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Nash Equilibria and Improvement Dynamics in Congestion Games

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Abstract

Communication infrastructures and markets are maintained and used by millions of entities each of them facing a private objective. The vast number of participants in conjunction with their individual goals to choose the best alternative gave rise to study such scenarios in the framework of game theory as it is rather unrealistic to assume that a centrally computed solution can be implemented. In this thesis, we follow this line of research and study congestion games as introduced by Rosenthal in 1973 and several modifications of his original approach.

Congestion games model scenarios in which a finite number of players individually strives to allocate resources maximizing their utility. Here, the resources can correspond to quite different types of objects, e. g. to edges in a network or to machines processing tasks. Given a set of resources a player's goal is to select a feasible subset of the resources, subsequently named a *strategy*, that minimizes the sum of the latencies of the resources in the set. Thereby, a subset is feasible if it possesses a predefined combinatorial structure, e. g., corresponds to a path or a tree in a network. The latency of a resource depends on the number of players sharing that resource, i. e., the congestion, as it increases the more player allocate the resource. Since the seminal presentation of this notion of games several modifications including weighted and player-specific congestion games have been proposed. In a weighted congestion game, the congestion on a resource depends on the weighted number of players, whereas players compute their latencies with respect to player-specific payoff functions in a player-specific congestion game.

Note that congestion games lie at the intersection of game theory and combinatorial optimization as from a global perspective we are concerned with a game, whereas from the local perspective of individual players we are concerned with a combinatorial optimization problem. Among others, one goal of this thesis is to apply results from combinatorial optimization in order to gain new insights into congestion games.

At first, we study the existence of Nash equilibria in weighted and in player-specific congestion games as every standard game without these additional requirements possesses a Nash equilibrium. We characterize those games with respect to the com-

binatorial structure of the players' strategy spaces in which a Nash equilibrium is guaranteed to exist. Namely, we show that the matroid property, i. e., if the strategy space of each player is the set of bases of a matroid, is the maximal property that guarantees the existence of Nash equilibria. If this property, however, is not satisfied we cannot guarantee the existence of Nash equilibrium without taking additional properties of the game into account.

We also study dynamics that arise if players actually play a congestion game and consider the time until they terminate at a stable configuration in which none of the players can improve its latency. In best response dynamics we assume that players sequentially switch to the best available strategy given fixed choices of the others. In case of standard congestion games, we show that the matroid property is the maximal property on the combinatorial structure of the players' strategy spaces that guarantees polynomial time convergence. In case of weighted and player-specific congestion games, however, we provide analytical and experimental evidence that even in singleton games, best response dynamics do not terminate quickly. Note that in singleton games, each strategy is a singleton set.

In case of standard congestion games, we also study concurrent imitation dynamics that arise if players imitate each other on the basis of a protocol we propose. We motivate to study such dynamics as the assumption that players have complete knowledge, which is usually applied when considering Nash equilibria, is likely not to be true in many real world applications. The protocol we propose guarantees pseudo-polynomial time convergence to an imitation-stable state in a monotonic fashion, that is, undesirable overshooting effects do not occur. We can also prove that an approximate equilibrium in which only a small fraction of the players sustains latency significantly above or below the average is reached quickly.

Finally, we propose to study a modification of player-specific singleton congestion games in which the resources can assign priorities to the players in order to foster some of them. In our model only the players with the highest priority gain access to a resource whereas the others are locked out. We analyze the existence of Nash equilibria in this class of games and discuss relationships to other existing models.

Zusammenfassung

Die heutigen Kommunikationsinfrastrukturen und Handelsplattformen werden von Millionen von Teilnehmern benutzt und betrieben. Dabei versucht jeder Benutzer seinen individuellen Nutzen zu optimieren. Auf Grund der großen Anzahl der Beteiligten in Kombination mit deren egoistischen Zielen, werden solche Szenarien häufig in spieltheoretischen Modellen untersucht, da es unrealistisch erscheint, eine zentral berechnete Lösung zu installieren. In dieser Arbeit folgen wir diesem Ansatz und untersuchen Auslastungsspiele, wie sie 1973 von Rosenthal vorgeschlagen wurden, sowie verschiedene Varianten dieses ursprünglichen Models.

Auslastungsspiele modellieren Szenarien, in denen endlich viele Spieler Ressourcen zwecks Optimierung Ihres Nutzens auswählen und belegen. Dabei können die Ressourcen ganz verschiedene Objekte sein, z.B. die Kanten eines Netzwerks oder Server. Von einer Menge von Ressourcen möchte jeder Spieler eine solche zulässige Teilmenge, im Folgenden auch Strategie genannt, belegen, die die Summe der Latenzen minimiert. Dabei ist eine Teilmenge zulässig, wenn sie eine bestimmte kombinatorische Struktur besitzt, z.B. wenn sie ein Pfad oder ein Baum in einem Netzwerk ist. Die Latenz einer Ressource hängt von ihrer Auslastung, also der Anzahl Spieler, die diese belegen, ab und steigt typischerweise mit steigender Anzahl von Spieler. Seit der ersten Präsentation von Auslastungsspielen wurden verschiedene andere Varianten präsentiert. Dazu zählen gewichtete Spiele, in denen die Auslastung einer Ressource von der gewichteten Anzahl Spieler abhängt und Spieler-spezifische Spiele, in denen die Spieler ihre Latenzen an Hand von Spieler-spezifischen Latenzfunktionen bestimmen.

Offensichtlich liegen Auslastungsspiele in der Schnittmenge von Spieltheorie und kombinatorischer Optimierung, da wir aus einer globalen Sicht ein Spiel betrachten, aus lokaler Sicht jedes einzelnen Spielers aber ein kombinatorisches Optimierungsproblem. Ein Ziel dieser Arbeit ist die Anwendung von Erkenntnissen der kombinatorischen Optimierung zur Vertiefung unseres Verständnisses von Auslastungsspielen.

Zunächst untersuchen wir die Existenz von Nash-Gleichgewichten in gewichteten und in Spieler-spezifischen Auslastungsspielen, da jedes Spiel ohne diese Zusätze ein Nash-

Gleichgewicht besitzt. Wir charakterisieren in Abhängigkeit der kombinatorischen Struktur der Strategiemengen der Spieler diejenigen Spiele, die immer ein Nash-Gleichgewicht haben. Dazu zeigen wir, dass jedes Spiel, in dem die Strategiemengen Mengen von Basen eines Matroids sind, immer ein Nash-Gleichgewicht hat. Falls diese Bedingung aber nicht erfüllt ist, dann können wir die Existenz eines Gleichgewichts nicht garantieren ohne weitere Eigenschaften des Spiels zu berücksichtigen. Die Matroid-Eigenschaft ist also die maximale Eigenschaft, die die Existenz von Nash-Gleichgewichten garantiert.

Wir betrachten auch Dynamiken, die auftreten, wenn Spieler ein Auslastungsspiel spielen und untersuchen wie lange es dauert, bis ein stabiler Zustand, in dem keiner der Spieler sich verbessern kann, erreicht wird. In der Besten-Antwort Dynamik nehmen wir an, dass jeweils ein Spieler zu seiner besten Strategie wechselt. Wir beweisen, dass im Falle von herkömmlichen Auslastungsspielen die Matroid-Eigenschaft die maximale Eigenschaft ist, die polynomielle Konvergenzzeit garantiert. Im Falle von gewichteten und von Spieler-spezifischen Auslastungsspielen präsentieren wir analytische und empirische Argumente, die untermauern, dass Beste-Anwort Dynamiken bereits in Spielen, in denen die Strategien nur ein-elementige Mengen sind, im Allgemeinen nicht schnell terminieren.

Im Fall von herkömmlichen Auslastungsspielen untersuchen wir auch parallele Imitationsdynamiken, die auftreten, wenn die Spieler ein von uns vorgestelltes Protokoll gleichzeitig anwenden. Motiviert wird dieser Ansatz durch die Beobachtung, dass die Spieler typischerweise kein vollständiges Wissen haben, die Definition des Nash-Gleichgewichts aber auf dieser Annahme beruht. Das von uns vorgestellte Protokoll garantiert, dass Imitationsdynamiken in einer monotonen Art und Weise zu einem Gleichgewicht konvergieren, ohne dass starke Schwankungen in den Latenzen auftreten. Wir zeigen außerdem, dass approximative Gleichgewichte, in denen nur eine kleine Anzahl von Spielern weit nach oben bzw. unten von der durchschnittlichen Latenz abweicht, schnell erreicht werden.

Abschließend stellen wir eine Modifikation von Spieler-spezifischen Auslastungsspielen vor, in den die Ressourcen den Spielern Prioritäten zuordnen, um einige von ihnen zu begünstigen. In unserem Modell erhalten unter allen Spielern die eine Ressource belegen wollen, nur diejenigen mit der höchsten Priorität Zugang zu dieser, die anderen Spieler aber nicht. Wir untersuchen die Existenz von Nash-Gleichgewichten in dieser Klasse von Spielen und diskutieren Verbindungen zu anderen Modellen.

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CHAPTER 1

Introduction

Congestion games are by now a well established approach to model scenarios in which selfish agents individually strive to allocate resources as effectively as possible. They have been introduced by Rosenthal [Ros73] in 1973 as a refinement of the previously studied Wardrop model [War52] and are motivated by the following examples. Suppose we are given a road network and a finite set of agents which want to spend about the same time traveling through the network. The goal of each agent is to select a route that minimizes the sum of the latencies of the road segments in it. Thereby, the latency of a segment depends on its congestion, i. e., on the number of agents using it. Another scenario somehow related to this one is the scenario of *network design games* as considered by Anshelevish et al. [ADK$^+$04]. In a network design game we are given a finite set of agents which strive to build a network satisfying certain connectivity requirements. Given a network in which every edge can be constructed at a certain cost each player can select a path connecting its source and its sink in the network. Now each player seeks to minimize the cost of the selected path which is the sum of the weighted costs of the edges in the path. Here, the term *weighted* stands for the fact that the costs of an edge are equally shared among all players using it. Both scenarios only differ in the definition of the latency and cost functions, respectively. However, especially the latter example can be defined with respect to many different combinatorial structures, e. g. with respect to spanning or Steiner trees.

In general, congestion games can be described as follows. In a congestion game we are given a finite set of resources and a finite set of players. Each player comes along with a set of strategies, henceforth called the player's strategy space. The strategies are the available options the player has and each of them corresponds to a subset of the set of resources. For each resource we are also given a payoff function which takes the congestion on the resource as input. Now each player individually strives to choose a strategy that optimizes its payoff given the choices of the other players. Thereby, the payoff of a strategy is the sum of the payoffs of the resources in it. In the terms of the examples discussed above, the set of resources is the set of edges in the network, a player's strategy is a path connecting its source and its sink, and a player's strategy space is the set of these paths.

From the local perspective of the players each of them faces a combinatorial optimization problem with varying payoffs of the resources. However, given fixed choices of the others, each player can solve this problem and choose the best available strategy. From a global perspective, however, we do not face a combinatorial optimization problem as there is no obvious approach to combine the individual players' quests for the best strategies into a single objective. For that reason, economists and game theorists have proposed to study *stable states* in which none of the players can individually switch to a different strategy that strictly improves on its payoff. This notion heavily relies on three assumptions. At first, players are *selfish*, i. e., they do not care about each other. Secondly, players have *complete knowledge* about the game, i. e., each of them knows its entire strategy space and the latency functions. Finally, players act rationally, i. e., their behavior is uniquely determined by the payoff functions. Due to the influential publication by John F. Nash [Nas50] on stable states in the general class of strategic games this situation is nowadays called *Nash equilibrium*. Rosenthal [Ros73] shows that every congestion game, according to the above description, possesses such an equilibrium.

In this thesis, we study two kinds of dynamics that arise when players actually "play" a congestion game. At first, we consider *best response dynamics* that emerge if players are permitted to play best responses. In this case, a single player is allowed to switch to the best available strategy given fixed choices of the others. We provide a characterization of games with respect to the combinatorial structure of the players' strategy spaces in which best response dynamics are guaranteed to reach a Nash equilibrium quickly. Additionally, we consider *imitation dynamics* in which players are permitted to imitate each other *concurrently* according to a protocol we propose. This approach is motivated by the observation that in many scenarios it is unlikely that players actually have complete knowledge. Instead, players are likely to imitate others which perform well. Again, we focus on the time until such dynamics reach a stable state. However, note that its definition needs to be adopted to the altered assumptions on the players' knowledge.

