On Hermitian theta series and modular forms

Michael Hentschel ⋆ On Hermitian theta series and modular forms

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A few words on the leitmotif of this thesis:
The main examples for Hermitian modular forms come from the classical case, from Hermitian Eisenstein series due to Hel Braun [Br3] or from liftings, see Gritsenko, Ikeda, Krieg, and lots of others. Cohen and Resnikoff [CoRe] introduced the method of constructing modular forms via theta-series to the theory of Hermitian modular forms and gave a construction for lattices which yield Hermitian modular forms for an arbitrary imaginary quadratic field. In [DeKr] then one finds an elementary method for the construction of those lattices. One is especially interested in the number of distinct isometry classes (in the genus) of lattices which then yield different theta-series, a problem which was already sketched in [CoRe], page 336, ”...its [the genus] class number remains unknown.”. So far, just the situation with respect to the Gaussian number field, see [KiMu] or [Sc1], was known. This thesis generalizes these results to arbitrary imaginary quadratic fields of class number one. In special, well-arranged cases we investigate the isometry classes of the lattices of interest. Together with Grabriele Nebe we have developed a mass formula which can be applied to imaginary quadratic fields of class number 1 easily and can be adopted to other class numbers. Then we compute the filtration of cusp forms analogous to [HeKr] where we considered the filtration of cusp forms arising from lattices over the Gaussian number field, but this is limited to very easy cases as bounds for dimension estimations get out of reach very soon. We give some information on the situation with respect to higher class numbers as 1. The natural continuation of this thesis then is to investigate the situation with respect to higher class numbers. But things seem to get very ugly very fast. On the other hand one can step deeper into the theory of modular forms itself and try to make some advance with respect to the filtration.

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Contents

1 The basic theory

1.1 Lattices

Let $d \in \mathbb{N}$ be a squarefree integer. Then $K := \mathbb{Q}(\sqrt{-d})$ is an imaginary quadratic number field. Obviously K is an algebraic $\mathbb{Q}$-field extension of degree 2. In case $d \equiv 1,2\ (4)$ we have $\mathrm{disc}(K) = -4d$ and in case $d \equiv 3\ (4)$ we have $\mathrm{disc}(K) = -d$. We consider the ring of integers $\mathcal{O}_K$ of the $\mathbb{Q}$-field extension K, we have $\mathcal{O}_K = \langle 1, \omega \rangle_{\mathbb{Z}}$, where

$$\omega = \begin{cases} i\sqrt{d} & d \equiv 1,2\ (4) \\ \frac{1+i\sqrt{d}}{2} & d \equiv 3\ (4). \end{cases}$$

$\mathcal{O}_K$ is a Dedekind domain.

Definition 1.1 A lattice of rank n over $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}$ is a subset $\Gamma \subset \mathbb{Q}(\sqrt{-d})^n$ which has the structure of an $\mathcal{O}_K$-submodule with respect to the vector space and fulfills $\mathbb{Q}(\sqrt{-d}) \cdot \Gamma = \mathbb{Q}(\sqrt{-d})^n$. In full generality one replaces $\mathbb{Q}(\sqrt{-d})^n$ by an abitrary vector space and $\mathcal{O}_K$ by a Dedekind domain within the underlying field. When the rank and the underlying field is not specified we just speak of a lattice.

Remark 1.2 a) Let K an algebraic number field, which is a $\mathbb{Q}$-extension of finite index, then the ideal class group cl_K is defined as the quotient J_K/P_K, the group of fractional ideals modulo the group of fractional principal ideals. The class number h_K is defined as the order $\sharp cl_K$. For

$$d \in \{1,2,3,7,11,19,43,67,163\}$$

the imaginary quadratic number field $K = \mathbb{Q}(\sqrt{-d})$ fulfills $h_K = 1$. In case of $h_K = 1$ a lattice Γ of rank n over $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}$ is a necessarily free $\mathcal{O}_K$-module, see [OMe], and so there exists a basis $(b_1, \ldots, b_n)$ of $\mathbb{Q}(\sqrt{-d})^n$ such that $\Gamma = \langle b_1, \ldots, b_n \rangle_{\mathcal{O}_K}$. Γ is a free $\mathcal{O}_K$-module of rank n in the vector space $\mathbb{Q}(\sqrt{-d})^n$.

b) Let Γ be a free lattice within an arbitrary vector space V. If V is equipped with a regular symmetric form $h : V \times V \to K$, where symmetry may be definied as convenient and regularity means that the form is linear in the first variable and K the underlying field, then one can introduce the Gram matrix of a lattice with a bilinear form (Γ, h) as

$$(\mathrm{Gram}(\Gamma, h))_{i,j} = h(b_i, b_j).$$

In the special case of imaginary quadratic fields, the vector space $\mathbb{Q}(\sqrt{-d})^n$, any squarefree d, is canonically equipped with the standard Hermitian form

$$\langle \cdot, \cdot \rangle : \mathbb{Q}(\sqrt{-d})^n \times \mathbb{Q}(\sqrt{-d})^n \to \mathbb{C},$$

which is of course linear in the first variable. So for $h_K = 1$ we can always consider Gram matrices of lattices. When we do not specify, we always will use the standard Hermitian form.

c) Let Λ an $\mathcal{O}_K$ lattice. In the case of arbitrary class number there are $y_i \in K^n$ and $\mathfrak{a}_i \in J_K$ with $\Lambda = \sum \mathfrak{a}_i y_i$, see [OMe].

Definition 1.3 a) Let (Γ, h) and (Λ, h') be two lattices of rank n. An $\mathcal{O}_K$-linear bijection

$$\varphi : \Gamma \to \Lambda, \quad \text{with} \quad h(x,y) = h'(\varphi(x), \varphi(y)) \text{ for all } x,y \in \Gamma,$$

is called an *isometry* of the lattices.

b) An isometry $\varphi : \Gamma \to \Gamma$ is called an *automorphism* of Γ. If Γ is a free lattice we have a lattice basis B, which also is a basis for the vector space, and we have $[\varphi]_B \subset \mathrm{Mat}(n \times n; \mathcal{O}_K)$. This will be called the *coordinate action*.

c) We set $\mathrm{Aut}(\Gamma) := \{\varphi : \Gamma \to \Gamma;\ \varphi \text{ is an isometry of } \Gamma\}$, the *automorphism group* of a lattice Γ.

When the underlying field is an imaginary quadratic number field, isometry and change of basis can be described via group actions. This is stated in a remark.

Remark 1.4 a) Via the isometry relation the set of all lattices in a given vector space is divided into classes. The *class* of a lattice Λ is denoted by $\mathrm{cls}(\Lambda)$. From the viewpoint of the coordinate action isometry is provided via the action of the unitary group $\mathrm{U}(n;K)$.

b) Let $\Gamma = \langle b_1, \ldots, b_n \rangle_{\mathcal{O}_K}$. A change of basis from the viewpoint of the coordinate action is provided via the action of $\mathrm{Gl}(n;\mathcal{O}_K)$. Thus a Gram matrix of a lattice is unique modulo the action of $\mathrm{Gl}(n;\mathcal{O}_K)$. On the other hand a lattice is uniquely determined by its $\mathrm{Gl}(n;\mathcal{O}_K)$-orbit in the set of symmetric matrices, where the concept of symmetry may be defined in a convenient way.

c) Obviously $\mathrm{Aut}(\Gamma) \subset U(n;\mathbb{C})$ is a group. It is well known, that $\mathrm{Aut}(\Gamma)$ is a finite group.

Definition 1.5 a) Let S the set of all non Archimedian prime spots of K and $K_\mathfrak{p}$ the completion of K at the spot $\mathfrak{p} \in S$. Furthermore let $I_\mathfrak{p}$ the group of fractional ideals of $K_\mathfrak{p}$ at $\mathfrak{p}$. We consider the canonical surjective homomorphism

$$I(S) \to I_\mathfrak{p}, \ \prod_{\mathfrak{q} \in S} \mathfrak{q}^{\nu_\mathfrak{q}} \mapsto \mathfrak{p}^{\nu_\mathfrak{p}}.$$

Then let $\mathfrak{a}_\mathfrak{p}$ the image under this mapping of $\mathfrak{a} \in J_K$. For a lattice $\Lambda = \sum \mathfrak{a}_i y_i$ we now define $\Lambda_\mathfrak{p} = \sum \mathfrak{a}_{\mathfrak{p},i} y_i$. We say that two $\mathcal{O}_K$ lattices Λ and Γ within K^n belong to the same *genus* if $\Lambda_\mathfrak{p}$ is isometric to $\Gamma_\mathfrak{p}$ for alle $\mathfrak{p} \in S$. The genus of Λ is denoted by $\mathrm{gen}(\Lambda)$.

b) We define the *mass of a genus* of lattices $\mathfrak{G}$ as

$$\mathrm{mass}(\mathfrak{G}) := \sum_{\mathrm{cls}(\Lambda) \in \mathfrak{G}} \frac{1}{\sharp \mathrm{Aut}(\Lambda)},$$

where the sum ranges over representatives of different isometry classes contained in the genus.

Remark 1.6 a) Let Γ and Λ be as chosen in the definition. Obviously $\mathrm{gen}(\Lambda) = \mathrm{gen}(\Gamma) \Leftrightarrow \mathrm{cls}(\Lambda_{\mathfrak{p}}) = \mathrm{cls}(\Gamma_{\mathfrak{p}})\ \forall \mathfrak{p} \in S$.

b) Considering free lattices Λ, Γ, then the genus condition reduces to the existence of $B_l \in \mathrm{Mat}(n \times n; \mathcal{O}_K)$ for all $l \in \mathbb{N}$ such that the matrix congruences

$$\overline{B_l}^{tr}\mathrm{Gram}(\Lambda)B_l \equiv \mathrm{Gram}(\Gamma)\ (l)$$

hold.

Definition 1.7 Let $\Lambda = \sum \mathfrak{a}_i y_i$ an $\mathcal{O}_K$ lattice, see Remark 1.2 c), together with a regular symmetric form h. We define the *scale* by

$$\mathrm{sc}(\Lambda) := h(\Lambda, \Lambda)$$

and the *volume* by

$$\mathrm{vol}(\Lambda) = \prod \mathfrak{a}_i^2 \cdot \det(h(y_i, y_j)_{i,j}).$$

Furthermore we call a lattice $\mathfrak{a}$-*modular* if $\mathrm{sc}(\Lambda) = \mathfrak{a}$ and $\mathrm{vol}(\Lambda) = \mathfrak{a}^r$. The *dual* of a lattice (of full rank) Λ with respect to a regular symmetric form h is defined as

$$\Lambda^\sharp = \{x \in V;\ h(x, \Lambda) \subset \mathcal{O}_K\}.$$

A lattice Λ is called *integral* if $\Lambda \subset \Lambda^\sharp$. If we have $\Lambda = \Lambda^\sharp$ then a lattice is called *unimodular* (with respect to the fixed regular symmetric form). $\Lambda^\sharp/\Lambda$ is called the *discriminant group* of a lattice.

Remark 1.8 a) If Λ is a free lattice the integrality can be read off from a Gram matrix. Furthermore Λ then is unimodular if the Gram matrix is unimodular.

b) In the case of free lattices the volume equals the discriminant of a lattice basis which is equal to the square root of the determinant of a Gram matrix.

Example 1.9 a) The *principal genus* of lattices of a vector space of dimension n is the genus that contains the lattice characterized by the Gram matrix E_n. The lattices contained in this genus are obviously unimodular.

b) An important genus for the theory of modular forms is the genus of even and unimodular $\mathbb{Z}$-lattices. Such lattices exist only if the dimension of the vector space $\mathbb{R}^n$ is divisible by 8. In dimension 8 the genus just consists of the E_8 class, in dimension 16 we have $E_8 \oplus E_8$, which is the orthogonal sum of two E_8 lattices, and $D_{16}^+ = \langle D_{16}, (\frac{1}{2})^{16} \rangle_{\mathbb{Z}}$, the lattice which is generated by the root lattice D_{16} together with the vector which components are all equal to $1/2$. In dimension 24 we have the 23 Niemeier lattices and the Leech lattice, see [CoSl]. We want to emphasize that all $\mathbb{Z}$-lattices are free lattices.

1.2 Hermitian modular forms

Our aim is to get structural results on lattices to apply them to the theory of modular forms. The connection is a theta-series construction. The classical (real) case starts with an even and integral unimodular lattice (in dimension $8\cdot k,\ k\in\mathbb{N}$) as an input for the theta-series. We present some facts of the theory of Hermitian modular forms.

Definition 1.10 The *Hermitian half-space* of degree $p\in\mathbb{N}$ is given by

$$\mathcal{H}_p(\mathbb{C}) := \left\{ Z\in \mathrm{Mat}(p\times p;\mathbb{C});\ \frac{1}{2i}(Z-\overline{Z}^{tr})>0 \right\}.$$

The *unitary symplectic group* with respect to an imaginary quadratic number field K is defined as

$$\mathrm{Sp}(p;K) := \left\{ M\in \mathrm{Mat}(2p\times 2p;K);\ MJ\overline{M}^{tr}=J \right\},$$

where $J=\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$. The *Hermitian modular group* of degree $p\in\mathbb{N}$ over the imaginary quadratic number field K is defined as the intersection

$$\Gamma_p(\mathcal{O}_K) := \mathrm{Sp}(p;K)\cap \mathrm{Mat}(2p\times 2p;\mathcal{O}_K).$$

Remark 1.11 The Hermitian modular group of degree p acts on the Hermitian half space of degree p via the usual fractional linear transformation

$$Z\mapsto M\langle Z\rangle := (AZ+B)(CZ+D)^{-1},$$

where A,B,C,D are the $p\times p$ sub-blocks of M.

From [Kl] we get explicit generators for the Hermitian modular group.

Theorem 1.12 *The modular group Γ_p of degree p with respect to an imaginary quadratic field K is generated by*

$$\left\langle J, T=\begin{pmatrix} E & S \\ 0 & E \end{pmatrix},\ S=\overline{S}^{\,tr}\in \mathrm{Sym}(p;\mathcal{O}_K),\ R=\begin{pmatrix} (U^{-1})^{tr} & 0 \\ 0 & \overline{U} \end{pmatrix},\ U\in \mathrm{Gl}(n;\mathcal{O}_K) \right\rangle,$$

where o_K ist the ring of integers of K.

In view of the theorem we turn to the definition of Hermitian modular forms.

Definition 1.13 A *Hermitian modular form* of weight $k\in\mathbb{Z}$ and degree p is a holomorphic function $f:\mathcal{H}_p\to\mathbb{C}$ satisfying

(i) $f(Z+S)=f(Z)$ for all Hermitian matrices S in $\mathrm{Mat}(n\times n;\mathcal{O}_K)$,

(ii) $f(Z[U])=f(Z)$ for all $U\in \mathrm{Gl}(n;\mathcal{O}_K)$,

(iii) $f(-Z^{-1}) = (\det Z)^k f(Z)$,

and for $p = 1$ the additional condition that

(iv) f is bounded in the domain $\{z \in \mathbb{C};\ \mathrm{Im}(z) \geq \beta\}$, $\beta > 0$.

The vector space of Hermitian modular forms of weight k and degree p is denoted by $\mathcal{M}_k(\Gamma_p)$. The subspace $\mathcal{S}_k(\Gamma_p)$ is characterized by the condition

$$f|\mathfrak{M}|\Phi = 0,$$

for all $\mathfrak{M} \in \mathrm{Sp}(p;K)$, where Φ is the Siegel Φ-operator, which is defined analogously to the Siegel case, the case of Siegel modular forms defined on the Siegel half space, see [Si].

Remark 1.14 a) Hermitian modular forms exhibit a Fourier expansion

$$f(Z) = \sum_{T \in \Lambda(p;\mathcal{O}_K)} \alpha_f(T) \exp(2\pi i \cdot \mathrm{trace}(TZ)),$$

where

$$\Lambda(p;\mathcal{O}_K) = \{T \in \mathrm{Mat}(p;K);\ T = \overline{T^{tr}}, (T)_{j,j} \in \mathbb{Z},\ (T)_{j,l} \in \mathcal{O}_K^\star,\ j \neq l\},$$

the dual lattice of the lattice of integral Hermitian matrices with respect to the trace form, see [Br1] or [Kr2].

b) Hermitian modular forms of negative weight vanish.

c) Analogous to the Siegel case, the vector spaces $\mathcal{M}_k(\Gamma_p)$ are finite dimensional.

d) For $n = 1$ and $K \notin \{\mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})\}$ the groups $\Gamma_1(\mathcal{O}_K)$ are equal to the classical modular group $\mathrm{Sl}(2;\mathbb{Z})$. For $K \in \{\mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})\}$ we have $\Gamma_1(\mathcal{O}_K) = \mathfrak{U} \cdot \mathrm{Sl}(2;\mathbb{Z})$, where $\mathfrak{U}$ is the group of units of $\mathcal{O}_K$. Thus for $n = 1$ the groups of modular transformations with respect to K, the modular group modulo the constants, are equal to the classical modular group for all imaginary quadratic number fields.

e) We want to emphasize that the invariance condition with respect to the slash operator

$$f|_k M(Z) := \det(CZ + D)^{-k} \cdot f(M\langle Z\rangle) = f(Z)$$

is equivalent to (i)-(iii) if we have $d \notin \{1,3\}$. Otherwise we have a problem with (ii) which comes from the existence of the additional roots of unity in the ring of integers of $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-3})$. In comparison to (i)-(iii) one looses many forms when using the latter version. Within the literature one finds both definitions.

f) For $h_K = 1$ we have a comfortable reduction theory, analogous to the real Minkowski reduction, and therefore the cusp form condition reduces to $f|\Phi = 0$ which is equal to the vanishing of all Fourier-coefficients with respect to the Hermitian matrices which are not positive definite, see [Br2].

Example 1.15 Let $K := \mathbb{Q}(\sqrt{-d})$, $p \in \mathbb{N}$, $k > 2p$ and

$$\widehat{\Gamma_p} := \begin{cases} \Gamma_p, & d \notin \{1,3\}, \\ \Gamma_p \cap \mathrm{Sl}(2p, \mathcal{O}_K), & d \in \{1,3\}. \end{cases}$$

Furthermore let

$$\widehat{\Gamma_{p,0}} := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \widehat{\Gamma_p};\ C = 0 \right\}$$

the subgroup of all those matrices which C-block is equal to 0. Then the *Hermitian Eisenstein series* is defined to be

$$E_n^k(Z) := \sum_{M:\widehat{\Gamma_{p,0}}\backslash\widehat{\Gamma_p}} \det(CZ+D)^{-k} = \sum_{M:\widehat{\Gamma_{p,0}}\backslash\widehat{\Gamma_p}} 1|_k M(Z),$$

where we have the usual decomposition of M into blocks. Then for $k > 2p$ the Hermitian Eisenstein series is a Hermitian modular form of weight k and degree p, see [Br3] and [Kr2]. In the latter reference you will also find further characterizations of the Hermitian Eisensteins series, including Fourier-coefficients, which are explicitly given for $p = 2$.

Definition 1.16 Let $f \in \mathcal{M}_k(\Gamma_p)$ be a Hermitian modular form with Fourier expansion

$$f(Z) = \sum_{T \in \Lambda(p;\mathcal{O}_K)} \alpha_f(T) \exp(2\pi i \cdot \mathrm{trace}(TZ)).$$

Such an f is called *singular* if $\alpha_f(T) = 0$ for all $T > 0$.

From [Va] we take the following lemma.

Lemma 1.17 *Let K be an imaginary quadratic number field of class number one.*

a) Let $f \in \mathcal{M}_k(\Gamma_p)$ a non-vanishing singular Hermitian modular form, then $k \equiv 0\ (4)$ and $k < p$. Furthermore $f \in \mathcal{M}_k(\Gamma_p)_\Theta$.

b) Every $f \in \mathcal{M}_k(\Gamma_p)$ with $p < k$ is singular.

c) All Hermitian modular forms of weight k and degree $k < p$ vanish identically if $k \not\equiv 0\ (4)$. For $k \equiv 0\ (4)$ we have the identity

$$\mathcal{M}_k(\Gamma_p) = \mathcal{M}_k(\Gamma_p)_\Theta.$$

Remark 1.18 a) Using the preceding lemma one can prove assertions concerning mapping characteristics of the Φ-operator.

b) Analogous results for Siegel modular forms and low degrees were obtained in [KoMa]. Most probably these results can be generalized to Hermitian modular forms.

1.3 Lattices and Hermitian modular forms

Our aim, following [CoRe], is to construct Hermitian modular forms using theta-series. At first we will restrict ourselves to free lattices, later in Section 5 we will discuss the case of non-free lattices. So let $\Lambda \subset K^n$ be a free $\mathcal{O}_K$-lattice of dimension n with Gram matrix H. Then the theta-series

$$\Theta^{(p)}_{\mathcal{O}_K}(Z,H) : \mathcal{H}_p(\mathbb{C}) \to \mathbb{C},\ Z \mapsto \mathrm{vol}(\mathcal{O}_K)^{p/2} \sum_{G \in \mathrm{Mat}(n\times p;\mathcal{O}_K)} \exp(\pi i\, \mathrm{trace}(ZH[G])),$$

satisfies

$$\Theta_{\mathcal{O}_K^\sharp}(-Z^{-1},H^{-1}) = \left(\det \frac{Z}{i}\right)^n (\det H)^p \Theta_{\mathcal{O}_K}(Z,H),$$

the theta-transformation formula, where

$$\mathrm{Mat}(n\times p;\mathcal{O}_K)^\sharp = \{M \in \mathrm{Mat}(n\times p;\mathbb{C});\ \mathrm{trace}(M\overline{N}^{tr}) \in \mathbb{Z},\ \text{for all } N \in \mathrm{Mat}(n\times p;\mathcal{O}_K)\}.$$

Using this theta-transformation formula we get explicit conditions on the $\mathcal{O}_K$-lattice Λ, so that $\Theta_{\mathcal{O}_K}(Z,H)$ fulfills the functional equation of Hermitian modular forms with respect to (partial) involutions in the modular group. Using

$$\mathrm{Mat}(n\times p;\mathcal{O}_K)^\sharp = \left(\frac{2}{\sqrt{-d}}\right)\mathrm{Mat}(n\times p;\mathcal{O}_K)$$

we get the condition, see [CoRe] p. 332,

$$\det H = 2^n d^{-n/2}.$$

Together with the other conditions from the functional equation one is interested in lattices $\Gamma \subset \mathbb{Q}(\sqrt{-d})^n$ together with a Hermitian form $\langle \cdot,\cdot \rangle :\ \mathbb{Q}(\sqrt{-d})^n \times \mathbb{Q}(\sqrt{-d})^n \to \mathbb{Q}$ fulfilling

(i) $\langle x,x\rangle \in \mathbb{Q}_+,$ for all $x \in \Gamma$ (positive definite),

(ii) $\langle x,x\rangle \in 2\mathbb{Z},$ for all $x \in \Gamma$ (integral even),

(iii) $\det(\mathrm{Gram}(\Gamma,\langle\cdot,\cdot\rangle)) = 2^n d^{-n/2}$ (determinant condition).

One can show that these conditions urge the theta-series with respect to such lattices to be a Hermitian modular form with respect to the modular group coming from the underlying field, see [CoRe]. We add a $\mathbb{C}$ in the notation of theta-series to stress the difference between the Hermitian and the classical Siegel case.

Theorem 1.19 *Let Γ be a free lattice of rank n over $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}$ fulfilling (i), (ii) and (iii). Then using the abbreviation $H := \mathrm{Gram}(\Gamma)$ we have*

$$\Theta^{(p)}(Z,H;\mathbb{C}) := \sum_{G \in \mathrm{Mat}(n\times p;\mathcal{O}_K)} \exp(\pi i \cdot \mathrm{trace}(Z \cdot H[G])$$

is a Hermitian modular form of weight n. We have $\Theta^{(n)}(\cdot,S;\mathbb{C})|\phi = \Theta^{(n-1)}\,(\cdot,S;\mathbb{C})$ with $\Theta^{(1)}(\cdot,S;\mathbb{C})|\phi = 1$.

The theta-subspace, the subspace of $\mathcal{M}_k(\Gamma_p)$ which is spanned by theta-series, is denoted by $\mathcal{M}_k(\Gamma_p)_\Theta$.

