and $|Q_2| = 1$ follows analogously.

Subcase 1.1.3.2. Let D' have a cycle-factor. According to Corollary 5.9 we have $\kappa(D') \geq 2$. Because of $\lfloor \frac{\alpha(D')}{2} \rfloor + 1 = \lfloor \frac{3}{2} \rfloor + 1 = 2$ Theorem 4.27 yields that D' is Hamiltonian, a contradiction.

Subcase 1.2. Let $(n_i) = 1, 1, 1, 1, 2, 3$. This yields $d^+(x) = d^-(x) = 4$ for $x \in V_1 \cup V_2 \cup V_3 \cup V_4$, $\{d^+(x), d^-(x)\} = \{3, 4\}$ for $x \in V_5$ and $d^+(x) = d^-(x) = 3$ for $x \in V_6$.

Subcase 1.2.1. Suppose that one of the vertices u and v is in V_6, say $v \in V_6$. In this case, we deduce that $D' = D - v$ has the partition-sequence $1, 1, 1, 1, 2, 2$. Since $\frac{|V(D')| - |V_5'| - 2|V_6'| + 2}{2} = 2 \geq i_g(D')$, Theorem 4.28 implies that D' has a Hamiltonian cycle, a contradiction.

Subcase 1.2.2. Assume that one of the vertices u and v is in V_5, say $v \in V_5$. It follows that $D' = D - v$ has the partition-sequence $1, 1, 1, 1, 1, 3$. Furthermore, we conclude that $\{d^+(x), d^-(x)\} = \{3, 4\}$ for $x \in V_1' \cup V_2' \cup \ldots \cup V_5'$ and $\{d^+(x), d^-(x)\} = \{2, 3\}$ for $x \in V_6'$. Because of $\frac{|V(D')| - |V_5'| - 2|V_6'| + 3}{2} = 2 > 1 = i_l(D')$ and Theorem 8.13, it remains to consider the case that D' has no cycle-factor and $Q_1 = \emptyset$ or $Q_2 = \emptyset$. If $|Y| \leq 2$ or $|Z| \leq |Y| - 2$, then $|V(D')| - 3|V_6'| + 4 = 3 > i_l(D')$ contradicts Theorem 6.4 with $t \geq 1$ or $k \geq 1$. Hence, let $|Y| = 3$ and $|Z| = 2$.

Firstly, let $Q_1 = \emptyset$. We observe that $Y = \{y_1, y_2, y_3\} \to Q_2 = \{q_1, q_2, q_3\} \to Z = \{z_1, z_2\} \to Y$, $v \to Y$, $q_1 \to q_2 \to q_3 \to q_1$, $z_1 \to z_2 \rightsquigarrow v \rightsquigarrow z_1$ and $Q_2 \rightsquigarrow v$, since otherwise we arrive at a contradiction to the degree-conditions or to (4.18). If $u \in Q_2$ (without loss of generality let $u = q_1$), then $q_1 v y_1 q_2 z_1 y_2 q_3 z_2 y_3$ is a Hamiltonian path, and if $u = z_2$, then $y_1 q_1 z_2 v y_2 q_2 z_1 y_3 q_3$ is a Hamiltonian path containing the arc e, in both cases a contradiction.

Secondly, let $Q_2 = \emptyset$. Analogously as above, we see that $Y = \{y_1, y_2, y_3\} \to v$, $Z = \{z_1, z_2\} \to Q_1 = \{q_1, q_2, q_3\} \to Y \to Z$, $z_1 \to z_2 \rightsquigarrow v \rightsquigarrow z_1$, $q_1 \to q_2 \to q_3 \to q_1$ and $v \rightsquigarrow Q_1$. Since $u \in V_1 \cup V_2 \cup \ldots \cup V_4$, we obtain that $u = z_2$. The vertex v has at least two outer neighbors in Q_1. Hence, without loss of generality, let $v \to q_1$. Now, $q_3 y_1 z_2 v q_1 y_2 z_1 q_2 y_3$ is a Hamiltonian path through e, a contradiction.

Subcase 1.2.3. Let $\{u, v\} \subseteq V_1 \cup V_2 \cup V_3 \cup V_4$. It follows that $D' = D - v$ has the partition-sequence $1, 1, 1, 2, 3$ such that $\{d^+(x), d^-(x)\} = \{3, 4\}$ for $x \in V_1' \cup V_2' \cup V_3'$, $d^+(x) = d^-(x) = 3$ or $\{d^+(x), d^-(x)\} = \{2, 4\}$ for $x \in V_4'$ and $\{d^+(x), d^-(x)\} = \{2, 3\}$ for $x \in V_5'$.

Subcase 1.2.3.1. Suppose that D' has a cycle-factor. In this case, let F be a minimum cycle-factor with the properties of the Theorems 8.10 and 8.11. Since $|V(D')| = 8$, F consists of at most two cycles. If F consists of one cycle, then D contains a Hamiltonian path with the terminal arc e, a contradiction. Hence, let F consist of the two cycles C_1 and C_2 such that $C_1 \Rightarrow C_2$. If $u \in V(C_2)$, then $C_1 u^+ \ldots u^- uv$ is a Hamiltonian path of D. Consequently, let $u \in V(C_1)$. If $v \to u^+$, then $uvu^+ \ldots u^- C_2$ is a Hamiltonian path of D and if there is a vertex $v_2 \in V(C_2)$ such that $v \to v_2$, then $u^+ \ldots u^- uvv_2 v_2^+ \ldots v_2^-$ is a Hamiltonian path of D, in both cases a contradiction. Hence, let $\{u, u^+\} \cup V(C_2) \to v$, which means $d^-(v) \geq 5$, also a contradiction.

Subcase 1.2.3.2. Assume that D' has no cycle-factor. If $Q_1 = \emptyset$ or $Q_2 = \emptyset$ and $|Y| \leq 2$ or $|Z| \leq |Y| - 2$, then we arrive at a contradiction to Theorem 6.4 with $k \geq 1$ or $t \geq 1$. Hence, let $|Y| = 3$ and $|Z| = 2$ in these cases.

Firstly, let $Q_1 = \emptyset$. We observe that $|Q_2| = |Q| = 3$. Because of $Y \to Q_2$ and $d^-(q) \leq 4$ for all $q \in Q_2$, it follows that $Q_2 \rightsquigarrow Z$, since otherwise, if $z_2 \to q_2$ with $z_2 \in Z$ and $q_2 \in Q_2$, then $V_4' \subseteq Q_2$, $d^-(q) = 4$ for all $q \in Q_2$ and $z_1 \to z_2$ with $z_1 \in Z$. This implies that $v \to (Y \cup Z)$, which means $d^+(v) \geq 5$, a contradiction. Hence, let $Y = \{y_1, y_2, y_3\} \to Q_2 = \{q_1, q_2, q_3\} \rightsquigarrow Z \to Y$ and $v \to Y$. If $u \in Z = \{z_1, z_2\}$ (without loss of generality let $u = z_2$), then $y_1q_3z_2vy_2q_2z_1y_3q_1$ is a Hamiltonian path containing e, and if $u \in Q_2$ (without loss of generality let $u = q_1$), then $y_1q_1vy_2q_2z_1y_3q_3z_2$ is a Hamiltonian path through e, if we enumerate the vertices of Z and Q_2 such that $q_2 \to z_1$ and $q_3 \to z_2$, in both cases a contradiction.

Secondly, let $Q_2 = \emptyset$. Since $|Q_1| = 3$, (4.18) and the degree-conditions imply that $Q_1 \to Y \to (Z \cup \{v\})$. Analogously as we have $Q_2 \rightsquigarrow Z$ in the case $Q_1 = \emptyset$, we conclude that the case $Q_2 = \emptyset$ yields $Z \rightsquigarrow Q_1$. If there is an arc $q_1 \to v$ with $q_1 \in Q_1$, then we arrive at $V_4' \subseteq Q_1$, $d^+(q) \geq 4$ for all $q \in Q_1$, $z_1 \to z_2$ for the vertices $z_1, z_2 \in Z$ and $v \to Z$. Summarizing our results, we see that $d^-(z_2) \geq |Y| + 2 = 5$, a contradiction. Hence, let $v \to Q_1$. Since $u \in V_1 \cup V_2 \cup \ldots \cup V_4$, it remains the case that $u = z_2$ with $z_2 \in Z$. If $Y = \{y_1, y_2, y_3\}$, $Z = \{z_1, z_2\}$ and $Q_1 = \{q_1, q_2, q_3\}$ and if we enumerate the vertices of Z and Q_1 such that $z_1 \to q_3$, then $q_1y_1z_2vq_2y_2z_1q_3y_3$ is a Hamiltonian path containing the arc e, a contradiction.

Thirdly, let $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. If $|Z| \leq |Y| - 2$, then Theorem 6.4 with $k \geq 1$ yields a contradiction. Hence let $|Z| = |Y| - 1$.

If $|Y| = 3$ and thus $|Z| = 2$, then firstly let $|Q_1| = 1$ and $|Q_2| = 2$. Because of (4.18) and the degree-conditions, we observe that $V_4' = \{q_1, q_2\}$ with $Q_1 = \{q_1\}$ and $Q_2 = \{q_2, q_2'\}$ such that $q_2' \to q_2$, $Q_1 \to Y$, $q_1 \to q_2'$, $Q_2 \to Z = \{z_1, z_2\} \to Q_1$ and $Q_2 \to v \to Q_1$. Notice that v has at least one outer neighbor in Y. Hence, let $v \to y_2$ with $y_2 \in Y$. Let $Y = \{y_1, y_2, y_3\}$. If $u = q_2'$, then $y_3q_2'vy_2q_2z_1z_2q_1y_1$ is a Hamiltonian path, and if $u \in Z$ (without loss of generality let $u = z_1$), then $y_1q_2z_1vy_2q_2'z_2q_1y_3$ is a Hamiltonian path through e, in both cases a contradiction. If $|Q_1| = 2$ and $|Q_2| = 1$, then analogously, we see that $V_4' = \{q_1, q_2\}$ with $Q_1 = \{q_1, q_1'\}$ and $Q_2 = \{q_2\}$, $Q_2 \to v \to Q_1$ and $Q_2 \to Z = \{z_1, z_2\} \to Q_1$. If we have $z \in Z$ arbitrary, then it follows that there is a vertex $y \in Y$ such that $y \to z$. Let $Y = \{y_1, y_2, y_3\}$. If $u \in Z$ (without loss of generality let $u = z_2$), then let $y_1 \to z_2$ and $y_1z_2vq_1y_2q_2z_1q_1'y_3$ is a Hamiltonian path containing the arc e, a contradiction. Since $u \in V_1 \cup V_2 \cup V_3 \cup V_4$, z_2 and z_1 are the only possible candidates for u.

Now, let $|Y| = 2$ and thus $|Z| = 1$. If $Y = V_4'$, then we deduce that $|Q_1| + |Q_2| = 5$ and there remain the cases that $|Q_1| = 2$ and $|Q_2| = 3$ or $|Q_1| = 3$ and $|Q_2| = 2$ or $|Q_1| = 1$ and $|Q_2| = 4$ or $|Q_1| = 4$ and $|Q_2| = 1$. Because of $Q_1 \to Y \to Q_2$ and $Q_1 \rightsquigarrow Q_2$ in all cases, it is a simple matter to verify that either there is a vertex $q_1 \in Q_1$ such that $d^+(q_1) \geq 5$ or $d^+(q_1) \geq 4$ and $q_1 \in V_5'$ or a vertex $q_2 \in Q_2$ such that $d^-(q_2) \geq 5$ or $d^-(q_2) \geq 4$ and $q_2 \in V_5'$, in all cases a contradiction. Hence, let $Y \subseteq V_5'$. If $Z \subseteq V_5'$, then we obtain $|Q_1| + |Q_2| = 5$ and analogously as above, we arrive at a contradiction. If $Z \not\subseteq V_5'$, then there exists a vertex $y \in V_5' - (Y \cup Z)$ and thus $|Q_1| + |Q_2| = 4$. If $|Q_1| = |Q_2| = 2$, then, without loss of generality, let $y \in R_1 \cap V_5'$. Since $y \to Q_2$, there is a vertex $q_2 \in Q_2$ such that $d^-(q_2) \geq 5$, a contradiction.