Obviously, the notion of congestion games, as introduced by Rosenthal, is a very general model capturing a variety of different scenarios in which players allocate resources. Still, many properties of particular real world scenarios are not taken into account. For that reason, researchers have proposed to study extensions and refinements of the original model. In this thesis, we also consider two refinements and propose to study a third one. The refinements we consider are motivated as follows. Consider a load balancing scenario in which each player strives to assign a job to one out of several machines that minimizes the completion time of the job. Obviously, such a model is related to job scheduling on related machines except that the jobs cannot be assigned to the machines by a central authority but jobs selfishly choose a machine. If the jobs all have the same size, then the load balancing scenario would be a congestion game as described above. However, it is very unlikely that different jobs have the same size. Hence, the congestion on a resource should be defined as the total weight of the jobs being assigned to it rather than their number only. This example suggests to study *weighted congestion games* in which we additionally associate a weight with each player and define the congestion of a resource as the sum of the players' weights allocating it. The second refinement we consider are *player-specific congestion games*. This class of games is motivated by the observation that in certain scenarios players are likely to compute their latencies with respect to their own private payoff functions instead of common payoff functions applied by everyone else. For example, different types of vehicles have to obey different speed limits. Hence, vehicle-specific speed limits have to be taken into account when computing the travel time along a road segment.

In both classes of games we study the existence of Nash equilibria and provide a characterization of games with respect to the combinatorial structure of the players' strategy spaces that are guaranteed to possess Nash equilibria. Furthermore, we consider best response dynamics as described above and present lower bounds on the time to reach an equilibrium.

Finally, we propose a new extension of congestion games and introduce *congestion games with priorities*. So far, all players allocating a resource gain access and suffer from the same amount of congestion. However, suppose that those authorities supplying the resources want to support some of the players. To this end, they can assign priorities to the players in which case only the players with the highest rank gain access to a resource whereas the other players with less priority are locked out. We study the existence of Nash equilibria in such games and discuss relationship to two-sided matching markets as introduced by Gale and Shapley [GS62].

The remainder of this chapter is organized as follows. At first, we provide formal definitions of Rosenthal's model of congestion games, of weighted and of player-specific congestion games. We proceed with a discussion of combinatorial structures in congestion games and provide a short introduction to matroids. Thereafter, we consider the notion of Nash equilibria in congestion games and discuss the potential function method. This method is frequently applied to prove the existence of Nash equilibria. Then, we give a summary of previously known results and point out to related models whenever appropriate. Finally, we give a more detailed outline of the results presented in this thesis.

1.1 Formal Definition of Congestion Games

In this section, we provide a formal introduction to congestion games. At first, we define congestion games as introduced by Rosenthal [Ros73]. Throughout this thesis, we refer to his definition by the term *standard congestion game*. We proceed with *weighted congestion games* in which the congestion of a resource depends on the players' weights. Then we define *player-specific congestion games* in which the players compute their payoffs with respect to player-specific payoff functions. Probably, the first to study these two classes of games is Milchtaich [Mil96]. Note that in correspondence to most existing literature on these models we refer to each player's objective function by the term *latency function* instead of payoff function which is usually employed in game theory. Furthermore, we change the direction of optimization as we assume that players want to minimize their latency instead of maximize their payoff.

Throughout this thesis, the term congestion game refers to any kind of game defined in this section.

1.1.1 Standard Congestion Games

A standard congestion game Γ is a tuple $(\mathcal{N}, \mathcal{R}, (\Sigma_i)_{i\in\mathcal{N}}, (\ell_r)_{r\in\mathcal{R}})$, where

$\mathcal{N} = \{1, \ldots, n\}$ is a set of n players,
$\mathcal{R} = \{r_1, \ldots, r_m\}$ a set of m resources,
$\Sigma_i \subseteq 2^{\mathcal{R}}$ the strategy space of player i, and
$\ell_r : \mathbb{N} \to \mathbb{N}$ a latency function associated with resource r.

In this thesis, $\mathbb{N}$ denotes the set of integers $\{0, 1, 2, 3, \ldots\}$. Furthermore, we assume that the latency functions ℓ_r are *non-decreasing.*

A *state* of the game Γ is a vector $S = (s_1, \ldots, s_n)$ where player $i \in \mathcal{N}$ chooses strategy $s_i \in \Sigma_i$. Given a state S, $S \oplus s_i'$ denotes the state

$$(s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n) \ ,$$

i. e., the state S except that player i chooses strategy s_i' instead of s_i. For every state $S \in \Sigma_1 \times \ldots \times \Sigma_n$, the *congestion* $x_r(S)$ on resource r is defined as $x_r(S) = |\{i \mid r \in s_i\}|$, that is, $x_r(S)$ is the number of players sharing resource r in state S. Given a state S, the *latency* $\ell_i(S)$ of player i is the sum of the latencies of the resources the player allocates, i. e.,

$$\ell_i(S) = \sum_{r\in s_i} \ell_r(x_r(S)) \ .$$

Furthermore, given a state S, we call a strategy $s_i^* \in \Sigma_i$ a *better response* of player i to S if $\ell_i(S \oplus s_i^*) < \ell_i(S)$. Additionally, we call a strategy $s_i^* \in \Sigma_i$ a *best response* of player i to S if s_i^* is a better response and if $\ell_i(S \oplus s_i^*) \le \ell_i(S \oplus s_i')$ for all $s_i' \in \Sigma_i$. Hence, a best response is the best available better response. Throughout this thesis, we say that *a player plays a better/best response in state* S if that player switches to a better/best response to S. That is, afterwards we obtain a new state S'.

1.1.2 Weighted Congestion Games

In a weighted congestion game, we additionally associate a *weight* $\omega_i \in \mathbb{N} \setminus \{0\}$ with every player $i \in \mathcal{N}$. In this case, we adopt the definition of the congestion of a resource in a particular state of the game in the following way. The congestion $x_r(S)$ on resource r in state S is the sum of the weights of all players sharing resource r in state S, i. e., $x_r(S) = \sum_{i:r\in s_i} \omega_i$.

1.1.3 Player-specific Congestion Games

In a player-specific congestion game, we assume that each player computes its latency according to *player-specific latency functions* instead of common latency functions. Formally, for every player $i \in \mathcal{N}$ and every resource $r \in \mathcal{R}$ we are given a non-decreasing, player-specific latency function $\ell_r^i \colon \mathbb{N} \to \mathbb{N}$. Then, the latency of player i in state S equals $\ell_i(S) = \sum_{r \in s_i} \ell_r^i(x_r(S))$.

1.2 Combinatorial Structures in Congestion Games

In the above definition of congestion games, we did not impose restrictions on the players' strategy spaces. Typically however, each strategy space possesses a combinatorial structure implicitly defined by an oracle determining if a given subset of the resources is feasible or not. For instance, a player's strategy space could be the set of bases of a matroid (see Definition 1.1). Still, in this case the combinatorial structures among different strategy spaces can differ as we did not assume that the resources have a common combinatorial interpretation. They can, for instance, be the set of edges of a graph, and the players' strategy spaces correspond to those subsets of the edges possessing a specific combinatorial structure, e. g. paths or trees. Below we mention three classes of congestion games in which each strategy space has a specific combinatorial structure, or in which the resources have a common combinatorial interpretation. We call a game Γ

singleton congestion game if for every player $i \in \mathcal{N}$ and every strategy $s_i \in \Sigma_i \colon |s_i| = 1$, i. e., if all strategies are singleton sets. Note that singleton congestion games are closely related to selfish load balancing scenarios in which players assign jobs to machines.

network congestion game if the set of resources $\mathcal{R}$ is the set of edges of an (un-)directed graph $G = (V, E)$, and if for every player i the strategy space Σ_i is the set of paths connecting a particular source $s_i \in V$ with a particular sink $t_i \in V$ in the network.

matroid congestion game if for every player $i \in \mathcal{N}$ the player's strategy space Σ_i is the set of bases of a matroid. Note that singleton congestion games are matroid congestion games. Another prominent example of this class of games are *spanning tree congestion games* in which players strive to allocate spanning trees of a graph. For an introduction to matroids and a formal definition of matroid congestion games we refer the reader to the next section.

We refer to a congestion game by the term *general* if we do not impose restrictions on the combinatorial structure of the game. In this case, the players' strategy spaces are given explicitly. Furthermore, we call a congestion game *symmetric* if $\Sigma_1 = \ldots = \Sigma_n$, i. e., if the strategy spaces are equal. Otherwise, we call it *asymmetric*.

1.3 A Short Introduction to Matroids

Matroids are a well known and extensively studied combinatorial structure. Among others, they are applicable to minimum spanning trees in undirected graphs. Matroids are exactly those structures possessing the *greedy property*, that is, an optimum solution can be computed with a greedy algorithm. Furthermore, they possess the *exchange property*. This property comes in various different characterizations of which we present some below. Almost all of these properties are common knowledge and proofs follow easily from the definition of matroids. Hence, we omit most of them and refer the reader to [Sch03].

Definition 1.1. *A tuple $\mathcal{M} = (\mathcal{R}, \mathcal{I})$ is called a* matroid *if $\mathcal{R} = \{r_1, \ldots, r_m\}$ is a finite set of resources and $\mathcal{I}$ is a nonempty collection of subsets of $\mathcal{R}$ such that*

1. *if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$.*
2. *if $I, J \in \mathcal{I}$ and $|J| < |I|$, then there exists a resource $r \in I \setminus J$ with $J \cup \{r\} \in \mathcal{I}$.*

Given a matroid $\mathcal{M} = (\mathcal{R}, \mathcal{I})$, we call a set $I \subseteq \mathcal{R}$ *independent* if $I \in \mathcal{I}$, otherwise we call it *dependent*. Furthermore, we call an inclusion-wise maximal independent set a *base* of $\mathcal{M}$. It follows easily from the definition of matroids that all bases of a matroid $\mathcal{M}$ have the same size. This size is usually denoted by the *rank* $\mathrm{rk}(M)$ of the matroid. Prominent examples of matroids are

k-Uniform Matroids A k-uniform matroid is determined by a set of resources $\mathcal{R}$ and an integer $k \leq |\mathcal{R}|$. The independent sets are the subsets of $\mathcal{R}$ of size at most k, the bases are the subsets of size exactly k.

Linear Matroids A linear matroid is determined by an $n \times m$ matrix A. The independent sets are the subsets of the columns of A which are linearly independent, the bases are the maximal linearly independent subsets of the columns.

Graphical Matroids A graphical matroid is determined by an undirected graph $G = (V, E)$. The independent sets are the subsets E' of the edges E such that $G' = (V, E')$ is a forest, i. e., $G' = (V, E')$ does not contain a cycle. The bases are the spanning trees of G.

Usually the *exchange property of matroids* refers to the following characterization of matroids. This corollary is essentially true due to the second condition in Definition 1.1.

Corollary 1.2. *Let $\mathcal{R}$ be a set of resources and let $\mathcal{S}$ be a nonempty collection of subsets of $\mathcal{R}$. Then, the following are equivalent:*

- *$\mathcal{S}$ is the collection of bases of a matroid over $\mathcal{R}$.*
- *if $B_1, B_2 \in \mathcal{S}$ and $r_1 \in B_1 \setminus B_2$, then there exists $r_2 \in B_2 \setminus B_1$ such that $B_1 \cup \{r_2\} \setminus \{r_1\} \in \mathcal{S}$.*

A more general formulation of the exchange property is the following one.

Corollary 1.3. *Let B_1, B_2 be bases of a matroid $\mathcal{M} = (\mathcal{R}, \mathcal{I})$. Consider the bipartite graph $G(B_1 \Delta B_2) = (V, E)$ with*

- $V = (B_1 \setminus B_2) \cup (B_2 \setminus B_1)$*, and*
- $E = \{\{r_1, r_2\} \mid r_1 \in B_1 \setminus B_2,\, r_2 \in B_2 \setminus B_1 \colon B_1 \cup \{r_2\} \setminus \{r_1\} \in \mathcal{I}\}$.

There exists a perfect matching in the graph $G(B_1 \Delta B_2)$.

Sometimes we consider the *k-truncation* of a matroid $\mathcal{M} = (\mathcal{R}, \mathcal{I})$. Given $k \in \mathbb{N}$ we call $\mathcal{M}' = (\mathcal{R}, \mathcal{I}')$ with $\mathcal{I}' = \{I \in \mathcal{I} \mid |I| \leq k\}$ the k-truncation of $\mathcal{M}$. It can easily be verified that $\mathcal{M}'$ is a matroid, too.

We call a matroid $\mathcal{M} = (\mathcal{R}, \mathcal{I})$ *weighted* if we are also given a weight function $w \colon \mathcal{R} \to \mathbb{N}$ and seek to find a basis of $\mathcal{M}$ of minimum weight. The weight of an independent set I is given by $w(I) = \sum_{r \in I} w(r)$. A minimum weight basis can be computed by a greedy algorithm. This algorithm works as follows. It starts with the empty set S, and iteratively adds minimum weight resources $r \in \mathcal{R} \setminus S$ to S such that $S \cup \{r\}$ is an independent set. Given a polynomial time algorithm determining whether a set is independent or not, this is a polynomial time algorithm. A useful characterization of a minimum weight basis is the next one.