Remark 1.20 a) Let $\Gamma = \langle b_1, \dots, b_n \rangle_{\mathcal{O}_K}$ be a free lattice of rank n over $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}$ fullfilling the conditions (i)-(iii) from above. Let again $\mathcal{O}_K = \langle 1, \omega \rangle_{\mathbb{Z}}$ then consider the $\mathbb{Z}$-module of rank $2n$

$$\Gamma_{\mathbb{Z}} = \langle b_1, \omega b_1 \dots, b_n, \omega b_n \rangle_{\mathbb{Z}}.$$

Easy calculations show that $\Gamma_{\mathbb{Z}}$ equipped with the (bilinear) form $\mathrm{Re}(\langle \cdot, \cdot \rangle)$ is an even and unimodular $\mathbb{Z}$-lattice. It is immediately clear that this is an equivalent characterization of lattices $\Gamma \subset (\mathbb{Q}(\sqrt{-d}))^n$ fulfilling (i)-(iii). This characterization allows a generalization to non-free lattices.

b) When Γ is mapped to $\Gamma_{\mathbb{Z}}$ we loose the information, which vectors are $\mathbb{C}$ multiples, the complex structure. The complex structure can be recovered from $\Gamma_{\mathbb{Z}}$ when knowing the action of the complex $\mathcal{O}_K$ generator of the ring of integers on $\Gamma_{\mathbb{Z}}$. On the other hand the complex structure of Γ can be recovered from $\Gamma_{\mathbb{Z}}$ by explicit knowledge of the mapping $\Gamma \to \Gamma_{\mathbb{Z}}$.

c) Obviously there is a canonical embedding $\mathrm{Aut}(\Gamma) \hookrightarrow \mathrm{Aut}(\Gamma_{\mathbb{Z}}) \subset \mathrm{O}(2n; \mathbb{Z})$. The embedded group will be denoted by $\mathrm{Aut}(\Gamma)_{\mathbb{Z}}$.

We take the preceding theorem and remark as an occasion for the next definition.

Definition 1.21 An $\mathcal{O}_K$ lattice $\Lambda \subset K^n$ of rank n which has the properties as in a) from the preceding remark is called ϑ-*lattice*.

Remark 1.22 As even and unimodular lattices are classified up to dimension 24, we can use the results on real unimodular lattices to obtain information on ϑ-lattices. For example one directly gets the information that ϑ-lattices only exist if the dimension is divisible by 4.

From [DeKr] one directly gets the following theorem.

Theorem 1.23 *For each genus of ϑ-lattices there exists a class of free lattices.*

Furthermore [DeKr] gives explicit constructions for ϑ-lattices. But in general one does not get representatives for all the isometry classes.

Example 1.24 As this thesis is dedicated to ϑ-lattices we want to give an example. Let $d = -11$, $w = i\sqrt{11}$ the complex $\mathcal{O}_K$ generator and $n = 4$. Then

$$\begin{pmatrix} 2 & 0 & \frac{6}{11}w & \frac{2}{11}w \\ 0 & 2 & \frac{2}{11}w & -\frac{6}{11}w \\ -\frac{6}{11}w & -\frac{2}{11}w & 2 & 0 \\ -\frac{2}{11}w & \frac{6}{11}w & 0 & 2 \end{pmatrix}$$

is a ϑ-lattice in $\mathbb{Q}(\sqrt{-11})^4$ and the order of its automorphism group equals 1920.

2 The case $d = 3$

2.1 Formulating a strategy

Remark 2.1 a) The cases $\mathbb{Q}(\sqrt{-1})$, the Gaussian number field, with field discriminant disc $=$ -4 and $\mathbb{Q}(\sqrt{-3})$, the Eisenstein number field, with field discriminant disc $= -3$ stand out due to the fact that the complex generator ω of their rings of integers can be choosen as a root of 1 and therefore is a unit.
b) The genus of ϑ-lattices in $\mathbb{Q}(\sqrt{-1})^n$ is special as the corresponding Gram matrices all have integral coefficients. For the ranks $n = 4, 8, 12$ the genera have been classified in [KiMu].

Now we will follow [KiMu] to classify the genera of ϑ-lattices over $\mathbb{Q}(\sqrt{-3})$ for the ranks $n = 4, 8, 12$. The basic idea, which will help to develop a strategy, is that the the complex generator ω acts as an automorphism on the ϑ-lattices and this action will become manifest in the automorphism group of the associated $\mathbb{Z}$-lattice of rank $2n$, see the last section.

Observation Let $\Gamma \subset \mathbb{Q}(\sqrt{-3})^n$ be a ϑ-lattice. We recall that $\Gamma_{\mathbb{Z}}$ (equipped with the bilinear form $\mathrm{Re} : \Gamma_{\mathbb{Z}} \to \mathbb{R}$) is an even and unimodular $\mathbb{Z}$-lattice. The group $U(n;\mathbb{Q}(\sqrt{-3}))$ acts transitively on every class of the ϑ-lattices, leaving the Gram matrix invariant. The coordinate action of $\mathrm{Gl}(n;\mathcal{O}_K)$ corresponds analogously to a change of the lattice basis and the automorphism group (coordinate action) fulfills $\mathrm{Aut}(\Gamma) = U(n;\mathbb{Q}(\sqrt{-3})) \cap \mathrm{Gl}(n;\mathcal{O}_K)$.

Now let $\Gamma, \Gamma^{'}$ be ϑ-lattices in $\mathbb{Q}(\sqrt{-3})^n$ with the additional property that $\Gamma_{\mathbb{Z}}$ and $\Gamma^{'}_{\mathbb{Z}}$ belong to the same class of even and unimodular $\mathbb{Z}$-lattices of rank $2n$. We recall that the complex structure of the ϑ-lattices can be recovered from the mapping $\Gamma \to \Gamma_{\mathbb{Z}}^{(')}$. An arbitrary $g \in \mathrm{O}(2n;\mathbb{R})$ applied to $\Gamma_{\mathbb{Z}}$ acts on the class of $\Gamma_{\mathbb{Z}}$ but in general one then cannot easily recover the complex structure of $g(\Gamma_{\mathbb{Z}})$ from the information encoded in the mapping. But the group $\mathrm{Aut}(\Gamma)_{\mathbb{Z}} \subset \mathrm{O}(2n;\mathbb{R})$ will respect the complex structure in the sense that the complex structure can be recovered via the mapping.

We turn back to the central idea of [KiMu], wich is that the complex $\mathcal{O}_K$ generator acts as an automorphism of the ϑ-lattices. In the case $K = \mathbb{Q}(\sqrt{-3})$ the complex $\mathcal{O}_K$ generator ω fulfills $\omega = \frac{1}{2}(1 + i\sqrt{3})$ and a calculation shows $\omega \cdot \overline{\omega} = 1$. So

$$(\omega E_n)\mathrm{Gram}(\Gamma)\overline{\omega E_n}^{tr} = \mathrm{Gram}(\Gamma)$$

which means $\omega E_n \in \mathrm{Aut}(\Gamma)$ and analogously $\omega E_n \in \mathrm{Aut}(\Gamma^{'})$. Let $\Gamma = \langle b_1, \ldots, b_n \rangle_{\mathcal{O}_K}$ and $\Gamma_{\mathbb{Z}} = \langle f_1, \ldots, f_{2n} \rangle_{\mathbb{Z}}$ with the property that $f_j = \omega f_{j-1}$ for $j \in 2\mathbb{N},\ j \leq 2n$. We compute the appearance of ωE_n embedded into $\mathrm{Aut}(\Gamma)_{\mathbb{Z}}$. Obviously we have

$$\omega \cdot f_j = \begin{cases} f_{j+1}, & 1 \leq j \leq 2n, 2 \nmid j, \\ \omega^2 \cdot f_{j-1} = \frac{-1+i\cdot\sqrt{3}}{2} f_{j-1} = \omega f_j - f_{j-1}, & 1 \leq j \leq 2n, 2 \mid j. \end{cases}$$

We get

$$\rho := (\omega E_n)_{\mathbb{Z}} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^{\oplus n},$$

and $\rho^3 = -E_{2n}$, so ρ is of order 6.

Then ρ is an automorphism of $\Gamma_{\mathbb{Z}}$ which respects the complex structure of Γ and the same holds for $\Gamma'_{\mathbb{Z}}$. We take $g \in \mathrm{O}(2n;\mathbb{R})$ with $g(\Gamma_{\mathbb{Z}}) = \Gamma'_{\mathbb{Z}}$. Then $\rho' := g \circ \rho \circ g^{-1}$ is an automorphism lying in $\mathrm{Aut}(\Gamma')_{\mathbb{Z}}$ which respects the complex Γ structure (in $\Gamma'_{\mathbb{Z}}$). If ρ' is in the same $\mathrm{Aut}(\Gamma')_{\mathbb{Z}}$ conjugacy class as ρ, for example $\rho' = f^{-1} \circ \rho \circ f,\ f \in \mathrm{Aut}(\Gamma')_{\mathbb{Z}}$, then $\eta := f \circ g$ fulfills $\rho = \eta \circ \rho \circ \eta^{-1}$ and we observe $\eta(\Gamma_{\mathbb{Z}}) = f(\Gamma'_{\mathbb{Z}}) = \Gamma'_{\mathbb{Z}}$. So η centralizes ρ and from this we get $\eta \in U(n;\mathbb{C})_{\mathbb{Z}}$, the unitary group embedded into $\mathrm{O}(2n;\mathbb{R})$. As $U(n;\mathbb{C})_{\mathbb{Z}}$ acts on the class of ϑ-lattices, we conclude that Γ and Γ' lie in the same class of ϑ-lattices. On the other hand it is then easy to see that if ρ and ρ' are not conjugated with respect to $\mathrm{Aut}(\Gamma')_{\mathbb{Z}}$ the lattices Γ and Γ' cannot be of the same class.

Remark 2.3 A lattice $\Gamma_{\mathbb{Z}}$, derived from a ϑ-lattice, may yield more than one complex structure. The number of different complex structures of $\Gamma_{\mathbb{Z}}$, which is the number of classes of ϑ-lattices Γ_j with $(\Gamma_j)_{\mathbb{Z}} = \Gamma_{\mathbb{Z}}$, equals the number of $\mathrm{O}(2n;\mathbb{R})$ conjugacy classes of ρ within $\mathrm{Aut}(\Gamma_{\mathbb{Z}})$.

This leads to the following strategy.

Strategy 2.4 For each even and unimodular lattice Γ in a given dimension, we construct the automorphism group. If Γ contains a complex structure as a ϑ-lattice over $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$ then $\mathrm{Aut}(\gamma)$ has to contain an automorphism of type ρ. The next corollary will show that the conjugacy classes with the invariant of having minimal polynomial $p(X) = X^2 - X + 1$ have a one to one correspondence to conjugacy classes of type ρ within the automorphism group of Γ. Together with our considerations from above, it establishes a bijective mapping from a ϑ-lattice into the the conjugacy classes that are having minimal polynomial $p(X) = X^2 - X + 1$ as invariant. Additionally using the next corollary one can explicitly construct Hermitian Gram matrices for ϑ-lattices: Let Γ be an even and unimodular lattice, END a representative of a conjugacy class of the automorphism group of γ with minimal polynomial $p(X) = X^2 - X + 1$ and $T \in \mathrm{Gl}(n;\mathbb{Z})$ with $T \cdot END \cdot T^{-1} = \rho$. Then compute

$$R := T \cdot \mathrm{Gram}(\Lambda) \cdot T^{tr}.$$

Now using the information encoded in the mapping $\Gamma \to \Gamma_{\mathbb{Z}}$ one can recover a Hermitian Gram matrix for the ϑ-lattice.

Now the crucial corollary.

Corollary 2.5 *Every matrix $B \in \mathbb{Z}^{2n\times 2n}$ with minimal polynomial $p(B) = X^2 - X + 1$ is conjugated to* $\mathrm{diag}(A,\dots,A)$ *with respect to* $\mathrm{Gl}(2n;\mathbb{Z})$*, where*

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Proof: From [Nw], Thm. III.12, we know that B is conjugated over $\mathbb{Z}$ to

$$\widetilde{B} = \begin{pmatrix} P & X_1 & X_2 & \dots & X_{n-2} & X_{n-1} \\ 0 & P & X_n & \dots & X_{2n-4} & X_{2n-3} \\ 0 & 0 & P & \dots & X_{2n-7} & X_{2n-6} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & P & X_{(n-1)n/2} \\ 0 & 0 & 0 & \dots & 0 & P \end{pmatrix},$$

where X_j is an arbitrary (integral) 2×2 matrix. Using excercise 5 from [Nw], p. 54, which is essentially the theorem by Latimer and McDuffe and the fact that the class number of $\mathbb{Q}(\sqrt{-3})$ is equal to 1, we can already assume $P = A$. Now take the first index j with non-vanishing X_j and annihilate X_j by conjugation in $\mathrm{Gl}(2n;\mathbb{Z})$. By iterating this procedure at most $(n-1)n/2$ steps we get the desired result. For simplicity we show the procedure for $j = 1$, a careful look generalizes the next steps to arbitrary j, when $X_k,\ 1 \leq k < j-1$, already vanish. The first step is to obtain more structural information on X_1. We compute

$$\widetilde{B}^2 = \begin{pmatrix} A^2 & AX_1 + X_1A & * & \dots & * & * \\ 0 & A^2 & * & \dots & * & * \\ 0 & 0 & A^2 & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & A^2 & * \\ 0 & 0 & 0 & \dots & 0 & A^2 \end{pmatrix},$$

and from $\widetilde{B}^2 - \widetilde{B} + E_{2n} = 0$, a condition arising from the minimal polynomial, the equation $AX_1 + X_1A - X_1 = 0$ has to hold. An explicit calculation, using

$$X_1 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

leads to the desired structural result on X_1,

$$X_1 = \begin{pmatrix} \alpha & \gamma + \alpha \\ \gamma & -\alpha \end{pmatrix}.$$

Now we conjugate by

$$\begin{pmatrix} E_2 & X & 0 & \dots & 0 & 0 \\ 0 & E_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & E_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & E_2 & 0 \\ 0 & 0 & 0 & \dots & 0 & E_2 \end{pmatrix},$$

where

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

So we get

$$\begin{pmatrix} E_2 & -X & 0 & \dots & 0 & 0 \\ 0 & E_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & E_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & E_2 & 0 \\ 0 & 0 & 0 & \dots & 0 & E_2 \end{pmatrix} \cdot \begin{pmatrix} A & X_1 & X_2 & \dots & X_{n-2} & X_{n-1} \\ 0 & A & X_n & \dots & X_{2n-4} & X_{2n-3} \\ 0 & 0 & A & \dots & X_{2n-7} & X_{2n-6} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & A & X_{(n-1)n/2} \\ 0 & 0 & 0 & \dots & 0 & A \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} E_2 & X & 0 & \dots & 0 & 0 \\ 0 & E_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & E_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & E_2 & 0 \\ 0 & 0 & 0 & \dots & 0 & E_2 \end{pmatrix} = \begin{pmatrix} A & -XA+AX+X_1 & * & \dots & * & * \\ 0 & A & * & \dots & * & * \\ 0 & 0 & A & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & A & * \\ 0 & 0 & 0 & \dots & 0 & A \end{pmatrix}.$$

Using the explicit forms of A, X and X_1 the matrix equation $-XA+AX+X_1 = 0$ looks like

$$\begin{pmatrix} -b-c & a-b-d \\ a+c-d & c+b \end{pmatrix} = -\begin{pmatrix} \alpha & \gamma+\alpha \\ \gamma & -\alpha \end{pmatrix}.$$

From this we finally get $b+c = \alpha,\ -a-c+d = \gamma$, which is easily solvable over $\mathbb{Z}$. □

Remark 2.6 Instead of ρ, respectively $\omega E_n \subset \mathrm{Aut}(\Gamma)$, we alternatively could have had a look at $(\omega-1)E_n \subset \mathrm{Aut}(\Gamma)$. Both mappings belong to the automorphism group of every ϑ-lattice, which is the essential condition. The order fulfills $\mathrm{ord}((\omega-1)E_n)_{\mathbb{Z}} = 3$ which could be an advantage as Sylow subgroups are better to handle. Nevertheless the line of argument will not change.

2.2 A mass formula

Before we now apply the strategy, we want to develop a mass formula for the ϑ-lattices. We proceed as in [BaNe] and so we are in the need of expressing those lattices in terms of modularity. Recall that a lattice L is called p-modular, if $L = pL^\sharp$, where p is an ideal and $L^\sharp$ is the dual lattice with respect to the associated form. In this subsection it is much more convenient, to consider the rescaled quadratic form $2\mathrm{Re}(\langle\cdot,\cdot\rangle) = \mathrm{trace}(\cdots + \overline{\cdots})$ instead of just $\mathrm{Re}(\langle\cdot,\cdot\rangle)$. Duality with respect to the trace-form is denoted by $\star$.

Remark 2.7 We fix an $\mathcal{O}_K$-unimodular lattice M, see [Fe]. It is well known, see for example [CoRe], p. 331, and rescale, that the dual with respect to the trace-form fulfills $M^\star = (1/\sqrt{-3})M$.

When using the counting argument from [BaNe] in the next steps, we consider the following chain of inclusion

$$\sqrt{-3}M \subset \sqrt{-3}L \subset M \subset L = L^\star \subset M^\star = (1/\sqrt{-3})M,$$

where the lattices L are unimodular with respect to trace and are having determinant $\det(L) = 3^{-n/2}$; the Hermitian form on $M^\star$ is $(1/3)\mathcal{O}_K$-valued, the same holds for the Hermitian form on L. Furthermore $\langle x,x\rangle \in (1/2)\mathbb{Z}$ for $x \in L$ as L is trace-unimodular, so we just count the trace-unimodular lattices fulfilling the condition that $\langle x,x\rangle \in (1/2)\mathbb{Z}\cap(1/3)\mathcal{O}_K = \mathbb{Z}$. After rescaling L (or equivalently the form) we get an even lattice $\sqrt{2}L$ with determinant $\det(\sqrt{2}L) = 2^n 3^{-n/2}$, a ϑ-lattice.

Let L be a trace-unimodular lattice. Analogous to [BaNe]

$$\overline{h} : L/\sqrt{-3}L \times L/\sqrt{-3}L \to \mathcal{O}_K/(\sqrt{-3}) \cong \mathbb{F}_3,\ (\tilde{x},\tilde{y}) \mapsto \sqrt{-3}\langle x,y\rangle/(\sqrt{-3}),$$

is a nondegenerate symplectic form on the vector space $\mathbb{F}_3^n$. We have $\langle l,l\rangle \in \mathbb{Z}$ for $l \in L$ and so all vectors are isotropic. Let now $x \in L$ with $\sqrt{-3}\langle x,y\rangle \in (\sqrt{-3})$ for all $y \in L$, then $\langle x,y\rangle \in \mathbb{Z}$ and so $x \in \sqrt{-3}L$. Furthermore let M be a $(\sqrt{-3})$-modular lattice with respect to trace, then

$$\phi : M/\sqrt{-3}M \times M/\sqrt{-3}M \to \mathcal{O}_K/(\sqrt{-3}),\ (\tilde{x},\tilde{y}) \mapsto \langle x,y\rangle/(\sqrt{-3}),$$

is neither a symplectic nor a Hermitian form and so ϕ induces an orthogonal geometry on the vector space $\mathbb{F}_3^n$. Take $x \in M$ with $\langle x,y\rangle \in (\sqrt{-3})$ for all $y \in M$. As M is $\mathcal{O}_K$-unimodular we conclude $x \in \sqrt{-3}M = 3M^\star$, the non-degeneracy.

Proposition 2.8 *Let M be a Hermitian lattice of rank n, which is unimodular with respect to the Hermitian form. The unimodular lattices with respect to trace containing M are the lattices L where $\sqrt{-3}L$ is a full preimage of a maximal isotropic subspace of the orthogonal $\mathbb{F}_3$ vector space $\mathbb{F}_3^n$.*

Proof: Let $\sqrt{-3}L \subset M$ with a trace-unimodular lattice L. As L is trace-unimodular and the trace-values of M lie in $\mathbb{Z}$, we have $M \subset L$ and $\sqrt{-3}M \subset \sqrt{-3}L$. We easily see that the image of $\sqrt{-3}L$ in $M/(\sqrt{-3})M$ is maximal isotropic. On the other hand let $\sqrt{-3}L$ be the full preimage of a maximal isotropic subspace of $M/(\sqrt{-3})M$. We see that L is a lattice and for $x \in L$ we have a look at $\langle x,y\rangle$, all $y \in L$; as $\sqrt{-3}x, \sqrt{-3}y \in \sqrt{-3}L$ we conclude that $\langle x,y\rangle \in 1/3(\sqrt{-3}) = (1/\sqrt{-3})\mathcal{O}_K$. The values of trace on $(1/\sqrt{-3})\mathcal{O}_K$ are integral, so $L \subset L^\star$. For $x \in L^\star \setminus L$ we have $x \in (1/\sqrt{-3})M \setminus L$. So $\sqrt{-3}x \in M$ and as there exist $y \in L$ with $\langle x,y\rangle \in 1/3(\mathcal{O}_K \setminus (\sqrt{-3}))$ (otherwise $\sqrt{-3}x \in \sqrt{-3}L$, a contradiction) we see that this is a contradiction, as the trace form on $1/3(\mathcal{O}_K \setminus (\sqrt{-3}))$ is not integral. □

Proposition 2.9 *Let L be an unimodular lattice with respect to trace. The $\mathcal{O}_K$-unimodular lattices contained in L are the preimages of maximal isotropic subspaces of the symplectic $\mathbb{F}_3$ vector space $L/\sqrt{-3}L$.*

Proof: Let M be an $\mathcal{O}_K$-unimodular lattice with $M \subset L$; let $l' \in \sqrt{-3}L$, then trace$\langle (1/\sqrt{-3})l', l\rangle \in \mathbb{Z}$ for all $l \in L$. So $\langle (1/\sqrt{-3})l', l\rangle \in (1/\sqrt{-3})\mathcal{O}_K$ and so $l' \in M$, as $M \subset L$. We easily see that the image of M in $L/\sqrt{-3}L$ is maximal isotropic. On the other hand let M be the full preimage of a maximal isotropic subspace of $L/\sqrt{-3}L$, then the Hermitian form is integral on M, hence $M \subset M'$ (where M' is the $\mathcal{O}_K$-dual of M with respect to the Hermitian form). Now suppose $m \in M' \setminus M \subset L \setminus M$. As $\langle m, l\rangle \in \mathcal{O}_K$ for all $l \in M$, we know that trace$(\langle (1/\sqrt{-3})m, l\rangle) \in \mathbb{Z}$, we conclude $(1/\sqrt{-3})m \in M^\star$, a contradiction as $m \notin M = \sqrt{-3}M^\star$. □

Now using the counting argument as in [BaNe], Proposition 2.4, we get the next proposition.

Proposition 2.10 *Let μ_n^* be the mass of the genus of the $\mathcal{O}_K$-unimodular $\mathcal{O}_K$-lattices in dimension n and μ_n the mass of the genus of the ϑ-lattices in dimension n. Then we have*

$$\mu_n^* = \mu_n \cdot \frac{c_n}{d_n},$$

where

$$d_n = \sharp\{\textit{max. isotr. subsp. of the orthogonal } \mathbb{F}_3^n \textit{ vector space }\}$$

and

$$c_n = \sharp\{\textit{max. isotr. subsp. of the symplectic } \mathbb{F}_3^n \textit{ vector space }\}.$$

From [Fe], Table V, $n = 4, 8, 12$, respectively [HaKo] we get the next lemma.

Lemma 2.11

$$\begin{aligned}
\mu_4^* &= \frac{1}{2^7 \cdot 3^5}, \\
\mu_8^* &= \frac{41}{2^{15} \cdot 3^{10} \cdot 5^2}, \\
\mu_{12}^* &= \frac{75.373.090.789}{2^{22} \cdot 3^{17} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13} = \frac{1847 \cdot 809 \cdot 691 \cdot 73}{2^{22} \cdot 3^{17} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13}, \\
\mu_{16}^* &= \frac{304816357180298316005264 3}{2^{31} \cdot 3^{22} \cdot 5^4 \cdot 11}.
\end{aligned}$$

Now we have to evaluate the factor $\frac{c_n}{d_n}$.