If $|Q_1| = 3$ and $|Q_2| = 1$, then it follows that $q_2 \in V'_4$, $y \in R_2 \cap V'_5$ and $Q_1 \cup \{q_2\} \to y$. Now, we see that $d^-(y) \geq 4$, a contradiction. Analogously the case that $|Q_1| = 1$ and $|Q_2| = 3$ yields a contradiction.

If $|Y| \leq 1$, then because of $\frac{|V(D')|-|V'_4|-2|V'_5|+5}{2} = \frac{5}{2} > i_l(D')$ we arrive at a contradiction to Theorem 6.4 with $t \geq 2$.

Subcase 1.3. Let $(n_i) = 1,1,1,1,1,2$. This implies that $d^+(x) = d^-(x) = 3$ for $x \in V_1 \cup V_2 \cup \ldots \cup V_5$ and $\{d^+(x), d^-(x)\} = \{2,3\}$ for $x \in V_6$.

Subcase 1.3.1. Assume that one of the vertices u and v is in V_6, say $v \in V_6$. It follows that $D' = D - v$ has the partition-sequence $1,1,1,1,1,1$. Since $\frac{|V(D')|-|V'_5|-2|V'_6|+2}{2} = \frac{5}{2} > i_g(D')$, Theorem 4.28 guarantees the existence of a Hamiltonian cycle in D', a contradiction.

Subcase 1.3.2. Let $\{u,v\} \subseteq V_1 \cup V_2 \cup \ldots \cup V_5$. This implies that $D' = D - v$ has the partition-sequence $1,1,1,1,2$ such that $\{d^+(x), d^-(x)\} = \{2,3\}$ for $x \in V'_1 \cup V'_2 \cup \ldots \cup V'_4$ and $d^+(x) = d^-(x) = 2$ or $\{d^+(x), d^-(x)\} = \{1,3\}$ for $x \in V'_5$.

Subcase 1.3.2.1. Suppose that D' has a cycle-factor. In this case, analogously to Subcase 1.2.3.1, we arrive at a contradiction.

Subcase 1.3.2.2. Assume that D' has no cycle-factor. If $|Y| \leq 1$ or $|Z| \leq |Y| - 2$, then we obtain a contradiction to Theorem 6.4 with $t \geq 1$ or $k \geq 1$. Hence, let $|Y| = 2$ and $|Z| = 1$. If $Q_1 = \emptyset$, then it follows that $|Q_2| = 3$. According to (4.18), we observe that $Y \to Q_2$ and the degree-conditions imply that $Z \to Y$. Since $d^-(q) \leq 3$ for all $q \in Q_2$, we obtain that $q_1 \to q_2 \to q_3 \to q_1$, if $Q_2 = \{q_1, q_2, q_3\}$. Now, we observe that $(Q_2 \cup Z) \to v$, which means $d^-(v) \geq 4$, a contradiction. Analogously, we arrive at a contradiction, if $Q_2 = \emptyset$.

Consequently, it remains to consider the case that $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. If $|Q_1| = 1$ and $|Q_2| = 2$, then because of $Q_1 \to (Y \cup Q_2)$ we observe that $d^+(q_1) \geq 4$, a contradiction to $d^+(x) \leq 3$ for all $x \in V(D)$. Analogously, we arrive at a contradiction, if $|Q_1| = 2$ and $|Q_2| = 1$.

Subcase 1.4. Assume that $(n_i) = 1,1,1,1,2,2$. In this case, it follows that $\{d^+(x), d^-(x)\} = \{3,4\}$ for $x \in V_1 \cup V_2 \cup V_3 \cup V_4$ and $d^+(x), d^-(x) = 3$ for $x \in V_5 \cup V_6$.

Subcase 1.4.1. Assume that one of the vertices u and v is in $V_5 \cup V_6$, say $v \in V_5 \cup V_6$. This yields that $D' = D - v$ has the partition-sequence $1,1,1,1,1,2$. Because of $\frac{|V(D')|-|V'_5|-2|V'_6|+2}{2} = 2 \geq i_g(D')$ Theorem 4.28 implies the existence of a Hamiltonian cycle of D', a contradiction.

Subcase 1.4.2. Let $\{u,v\} \subseteq V_1 \cup V_2 \cup V_3 \cup V_4$. It follows that $D' = D - v$ has the partition-sequence $1,1,1,2,2$ such that $d^+(x) = d^-(x) = 3$ or $\{d^+(x), d^-(x)\} = \{2,4\}$ for $x \in V'_1 \cup V'_2 \cup V'_3$ and $\{d^+(x), d^-(x)\} = \{2,3\}$ for $x \in V'_4 \cup V'_5$.

Subcase 1.4.2.1. Suppose that D' has a cycle factor. In this case, analogously to Subcase 1.2.3.1, we arrive at a contradiction.

Subcase 1.4.2.2. Let D' have no cycle-factor. Since $|V(D')| - 3|V'_5| + 2 = 3 > i_l(D')$ Theorem 6.4 yields that $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. If $|Y| \leq 1$ or $|Z| \leq |Y| - 2$, then we obtain a contradiction to Theorem 4.28 with $t \geq 1$ or $k \geq 1$. Hence, let $|Y| = 2$ and $|Z| = 1$. Without loss of generatity, let $Y = V'_5$.

Firstly, let $|Q_1| = |Q_2| = 2$. If $Q_1 \cap V_4' = \emptyset$, then there is an arc $q_1 \to q_1'$ in $E(D'[Q_1])$ and we conclude that $d^+(q_1) \geq |Y| + |Q_2| + 1 = 5$, a contradiction. Analogously, we arrive at a contradiction, if $Q_2 \cap V_4' = \emptyset$. Let $q_1q_1' \in E(D'[Q_1])$. Then either $d^+(q_1) \geq 4$ and $q_1 \in V_4'$ or $d^+(q_1) \geq 5$, in both cases a contradiction.

Secondly, let $|Q_1| = 1$ and $|Q_2| = 3$. If $q_1 \in Q_1$, then we observe that either $d^+(q_1) \geq 4$ and $q_1 \in V_4'$ or $d^+(q_1) \geq 5$, in both cases a contradiction. Analogously, we obtain a contradiction, if $|Q_1| = 3$ and $|Q_2| = 1$.

Case 2. Let $c = 5$. If $|V_3| \geq 4$ or $|V_2| \geq 3$ or $|V_3| = 3$ and $|V_2| \geq 2$ or $|V_5| \leq 3$ and $|V_1| \geq 2$ or $|V_5| = |V_2| = 2$, then, using Corollary 8.8 or similar methods, it follows that D contains a Hamiltonian path starting with the arc $e = uv$, a contradiction. Since $i_g(D) = 1$, the partition-sequences $1,1,1,2,3$; $1,1,2,3,3$; $1,2,2,2,3$ and $2,2,2,3,4$ are impossible. Consequently, there remain to consider the following 10 partition-sequences.

Subcase 2.1. Let $(n_i) = 2,2,2,4,4$. It follows that $d^+(x) = d^-(x) = 6$ for $x \in V_1 \cup V_2 \cup V_3$ and $d^+(x) = d^-(x) = 5$ for $x \in V_4 \cup V_5$.

Subcase 2.1.1. Assume that one of the vertices u and v is part of $V_4 \cup V_5$, say $v \in V_4 \cup V_5$. In this case $D' = D - v$ has the partition-sequence $2,2,2,3,4$, and according to Theorem 4.28, $\frac{|V(D')|-|V_4'|-2|V_5'|+2}{2} = 2 \geq i_g(D')$ implies that D' is Hamiltonian, a contradiction.

Subcase 2.1.2. Let $\{u,v\} \subseteq V_1 \cup V_2 \cup V_3$. Then $D' = D - v$ has the partition-sequence $1,2,2,4,4$ such that $d^+(x) = d^-(x) = 6$ for $x \in V_1'$, $\{d^+(x), d^-(x)\} = \{5,6\}$ for $x \in V_2' \cup V_3'$ and $\{d^+(x), d^-(x)\} = \{4,5\}$ for $x \in V_4' \cup V_5'$. Because of $\frac{|V(D')|-|V_4'|-2|V_5'|+2}{2} = \frac{3}{2} > i_l(D')$ and $|V(D')| - 3|V_5'| + 1 = 2 > i_l(D')$, Theorem 4.28 yields that D' is Hamiltonian, a contradiction.

Subcase 2.2. Let $(n_i) = 2,2,2,2,4$. It follows that $d^+(x) = d^-(x) = 5$ for $x \in V_1 \cup V_2 \cup V_3 \cup V_4$ and $d^+(x) = d^-(x) = 4$ for $x \in V_5$.

Subcase 2.2.1. Suppose that one of the vertices u and v is in V_5, say $v \in V_5$. We observe that $D' = D - v$ has the partition-sequence $2,2,2,2,3$. According to Theorem 4.28, the fact that $\frac{|V(D')|-|V_4'|-2|V_5'|+2}{2} = \frac{5}{2} > i_g(D')$ implies that D' is Hamiltonian, a contradiction.

Subcase 2.2.2. Assume that $\{u,v\} \subseteq V_1 \cup V_2 \cup V_3 \cup V_4$. We conclude that $D' = D - v$ has the partition-sequence $1,2,2,2,4$ such that $d^+(x) = d^-(x) = 5$ for $x \in V_1'$, $\{d^+(x), d^-(x)\} = \{4,5\}$ for $x \in V_2' \cup V_3' \cup V_4'$ and $\{d^+(x), d^-(x)\} = \{3,4\}$ for $x \in V_5'$. This shows that $i_l(D') = 1$.

Subcase 2.2.2.1. Let D' have a cycle-factor. Since $\frac{|V(D')|-|V_4'|-2|V_5'|+3}{2} = 2 > i_l(D')$, Lemma 8.13 implies that D' is Hamiltonian, a contradiction.

Subcase 2.2.2.2. Let D' have no cycle-factor. Since $\frac{|V(D')|-|V_4'|-2|V_5'|+3}{2} = 2 > i_l(D')$ and Theorem 6.4, we conclude that $Q_1 = \emptyset$ or $Q_2 = \emptyset$. If $|Y| \leq 3$ or $|Z| \leq |Y| - 2$, then we arrive at a contradiction to Theorem 6.4. Hence, let $|Y| = 4$ and $|Z| = 3$.