Corollary 1.4. *Let $\mathcal{M} = (\mathcal{R}, \mathcal{I})$ be a weighted matroid. A basis $B \in \mathcal{I}$ is a minimum weight basis of $\mathcal{M}$ if and only if there exists no basis $B^* \in \mathcal{I}$ with $|B \setminus B^*| = 1$ and $w(B^*) < w(B)$.*

The next two corollaries extend the exchange property to weighted matroids in a natural way.

Corollary 1.5. *Let $\mathcal{M} = (\mathcal{R}, \mathcal{I})$ be a weighted matroid with weights $w \colon \mathcal{R} \to \mathbb{N}$ and let B_{opt} be a basis of $\mathcal{M}$ with minimum weight. Suppose that the weight of a single resource $r_{opt} \in B_{opt}$ is increased such that B_{opt} is no longer of minimum weight. In order to obtain a minimum weight basis again, it suffices to exchange r_{opt} with a resource $r^* \in \mathcal{R}$ of minimum weight such that $B_{opt} \cup \{r^*\} \setminus \{r_{opt}\}$ is a basis.*

Proof. Let B'_{opt} be a minimum weight basis with respect to the increased weight of r_{opt}. Let P be a perfect matching of the graph $G(B_{opt} \Delta B'_{opt})$ and denote by e the edge from P that contains r_{opt}. Recall that such a matching exists due to Corollary 1.3. For every edge $\{r, r'\} \in P \setminus \{e\}$, it holds $w(r) \leq w(r')$ as, otherwise, if $w(r) > w(r')$, the basis $B_{opt} \cup \{r'\} \setminus \{r\}$ would have smaller weight than B_{opt}.

Now denote by r'_{opt} the resource that is matched with r_{opt}, i.e., the resource such that $e = \{r_{opt}, r'_{opt}\} \in P$. Since $w(r) \leq w(r')$ for every $\{r, r'\} \in P \setminus \{e\}$, the weight of $B_{opt} \setminus \{r_{opt}\}$ is bounded from above by the weight of $B'_{opt} \setminus \{r'_{opt}\}$. By the definition of the matching P, $B_{opt} \cup \{r'_{opt}\} \setminus \{r_{opt}\}$ is a basis. By our arguments above, the weight of this basis is bounded from above by the weight of B'_{opt}. Hence, this basis is optimal with respect to the increased weight of r_{opt}. □

Corollary 1.6. *Let $\mathcal{M} = (\mathcal{R}, \mathcal{I})$ be a matroid with weights $w \colon \mathcal{R} \to \mathbb{N}$ and let B_{opt} be a basis of $\mathcal{M}$ with minimum weight. Suppose that the weight of a single resource $r^* \in \mathcal{R} \setminus B_{opt}$ is decreased such that B_{opt} is no longer of minimum weight. In order to obtain a minimum weight basis again, it suffices to exchange r^* with a resource $r_{opt} \in B_{opt}$ of maximum weight such that $B_{opt} \cup \{r^*\} \setminus \{r_{opt}\}$ is a basis.*

The proof of Corollary 1.6 follows the same arguments as the proof of Corollary 1.5. Hence, we omit it.

1.3.1 Matroid Congestion Games

We are now ready to state a formal definition of matroid congestion games.

Definition 1.7. *We call a standard congestion game $\Gamma = (\mathcal{N}, \mathcal{R}, (\Sigma_i)_{i \in \mathcal{N}}, (\ell_r)_{r \in \mathcal{R}})$ matroid congestion game if for every player $i \in \mathcal{N}$, $M_i := (\mathcal{R}_i, \mathcal{I}_i)$ with $\mathcal{R}_i = \cup_{s \in \Sigma_i} s$ and $\mathcal{I}_i = \{I \subseteq S \mid S \in \Sigma_i\}$ is a matroid, and if Σ_i is the set of bases of M_i. Additionally, we denote by* $\mathrm{rk}(\Gamma) = \max_{i \in \mathcal{N}} \mathrm{rk}(M_i)$ *the* rank of the matroid congestion game *Γ.*

Occasionally, we assume that players only play best responses that exchange the least number of resources compared to their current strategies. We call such best responses *lazy best responses* and define them formally as follows.

Definition 1.8. *Given a state S of a standard matroid congestion game, we call a best response s_i^* of player i* lazy *if it can be decomposed into a sequence of strategies $s_i = s_i^0, s_i^1, \ldots, s_i^k = s_i^*$ with $|s_i^{j+1} \setminus s_i^j| = 1$ and $\ell_i(S \oplus s_i^{j+1}) < \ell_i(S \oplus s_i^j)$, for $0 \leq j < k$.*

From Corollary 1.4 we can conclude that whenever a player can play a best response, then there exists a lazy best response for this player, too.

The above definitions are formulated in terms of standard congestion games. However, both definitions obviously extend to weighted or player-specific congestion games.

1.3.2 A Characterization of Non-Matroid Set Systems

Matroids are uniquely characterized by their exchange properties. Referring to Corollaries 1.5 and 1.6 one might also want to call them $(1,1)$-exchange properties. Next, we present a *novel* characterization of *non-matroid set systems* in which $(1,2)$-exchanges need to be performed in order to obtain an optimum solution again.

Let $\mathcal{S}$ be a set system on a set $\mathcal{R}$ of resources, i. e., $\mathcal{S}$ is a collection of subsets of $\mathcal{R}$. The set system $\mathcal{S}$ is called an *antichain* if for every $X \in \mathcal{S}$, no proper superset $Y \supset X$ belongs to $\mathcal{S}$. Moreover, we call $\mathcal{S}$ a *non-matroid set system* if the tuple $(\mathcal{R}, \{X \subseteq S \mid S \in \mathcal{S}\})$ is not a matroid.

Definition 1.9 ($(1,2)$-exchange property)**.** *Let $\mathcal{S}$ be an antichain on a set of resources $\mathcal{R}$. We say that $\mathcal{S}$ satisfies the* $(1,2)$-exchange property *if we can identify three distinct resources $a, b, c \in \mathcal{R}$ with the property that for every given $k \in \mathbb{N}$ with $k > |\mathcal{R}|$, we* *can choose a weight* $w(r) \in \{1, k + |\mathcal{R}|\}$ *for every* $r \in \mathcal{R} \setminus \{a, b, c\}$ *such that for every choice of the weights of* a*,* b*, and* c *with* $|\mathcal{R}| \leq w(a), w(b), w(c) \leq k$*, the following property is satisfied:*

- *If* $w(a) + |\mathcal{R}| \leq w(b) + w(c)$*, then for every set* $S \in \mathcal{S}$ *with minimum weight,* $a \in S$ *and* $b, c \notin S$*.*
- *If* $w(a) \geq w(b) + w(c) + |\mathcal{R}|$*, then for every set* $S \in \mathcal{S}$ *with minimum weight,* $a \notin S$ *and* $b, c \in S$*.*

We now prove that every non-matroid set system possesses the $(1, 2)$-exchange property.

Lemma 1.10. *Let* $\mathcal{S}$ *be an antichain on a set of resources* $\mathcal{R}$*. Furthermore, let* $\mathcal{I} = \{X \subseteq S \mid S \in \mathcal{S}\}$*, and assume that* $(\mathcal{R}, \mathcal{I})$ *is not a matroid, i. e.,* $\mathcal{S}$ *is not the set of bases of some matroid. Then* $\mathcal{S}$ *possesses the* $(1, 2)$*-exchange property.*

Proof. Since $(\mathcal{R}, \mathcal{I})$ is not a matroid, there exist, due to Corollary 1.2, two sets $X, Y \in \mathcal{S}$ and a resource $x \in X \setminus Y$ such that for every $y \in Y \setminus X$, the set $X \setminus \{x\} \cup \{y\}$ is not contained in $\mathcal{S}$.

Let X and Y be such sets and let $x \in X$ be such a resource. Consider all subsets Y' of the set $X \cup Y \setminus \{x\}$ with $Y' \in \mathcal{S}$. Every such set Y' can be written as $Y' = X \setminus \{x = x_1, \ldots, x_l\} \cup \{y_1, \ldots, y_{l'}\}$ with $x_i \in X \setminus Y$ and $y_i \in Y \setminus X$ and $l + l' > 2$. This is true since $l \geq 1$ holds per definition and $l' \geq 1$ holds because $\mathcal{S}$ is an antichain. Furthermore l and l' cannot both equal 1 as otherwise we obtain a contradiction to the choice of X, Y, and x. Among all these sets Y', let $Y_{\min}$ denote one set for which l' is minimal. Observe that we can replace Y by $Y_{\min}$ without changing the aforementioned properties of X, Y, and x. Hence, in the following, we assume that $Y = Y_{\min}$, that is, we assume that $Y \setminus X = Y' \setminus X$ for all of the aforementioned sets Y'.

We claim that we can always identify resources $a, b, c \in X \cup Y$ such that either $a \in X \setminus Y$ and $b, c \in Y \setminus X$ or $a \in Y \setminus X$ and $b, c \in X \setminus Y$ with the property that for every $Z \subseteq X \cup Y$ with $Z \in \mathcal{S}$, if $a \notin Z$, then $b, c \in Z$. In order to see this, we distinguish between the cases $l' = 1$ and $l' \geq 2$:

1. Let $Y \setminus X = \{y_1\}$ and hence $X \setminus Y = \{x = x_1, \ldots, x_l\}$ with $l \geq 2$. Then we set $a = y_1$, $b = x_1$, and $c = x_2$. Consider a set $Z \subseteq X \cup Y$ with $Z \in \mathcal{S}$ and $a \notin Z$. Then $Z = X$ since $\mathcal{S}$ is an antichain, and hence $b, c \in Z$.
2. Let $Y \setminus X = \{y_1, \ldots, y_{l'}\}$ with $l' \geq 2$. Then we set $a = x$, $b = y_1$, and $c = y_2$. Consider a set $Z \subseteq X \cup Y$ with $Z \in \mathcal{S}$ and $a \notin Z$. Since we assumed that $Y = Y_{\min}$, it must be $b, c \in Z$ as otherwise $Z \setminus X \neq Y \setminus X$.

Now we define weights for the resources in $\mathcal{R} \setminus \{a, b, c\}$ such that the properties as stated in Definition 1.9 are satisfied. Let $k \in \mathbb{N}$ be chosen as in Definition 1.9, that is, $w(a), w(b), w(c) \in \{|\mathcal{R}|, \ldots, k\}$. We set $w(r) = k + |\mathcal{R}|$ for every resource $r \notin X \cup Y$ and $w(r) = 1$ for every resource $r \in (X \cup Y) \setminus \{a, b, c\}$. First of all, observe that in the first case the weight of Y equals $w(a) + |Y| - 1 < k + |\mathcal{R}|$ and that in the second case the weight of X equals $w(a) + |X| - 1 < k + |\mathcal{R}|$. Hence, a set $Z \in \mathcal{S}$ that contains a resource $r \notin X \cup Y$ can never have minimum weight as its weight is at least $k + |\mathcal{R}|$. Thus, only sets $Z \in \mathcal{S}$ with $Z \subseteq X \cup Y$ can have minimum weight. Since for such sets, $a \notin Z$ implies $b, c \in Z$, we know that every set with minimum weight must contain a or it must contain b and c.

Consider the case $w(a) + |\mathcal{R}| \leq w(b) + w(c)$ and assume for contradiction that there exists an optimal set Z^* with $a \notin Z^*$. Due to the choice of a, b, and c, the set Z^* must then contain b and c. Hence $w(Z^*) \geq w(b) + w(c)$. Furthermore, again due to the choice of a, b, and c, there exists a set $Z' \subseteq X \cup Y$ with $a \in Z'$ and $b, c \notin Z'$. The weight of Z' is $w(Z') = w(a) + |Z'| - 1 < w(a) + |\mathcal{R}| \leq w(b) + w(c) \leq w(Z^*)$, contradicting the assumption that Z^* has minimum weight. Hence, every optimal set Z^* must contain a. If Z^* additionally contains b or c, then its weight is at least $w(a) + |\mathcal{R}| > w(Z')$. Hence, in the case $w(a) + |\mathcal{R}| \leq w(b) + w(c)$ every optimal set Z^* contains a but it does not contain b and c.