Lemma 2.12 *With respect to the vector space $\mathbb{F}_3^n$ the quotient c_n/d_n equals:*

	$n=4$	$n=8$	$n=12$	$n=16$
c_n/d_n	5	41	$73 \cdot 5$	$193 \cdot 17$

Proof: Due to [Ta], Exercise (8.1) from p. 78, we have (symplectic geometry)

$$c_n = \prod_{i=0}^{m-1}(q^{2m-2i}-1)/(q^{i+1}-1) = \prod_{i=0}^{m-1}(q^{m-i}-1)(q^{m-i}+1)/(q^{m-i}-1),$$

where $n=2m$ and due to [Ta], Exercise (11.3) from p. 174, we have (orthogonal geometry)

$$d_n = \prod_{i=0}^{m-1}(q^{m-1-i}+1).$$

The quotient equals

$$\frac{d_n}{c_n} = \frac{q^m+1}{2}.$$

If we specialize to $q=3$ we get the tabular from above. □

And finally:

Theorem 2.13 *For the masses of the genera of the ϑ-lattices of* rank 4 *over $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$ the following holds*

$$\begin{aligned}
\mu_4 &= \frac{1}{2^7\cdot 3^5\cdot 5},\\
\mu_8 &= \frac{1}{2^{15}\cdot 3^{10}\cdot 5^2},\\
\mu_{12} &= \frac{1847\cdot 809\cdot 691}{2^{22}\cdot 3^{17}\cdot 5^3\cdot 7\cdot 11\cdot 13},\\
\mu_{16} &= \frac{16519\cdot 3617\cdot 1847\cdot 809\cdot 691\cdot 419\cdot 47\cdot 13}{2^{31}\cdot 3^{22}\cdot 5^4\cdot 11\cdot 17}.
\end{aligned}$$

2.3 The computation of the ϑ-lattices

Within the next subsections we use MAGMA, see [MAGMA], to determine the ϑ-lattices over $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$ by using our strategy. We also give the orders of the automorphism groups.

2.3.1 ϑ-lattices of rank $n=4$ and $n=8$

Remark 2.14 There is exactly one even and unimodular $\mathbb{Z}$-lattice in dimension 8, the famous E_8 lattice, and exactly two even and unimodular $\mathbb{Z}$-lattice in dimension 16, the lattices $E_8\oplus E_8$, which is the orthogonal sum of two E_8 lattices, and $D_{16}^+ = \langle D_{16}, (\frac{1}{2})^{16}\rangle_{\mathbb{Z}}$, the lattice which is generated by the root lattice D_{16} together with the vector which components are alle equal to $1/2$. Now we determine the conjugacy classes of those three lattices and search for classes with invariant minimal polynomial $p(X)=X^2-X+1$. Obviously the situation carries over form E_8 to $E_8\oplus E_8$.

Remark 2.15 a) The group $\mathrm{Aut}(E_8)$ contains 696.729.600 elements and splits into 112 conjugacy classes, exactly one of the classes fulfills the minimal polynomial condition.
b) The group $\mathrm{Aut}(E_8 \oplus E_8)$ fulfills $\sharp\mathrm{Aut}(E_{16}) = 970.864.271.032.320.000 = 2^{29}3^{10}5^4 7^2$ and the number of conjugacy classes of $\mathrm{Aut}(E_8 \oplus E_8)$ equals 6440, exactly one of them fulfilling the minimal polynomial condition.
c) The group $\mathrm{Aut}(D_{16}^+)$ fulfills $\sharp\mathrm{Aut}(D_{16}^+) = 685.597.979.049.984.000 = 2^{30}3^6 5^3 7^2 \cdot 11 \cdot 13$ and the number of conjugacy classes of $\mathrm{Aut}(D_{16}^+)$ equals 2944, none of the classes fulfilling the minimal polynomial condition.

This proves the following theorem. A construction for these lattices in the language of Gram matrices will be pointed out in a subsequent remark.

Theorem 2.16 *In the cases of rank $n = 4$ and $n = 8$ there exists exactly one isometry class of ϑ-lattices over $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$.*

Remark 2.17 a) Let Γ_4 denote a representative of the only class of ϑ-lattices of rank $n = 4$ over $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$. [DeKr] or the strategy from above provides us with a Hermitian Gram matrix. We compute $\mathrm{Aut}(\Gamma_4)$ as the normalizer of an element of the conjugacy class fulfilling the minimal polynomial condition. Alternatively take the E_8 automorphisms which preserve the complex structure of E_8 with respect to $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$. We find $\sharp\mathrm{Aut}(\Gamma_4) = 155.520 = 2^7 \cdot 3^5 \cdot 5$.
b) Let Γ_8 denote a representative of the only class of ϑ-lattices of rank 8 over $\mathbb{Q}(\sqrt{-3})$. Obviously $\Gamma_8 = \Gamma_4 \oplus \Gamma_4$ which provides a Hermitian Gram matrix. We compute $\mathrm{Aut}(\Gamma_8)$ as the normalizer of an element of the conjugacy class fulfilling the minimal polynomial condition. We have $\sharp\mathrm{Aut}(\Gamma_8) = 2^{15} \cdot 3^{10} \cdot 5^2$. This is the square of $\mathrm{Aut}(\Gamma_4)$ times a factor 2 from the component interchanging, which gives another way to compute to automorphism group.
c) The computations in the cases of rank $n = 4$ and $n = 8$ just take a few seconds.

2.3.2 ϑ-lattices of rank $n = 12$

We now turn to the interesting case of rank $n = 12$. The situation is more complicated, as we have 24 unimodular even integral $\mathbb{Z}$-lattices of rank 24. We apply our strategy onto these $\mathbb{Z}$-lattices of rank 24 and try to determine the conjugacy classes of the automorphism groups which fulfill the minimal polynomial condition of those lattices. We indicate the Niemeier lattices by their root system (see [CoSl]).

Lemma 2.18 *The Niemeier lattice $N23$, which corresponds to the root system $D_{16}E_8$ has no complex structure over $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$.*

Proof: From [CoSl] we get $\mathrm{Aut}(N23) = \mathrm{Aut}(E_8) \times \mathrm{Aut}(D_{16}^+)$. As $\mathrm{Aut}(D_{16}^+)$ does not contain elements with minimal polynomial $X^2 - X + 1$ the same holds for $\mathrm{Aut}(N23)$. We conclude that $N23$ has no complex structure as a ϑ-lattice over $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$. □

Remark 2.19 The other lattices were processed using computer calculations. We give the total number of conjugacy classes ($\sharp CC$) and the number of classes with minimal polynomial $p(X) = X^2 - X + 1$ ($\sharp RCC$). The Niemeier lattices N01-N24 are indicated by their root lattices.

	D_{24}	$3E_8$	A_{24}	$2D_{12}$	$A_{17}E_7$	$D_{10}2E_7$	$A_{15}D_9$	$3D_8$
♯ CC	47233	253120	3916	180299	46200	473130	69300	181800
♯ RCC	-	1	-	-	-	-	-	-
	$2A_{12}$	$A_{11}D_7E_6$	$4E_6$	$2A_9D_6$	$4D_6$	$3A_8$	$2A_72D_5$	$4A_6$
♯ CC	10504	211750	58200	69174	107882	11780	102762	9975
♯ RCC	-	-	1	-	-	-	-	-
	$4A_5D_4$	$6D_4$	$6A_4$	$8A_3$	$12A_2$	$24A_1$	$D_{16}E_8$	$\varnothing$
♯ CC	37565	19857	5418	7035	1816	814	329728	167
♯ RCC	-	1	-	-	1	-	-	1

Theorem 2.20 *There exist exactly five classes of ϑ-lattices of rank 12, corresponding to conjugacy classes having as invariant minimal polynomial $p(X) = X^2 - X + 1$ within the automorphism groups of the Niemeier lattices. These lattices have the root systems*

$$3E_8, 4E_6, 6D_4, 12A_2 \ \textit{and} \ \varnothing.$$

Proof: From our strategy we know that there is a bijective mapping from every rank 12 ϑ-lattice into the conjugacy classes with minimal polynomial $p(X) := X^2 - X + 1$, of the automorphism groups of the Niemeier lattices. There exist exactly five of those conjugacy classes, which can be found in the automorphism groups of the Niemeier lattices with root system $3E_8, 4E_6, 6D_4, 12A_2$ and the Leech lattice. □

Remark 2.21 Our strategy provides us with Hermitian Gram matrices for the five ϑ-lattices of rank 12. In case of $3E_8$ one can get a Gram matrix by summarizing $\Gamma_4 \oplus \Gamma_4 \oplus \Gamma_4$. All the lattices are listed at the end of this thesis in the Appendix.

From the description of the root lattices [Qu] we get the next theorem.

Theorem 2.22 *The ϑ-lattices with respect to $3E_8$, $6D_4$ and the Leech lattice yield a quaternionic structure over the Hurwitz quaternions.*

We compute the automorphism groups of the ϑ-lattices as the centralizer of a representative of the conjugacy classes fulfilling the minimal polynomial condition. As pointed out one can also take the automorphisms of the associated $\mathbb{Z}$-lattice of rank $2n$ which preserve the imaginary part of the Gram matrix. We give the orders of the automorphism groups.

Theorem 2.23

root sys.	E_8	$2E_8$	$3E_8$	$4E_6$	$6D_4$	$12A_2$	*Leech*
♯*Auto*	$2^7 3^5 \cdot 5$	$2^{15} 3^{10} 5^2$	$2^{22} 3^{16} 5^3$	$2^{16} 3^{17}$	$2^{21} 3^9 \cdot 5$	$2^7 3^{15} \cdot 5 \cdot 11$	$2^{14} 3^8 5^2 \cdot 7 \cdot 11 \cdot 13$
♯*index*	$2^7 \cdot 5 \cdot 7$	$2^{14} 5^2 7^2$	$2^{21} 5^3 7^3$	$2^{16} 5^4$	2^{19}	2^{12}	$2^8 \cdot 3 \cdot 5^2 \cdot 7 \cdot 23$

We compute the mass in the case of the 12*-dimensional* ϑ*-lattices:*

$$\mu_{12} = \frac{1032508093}{6777434441672294400} = 1847 \cdot 809 \cdot 691 \cdot 2^{-22} \cdot 3^{-17} \cdot 5^{-3} \cdot 7^{-1} \cdot 11^{-1} \cdot 13^{-1},$$

in accordance with the mass formula from Subsection 2.2.

Finally we compute the index within the quaternionic automorphism groups.

Theorem 2.24 *The index of the automorphism groups of the quaternionic lattices within the automorphism groups of the* ϑ*-lattices equals:*

root system	E_8	$2E_8$	$3E_8$	$6D_4$	*Leech*
$\sharp\mathbb{H}Auto$	$2^7 \cdot 3 \cdot 5$	$(2^7 \cdot 3 \cdot 5)^2 \cdot 2$	$(2^7 \cdot 3 \cdot 5)^3 \cdot 6$	$2^{21}3^3 \cdot 5$	$2^{13}3^3 5^2 \cdot 7 \cdot 13$
$\sharp index$	3^4	3^8	3^{12}	3^6	$2 \cdot 3^5 \cdot 11$

Remark 2.25 Although not so pure and elegant this procedure reproduces the results of Kitazume and Munemasa [KiMu] easily and also gives Gram matrices for the 28 complex lattices in $\mathbb{Q}(i)^{12}$.

2.4 Another approach to the ϑ-lattices of rank $n = 4$, $n = 8$ and $n = 12$

Instead of using computer calculations one can follow [KiMu] more directly. This was heavily supported by Gabriele Nebe.

Well known is the next lemma.

Lemma 2.26 *Let L be a root lattice which has a decompostion into irreducible and non-isomorphic root lattices* $L = R_1^{n_1} \oplus \ldots \oplus R_s^{n_s}$*. Then we have:*

(i) $\mathrm{Aut}(L) = \times_i \mathrm{Aut}(R_i^{n_i})$,

(ii) $\mathrm{Aut}(R_i^{n_i}) = \{\mathrm{diag}(\phi_1, \ldots, \phi_{n_i})\sigma;\ \phi_i \in \mathrm{Aut}(R_i),\ \sigma \in S_{n_i}\}$.

Lemma 2.27 *Let L be a root lattice which has a decomposition into irreducible and non-isomorphic root lattices* $L = R_1^{n_1} \oplus \ldots \oplus R_s^{n_s}$ *and* $\phi \in \mathrm{Aut}(L)$ *fulfilling the property of having minimal polynomial* $p(X) = X^2 - X + 1$*. Then all the component interchanging* σ *are of order* 1*.*

Proof: Without loss of generality let $s = 1$. From the minimal polynomial we derive that σ has order 1 or 3. If the order equals 3 then σ contains a 3-cycle. After changing the numbering of the irreducible root lattices we can assume that the 3-cycle equals $(1,2,3)$ and ϕ is $R_1 \oplus R_2 \oplus R_3$-invariant. Now an easy calculation with block-matrices shows that $\phi|_{R_1 \oplus R_2 \oplus R_3}$ cannot fulfill the minimal polynomial condition, therefore ϕ cannot fulfill the minimal polynomial condition. □

From [CoSl] Chapter 4 we get some information on automorphism groups.

Lemma 2.28 *(i)* $\mathrm{Aut}(A_n) = G_0 \cdot G_1$, *where* $G_0 \cong S_{n+1}$, *the Weyl group, which is the permutation of the coordinates, and* $G_1 = 1,\ n = 1$, *respectively* $G_n = \pm 1,\ n \geq 2$, *the negation of all coordinates.*

(ii) $\mathrm{Aut}(D_n) = G_0 \cdot G_1$, *where* G_0 *is generated by all permutations of the coordinates together with the sign changes of evenly many coordinates, and* G_1 *contains the sign change of the last coordiante and, for* $n = 4$ *only, the Hadamard graph-automorphism.*

Lemma 2.29 *(i) For* $n \neq 2$ *the lattices* A_n *do not have automorphisms satisfying the minimal polynomial condition.*

(ii) For $n \neq 4$ *the lattices* D_n *do not have automorphisms satisfying the minimal polynomial condition.*

(iii) E_7 *does not have any automorphism satisfying the minimal polynomial condition.*

Proof: (ii) The group $\mathrm{Aut}(D_2)$ does not contain an element of order 3. For $n = 3$ or $n \geq 5$ an element of order 3 in $\mathrm{Aut}(D_n)$ contains a 3-cycle. After a change of coordinates and basis we can assume that the 3-cycle is $(1,2,3)$ and the upper 3×3 block of a generating matrix is equal to

$$\begin{pmatrix} -1 & -1 & 0 & \dots \\ -1 & 1 & 0 & \dots \\ 0 & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

An easy calculation shows that the action of $(1,2,3)$ on the first three basis-vectors is as follows:

$$\begin{aligned} &d_1 \mapsto d_1 - d_2 - d_3 \mapsto d_1 - d_3 \mapsto d_1, \\ &d_2 \mapsto d_3 \mapsto -d_2 - d_3 \mapsto d_2, \\ &d_3 \mapsto -d_2 - d_3 \mapsto d_2 \mapsto d_3. \end{aligned}$$

An automorphism acting this way cannot fulfill the minimal polynomial condition.
(i) For $n = 1$ we have $A_n \cong \mathbb{Z}$. For $n \geq 3$ an element of order 3 in $\mathrm{Aut}(A_n)$ contains a 3-cycle. After a change of coordinates and basis we can assume that the 3-cycle is $(1,2,3)$ and the upper 3×4 block of a generating matrix is equal to

$$\begin{pmatrix} -1 & 1 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & \dots \\ 0 & 0 & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

A calculation shows that the action of $(1,2,3)$ on the three basis-vectors is as follows:

$$\begin{aligned} &d_1 \mapsto d_2 \mapsto -d_1 - d_2 \mapsto d_1 \\ &d_2 \mapsto -d_1 - d_2 \mapsto d_1 \mapsto d_2 \\ &d_3 \mapsto d_1 + d_2 + d_3 \mapsto d_2 + d_3 \mapsto d_3. \end{aligned}$$

An automorphism acting this way cannot fulfill the minimal polynomial condition.
(iii) As the dimension is odd and the characteristic polynomial has integer coefficients, we get a real eigenvalue of any automorphism of E_7, a contradiction to the minimal polynomial condition. □

The next lemma can easily be verified.

Lemma 2.30 *The automorphism groups of the root lattices A_2, D_4, E_6 and E_8 contain an automorphism with minimal polynomial $p(X) = X^2 - X + 1$, which is unique with respect to conjugation. In case of D_4 this automorphism is a graph automorphism.*

Observation A Niemeier lattice Λ is the span of the orthogonal sum of root-lattices together with certain glue vectors, whose components lie in the duals of the root-lattices. Niemeier lattices are uniquely determined by the underlying root-lattices, which again are uniquely determined by their vectors of norm 2. Considering the action of $\mathrm{Aut}(\Lambda)$ on the vectors of norm 2, who form a finite set, together with the proven fact, that isomorphic irreducible root-lattices are not shuffeld by an automorphism fulfilling the minimal polynomial condition, we see that the irreducible root lattices are fixed with respect to automorphisms fulfilling the minimal polynomial condition. Together with the lemmas from above we conclude that an automorphism of any Niemeier lattice fulfilling the minimal polynomial condition requires the Niemeier lattice to consist of irreducible root-lattices of type A_2, D_4, E_6 or E_8. Furthermore this automorphism is unique with respect to conjugation.

The groups $\mathrm{Aut}(E_6)$ and $\mathrm{Aut}(E_8)$ are generated by the reflections in the minimal vectors (see [CoSl], Chapter 4) and so are equal to the Weyl groups of the corresponding lattices; from the description of $\mathrm{Aut}(A_2)$ we see that an automorphism fulfilling the minimal polynomial condition lies in the Weyl subgroup of $\mathrm{Aut}(A_2)$. The direct product of the Weyl groups of the irreducible root-lattices is a subgroup of the automorphism group of the corresponding Niemeier lattice. Furthermore (see [CoSl] Chapter 16) the automorphism group of the Niemeier lattice denoted by $6D_4$ is the product of the Weyl groups of the irreducible D_4 lattices, the component interchanging group of order $6! = 720$ and a group of order 3 which acts nontrivial on the glue vectors; this subgroup of order 3 fixes the irreducible components and is not contained in the Weyl group and so is the direct product of the graph automorphism of D_4 of order 3. Thus we have proven that a Niemeier lattice which has just irreducible root-lattice components of type A_2, D_4, E_6 or E_8 has an automorphism, which is unique with respect to conjugation, with minimal polynomial $p(X) = X^2 - X + 1$. Together with the explicit description of the root lattices of the Niemeier lattices this is an essential part of the proof of the following theorem.

From the explicit description of Conway zero C_0, the automorphism group of the Leech lattice, see the Atlas of Finite Groups [Co], one gets that there is exactly one automorphism of the Leech lattice corresponding to conjugacy classes with minimal polynomial $p(X) = X^2 - X + 1$.

Theorem 2.32 *The Niemeier lattices which are denoted by $3E_8, 4E_6, 6D_4$ and $12A_2$ and the Leech lattice have a structure over $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$ which is unique up to an isomorphism.*

2.5 Application to Hermitian modular forms

In the following we denote the five classes of ϑ-lattices of rank 12 over $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$ by $3E_8^\vartheta, 4E_6^\vartheta, 6D_4^\vartheta, 12A_2^\vartheta, \varnothing^\vartheta$.

As there is at most just one class of ϑ-lattices with respect to each of the 24-dimensional $\mathbb{Z}$-lattices, the transposed Gram Matrix describes the same lattice as the original one. Therefore we get the next corollary.

Corollary 2.33 *The theta-series with respect to the ϑ-lattices are symmetric Hermitian modular forms (of weight* 4, 8 *respectively* 12*).*

We want to compute the filtration of the cusp forms of the theta-subspace of weight 12 provided by those five ϑ-lattices.

From [DeKr] we get the lemma.

Lemma 2.34

$$\dim(M_{12}(\Gamma_2(\mathcal{O}_K))) = 3.$$

To be more precise

$$M_{12}(\Gamma_2(\mathcal{O}_K))) = \langle E_4^3, E_6^2, E_{12}\rangle_{\mathbb{Q}(\sqrt{-3})},$$

where E_k is an Hermitian Eisenstein-series of weight k.

From [BoFrWe] we get the next tabular containing the number of sublattices of certain lattices as suitably normalized Fourier-coefficients.

	Leech	$12A_2$	$6D_4$	$4E_6$	$3E_8$
0	1	1	1	1	1
A01		36	72	144	360
A02		12	96	480	3360
A03			72	1080	22680
A04				864	72576

We conclude that the theta-series with respect to our five lattices are linearly independent in degree 4, as the same holds for the projection to the Siegel case.

Remark 2.35 In view of [NeVe] we consider

$$\mathcal{V} := \langle 3E_8^\vartheta, 4E_6^\mathbb{C}, 6D_4^\vartheta, 12A_2^\vartheta, \varnothing^\vartheta\rangle_{\mathcal{O}_K}$$

and the mapping

$$\Theta^{(p)} : \mathcal{V} \to \mathcal{M}_{12}(\Gamma_p(\mathcal{O}_K))_\Theta.$$

From the computation of the Fourier-coefficients we derive the following theorem.

Theorem 2.36 *The form*

$$F^{(4)} := \Theta^{(4)}\left(3E_8^{\vartheta} - 30\cdot 4E_6^{\vartheta} + 135\cdot 6D_4^{\vartheta} - 160\cdot 12A_2^{\vartheta} + 54\cdot \varnothing^{\vartheta}\right)$$

is a non-trivial Hermitian form of degree 4*. We either have* $F^{(3)} \equiv 0$ *or* $F^{(3)}$ *is a non-trivial cusp form of degree* 3*. For the dimension of cusp forms of degree* 4 *it holds*

$$\dim(S_{12}(\Gamma_4(\mathcal{O}_K))_\Theta) \in \{0,1\}.$$

The form

$$G^{(3)} := \Theta^{(3)}\left(4E_6^{\vartheta} - 6\cdot 6D_4^{\vartheta} + 8\cdot 12A_2^{\vartheta} - 3\cdot \varnothing^{\vartheta}\right)$$

is a non-trivial cusp form of degree 3*. For the dimension of cusp forms of degree* 3 *it holds*

$$\dim(S_{12}(\Gamma_3(\mathcal{O}_K))_\Theta) \in \{1,2\}.$$

The form

$$H^{(2)} := \Theta^{(2)}\left(6D_4^{\vartheta} - 2\cdot 12A_2^{\vartheta} + \varnothing^{\vartheta}\right)$$

is a non-trivial cusp form of degree 2*. For the dimension of cusp forms of degree* 2 *it holds*

$$\dim(S_{12}(\Gamma_2(\mathcal{O}_K))_\Theta) = 1.$$

The form

$$J^{(1)} := \Theta^{(1)}\left(12A_2^{\vartheta} - \varnothing^{\vartheta}\right)$$

is a non-trivial cusp form of degree 1*. For the dimension of cusp forms of degree* 1 *it holds*

$$\dim(S_{12}(\Gamma_1(\mathcal{O}_K))_\Theta) = 1.$$

As the just mentioned linear combination of lattices build a basis of $\mathcal{V}$*, we have:*

degree	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n\geq 5$
$\dim(S_{12}(\Gamma_n(\mathcal{O}_K))_\Theta)$	*1*	*1*	*1*	*1-2*	*0-1*	*0*

Proof: Using the [BoFrWe] tabular ($A04$) we see that the restriction of $F^{(4)}$ onto the Siegel half space is non-vanishing. So $F^{(4)}$ is not vanishing.

In the other cases one can also check the [BoFrWe] tabular, but we will give explicit examples of Fourier-coefficients.

As we have an example for a non-vanishing Fourier-coefficient,

$$\alpha\left(\frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right) = 3545856,$$

we have $G^{(3)} \not\equiv 0$.

We find the same situation for $H^{(2)}$ and give a non-vanishing Fourier-coefficient,

$$\alpha\left(\frac{1}{2}\begin{pmatrix} 4 & w-3 & 0 \\ \overline{w-3} & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = 7776,$$

so $H^{(2)} \not\equiv 0$.

Finally $J^{(1)}$ is non-vanishing, check the root system of the corresponding lattices A_2^{ϑ} and $\varnothing^{\vartheta}$.

Using $\dim(M_{12}(\Gamma_2(\mathcal{O}_K))_\Theta) = 3$ together with $J^{(2)} \not\equiv 0$ and $H^{(2)} \not\equiv 0$ we conclude $G^{(2)}(Z) \equiv 0$ and $F^{(2)}(Z) \equiv 0$ by checking a few more Fourier-coefficients.