Firstly, let $Q_1 = \emptyset$. It follows that $|Q| = |Q_2| = 4$. Using Lemma 4.29 and $d^-(x) \leq 5$ for all $x \in V(D)$ we see that $Y = \{y_1, y_2, y_3, y_4\} \to Q_2 = \{q_1, q_2, q_3, q_4\} \to Z = \{z_1, z_2, z_3\}$, $q_1 \to q_2 \to q_3 \to q_4 \to q_1$ such that $V(q_1) = \{q_1, q_3\}$ and $V(q_2) = \{q_2, q_4\}$, $z_1 \to z_2 \to z_3$ such that $V(z_1) = \{z_1, z_3\}$ and $V(v) = \{v, z_2\}$, $Q_2 \to v \to Y$ and $z_3 \to v \to z_1$. If $u \in Q_2$, without loss of generality $u = q_1$, then $y_1q_1vy_2q_2z_1y_3q_3z_2y_4q_4z_3$ is a Hamiltonian path through e, and if $u = z_3$, then $y_1q_1z_3vy_2q_2z_1y_3q_3z_2y_4q_4$ is a Hamiltonian path through e, in both cases a contradiction.

Secondly, let $Q_2 = \emptyset$. Analogously as above it follows $Y = \{y_1, y_2, y_3, y_4\} \to Z = \{z_1, z_2, z_3\} \to Q_1 = \{q_1, q_2, q_3, q_4\} \to Y$, $q_1 \to q_2 \to q_3 \to q_4 \to q_1$ such that $V(q_1) = \{q_1, q_3\}$ and $V(q_2) = \{q_2, q_4\}$, $z_1 \to z_2 \to z_3$ such that $V(z_1) = \{z_1, z_3\}$ and $V(v) = \{v, z_2\}$, $Y \to v \to Q_1$ and $z_3 \to v \to z_1$. Since $u \in V_1 \cup V_2 \cup V_3 \cup V_4$, it follows that $u = z_3$. But now $q_4y_1z_3vq_1y_2z_1q_2y_3z_2q_3y_4$ is a Hamiltonian path containing e, a contradiction.

Subcase 2.3. Suppose that $(n_i) = 1, 1, 2, 2, 2$. In this case, we deduce that $\{d^+(x), d^-(x)\} = \{3, 4\}$ for $x \in V_1 \cup V_2$ and $d^+(x) = d^-(x) = 3$ for $x \in V_3 \cup V_4 \cup V_5$.

Subcase 2.3.1. Assume that one of the vertices u and v is part of $V_3 \cup V_4 \cup V_5$, say $v \in V_3 \cup V_4 \cup V_5$. It follows that $D' = D - v$ has the partition-sequence $1, 1, 1, 2, 2$ such that $d^+(x) = d^-(x) = 3$ or $\{d^+(x), d^-(x)\} = \{2, 4\}$ for $x \in V_1' \cup V_2'$, $d^+(x) = d^-(x) = 3$ for $x \in V_3'$ and $\{d^+(x), d^-(x)\} = \{2, 3\}$ for $x \in V_4' \cup V_5'$.

Subcase 2.3.1.1. Suppose that D' has a cycle-factor. Then let F be a minimal cycle-factor of D' with the properties of the Theorems 8.10 and 8.11. If F consists of one cycle, then D has a Hamiltonian path with the terminal arc e, a contradiction. Since $|V(D')| = 7$, it is impossible that F consists of more then two cycles. Hence, let F consist of the cycles C_1 and C_2 such that $C_1 \Rightarrow C_2$. As in Subcase 1.2.3.1, we conclude that $u \in V(C_1)$ and $(\{u, u^+\} \cup V(C_2)) \rightsquigarrow v$, and thus $d^-(v) \geq 4$, a contradiction to $v \in V_3 \cup V_4 \cup V_5$.

Subcase 2.3.1.2. Let D' have no cycle-factor. Because of $|V(D')| - 3|V_5'| + 2 = 3 > i_l(D')$ and Theorem 6.4, it remains to consider the case that $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. If $|Y| \leq 1$ or $|Z| \leq |Y| - 2$, then we arrive at a contradiction to Theorem 6.4. Hence, let $|Y| = 2$ and $|Z| = 1$, and thus $|Q| = 4$.

Firstly, let $|Q_1| = |Q_2| = 2$. Since $Q_1 \to Y \to Q_2$ and $Q_1 \rightsquigarrow Q_2$, it is easy to see that there exists a vertex $q_1 \in Q_1$ such that either $d^+(q_1) \geq 5$ or $d^+(q_1) \geq 4$ and $|V(q_1)| = 2$, in both cases a contradiction.

Secondly, let $|Q_1| = 1$ and $|Q_2| = 3$. Analogously as above we conclude that either $d^+(q_1) \geq 5$ or $d^+(q_1) \geq 4$ and $|V(q_1)| = 2$, if $Q_1 = \{q_1\}$, in all cases a contradiction.

Analogously, the case that $|Q_1| = 3$ and $|Q_2| = 1$ leads to a contradiction.

Subcase 2.3.2. Let $\{u, v\} \subseteq V_1 \cup V_2$. It follows that $D' = D - v$ has the partition-sequence $1, 2, 2, 2$ such that $d^+(x) = d^-(x) = 3$ or $\{d^+(x), d^-(x)\} = \{2, 4\}$ for $x \in V_1'$ and $\{d^+(x), d^-(x)\} = \{2, 3\}$ for $x \in V_2' \cup V_3' \cup V_4'$.

Subcase 2.3.2.1. Assume that D' has a cycle-factor. In this case, analogously as in Subcase 1.2.3.1, we arrive at a contradiction.

Subcase 2.3.2.2. Suppose that D' has no cycle-factor. Because of $|V(D')| - 3|V_4'| + 2 = 3 > i_l(D')$ and Theorem 6.4, it remains to consider the case that $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. If $|Y| \leq 1$ or $|Z| \leq |Y| - 2$, then we arrive at a contradiction to Theorem 6.4 with $t \geq 1$ or $k \geq 1$. Hence, let $|Y| = 2$ and $|Z| = 1$ and thus $|Q| = 4$. If $|Q_1| = |Q_2| = 2$, then it is a simple matter to verify that there is a vertex $q_1 \in Q_1$ such that either $d^+(q_1) \geq 5$ or $d^+(q_1) \geq 4$ and $|V'(q_1)| = 2$, in both cases a contradiction. If $|Q_1| = 1$ and $|Q_2| = 3$ or

$|Q_1| = 3$ and $|Q_2| = 1$, then analogously we arrive at a contradiction.

Subcase 2.4. Let $(n_i) = 1, 1, 1, 2, 2$. It follows that $d^+(x) = d^-(x) = 3$ for $x \in V_1 \cup V_2 \cup V_3$ and $\{d^+(x), d^-(x)\} = \{2, 3\}$ for $x \in V_4 \cup V_5$.

Subcase 2.4.1. Assume that one of the vertices u and v is part of $V_4 \cup V_5$, say $v \in V_4 \cup V_5$. In this case we conclude that $D' = D - v$ has the partition-sequence $1, 1, 1, 1, 2$ such that $\{d^+(x), d^-(x)\} = \{2, 3\}$ for $x \in V_1' \cup V_2' \cup V_3' \cup V_4'$ and $d^+(x) = d^-(x) = 2$ or $\{d^+(x), d^-(x)\} = \{1, 3\}$ for $x \in V_5'$.

Subcase 2.4.1.1. Let D' have a cycle-factor. Analogously as in Subcase 2.3.1.1 we arrive at a contradiction.

Subcase 2.4.1.2. Suppose that D' has no cycle-factor. If $|Y| \leq 1$ or $|Z| \leq |Y| - 2$, then using Theorem 6.4 with $t \geq 1$ or $k \geq 1$ we obtain a contradiction. Hence, let $|Y| = 2$ and $|Z| = 1$ and thus $|Q| = 3$.

Firstly, let $Q_1 = \emptyset$. Because of $Y \to Q_2$ and the degree-conditions we observe that $Q_2 = \{q_1, q_2, q_3\} \to Z = \{z\} \to Y$, $q_1 \to q_2 \to q_3 \to q_1$ and $(Q_2 \cup Z) \rightsquigarrow v \to Y$. If $u \in Q_2$, without loss of generality $u = q_1$, then $q_1 v y_1 q_2 z y_2 q_3$ is a Hamiltonian path through e and if $u = z$, then $y_1 q_1 z v y_2 q_2 q_3$ is a Hamiltonian path through e, in both cases a contradiction.

Secondly, let $Q_2 = \emptyset$. Analogously as above we see that $Q_1 = \{q_1, q_2, q_3\} \to Y = \{y_1, y_2\} \to Z = \{z\}$, $q_1 \to q_2 \to q_3 \to q_1$ and $Y \to v \rightsquigarrow (Q_1 \cup Z)$. Let $v \to q_1$. It follows that $u \in Y$, without loss of generality $u = y_1$, and $y_1 v q_1 q_2 q_3 y_2 z$ is a Hamiltonian path of D containing the arc e, if the vertices of Q_1 are numerated such that $v \to q_1$, a contradiction.

Thirdly, let $|Q_1| = 1$ and $|Q_2| = 2$. Since $Q_1 = \{q_1\} \to (Y \cup Q_2)$ we conclude that $d^+(q_1) \geq 4$, a contradiction.

If $|Q_1| = 2$ and $|Q_2| = 1$, then analogously, we arrive at a contradiction.

Subcase 2.4.2. Let $\{u, v\} \subseteq V_1 \cup V_2 \cup V_3$. This implies that $D' = D - v$ has the partition-sequence $1, 1, 2, 2$ such that $\{d^+(x), d^-(x)\} = \{2, 3\}$ for $x \in V_1' \cup V_2'$ and $d^+(x) = d^-(x) = 2$ or $\{d^+(x), d^-(x)\} = \{1, 3\}$ for $x \in V_3' \cup V_4'$.

Subcase 2.4.2.1. Assume that D' has a cycle-factor. In this case, analogously as in Subcase 1.2.3.1 we arrive at a contradiction.

Subcase 2.4.2.2. Let D' have no cycle-factor. If $Q_1 = \emptyset$ or $Q_2 = \emptyset$ and $|Y| \leq 1$ or $|Z| \leq |Y| - 2$, then this contradicts Theorem 6.4 with $t \geq 1$ or $k \geq 1$. Hence, let $|Y| = 2$ (without loss of generality let $Y = V_4'$) and $|Z| = 1$ and thus $|Q| = 3$ in these cases.

Firstly, let $Q_1 = \emptyset$. According to (4.18) and the degree-conditions, we have $(v \cup (Z = \{z\})) \to Y = \{y_1, y_2\} \to Q_2 = \{q_1, q_2, q_3\}$. Furthermore, only one of the following three conditions can be fulfilled at the same time: 1. $V_3' \cap Q_2 \neq \emptyset$ and $V_3' \cap Z \neq \emptyset$, 2. There is an arc leading from $v \to Q_2$, 3. There is an arc leading from Z to Q_2. Furthermore, there exists an arc $q_1 q_2 \in E(D[Q_2])$ such that $q_3 \to z$. If $u \in Q_2$, without loss of generality $u = q_1$, then the vertices of Q_2 can be numerated such that $q_1 v y_1 q_3 z y_2 q_2$ is a Hamiltonian path through e, and if $u = z$, then $y_1 q_3 z v y_2 q_1 q_2$ is a Hamiltonian path containing the arc $e = uv$, in all cases a contradiction.

Secondly, let $Q_2 = \emptyset$. Analogously as above, we see that $Q_1 = \{q_1, q_2, q_3\} \to Y = \{y_1, y_2\} \to (v \cup (Z = \{z\}))$. Since $d^-(v) \leq 3$, there is only one further inner neighbor of v except the two vertices of Y. This vertex has to be u. If $u = z$, then, with a suitable numbering of the vertices of Q_1, $q_1 y_1 z v q_2 q_3 y_2$ is a Hamiltonian path of D, and if $u \in Q_1$, without loss of generality $u = q_3$, then $z \to Q_1$ and $q_1 \in V(q_2)$ and $q_1y_1zq_3vq_2y_2$ is a Hamiltonian path of D containing the arc e, in all cases a contradiction.