Consider the case $w(a) \geq w(b) + w(c) + |\mathcal{R}|$ and assume for contradiction that there exists an optimal set Z^* with $b \notin Z^*$ or $c \notin Z^*$. Then Z^* must contain a and hence its weight is at least $w(a)$. Due to the choice of a, b, and c, there exists a set $Z' \subseteq X \cup Y$ with $a \notin Z'$ and $b, c \in Z'$. The weight of Z' is $w(Z') = w(b) + w(c) + |Z'| - 2 < w(b) + w(c) + |\mathcal{R}| \leq w(a) \leq w(Z^*)$, contradicting the assumption that Z^* has minimum weight. Hence, every optimal set Z^* must contain b and c. If Z^* additionally contains a, then its weight is at least $w(b) + w(c) + |\mathcal{R}| > w(Z')$. Hence, in the case $w(a) \geq w(b) + w(c) + |\mathcal{R}|$ every optimal set Z^* contains b and c but it does not contain a. □

1.4 Nash Equilibria in Congestion Games

As we already mentioned in the introduction, Nash equilibria are the predominant solution concept to congestion games. Intuitively, a state is a Nash equilibrium if none of the players can switch to a different strategy in order to decrease its latency. Below, we give a formal definition of such states. Before, however, we need to revisit some assumptions usually made about the players' knowledge and their behavior.

In congestion games, it is assumed that players only choose *pure strategies* instead of *mixed strategies*, i. e., probability distributions over all strategies. In the latter case, the latency of a player in a particular state is its expected latency with respect to the chosen distributions. In computer science the assumption that the players only choose pure strategies is quite natural and generally accepted, as for various scenarios "randomizing over strategies is not a realistic option" [ADTW03]. This is especially true in the presence of pure Nash equilibria.

Additionally, it is assumed that players have *complete knowledge* about the game meaning each player knows the entire set of resources, its complete strategy space, its latency functions and in every state of the game the choices of its opponents. In this case, each player has access to the information required to determine whether the selected strategy is the best one to choose or if a better one is available.

Furthermore, it is assumed that the players act *rationally* meaning that their behavior is uniquely determined by their latencies. That is, a player which changes its strategy always switches to a strategy strictly decreasing its latency. Players could also make their choices on the basis of additional likes and dislikes not being captured by the latency functions. For example, a player might not want to share a resource with a particular other player. Or, players could behave strategically and anticipate the behavior of others in order to gain advantages in the long run. Obviously, such constraints are not captured by the payoff functions.

Under these assumptions, a *pure Nash equilibrium* is defined as a state in which none of the players can unilaterally decrease its latency by changing its strategy given fixed choices of the others.

Definition 1.11. *Consider a standard, weighted or player-specific congestion game Γ and let S be a state of Γ. We call S* pure Nash equilibrium *if and only if for every player $i \in \mathcal{N}$ and every strategy $s'_i \in \Sigma_i$*

$$\ell_i(S) \leq \ell_i(S \oplus s'_i) \ .$$

In terms of better responses, none of the players can play a better response.

In this thesis, we only consider pure Nash equilibria. One might also consider mixed Nash equilibria which are defined in a similar way except that players also play mixed strategies. Therefore, we omit the term *pure* and refer by the term Nash equilibrium to a pure one.

Approximate Nash Equilibrium A slight relaxation of the notion of a Nash equilibrium is that of an *approximate Nash equilibrium.* It refers to scenarios in which players are ε-greedy meaning that each of them plays a better response if and only if its latency would decrease by more than a factor of $1 + \varepsilon$ for some $\varepsilon > 0$.

Definition 1.12. *Let $\varepsilon > 0$. Consider a standard, weighted or player-specific congestion game Γ and let S be a state of Γ. We call S* approximate Nash equilibrium *if and only if for every player $i \in \mathcal{N}$ and every strategy $s'_i \in \Sigma_i$*

$$\frac{\ell_i(S)}{\ell_i(S \oplus s'_i)} \quad < \quad 1 + \varepsilon \ .$$

In terms of better responses, none of the ε-greedy players can play a better response, i. e., switch to a strategy that decreases its latency by a factor of more than $1 + \varepsilon$.

The notion of an approximate Nash equilibrium is motivated by the following observation. If the anticipated latency gain of a player is small compared to its current latency the player might not want to change its strategy. This is especially true if changing a strategy is not for free.

1.4.1 The Potential Function Method

A frequently applied technique to prove the existence of Nash equilibria in congestion games is the *potential functions method.* A potential function Φ maps every state of a congestion game to an element from a totally ordered set. Furthermore, the potential function has the property that there exists a better state with less potential in the neighborhood of the state currently considered if and only if this state can be generated by letting one player play a better response. Throughout this thesis we apply this method to various classes of congestion games in order to prove the existence of Nash equilibria. Subsequently, we give a short introduction to this technique which is studied in detail by Monderer and Shapley [MS96].

Let $\mathcal{X}$ be a totally ordered set and let Γ be a congestion game. Monderer and Shapely call a function $\Phi\colon \Sigma_1, \dots, \Sigma_n \to \mathcal{X}$ *ordinal potential function* if and only if for every state S of the game, every player $i \in \mathcal{N}$ and every strategy $s_i^* \in \Sigma_i$

$$\Phi(S \oplus s_i^*) - \Phi(S) < 0 \quad \Leftrightarrow \quad \ell_i(S \oplus s_i^*) - \ell_i(S) < 0 \ . \tag{1.1}$$

Obviously, if such an ordinal potential function exists, then Γ possesses a Nash equilibrium since the Cartesian product of the players' strategy spaces is finite and hence there exists a minimum in the range of Φ. Monderer and Shapely call a congestion game *ordinal potential game* if an ordinal potential function exists. Furthermore, they call a congestion game *weighted potential game* if a potential function $\Phi\colon \Sigma_1, \dots, \Sigma_n \to \mathbb{R}_{\geq 0}$ and a vector $(w_i)_{i \in \mathcal{N}}$, $w_i \in \mathbb{R}_{>0}$, exists such that for every player $i \in \mathcal{N}$ and every strategy $s_i^* \in \Sigma_i$

$$\Phi(S \oplus s_i^*) - \Phi(S) = w_i \cdot (\ell_i(S \oplus s_i^*) - \ell_i(S)) \ . \tag{1.2}$$

The reader should note the difference between a player's weight ω_i in a weighted congestion game and the weight w_i associated with player i in the weighted potential function.

Finally, Monderer and Shapely call a weighted potential game *exact potential game* if $w_i = 1$ for every player $i \in \mathcal{N}$. A prominent class of exact potential games is the class of standard congestion games.

Proposition 1.13 (Rosenthal [Ros73])**.** *Every standard congestion game Γ is an exact potential game.*

Proof. Consider the potential function $\Phi : \Sigma_1 \times \cdots \times \Sigma_n \to \mathbb{N}$ with

$$\Phi(S) = \sum_{r \in \mathcal{R}} \sum_{i=1}^{x_r(S)} \ell_r(i) \ . \tag{1.3}$$

Let $x_r^{(i)}(S)$ denote the number of players from the subset $\{1, \dots, i\}$ of the players using resource r in state S, and let $\ell'_i(S) = \sum_{r \in s_i} \ell_r(x_r^{(i)}(S))$. Hence, we can rewrite $\Phi(S)$ as

$$\Phi(S) = \sum_{i \in \mathcal{N}} \sum_{r \in s_i} \ell_r(x_r^{(i)}(S)) = \sum_{i \in \mathcal{N}} \ell'_i(S) \ .$$

Observe that especially for the last player $\ell'_n(S) = \ell_n(S)$ holds. Hence, whenever player n switches to a different strategy s^*_n: $\Phi(S \oplus s^*_n) - \Phi(S) = \ell_n(S \oplus s^*_n) - \ell_n(S)$. However, since the players can be considered in an arbitrary order $\Phi(S \oplus s^*_i) - \Phi(S) = \ell_i(S \oplus s^*_i) - \ell_i(S)$ holds for every player i switching to a different strategy s^*_i. We conclude that Φ is an exact potential function. $\square$

In this thesis, we refer to the potential function as defined in Equation 1.3 by *Rosenthal's potential function*. Potential functions are a very elegant way to prove the existence of Nash equilibria. From the existence of a potential function we can also conclude that the local search algorithm, which sequentially selects the next player to play a better or best response, computes a Nash equilibrium after a finite number of iterations. In other words better and best response dynamics in which one after the other player plays a better or best response are guaranteed to converge to a Nash equilibrium.

Unfortunately, not every class of congestion games is a class of potential games although every game from the class possesses a Nash equilibrium. This is especially true for weighted or for player-specific matroid congestion games. The obvious way to prove that a class of games is not a class of potential games is to present games in which the local search algorithm can cycle, i. e., return to a state which it already visited. In order to prove the existence of Nash equilibria in these games different techniques are required. Such techniques are presented in Chapter 2.

1.5 State of the Art

Congestion games have a long history in economics as they capture the essence of many environments in which selfish players independently allocate resources. Probably, researchers have spent most attention to *selfish routing*, i. e., to network congestion games in which players strive to select shortest paths in a network. First results in a slightly different scenario date back to 1920, when Pigou [Pig20] considered transportation networks with selfish players and observed that selfish behavior can reduce the global performance. This model assumes an infinite number of players each of which has an infinitesimal impact on the congestion of the edges it allocates. Probably, Wardorp [War52] was the first to study such games systematically. Roughgarden [Rou05] provides a detailed introduction to selfish routing and discusses relationships between selfish routing and routing on the Internet.

Only recently, congestion games have attracted much attention among computer scientists. At first, various computational questions arise when considering congestion games. Probably the most obvious question asks for the computational complexity of Nash equilibria. Secondly, since the design and use of communication networks play an important role in the computer science community, researchers have applied congestion games to such networks in order to gain more insights on the impact of selfish behavior into the global performance. For instance, researchers successfully applied congestion games to network design problems [ADK$^+$04] as described in the introduction and to selfish load balancing scenarios [EDKM03, FPMV07, GLMT06, KP99] as frequently observed in the Internet or in wireless networks.

Subsequently, we will summarize previously known results about congestion games. We focus on the existence and computational complexity of Nash equilibria and on sequential and concurrent improvement dynamics. Furthermore, we comment on several extensions and refinements of congestion games that have been considered.

1.5.1 Existence of Nash Equilibria

It is well known that congestion games do not posses Nash equilibria in general. In this paragraph, we summarize all known results regarding the existence of Nash equilibria in standard, weighted and player-specific congestion games.

Standard Congestion Games. Applying an exact potential function argument Rosenthal [Ros73] proves that every standard congestion game possesses a Nash equilibrium. This potential function neither relays on the assumption that the latency functions are non-decreasing nor that the players' strategy spaces have a specific combinatorial structure. Proposition 1.13 states this result formally.

Player-specific Congestion Games. In general, player-specific congestion games may not possess Nash equilibria [Mil96, Mil06]. On the other hand, Milchtaich [Mil96] proves that every player-specific congestion game possesses an equilibrium if the players' strategies are *singleton sets*, although these games are not potential games. In Section 2.2 we extend his proof towards player-specific matroid congestion games. Gairing et al. [GMT06] consider a restricted class of player-specific singleton congestion games. Namely, they consider such games with linear latency functions. They show that in contrast to general player-specific singleton games these games are potential games. Additionally, Mavronicolas et al. [MMMT07] consider the existence of potential functions in standard singleton congestion games with player-specific constants.

Milchtaich [Mil06] also considers player-specific network congestion games and presents a game that does not possess an equilibrium. Because of that, he proposes to characterize those player-specific network congestion games with respect to the *topology* of the network which always possess a Nash equilibrium. First results concerning this issue are presented in [Mil06].

Weighted Congestion Games. Similar to player-specific congestion games weighted congestion games do not necessarily possess Nash equilibria [FKS05, GMV05]. Again, on the other hand, several authors independently observe that weighted singleton congestion games are ordinal potential games, and hence every such game possesses a Nash equilibrium [EDKM03, FPT04, FKK+02]. Furthermore, Fotakis et al. [FKS05] prove that every weighted network congestion game with *linear latency functions* possesses a Nash equilibrium, although weighted network congestion games with arbitrary non-decreasing latency functions do not in general [FKS05, GMV05]. The proof of the first result relies on a weighted potential function that applies to every weighted congestion game with affine latency functions regardless of the combinatorial structure of the players' strategy spaces. A similar proof is presented in [Mey06].

Player-Specific Congestion Games with Weighted Players. Milchtaich addresses the existence of Nash equilibria in congestion games which are both player-specific and weighted [Mil96]. He observes that Nash equilibria do not necessarily exist even in such singleton games. However, Georgiou et al. [GPP06] and Gairing et al. [GMT06] conjecture that every such game possesses a Nash equilibrium if the player-specific latency functions are linear functions without offsets. Georgiou et al. [GPP06] prove this conjecture for games with three players and for games with two resources, whereas Gairing et al. [GMT06] show that potential functions are no suitable tool to solve this open problem for general games.