From the root systems of the corresponding lattices one gets that the Fourier-coefficients with respect to $\frac{1}{2}[1], \frac{1}{2}[2], \frac{1}{2}[3]$ of $H^{(1)}$ vanish. Using [Br1], p. 142, Satz 4, we get $H^{(1)}(Z) \equiv 0$. Alternatively one could have had a look at the classical case, see Remark 1.14.

Furthermore it is obvious that $J^{(1)}(Z)$ is non vanishing and $J^{(0)}(Z) \equiv 0$. □

Remark 2.37 a) All the Fourier coefficients of $F^{(3)}(Z)$ with respect to

$\frac{1}{2}\cdot\begin{pmatrix} 2 & * & * \\ * & 2 & * \\ * & * & 2 \end{pmatrix}$	$\frac{1}{2}\cdot\begin{pmatrix} 4 & * & * \\ * & 2 & * \\ * & * & 2 \end{pmatrix}$	$\frac{1}{2}\cdot\begin{pmatrix} 4 & * & * \\ * & 4 & * \\ * & * & 2 \end{pmatrix}$	$\frac{1}{2}\cdot\begin{pmatrix} 2 & * & * \\ * & 2 & * \\ * & * & 0 \end{pmatrix}$
$\frac{1}{2}\cdot\begin{pmatrix} 4 & * & * \\ * & 2 & * \\ * & * & 0 \end{pmatrix}$	$\frac{1}{2}\cdot\begin{pmatrix} 4 & * & * \\ * & 4 & * \\ * & * & 0 \end{pmatrix}$	$\frac{1}{2}\cdot\begin{pmatrix} 2 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}$	$\frac{1}{2}\cdot\begin{pmatrix} 4 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}$

vanish.

b) All the Fourier coefficients of $G^{(2)}(Z)$ with respect to

$\frac{1}{2}\cdot\begin{pmatrix} 2 & * & * \\ * & 2 & * \\ * & * & 0 \end{pmatrix}$	$\frac{1}{2}\cdot\begin{pmatrix} 4 & * & * \\ * & 2 & * \\ * & * & 0 \end{pmatrix}$	$\frac{1}{2}\cdot\begin{pmatrix} 4 & * & * \\ * & 4 & * \\ * & * & 0 \end{pmatrix}$	$\frac{1}{2}\cdot\begin{pmatrix} 2 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}$

and

$\frac{1}{2}\cdot\begin{pmatrix} 4 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}$

vanish.

c) As a verification one easily checks that our results fit the [BoFrWe] tabular.

We conjecture the behavior of the filtration.

Conjecture 2.38 *The filtration of cusp forms is as follows.*

degree	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n\geq 5$
$\dim(S_{12}(\Gamma_n(\mathcal{O}_K))_\Theta)$	*1*	*1*	*1*	*1*	*1*	*0*

Remark 2.39 a) We fail to prove $F^{(3)} \equiv 0$. Nevertheless there is some incidence for the vanishing of $F^{(3)}$, like thousands of vanishing Fourier-coefficients and the vanishing of the restriction to the Siegel case which follows from [NeVe] (Theorem 3.7.) and the theorem from above.

b) To prove the open case, essentially $F^{(3)} \equiv 0$, one could follow [HeKr], which means checking if all Fourier-coefficients corresponding to matrices T with

$$T = \begin{pmatrix} * & * \\ * & m \end{pmatrix} \quad \text{and} \quad \operatorname{trace}(T) \leq r := 2m + \frac{16+2\cdot 12}{\pi\sqrt{3}},$$

where $m \in \{0,1,2\}$ is the index of the Fourier-Jacobi-coefficient, are equal to zero. A calculation using $m=2$ and $k=12$, which is the weight of the Hermitian modular form, shows $r = 11,35\ldots$. This implies determining all the vectors of a 24-dimensional $\mathbb{Z}$-lattice with norm up to $16 = 22-2-4$. As determining all vectors with norm 8 of our five 24-dimensional $\mathbb{Z}$-lattices of interest is already out of reach, one can imagine, noticing that the effort increases exponentially, that the same for norm 16 will be out of reach for some time.

c) In [NeVe] Nebe and Venkov use the multiplication $[\Gamma_i] \circ [\Gamma_j] := \sharp \operatorname{Aut}(\Gamma_i)\delta_{[\Gamma_i],[\Gamma_j]}[\Gamma_i]$ on the formal vector space of lattices to conclude whether their cusp forms vanish or not. But the main tool for their argument is a non-vanishing cusp form of high degree, which they get from [BoFrWe]. But in our case it is just the cusp form of high degree which is in question. So the argument of [NeVe] will not work in our case.

d) Another method worth trying seemed the use of the dyadic trace due to Poor and Yuen [PoYu] or the determinant. But the computed bounds were also out of reach.

Corollary 2.40 *From [DeKr], Corollary 2, and Theorem 2.37. we derive the identitiy*

$$H^{(2)} = \Theta^{(2)}\left(6D_4^{\vartheta} - 2\cdot 12A_2^{\vartheta} + \varnothing^{\vartheta}\right) = c\cdot\left(E_{12} - \frac{441}{691}E_4^3 - \frac{250}{691}E_6^2\right),$$

where $c = -\frac{109835360}{11486493}$ *using [Kr2].*

2.6 A first look onto situation of rank $n = 16$

Here we take a first short look at the situation of rank $n = 16$. The corresponding case of even and unimodular lattices in dimension $2n = 32$ is quite difficult as we have at least one billion classes of lattices. Our hope is that the restrictions coming from the complex structure of the ϑ-lattices are strong enough to get a nice classification. A first not very encouraging result is stated in a remark.

Remark 2.41 a) We repeat the mass of the genus of the ϑ-lattices over $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$.

$$\mu_{16} = \frac{\mu_{16}}{17 \cdot 193} = \frac{13 \cdot 47 \cdot 419 \cdot 691 \cdot 809 \cdot 1847 \cdot 3617 \cdot 16519}{2^{31} \cdot 3^{22} \cdot 5^{4} \cdot 17 \cdot 11} < 0.0020053$$

b) The first idea to construct ϑ-lattices of rank 16 is to construct them via orthogonal summation from a rank 12 and a rank 4 ϑ-lattice. We compute the order of the automorphism groups.

root system	$4E_8$	$4E_6E_8$	$6D_4E_8$	$12A_2E_8$	$\varnothing E_8$
$\sharp$ Auto	$2^{31}3^{21}5^{4}$	$2^{23}3^{22} \cdot 5$	$2^{28}3^{14} \cdot 5^{2}$	$2^{14}3^{20} \cdot 5^{2} \cdot 11$	$2^{21}3^{13}5^{3} \cdot 7 \cdot 11 \cdot 13$

c) From this one can compute the partial mass of the decomposable lattice with respect to the total mass. We find

$$\sum_{\Gamma:\ decomp.} \frac{1}{\mathrm{Aut}(\Gamma)} \frac{1}{\mu_{16}} < 1,5 \cdot 10^{-17}.$$

3 The neighbourhood method

The neighbourhood method was originally developed by Kneser for integral $\mathbb{Z}$-lattices. Here we will follow Alexander Schiemann and will present the main facts roughly in the case of an imaginary quadratic number field. For the details we refer to [Sc2].

Definition 3.1 Let Λ be an $\mathcal{O}_K$-lattice over $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}$, h the corresponding Hermitian form and $\mathfrak{p} \subset \mathcal{O}_K$ a prime ideal which does not divide the discriminant of Λ.

a) An integral $\mathcal{O}_K$-lattice Γ with

$$\Gamma/(\Lambda\cap\Gamma) \simeq \mathcal{O}_K/\mathfrak{p} \quad \text{and} \quad \Lambda/(\Lambda\cap\Gamma) \simeq \mathcal{O}_K/\overline{\mathfrak{p}}$$

is called $\mathfrak{p}$-*neighbour* of Λ. Both lattices are then called neighboured.

b) An $v \in \Lambda \setminus \mathfrak{p}\Lambda$ with norm in $\mathfrak{p}\overline{\mathfrak{p}}$ is called *admissible*.

c) The $\mathfrak{p}$-neighbour at an admissible vector v is defined as

$$\Lambda(\mathfrak{p}, v) := \mathfrak{p}^{-1}v + \{y \in \Lambda;\ (v,y) \in \mathfrak{p}\}.$$

The connection between neighbours and admissible vectors is given in the next lemma.

Lemma 3.2 *Let Λ be a lattice as in the definition given above and Γ another $\mathcal{O}_K$-lattice in $\mathbb{Q}(\sqrt{-d})^n$. Then the following assertions are equivalent:*

a) Γ is a $\mathfrak{p}$-neighbour of Λ.

b) There is an $v \in \Lambda \setminus \mathfrak{p}\Lambda$ with norm in $\mathfrak{p}\overline{\mathfrak{p}}$ and $\Gamma = \Lambda(\mathfrak{p}, v)$.

Lemma 3.3 *Let Λ again an $\mathcal{O}_K$-lattice in $\mathbb{Q}(\sqrt{-d})^n$ and $\mathfrak{p}$ a prime ideal not dividing the discriminant of Λ. If $\mathfrak{p}$ ist split, inert or $\mathfrak{p} \nmid 2$ then all the $\mathfrak{p}$-neighbours of Λ lie in the same genus as Λ. If $\mathfrak{p}$ ist ramified with $\mathfrak{p} \mid 2$ then a $\mathfrak{p}$-neigbour of Λ lies in gen(Λ) if and only if the norm ideals, the $\mathcal{O}_K$-ideal generated by all norms of lattice vectors, of Λ and its neighbour coincide.*

Definition 3.4 Let Λ an integral $\mathcal{O}_K$-lattice and $\mathfrak{p} \subset \mathcal{O}_K$ a prime ideal. We define the neighbourhood of Λ as

$$\mathfrak{N}(\Lambda,\mathfrak{p}) := \{\Gamma \text{ is an } \mathcal{O}_K\text{-lattice}\ ;\ \Lambda = L_0, L_1, \ldots L_k, \text{ s.t. } L_K \in \mathrm{cl}(\Gamma),\ L_i, L_{i+1} \text{ are } \mathfrak{p}\text{-neighboured}\}.$$

Now the crucial theorem.

Theorem 3.5 *Let Λ and $\mathfrak{p}$ as in the definition.*

a) Furthermore let $\mathfrak{p} \mid \mathfrak{p}$ a prime. Then

$$\mathfrak{N}(\Lambda,\mathfrak{p}) = \{\Gamma \textit{ is an } \mathcal{O}_K\textit{-lattice}\ ;\ disc(M) = disc(L) \textit{ and there is a } \beta \textit{ in the unitary group } s.t.\ \Lambda_{\mathfrak{q}} = \beta\Gamma_{\mathfrak{q}}\}$$

for all spots $\mathfrak{q}$ but $\mathfrak{p}$.

b) *Let $n \geq 2, \Gamma \in gen(\Lambda)$ and T a finite set of spots in $\mathbb{Q}$. Then there is $\Gamma' \in cl(\Gamma)$ with $\Gamma'_{\mathfrak{q}} = \Lambda_{\mathfrak{q}}$ for all $\mathfrak{q} \in T$.*

For the explicit application of the neighbourhood method it is helpful to determine sets of admissible vectors with respect to the relation $v \sim v' \Leftrightarrow \Lambda(x) = \Lambda(x')$. As we are dealing with imaginary quadratic fields we are interested in the split case.

Lemma 3.6 *Let $\mathfrak{P} \subset \mathcal{O}_K$ a prime ideal and $\mathfrak{p} = \mathfrak{P} \cap \mathbb{Z}$. Consider the space $\boldsymbol{P}(\Lambda/\mathfrak{P}\Lambda)$ and for $x \in \Lambda \setminus \mathfrak{P}\Lambda$ let $[x]$ denote the class of x in $\boldsymbol{P}(\Lambda/\mathfrak{P}\Lambda)$. Then every class $[x]$ contains an admissible vector and all admissible vectors in $[x]$ lead to the same neigbour.*

Example 3.7 The following easy computations show an explicit application of the neighbourhood method and illustrate our implementation. As we focus on free lattices the implementation is easier than in [Sc1,Sc2] as we can omit a lot of ideal computations and are endued with Gram matrices.

Consider $\mathbb{Q}(\sqrt{-7})$. Let $w := \sqrt{-7}$, then the Gram matrix

$$\begin{pmatrix} 14 & -4\cdot w & 4\cdot w & 2\cdot w \\ 4\cdot w & 14 & -w-7 & 0 \\ -4\cdot w & w-7 & 14 & 2\cdot w \\ -2\cdot w & 0 & -2\cdot w & 14 \end{pmatrix}$$

determines a ϑ-lattice Λ in $\mathbb{Q}(\sqrt{-7})^4$. Let $v = (4,1,1,4) \in \Lambda \setminus (2+w)\Lambda$ with respect to the basis $B = (b_1, \dots b_4)$ induced by the Gram matrix. We compute $h(vB, vB) = 462$, h the standard Hermitian form, which is divisible by $(2+w)\overline{(2+w)} = 11$. We change b_3 to vB. And get

$$\begin{pmatrix} 14 & -4\cdot w & 8\cdot w+56 & 2\cdot w \\ 4\cdot w & 14 & 15\cdot w+7 & 0 \\ -8\cdot w+56 & -15\cdot w+7 & 462 & 10\cdot w+56 \\ -2\cdot w & 0 & -10\cdot w+56 & 14 \end{pmatrix}.$$

Let $(b'_1, \dots b'_4)$ be the basis induced by the new Gram matrix. Then replace b'_1 by $b'_1 - 2b'_4$ and get

$$\begin{pmatrix} 70 & -4\cdot w & 28\cdot w-56 & 2\cdot w-28 \\ 4\cdot w & 14 & 15\cdot w+7 & 0 \\ -28\cdot w-56 & -15\cdot w+7 & 462 & 10\cdot w+56 \\ -2\cdot w-28 & 0 & -10\cdot w+56 & 14 \end{pmatrix}.$$

Again let $(b''_1, \dots b''_4)$ the basis induced by the new Gram matrix. Replace b''_2 by $b''_2 - 5b''_4$ which leads to

$$\begin{pmatrix} 70 & -14\cdot w+140 & 28\cdot w-56 & 2\cdot w-28 \\ 14\cdot w+140 & 364 & 65\cdot w-273 & -70 \\ -28\cdot w-56 & -65\cdot w-273 & 462 & 10\cdot w+56 \\ -2\cdot w-28 & -70 & -10\cdot w+56 & 14 \end{pmatrix}.$$

Now we compute the neighbour. Therefor we divide the coefficients $(3,1),(3,2)$ and $(3,3)$ by $2+w$ and multiply $(3,4)$ by $2+w$ and treat the columns analogously with $2-w$

$$\begin{pmatrix} 70 & -14\cdot w+140 & -28 & -24\cdot w-70 \\ 14\cdot w+140 & 364 & -13\cdot w-91 & -70\cdot w-140 \\ -28 & 13\cdot w-91 & 42 & 10\cdot w+56 \\ 24\cdot w-70 & 70\cdot w-140 & -10\cdot w+56 & 154 \end{pmatrix}.$$

The last Gram matrix represents a $2+w$-neighbour of Λ_4 at vB.

3.1 A closer look onto ϑ-lattices of rank 16 over $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$

We already have taken a short look onto the situation in the case of rank 16. Now we want to apply the neighbourhood method onto ϑ-lattices of rank 16 over $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$. As starting lattices we take the orthogonal sums of a ϑ-lattice of rank 12 and a ϑ-lattice of rank 4.

Remark 3.8 a) Using the neigbhourhood method we constructed via random choice of admissible vectors parts of the neighbourhood from the ϑ-lattices given so far at the ideals $I_1 = 1-w$, $I_2 = 2-w$ and $I_3 = 4-w$.

b) Furthermore we constructed 16-dimensional ϑ-lattices from

(i) self-dual codes in $\mathbb{F}_3^{16}$,

(ii) Hermitian self-dual codes in $\mathbb{F}_4^8$,

(iii) quaternionic matrix groups (qm-lib. in MAGMA),

(iv) the 15 Koch-Venkov extremal 32-dimensional unimodular lattices,

(v) some of the 28-dimensional unimodular lattices.

The construction of ϑ-lattices via codes will explicitly be explained on the next pages. In (iii) and (iv) we look for an automorphism that corresponds to the action of ω, the complex $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$ generator. In (v) we take the even sublattice T of a 28-dimensional unimodular lattice and check if T has a structure with respect to $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$. If this is true we consider $\langle T \perp D_4, (x \perp y), \zeta_3(x \perp y)\rangle$ which then is a 16-dimensional ϑ-lattice.

c) So far we have constructed 79 ϑ-lattices whose partial mass is more than $99,978\%$ of the mass of the genus.

d) You will find the ϑ-lattices we have already constructed at [MathA]. At the end of this thesis we give the orders of the automorphism groups of these lattices together with their real root-systems. We use the same ordering as in [MathA] and remark which construction yields the lattice. Furthermore we give the generators of some of the automorphism groups at [MathA].

e) We have some examples of different isometry classes of ϑ-lattices having the same automorphism group order, for example the classes of lattices denoted by 39 and 46. All the lattices found so far are symmetric, which means that for each class the representatives are isometric to their Galois conjugates. In terms of Gram matrices this means that for a lattice given by its Gram matrix H, this lattice is isometric to H^{tr}.

Now we turn to the construction of ϑ-lattices from codes which yield the ϑ-lattices of rank 16 whose root-systems are of full rank. From [Kg] (respectively [Ke]) one gets the next corollary.

Corollary 3.9 *a) There are 143 isometry classes of indecomposable even and unimodular $\mathbb{Z}$-lattices of rank* 32 *whose root-lattices have full rank, 119 of them are indecomposable.*

b) From the explicit description of the root lattices one finds that there are only 9 isometry classes of even and unimodular $\mathbb{Z}$-lattices of rank 32 *with root-lattices of full rank which have a structure over $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$, 5 of them are indecomposable and the 4 decomposable lattices have E_8 as orthogonal summand. We give these lattices abbreviated by their root lattices:*

$$16A_2, 13A_2+E_6, 12A_2+E_8, 10A_2+2E_6, 8D_4, 6D_4+E_8, 4A_2+4E_6, 4E_6+E_8, 4E_8.$$

Remark 3.10 As can be seen from the Appendix, these 9 classes are contained in the list of 79 lattices which have been found. But we will give explicit constructions for these lattices also.

Construction 3.11 For convenience we take the trace form.

a) Let L be a ϑ-lattice whose real root-system contains A_2^{16}. These lattices are exactly the seven lattices abbreviated by

$$16A_2, 13A_2+E_6, 12A_2+E_8, 10A_2+2E_6, 4A_2+4E_6, 4E_6+E_8, 4E_8.$$

We consider

$$\frac{1}{\sqrt{-3}}A_2^{16} = \left(A_2^{16}\right)^{\star} \supset L \supset A_2^{16},$$

where, as usual, $\star$ denotes the dual with respect to the trace form. It is convenient to consider A_2 as the hexagonal lattice, and if we consider A_2 as a lattice of rank 1 over $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$ we have $A_2 = \langle 1 \rangle_{\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}}$, so we have $A_2 = \mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$. Now $\left(A_2^{16}\right)^{\star}/A_2^{16} \cong \mathbb{F}_3^{16}$. Then the self-dual codes in $\mathbb{F}_3^{16}$ correspond to ϑ-lattices L_C of rank 16 which appear as the preimage of the reduction modulo A_2^{16}: The complex A_2 equipped with the trace form has determinant $\det(A_2, \mathrm{trace}) = 3$

and analogously $\det(A_2^\star, \mathrm{trace}) = 1/3$. The self-dual codes in $\mathbb{F}_3^{16}$ have dimension 8 and as the reduction is surjective, we then have $[(A_2^{16})^\star : L_C] = 3^8$. Therefore we see $\det(L_C, \mathrm{trace}) = 3^{16-2\cdot 8}$ from the invariant factors. Let $v \in C$ be a word of the self dual code and for example $1/\sqrt{-3}$ the preimage of 1 and $-1/\sqrt{-3}$ the preimage of 2. From the explicit construction of the preimage and the self duality one concludes that the values of the trace form on the preimage are integral, that the norms of preimage vectors are integral and therefore even with respect to the trace form.

b) Let L be a ϑ-lattice whose real root-system contains D_4^8. These lattices are exactly the three lattices abbreviated by

$$8D_4, 6D_4 + E_8, 4E_8.$$

Using

$$D_4 = \sqrt{\frac{1}{3}} \begin{pmatrix} \sqrt{-3} & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}_{\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}}$$

we find $(D_4^8)^\star / D_4^8 \cong \mathbb{F}_4^8 = \{0, 1, \omega, \omega^2\}$ as $\mathbb{F}_2[\omega]$-modules, where $\omega = \frac{-1+i\sqrt{3}}{2}$. The construction of lattices then is analogous to the case of ternary codes.

Remark 3.12 a) (i) From [Mu] one gets the seven classes of ternary self-dual codes of length 16.

(ii) The classification of the classes of quaternary hermitian self-dual codes of length 8 is as follows:

$$\langle [1,1,0,0,0,0,0,0], [0,0,1,1,0,0,0,0], [0,0,0,0,1,1,0,0], [0,0,0,0,0,0,1,1] \rangle,$$
$$\langle [1,0,0,1,0,1,1,0], [0,1,0,1,0,1,0,1], [0,0,1,1,0,0,1,1], [0,0,0,0,1,1,1,1] \rangle,$$
$$\langle [1,1,0,0,0,0,0,0], [0,0,1,0,0,1,w^2,w], [0,0,0,1,0,1,w,w^2], [0,0,0,0,1,1,1,1] \rangle.$$

b) Using the given construction together with the classification of the root systems of the ϑ-lattices of rank 16 whose root lattices have full rank, one can explicitly construct these 9 ϑ-lattices from the given ternary and quaternary codes. Notice that both types of codes yield the ϑ-lattice abbreviated by $4E_8$.

Example 3.13 From [Mu] one finds that $\langle (1,2,1,0)^{tr}, (0,1,1,1)^{tr} \rangle_{\mathbb{F}_3}$ is a representative of a self dual code in $\mathbb{F}_3^4$. Let $1/\sqrt{-3} \notin A_2^{16}$ be the preimage of 1 and $-1/\sqrt{-3}$ the preimage of 2 which are incongruent modulo the reduction with respect to A_2. We get the two generating vectors of the lattice L as $(1/\sqrt{-3}, -1/\sqrt{-3}, 1/\sqrt{-3}, 0)^{tr}$ and $(0, 1/\sqrt{-3}, 1/\sqrt{-3}, 1/\sqrt{-3})^{tr}$. Adding $(1,0,0,0)^{tr}$ and $(0,1,0,0)^{tr}$ guarantees $L \supset A_2^4 \cong \mathcal{O}^4_{\mathbb{Q}(\sqrt{-3})}$. This gives us a generating matrix for L from which one gets a Gram matrix.

4 Lattices with respect to other imaginary quadratic fields

4.1 The case of prime discriminants

Possibly one of the first ideas to extend our results to other imaginary quadratic fields is to start with a generalization of the mass formula. This gives a natural breakdown into two steps. At first we need analogous results on unimodular masses as in [Fe], then we have to generalize the counting argument due to Nebe and Venkov as in [NeBa].
From [HaKo] we get the following formula in case of even dimension.

Proposition 4.1 *For the mass of the odd unimodular lattices in even dimension m we have*

$$\mu_m^\star = 2^{1-t}\prod_{j=1}^{m}\frac{|B_{j,\chi^j}|}{2j}\prod_{p|disc(K),\ p\neq 2\ prime}\left(p^{m/2}+\left(\frac{-1}{p}\right)_L^{m/2}\right)\cdot\begin{cases}2^m-1, & 4\parallel disc(K),\\ 2^{m/2}(2^m-1), & 8\parallel disc(K),\\ 1, & 2\nmid disc(K),\end{cases}$$

where t is the number of distinct prime divisors of the field discriminant and B_{j,χ^j} is the jth generalized Bernoulli number with respect to the jth power of the character $\chi=\chi_{disc(K)}$ which is attached to the imaginary quadratic number field (see [Za], p. 38) using the additional convention $\chi(p)^j=1$ if $\chi(p)=0,\ 2|j$, and $\chi(p)^j=0$ if $\chi(p)=0,\ 2\nmid j$.