Thirdly, let $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. According to Theorem 6.4 we have $|Y| = |Z|+1$. At first, let $|Y| = 2$ and thus $|Z| = 1$. Without loss of generality, let $Y = V_4'$. If $|Q_1| = 1$ and $|Q_2| = 2$, then the degree-conditions and (4.18) imply that $Q_1 = \{q_1\} \to Y = \{y_1, y_2\} \to Q_2 = \{q_2, q_2'\} \to Z = \{z\} \to Q_1$, $q_2 \to q_2'$, $Q_2 \to v \to Q_1$, $q_1 \leadsto Q_2$ and $V_3' = \{q_1, q_2'\}$. Observing that v has at least one outer neighbor in Y, say $v \to y_2$, we deduce that $y_1q_2vy_2q_2'zq_1$ is a Hamiltonian path through e, if $u = q_2$ and $y_1q_2q_2'zvq_1y_2$ is a Hamiltonian path containing e, if $u = z$, in all cases a contradiction. If $|Q_1| = 2$ and $|Q_2| = 1$, then analogously, we obtain that $Y = \{y_1, y_2\} \to Q_2 = \{q_2\} \to Z = \{z\} \to Q_1 = \{q_1, q_1'\}$, $Q_1 \leadsto Q_2 \to v \to Q_1$, $q_1 \to q_1'$ and $V_3' = \{q_1, q_2\}$. This yields $u = z$, $Y \to Z$ and $y_1zvq_1q_1'y_2q_2$ is a Hamiltonian path containing $e = uv$, a contradiction. If finally $|Y| = 1$ and $|Z| = 0$, then we arrive at a contradiction to the degree-conditions.

Subcase 2.5. Let $(n_i) = 1, 1, 1, 1, 2$. This implies that $\{d^+(x), d^-(x)\} = \{2, 3\}$ for $x \in V_1 \cup V_2 \cup V_3 \cup V_4$ and $d^+(x) = d^-(x) = 2$ for $x \in V_5$.

Subcase 2.5.1. Let one of the vertices u and v be in V_5, say $v \in V_5$. It follows that $D' = D - v$ has the partition-sequence $1, 1, 1, 1, 1$. Since $\frac{|V(D')|-|V_4'|-2|V_5'|+2}{2} = 2 \geq i_g(D')$, Theorem 4.28 yields that D' is Hamiltonian, a contradiction.

Subcase 2.5.2. Assume that $\{u, v\} \subseteq V_1 \cup V_2 \cup V_3 \cup V_4$. This yields that D' has the partition-sequence $1, 1, 1, 2$ such that $d^+(x) = d^-(x) = 2$ or $\{d^+(x), d^-(x)\} = \{1, 3\}$ for $x \in V_1' \cup V_2' \cup V_3'$ and $\{d^+(x), d^-(x)\} = \{1, 2\}$ for $x \in V_4'$.

Subcase 2.5.2.1. Suppose that D' has a cycle-factor. In this case D' is Hamiltonian, a contradiction.

Subcase 2.5.2.2. Let D' have no cycle-factor. If $Q_1 = \emptyset$ or $Q_2 = \emptyset$, then this implies that $|Y| = 2$ and $|Z| = 1$, since otherwise we arrive at a contradiction to Theorem 6.4. Analogously, we see that $|Z| = |Y|-1$, if $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$.

Firstly, let $Q_1 = \emptyset$. It follows that $(\{v\} \cup (Z = \{z\})) \to Y = \{y_1, y_2\} \to Q_2 = \{q_1, q_2\}$. If $u \in Q_2$, say $u = q_1$, then, using the degree-conditions, it is a simple matter to show that either $y_1q_1vy_2q_2z$ or $y_1q_1vzy_2q_1$ is a Hamiltonian path containing the arc e, a contradiction. If $u = z$, then $Q_2 \to z$ and $y_1q_1zvy_2q_2$ is a Hamiltonian path, a contradiction.

Secondly, let $Q_2 = \emptyset$. Analogously, we observe that $Q_1 = \{q_1, q_2\} \to Y \to \{v\} \cup (Z = \{z\})$. Without loss of generality, let $q_1 \to q_2$. If $u = q_2$, then $v \to q_1$ and $z \to Q_1$ and $y_1zq_2vq_1y_2$ is a Hamiltonian path through e, a contradiction. If $u = z$, then $v \to Q_1$ and $y_1zvq_1q_2y_2$ is a Hamiltonian path, again a contradiction.

Thirdly, let $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. At first, let $|Y| = 2$, and thus $|Z| = 1$. In this case (4.18) and the degree-conditions imply that $Q_1 = \{q_1\} \to Y = \{y_1, y_2\} \to Q_2 = \{q_2\} \to (\{v\} \cup (Z = \{z\})) \to Q_1$. If $u = q_2$, then either $zy_1q_2vq_1y_2$ or $y_2q_2vq_1y_1z$ is a Hamiltonian path through e, and if $u = z$, then $y_1q_2zvq_1y_2$ is a Hamiltonian path containing e, in all cases a contradiction. If finally $|Y| = 1$, and thus $|Z| = 0$, then it is straightforward to show that there is a vertex $q_1 \in R_1$ such that $d^-(q_1) = 0$ or a vertex $q_2 \in R_2$ such that $d^+(q_2) = 0$, which contradicts the degree-conditions.

Subcase 2.6. Let $(n_i) = 1, 1, 3, 3, 3$. This implies that $d^+(x) = d^-(x) = 5$ for $x \in V_1 \cup V_2$ and $d^+(x) = d^-(x) = 4$ for $x \in V_3 \cup V_4 \cup V_5$.

Subcase 2.6.1. Assume that one of the vertices u and v is in $V_3 \cup V_4 \cup V_5$, say $v \in V_3 \cup V_4 \cup V_5$. In this case, we deduce that $D' = D - v$ has the partition-sequence $1, 1, 2, 3, 3$ such that $i_l(D') = 1$. Because of $\frac{|V(D')|-|V_4'|-2|V_5'|+2}{2} = \frac{3}{2} > i_l(D') = 1$ and $|V(D')| - 3|V_5'| + 1 = 2 > i_l(D')$, Theorem 4.28 yields that D' is Hamiltonian, a contradiction.

Subcase 2.6.2. Suppose that $\{u, v\} \subseteq V_1 \cup V_2$. This implies that $D' = D - v$ has the partition-sequence $1, 3, 3, 3$ with $i_l(D') = 1$. Since $\frac{|V(D')|-|V_3'|-2|V_4'|+2}{2} = \frac{3}{2} > i_l(D') = 1$ and $|V(D')| - 3|V_4'| + 1 = 2 > i_l(D')$, we arrive at a contradiction to Theorem 4.28.

Subcase 2.7. Let $(n_i) = 1, 1, 2, 2, 3$. This implies that $d^+(x) = d^-(x) = 4$ for $x \in V_1 \cup V_2$, $\{d^+(x), d^-(x)\} = \{3, 4\}$ for $x \in V_3 \cup V_4$ and $d^+(x) = d^-(x) = 3$ for $x \in V_5$.

Subcase 2.7.1. Assume that one of the vertices u and v is in V_5, say $v \in V_5$. It follows that $D' = D - v$ has the partition-sequence $1, 1, 2, 2, 2$, and because of $\frac{|V(D')|-|V_4'|-2|V_5'|+2}{2} = 2 \geq i_g(D')$, Theorem 4.28 yields that D' is Hamiltonian, a contradiction.

Subcase 2.7.2. Suppose that one of the vertices u and v is in $V_3 \cup V_4$, say $v \in V_3 \cup V_4$. In this case, $D' = D - v$ has the partition-sequence $1, 1, 1, 2, 3$ such that $\{d^+(x), d^-(x)\} = \{3, 4\}$ for $x \in V_1' \cup V_2' \cup V_3'$, $d^+(x), d^-(x) = 3$ or $\{d^+(x), d^-(x)\} = \{2, 4\}$ for $x \in V_4'$ and $\{d^+(x), d^-(x)\} = \{2, 3\}$ for $x \in V_5'$.

Subcase 2.7.2.1. Let D' have a cycle-factor. Theorem 4.16 implies that $\kappa(D') \geq \frac{|V(D')|-\alpha(D')-2i_l(D')}{3} = \frac{1}{3}$, which means $\kappa(D') \geq 1$. If even $\kappa(D') \geq 2 = \lfloor \frac{\alpha(D')}{2} \rfloor + 1$, then Theorem 4.27 yields that D' is Hamiltonian, a contradiction. Hence let $\kappa(D') = 1$ and let $\hat{v}$ be a cut-vertex of D'. Now $D'' = D' - \hat{v}$ consists of the strong components $P_1, P_2, \ldots, P_t$ with $t \geq 2$ and $P_i \rightsquigarrow P_j$, if $i < j$. If $|V(P_1)| = 1$ and $v_1 \in V(P_1)$, then $d^-_{D'}(v_1) \leq 1$, a contradiction. Consequently, let $|V(P_1)| \geq 3$ and analogously, it follows that $|V(P_t)| \geq 3$. Because of $|V(D'')| = 7$, we obtain that $|V(P_1)| = 3$ or $|V(P_t)| = 3$, say $|V(P_1)| = 3$. This implies that $d^-_{D'}(v_1) \leq 2$ for all $v_1 \in V(P_1)$. Since all the three vertices of P_1 are in different partite sets, this is a contradiction.

Subcase 2.7.2.2. Let D' have no cycle-factor. To get no contradiction to Theorem 6.4, we conclude that $|Y| = 3$ and $|Z| = 2$, if $Q_1 = \emptyset$ or $Q_2 = \emptyset$ and $|Y| = |Z| + 1$ with $|Y| \geq 2$, if $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$.

Firstly, let $Q_1 = \emptyset$, and thus $|Q_2| = 3$. The degree-conditions and (4.18) imply that $(\{v\} \cup (Z = \{z_1, z_2\})) \to Y = \{y_1, y_2, y_3\} \to Q_2 = \{q_1, q_2, q_3\}$, every vertex of Z has at most one outer neighbor in Q_2 and every vertex of Q_2 has at most one inner neighbor in Z. If there are vertices $z_1 \in Z$ and $q_1, q_2 \in Q_2$ such that $z_1 \to q_1$ and $q_2 \in V(z_1)$, then it is straightforward to show that we arrive at a contradiction. Summarizing our results, we see that each vertex of Z has at least two inner neighbors in Q_2. If $u \in Q_2$, without loss of generality $u = q_1$, then the vertices of Q_2 and Z can be numerated such that $y_1 q_1 v y_2 q_2 z_1 y_3 q_3 z_2$ is a Hamiltonian path containing the arc e, a contradiction. If $u \in Z$, say

$u = z_2$, then the vertices can be numerated such that $y_1q_1z_2vy_2q_2z_1y_3q_3$ is a Hamiltonian path through e, again a contradiction.