1.5.2 Computational Complexity of Nash Equilibria

There are two fundamental computational questions related to Nash equilibria in congestion games. Given a class of congestion games which, in general, do not possess Nash equilibria we like to determine efficiently if a particular game possesses an equilibrium. Subsequently, we refer to this problem by the term *existence problem.* Furthermore, given a class of games which are guaranteed to possess Nash equilibria, we like to compute an equilibrium efficiently. Below, we refer to this problem by the term *search problem.* In this section, we summarize previously known results regarding both problems.

The Existence Problem. For various classes of congestion games which, in general, do not possess Nash equilibria it is known that the related decision problem is NP-complete. Namely, for weighted network congestion games with arbitrary non-decreasing latency functions and for singleton congestion games which are both weighted and player-specific it is NP-complete to determine if a given game possesses an equilibrium [DS06]. In Section 2.4 we extend these completeness results towards player-specific network congestion.

The Search Problem in Standard Congestion Games. Obviously, the complexity of computing Nash equilibria in standard congestion games is equivalent to the complexity of computing a local optimum of Rosenthal's potential function [FPT04]. Subsequently, we first give a short introduction to local search problems, and summarize results about the complexity of computing Nash equilibria in standard congestion games afterwards.

A *local search problem* Π is given by its set of instances $\mathcal{I}_\Pi$. For every instance $I \in \mathcal{I}_\Pi$, we are given a finite set of feasible solutions $\mathcal{F}(I)$, an objective function $c\colon \mathcal{F}(I) \to \mathbb{N}$, and for every feasible solution $S \in \mathcal{F}(I)$, a neighborhood $\mathcal{N}(S, I) \subseteq \mathcal{F}(I)$. Given an instance I of a local search problem, we seek for a *locally optimal solution* S^*, i. e., a solution that does not have a strictly better neighbor. A neighbor S' of a solution S

is strictly better if the objective value $c(S')$ is larger or smaller than $c(S)$ in the case of a maximization or minimization problem, respectively. The class PLS (polynomial time local search) is defined by Johnson et al. [JPY88]. It contains all local search problems with polynomial time-searchable neighborhoods. Johnson et al. also define the notion of PLS-reduction and prove that complete problems with respect to this kind of reduction exist. For example, they prove that computing a partition of a graph that is locally optimal with respect to the Kernighan-Lin local search algorithm is PLS-complete. Later, Schaeffer and Yannakakis [SY91] introduce a more restricted kind of PLS-reduction called *tight* PLS-reduction. This kind of reduction preserves lower bounds on the length of local search paths.

Fabrikant et al. [FPT04] notice the close relationship between the complexity of computing Nash equilibria in standard congestion games and the complexity of computing locally optimal solutions. The natural local search algorithm for computing Nash equilibria in a congestion game works as follows. As long as the current state is not a Nash equilibrium select an unsatisfied player and let this player play a better or best response. If best responses can be computed efficiently than the search problem belong to PLS. As their main result Fabrikant et al. [FPT04] prove that it is PLS-complete to compute a Nash equilibrium of standard asymmetric network congestion games. However, their reduction is quite involved as it reconsiders a very complicated PLS-reduction which shows that computing local optimal assignments of a certain kind of weighted SAT formulas is PLS-complete [SY91]. A much simpler reduction has been presented in [ARV08] which even holds in case of networks with linear latency functions without offsets. This proof relies on a reduction from *threshold games* for which PLS-completeness follows easily by a reduction from the local search variant of MaxCut. In a threshold game each player either allocates a single resource on its own, or shares a unique bunch of resources with the other players. A formal definition of these games and more details about the reductions are given in Section 4.4. The authors of [ARV08] also prove PLS-completeness for other classes of structured standard congestion games including market sharing games with polynomially bounded costs [Mir05] and overlay network design games.

Besides the previously mentioned negative results, there are also some positive ones. Fabrikant et al. [FPT04] present a polynomial time algorithm computing a Nash equilibrium in standard *symmetric* network congestion games. However, this algorithm is not a local search algorithm as it reduces the problem to a min-cost flow problem. Additionally, Ieong et al. [IMN$^+$05] prove that in case of singleton strategy spaces the local search algorithm is guaranteed to terminate quickly. In Section 3.1 we extend their proof towards matroid congestion games.

One might ask whether the complexity of computing Nash equilibria changes if one relaxes the notion of equilibria and considers *approximate Nash equilibria*. In general, this is not the case, as Skopalik and Vöcking [SV08] prove that computing an approximate Nash equilibrium in an asymmetric standard congestion game is PLS-complete. Again, this proof is quite involved as the players' strategy spaces and the latency functions have to be defined very carefully. On the other hand, Chien and Sinclair [CS07] prove that approximate equilibria in symmetric congestion games can be computed efficiently if the latency functions satisfy the α-bounded jump condition.

A latency function satisfies the α-bounded jump condition if $\ell(x+1)/\ell(x) \leq \alpha$ for every $1 \leq x \leq n$.

The Search Problem in Player-specific or Weighted Singleton Congestion Games. The complexity of computing a Nash equilibria in player-specific singleton congestion games is addressed by Milchtaich [Mil96]. His existence proof is constructive and implicitly describes an efficient algorithm computing an equilibrium. In contrast to this positive result the complexity of computing Nash equilibria in weighted singleton congestion games is a challenging open problem. Positive results are only known in two special cases. In the case of resources with linear latency functions and symmetric players a Nash equilibrium can be computed using Graham's LPT scheduling algorithm [FKK$^+$02]. In the same model, but with asymmetric strategy spaces, Gairing et al. [GLMM04] present an efficient algorithm computing a Nash equilibrium.

Finally, the authors of [AS07] consider standard network congestion games in which a player is prohibited to use certain edges. Obviously, such games can be formulated in terms of player-specific network congestion games however, they are potential games. In [AS07] it is shown that computing an equilibrium is PLS-complete even in games with only three players.

1.5.3 Sequential Best Response Dynamics

Best response dynamics arise if players sequentially play best responses. It is of particular interest when considering such dynamics whether they converge to an equilibrium at all, and if so, how long it takes. Obviously, in case of potential games they are guaranteed to converge, whereas they can cycle, i. e., return to states already visited, in non-potential games. Below, we summarize previously known results on the convergence time of best response dynamics. Note that these results are closely related to the computational complexity of Nash equilibria in the case of potential games.

Standard Congestion Games. Fabrikant et al. [FPT04] observe that the convergence time of best response dynamics in standard congestion games can be exponential. From their PLS-completeness result about the computational complexity of Nash equilibria they conclude that there exist games with appropriately chosen initial states from which the number of best responses until players finally reach an equilibrium is exponentially. Note that these results hold, regardless of the next player to play a best response, as in every intermediate state there is exactly one player which can play a best response. This negative result still holds in case of standard symmetric network congestion games, although an equilibrium can be computed efficiently [ARV08, FPT04]. On the positive side, best response dynamics converge quickly in standard singleton congestion games [IMN$^+$05]. In Section 3.1 we extend this result towards standard matroid congestion games. Additionally, best response dynamics in standard network congestion games with linearly independent paths are guaranteed to terminate quickly [Fot08]. In such games each path contains an edge that is not contained in any other path.

The negative result about the convergence time also holds in general standard congestion games with ε-greedy players, i. e., in games in which players only deviate if their latency decreases by a relative factor of at least $1 + \varepsilon$ [ARV08, CS07, SV08]. If, however, the latency functions satisfy the α-bounded jump condition then best response dynamics are guaranteed to terminate quickly in symmetric congestion games [CS07].

Weighted Congestion Games. Even-Dar et al. [EDKM03] consider better and best response dynamics in weighted singleton congestion games under different assumptions on the players' weights, the latency functions, and the schedule selecting the next player to play a best response. For most scenarios they prove pseudo-polynomial upper bounds and corresponding lower bounds on the convergence time. Goldberg [Gol04] considers a schedule which selects the next player to act at random. This player then migrates to a randomly selected resource if this improves its latency. In general, the expected time to reach a Nash equilibrium is pseudo-polynomial whereas it is polynomial in the case of unweighted players.

Player-Specific Congestion Games. Milchtaich observes that best response dynamics in player-specific singleton congestion games can cycle. However, he also shows that from every state of such a game there exists a polynomially long sequence of best responses leading to a Nash equilibrium. Hence, random best response dynamics which select the next player to deviate uniformly at random are guaranteed to reach an equilibrium with probability one. In Section 3.3 we extend this result towards player-specific matroid congestion games and better responses. We also address the expected time until random best response dynamics in player-specific singleton congestion games terminate at an equilibrium.

At this point, we like to mention that better and best response dynamics in two-sided matching markets possess similar properties as such dynamics in player-specific singleton congestion games do. For an introduction to two-sided matching markets, we refer the reader to Section 1.5.7.

1.5.4 Concurrent Improvement Dynamics

In concurrent improvement dynamics players are permitted to change their strategies concurrently. Obviously, in this case it might happen that several players choose to select the same strategy at the same point in time. In this case, the latency observed after the strategy change might be larger than the anticipated latency before the strategy change. In order to compensate such effects various randomized protocols concurrently applied by the players have been proposed and analyzed with respect to their expected convergence time to Nash equilibria. They differ in the assumptions made about the players' knowledge and about the latency functions. However, all these protocols are specially designed for singleton congestion games. In Chapter 4 we propose a protocol that applies to arbitrary symmetric strategy spaces, too.

Even-Dar and Mansour [EDM05] consider concurrent protocols in a setting where the resources have linear latency functions without offsets. Their protocols require global knowledge in the sense that the players must be able to determine the set of under- and overloaded resources. Given this knowledge, the convergence time is doubly logarithmic in the number of players. Berenbrink et al. [BFG$^+$06] consider a protocol for the case that the latency on a resource equals the load. Note that in this case Nash equilibria are essentially unique. This protocol does not rely on global knowledge as in each round every player samples a resource uniformly at random and migrates with probability $\max\{0, 1 - y/x\}$ to that resource. Here, x denotes the congestion of the resource the player currently allocates and y denotes the congestion of the sampled resource. The convergence time of this protocol is also doubly logarithmic in the number of players but polynomial in the number of links. Berenbrink et al. [BFHH07] also generalize the protocol described above to the case of weighted players. In this case, the convergence time is only pseudo-polynomial, i. e., polynomial in the number of players, resources, and in the maximum weight. Fotakis et al. [Fot08] consider a scenario with non-decreasing latency functions. Their protocol involves local coordination among the players sharing a resource in the sense that in every round at most one player leaves a resource. For the family of games in which the number of players asymptotically equals the number of resources they prove fast convergence to almost Nash equilibria. Intuitively, an almost Nash equilibrium is a state in which there are not too many too expensive or too cheap resources. We consider a similar notion of equilibria in Section 4.2.2. Finally, Fischer et al. [FMSV08] consider a scenario in which no information about potential target resources is available. The authors present an efficient protocol in which the probability to migrate depends purely on the latency of the currently selected strategy.

1.5.5 Inefficiency of Equilibria

Most of the existing literature on congestion games investigates the impact of selfish behavior on the global performance of the system. For that reason, we need to define the global performance or social cost first. The social cost of a state of a congestion game is either the sum of the players latencies or the maximum taken over the players' latencies. In both cases, the *price of anarchy* of a game is defined as the ratio of the social cost of the worst Nash equilibrium to that of a state minimizing the social cost. This quantity is related to the approximation ratio of approximation algorithms which measures the lack of unbounded computational power [Vaz01], and to the competitive ratio of online algorithms which measures the lack of complete information [Bor98]. A high-level summary of the various results on the price of anarchy is that under reasonable assumptions on the latency functions selfish behavior does not reduce the global performance too much, i. e., by constant factors only. Below we summarize the most important results on the price of anarchy in congestion games. For a detailed discussion and further results we refer the reader the Chapters 17-20 in [NRTV07] and the references given therein.

A number of publications consider the price of anarchy in the KP-model which was presented first by Koutsoupias and Papadimitriou [KP99]. This model coincides with our definition of weighted singleton congestion games, except that the resources always have linear latency functions, and that the players may also choose mixed strategies. Czumaj and Vöcking [CV07] prove a tight bound of $\Theta(\log m/ \log\log\log m)$

on the price of anarchy for such games. Here the social cost is the maximum over the players' latencies. For a summary of all results in the KP-model we refer the reader to Chapter 20 in [NRTV07].