As the formula from above is quite hard to evaluate we give an example, reproducing a result of [Fe], disc(K)$=-3$, for $m=4$.

Example 4.2 At first we have to evaluate the Bernoulli numbers. Let χ be a Dirichlet character mod N, then

$$B_{n,\chi}=N^{n-1}\sum_{k=0}^{N-1}\chi(k)B_n\left(\frac{k}{N}\right),$$

where the Bernoulli polynomials are denoted by B_n. Focus on the fact that the convention in [HaKo] forces us to start the summation with $k=0$. Furthermore with [Za] we get

$$\chi_{\mathrm{disc}(K)}(1)=1,\ \chi_{\mathrm{disc}(2)}=-1,$$

and for $\chi_{\mathrm{disc}(K)}(0)$ we take $\chi_{\mathrm{disc}(K)}(0)=0$. As $t=1$ and Legendre's symbol is equal to -1, we get

$$\begin{aligned}\mu_4 =& 1\cdot(B_1(1/3)-B_1(2/3))\cdot 3\cdot(B_2(0)+B_2(1/3)+B_2(2/3))\cdot\\ &\cdot 3^2\cdot(B_3(1/3)-B_3(2/3))+3^3\cdot(B_4(0)+B_4(1/3)+B_4(2/3))\cdot\\ &\cdot\frac{3^2+(-1)^2}{4!\cdot 2^4}=\frac{1}{31104},\end{aligned}$$

just as expected.

We compute the masses of the principal genera for the dimensions $m = 4, 8$ and $m = 12$ with respect to the imaginary quadratic fields with class number 1. Notice that results for disc $= -4$ can be found in [KiMu], [HaKo] or [Sc1]; for disc $= -3$ see Section 2.

Remark 4.3 We will stick to the notation $\mu^\star$ for the masses of the principal genera.

m	$m=4$	$m=8$	$m=12$
$d=2$	$\mu^\star = \frac{1}{128}$	$\mu^\star = \frac{99161}{1146880}$	$\mu^\star = \frac{373435015066676747}{734003200}$

m	$m=4$	$m=8$	$m=12$
$d=7$	$\mu^\star = \frac{5}{1008}$	$\mu^\star = \frac{87673}{9525600}$	$\mu^\star = \frac{4126009705493629}{1260236880}$

m	$m=4$	$m=8$	$m=12$
$d=11$	$\mu^\star = \frac{61}{1920}$	$\mu^\star = \frac{150219599}{7569408}$	$\mu^\star = \frac{20840938257308057862000175}{223812255744}$

m	$m=4$	$m=8$	$m=12$
$d=19$	$\mu^\star = \frac{1991}{5760}$	$\mu^\star = \frac{10433603234087}{39813120}$	$\mu^\star = \frac{2171624469562764970245227905006271863}{18300020981760}$

$d=43$	$\mu_4^\star = \frac{15355}{1152}$
	$\mu_8^\star = \frac{576192760005014764789}{1393459200}$
	$\mu_{12}^\star = \frac{35498439321428958000013789177199047526879436293659}{6742112993280}$

$d=67$	$\mu_4^\star = \frac{112699}{1152}$
	$\mu_8^\star = \frac{1349743051608803391092821}{1393459200}$
	$\mu_{12}^\star = \frac{50697745543869727276270651025966656426580351523683788188589}{370816214630400}$

$d=163$	$\mu_4^\star = \frac{6150955}{1152}$
	$\mu_8^\star = \frac{308233067846924033623202086525}{55738368}$
	$\mu_{12}^\star = \frac{1485886484492896056219616712721815545227280475824084950068223756214361125}{14832648585216}$

Remark 4.4 The counting argument from [BaNe], see Section 2, can easily be adopted to the case of imaginary quadratic fields with prime discriminants $\equiv 1(4)$. Analogous to Section 2 we denote by μ the mass of the genus of ϑ-lattices and we get

$$\mu_n^* = \mu_n \cdot \frac{c_n(q)}{d_n(q)}, \text{ where } \frac{c_n}{d_n} = \frac{q^{m/2}+1}{2}, \; q = \sharp(\mathcal{O}_K/(\sqrt{-d})).$$

Corollary 4.5 *Let $d \in \mathbb{N}$ be a prime with $(-1/d)_L = -1$ and $K = \mathbb{Q}(\sqrt{-d})$ with the property that there is only one genus of unimodular lattices (the genus does not split in an odd and even part); this holds for $d \in \{3,7,11,19,43,67,163\}$. For the mass of the ϑ-lattices of rank $m \in \{4,8,12\}$ over $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}$ we have*

$$\mu_m = \frac{1}{2^{m-1}\cdot m!}\prod_{j=1}^{m}|B_{j,\chi^j}|\cdot\frac{d^{m/2}-1}{d^{m/2}+1}$$

where B_{j,χ^j} is the jth generalized Bernoulli number, see the proposition from above for further details.

Back to class number 1. At first we focus on $m = 4$.

Lemma 4.6 *For the masses μ of the genera of ϑ-lattice of rank $m = 4$ with respect to the imaginary quadratic fields with $d \in \{7,11,19,43,67,163\}$ it holds:*

	$d=7$	$d=11$	$d=19$	$d=43$
$m=4$	$\mu_4 = \frac{1}{5040}$	$\mu_4 = \frac{1}{1920}$	$\mu_4 = \frac{11}{3}\cdot\frac{1}{1920}$	$\mu_4 = \frac{83}{8\cdot 720}$

	$d=67$	$d=163$
$m=4$	$\mu_4 = \frac{251}{5760}$	$\mu_4 = \frac{463}{1152}$

Remark 4.7 We use [DeKr] to construct ϑ-lattices of rank $m = 4$ and compute the automorphism groups of these lattices. For the order of these automorphism groups it holds:

	$d=7$	$d=11$	$d=19$	$d=43$	$d=67$	$d=163$
$\sharp$Auto	5040	1920	1920	720	1920	1920

This proves part of the next lemma.

Lemma 4.8 *The numbers of isometry classes of the ϑ-lattices of rank 4 with respect to imaginary quadratic fields of class number one equals 1 if $d \in \{1,3,7,11\}$. In the case of $d \in \{19,43,67,163\}$ the number of isometry classes of ϑ-lattices is greater than one. More precisely:*

	$d=19$	$d=43$	$d=67$	$d=163$
$\sharp$ *iso. cl.*	2	4	6	16

Proof: The results for $d \in \{1,3\}$ can be found in Section 2 respectively [Sc1] or in [KiMu]. The rest of the tabular can be achieved using neighbour stepping from Section 3 applied to a starting-lattice from [DeKr] at the ideal $(1+\sqrt{-d}) \subset \mathcal{O}_K$. $\square$

Remark 4.9 Representatives of the isometry classes of the ϑ-lattices given as Gram matrices can be found in the Appendices.

Remark 4.10 For rank $m = 8$ the situation is much more difficult, as the masses grow exponentially. Furthermore it showed up that isometry testing is the most time consuming component of neighbour stepping and as the numbers of lattices grow radpidly for higher ranks and greater absolut values of discriminants, the computing time increases rapidly. The next lemma describes the situation for rank $m = 8$. The results were analogously achieved as for the rank $m = 4$ case together with the fact that $\{-E_m\}$ is a subgroup of the automorphism group of the lattices for the estimations.

Lemma 4.11 *For the numbers of the isometry classes of* rank 8 ϑ*-lattices with respect to imaginary fields of class number* 1 *(except $d = 2$) it holds:*

rank 8	$d = 1$	$d = 3$	$d = 7$	$d = 11$	$d = 19$
$\sharp$ *iso. cl.*	3	1	3	7	83

rank 8	$d = 43$	$d = 67$	$d = 163$
$\sharp$ *iso. cl.*	> 480.000	$> 22 \cdot 10^6$	$> 3 \cdot 10^{13}$

Remark 4.12 Representatives of the isometry classes of the ϑ-lattices, with respect to the first tabular, given as Gram matrices can be found in Section A respectively [MathA] for $d = 19$.

Remark 4.13 The difficulties arising from the rapidly growing masses and orders of the automorphism groups grow worse when one considers lattices of rank $m = 12$. The situation for $d = 1$ was considered in [KiMu] and for $d = 3$ see Section 2. For $d \in \{11, 19, 43, 67, 163\}$ the estimations were achieved using that $\{\pm E_m\}$ is a subgroup of the automorphism group. For $d = 7$ the simple estimation yields that the number of isometry classes is greater than 110 (mass=$55,6565$). Application of neighbour stepping yielded 464 isometry classes of lattices in the genus.

This yiels the next lemma

Lemma 4.14 *For the numbers of the isometry classes of* rank 12 ϑ*-lattices with respect to imaginary quadratic fields of class number* 1 *(except $d = 2$) it holds:*

$m = 12$	$d = 1$	$d = 3$	$d = 7$	$d = 11$	$d = 19$
$\sharp$ *iso .cl.*	28	5	464	$> 2,1 \cdot 10^8$	$> 10^{16}$

$m = 12$	$d = 43$	$d = 67$	$d = 163$
$\sharp$ *iso. cl.*	$> 3,3 \cdot 10^{27}$	$> 6,0 \cdot 10^{33}$	$> 2,1 \cdot 10^{46}$

4.2 The case $d = 2$

We consider the case of the imaginary quadratic field $\mathbb{Q}(\sqrt{-2})$ with discriminant $disc = -8$.

The situation varies from the prime discriminant case. We now consider $\mathcal{O}_K$-lattices M which are even and unimodular with respect to the Hermitian form and $\mathcal{O}_K$-lattices L which are $\mathbb{Z}$-unimodular with respect to the trace form, the determinant condition is $\det(L) = 8^{-n/2}$, n the rank of the lattice, so lattices of type L are suitably scaled ϑ-lattices. We consider the chain of inclusion

$$(\sqrt{-2})L \subset \frac{1}{(\sqrt{-2})}M \subset L = L^* \subset \frac{1}{2}M.$$

We consider

$$\overline{h} : L/\sqrt{-2}L \times L/\sqrt{-2}L \to \frac{1}{\sqrt{-2}}\mathcal{O}_K/\mathcal{O}_K \cong \mathbb{F}_2,\ (\tilde{x},\tilde{y}) \mapsto 2\langle x,y\rangle + \mathcal{O}_K,$$

which is a nondegenerate symplectic form on the vector space $\mathbb{F}_2^n$. We have $\langle l,l\rangle \in \frac{1}{\sqrt{-2}}\mathcal{O}_K \cap \mathbb{Q}$ for $l \in L$ and so all vectors are isotropic. Let now $x \in L$ with $2\langle x,y\rangle \in \mathcal{O}_K$ for all $y \in L$, then $x \in L'$, the dual with respect to twice the Hermitian form. The non-degeneracy follows from $L' = \sqrt{-2}L$. Furthermore let M be an even unimodular lattice with respect to the Hermitian form, then

$$\phi : \frac{1}{2}M/\left(\frac{1}{\sqrt{-2}}\right)M \times \frac{1}{2}M/\left(\frac{1}{\sqrt{-2}}\right)M \to \frac{1}{2}\mathbb{Z}/\mathbb{Z},\ (\tilde{x},\tilde{y}) \mapsto \mathrm{trace}\,(\langle x,y\rangle) + \mathbb{Z},$$

is a nondegenerate symmetric bilinear form on on the vector space $\mathbb{F}_2^n$. Take $x \in 1/2M$ with $\langle x,y\rangle \in \mathbb{Z}$ for all $y \in 1/2M$. As the trace of the Hermitian form on $1/2M \times 1/2M$ lies in $1/2\mathbb{Z}$, we get $y \in \sqrt{-2}(1/2)M = (1/\sqrt{-2})M$, the non-degeneracy. The associated quadratic form

$$q_\phi : \frac{1}{2}M/\left(\frac{1}{\sqrt{-2}}\right)M \to \frac{1}{2}\mathbb{Z}/\mathbb{Z},\ \tilde{x} \mapsto \frac{1}{2}\mathrm{trace}\,(\langle x,x\rangle) + \mathbb{Z},$$

is nondegenerate. Take $x \in (1/2)M$, then as M is even we have the inclusion $q_\phi(x) \in (1/2)\mathrm{trace}\,((1/4)\cdot 2\mathbb{Z}) + \mathbb{Z} = (1/2)\mathbb{Z} + \mathbb{Z}$. This induces an orthogonal geometry on $\frac{1}{2}M/(1/\sqrt{-2})M \cong \mathbb{F}_2^n$. Remark that the non-degeneracy of the quadratic form and the non-degeneracy of the symmetric bilinear form correspond if and only if the characteristic of the underlying field is not 2.

Proposition 4.15 ***Situation:*** $(1/\sqrt{-2})M \subset L \subset 1/2M$

Let M be an even $\mathcal{O}_K$-lattice of rank n, which is unimodular with respect to the Hermitian form. The unimodular lattices with respect to trace contained in $(1/2)M$ are the lattices L where L is a full preimage of a maximal isotropic subspace of the orthogonal $\mathbb{F}_2$ vector space $(1/2)M/(1/\sqrt{-2})M$.

Proof: (i) Let $L \subset (1/2)M$ with a trace-unimodular lattice L. As $1/2M$ is the trace-dual of $(1/\sqrt{-2})M$ and $L \subset 1/2M$ we have $(1/\sqrt{-2})M \subset L$. We easily see that the image of L in $(1/2)M/(1/\sqrt{-2})M$ is maximal isotropic.

(ii) On the other hand let L be the full preimage of a maximal isotropic subspace of the $(1/2)M/(1/\sqrt{-2})M$-space. Obviously the trace-values on L lie in $\mathbb{Z}$ and therefore $L \subset L^\star$. Now let $l' \in L^\star \setminus L$ then $l' \in (1/2)M$, and as M is even we have $\mathrm{trace}(l',l') \in \mathbb{Z}$. As $\mathrm{trace}(l',l) \in \mathbb{Z}$ by definition of $L^\star$ we have $\langle L, l' \rangle/(1/\sqrt{-2})M$ is a maximal isotropic subspace containing $L/(1/\sqrt{-2})M$, a contradiction. □

Proposition 4.16 ***Situation:*** $\sqrt{-2}L \subset (1/\sqrt{-2})M \subset L$.

Let L be an unimodular lattice with respect to trace. Then the even $\mathcal{O}_K$-unimodular lattices scaled by $(1/\sqrt{-2})$ contained in L are the preimages of maximal isotropic subspaces of the symplectic $\mathbb{F}_2$ vector space $L/\sqrt{-2}L$.

Proof: (i) Let M be an $\mathcal{O}_K$-unimodular lattice with $(1/\sqrt{-2})M \subset L$. $(1/\sqrt{-2})M$ is self dual with respect to twice the Hermitian form. As furthermore $\sqrt{-2}L$ is the dual of L with respect to twice the Hermitian form and $(1/\sqrt{-2})M \subset L$ we have $\sqrt{-2}L \subset (1/\sqrt{-2})M$. As the Hermitian form on $(1/\sqrt{-2})M$ lies in $1/2\mathcal{O}_K$, the image of $(1/\sqrt{-2})M$ in the symplectic $L/\sqrt{-2}L$-space is maximal isotropic.

(ii) On the other hand let $(1/\sqrt{-2})M$ be the full preimage of a maximal isotropic subspace of the symplectic $L/\sqrt{-2}L$-space, then the form $\overline{h}$ is $\mathcal{O}_K$-valued on $(1/\sqrt{-2})M$, so $M \subset M'$ (where M' is the $\mathcal{O}_K$-dual of M with respect to the Hermitian form) as $\overline{h}$ is essentially twice the Hermitian form.
Now let $m' \in M' \setminus M$ then $(1/\sqrt{-2})m' \in L \setminus (1/\sqrt{-2}M)$ so $\langle (1/\sqrt{-2})m', (1/\sqrt{-2})m' \rangle \in (1/2)\mathbb{Z}$, therefore $\langle (1/\sqrt{-2})M, (1/\sqrt{-2})m' \rangle/\sqrt{-2}L$ is isotropic with respect to twice the Hermitian form, a contradiction. From $\sqrt{-2}L \subset (1/\sqrt{-2})M$ and $\langle x,x \rangle \in (1/2)\mathbb{Z}$ for $x \in L$ one gets that M is even. □

Remark 4.17 a) Using the counting argument one gets the factor $\frac{2^{n/2}+1}{2}$, see Section 1. It is well known that there are two genera of $\mathcal{O}_K$-unimodular lattices, the even and the odd genus. The mass of the principal genus equals $\mu_4^\star = 1/128$, see [HaKo], and the mass of the genus of the even unimodular lattices, the one we are interested in, equals $\mu_4^\star = 1/(12 \cdot 128)$, [Sc1]. So we get

$$\mu_4 = \frac{13}{12 \cdot 128} \frac{2}{5 \cdot 13} = \frac{1}{3840}$$

b) Again from [Sc1] we take the mass of the genus of the even and unimodular lattices $\mu_{even}^\star = 99161/275251200$. We compute the mass of the genus of ϑ-lattices of rank 8 over $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$ as

$$\mu_8 = \frac{99161}{275251200} \frac{2}{2^4+1} = \frac{5833}{137625600}.$$

c) Some calculations lead to the conjecture that the unimodular masses are connected via $(2^{n/2}(2^{n/2}-1))\mu^\star_{\text{even}} = \mu^\star_{\text{odd}}$. Using $\mu^\star_{\text{odd}} = 373435015066676747/734003200$ and the trivial estimation for the order of the automorphism groups, one therefore conjectures more than 3882 isometry classes of lattices in case of rank 12.

Theorem 4.18 *The number of isometry classes of ϑ-lattices in $\mathbb{Q}(\sqrt{-2})^m$ where $m \in \{4,8\}$ can be read off from the following tabular. We give a conjecture for $m = 12$, see the remark from just above.*

$\mathbb{Q}(\sqrt{-2})$	$m=4$	$m=8$	$m=12$
$\sharp$ *iso. cl*	1	6	*conj.* > 3882

Proof: Using [DeKr] we construct a ϑ-lattice of rank $m = 4$ with order of the automorphism group $\sharp\text{Auto} = 3840$. The mass shows that there is only one isometry class of ϑ-lattices. The case of rank $m = 8$ can be revealed by using neighbour stepping. □

4.3 Application to Hermitian modular forms of low weight

Using the ϑ-lattices we construct Hermitian modular forms. We want to determine the filtration of cusp forms arising from rank 4 lattices and give some information of the forms arising from rank 8 lattices.

4.3.1 Weight $k = 4$

Theorem 4.19 *From the computation of Fourier-coefficients in the case of rank 4 ϑ-lattices one gets the filtration for the cusp forms in the cases $\mathbb{Q}(\sqrt{-19})$, $\mathbb{Q}(\sqrt{-43})$ and $\mathbb{Q}(\sqrt{-67})$, which is in these cases the trivial one. The method is described explicitly in [He].*

degree	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
$\dim(S_4(\Gamma_n(\mathcal{O}_{\mathbb{Q}(\sqrt{-19}))})_\Theta)$	*1*	*0*	*1*	*0*	*0*

degree	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
$\dim(S_4(\Gamma_n(\mathcal{O}_{\mathbb{Q}(\sqrt{-43})}))_\Theta)$	*1*	*0*	*3*	*0*	*0*

degree	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
$\dim(S_4(\Gamma_n(\mathcal{O}_{\mathbb{Q}(\sqrt{-67})}))_\Theta)$	*1*	*0*	*5*	*0*	*0*

The case left, $\mathbb{Q}(\sqrt{-163})$, is the only case showing more interesting behaviour. From the computation of the Fourier-coefficients one gets the next lemma.

Lemma 4.20 *The filtration of cusp forms with respect to rank* 4 *lattices over* $\mathbb{Q}(\sqrt{-163})$ *holds:*

degree	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
$\dim(S_4(\Gamma_n(\mathcal{O}_{\mathbb{Q}(\sqrt{-163})}))_\Theta)$	*1*	*0*	*13-15*	*0-2*	*0*

Remark 4.21 a) From the computation of some hundred thousand Fourier-coefficients we conjecture

degree	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
$\dim(S_4(\Gamma_n(\mathcal{O}_{\mathbb{Q}(\sqrt{-163})}))_\Theta)$	1	0	13	2	0

b) Unfortunately we cannot prove the conjectured behaviour. Constants in the classical dimension formula, see [Br1], are getting far out of reach for high discriminants.

c) Other estimations work with dyadic trace or determinant instead of trace [PoYu], [Kn]. But these estimations are inconvenient for our purpose. The explicit computation of Fourier-coefficients of theta-series starts with the computation of the lattice vectors of given length and then one just counts the scalar products. So working with trace is the natural way to do the computations. If one takes for example the determinant, one has to check all the possible combinations of matrix entries, compute the vectors with respect to possible diagonals and search for the scalar products in the subdiagonals. Essentially this search is the same then as computing the scalar products with respect to the classical trace, but wastes a lot of information. At present state of time one can compute Fourier-coefficients with respect to matrices of trace up to 14.

4.3.2 Weight $k=8$

From [HeKr] we get an example of a filtration of Hermitian modular forms which are constructed from rank 8 ϑ-lattices in the case of the Gaussian integers.

Corollary 4.22 *We have* $\dim \mathcal{S}_8(\Gamma_p) = 1$ *for* $p \in \{0,2,4\}$ *and* $\dim \mathcal{S}_8(\Gamma_p) = 0$ *otherwise.*

Furthermore we have a look a the cases $d \in \{2,7,11\}$ where the number of isometry classes of lattices of rank 8 is small. But as the constants within the dimension estimations are inoperable, see the remark from above, we only can give little information on the filtration.

Lemma 4.23 *a)* ***d=7****: From the explicit computation of the Fourier-coefficients with diagonals* $1/2\cdot(2,2,2,2)$ *we get* $\dim(M_8(\Gamma_4(\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}))) = 3$*, the forms arising from theta series are linearly independent in degree* 4*.*

b) ***d=2****: From the explicit compuation of the Fourier-coefficients with diagonals* $1/2\cdot(2,2,2,2)$ *we get the linear independency of the forms in degree* 4*. From the computation of the Fourier-coefficients with diagonals* $1/2\cdot(2,2)$ *we get* $\dim(M_8(\Gamma_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}))) \geq 3$ *and* *from the computation of some more Fourier-coefficients and the classical case we conclude* $\dim(S_8(\Gamma_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}))) \geq 2$. *From [De] one finally gets the equality in both estimations.*

degree	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
$\dim(S_8(\Gamma_n(\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}))_\Theta)$	*1*	*0*	*2*	*0-3*	*0-3*

From the computation of some thousand Fourier-coefficients we conjecture.

Conjecture 4.24 *a)* ***d=7****:*

degree	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
$\dim(S_8(\Gamma_n(\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}))_\Theta)$	*1*	*0*	*1*	*0*	*1*

b) ***d=2****:*

degree	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
$\dim(S_8(\Gamma_n(\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}))_\Theta)$	*1*	*0*	*2*	*0*	*3*

c) ***d=11****:*

degree	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
$\dim(S_8(\Gamma_n(\mathcal{O}_{\mathbb{Q}(\sqrt{-11})}))_\Theta)$	*1*	*0*	*2*	*0*	*4*

5 Non-free lattices

From Section 1 we repeat some facts on lattices. Let V be an arbitrary vector space over the field F and $\mathcal{O} \subset F$ a Dedekind domain. A lattice Λ in the sense of O' Meara [OMe] is a subset $M \subset V$ together with an $\mathcal{O}$-module structure. Such a lattice needs not to be free. We want to show that the theta-series with respect to non-free ϑ-lattices are Hermitian modular forms as well. These non-free ϑ-lattices correspond to imaginary quadratic number fields of class number greater 1. At first we check the transformation with respect to the involution J. We will follow [Kr1], p. 111. Essentially the proof requires just a careful look at the free case. But at first we collect some information on non-free lattices.