Secondly, let $Q_2 = \emptyset$, and thus $|Q_1| = 3$. Analogously as in the case $Q_1 = \emptyset$, we observe that $Q_1 = \{q_1, q_2, q_3\} \to Y = \{y_1, y_2, y_3\} \to (\{v\} \cup (Z = \{z_1, z_2\}))$ and each vertex of Z has at least one outer neighbor in Q_1. If $u \in Q_1$, say $u = q_1$, then it is straightforward to prove that $v \to Q_1 - \{q_1\}$, $z_1 \to z_2$ with $z_2 \in V(v)$, $z_2 \to Q_1$ and $z_1 \rightsquigarrow Q_1$. Therefore, the vertices in Q_1 can be numerated such that $y_1z_2q_1vq_2y_2z_1q_3y_3$ is a Hamiltonian path containing e, a contradiction. If $u \in Z$, say $u = z_2$, then the vertices of Q_1 can be numerated such that $q_1y_1z_2vq_2y_2z_1q_3y_3$ is a Hamiltonian path of D, a contradiction.

Thirdly, let $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. At first, let $|Y| = 3$ and thus $|Z| = 2$ and $|Q_1| + |Q_2| = 3$. If $|Q_1| = 1$ and $|Q_2| = 2$, then it is easy to see that $Q_1 = \{q_1\} \to Y = \{y_1, y_2, y_3\} \to Q_2 = \{q_2, q_2'\} \to Z = \{z_1, z_2\} \to Q_1$, $q_1 \to q_2$, $V_4' = \{q_1, q_2'\}$, $q_2 \to q_2'$, without loss of generality $z_1 \to z_2$ and $Q_2 \rightsquigarrow v \to Q_1$. Furthermore, we deduce that each vertex of Z has at least one outer neighbor in Y. If $u = q_2$, then the vertices of Y can be numerated such that $y_1q_2vq_1y_2q_2'z_1z_2y_3$ is a Hamiltonian path containing the arc e, a contradiction. If $u = z_1$, then the vertices of Y can be numerated such that $y_1q_2z_1vq_1y_2q_2'z_2y_3$ is a Hamiltonian path, again a contradiction. If $u = z_2$, then analogously, we arrive at a contradiction. Now, let $|Q_1| = 2$ and $|Q_2| = 1$. Following the same lines as above, we observe that $Q_1 = \{q_1, q_1'\} \to Y = \{y_1, y_2, y_3\} \to Q_2 = \{q_2\} \to Z = \{z_1, z_2\} \to Q_1$, $q_1 \to q_1' \to q_2$, $V_4' = \{q_1, q_2\}$ and $Q_2 \to v \rightsquigarrow Q_1$. Moreover, there are two different vertices , say $y_2, y_3 \in Y$ such that $y_2 \to z_1$ and $y_3 \to z_2$. If $u = q_2$, then $y_1q_2vq_1y_2z_1q_1'y_3z_2$ is a Hamiltonian path through e, and if $u \in Z$, say $u = z_1$, then $y_2z_1vq_1y_1q_2z_2q_1'y_3$ is a Hamiltonian path containing the arc $e = uv$, in both cases a contradiction.

Suppose that $|Y| = 2$ and thus $|Z| = 1$. Since $|V(D')| - |Y| - |Z| = 5$, we either have $|R_1| \leq 2$ or $|R_2| \leq 2$. If $|R_1| \leq 2$, then there is a vertex $r_1 \in R_1$ such that $d^-_{D'}(r_1) \leq 1$ and if $|R_2| \leq 2$, then there is a vertex $r_2 \in R_2$ such that $d^+_{D'}(r_2) \leq 1$, in both cases a contradiction.

Subcase 2.7.3. Assume that $\{u, v\} \subseteq V_1 \cup V_2$. This implies that $D' = D - v$ has the partition-sequence $1, 2, 2, 3$ such that $\{d^+(x), d^-(x)\} = \{3, 4\}$ for $x \in V_1'$, $d^+(x) = d^-(x) = 3$ or $\{d^+(x), d^-(x)\} = \{2, 4\}$ for $x \in V_2' \cup V_3'$ and $\{d^+(x), d^-(x)\} = \{2, 3\}$ for $x \in V_4'$.

Subcase 2.7.3.1. Let D' have a cycle-factor. Analogously as in Subcase 1.2.3.1, we arrive at a contradiction.

Subcase 2.7.3.2. Suppose that D' has no cycle-factor. According to Theorem 6.4, we conclude that $|Y| = 3$ and $|Z| = 2$, if $Q_1 = \emptyset$ or $Q_2 = \emptyset$, and $|Z| + 1 = |Y| \geq 2$, if $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$.

Firstly, let $Q_1 = \emptyset$ and thus $|Q_2| = |Q| = 3$. This yields $(\{v\} \cup (Z = \{z_1, z_2\})) \to Y = \{y_1, y_2, y_3\} \to Q_2 = \{q_1, q_2, q_3\}$. If there are vertices $z_2 \in Z$ and $q_1 \in Q_2$ such that $z_2 \to q_1$, then it follows that $z_1 \to z_2$ or $z_1 \in V(z_2)$. If $z_1 \to z_2$, then we have $v \to Z$, a contradiction to $d^+(v) \leq 4$. If $z_1 \in V(z_2)$, then we conclude that $z_1 \to v \to Q_2 \to z_1$ and $Q_2 - \{q_1\} \to z_2$. The case that $u = z_1$ yields the Hamiltonian path $y_1q_1z_1vy_2q_2z_2y_3q_3$, and if $u = q_1$, then $y_1q_2vy_2q_1z_1y_3q_3z_2$ is a Hamiltonian path containing e, in both cases a contradiction. Hence, $Q_2 \rightsquigarrow Z$ and for each fixed $q_1 \in Q_2$, the other two vertices q_2, q_3 of Q_2 can be numerated such that $q_2 \rightarrow z_1$ and $q_3 \rightarrow z_2$. If $u \in Q_2$, say $u = q_1$, then $y_1q_1vy_2q_2z_1y_3q_3z_2$ is a Hamiltonian path and if $u \in Z$, say $u = z_1$, then $y_1q_2z_1vy_2q_3z_2y_3q_1$ is a Hamiltonian path containing the arc $e = uv$, in all cases a contradiction.

Secondly, let $Q_2 = \emptyset$ and thus $|Q_1| = 3$. Following the same lines as above, we see that $Q_1 = \{q_1, q_2, q_3\} \rightarrow Y = \{y_1, y_2, y_3\} \rightarrow (\{v\} \cup (Z = \{z_1, z_2\}))$. Furthermore, for each fixed vertex $q_1 \in Q_1$, it is straightforward to show that the vertices of Z can be numerated such that $z_1 \rightarrow q_2$ and $z_2 \rightarrow q_3$. Since $u \in V_1'$, it is impossible that $u \in Y$. If $u \in Q_1$, say $u = q_3$, then $y_1z_2q_3vq_1y_2z_1q_2y_3$ is a Hamiltonian path containing e, and if $u \in Z$, say $u = z_1$, then $q_1y_1z_1vq_2y_2z_2q_3y_3$ is a Hamiltonian path, in all cases a contradiction.

Thirdly, let $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. At first, we assume that $|Y| = 3$ and thus $|Z| = 2$ and $|Q_1| + |Q_2| = 3$. If $|Q_1| = 1$ and $|Q_2| = 2$, then it follows that $Q_1 = \{q_1\} \rightarrow Y = \{y_1, y_2, y_3\} \rightarrow Q_2 = \{q_2, q_2'\} \rightsquigarrow Z = \{z_1, z_2\} \rightarrow Q_1$, $q_1 \rightarrow q_2 \rightarrow q_2'$, $q_2' \in V(q_1)$ and $Q_2 \rightarrow v \rightarrow Q_1$. Furthermore, v has an outer neighbor in Y, say $v \rightarrow y_2$. If $u = q_2$, then $z_1y_1q_2vy_2q_2'z_2q_1y_3$ or $y_3q_2vy_2q_2'z_2q_1y_1z_1$ is a Hamiltonian path and if $u \in Z$, say $u = z_1$, then either $y_1q_2z_1vy_2q_2'z_2q_1y_3$ or $y_1q_2'z_1vy_2q_2z_2q_1y_3$ is a Hamiltonian path containing e, in all cases a contradiction. If $|Q_1| = 2$ and $|Q_2| = 1$, then analogously, we observe that $Q_1 = \{q_1, q_1'\} \rightarrow Y = \{y_1, y_2, y_3\} \rightarrow Q_2 = \{q_2\} \rightarrow Z = \{z_1, z_2\} \rightsquigarrow Q_1$, $q_1 \rightarrow q_1' \rightarrow q_2$, $q_1 \in V(q_2)$ and $Q_2 \rightarrow v \rightarrow Q_1$. Since $u \in V_1'$, it follows that $u \in Z$. If $u = z_1$, then it is straightforward to show that z_2 has an inner neighbor in Y, say $y_2 \rightarrow z_2$. But now, $y_1q_2z_1vq_1'y_2z_2q_1y_3$ is a Hamiltonian path containing e, a contradiction. If $u = z_2$, then analogously, we arrive at a contradiction.

Now, let $|Y| = 2$ and thus $|Z| = 1$. Since $|V(D')| - |Y| - |Z| = 5$, we either have $|R_1| \leq 2$ or $|R_2| \leq 2$. If $|R_1| \leq 2$, then there is a vertex $r_1 \in R_1$ such that $d^-_{D'}(r_1) \leq 1$ and if $|R_2| \leq 2$, then there is a vertex $r_2 \in R_2$ such that $d^+_{D'}(r_2) \leq 1$, in both cases a contradiction.

Subcase 2.8. Let $(n_i) = 1, 1, 1, 3, 3$. This implies that $d^+(x) = d^-(x) = 4$ for $x \in V_1 \cup V_2 \cup V_3$ and $d^+(x) = d^-(x) = 3$ for $x \in V_4 \cup V_5$.

Subcase 2.8.1. Suppose that one of the vertices u and v is in $V_4 \cup V_5$, say $v \in V_4 \cup V_5$. This yields that $D' = D - v$ has the partition-sequence $1, 1, 1, 2, 3$ such that $\{d^+(x), d^-(x)\} = \{3, 4\}$ for $x \in V_1' \cup V_2' \cup V_3'$, $d^+(x) = d^-(x) = 3$ for $x \in V_4'$ and $\{d^+(x), d^-(x)\} = \{2, 3\}$ for $x \in V_5'$.

Subcase 2.8.1.1. Let D' have a cycle-factor. Since $\frac{|V(D')|-2|V_5'|-|V_4'|+3}{2} = \frac{3}{2} > 1 = i_l(D')$, Theorem 8.13 implies that D' is Hamiltonian, a contradiction.

Subcase 2.8.1.2. Let D' have no cycle-factor. Since $\frac{|V(D')|-2|V_5'|-|V_4'|+3}{2} = \frac{3}{2} > 1 = i_l(D')$, Theorem 6.4 yields that $Q_1 = \emptyset$ or $Q_2 = \emptyset$. Furthermore, Theorem 6.4 implies that $|Y| = 3$ and $|Z| = 2$.

Firstly, let $Q_1 = \emptyset$. It follows that $Y = \{y_1, y_2, y_3\} \rightarrow Q_2 = \{q_1, q_2, q_3\} \rightarrow Z = \{z_1, z_2\} \rightarrow Y$ $q_1 \rightarrow q_2 \rightarrow q_3 \rightarrow q_1$, $Q_2 \rightarrow v \rightarrow Y$ and $Z \subseteq V(v)$. Hence, $u \in Q_2$, say $u = q_1$ and $y_1q_1vy_2q_2z_1y_3q_3z_2$ is a Hamiltonian path through e, a contradiction.