Among others, the price of anarchy in standard congestion games is considered in [AAE06, CK05]. Both papers present almost identical results and focus on the sum of the players' latencies as the measure of social cost. In the case of linear latency functions they prove an upper bound of 5/2. Additionally, in the case of polynomial latency functions with maximum degree d an upper bound of $d^{\Theta(d)}$ is shown. Both results also hold in the case of mixed strategies. Furthermore, Awerbuch et al. [AAE06] also consider weighted congestion games with linear latency functions and prove an upper bound of 2.618 on the price of anarchy for such games. However, as there exist such games which do not possess Nash equilibria Goemans et al. [GMV05] propose to study the *price of sinking* in such games. Sink equilibria are guaranteed to exist in every congestion game, as they correspond to sinks of the best response graphs, i. e., to subsets of the states that cannot be left again once the dynamics has reached one of them.

Another series of papers considers the convergence time of sequential best response dynamics to states which are approximately socially optimal with respect to the sum of the players latencies [AAE$^+$08, CMS06, GMV05, FFM08, Mir05, MV04]. Mirrokni [Mir05] gives an extensive summary of the early results in this branch of research. Probably the most interesting result is due to Fanelli et al. [FFM08]. If players are activated in a round robin fashion and if the latency functions are linear, then $\log\log n$ many rounds, i. e., at most $n\log\log n$ best responses, suffice to reach a state which is socially optimal up to a constant factor. In Section 4.3 we follow a similar approach, though, we consider concurrent dynamics instead of sequential ones.

Finally, we like to mention that the price of anarchy is also considered in player-specific singleton congestion games with linear latency functions without offset [GMT06].

Complexity of Computing Socially Optimal States The computational complexity of socially optimal states is a matter of interest in its own right. In weighted singleton congestion games this problem is obviously NP-hard. The same holds if we seek for a socially optimal Nash equilibrium [GLM$^+$05, FKK$^+$02]. Meyers [Mey06] fully categorizes the computational complexity of computing socially optimal states in standard network congestion games as she considers games with respect to different assumptions on the structure of the game and on the latency functions. In almost all cases she proves NP-hardness even if we care about approximate solutions only. Note that she considers the sum of the players' latencies.

Chakrabarty et al. [CMN05] consider the same problem in player-specific singleton congestion games with general non-decreasing latency functions. They prove that under various assumptions computing socially optimal states is NP-hard although for certain special cases polynomial time algorithms exist. Finally, Blumrosen and Dobzinski [BD07] consider the same model as Chakrabarty et al., however they assume that the players strive to maximize their latencies and hence they are interested in maximizing this sum. They provide hardness results, approximation algorithms, and point out relationships to combinatorial auctions.

1.5.6 Extensions and Refinements of Congestion Games

Since the first presentation of congestion games, various extensions and refinements have been proposed. Subsequently, we give a summary of those we are aware of.

Splittable Demands In weighted network congestion games it is often unrealistic to assume that players only strive to select a single path. For that reason several models in which players can split their traffic over several paths have been considered. Note that in all these models the latency of a player on a particular edge is weighted by the amount of flow sent by that player along the edge.

Meyers [Mey06] considers a model in which each player can assign traffic up to k paths (k-splittable). Under different assumptions on the amount of traffic that must be assigned to a path she considers the existence of Nash equilibria and proves that they are guaranteed to exist if the latency functions are linear. Meyers also considers the price of anarchy in such games. Dunkel and Schulz [DS06] consider a closely related model. However, in their model Nash equilibria do not necessarily exist, and determining if they exist is NP-hard.

Other works consider weighted network congestion games in which players can split their traffic arbitrarily, i. e., they strive to minimize the latency of a flow [ABJS93, CCM06, ORS93, RS05]. Under the assumptions that the latency functions are convex the existence of Nash equilibria in such games follows immediately from a very general result of Rosen [Ros65]. Orda et al. [ORS93] study conditions on the latency functions that guarantee uniqueness of Nash equilibria in singleton games, whereas Altman et al. [ABJS93] prove that in the case of monomial latency functions Nash equilibria are guaranteed to be unique even in arbitrary networks. Furthermore, Richman and Shimkin [RS05] present a characterization of networks with one source and one sink such that Nash equilibria are unique. Their result even holds in the case of player-specific latency functions. Finally, Cominetti et al. [CCM06] study the price of anarchy in such games.

Non-Atomic Players. In our definition of congestion games the number of players is finite, hence, each of them has a non-negligible impact on the congestion of the resources it allocates. In some scenarios, however, there are millions of participants, each of them having a negligible impact on the congestion. For example, this is true in road traffic. In order to study such scenarios Wardrop [War52] proposed to study non-atomic congestion games with an infinite number of players each carrying an infinitesimal small amount of traffic. Much of the existing literature on selfish routing is dedicated to the Wardrop model and many questions considered in atomic congestion games have previously been studied in non-atomic games. For an introduction to this model taking recent results into account we refer the reader to [Rou05].

Of particular interest for this thesis are the results presented by Fischer [Fis07] and Fischer et al. [FRV06]. They propose and analyze convergence properties of a concurrent imitation protocol according to which players strive to improve on their latency over time. The approach presented in Chapter 4 is inspired by their work.

Bottleneck Games. In congestion games, the latency a player sustains is the sum of the latencies of the resources the player allocates. In contrast to this definition, Banner and Orda [BO07] consider weighted network congestion games in which the latency of a player is determined by the slowest edge. Banner and Orda distinguish between games in which the players may or may not split their traffic among several paths. They focus on the existence of Nash equilibria and show that they always exist. Furthermore, they care about convergence properties and the price of anarchy.

Selfishness, Altruism and Spite. The notion of Nash equilibria in congestion games relies on the assumption that players act selfishly, i. e., they neither care about the latency of others nor try to harm others explicitly. Only recently, several models taking altruistic and spiteful behavior into account have been proposed, see e. g. [CK08, HS08] and the references therein. Most related to our model of congestion games is the one by Hoefer and Skopalik [HS08] who consider singleton congestion games in which each player strives to minimize a player-specific linear combination of its individual latency and the social cost. They present results about the existence and computational complexity of Nash equilibria in such games.

Local-Effect Games. In standard congestion games, the latency of a resource solely depends on the congestion on that resource. In order to study games with local effects between resources, Leyton-Brown and Tennenholtz [LBT03] propose to study standard singleton congestion games in which for every ordered pair of resources there also exists a latency function accounting for the effect of the congestion of the first resource on the latency of the second one. They focus on the existence of Nash equilibria in such games and deduce conditions for the existence of potential functions. Dunkel and Schulz [DS06] consider a special case of local effect games and prove that computing Nash equilibria is PLS-complete for such games.

1.5.7 Games with Priorities

So far resources treat players as equal, that is, a resource cannot refuse a player to use it. Even more, in standard congestion games all players allocating a resource observe the same latency. In other words, all players sharing a resource are processed in round-robin with infinitesimally small rounds. Immorlica et al. [ILMS05] and Farzad et al. [FOV08a] propose models in which the order according to which the players sharing a resource are processed is determined by a mechanism.

Immorlica et al. [ILMS05] consider weighted singleton congestion games and propose to study coordination mechanisms that define in which order players sharing a resource are processed. Hence, the latency of a player only depends on the weight of the players with higher priority. They focus on the price of anarchy with respect to the maximum of the players' latencies in such games. Farzad et al. [FOV08a] take a similar approach, however, they consider standard network congestion games in which the order of the players may also depend on the time needed to reach a resource. Again, they focus on the price of anarchy in such games.

Another model that solely relies on priorities is the notion of two-sided matching markets which we discuss next.

Two-Sided Matching Markets Actually, two-sided matching markets are no refinement of congestion games, but a completely different notion of games. In Chapter 5, however, we present refinements of player-specific singleton congestion games and of two-sided matching markets that show strong relations between the two models.

Two sided-matching markets were introduced by Gale and Shapley [GS62] to model markets on which different kinds of agents are matched to one another, for example men and women, students and colleges [GS62], interns and hospitals [Rot84]. Using the same terms as for congestion games, we say that the goal of a two-sided matching market is to match players and resources (or markets). In contrast to congestion games, each resource can only be matched to one player. With each pair of player and resource a payoff is associated, and players are interested in maximizing their payoffs. Hence, the payoffs implicitly define a preference list over the resources for each player. Additionally, each resource has a preference list over the players that is independent of the profits. Every player can strive to allocate a resource and if several players strive to allocate the same resource, only the most preferred player is *assigned* to that resource and receives the corresponding payoff. This way, every set of proposals corresponds to a bipartite matching between players and resources. A matching is *stable* if no player can be assigned to a resource from which it receives a higher payoff than from its current resource given the proposals of the other players. Gale and Shapley [GS62] show that stable matchings always exist and can be found in polynomial time. Since the seminal work of Gale and Shapley there has been a significant amount of work in studying two-sided matching markets. See for example the book by Knuth [Knu76], by Gusfield and Irving [GI89] or by Roth and Sotomayor [RS90].

Knuth [Knu76] proposes to study better and best response dynamics in two-sided matching markets and observes that they can cycle. However, Roth and Vande Vate [RV90] observe that short better response paths to stable matchings always exists. The authors of [AGM$^+$08] follow this line of research and prove an exponential lower bound on the expected time until random better (best) response dynamics terminate. These results are related to the convergence time of random best response dynamics in player-specific singleton congestion games as presented in Section 3.3.

1.5.8 Other Equilibrium Concepts

Besides the notion of Nash equilibria, game theory provides several other solution concepts to strategic games. For an introduction to these concepts we refer the reader to his or her favorite book about game theory. Here we comment on those concepts that have been studied in congestion games.

So far we assumed that players are selfish and do not care about others. However, it is also natural to consider games in which players can join and leave coalitions. Fotakis et al. [FKS08b] consider weighted singleton congestion games with identical resources and with static coalitions. That is, players decide which coalition to join in advance and never leave it again. They provide results on the existence and computational complexity of equilibria in such games and consider the price of anarchy.

Holzman and Law-Yone [HLY97] consider standard congestion games in which players can join and leave coalitions dynamically. In this case, a strong equilibrium is a state in which no subset of the players can improve on their individual latency if they would form a coalition and change their strategies in a cooperative way. Holzman and Law-Yone provide a characterization of the combinatorial structure of the players strategy spaces that guarantees the existence of strong equilibrium. This result is similar to our results presented in Section 2.3 and Section 3.1.2.

Finally, Papadimitriou [Pap05] studies the computational complexity of correlated equilibria in congestion games and proves that they can be computed efficiently if the players strategy spaces are given explicitly. Last but not least, among others, Gairing [Gai06] and Tiemann [Tie07] study Bayesian equilibria in congestion games.

1.6 Outline and Bibliographical Notes

In this thesis we consider two fundamental problems related to congestion games and the notion of Nash equilibria. At first, we consider the existence of Nash equilibria in player-specific and in weighted congestion games. Secondly, we study dynamics that arise when players actually play a congestion game. We also propose a refinement of player-specific congestion games in which the resources assign priorities to the players. Each of the following paragraphs motivates and states the results presented in the chapter with the same title. However, more intensive motivations and discussions can be found at the beginning of each chapter.

Existence of Nash Equilibria It is well known that every standard congestion game possesses a Nash equilibrium [Ros73], whereas, in general, weighted congestion games and player-specific congestion games do not [FKK+02, Mil96]. On the other hand, Fotakis et al. [FKK+02] and Milchtaich [Mil96] prove that every weighted congestion game and every player-specific congestion game possesses a Nash equilibrium if the strategies are singleton sets only. In Chapter 2 we extend their proofs towards weighted and towards player-specific *matroid* congestion games. Furthermore, we show that the matroid property is the maximal property of the combinatorial structure of the strategy spaces of individual players that guarantees the existence of Nash equilibria in such games. Our characterization is based on the characterization of non-matroid set systems as presented in Section 1.3.2. Additionally, we consider the computational complexity of determining whether a player-specific network congestion game possesses a Nash equilibrium. We prove that this problem is NP-complete.

The results are presented in Chapter 2. In preliminary form they already appeared in [ARV06, AS07].

Best Response Dynamics Rosenthal [Ros73] shows that standard congestion games are potential games. Besides the fact that every such game possesses a Nash equilibrium this also implies that best response dynamics are guaranteed to terminate at a Nash equilibrium after a pseudo-polynomial number of steps. Such dynamics arise if at each point in time a single player plays a best response. In this thesis, we consider the maximum number of steps until best response dynamics terminate at a Nash equilibrium. Subsequently, we refer to this quantity by the convergence time of best response dynamics. Previous work shows an exponential lower bound on the convergence time in standard network congestion games [FPT04]. On the other hand, there exists a polynomial upper bound on the convergence time of best response dynamics in standard singleton congestion games. At first, we extend this positive result towards matroid congestion games and prove that best response dynamics in standard matroid congestion games are guaranteed to terminate quickly. Additionally, we show that the matroid property is the maximal property of the combinatorial structure of the strategy spaces of individual players that guarantees polynomial convergence time.