Remark 5.1 a) As there is no basis for a non-free $\mathcal{O}$-lattice Λ we cannot give a Gram matrix in the sense of Section 1. This is the reason why one has to pay some more attention to the non-free case.

b) The result for free lattices was proven in [CoRe] and claimed for non-free lattices also but not proven „...for convenience...“. But their proof is based on the existence of a Gram matrix, so it is worth to elaborate the proof in the general case. A short version of the proof was already published in [HeNe].

c) An $\mathcal{O}$-lattice Λ is nearly free, which means that there is a basis $z_1,\dots,z_m$ for V and a fractional Ideal $\mathfrak{a}$ such that

$$\Lambda = \mathcal{O}z_1 + \ldots + \mathcal{O}z_{m-1} + \mathfrak{a}z_m,$$

see [OMe].

d) As usual a possibly non-free ϑ-lattice over an imaginary quadratic number field K can be considered as a $\mathbb{Z}$-module of rank $2m$.

For convenience we take a definition for theta-series that slightly differs from the definition in Section 1.

Definition 5.2 Let K now be an imaginary quadratic field, $\mathcal{O}_K$ the ring of integers of K, $Z \in \mathcal{H}_n$, Λ_0 an $\mathcal{O}_K$-lattice in $\mathbb{C}^m$ and $\Lambda = \Lambda_0^n$. We define

$$\Theta_\Lambda^{(n)}(Z) = \sum_{(b_1,\ldots,b_n)\in\Lambda_0^n} \exp((\pi i \,\mathrm{trace}\langle b_k, b_l\rangle_{k,l} Z)).$$

This theta-series is essentially the same as the one introduced in Section 1.

Theorem 5.3 *Take the setting from above and let $\Lambda_0 \subset \mathbb{C}^m$ together with the* trace-*form be $\mathbb{Z}$-unimodular. Then we have*

$$\Theta_\Lambda^{(n)}(J\langle Z\rangle) = \Theta_\Lambda^{(n)}(-Z^{-1}) = \det(Z/i)^m \Theta_\Lambda^{(n)}(Z),$$

which means as $m \equiv 0(4)$ that Θ behaves as a modular form under the transformation by the involution J.

Proof: Using the identity theorem we can restrict to the case $Z = iY$, $Y > 0$. At first we consider the mapping

$$\begin{aligned}\varphi : \mathbb{R}^{2mn} &\to \mathrm{Mat}(m \times n; \mathbb{C}),\\ &x = (x^{\mathrm{Re}}_{1,1}, x^{\mathrm{Im}}_{1,1}, \ldots x^{\mathrm{Re}}_{1,m}, x^{\mathrm{Im}}_{1,m}, x^{\mathrm{Re}}_{2,1}, x^{\mathrm{Im}}_{2,1}, \ldots, x^{\mathrm{Re}}_{2,m}, x^{\mathrm{Im}}_{2,m}, \ldots x^{\mathrm{Re}}_{n,1}, x^{\mathrm{Im}}_{n,1}, \ldots, x^{\mathrm{Re}}_{n,m}, x^{\mathrm{Im}}_{n,m})^{tr} \mapsto\\ &\quad \mapsto (x^{\mathrm{Re}}_{k,l} + i \cdot x^{\mathrm{Im}}_{k,l})_{l,k}.\end{aligned}$$

Let $\Lambda = \Lambda_0^n$ then $\varphi^{-1}(\Lambda) \subset \mathbb{R}^{2mn}$ is a lattice, so there exists $F \in \mathrm{Mat}(2mn \times 2mn; \mathbb{R})$ with $\varphi(F\mathbb{Z}^{2mn}) = \Lambda$. We have $|\det F| = \mathrm{vol}(\Lambda) = (\mathrm{vol}(\Lambda_0))^n$. We know that Λ_0 is trace unimodular and essentially $\varphi^{-1}(\Lambda) \subset \mathbb{R}^{2mn}$ is n-copies of the corresponding real lattice. So $\mathrm{vol}(\Lambda) = 1$ and furthermore $\Lambda = \Lambda^\tau$, where $\tau(A,B) := (1/2)\,\mathrm{trace}(A\overline{B}^{tr} + \overline{A}B^{tr})$. Now define

$$\psi : \mathbb{R}^{2mn} \to \mathbb{C},\ x \mapsto \exp(-\pi\tau(\overline{\varphi(Fx)}^{tr} \cdot \varphi(Fx)), Y).$$

Then

$$\Theta^{(n)}_\Lambda(iY) = \sum_{g \in \mathbb{Z}^{2mn}} \psi(g).$$

Application of the classical Poisson summation yields

$$\begin{aligned}\Theta^{(n)}_\Lambda(iY) &= \sum_{g \in \mathbb{Z}^{2mn}} \psi(g) = \sum_{h \in \mathbb{Z}^{2mn}} \int\limits_{x:\mathbb{R}^{2mn}} \psi(x) \exp(-2\pi i h^{tr} x))\, dx\\ &= \sum_{h \in \mathbb{Z}^{2mn}} \int\limits_{x:\mathbb{R}^{2mn}} \exp(-\pi\tau(\overline{\varphi(Fx)}^{tr} \cdot \varphi(Fx)), Y) \exp(-2\pi i h^{tr} x))\, dx\\ &\underbrace{=}_{y=Fx} \sum_{h \in \mathbb{Z}^{2mn}} \int\limits_{y:\mathbb{R}^{2mn}} \exp(-\pi\tau(\overline{\varphi(y)}^{tr} \cdot \varphi(y)), Y) \exp(-2\pi i h^{tr}(F^{-1}y))\, dy \cdot \frac{1}{|\det F|}\\ &\underbrace{=}_{|\det F|=1} \sum_{h \in \mathbb{Z}^{2mn}} \int\limits_{y:\mathbb{R}^{2mn}} \exp(-\pi\tau(\overline{\varphi(y)}^{tr} \cdot \varphi(y)), Y) \exp(-2\pi i((F^{tr})^{-1}h)^{tr} y)))\, dy.\end{aligned}$$

Now

$$\tau(\varphi((F^{tr})^{-1}h), \varphi(Fl)) = (1/2)\,\mathrm{trace}\Big((\varphi((F^{tr})^{-1}h)\overline{\varphi(Fl)}^{tr} + \overline{(F^{tr})^{-1}h)}\varphi(Fl)^{tr}\Big).$$

Recalling the definition of φ and the fact that trace is essentially the canonical scalar product with respect to $\mathrm{Mat}(m \times n; K)$ the right hand side equals $h^{tr}l$. So $(\varphi((F^{tr})^{-1}\mathbb{Z}^{2mn}))$ equals Λ^τ. Therefore

$$\begin{aligned}\Theta^{(n)}_\Lambda(iY) &= \sum_{H \in \Lambda^\tau = \Lambda} \int\limits_{y:\mathbb{R}^{2mn}} \exp(-\pi\tau(\overline{\varphi(y)}^{tr} \cdot \varphi(y)), Y) \exp(-2\pi i((\varphi^{-1}H)^{tr} \cdot y)))\, dy\\ &\underbrace{=}_{U=\varphi(Y)} \sum_{H \in \Lambda^\tau = \Lambda} \int\limits_{U:\mathrm{Mat}(m \times n;\mathbb{C})} \exp(-\pi\tau(\overline{U}^{tr} \cdot U, Y) \exp(-2\pi i\tau(H,U))\, dU.\end{aligned}$$

Furthermore we have the identity

$$-\tau(\overline{U}^{tr}U,Y) - 2i\tau(H,U) = -\tau(E[U+iHY^{-1}],Y) - \tau(E[H],Y^{-1}).$$

Using elementary matrix arithmetic

$$\begin{aligned}
-\tau(E[U+iHY^{-1}],Y) &= -\tau((\overline{U}^{tr} + i\cdot\overline{(Y^{-1})}^{tr}\overline{H}^{tr})(U+iHY^{-1}),Y) \\
&= -\tau(\overline{U}^{tr}U + i\cdot\overline{(Y^{-1})}^{tr}\overline{H}^{tr}U + i\cdot\overline{U}^{tr}HY^{-1} - \overline{Y^{-1}}^{tr}\overline{H}^{tr}HY^{-1},Y) \\
&= -\tau(\overline{U}^{tr}U,Y) - i/2\,\mathrm{trace}\left(\overline{(Y^{-1})}^{tr}\overline{H}^{tr}U\overline{Y}^{tr} + \overline{U}^{tr}HY^{-1}Y\right) - \\
&\quad - i/2\,\mathrm{trace}\left(\overline{U}^{tr}HY^{-1}\overline{Y}^{tr} + \overline{Y^{-1}}^{tr}\overline{H}^{tr}UY\right) + \mathrm{trace}\left(\overline{Y^{-1}}^{tr}\overline{H}^{tr}HY^{-1}Y\right).
\end{aligned}$$

One directly verifies the indentity using $\overline{Y^{-1}}^{tr} = Y^{-1}$. So

$$\begin{aligned}
\Theta_\Lambda^{(n)}(iY) &= \sum_{H\in\Lambda^\tau=\Lambda}\ \int_{U:\mathrm{Mat}(m\times n;\mathbb{C})} \exp(-\pi\tau(E[U+iHY^{-1}],Y)\exp(-\pi\tau(E[H],Y^{-1}))\,dU \\
&\underbrace{=}_{W:=iHY^{-1}} \sum_{H\in\Lambda}\exp(-\pi\tau(E[H],Y^{-1}))\int_{U:\mathrm{Mat}(m\times n;\mathbb{C})}\exp(-\pi\tau(E[U+W],Y)\,dU
\end{aligned}$$

As the first factor is just $\Theta_\Lambda^{(n)}(-(1/i)Y^{-1})$ and the last integral equals $\det Y^{-m}$, see [Kr], p. 110, the assertion holds. □

Theorem 5.4 *Using the setting from above $\Theta_\Lambda^{(n)}$ is a Hermitian modular form.*

Proof: Due to a result from Klingen [Kl], which we already used in Section 1, the modular group Γ_n of degree n with respect to an imaginary quadratic field K is generated by

$$\Gamma = \left\langle J, T = \begin{pmatrix} E & S \\ 0 & E \end{pmatrix},\ S\in\mathrm{Sym}(n;\mathcal{O}_K),\ R = \begin{pmatrix} (U^{-1})^{tr} & 0 \\ 0 & \overline{U} \end{pmatrix},\ U\in\mathrm{Gl}(n;\mathcal{O}_K)\right\rangle,$$

where $\mathcal{O}_K$ ist the ring of integers of K. As the lattice Λ_0 is even, this also holds for $\mathrm{trace}(\langle b_k,b_l\rangle_{k,l}S)$ which means $\exp(\pi i\,\mathrm{trace}(\langle b_k,b_l\rangle_{k,l}S) = 1$ and as furthermore U just permutes the lattice vectors, see the definition, $\Theta_\Lambda^{(n)}(Z)$ transforms like a modular form if one takes the theorem from above into account. □

Remark 5.5 a) The mass of the odd integral and unimodular lattices of rank 4 over $\mathbb{Q}(\sqrt{-5})$ equals $65/128$. Using the counting argument we get $5/128$ as a lower bound for the mass. From [DeKr] we construct a ϑ-lattice whose automorphism group is of order $1/384$. After constructing 100.000 free neighbours of the ϑ-lattice from [DeKr] which were all isometric to the lattice itself we guess that the hope of just finding free representatives in this case cannot be fulfilled. The representative from [DeKr] yields 6.66% of the mass-bound.

b) The situation with respect to $\mathbb{Q}(\sqrt{-23})$ is analogously to the just described situation with respect to $\mathbb{Q}(\sqrt{-5})$. We have $\mu = 53/16 \cdot 2/(23^2+1) = 1/80$. The order of the automorphism group of the ϑ-lattice contributed by [DeKr] equals $1/240$, this is $1/3$ of the mass. But construction of free neighbours did not yield another isometry class.

d) We conjecture that there is no imaginary quadratic field of class number greater than 1 which admits a genus of ϑ-lattices of rank $m = 4$ which consists of just one isometry class.

A Appendix

A.1 Numbers of isometry classes of ϑ-lattices

We give an overview of the number of isometry classes of ϑ-lattices with respect to imaginary quadratic fields of class number 1 and field discriminant D. Later we will give Gram matrices for the representatives of the isometry classes of lattices of rank $m = 4$ for $D \in \{-3, -7, -8, -19 - 43, -67\}$, for $D = -4$ see [Sc1] or [HaKo], for $D = -11$ see Section 1, Example 1.24, and for $d = -163$ see [MathA]. For rank $m = 8$ and $D \in \{-7, -8, -11\}$ see the next pages, for $D = -4$ see [Sc1] or [HaKo] and for $D = -19$ see [MathA]. For rank $m = 12$ and $D = -3$ see the next pages.

The situation for rank 4.

$m = 4$	$D = -3$	$D = -4$	$D = -7$	$D = -8$	$D = -11$	$D = -19$
$\sharp$ *iso. cl.*	1	1	1	1	1	2

$m = 4$	$D = -43$	$D = -67$	$D = -163$
$\sharp$ *iso. cl.*	4	6	16

The situation in for rank 8.

$m = 8$	$D = -3$	$D = -4$	$D = -7$	$D = -8$	$D = -11$	$D = -19$
$\sharp$ *iso. cl.*	1	3	3	6	7	83

$m = 8$	$D = -43$	$D = -67$	$D = -163$
$\sharp$ *iso. cl.*	> 480.000	$> 2,2 \cdot 10^7$	$> 3 \cdot 10^{13}$

The situation for rank 12.

$m = 12$	$D = -3$	$D = -4$	$D = -7$	$D = -8$	$D = -11$	$D = -19$
$\sharp$ *iso.cl.*	5	28	464	conj. > 3882	$> 2,1 \cdot 10^8$	$> 10^{16}$

$m = 12$	$D = -43$	$D = -67$	$D = -163$
$\sharp$ *iso.cl.*	$> 3,3 \cdot 10^{27}$	$> 6,0 \cdot 10^{33}$	$> 2,1 \cdot 10^{46}$

A.2 The rank 4 ϑ-lattice over $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$

$$G01 := \begin{pmatrix} 2 & 0 & w+1 & 1/2w \\ 0 & 2 & 1/2w & -w+1 \\ -w+1 & -1/2w & 2 & 0 \\ -1/2w & w+1 & 0 & 2 \end{pmatrix}$$

With $w = i\sqrt{2}$ and automorphism group order $\sharp\,\mathrm{Aut}(G01) = 3840$.

A.3 The rank 4 ϑ-lattice over $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$

$$G01 := \frac{1}{3}\begin{pmatrix} 6 & 0 & 2w & 2w \\ 0 & 6 & 2w & -2w \\ -2w & -2w & 6 & 0 \\ -2w & 2w & 0 & 6 \end{pmatrix}$$

With $w = i\sqrt{3}$ and automorphism group order $\sharp\,\mathrm{Aut}(G01) = 155520$.

A.4 The rank 4 ϑ-lattice over $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$

$$G01 := \frac{1}{7}\begin{pmatrix} 14 & 0 & 6w & 4w \\ 0 & 14 & 4w & -6w \\ -6w & -4w & 28 & 0 \\ -4w & 6w & 0 & 28 \end{pmatrix}$$

With $w = i\sqrt{7}$ and automorphism group order $\sharp\,\mathrm{Aut}(G01) = 5040$.

A.5 The two rank 4 ϑ-lattices over $\mathcal{O}_{\mathbb{Q}(\sqrt{-19})}$

$$G01 := \frac{1}{19}\begin{pmatrix} 38 & 0 & 6w & 6w \\ 0 & 38 & 6w & -6w \\ -6w & -6w & 38 & 0 \\ -6w & 6w & 0 & 38 \end{pmatrix}$$

With $w = i\sqrt{7}$ and automorphism group order $\sharp\,\mathrm{Aut}(G01) = 1920$.

$$G02 := \frac{1}{19}\begin{pmatrix} 76 & -7w-19 & -6w+76 & -16w-76 \\ 7w-19 & 38 & 17w+19 & -6w+76 \\ 6w+76 & -17w+19 & 190 & -38w-38 \\ 16w-76 & 6w+76 & 38w-38 & 190 \end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(G02) = 720$.

A.6 The four rank 4 ϑ-lattices over $\mathcal{O}_{\mathbb{Q}(\sqrt{-43})}$

$$G01 := \frac{1}{43}\begin{pmatrix} 86 & 0 & 14w & 12w \\ 0 & 86 & 12w & -14w \\ -14w & -12w & 172 & 0 \\ -12w & 14w & 0 & 172 \end{pmatrix}$$

With $w = i\sqrt{43}$ and automorphism group order $\sharp\,\mathrm{Aut}(G01) = 720$.

$$G02 := \frac{1}{43}\begin{pmatrix} 774 & 66w-86 & 74w+1720 & 178w-86 \\ -66w-86 & 258 & -159w+43 & -14w+688 \\ -74w+1720 & 159w+43 & 4472 & 430w+430 \\ -178w-86 & 14w+688 & -430w+430 & 1892 \end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(G02) = 240$.

$$G03 := \frac{1}{43}\begin{pmatrix} 4386 & -273w-301 & -34w+3440 & -424w-688 \\ 273w-301 & 774 & 222w-86 & -14w+1204 \\ 34w+3440 & -222w-86 & 2924 & -344w-344 \\ 424w-688 & 14w+1204 & 344w-344 & 1892 \end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(G03) = 120$.

$$G04 := \frac{1}{43}\begin{pmatrix} 516 & -129w+129 & 64w & -2w+688 \\ 129w+129 & 2838 & 54w-2064 & 337w+645 \\ -64w & -54w-2064 & 1720 & -258w-258 \\ 2w+688 & -337w+645 & 258w-258 & 1892 \end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(G04) = 1920$.

A.7 The six rank 4 ϑ-lattices over $\mathcal{O}_{\mathbb{Q}(\sqrt{-67})}$

$$G01 := \frac{1}{67}\begin{pmatrix} 134 & 0 & 20w & 20w \\ 0 & 134 & 20w & -20w \\ -20w & -20w & 402 & 0 \\ -20w & 20w & 0 & 402 \end{pmatrix}$$

With $w = i\sqrt{67}$ and automorphism group order $\sharp\,\mathrm{Aut}(G01) = 1920$.

$$G02 := \frac{1}{67}\begin{pmatrix} 19832 & 260w+16884 & 334w+134 & 1417w+737 \\ -260w+16884 & 14606 & 282w+402 & 1196w+1876 \\ -334w+134 & -282w+402 & 402 & 1608 \\ -1417w+737 & -1196w+1876 & 1608 & 6834 \end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(G02) = 240$.

$$G03 := \frac{1}{67}\begin{pmatrix} 19832 & 506w-134 & 100w+11256 & 1417w+737 \\ -506w-134 & 1072 & -321w+201 & -20w+2412 \\ -100w+11256 & 321w+201 & 6834 & 804w+804 \\ -1417w+737 & 20w+2412 & -804w+804 & 6834 \end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(G03) = 120$.

$$G04 := \frac{1}{67}\begin{pmatrix} 10184 & -645w-1005 & -100w+12060 & -995w-1675 \\ 645w-1005 & 3082 & 749w-469 & -20w+4422 \\ 100w+12060 & -749w-469 & 14874 & -1206w-1206 \\ 995w-1675 & 20w+4422 & 1206w-1206 & 6834 \end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(G04) = 48$.

$$G05 := \frac{1}{67}\begin{pmatrix} 134 & -27w+67 & 13w+67 & 804 \\ 27w+67 & 536 & 40w-402 & 191w+871 \\ -13w+67 & -40w-402 & 804 & -201w-201 \\ 804 & -191w+871 & 201w-201 & 6834 \end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(G05) = 120$.

$$G06 := \frac{1}{67}\begin{pmatrix} 130918 & -6705w-16817 & 14882w-73968 & 6201w+112359 \\ 6705w-16817 & 25192 & -5698w-41540 & 4953w-35711 \\ -14882w-73968 & 5698w-41540 & 155172 & -16281w-16281 \\ -6201w+112359 & -4953w-35711 & 16281w-16281 & 116178 \end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(G06) = 720$.

The mass of the genus ist $251/5760$.

A.8 The six rank 8 ϑ-lattices over $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$

$$G01 := \frac{1}{2}\begin{pmatrix} 4 & 0 & 2w+2 & w & 0 & 0 & 0 & 0 \\ 0 & 4 & w & -2w+2 & 0 & 0 & 0 & 0 \\ -2w+2 & -w & 4 & 0 & 0 & 0 & 0 & 0 \\ -w & 2w+2 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 2w+2 & w \\ 0 & 0 & 0 & 0 & 0 & 4 & w & -2w+2 \\ 0 & 0 & 0 & 0 & -2w+2 & -w & 4 & 0 \\ 0 & 0 & 0 & 0 & -w & 2w+2 & 0 & 4 \end{pmatrix}$$

With $w = i\sqrt{2}$ and automorphism group order $\sharp\,\mathrm{Aut}(G01) = 29491200$. The lattice $G01$ corresponds to the real $E_8 \oplus E_8$ lattice.

$$G02 := \frac{1}{2}\begin{pmatrix} 8 & 0 & 2w+2 & -w-2 & w-4 & -2w+2 & 4 & -4w-4 \\ 0 & 4 & w & 2 & 0 & 0 & 0 & 0 \\ -2w+2 & -w & 4 & 0 & 0 & 0 & 0 & 0 \\ w-2 & 2 & 0 & 4 & -3w & w & 0 & 4 \\ -w-4 & 0 & 0 & 3w & 8 & w-2 & w-2 & 5w+2 \\ 2w+2 & 0 & 0 & -w & -w-2 & 4 & 3w+2 & -4w+2 \\ 4 & 0 & 0 & 0 & -w-2 & -3w+2 & 8 & -4w-4 \\ 4w-4 & 0 & 0 & 4 & -5w+2 & 4w+2 & 4w-4 & 12 \end{pmatrix}$$

With automorphism group order $\sharp\mathrm{Aut}(G02) = 147456$. The lattice $G02$ corresponds to the real $E_8 \oplus E_8$ lattice.

$$G03 := \frac{1}{2}\begin{pmatrix} 28 & -4w-16 & 2w+2 & -w+4 & 2w-22 & -2w+16 & -8w-8 & -20w-8 \\ 4w-16 & 16 & w & 2w-2 & -5w+14 & 4w-10 & 4w+8 & 12w+12 \\ -2w+2 & -w & 4 & 0 & 0 & 0 & 0 & 0 \\ w+4 & -2w-2 & 0 & 4 & -2w-6 & 2 & -4w & -4w \\ -2w-22 & 5w+14 & 0 & 2w-6 & 24 & -14 & 10w+8 & 19w+4 \\ 2w+16 & -4w-10 & 0 & 2 & -14 & 12 & -5w-4 & -14w-2 \\ 8w-8 & -4w+8 & 0 & 4w & -10w+8 & 5w-4 & 12 & 4w+16 \\ 20w-8 & -12w+12 & 0 & 4w & -19w+4 & 14w-2 & -4w+16 & 36 \end{pmatrix}$$

With automorphism group order $\sharp\mathrm{Aut}(G03) = 786432$. The lattice $G03$ corresponds to the real D_{16}^{+} lattice.

$$G04 := \frac{1}{2}\begin{pmatrix} 8 & -4 & -2 & -w-2 & 2w+2 & w+4 & -4w-4 & 0 \\ -4 & 12 & 3w+2 & 2 & -w-10 & -3w-6 & 8w+8 & -4w-4 \\ -2 & -3w+2 & 4 & 0 & 3w-4 & w-4 & 8 & -4 \\ w-2 & 2 & 0 & 4 & -3w & w & 0 & 4 \\ -2w+2 & w-10 & -3w-4 & 3w & 20 & 3w+6 & -9w-12 & 9w+6 \\ -w+4 & 3w-6 & -w-4 & -w & -3w+6 & 8 & -3w-12 & 6 \\ 4w-4 & -8w+8 & 8 & 0 & 9w-12 & 3w-12 & 24 & -12 \\ 0 & 4w-4 & -4 & 4 & -9w+6 & 6 & -12 & 12 \end{pmatrix}$$

With automorphism group order $\sharp\mathrm{Aut}(G04) = 92160$. The lattice $G04$ corresponds to the real $E_8 \oplus E_8$ lattice.

$$G05 := \frac{1}{2}\begin{pmatrix} 8 & 4 & -w-4 & -w-6 & 2w+2 & w & -4w-4 & 0 \\ 4 & 12 & -2w-10 & -6 & w+10 & 3w-2 & -8w-8 & 4w+4 \\ w-4 & 2w-10 & 16 & -4w+10 & w-16 & -4w+2 & 7w+14 & -6w-6 \\ w-6 & -6 & 4w+10 & 12 & -4w-10 & -2w+2 & 8w+8 & -4w \\ -2w+2 & -w+10 & -w-16 & 4w-10 & 20 & 4w-4 & -9w-12 & 9w+6 \\ -w & -3w-2 & 4w+2 & 2w+2 & -4w-4 & 4 & 5w-4 & -4w+2 \\ 4w-4 & 8w-8 & -7w+14 & -8w+8 & 9w-12 & -5w-4 & 24 & -12 \\ 0 & -4w+4 & 6w-6 & 4w & -9w+6 & 4w+2 & -12 & 12 \end{pmatrix}$$

With automorphism group order $\sharp\mathrm{Aut}(G05) = 43008$. The lattice $G05$ corresponds to the real D_{16}^{+} lattice.

$$G06 := \frac{1}{2}\begin{pmatrix} 24 & -10w+2 & 2w+14 & -w+10 & 6 & -5w-8 & 12w+12 & -8w-8 \\ 10w+2 & 28 & 4w+6 & 4w+8 & -w & -4w+4 & 9w-10 & -7w+12 \\ -2w+14 & -4w+6 & 12 & 8 & -w+6 & -3w-6 & 8w+8 & -4w-4 \\ w+10 & -4w+8 & 8 & 12 & -4w+6 & -2w-6 & 8w+8 & -4w \\ 6 & w & w+6 & 4w+6 & 12 & -3w-6 & 7w+4 & w-2 \\ 5w-8 & 4w+4 & 3w-6 & 2w-6 & 3w-6 & 8 & -3w-12 & 6 \\ -12w+12 & -9w-10 & -8w+8 & -8w+8 & -7w+4 & 3w-12 & 24 & -12 \\ 8w-8 & 7w+12 & 4w-4 & 4w & -w-2 & 6 & -12 & 12 \end{pmatrix}$$

With automorphism group order $\sharp\mathrm{Aut}(G06) = 5160960$. The lattice $G06$ corresponds to the real $E_8 \oplus E_8$ lattice.