Secondly, let $Q_2 = \emptyset$. Analogously, we observe that $Y = \{y_1, y_2, y_3\} \rightarrow Z = \{z_1, z_2\} \rightarrow Q_1 = \{q_1.q_2, q_3\} \rightarrow Y \rightarrow v \rightarrow Q_1$, $q_1 \rightarrow q_2 \rightarrow q_3 \rightarrow q_1$ and $Z \subseteq V(v)$. This yields $u \in Y$, say $u = y_1$ and $y_1vq_1y_2z_1q_2y_3z_2q_3$ is a

Hamiltonian path containing e, a contradiction.

Subcase 2.8.2. Suppose that $\{u,v\} \subseteq V_1 \cup V_2 \cup V_3$. In this case, $D' = D - v$ has the partition-sequence $1,1,3,3$ such that $\{d^+(x), d^-(x)\} = \{3,4\}$ for $x \in V_1' \cup V_2'$ and $\{d^+(x), d^-(x)\} = \{2,3\}$ for $x \in \{2,3\}$ for $x \in V_3' \cup V_4'$.

Subcase 2.8.2.1. Let D' have a cycle-factor. Then analogously as in Subcase 1.2.3.1, we arrive at a contradiction.

Subcase 2.8.2.2. Assume that D' has no cycle-factor. According to Theorem 6.4, we have $|Y| = 3$ and $|Z| = 2$.

Firstly, suppose that $Q_1 = \emptyset$. This implies that $Y = \{y_1, y_2, y_3\} \to Q_2 = \{q_1, q_2, q_3\} \to Z = \{z_1, z_2\} \to Y$, $z_1 \to z_2$, $Q_2 = V(q_1)$ and $Q_2 \cup \{z_2\} \to v \to Y \cup \{z_1\}$. Since $u \in V_1' \cup V_2'$, it follows that $u = z_2$ and $y_1q_1z_2vy_2q_2z_1y_3q_3$ is a Hamiltonian path containing the arc e, a contradiction.

Secondly, let $Q_2 = \emptyset$. Analogously, we observe that $Y = \{y_1, y_2, y_3\} \to Z = \{z_1, z_2\} \to Q_1 = \{q_1, q_2, q_3\} \to Y$, $z_1 \to z_2$, $Q_1 = V(q_1)$ and $Y \cup \{z_2\} \to v \to Q_1 \cup \{z_1\}$. Hence, $u = z_2$ and $q_1y_1z_2vq_2y_2z_1q_3y_3$ is a Hamiltonian path through e, a contradiction.

Thirdly, let $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$, and thus $|Q_1| + |Q_2| = 3$. If $|Q_1| = 1$ and $|Q_2| = 2$, then it follows that $Q_1 = \{q_1\} \to Y = \{y_1, y_2, y_3\} \to Q_2 = \{q_2, q_2'\} \to Z = \{z_1, z_2\} \to Q_1$, $z_1 \to z_2$, $V(q_1) = \{q_1, q_2, q_2'\}$ and $Q_2 \to v \to Q_1$. Furthermore, each vertex of Z has an outer neighbor in Y. Since $u \in V_1' \cup V_2'$, we deduce that $u \in Z$, say $u = z_2$. But now, the vertices of Y can be numerated such that $y_1q_2z_2vq_1y_2q_2'z_1y_3$ is a Hamiltonian path containing the arc e, a contradiction. If $|Q_1| = 2$ and $|Q_2| = 1$, then analogously, we see that $Y = \{y_1, y_2, y_3\} \to Q_2 = \{q_2\} \to Z = \{z_1, z_2\} \to Q_1 = \{q_1, q_1'\} \to Y$, $z_1 \to z_2$, $\{q_1, q_1', q_2\} = V(q_1)$ and $Q_2 \to v \to Q_1$. Moreover, each vertex of Z has an inner neighbor in Y. Since $u \in V_1' \cup V_2'$, we conclude that $u \in Z$, say $u = z_2$. But now, the vertices of Y can be numerated such that $y_1z_2vq_1y_2q_2z_1q_1'y_3$ is a Hamiltonian path, again a contradiction.

Subcase 2.9. Let $(n_i) = 1,1,1,1,3$. This implies that $d^+(x) = d^-(x) = 3$ for $x \in V_1 \cup V_2 \cup V_3 \cup V_4$ and $d^+(x) = d^-(x) = 2$ for $x \in V_5$.

Subcase 2.9.1. Suppose that one of the vertices u and v is in V_5, say $v \in V_5$. It follows that $D' = D - v$ has the partition-sequence $1,1,1,1,2$ such that $\{d^+(x), d^-(x)\} = \{2,3\}$ for $x \in V_1' \cup V_2' \cup V_3' \cup V_4'$ and $d^+(x) = d^-(x) = 2$ for $x \in V_5'$. Because of $\frac{|V(D')| - |V_4'| - 2|V_5'| + 2}{2} = \frac{3}{2} > 1 = i_l(D')$ and $|V(D')| - 3|V_5'| + 1 = 1 = i_l(D')$, Theorem 4.28 yields that D' is Hamiltonian, a contradiction.

Subcase 2.9.2. Assume that $\{u,v\} \subseteq V_1 \cup V_2 \cup V_3 \cup V_4$. In this case, $D' = D - v$ has the partition-sequence $1,1,1,3$ such that $\{d^+(x), d^-(x)\} = \{2,3\}$ for $x \in V_1' \cup V_2' \cup V_3'$ and $\{d^+(x), d^-(x)\} = \{1,2\}$ for $x \in V_4'$.

Subcase 2.9.2.1. Let D' have a cycle-factor. This implies that D' is Hamiltonian, a contradiction.

Subcase 2.9.2.2. Suppose that D' has no cycle-factor. It follows that $|Y| = |Z| + 1 \geq 2$, if $Q_1 = \emptyset$ or $Q_2 = \emptyset$, since otherwise we arrive at a contradiction to Theorem 6.4 with $t \geq 2$ or $k \geq 1$. If $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$, then we deduce from Theorem 6.4 that $|Y| = 3$ and $|Z| = 2$. But this yields that either $Q_1 = \emptyset$ or $Q_2 = \emptyset$, in both cases a contradiction.

Firstly, let $Q_1 = \emptyset$. Let $|Y| = 3$ and $|Z| = 2$. In this case, it follows that $Y = \{y_1, y_2, y_3\} \to Q_2 = \{q_2\} \to (\{v\} \cup (Z = \{z_1, z_2\}))$ and $z_1 \to z_2$.

Suppose that $v \to Y$. Then, we conclude that $Z \to v$ and the vertices of Y can be numerated such that $z_2 \to \{y_1, y_2\} \to z_1 \to y_3 \to z_2$. If $u = q_2$, then $y_3q_2vy_2z_1z_2y_1$ is a Hamiltonian path, if $u = z_1$, then $y_1z_1vy_3q_2z_2y_2$ is a Hamiltonian path and if $u = z_2$, then $y_1q_2z_2vy_2z_1y_3$ is a Hamiltonian path containing e, in all cases a contradiction. Now, let v have two outer neighbors in Y, say $y_3 \to v \to \{y_1, y_2\}$. If $z_2 \to v \to z_1$, then we deduce that $Z \to y_3$ and without loss of generality $z_1 \to y_1 \to z_2 \to y_2 \to z_1$. If $u = q_2$, then $y_1q_2vy_2z_1z_2y_3$ is a Hamiltonian path and if $u = z_2$, then $y_1z_2vy_2q_2z_1y_3$ is a Hamiltonian path, in both cases a contradiction. Hence, let $z_1 \to v \to z_2$. Now, we see that $Z \to y_3$ and $z_2 \to \{y_1, y_2\} \to z_1$. If $u = q_2$, then $y_1q_2vy_2z_1z_2y_3$ is a Hamiltonian path and if $u = z_1$, then $y_2z_1vy_1q_2z_2y_3$ is a Hamiltonian path through e, in both cases a contradiction. Hence, let v have only one outer neighbor in Y, say $\{y_2, y_3\} \to v \to y_1$. This implies that $z_2 \to Y$ and $y_1 \to z_1 \to \{y_2, y_3\}$. If $u = q_2$, then $y_2q_2vy_1z_1z_2y_3$ is a Hamiltonian path containing e, also a contradiction.

Consequently, let $|Y| = 2$, $|Z| = 1$ and $|Q_2| = 2$. The degree-conditions and (4.18) imply that $V_4' - Y = \{y_3\} \subseteq R_2$, $(\{v\} \cup (Z = \{z\})) \to Y = \{y_1, y_2\} \to Q_2 = \{q_1, q_2\}$, $q_1 \to q_2 \to y_3$ and $q_2 \to \{z, v\}$. Since $d^+(v) \leq 3$, we conclude that v has only one further outer neighbor except the two vertices of Y. If this vertex is q_1, then it follows that $q_1 \to y_3$ and $\{q_1, y_3\} \to z$. If $u = q_2$, then $y_1q_2vq_1y_3zy_2$ is a Hamiltonian path, and if $u = z$, then $y_1q_1y_3zvy_2q_2$ is a Hamiltonian path containing the arc e, in both cases a contradiction. If the third outer neighbor is z, then we deduce that either $z \to q_1$ or $z \to y_3$. If $z \to q_1$, then we have $q_1 \to y_3 \to z$. If $u = q_1$, then $q_1vy_1q_2y_3zy_2$ is a Hamiltonian path, and if $u = q_2$, then $q_2vy_1q_1y_3zy_2$ is a Hamiltonian path through e, in both cases a contradiction. Hence, let $z \to y_3$. This yields that $y_3 \to q_1 \to z$. If $u = q_1$, then $y_1q_1vzy_2q_2y_3$ is a Hamiltonian path and if $u = q_2$, then $y_1q_2vy_2q_1zy_3$ is a Hamiltonian path containing the arc e, in both cases a contradiction. Finally, let $v \to y_3$, which implies that $y_3 \to q_1$ and $\{y_3, q_1\} \to z$. If $u = q_1$, then $q_1vy_1q_2y_3zy_2$ is a Hamiltonian path, if $u = q_2$, then $y_1q_2vy_3zy_2q_1$ is a Hamiltonian path, and if $u = z$, then $y_1q_2y_3zvy_2q_1$ is a Hamiltonian path containing the arc e, in all cases a contradiction.

Secondly, let $Q_2 = \emptyset$. Let $|Y| = 3$ and $|Z| = 2$. It follows that $Z = \{z_1, z_2\} \to Q_1 = \{q_1\} \to Y = \{y_1, y_2, y_3\}$, $v \to q_1$ and $z_1 \to z_2$. If $Y \to v$, then we arrive at a contradiction to $u \in V_1' \cup V_2' \cup V_3'$. If v has only one inner neighbor in Y, say $y_1 \to v \to \{y_2, y_3\}$, then we deduce that $Z \to v$, $\{y_2, y_3\} \to Z$ and $z_2 \to y_1 \to z_1$. If $u = z_1$, then $y_2z_1vq_1y_3z_2y_1$ is a Hamiltonian path and if $u = z_2$, then $y_2z_2vy_3z_1q_1y_1$ is a Hamiltonian path through e, in both cases a contradiction. Hence, let v have two inner neighbors in Y, say $\{y_1, y_2\} \to v \to y_3$. If $z_2 \to v \to z_1$, then it follows that $y_3 \to Z$ and without loss of generality $z_2 \to y_1 \to z_1 \to y_2 \to z_2$. This implies that $u = z_2$ and $y_2z_2vy_3z_1q_1y_1$ is a Hamiltonian path containing e, a contradiction. On the other hand, if $z_1 \to v \to z_2$, then this implies that $z_2 \to \{y_1, y_2\} \to z_1$ and $y_3 \to Z$. This yields $u = z_1$ and $y_1z_1vq_1y_3z_2y_2$ is a Hamiltonian path containing the arc e, a contradiction.