In case of weighted singleton congestion games, which are known to be potential games, too, we present an exponential lower bound on the convergence time of best response dynamics.

Additionally, we consider best response dynamics in player-specific singleton congestion games. It is well known that there exist games in which best response dynamics can cycle. On the other hand, it is also known that from every state of such a game there exists a polynomially long sequence of best responses leading to a Nash equilibrium [Mil96]. Hence, random best response dynamics in which the next player to act is selected uniformly at random terminate after a finite number of steps with probability one. We present empirical evidence supporting our conjecture that there exists an exponential lower bound on the expected convergence time of random best response dynamics in player-specific singleton congestion games. Our conjecture is motivated by a careful analysis of games in which each player chooses between two resources.

The results are presented in Chapter 3. In preliminary form they already appeared in [Ack07, AR08, ARV08].

Imitation Dynamics In unknown scenarios players often have only little or no experience at all upon which they can base their decisions. This is in contrast to the notion of Nash equilibria in which players have complete knowledge about the game. An obvious approach to act in unknown scenarios is to imitate successful behavior. In order to take such behavior into account, we propose to study imitation dynamics in standard congestion games. In such dynamics players imitate each other in a round based fashion according to the following protocol. At first, each player samples another player uniformly at random. Then it considers the latency gain that it would have by adopting the strategy of the sampled player, under the assumption that no one else changes its strategy. If this latency gain is not too small the player adopts the sampled strategy with a *migration probability* mainly depending on the anticipated latency gain. We focus on convergence properties of such dynamics. Using a potential function argument, we show that imitation dynamics converge in a monotonic fashion to stable states. In such a state none of the players can improve its latency by imitating somebody else.

Furthermore, we show rapid convergence to approximate equilibria. In an approximate equilibrium only a small fraction of players sustains a latency significantly above or below average. In particular, imitation dynamics behave like fully polynomial time approximation schemes (FPTAS). Fixing all other parameters, the convergence time depends only in a logarithmic fashion on the number of players.

Obviously, imitation processes are not innovative, hence they cannot discover unused strategies. Furthermore, strategies may become extinct with non-zero probability. For the case of singleton games, we show that the probability of this event occurring is negligible. Additionally, we prove that the expected social cost of a stable state reached by our dynamics is not much worse than an optimal state. This result applies to singleton congestion games with linear latency function. Finally, we discuss how the protocol can be extended such that, in the long run, the dynamics converges to a Nash equilibrium.

The results are presented in Chapter 4. In preliminary form they already appeared in [ABFH08].

Congestion Games with Priorities So far, the notion of congestion games does not capture the fact that in some scenarios resources might want to foster certain players. For this reason, we propose to study player-specific singleton congestion games with priorities in which each resource assigns a rank to every player. Suppose that several players strive to allocate the same resource. In our model only the players which are ranked highest by the resource observe finite latency depending on their number. The other players, however, observe infinite latency on it. Intuitively, they are displaced by the players with higher priority. We prove that every player-specific singleton congestion game with priorities possesses a Nash equilibrium.

Furthermore, we observe that our model is closely related to two-sided matching markets. In a two sided market we are given a set of players and a set of markets. Each player strives to allocate a market maximizing its revenue. Additionally, each market has a preference list of the players such that each player receives a unique rank. The goal is to match players to markets such that none of the players wants to switch to another market. It is well known that such a stable matching always exists [GS62]. We extend the notion of two-sided matching markets towards games in which the markets' preference lists can contain ties. In this case all players allocating the same market and having the same maximal rank receive a payoff from the market that depends on the number of players sharing that market.

The results are presented in Chapter 5. In preliminary form they already appeared in [AGM$^+$07].

CHAPTER 2

Existence of Nash Equilibria

As we already discussed in the introduction of this thesis, Nash equilibria are the predominant solution concept for congestion games if players have complete knowledge and act fully rational. They are stable in the sense that no player unilaterally wants to change its strategy in order to decrease its latency. In this chapter, we focus on the existence of such stable states in weighted and in player-specific congestion games, as in contrast to standard congestion games, these games do not possess Nash equilibria in general [FKS05, Mil96, Mil06]. This is especially true in network congestion games in which players want to select shortest paths in a network. It is known, however, that there exist equilibria for both of these variants if the players' strategies are singleton sets only [FKK$^+$02, Mil96]. Motivated by these positive results, we study conditions on the combinatorial structure of the strategy spaces of individual players that guarantee the existence of Nash equilibria. The assumption that each strategy space is a set of singleton sets can be seen as one such condition. We extend the positive results from singleton congestion games towards matroid congestion games, and show that both weighted and player-specific congestion games admit Nash equilibria if the strategy space of each player corresponds to the set of bases of a matroid. Both results hold, regardless of the global structure of the game and for any kind of non-decreasing latency functions. In the case of player-specific matroid congestion games, our analysis also yields a polynomial time algorithm for computing Nash equilibria. For weighted matroid congestion games, however, we do not obtain an efficient algorithm for computing Nash equilibria, but we show that players playing lazy best responses reach a Nash equilibrium after a finite number of strategy changes. Recall that a best response is lazy if it exchanges the least number of resources compared to the current strategy (see Definition 1.8).

We can also show that the matroid property is the *maximal property* of the combinatorial structure of the strategy spaces of individual players that guarantees the existence of Nash equilibria in weighted and in player-specific congestion games without taking into account how the strategy spaces of different players are interweaved. To this end, given a non-matroid set system we show how to construct a weighted and a player-specific congestion game such that the players' strategy spaces are isomorphic to the non-matroid set system and such that the game does not possess a Nash equilibrium.

Finally, given a class of congestion games which do not possess Nash equilibria in general, it is natural to ask whether the related decision problem can be solved efficiently. At the end of this chapter, we prove that deciding if a player-specific network congestion game with two players only possesses a Nash equilibrium is NP-complete. In contrast to this negative result, we also present an efficient algorithm which determines whether a player-specific network congestion game with a constant number of edges possesses a Nash equilibrium.

2.1 Weighted Matroid Congestion Games

In the following, we prove that every weighted matroid congestion game possesses a Nash equilibrium. Moreover, we show that players who are only permitted to play lazy best responses that exchange the least number of resources compared to their current strategies eventually reach a Nash equilibrium (c.f. Definition 1.8). In Section 3.2, we also show that players sequentially playing arbitrary best responses do not necessarily reach an equilibrium. Hence, in general, weighted matroid congestion games are not potential games.

Theorem 2.1. *Every weighted matroid congestion game Γ with non-decreasing latency functions possesses a Nash equilibrium. Furthermore, players reach an equilibrium after a finite number of lazy best responses.*

Proof. Let S be a state of Γ. With each resource r, we associate a pair $z_r(S) = (\ell_r(x_r(S)), x_r(S))$ consisting of the latency and the congestion of r in state S. For two resources r and r' and states S and S', let $z_r(S) \geq z_{r'}(S')$ if and only if $\ell_r(x_r(S)) > \ell_{r'}(x_{r'}(S'))$ or $\ell_r(x_r(S)) = \ell_{r'}(x_{r'}(S'))$ and $x_r(S) \geq x_{r'}(S')$. Let $z_r(S) > z_{r'}(S')$ if and only if $z_r(S) \geq z_{r'}(S')$ and $z_r(S) \neq z_{r'}(S')$. Let $\bar{z}(S)$ denote a vector containing the pairs $z_r(S)$ of all resources $r \in \mathcal{R}$ in non-increasing order, that is, $\bar{z}_j(S) \geq \bar{z}_{j+1}(S)$, where $\bar{z}_j(S)$ denotes the j-th component of $\bar{z}(S)$, for $1 \leq j < |\mathcal{R}|$. We denote by $<_{\text{lex}}$ the lexicographic order among the vectors $\bar{z}(S)$, i. e., $\bar{z}(S_1) <_{\text{lex}} \bar{z}(S_2)$ if there exists an index l such that $\bar{z}_k(S_1) = \bar{z}_k(S_2)$, for all $k < l$, and $\bar{z}_l(S_1) < \bar{z}_l(S_2)$.

Due to Corollary 1.4, in every state S which is not a Nash equilibrium there exists at least one player i who can decrease its latency by playing a lazy best response s_i^*. Since s_i^* is a lazy best response, there exists a sequence of strategies $s_i = s_i^0, \dots, s_i^k = s_i^*$ such that, for every $0 \leq j < k$, $|s_i^{j+1} \setminus s_i^j| = 1$ and

$$\ell_i(S) = \ell_i(S \oplus s_i^0) > \ell_i(S \oplus s_i^1) > \cdots > \ell_i(S \oplus s_i^k) = \ell_i(S \oplus s_i^*) \ .$$

We now claim that $\bar{z}(S \oplus s_i^{j+1}) <_{\text{lex}} \bar{z}(S \oplus s_i^j)$, for every $0 \leq j < k$. Let r_j be the unique resource in s_i^j that is not contained in s_i^{j+1} and let r_j^* be the unique resource that is contained in s_i^{j+1} but not in s_i^j. Since the latency decreases strictly with the exchange, we have

$$\ell_{r_j}(x_{r_j}(S \oplus s_i^j)) > \ell_{r_j^*}(x_{r_j^*}(S \oplus s_i^{j+1})) \ .$$

Additionally, since we assume non-decreasing latency functions,

$$\ell_{r_j}(x_{r_j}(S \oplus s_i^j)) \geq \ell_{r_j}(x_{r_j}(S \oplus s_i^j) - \omega_i) = \ell_{r_j}(x_{r_j}(S \oplus s_i^{j+1})) \ .$$

Furthermore, $x_{r_j}(S \oplus s_i^j) > x_{r_j}(S \oplus s_i^{j+1})$. Combining these inequalities implies $z_{r_j}(S \oplus s_i^j) > z_{r_j}(S \oplus s_i^{j+1})$ and $z_{r_j}(S \oplus s_i^j) > z_{r_j^*}(S \oplus s_i^{j+1})$. This yields

$$\max\left\{z_{r_j}(S \oplus s_i^{j+1}), z_{r_j^*}(S \oplus s_i^{j+1})\right\} < \max\left\{z_{r_j}(S \oplus s_i^j), z_{r_j^*}(S \oplus s_i^j)\right\}$$

and hence $\bar{z}(S \oplus s_i^j) >_{\text{lex}} \bar{z}(S \oplus s_i^{j+1})$. That is, the lexicographic order decreases with every exchange and, hence, with every lazy best response. This concludes the proof of the theorem. □

Note, that the above proof does not provide an efficient algorithm for computing Nash equilibria in weighted matroid congestion games as the number of best responses can be exponential (c.f. Even-Dar et al. [EDKM03] and Section 3.2). The invention of such an algorithm remains a challenging open problem.

2.2 Player-Specific Matroid Congestion Games

Next we consider player-specific matroid congestion games and prove that every such game possesses a Nash equilibrium. Our proof extends techniques invented for singleton congestion games [Mil96] towards matroid congestion games, and implicitly describes an efficient algorithm to compute an equilibrium of such games.

Theorem 2.2. *Every player-specific matroid congestion game* Γ *with non-decreasing latency functions possesses a Nash equilibrium.*

Proof. Recall that since the strategy space of player i corresponds to the set of bases of a matroid $\mathcal{M}_i$, all strategies of player i have the same size $\text{rk}(\mathcal{M}_i)$. In the following, we represent a strategy of player i by $\text{rk}(\mathcal{M}_i)$ tokens that the player places on the resources it allocates. Suppose that we reduce the number of tokens of some of the players, that is, player i has $k_i \leq \text{rk}(\mathcal{M}_i)$ tokens that it places on the resources of an independent set of cardinality k_i. Observe that the independent sets of cardinality k_i form the bases of a matroid $\mathcal{M}'_i$ whose independent sets correspond to those independent sets of $\mathcal{M}_i$ with cardinality at most k_i. The matroid $\mathcal{M}'_i$ is also called the k_i-*truncation* of the matroid $\mathcal{M}_i$ (c.f. Section 1.3). Hence, a game in which some of the players have a reduced number of tokens is also a matroid congestion game.

We prove the theorem by induction on the total number of tokens $\tau = \sum_{i \in \mathcal{N}} \text{rk}(\mathcal{M}_i)$ that the players are allowed to place, that is, we prove the existence of Nash equilibria for a sequence of games $\Gamma_0, \Gamma_1, \ldots, \Gamma_\tau$, where Γ_{l+1} is obtained from Γ_l by giving one more token to one of the players. Γ_0 is the game in which each player has only the empty strategy. Obviously, Γ_0 has only one state and this state is a Nash equilibrium.