The mass of the genus is $5833/137625600$.

A.9 The rank 8 ϑ-lattice over $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$

$$G01 := \frac{1}{3}\begin{pmatrix} 6 & 0 & 2w & 2w & 0 & 0 & 0 & 0 \\ 0 & 6 & 2w & -2w & 0 & 0 & 0 & 0 \\ -2w & -2w & 6 & 0 & 0 & 0 & 0 & 0 \\ -2w & 2w & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 2w & 2w \\ 0 & 0 & 0 & 0 & 0 & 6 & 2w & -2w \\ 0 & 0 & 0 & 0 & -2w & -2w & 6 & 0 \\ 0 & 0 & 0 & 0 & -2w & 2w & 0 & 6 \end{pmatrix}$$

With $w = i\sqrt{3}$ and automorphism group order $\sharp\,\mathrm{Aut}(G01) = 155520$. The lattice $G01$ corresponds to the real $E_8 \oplus E_8$ lattice.

A.10 The three rank 8 ϑ-lattices over $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$

$$G01 := \frac{1}{7}\begin{pmatrix} 14 & 0 & 6w & 4w & 0 & 0 & 0 & 0 \\ 0 & 14 & 4w & -6w & 0 & 0 & 0 & 0 \\ -6w & -4w & 28 & 0 & 0 & 0 & 0 & 0 \\ -4w & 6w & 0 & 28 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 14 & 0 & 6w & 4w \\ 0 & 0 & 0 & 0 & 0 & 14 & 4w & -6w \\ 0 & 0 & 0 & 0 & -6w & -4w & 28 & 0 \\ 0 & 0 & 0 & 0 & -4w & 6w & 0 & 28 \end{pmatrix}$$

With $w = i\sqrt{7}$ and automorphism group order $\sharp\,\mathrm{Aut}(G01) = 50803200$. The lattice $G01$ corresponds to the real $E_8 \oplus E_8$ lattice.

$$G02 := \frac{1}{7}\begin{pmatrix} 14 & 0 & 6w & w-7 & 0 & 0 & 0 & 0 \\ 0 & 28 & 4w & -3w+21 & 0 & 7w+7 & 4w & -6w \\ -6w & -4w & 28 & 0 & 0 & 0 & 0 & 0 \\ -w-7 & 3w+21 & 0 & 28 & -w-7 & 6w & 0 & -7w+7 \\ 0 & 0 & 0 & w-7 & 14 & 0 & 6w & 4w \\ 0 & -7w+7 & 0 & -6w & 0 & 28 & 2w+14 & -3w-21 \\ 0 & -4w & 0 & 0 & -6w & -2w+14 & 28 & 0 \\ 0 & 6w & 0 & 7w+7 & -4w & 3w-21 & 0 & 28 \end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(G02) = 225792$. The lattice $G02$ corresponds to the real D_{16}^{+} lattice.

$$G03 := \frac{1}{7}\begin{pmatrix} 28 & 0 & 3w-21 & 2w-14 & 7w+7 & 0 & 6w & 4w \\ 0 & 28 & 2w-14 & -3w+21 & 0 & 7w+7 & 4w & -6w \\ -3w-21 & -2w-14 & 28 & 0 & -6w & -4w & -7w+7 & 0 \\ -2w-14 & 3w+21 & 0 & 28 & -4w & 6w & 0 & -7w+7 \\ -7w+7 & 0 & 6w & 4w & 28 & 0 & 3w+21 & 2w+14 \\ 0 & -7w+7 & 4w & -6w & 0 & 28 & 2w+14 & -3w-21 \\ -6w & -4w & 7w+7 & 0 & -3w+21 & -2w+14 & 28 & 0 \\ -4w & 6w & 0 & 7w+7 & -2w+14 & 3w-21 & 0 & 28 \end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(G03) = 311040$. The lattice $G03$ corresponds to the real $E_8 \oplus E_8$ lattice.

The mass of the genus is $73/9525600$.

A.11 The seven rank 8 ϑ-lattices over $\mathcal{O}_{\mathbb{Q}(\sqrt{-11})}$

$$G01 := \frac{1}{11}\begin{pmatrix} 22 & 0 & 6w & 2w & 0 & 0 & 0 & 0 \\ 0 & 22 & 2w & -6w & 0 & 0 & 0 & 0 \\ -6w & -2w & 22 & 0 & 0 & 0 & 0 & 0 \\ -2w & 6w & 0 & 22 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 22 & 0 & 6w & 2w \\ 0 & 0 & 0 & 0 & 0 & 22 & 2w & -6w \\ 0 & 0 & 0 & 0 & -6w & -2w & 22 & 0 \\ 0 & 0 & 0 & 0 & -2w & 6w & 0 & 22 \end{pmatrix}$$

With $w = i\sqrt{11}$ and automorphism group order $\sharp\,\mathrm{Aut}(G01) = 7372800$. The lattice $G01$ corresponds to the real $E_8 \oplus E_8$ lattice.

$$G02 := \frac{1}{11}\begin{pmatrix} 44 & 0 & 6w & 4w & -2w+22 & 6w+22 & -22 & 11w+11 \\ 0 & 22 & 2w & -w+11 & 0 & 0 & 0 & 0 \\ -6w & -2w & 22 & 0 & 0 & 0 & 0 & 0 \\ -4w & w+11 & 0 & 22 & -5w-11 & -3w+11 & 0 & 22 \\ 2w+22 & 0 & 0 & 5w-11 & 44 & 8w+22 & 4w-22 & 12w \\ -6w+22 & 0 & 0 & 3w+11 & -8w+22 & 44 & 8w-22 & 8w+44 \\ -22 & 0 & 0 & 0 & -4w-22 & -8w-22 & 44 & -11w-11 \\ -11w+11 & 0 & 0 & 22 & -12w & -8w+44 & 11w-11 & 66 \end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(G02) = 9216$. The lattice G corresponds to the real $E_8 \oplus E_8$ lattice.

$$G03 := \frac{1}{11}\begin{pmatrix} 132 & 66 & -10w-22 & 4w & 2w+44 & 14w-22 & -33w-33 & 22w+22 \\ 66 & 66 & -7w-11 & -w+11 & 4w+22 & 8w-22 & -22w-22 & 11w+11 \\ 10w-22 & 7w-11 & 22 & 0 & 3w-11 & -5w-11 & 44 & -22 \\ -4w & w+11 & 0 & 22 & -5w-11 & -3w+11 & 0 & 22 \\ -2w+44 & -4w+22 & -3w-11 & 5w-11 & 44 & 12w & -9w-33 & 12w \\ -14w-22 & -8w-22 & 5w-11 & 3w+11 & -12w & 44 & 15w-33 & -3w+33 \\ 33w-33 & 22w-22 & 44 & 0 & 9w-33 & -15w-33 & 132 & -66 \\ -22w+22 & -11w+11 & -22 & 22 & -12w & 3w+33 & -66 & 66 \end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(G03) = 10080$. The lattice $G03$ corresponds to the real $E_8 \oplus E_8$ lattice.

$$G04 := \frac{1}{11}\begin{pmatrix} 132 & -66 & -21w-33 & 4w & 2w+44 & 14w-22 & -33w-33 & 22w+22 \\ -66 & 66 & 15w+11 & -w+11 & -4w-22 & -8w+22 & 22w+22 & -11w-11 \\ 21w-33 & -15w+11 & 66 & 0 & 6w-22 & -10w-22 & 88 & -44 \\ -4w & w+11 & 0 & 22 & -5w-11 & -3w+11 & 0 & 22 \\ -2w+44 & 4w-22 & -6w-22 & 5w-11 & 44 & 12w & -9w-33 & 12w \\ -14w-22 & 8w+22 & 10w-22 & 3w+11 & -12w & 44 & 15w-33 & -3w+33 \\ 33w-33 & -22w+22 & 88 & 0 & 9w-33 & -15w-33 & 132 & -66 \\ -22w+22 & 11w-11 & -44 & 22 & -12w & 3w+33 & -66 & 66 \end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(G04) = 1200$. The lattice $G04$ corresponds to the real $E_8 \oplus E_8$ lattice.

$$G05 := \frac{1}{11}\begin{pmatrix} 44 & 22 & 2w-22 & 4w+22 & -2w+22 & 6w+22 & -11w-11 & 11w+11 \\ 22 & 66 & -2w-22 & -w+55 & 4w+22 & 8w+22 & -22w-22 & 11w+11 \\ -2w-22 & 2w-22 & 22 & w-11 & -2w-22 & -5w-11 & 8w+22 & -6w \\ -4w+22 & w+55 & -w-11 & 66 & -w+11 & 5w+33 & -22w-22 & 11w+33 \\ 2w+22 & -4w+22 & 2w-22 & w+11 & 44 & 8w+22 & -9w-33 & 12w \\ -6w+22 & -8w+22 & 5w-11 & -5w+33 & -8w+22 & 44 & -7w-55 & 8w+44 \\ 11w-11 & 22w-22 & -8w+22 & 22w-22 & 9w-33 & 7w-55 & 132 & -66 \\ -11w+11 & -11w+11 & 6w & -11w+33 & -12w & -8w+44 & -66 & 66 \end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(G05) = 672$. The lattice $G05$ corresponds to the real D_{16}^{+} lattice.

$$G06 := \frac{1}{11}\begin{pmatrix} 132 & -11w-77 & 6w & 19w-11 & -14w+22 & -2w+66 & 22w+44 & 44w-22 \\ 11w-77 & 88 & 2w & -13w-11 & 12w & 8w-44 & -11w-55 & -33w-33 \\ -6w & -2w & 22 & 0 & 0 & 0 & 0 & 0 \\ -19w-11 & 13w-11 & 0 & 44 & -w-33 & -11w-11 & -11w+33 & 88 \\ 14w+22 & -12w & 0 & w-33 & 44 & 8w+22 & 16w-22 & 6w-66 \\ 2w+66 & -8w-44 & 0 & 11w-11 & -8w+22 & 44 & 16w+22 & 26w-22 \\ -22w+44 & 11w-55 & 0 & 11w+33 & -16w-22 & -16w+22 & 88 & 22w+88 \\ -44w-22 & 33w-33 & 0 & 88 & -6w-66 & -26w-22 & -22w+88 & 198 \end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(G06) = 10560$. The lattice $G06$ corresponds to the real D_{16}^{+} lattice.

$$G07 := \frac{1}{11}\begin{pmatrix} 44 & 8w & 6w-22 & 4w+22 & -6w & -2w & 11w+11 & 0 \\ -8w & 176 & -w-55 & -13w+77 & -3w-77 & -20w-44 & 30w+44 & -28w \\ -6w-22 & w-55 & 66 & -44 & 4w+22 & 8w+22 & -22w-22 & 11w+11 \\ -4w+22 & 13w+77 & -44 & 66 & -9w-33 & -11w-11 & 22w+22 & -11w+11 \\ 6w & 3w-77 & -4w+22 & 9w-33 & 44 & 8w+22 & -9w-33 & 12w \\ 2w & 20w-44 & -8w+22 & 11w-11 & -8w+22 & 44 & -7w-55 & 8w+44 \\ -11w+11 & -30w+44 & 22w-22 & -22w+22 & 9w-33 & 7w-55 & 132 & -66 \\ 0 & 28w & -11w+11 & 11w+11 & -12w & -8w+44 & -66 & 66 \end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(G07) = 11520$. The lattice $G07$ corresponds to the real $E_8 \oplus E_8$ lattice.

The mass of the genus is $20519/7569408$.

A.12 The five rank 12 ϑ-lattices over $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$

$$Gram02 := \frac{1}{3}\begin{pmatrix}
6 & -w+3 & -w+3 & w+3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
w+3 & 6 & w+3 & w+3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
w+3 & -w+3 & 6 & w+3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-w+3 & -w+3 & -w+3 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & w-3 & -w+3 & -w+3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -w-3 & 6 & -w-3 & w-3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & w+3 & w-3 & 6 & -w+3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & w+3 & -w-3 & w+3 & 6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & w+3 & w-3 & w-3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -w+3 & 6 & w-3 & w-3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -w-3 & -w-3 & 6 & -w+3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -w-3 & -w-3 & w+3 & 6
\end{pmatrix}$$

With $w = i\sqrt{3}$ and automorphism group order $\sharp\,\mathrm{Aut}(Gram02) = 2^{22}3^{16}5^{3}$. The lattice $Gram02$ corresponds to the real $3E_8$ lattice.

$$Gram11 := \frac{1}{3}\begin{pmatrix}
6 & w+3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -w+3 & 2w & 2w \\
-w+3 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2w & 2w & 2w \\
0 & 0 & 6 & -w+3 & 0 & 0 & 0 & 0 & 0 & 0 & -w-3 & 0 \\
0 & 0 & w+3 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & w-3 & -w+3 & 0 & 0 & -w-3 & w-3 & -w-3 \\
0 & 0 & 0 & 0 & -w-3 & 6 & -w-3 & 0 & 0 & w+3 & -w+3 & w+3 \\
0 & 0 & 0 & 0 & w+3 & w-3 & 6 & 0 & 0 & -w-3 & w-3 & -w-3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & -w+3 & -w-3 & 0 & -w-3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & w+3 & 6 & 0 & 0 & -w-3 \\
w+3 & 2w & 0 & 0 & w-3 & -w+3 & w-3 & w-3 & 0 & 12 & -2w & w+3 \\
-2w & -2w & w-3 & 0 & -w-3 & w+3 & -w-3 & 0 & 0 & 2w & 12 & 2w+6 \\
-2w & -2w & 0 & 0 & w-3 & -w+3 & w-3 & w-3 & w-3 & -w+3 & -2w+6 & 12
\end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(Gram11) = 2^{16}3^{17}$. The lattice $Gram11$ corresponds to the real $4E_6$ lattice.

$$Gram18 := \frac{1}{3}\begin{pmatrix}
6 & w+3 & 0 & 0 & 0 & 0 & 0 & 0 & -w-3 & -w-3 & w+3 & -w+3 \\
-w+3 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & -w-3 & -w-3 & -w+3 & -2w \\
0 & 0 & 6 & w+3 & 0 & 0 & 0 & 0 & w-3 & -w-3 & w-3 & 0 \\
0 & 0 & -w+3 & 6 & 0 & 0 & 0 & 0 & w-3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & w+3 \\
0 & 0 & 0 & 0 & 0 & 6 & -w+3 & 0 & w-3 & 2w & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & w+3 & 6 & 0 & w-3 & 0 & w+3 & -w-3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & w-3 & -w+3 \\
w-3 & w-3 & -w-3 & -w-3 & 0 & -w-3 & -w-3 & 0 & 12 & w+3 & 0 & 2w \\
w-3 & w-3 & w-3 & 0 & 0 & -2w & 0 & 0 & -w+3 & 12 & 0 & 0 \\
-w+3 & w+3 & -w-3 & 0 & 0 & 0 & -w+3 & -w-3 & 0 & 0 & 12 & -w-3 \\
w+3 & 2w & 0 & 0 & -w+3 & 0 & w-3 & w+3 & -2w & 0 & w-3 & 12
\end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(Gram18) = 2^{21}3^{9}\cdot 5$. The lattice $Gram18$ corresponds to the real $6D_4$ lattice.

$$Gram21 := \frac{1}{3}\begin{pmatrix}
6 & 0 & 0 & 0 & 0 & w+3 & w+3 & w+3 & w+3 & w+3 & 0 & 0 \\
0 & 6 & 0 & 0 & 0 & w+3 & w+3 & -w+3 & w-3 & 0 & w+3 & -w+3 \\
0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & -w-3 & -w-3 & 0 \\
0 & 0 & 0 & 6 & 0 & -w-3 & 0 & w+3 & w+3 & 2w & -w-3 & -w-3 \\
0 & 0 & 0 & 0 & 6 & -w+3 & w-3 & -w+3 & 0 & w-3 & 0 & -w-3 \\
-w+3 & -w+3 & 0 & w-3 & w+3 & 12 & -w+3 & -w+3 & w-3 & -w-3 & 6 & -3w+3 \\
-w+3 & -w+3 & 0 & 0 & -w-3 & w+3 & 12 & -2w & 2w & 6 & w+3 & w+3 \\
-w+3 & w+3 & 0 & -w+3 & w+3 & w+3 & 2w & 12 & 0 & 2w & w-3 & -w-3 \\
-w+3 & -w-3 & 0 & -w+3 & 0 & -w-3 & -2w & 0 & 12 & 6 & -w-3 & w-3 \\
-w+3 & 0 & w-3 & -2w & -w-3 & w-3 & 6 & -2w & 6 & 12 & 2w & 4w \\
0 & -w+3 & w-3 & w-3 & 0 & 6 & -w+3 & -w-3 & w-3 & -2w & 12 & -2w+6 \\
0 & w+3 & 0 & w-3 & w-3 & 3w+3 & -w+3 & w-3 & -w-3 & -4w & 2w+6 & 12
\end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(Gram21) = 2^7 3^{15} \cdot 5 \cdot 11.$ The lattice $Gram21$ corresponds to the real $12A_2$ lattice.

$$Gram24 := \frac{1}{3} \begin{pmatrix} 12 & -6 & -2w+6 & w-3 & -w-3 & -2w-6 & -6 & w-3 & w+3 & 0 & 2w & 2w-6 \\ -6 & 12 & 2w-6 & -w-3 & 6 & 2w+6 & -w+3 & -w-3 & -2w-6 & 3w-3 & -w+3 & -w+3 \\ 2w+6 & -2w-6 & 12 & w-3 & -w-3 & -3w-3 & w-3 & 0 & 2w & -2w & 2w-6 & 2w-6 \\ -w-3 & w-3 & -w-3 & 12 & w+3 & 0 & 6 & 0 & 6 & 0 & -w-3 & 6 \\ w-3 & 6 & w-3 & -w+3 & 12 & 6 & -w+3 & -3w-3 & -w-3 & 3w-3 & -w-3 & -w+3 \\ 2w-6 & -2w+6 & 3w-3 & 0 & 6 & 12 & -w+3 & -2w & w-3 & 2w & -2w & -3w+3 \\ -6 & w+3 & -w-3 & 6 & w+3 & w+3 & 12 & -w-3 & w+3 & w-3 & 0 & 6 \\ -w-3 & w-3 & 0 & 0 & 3w-3 & 2w & w-3 & 12 & -w-3 & -2w & -2w & 0 \\ -w+3 & 2w-6 & -2w & 6 & w-3 & -w-3 & -w+3 & w-3 & 12 & -w+3 & 0 & 0 \\ 0 & -3w-3 & 2w & 0 & -3w-3 & -2w & -w-3 & 2w & w+3 & 12 & -2w & -w-3 \\ -2w & w+3 & -2w-6 & w-3 & w-3 & 2w & 0 & 2w & 0 & 2w & 12 & -w+3 \\ -2w-6 & w+3 & -2w-6 & 6 & w+3 & 3w+3 & 6 & 0 & 0 & w-3 & w+3 & 12 \end{pmatrix}$$

With automorphism group order $\sharp\,\mathrm{Aut}(Gram24) = 2^{14} 3^8 5^2 \cdot 7 \cdot 11 \cdot 13.$ The lattice $Gram24$ corresponds to the real Leech lattice.

The mass of the genus is $1032508093/67774344416722944000$.