Consequently, it remains to consider the case that $|Y| = 2$, $|Z| = 1$ and $|Q_1| = 2$. As in the case that $Q_1 = \emptyset$, we see that $V_4' - Y = \{y_3\} \subseteq R_1$. The degree-conditions and (4.18) yield $Q_1 = \{q_1, q_2\} \to Y = \{y_1, y_2\} \to (\{v\} \cup (Z = \{z\}))$, $y_3 \to q_1 \to q_2$ and $\{v, z\} \to q_1$. Since $d^-(v) = 3$, we conclude that v has one further inner neighbor except the two vertices of Y. This neighbor has to be u. If $u = q_2$, then it follows that $y_3 \to q_2$ and $z \to \{y_3, q_2\}$ and $y_1 z q_2 v y_3 q_1 y_2$ is a Hamiltonian path containing e, a contradiction. If $u = z$, then $y_1 z v y_3 q_1 q_2 y_2$ is a Hamiltonian path through e, again a contradiction.

Subcase 2.10. Let $(n_i) = 1, 2, 2, 3, 3$. It follows that $d^+(x) = d^-(x) = 5$ for $x \in V_1$, $\{d^+(x), d^-(x)\} = \{4, 5\}$ for $x \in V_2 \cup V_3$ and $d^+(x) = d^-(x) = 4$ for $x \in V_4 \cup V_5$.

Subcase 2.10.1. Assume that one of the vertices u and v is in $V_4 \cup V_5$, say $v \in V_4 \cup V_5$. In this case, $D' = D - v$ has the partition-sequence $1, 2, 2, 2, 3$. Because of $\frac{|V(D')| - 2|V_5'| - |V_4'| + 2}{2} = 2 \geq i_g(D')$, Theorem 4.28 yields that D' is Hamiltonian, a contradiction.

Subcase 2.10.2. Suppose that one of the vertices u and v is in $V_2 \cup V_3$, say $v \in V_2 \cup V_3$. This implies that $D' = D - v$ has the partition-sequence $1, 1, 2, 3, 3$ such that $\{d^+(x), d^-(x)\} = \{4, 5\}$ for $x \in V_1' \cup V_2'$, $d^+(x) = d^-(x) = 4$ or $\{d^+(x), d^-(x)\} = \{3, 5\}$ for $x \in V_3'$ and $\{d^+(x), d^-(x)\} = \{3, 4\}$ for $x \in V_4' \cup V_5'$.

Subcase 2.10.2.1. Let D' have a cycle-factor. According to Corollary 5.9, we have $\kappa(D') \geq 2 = \lfloor \frac{\alpha(D')}{2} \rfloor + 1$. Now Theorem 4.27 yields a contradiction.

Subcase 2.10.2.2. Let D' have no cycle-factor. Since $|V(D')| - 3|V_5'| + 2 = 3 > i_l(D')$, with Theorem 6.4, it follows that $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. To get no contradiction to Theorem 6.4, we also conclude that $|Y| = 3$ and $|Z| = 2$ and thus $|Q_1| + |Q_2| = 5$. If $|Q_1| = 2$ and $|Q_2| = 3$, then there is a vertex $q_1 \in Q_1$ such that either $d^+(q_1) \geq 6$ or $d^+(q_1) \geq 5$ and $|V(q_1)| \geq 3$, in both cases a contradiction. Analogously, we arrive at a contradiction, if $|Q_1| = 3$ and $|Q_2| = 2$ or $|Q_1| = 1$ and $|Q_2| = 4$ or $|Q_1| = 4$ and $|Q_2| = 1$.

This completes the proof of the theorem. $\square$

In the case that $c = 4$, the statement of the last theorem becomes false as the following example demonstrates.

Example 8.16 (Volkmann, Winzen [41]) *Let $V_1 = \{u\}$, $V_2 = \{v\}$, $V_3 = \{q_1, q_2\}$ and $V_4 = \{y_1, y_2, y_3\}$ be the partite sets of a multipartite tournament D such that $V_4 \to q_2 \to (V_1 \cup V_2) \to q_1 \to V_4$ and $u \to v \to \{y_1, y_2\} \to u \to y_3 \to v$ (see Figure 8.1). Then D is an almost regular 4-partite tournament with the property that the arc uv is not contained in a Hamiltonian path of D.*

Remark 8.17 (Volkmann, Winzen [41]) *With similar methods as in the proof of Theorem 8.15 we have shown that the assertion of this theorem also holds, if $c = 4$ and D does not have the partition-sequence $1, 1, 2, 3$, but the improvement may not be worth the additional effort.*

Theorem 8.14, Theorem 8.15, Example 8.16 and Remark 8.17 lead immediately to our main result, which contains the statement that $h(1) = 5$ with h defined as in Problem 8.2.

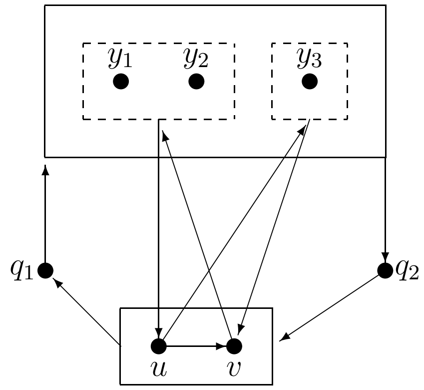

Figure 8.1: An almost regular 4-partite tournament D with the property that the arc uv is not contained in a Hamiltonian path of D

Theorem 8.18 (Volkmann, Winzen [41]) *Let D be an almost regular c-partite tournament. If $c \geq 5$ or if $c = 4$ and D does not have the partition-sequence $1, 1, 2, 3$, then every arc of D is contained in a Hamiltonian path of D.*

8.3 Almost regular 3-partite tournaments

As seen in Corollary 8.7 and in Remark 8.17, there are only finitely many almost regular 4-partite tournaments with the property that not all arcs are contained in a Hamiltonian path. In the class of almost regular 3-partite tournaments, there are infinitely many digraphs with this property as the following two examples demonstrate.

Example 8.19 (Volkmann, Winzen [41]) *Let $V_1 = \{x_1, x_2, \ldots, x_r\}$, $V_2 = \{y_1, y_2, \ldots, y_r\}$ and $V_3 = \{z_1, z_2, \ldots, z_{r+1}\}$ be the partite sets of a 3-partite tournament D' such that $V_1 \to V_2 \to V_3 \to V_1$. If we reverse the cycle $x_1y_1z_1x_1$ of D', then the resulting 3-partite tournament D is almost regular with the property that the arc $y_1 \to x_1$ is not contained in a Hamiltonian path of D.*

Example 8.20 (Volkmann, Winzen [41]) *Let $V_1 = \{x_1, x_2, \ldots, x_r\}$, $V_2 = \{y_1, y_2, \ldots, y_r\}$ and $V_3 = \{z_1, z_2, \ldots, z_{r+2}\}$ be the partite sets of a 3-partite tournament D' such that $V_1 \to V_2 \to \{z_1, z_2, \ldots, z_r\} \to V_1$, $V_1 \to z_{r+1} \to V_2$ and $V_2 \to z_{r+2} \to V_1$. If we reverse the cycle $x_1y_1z_{r+2}x_1$ of D', then the resulting 3-partite tournament D is almost regular with the property that the arc $y_1 \to x_1$ is not contained in a Hamiltonian path of D.*

Nevertheless, we will present a sufficient condition for an arc of an almost regular 3-partite tournament D to be part of a Hamiltonian path of D.

Theorem 8.21 (Volkmann, Winzen [41]) *Let D be an almost regular 3-partite tournament. Then every arc that is contained in a cycle-factor of D belongs to a Hamiltonian path of D.*

Proof. Let $e = uv$ be an arbitrary arc that is contained in a cycle-factor F. Suppose that there is no Hamiltonian path in D including e. If F consists of only one cycle, then D is Hamiltonian, a contradiction. Hence, let F consist of at least two cycles. Now let C be the cycle containing the arc e and $F' = F - V(C)$. According to Theorem 8.10, we may assume that $F' = C_1 \cup C_2 \cup \ldots \cup C_p$ with the properties given in Theorem 8.10. Since D is almost regular, Lemma 4.10 implies that

$$d(C_1, F' - V(C_1)) + d(C_1, C) \leq |C_1| + d(F' - V(C_1), C_1) + d(C, C_1). \tag{8.2}$$

If there is a vertex in C except u with an outer neighbor in C_1, then, according to Theorem 8.10, it is a simple matter to find a Hamiltonian path containing the arc $e = uv$, by using this arc, and first picking up all vertices in C, then all vertices in C_1, then all in C_2, etc.

Thus, it remains the case that $V(C_1) \rightsquigarrow (V(C) - \{u\})$. Since $|V(C)| \geq 3$, we conclude that $d(C_1, C) \geq d(C, C_1)$.

Firstly, we assume that F' consists of only one cycle. As seen above, we have $V(C_1) \rightsquigarrow v^+$. If $w \in V(C_1)$ is a vertex such that $w \rightarrow v^+$, then $w^+ \ldots w^- w v^+ \ldots uv$ is a Hamiltonian path with $e = uv$ as the last arc, a contradiction.

Secondly, let F' consist of $p \geq 2$ cycles. If $V(C_1) \rightsquigarrow V(C_j)$ for some $j \in \{2, 3, \ldots, n\}$, then with Theorem 8.11, we conclude that $d(C_1, F' - (V(C_1) \cup V(C_j))) \geq 2d(F' - (V(C_1) \cup V(C_j)), C_1)$. Because of $d(C_1, C_j) \geq 2|C_1| > |C_1|$, $d(C_j, C_1) = 0$ and $d(C_1, C) \geq d(C, C_1)$, we obtain

$$\begin{aligned}
& d(C_1, F' - V(C_1)) + d(C_1, C) \\
= \ & d(C_1, C_j) + d(C_1, F' - (V(C_1) \cup V(C_j))) + d(C_1, C) \\
> \ & |C_1| + d(C_j, C_1) + d(F' - (V(C_1) \cup V(C_j)), C_1) + d(C, C_1) \\
= \ & |C_1| + d(C, C_1) + d(F' - V(C_1), C_1),
\end{aligned}$$

a contradiction to (8.2). Hence, for all $2 \leq j \leq p$, there exist vertices $v_j \in V(C_j)$ and $v_{1,j} \in V(C_1)$ such that $v_j \rightarrow v_{1,j}$. According to Theorem 8.11, we have $v_{1,j} \rightarrow v_j^+$ and $v_{1,j}^- \rightarrow v_j$ and there is a partite set $V^*(1,j)$ such that $\{v_{1,j}^-, v_j^+\} \subseteq V^*(1,j)$. Theorem 8.10 implies that every vertex of C_{i+1} has an inner neighbor in $V(C_i)$. Using the arc $v_p v_{1,p}$, we conclude that $|V(C)| = 3$, which means $C = uvwu$, and $V^*(1,p) = V(w)$, since otherwise

$$\begin{array}{rl}
C_2 C_3 \ldots C_{p-1} v_p^+ \ldots v_p^- v_p v_{1,p} v_{1,p}^+ \ldots v_{1,p}^- v^+ \ldots uv & \text{or} \\
C_2 C_3 \ldots C_{p-1} v_p^+ \ldots v_p^- v_p v_{1,p} v_{1,p}^+ \ldots v_{1,p}^- (v^+)^+ \ldots uvv^+ &
\end{array} \tag{8.3}$$

is a Hamiltonian path through $e = uv$, if $p \geq 3$, a contradiction. In the case that $p = 2$ omiting the terms $C_2 C_3 \ldots C_{p-1}$ in (8.3), we analogously arrive at a contradiction.