A.13 The non-isometric rank 16 ϑ-lattices over $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$

From the isometry classes of the rank 16 ϑ-lattices over $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$ we give the orders of the automorphism groups, the corresponding real root lattices and additional information. The abbreviation "f.r." stands for full rank and indicates if the corresponding real root lattice has full rank. Generators for the automorphism group and Gram matrices of the ϑ-lattices can be found at [MathA].

latt.nr.	# Aut	**root system**	**f.r.**	**comment**
1	1403964840984182784000	4E8	+	lattice from ternary code, decomposable
2	40122452017152	4D4+2E6		-
3	2742118830047232	4A2+4E6	+	lattice from ternary code
4	1316217038422671360	4E6+E8	+	lattice from ternary code, decomposable
5	443823666757632	8D4	+	lattice from quaternionic matrix group, den. L8P8
6	32097961613721600	6D4+E8	+	decomposable
7	1451188224	4A2+3D4		-
8	15479341056	4A2+4D4		-
9	48977602560	6A2+D4+E6		-
10	134369280	2A2+2D4		-
11	1277045637120	D4+E6		-
12	8707129344	8A2+2D4		-
13	107495424	4A2+2D4		-
14	22674816	6A2+D4		-
15	31345665638	4A2+3D4+E6		-
16	423263232	8A2		-
17	302330880	6A2+2D4		-
18	825564856320	6D4		-
19	408146688	10A2		-
20	1679616	6A2		-
21	4478976	4A2+D4		-
22	7644119040	2D4		lattice from quaternionic matrix group, den. L8P2
23	161243136	4A2		-
24	82556485632	4D4		lattice from quaternionic matrix group, den. L8P4
25	9795520512	4A2+E6		-
26	313456656384	4A2+3D4+E6		-
27	71409344532480	10A2+2E6	+	lattice from ternary code
28	1851353376768	16A2	+	lattice from ternary code
29	303216721920	$\varnothing$		lattice from quaternionic matrix group, den. LBW32,L32ss
30	15710055797145600	12A2+E8	+	lattice from ternary code, decomposable
31	52907904	7A2+D4		-
32	314928	4A2		-
33	2519424	3A2+D4		-
34	1710720	2A2+D4		-
35	8398080	5A2+D4		-
36	3265920	A2+D4		-
37	186624	2A2		-
38	15552	3A2		-
39	139968	5A2		-

latt.nr.	# Aut	**root system**	**f.r.**	**comment**
40	58320	3A2		-
41	2426112	A2+D4		-
42	108864	A2		-
43	1451188224	4A2+3D4		-
44	18144	2A2		-
45	3888	A2		-
46	139968	4A2		-
47	2916	A2		-
48	68024448	4A2		-
49	69984	4A2		-
50	1836660096	9A2+D4		-
51	2834352	7A2		-
52	629856	6A2		-
53	16200	2A2		-
54	418360150720512000	E8		decomposable
55	21427701120	7A2+E6		-
56	1990656	D4		-
57	4608	∅		-
58	4199040	4A2		-
59	25920	2A2		-
60	41472	2A2		-
61	18144	2A2		-
62	7558272	8A2		-
63	22448067840	A2+E6		-
64	113374080	4A2		-
65	1399680	4A2		-
66	387072	∅		-
67	2592	∅		-
68	29376	∅		-
69	9289728	∅		lattice from quaternionic matrix group, den. L32
70	15552000	∅		lattice from quaternionic matrix group, den. L32s
71	87091200	D4		lattice from quaternionic matrix group, den. L8P
72	656916480	2D4		-
73	11604018486528	13A2+E6	+	lattice from ternary code
74	10368	∅		-
75	8064	∅		-
76	5760	∅		-
77	606528	2A2		-
78	660290641920	D4		-
79	1658880	∅		-

A.14 The non-isometric rank 12 ϑ-lattices over $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$

The following tabular contains an enumeration of the alle the non-isometric rank 12 ϑ-lattices over $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$, ordered via automorphism group order and root system. Furthermore the column „sc“ indicates if a lattice is isomorphic to its Galois conjugated lattice $(\mathrm{Re}, -\mathrm{Im})$, for details and the explicit list, given as Gram matrices, see again [MathA].

latt.nr.	# Aut	**root system**	**sc**	**latt.nr.**	# Aut	**root system**	**sc**
1	768144384000	3E8	+	2	1567641600	3E8	+
3	1137991680	E8+D16	+	4	227598336	3D8	
5	227598336	3D8		6	5419008	2D12	+
7	4667544	4A6	+	8	3359232	4E6	+
9	2419200	∅	+	10	870912	2E7+D10	+
11	846720	∅	+	12	774144	3D8	
13	774144	3D8		14	592704	A15+D9	+
15	497664	6D4		16	497664	6D4	
17	331776	6D4	+	18	174960	12A2	+
19	111132	3A8	+	20	84672	2A7+2D5	+
21	82944	3D8	+	22	15876	2A12	+
23	15876	2A9+D6	+	24	15600	∅	+
25	14400	∅		26	14400	∅	
27	13824	∅	+	28	13824	4E6	+
29	10368	6D4	+	30	9216	4D6	
31	9216	6D4	+	32	9216	4D6	
33	8064	24A1		34	8064	24A1	
35	7056	∅	+	36	5760	∅	+
37	4608	24A1		38	4608	6D4	+
39	4608	24A1		40	4368	∅	+
41	3888	8D3		42	3888	8D3	+
43	3888	8D3		44	3456	4D6	+
45	3402	4A6		46	3402	4A6	
47	3024	E6+A11+D7	+	48	2592	4A5+D4	+
49	1944	6D4	+	50	1728	2A7+2D5	+
51	1728	2A7+2D5		52	1728	2A7+2D5	
53	1440	12A2	+	54	1296	4D6	+
55	1296	8D3	+	56	1008	2A7+2D5	+
57	960	24A1		58	960	24A1	
59	882	4A6	+	60	864	12A2	+
61	864	6D4	+	62	864	12A2	+
63	756	2A9+D6	+	64	756	2A7+2D5	+
65	756	3A8	+	66	648	6A4	
67	648	6A4		68	648	4A6	
69	648	6A4		70	648	4A6	
71	648	4A6	+	72	648	6A4	
73	576	24A1		74	576	24A1	
75	432	6D4	+	76	384	24A1	
77	384	6D4		78	384	6D4	
79	384	24A1		80	360	12A2	

latt.nr.	#Aut	**root system**	**sc**	**latt.nr.**	#Aut	**root system**	**sc**
81	360	12A2		82	336	24A1	
83	336	24A1		84	324	6A4	
85	324	2A9+D6	+	86	324	6A4	
87	288	8D3		88	288	6D4	+
89	288	8D3		90	240	24A1	
91	240	24A1	+	92	240	24A1	
93	216	8D3	+	94	216	6A4	
95	216	6A4		96	192	8D3	
97	192	8D3	+	98	192	8D3	
99	162	3A8	+	100	160	24A1	+
101	144	6D4	+	102	144	12A2	+
103	144	6D4	+	104	144	6D4	+
105	144	12A2	+	106	144	24A1	+
107	144	12A2	+	108	128	8D3	+
109	120	24A1	+	110	120	12A2	+
111	120	6A4		112	120	6A4	
113	108	4A6		114	108	4A5+D4	
115	108	2A7+2D5		116	108	4A5+D4	
117	108	2A7+2D5		118	108	4A6	
119	96	4A5+D4	+	120	96	24A1	
121	96	12A2	+	122	96	24A1	
123	96	2A7+2D5	+	124	96	24A1	
125	96	8D3	+	126	96	24A1	
127	72	24A1		128	72	12A2	+
129	72	12A2		130	72	4A6	
131	72	4A5+D4	+	132	72	8D3	+
133	72	4A5+D4		134	72	4A5+D4	+
135	72	8D3	+	136	72	2A7+2D5	+
137	72	6D4	+	138	72	12A2	+
139	72	4A5+D4		140	72	4A6	
141	72	8D3	+	142	72	12A2	
143	72	24A1		144	64	8D3	+
145	60	12A2		146	60	12A2	
147	60	12A2		148	60	12A2	
149	60	12A2		150	60	12A2	
151	56	4A6	+	152	54	4A5+D4	+
153	48	12A2	+	154	48	24A1	
155	48	24A1		156	48	12A2	+
157	48	12A2	+	158	48	6D4	+
159	48	12A2		160	48	24A1	
161	48	24A1		162	48	24A1	+
163	48	12A2	+	164	48	24A1	
165	48	24A1		166	48	24A1	+
167	48	12A2		168	40	6A4	+
169	36	6A4		170	36	6A4	+
171	36	4A6	+	172	36	8D3	
173	36	8D3		174	36	6A4	+
175	36	8D3	+	176	36	8D3	
177	36	2A7+2D5	+	178	36	8D3	
179	36	4A6	+	180	36	8D3	
181	36	8D3		182	36	6A4	+
183	36	8D3		184	36	6A4	+
185	36	2A7+2D5	+	186	36	8D3	
187	36	2A7+2D5	+	188	36	6A4	
189	32	24A1		190	32	24A1	+

latt.nr.	#Aut	**root system**	**sc**	**latt.nr.**	#Aut	**root system**	**sc**
191	32	8D3		192	32	8D3	
193	32	24A1		194	24	24A1	+
195	24	8D3		196	24	6A4	
197	24	4A5+D4		198	24	4A5+D4	
199	24	6A4		200	24	12A2	
201	24	8D3		202	24	8D3	
203	24	12A2	+	204	24	6A4	+
205	24	12A2		206	24	12A2	
207	24	6A4		208	24	6A4	
209	24	12A2		210	24	12A2	+
211	24	6A4	+	212	24	8D3	+
213	24	4A5+D4	+	214	24	8D3	+
215	24	12A2		216	24	8D3	
217	24	12A2		218	20	24A1	
219	20	24A1		220	18	12A2	+
221	18	4A5+D4	+	222	18	4A5+D4	
223	18	8D3	+	224	18	4A5+D4	
225	16	12A2		226	16	12A2	+
227	16	8D3		228	16	8D3	
229	16	8D3		230	16	8D3	
231	16	8D3		232	16	8D3	+
233	16	8D3		234	16	8D3	
235	16	8D3	+	236	16	24A1	+
237	16	8D3		238	16	24A1	+
239	16	8D3		240	16	8D3	
241	16	8D3		242	16	8D3	
243	16	12A2		244	12	4A5+D4	+
245	12	6A4		246	12	12A2	+
247	12	6A4		248	12	12A2	
249	12	6A4		250	12	12A2	
251	12	6A4		252	12	6A4	
253	12	12A2		254	12	12A2	+
255	12	8D3	+	256	12	6A4	
257	12	12A2		258	12	6A4	+
259	12	6A4		260	12	8D3	+
261	12	24A1		262	12	6A4	
263	12	8D3		264	12	12A2	
265	12	12A2		266	12	6A4	
267	12	6A4		268	12	8D3	+
269	12	4A6		270	12	24A1	+
271	12	6A4		272	12	12A2	
273	12	4A6		274	12	8D3	+
275	12	24A1		276	12	8D3	
277	12	6A4		278	12	6A4	
279	12	8D3		280	12	4A5+D4	+
281	12	4A5+D4		282	12	12A2	
283	12	4A5+D4	+	284	12	8D3	
285	12	6A4	+	286	12	6A4	
287	12	4A5+D4		288	12	6A4	+
289	12	6A4		290	12	6A4	
291	12	12A2	+	292	8	8D3	+
293	8	8D3		294	8	4A5+D4	+
295	8	12A2		296	8	8D3	
297	8	8D3		298	8	8D3	
299	8	12A2		300	8	12A2	

latt.nr.	# Aut	**root system**	**sc**	**latt.nr.**	# Aut	**root system**	**sc**
301	8	8D3		302	8	24A1	+
303	8	12A2		304	8	12A2	
305	8	12A2		306	8	8D3	+
307	8	8D3	+	308	8	4A6	+
309	8	8D3	+	310	8	12A2	+
311	8	8D3		312	8	4A5+D4	+
313	8	12A2		314	8	12A2	
315	8	8D3	+	316	8	6A4	+
317	8	12A2		318	8	8D3	+
319	8	8D3	+	320	8	24A1	
321	8	8D3	+	322	8	12A2	
323	8	8D3		324	8	12A2	
325	8	12A2		326	8	12A2	
327	8	24A1	+	328	8	24A1	
329	8	12A2		330	8	12A2	+
331	8	24A1	+	332	8	12A2	
333	8	4A5+D4	+	334	8	8D3	
335	8	24A1	+	336	8	12A2	
337	6	4A5+D4	+	338	6	6A4	+
339	6	8D3	+	340	6	4A5+D4	+
341	6	6A4	+	342	6	6A4	
343	6	6A4		344	4	6A4	+
345	4	8D3		346	4	8D3	
347	4	8D3		348	4	6A4	
349	4	8D3	+	350	4	12A2	+
351	4	4A5+D4	+	352	4	12A2	+
353	4	8D3		354	4	12A2	
355	4	6A4		356	4	12A2	
357	4	6A4		358	4	8D3	+
359	4	8D3		360	4	12A2	
361	4	8D3		362	4	12A2	
363	4	8D3	+	364	4	12A2	
365	4	12A2		366	4	12A2	
367	4	8D3		368	4	8D3	
369	4	12A2	+	370	4	8D3	+
371	4	12A2		372	4	8D3	
373	4	12A2		374	4	12A2	
375	4	8D3	+	376	4	8D3	
377	4	12A2		378	4	12A2	
379	4	12A2		380	4	12A2	
381	4	12A2	+	382	4	6A4	
383	4	6A4		384	4	12A2	+
385	4	8D3		386	4	12A2	
387	4	4A5+D4	+	388	4	8D3	
389	4	12A2		390	4	8D3	+
391	4	8D3	+	392	4	8D3	
393	4	6A4		394	4	12A2	
395	4	12A2	+	396	4	8D3	
397	4	8D3		398	4	6A4	+
399	4	8D3		400	4	6A4	+
401	4	8D3	+	402	4	6A4	+
403	4	8D3	+	404	4	6A4	
405	4	6A4		406	4	8D3	
407	4	8D3		408	4	8D3	

latt.nr.	#Aut	**root system**	**sc**	**latt.nr.**	#Aut	**root system**	**sc**
409	4	12A2	+	410	4	12A2	
411	4	12A2	+	412	4	8D3	+
413	4	6A4		414	4	12A2	+
415	4	8D3		416	4	12A2	+
417	4	12A2	+	418	4	4A5+D4	+
419	4	8D3		420	4	8D3	
421	4	6A4		422	4	12A2	
423	4	12A2		424	2	8D3	+
425	2	8D3	+	426	2	8D3	
427	2	8D3	+	428	2	6A4	+
429	2	6A4		430	2	8D3	
431	2	12A2		432	2	12A2	+
433	2	12A2	+	434	2	6A4	
435	2	8D3	+	436	2	8D3	+
437	2	12A2		438	2	8D3	
439	2	8D3		440	2	12A2	+
441	2	8D3		442	2	6A4	+
443	2	8D3		444	2	4A5+D4	+
445	2	8D3		446	2	8D3	
447	2	12A2		448	2	8D3	
449	2	12A2		450	2	6A4	
451	2	12A2		452	2	8D3	+
453	2	8D3		454	2	8D3	+
455	2	6A4		456	2	12A2	+
457	2	12A2		458	2	8D3	
459	2	8D3	+	460	2	8D3	
461	2	8D3	+	462	2	8D3	
463	2	6A4	+	464	2	8D3	

A.15 The Magma Code for neighbour stepping

Remark A.1 a) On the following pages you will find our implementation of the neighbourhood method. This implementation is much simpler than the implementation of [Sc1/2], as we only deal with class number $h_K = 1$.

b) The most time consuming part is the testing for isometry. Sometimes it can be useful to compute the order of the automorphism group as an invariant first. This has been implemented but is not contained in the following version.

c) Furthermore notice that the Hermitian Gram matrices have to be provided in the variable ”Erg” which is an ordered set. Additionally we use ”ErgEXT”, an ordered set, which contains lists which contain (i) the Hermitian Gram matrix, (ii) the Gram matrix of the associated $\mathbb{Z}$-module, (iii) the complex structure and (iv) some numbers of short vectors. In addition the number field and the complex $\mathcal{O}_K$ generator have to be provided ”`S<w>:=QuadraticField(-d);`”.

The code in MAGMA syntax:

```
/***------important parameters------***/
Dimension:=16;
IdealP:=1-w;
/***------important parameters------***/
b:=[0];
for j:=1 to Dimension-1 do
                        Append(~b,0);
                        end for;
SetMemoryLimit(50*1024^3);
IdealPP:=Integers() ! (IdealP*Conjugate(IdealP))[1];
divisor:=1;
if IdealPP mod 2 eq 0 then
                       divisor:=2;
                      end if;

/*--->additional procedures<---*/
helps:=function(n)      /*<-----------------------------*/
h:=[0];
for j:=1 to Dimension-2 do
                        Append(~h,0);
                        end for;
for t:=1 to (Dimension-2) do
                          h[t] :=n mod IdealPP;n:=ExactQuotient(n-h[t] ,IdealPP);
                          end for;
h[Dimension-1]:=n mod IdealPP;
return h;
end function;

vorz:=function(h,vz,e,fixvecnr)       /*<-----------------------------*/
for t:=1 to Dimension-3 do
                        h[t]:=h[t]*(-1)^(vz mod 2);vz:=ExactQuotient(vz-(vz mod 2),2);
                        end for;
h[Dimension-2]:=h[Dimension-2]*(-1)^(vz);
ewert:=h[Dimension-1]+1;Remove(~h,Dimension-1);
if e lt fixvecnr then
                 Insert(~h,e,ewert);
                 Insert(~h,fixvecnr,0);     /*fixvecnr hat keinen Einfluss*/
                else
```

```
                 Insert(~h,fixvecnr,0);
                 Insert(~h,e,ewert);
                 end if;
return h;
end function;

Umm:=function(g);        /*<-----------------------------*/
U:=MatrixRing(Integers(),Dimension) ! 1;
space:=KModule(Rationals(),Dimension);
g:=space ! g;
for idxx:=1 to Dimension do
                         if g[Dimension+1-idxx] ne 0 then
                                                            sbg:=Dimension+1-idxx;
                                                            end if;
                  end for;
for idxx:=sbg to (Dimension-1) do
                             tux,sux:=XGCD([Integers() ! g[idxx],Integers() ! g[Dimension]]);
                             V:=MatrixRing(Integers(),Dimension) ! 1;
                             V[Dimension,idxx]:=sux[1];V[Dimension,Dimension]:=sux[2];
                             V[idxx,idxx]:=ExactQuotient(-g[Dimension],tux);
                             V[idxx,Dimension]:=ExactQuotient(g[idxx],tux);
                             U:=U*Transpose(V);
                             g:=g * (Hom(space,space) ! Transpose(V));
                             end for;
return U;
end function;

swap:=function(lin,jin)        /*<-----------------------------*/
mat:=MatrixRing(Integers(),Dimension) ! 0;
for lauf:=1 to Dimension do
                         if lauf eq lin then mat[lin,jin]:=1; end if;
                         if lauf eq jin then mat[jin,lin]:=1; end if;
                         if (lauf ne jin) and (lauf ne lin) then mat[lauf,lauf]:=1; end if;
                         end for;
return mat;
end function;
/*--->additional procedures<---*/


rndzahler:=1000;
for lauff:=1 to rndzahler do               /*<----------------starts main loop*/
                   richtig:=false;
                   idgi:=Random(#Erg-1)+1;
                   while richtig eq false do
                                          zero:=true;
                                          for j:=1 to Dimension do
                                                                b[j]:=Random(IdealPP-1);
                                                                /*chooses adm. vector*/
                                                                if b[j] ne 0 then zero:=false; end if;
                                                                end for;
                                          if zero eq true then b[1]:=1; end if;
                                          gcdv:=XGCD(b);
                                          for j:=1 to Dimension do
                                                                b[j]:=ExactQuotient(b[j],gcdv);
                                                                end for;
                                          U:=Umm(b);
                                          START:=(U^-1)*Erg[idgi]*d*Transpose(U^-1);
                                          fixvecnr:=Dimension;
                                          wert:=START[fixvecnr,fixvecnr];
                                          rest:=(Integers() ! wert) mod (divisor*IdealPP);
                                          if rest eq 0 then
                                                       richtig:=true;
                                                       /*h(x,x) \in P\overline(P)???*/
                                                       end if;
                                          end while;
```

```
print "Starts with adm. vector:", b,".  Constr. neighb. of:",idgi; ; /* " ,Fixvecnr=",fixvecnr;*/

/********************determines rows/columns which will be mult. with the ideal*******************/
f:=0;
for s:=1 to Dimension do
                      if s ne fixvecnr then
                                       if START[fixvecnr,s] ne 0 then
                                                                 wert:=START[fixvecnr,s]*IdealP/IdealPP;
                                                                 if IsIntegral(wert[1]) eq false then
                                                                                                f:=s;
                                                                                                end if;
                                                                 if IsIntegral(wert[2]) eq false then
                                                                                                f:=s;
                                                                                                end if;
                                                                 end if;
                                end if;
               end for;
/********************determines rows/columns which will be mult. with the ideal*******************/

/********************bring matrix in shape to allow neighbour construction*******************/
for eintrag:=1 to Dimension do
                            if (eintrag ne f) and (eintrag ne fixvecnr) then
/**/
ganz:=false;
zaehler:=0;
wert:=0;
/**/print "e:",eintrag;
idd:=Minimum({f,eintrag});
if eintrag gt fixvecnr then
                       idd:=idd-1;
                       end if;
while ganz eq false do
                    hz:=helps(zaehler);
                    t:=Dimension-2;vmod:=t;
                    while (hz[t] eq 0) and (t ge 2) do
                                                  t:=t-1;
                                                  vmod:=t;
                                                  end while;
                    vz:=1;
                    while (vz le 2^vmod) and (ganz eq false) do
                            h:=vorz(hz,vz,eintrag,fixvecnr);
                            wert:=0;
                            for idG:=idd to Dimension do
                                                    wert:=wert+START[fixvecnr,idG]*h[idG];
                                                    end for;
                            wert:=wert*IdealP/IdealPP;
                            if (IsIntegral(wert[1]) eq true) and (IsIntegral(wert[2]) eq true) then
                                                                                              ganz:=true;
                                                                                              r:=h;
                                                                                              end if;
                            vz:=vz+2^(idd-1);
                                                          end while;
                    zaehler:=zaehler+IdealPP^(idd-1);
                    end while;
V:=Umm(r)*swap(eintrag,fixvecnr);
START:=(V^-1)*START*Transpose(V^-1);

end if;
end for;
/********************bring matrix in shape to allow neighbour construction*******************/

/********************neighbour construction*******************/
for t:=1 to Dimension do
                      START[fixvecnr,t] :=START[fixvecnr,t]*IdealP/IdealPP;             /***Ideal***/
                      START[t,fixvecnr] :=START[t,fixvecnr ]*Conjugate(IdealP)/IdealPP; /***Ideal***/
```

```
                      START[f,t]        :=START[f,t]*IdealP;                          /***Ideal***/
                      START[t,f]        :=START[t,f]*Conjugate(IdealP);               /***Ideal***/
                      end for;
print "Nachbar erzeugt...";
/*neighbour construction*/

/*******************Z-modules*******************/
VGRAMS:=START/d;
RealGRAMS:=ScalarMatrix(Rationals(),2*Dimension,0);
ImGRAMS:=ScalarMatrix(Rationals(),2*Dimension,0);
for j:=1 to Dimension do
                      for k:=1 to Dimension do
            RealGRAMS[j,k]:=VGRAMS[j,k][1];
            RealGRAMS[Dimension+j,Dimension+k]:=mul*VGRAMS[j,k][1];
            RealGRAMS[j,Dimension+k]:=(1/2)*VGRAMS[j,k][1]+(1/2)*d*VGRAMS[j,k][2];
            RealGRAMS[Dimension+j,k]:=(1/2)*VGRAMS[j,k][1]-(1/2)*d*VGRAMS[j,k][2];
                                           end for;
                      end for;

for j:=1 to Dimension do
                      for k:=1 to Dimension do
            ImGRAMS[j,k]:=VGRAMS[j,k][2];
            ImGRAMS[Dimension+j,Dimension+k]:=mul*VGRAMS[j,k][2];
            ImGRAMS[j,Dimension+k]:=(1/2)*VGRAMS[j,k][2]-(1/2)*VGRAMS[j,k][1];
            ImGRAMS[Dimension+j,k]:=(1/2)*VGRAMS[j,k][2]+(1/2)*VGRAMS[j,k][1];
                                            end for;
                      end for;
ImGRAMS:=d*ImGRAMS;
/*******************Z-modules*******************/

/*---even neighbour?---*/
isot:=false;
for lj1:=1 to (2*Dimension) do
                            for lj2:=1 to Dimension do
                                                    if IsIntegral(RealGRAMS[lj1,lj2]) eq false then
                                                                                       isot:=true;
                                                                                       end if;
                                                    if IsIntegral(ImGRAMS[lj1,lj2]) eq false then
                                                                                     isot:=true;
                                                                                     end if;
                                                    end for;
                            end for;
if isot eq true then
                 print "Not even...";
                end if;
/*---even neighbour?---*/


if isot eq false then

/********************reduce value of matrix entries (better for isometry testing)********************/
for trid:=1 to 10 do
                  RealGRAMS, TrafoS:=SeysenGram(RealGRAMS);
                  RealGRAMS, TrafoL:=LLLGram(RealGRAMS);
                  RealGRAMS, TrafoP:=PairReduceGram(RealGRAMS);
                  ImGRAMS:=TrafoP*TrafoL*TrafoS*ImGRAMS*Transpose(TrafoP*TrafoL*TrafoS);
                  end for;
/********************reduce value of matrix entries)********************/

for p:=1 to #Erg do
               numba:=#Erg+1-p;
               VGRAMST:=Erg[numba];
               RealGRAMST:=ErgEXT[numba,2];
               ImGRAMST:=ErgEXT[numba,3];
               if isot eq false then
```

```
                                print "Determining isometry class (",numba," ?)...";
                                svinvariante:=true;
                                SS22:=#ShortVectors(LatticeWithGram(RealGRAMS),2,2);
                                if SS22 ne ErgEXT[numba,4] then
                                                svinvariante:=false;
                                                print "#SV not equal (",SS22,"|",ErgEXT[numba,4],")!";
                                                end if;
                                if svinvariante eq true then

/********************---main isometry test---********************/
                                print "#SV equal => isometry testing...";
if IsIsomorphic([MatrixRing(Integers(),2*Dimension) ! RealGRAMST,
                 MatrixRing(Integers(),2*Dimension) ! ImGRAMST],
                [MatrixRing(Integers(),2*Dimension) ! RealGRAMS,
                 MatrixRing(Integers(),2*Dimension) ! ImGRAMS]: ShortVectorsLimit:=10^9) eq true then
                                                                                isot:=true;
                           print "                     Isometric to lattice: ---> ", numba," <---";
                                                                                end if;
/********************---main isometry test---********************/
                                                        end if;
                                end if;
end for; /*all present lattices tested*/

if isot eq false then
                 print "New lattice found...";
                 Include(~Erg,START/d);
                 START; PrintMagmaMatrix(START);
                 Liste:=[* *];Append(~Liste,VGRAMS);Append(~Liste,RealGRAMS);Append(~Liste,ImGRAMS);
                 Append(~Liste,S22);Append(~Liste,#AGS);
                 Append(~ErgEXT,Liste);
                 end if;
end if;

print "Loop number= ",lauff; print " ";
end for; /*rnd-loop*/
```

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