Altogether we observe that for all vertices $v_1 \in V(C_1)$ having an inner neighbor in $V(C_j)$ it has to be $v_1^- \rightsquigarrow V(C_j)$, since otherwise $(v_1^-)^- \in V^*(1,j)$, a contradiction to the existence of the arc $(v_1^-)^- \rightarrow v_1^-$. According to Theorem 8.10, every vertex of $V(C_1)$ has an outer neighbor in $V(C_2)$. By the results above, we conclude that for the predecessor v_1^- of each vertex $v_1 \in V(C_1)$ with $|N^-(v_1) \cap V(C_j)| = i$ $(2 \leq j \leq p,\ 1 \leq i)$ we have the estimation

$$|N^+(v_1^-) \cap V(C_j)| \geq \begin{cases} 2 & \text{, if} \quad i = 1 \\ i & \text{, if} \quad i \geq 2 \end{cases}.$$

Furthermore, we observe that $|N^+(v_1) \cap V(C_2)| \geq 2$, if $|N^-(v_1) \cap V(C_2)| \geq 2$. Altogether, we observe the following: Every vertex of $V(C_1)$ has an outer neighbor in $V(C_2)$. Since $v_1^- \rightsquigarrow V(C_2)$, if there is an arc $v_2 \to v_1$ with $v_1 \in V(C_1)$ and $v_2 \in V(C_2)$, the observations above imply that for each arc leading from $V(C_2)$ to $V(C_1)$, there is an additional arc leading from $V(C_1)$ to $V(C_2)$. This means $d(C_1, C_2) \geq |C_1| + d(C_2, C_1)$. Using this and Theorem 8.10, we arrive at a contradiction to (8.2) by

$$\begin{aligned} & d(C_1, F' - V(C_1)) + d(C_1, C) \\ = \ & d(C_1, F' - (V(C_1) \cup V(C_2))) + d(C_1, C_2) + d(C_1, C) \\ \geq \ & 2d(F' - (V(C_1) \cup V(C_2)), C_1) + |C_1| + d(C_2, C_1) + d(C, C_1) \\ > \ & d(F' - V(C_1), C_1) + |C_1| + d(C, C_1), \end{aligned}$$

if $p \geq 3$.

Hence, let $p = 2$. In this case, it is easy to see that $(V(C) - \{v\}) \rightsquigarrow V(C_2)$, since otherwise there can be found a Hamiltonian path containing the arc e, a contradiction. Especially, it has to be $u \rightsquigarrow C_2$. If additionally there is at most one vertex $\hat{v}_1 \in V(C_1)$ such that $\hat{v}_1 \to u$, then because of $|V(C)| = 3$, it follows that $d^-(u) \leq 2$ and $d^+(u) \geq 4$, a contradiction to $i_g(D) \leq 1$. Hence, there exist vertices $\hat{v}_1, \tilde{v}_1 \in V(C_1)$ such that $\{\hat{v}_1, \tilde{v}_1\} \to u$, and thus we have $d(C_1, C) \geq d(C, C_1) + 4$. Summarizing our results, we arrive at

$$d(C_1, F' - V(C_1)) + d(C_1, C) \geq |C_1| + d(F' - V(C_1), C_1) + d(C, C_1) + 4,$$

a contradiction to (8.2). This completes the proof of the theorem. □

Bibliography

[1] B. Alspach, Cycles of each length in regular tournaments, *Canad. Math. Bull.* **10** (1967), 283-286.

[2] J. Bang-Jensen, G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer, London, 2000.

[3] J. Bang-Jensen, G. Gutin, J. Huang, A sufficient condition for a semicomplete multipartite digraph to be Hamiltonian, *Discrete Math.* **161** (1996), 1-12.

[4] J. Bang-Jensen, G. Gutin, A. Yeo, On k-strong and k-cyclic digraphs, *Discrete Math.* **162** (1996), 1-11.

[5] L.W. Beineke, C. Little, Cycles in bipartite tournaments, *J. Combinat. Theory Ser. B* **32** (1982), 140-145.

[6] J.A. Bondy, Diconnected orientation and a conjecture of Las Vergnas, *J. London Math. Soc.* **14** (1976), 277-282.

[7] P. Camion, Chemins et circuits hamiltoniens des graphs complets, *C. R. Acad. Sci. Paris* **249** (1959), 2151-2152.

[8] W.D. Goddard, O.R. Oellermann, On the cycle structure of multipartite tournaments, in: Y. Alavi, G. Chartrand, O.R. Oellermann, A.J. Schenk (Eds.), *Graph Theory Combinat. Appl. 1*, Wiley-Interscience, New York (1991), 525-533.

[9] Y. Guo, Semicomplete Multipartite Digraphs: A Generalization of Tournaments, *Habilitation thesis*, RWTH Aachen, Germany, (1998).

[10] Y. Guo, J.H. Kwak, The cycle structure of regular multipartite tournaments, *Discrete Appl. Math.* **120** (2002), 107-114.

[11] Y. Guo, A. Pinkernell, L. Volkmann, On cycles through a given vertex in multipartite tournaments, *Discrete Math.* **164** (1997), 165-170.

[12] Y. Guo, M. Tewes, L. Volkmann, A. Yeo, Sufficient conditions for semicomplete multipartite digraphs to be Hamiltonian, *Discrete Math.* **212** (2000), 91-100.

[13] Y. Guo, L. Volkmann, Cycles in multipartite tournaments, *J. Combinat. Theory Ser. B* **62** (1994), 363-366.

[14] G. Gutin, Characterization of complete n-partite digraphs that have a Hamiltonian path, *Kibernetica* **1** (1998), 107-108 (in Russian).

[15] G. Gutin, Cycles and paths in semicomplete multipartite digraphs, theorems and algorithms: a survey, *J. Graph Theory* **19** (1995), 481-505.

[16] G. Gutin, A. Yeo, Note on the path covering number of a semicomplete multipartite digraph, *J. Comb. Math. Comb. Comput.* **32** (2000), 231-237.

[17] F. Harary, L. Moser, The theory of round robin tournaments, *Amer. Math. Monthly* **73** (1966), 231-246.

[18] B. Jackson, Long paths and cycles in oriented graphs, *J. Graph Theory* **5** (1981), 145-157.

[19] O.S. Jakobsen, Cycles and paths in tournaments, *Ph. D. thesis*, Aarhus University (1972).

[20] J.W. Moon, On subtournaments of a tournament, *Canad. Math. Bull.* **9** (1966), 297-301.

[21] L. Rédei, Ein kombinatorischer Satz, *Acta Litt. Sci. Szeged* **7** (1934), 39-43.

[22] I. Stella, L. Volkmann, S. Winzen, How close to regular must a multipartite tournament be to secure a Hamiltonian path? submitted.

[23] M. Tewes, In-tournaments and semicomplete multipartite digraphs, *Ph. D. thesis*, RWTH Aachen, Germany, (1999).

[24] M. Tewes, L. Volkmann, A. Yeo, Almost all almost regular c-partite tournaments with $c \geq 5$ are vertex pancyclic, *Discrete Math.* **242** (2002), 201-228.

[25] C. Thomassen, Hamiltonian-connected tournaments, *J. Combin. Theory* **B 28** (1980), 142-163.

[26] C. Thomassen, Whitney's 2-switching theorem, cycle spaces, and arc mappings of directed graphs, *J. Comb. Theory, ser. B* **46** (1989), 257-291.

[27] P. Turán, An extremal problem in graph theory, *Mat.-Fiz. Lapok* **48** (1941), 436-452 (in Hungarian).

[28] L. Volkmann, *Fundamente der Graphentheorie*, Springer-Verlag, Vienna, New York (1996).

[29] L. Volkmann, Strong subtournaments of multipartite tournaments, *Australas. J. Combin.* **20** (1999), 189-196.

[30] L. Volkmann, Cycles through a given arc in certain almost regular multipartite tournaments, *Australas. J. Combin.* **26** (2002), 121-133.

[31] L. Volkmann, Cycles in multipartite tournaments: results and problems, *Discrete Math.* **245** (2002), 19-53.

[32] L. Volkmann, Cycles of length four through a given arc in almost regular multipartite tournaments, *Ars Combin.* **68** (2003), 181-192.

[33] L. Volkmann, Hamiltonian paths containing a given arc, in almost regular bipartite tournaments, *Discrete Math.* **285** (2004), 359-364.

[34] L. Volkmann, Complementary cycles in regular multipartite tournaments, where one cycle has length four, *Kyungpook Math. J.* **44** (2004), 219-247.

[35] L. Volkmann, All regular multipartite tournaments that are cycle complementary, *Discrete Math.* **281** (2004), 255-266.

[36] L. Volkmann, On the connectivity of regular multipartite tournaments, unpublished manuscript (2002).

[37] L. Volkmann, S. Winzen, Cycles through a given arc in almost regular multipartite tournaments, *Australas. J. Combin.* **27** (2003), 223-245.

[38] L. Volkmann, S. Winzen, Cycles through a given arc and certain partite sets in almost regular multipartite tournaments, *Discrete Math.* **283** (2004), 217-229.

[39] L. Volkmann, S. Winzen, Almost regular c-partite tournaments contain a strong subtournament of order c when $c \geq 5$, submitted.

[40] L. Volkmann, S. Winzen, Close to regular multipartite tournaments containing a Hamiltonian path, *J. Comb. Math. Comb. Comp.* **49** (2004), 195-210.

[41] L. Volkmann, S. Winzen, Almost regular multipartite tournaments containing a Hamiltonian path through a given arc, *Discrete Math.* **285** (2004), 267-278.

[42] L. Volkmann, S. Winzen, Almost regular c-partite tournaments with $c \geq 8$ contain an n-cycle through a given arc for $4 \leq n \leq c$, *Australas. J. Combin.* **31** (2005), 61-84, to appear.

[43] L. Volkmann, S. Winzen, Cycles with a given number of vertices from each partite set in regular multipartite tournaments, submitted.

[44] L. Volkmann, S. Winzen, On the connectivity of close to regular multipartite tournaments, submitted.

[45] L. Volkmann, S. Winzen, Paths with a given number of vertices from each partite set in regular multipartite tournaments, submitted.

[46] L. Volkmann, A. Yeo, Hamiltonian paths, containing a given path or collection of arcs, in close to regular multipartite tournaments, *Discrete Math.* **281** (2004), 267-276.

[47] S. Winzen, Strong subtournaments of close to regular multipartite tournaments, *Australas. J. Combin.* **29** (2004), 49-57.

[48] A. Yeo, One-diregular subgraphs in semicomplete multipartite digraphs, *J. Graph Theory* **24** (1997), 175-185.

[49] A. Yeo, Semicomplete Multipartite Digraphs, *Ph. D. thesis*, Odense University, (1998).

[50] A. Yeo, Diregular c-partite tournaments are vertex-pancyclic when $c \geq 5$, *J. Graph Theory* **32** (1999), 137-152.

[51] A. Yeo, How close to regular must a semicomplete multipartite digraph be to secure Hamiltonicity? *Graphs Combin.* **15** (1999), 481-493.

[52] C.-Q. Zhang, Hamilton paths in multipartite oriented graphs, *Ann. Discrete Math.* **41** (1989), 499-514.

[53] K.-M. Zhang, Vertex even-pancyclicity in bipartite tournaments, *Nanjing Daxue Xuebao Shuxue Bannian Kan* **1**, (1984), 85-88.

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