Assume as induction hypothesis that player i has placed $k_i \geq 0$ tokens, for $1 \leq i \leq n$, and this placement corresponds to a Nash equilibrium of the player-specific matroid congestion game $\Gamma_l = (\mathcal{N}, \mathcal{R}, (\Sigma_i^{k_i})_{i\in\mathcal{N}}, (\ell_r^i)_{i\in\mathcal{N}, r\in\mathcal{R}})$ with $l = \sum_{i\in\mathcal{N}} k_i$, in which the set of strategies $\Sigma_i^{k_i}$ coincides with the k_i-truncation of $\mathcal{M}_i$.

Now assume that some player i_0 has to place an additional token t_0. We show how to compute a Nash equilibrium for the game Γ_{l+1} obtained from a Nash equilibrium of Γ_l by changing i_0's strategy space to the set of independent sets of size $k_{i_0} + 1$. Since an optimal basis of a matroid can be computed by a greedy algorithm, there exists a resource r_0 such that placing the token t_0 on r_0 gives an independent set of size $k_{i_0} + 1$ with minimum latency among all independent sets of the same size. Thus, assuming that the tokens of the other players are fixed, an optimal strategy for player i_0 is to place t_0 on r_0 and leave all other tokens unchanged. However, as the congestion on r_0 is increased by one, other players may want to move their tokens from r_0 in order to obtain a better independent set. We use matroid properties to show that a Nash equilibrium of Γ_{l+1} can be reached with at most $n\,m\,\mathrm{rk}(\Gamma)$ moves of tokens.

After placing token t_0 of player i_0 on resource r_0, resource r_0 has one additional token in comparison to the initial Nash equilibrium S_l of the game Γ_l. Since we assume non-decreasing latency functions, only the players with a token on r_0 might now have an incentive to change their strategies. Let i_1 be one of these players. It follows from Corollary 1.5 that i_1 has a best response in which it moves a token t_1 from resource r_0 to another resource that we call r_1. Now r_1 is the only resource with one additional token in comparison to S_l. Suppose we have not yet reached a Nash equilibrium. Only those players with a token on r_1 might have an incentive to change their strategies. Again, by applying Corollary 1.5, we can identify a player i_2 that has a best response in which it moves a token t_2 from r_1 to a resource r_2, which then is the only resource with one additional token.

The token migration process described above can be continued in the same way until it reaches a Nash equilibrium of the game Γ_{l+1}. The correctness of the process is ensured by the following invariant.

Invariant 2.3. *For every $j \geq 0$, after player i_j moves token t_j onto resource r_j,*

a) only players with a token on r_j may have an incentive to change their strategy,

b) the Nash equilibrium condition of all players would be satisfied if one ignores the additional token on r_j, that is, if each player calculates the latency on r_j as if there were one token less on this resource.

The invariant follows by induction on j: For player i_j the invariant is satisfied as this player plays a best response according to Corollary 1.5. Thus it satisfies the Nash equilibrium condition even without virtually reducing the congestion on r_j. For all other players, the validity of the invariant for j follows directly from the validity of the invariant for $j - 1$ as these players do not move their tokens.

Thus, in order to show the existence of a Nash equilibrium for Γ_{l+1}, it suffices to show that the token migration process is finite. Consider an arbitrary token t of player i. For a resource r, let $\ell_i^+(r)$ denote the latency of i on r if r has one more token than in the initial state S. Whenever t is moved by the migration process from a resource r to a resource r', it must be $\ell_i^+(r) > \ell_i^+(r')$. Hence, the token t can visit each resource at most once during the token migration process. As there are at most $n \cdot \mathrm{rk}(\Gamma)$ tokens, the migration process terminates after at most $nm\,\mathrm{rk}(\Gamma)$ steps in a Nash equilibrium of Γ_{l+1}. □

Observe that the proof of Theorem 2.2 implicitly describes an efficient algorithm to compute a Nash equilibrium with at most $n^2\, m\, \mathrm{rk}^2(\Gamma)$ moves of tokens.

Corollary 2.4. *There exists a polynomial time algorithm to compute a Nash equilibrium of a player-specific matroid congestion game with non-decreasing player-specific latency functions.*

2.3 Non-Matroid Strategy Spaces

In the previous two sections, we showed that the matroid property is a condition of the combinatorial structure of the players' strategy spaces that guarantees the existence of Nash equilibria in weighted and in player-specific matroid congestion games. In this section, we show that the matroid property is also the maximal property of the combinatorial structure of the strategy spaces of individual players that guarantees the existence of Nash equilibria in such games if one does not take into account how the strategy spaces of different players are interweaved. Our negative result shows that for every non-matroid set system there exists a weighted and a player-specific congestion game with the following properties. The strategy space of each player is isomorphic to the given non-matroid set system, and the game does not possess a Nash equilibrium. In our construction we assume that the strategy spaces of different players can be interweaved appropriately, that is, there does not exist a common combinatorial interpretation of the resources. Furthermore, we require the latency functions to be positive instead of non-negative.

If one drops these assumptions and considers special classes of congestion games in which the latency functions or the way in which the strategy spaces can be interweaved are restricted, then one can identify larger classes of weighted or player-specific congestion games that possess Nash equilibria. For instance, Fotakis et al. [FKS05] prove that every weighted congestion game possesses a Nash equilibrium if one additionally assumes that the latency functions are linear. Additionally, Milchtaich [Mil06] shows that every player-specific network congestion game possesses an equilibrium if the network graph belongs to a certain restricted class of graphs.

Theorem 2.5. *For every non-matroid antichain Σ on a set of resources $\mathcal{R}$ there exists a weighted congestion game Γ with two players whose strategy spaces are isomorphic to Σ that does not possess a Nash equilibrium. The latency functions in Γ are positive and non-decreasing.*

Proof. Given a non-matroid antichain we describe how to construct a weighted congestion game with the properties stated in the theorem. We first describe how the strategy spaces are defined and then how the latency functions are chosen.

Let Σ_1 and Σ_2 be two set systems on sets of resources $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. In the following we assume that both sets are isomorphic to Σ and that Σ_i is the strategy space of player i, for $i = 1, 2$. Due to the $(1, 2)$-exchange property we can, for every player i, identify three distinct resources $a_i, b_i, c_i \in \mathcal{R}_i$ with the properties as in Definition 1.9. Since we have not made any assumption on the global structure of the game, we can arbitrarily decide which resources from $\mathcal{R}_1$ and $\mathcal{R}_2$ coincide. The resources $\mathcal{R}_i \setminus \{a_i, b_i, c_i\}$ are exclusively used by player i. Hence, we can assume that their latencies are chosen such that the $(1, 2)$-exchange property is satisfied. Thus, to simplify matters we can assume that

$$\Sigma_1 = \{\underbrace{\{a_1\}}_{s_1^1}, \underbrace{\{b_1, c_1\}}_{s_1^2}\} \text{ and } \Sigma_2 = \{\underbrace{\{a_2\}}_{s_2^1}, \underbrace{\{b_2, c_2\}}_{s_2^2}\} .$$

In the following, we assume that $a_1 = b_2$, $b_1 = a_2$ and $c_1 = c_2$. Thus we can rewrite the strategy spaces as follows: $\Sigma_1 = \{r_1\}, \{r_2, r_3\}\}$ and $\Sigma_2 = \{\{r_2\}, \{r_1, r_3\}\}$.

We set the players' weights $\omega_1 = 2$ and $\omega_2 = 1$ and define the following increasing latency functions for the resources r_1, r_2 and r_3, where $m = |\mathcal{R}|$:

	$x_r = 1$	$x_r = 2$	$x_r = 3$
$\ell_{r_1}(x_{r_1})$	m	$20 \cdot m$	$21 \cdot m$
$\ell_{r_2}(x_{r_2})$	$5 \cdot m$	$12 \cdot m$	$15 \cdot m$
$\ell_{r_3}(x_{r_3})$	$3 \cdot m$	$4 \cdot m$	$10 \cdot m$

One can easily verify that $|\ell_i(S \oplus s_i^1) - \ell_i(S \oplus s_i^2)| \geq m$, for $i = 1, 2$, regardless of the choice of the other player. Hence, for every player, one of the inequalities in Definition 1.9 is always satisfied. This game does not possess a Nash equilibrium since player 1 prefers to play strategy s_1^2 if player 2 plays strategy s_2^1, and s_1^1 if player 2 plays strategy s_2^2. Additionally, player 2 prefers to play strategy s_2^2 if player 1 plays strategy s_1^2, and s_2^1 if player 1 plays strategy s_1^1. □

Theorem 2.6. *For every non-matroid antichain Σ on a set of resources $\mathcal{R}$ there exists a player-specific congestion game Γ with two players whose strategy spaces are isomorphic to Σ that does not possess a Nash equilibrium. The latency functions in Γ are positive and non-decreasing.*

Proof. The proof is similar to the proof of Theorem 2.5. In particular, the construction of the strategy spaces of the players is identical. The player-specific latency functions are obtained from the latency functions in the proof of Theorem 2.5 as follows: For the first player $\ell_r^1(x_r) = \ell_r(x_r + 1)$, for every resource $r \in \{r_1, r_2, r_3\}$ and every congestion $x_r \in \{1, 2\}$. For the second player $\ell_r^2(1) = \ell_r(1)$ and $\ell_r^2(2) = \ell_r(3)$, for every resource $r \in \{r_1, r_2, r_3\}$. □

In Theorems 2.1 and 2.2, we showed that every weighted and every player-specific congestion game possesses a Nash equilibrium if the strategy space of each player corresponds to the bases of a matroid. Both results are true regardless of how the strategy spaces of different players are interweaved and for every choice of non-decreasing latency functions. The previous proofs show that every non-matroid antichain can be used to construct a weighted and a player-specific congestion game with positive, non-decreasing latency functions, that does not possess a Nash equilibrium. In both proofs we assumed that there is no restriction on how the players' strategy spaces can be interweaved.

Observe that our negative result also holds if the system is not an antichain but the *pruned set system*, i. e., the set system obtained after removing all supersets, is not the set of bases of a matroid. This is because supersets cannot occur in a Nash equilibrium in the case of positive latency functions. Correspondingly, our results presented in Theorems 2.1 and 2.2 show that a weighted or a player-specific congestion game in which all pruned strategy spaces correspond to bases of matroids possesses a Nash equilibrium with respect to the pruned and, hence, also with respect to the original strategy spaces because supersets are weakly dominated by subsets in the case of non-negative latency functions. Thus, the matroid property (applied to the pruned strategy spaces) is the maximal property of the combinatorial structure of the strategy spaces of individual players that guarantees the existence of Nash equilibria in weighted and in player-specific congestion games.

Corollary 2.7. *The matroid property is the maximal property of the combinatorial structure of the pruned strategy spaces of individual players that guarantees the existence of Nash equilibria in weighted and in player-specific congestion games with non-negative, non-decreasing latency functions.*

2.3.1 An Embedding into Networks

Our negative results in Theorems 2.5 and 2.6 assume that it is possible to interweave the strategy spaces of the players in a specific manner. A legitimate question is whether our construction can nevertheless be embedded into natural classes of congestion games in which the resources have a common combinatorial interpretation. Here we demonstrate that our construction can, for instance, easily be embedded into network congestion games. However, note that we are not the first to present weighted or player-specific network congestion games which do not possess Nash equilibria [FKS05, Mil06].

Consider the network depicted in Figure 2.1. The first player likes to route its traffic from s_1 to t_1, the second player from s_2 to t_2. Observe that the sets of paths of player 1 and 2 coincide with the strategy spaces as defined above. We conclude the following corollary.

Corollary 2.8. *There exist instances of player-specific and of weighted network congestion games with non-decreasing latency functions which do not possess Nash equilibria.*

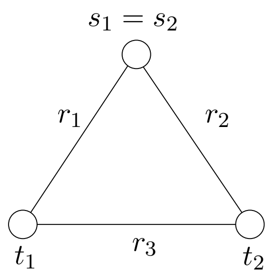


Figure 2.1: An example of a network congestion game with the strategy spaces as defined in the proofs of Theorems 2.5 and 2.6.

Observe that the players are not symmetric, i. e., they like to connect the source to different sinks. However, it is not difficult to make the game symmetric by introducing a common sink t which is connected to t_1 and t_2 and by appropriately defining the latency functions of the edges $\{t_1, t\}$ and $\{t_2, t\}$.

2.4 Player-Specific Network Congestion Games

In this section, we consider the complexity of deciding whether a player-specific network congestion game possesses a Nash equilibrium since, in general, these games do not possess Nash equilibria [Mil06]. We prove that this problem is NP-complete even in the case of two players. At first, we consider networks with directed edges and present a fairly simple reduction from the Directed-Edge-Disjoint Path problem. Unfortunately, our reduction cannot be extended towards networks with undirected edges as the Undirected-Edge-Disjoint Path problem admits a polynomial time algorithm in the case of constant number of source-sink pairs [RS95]. We therefore present a reduction from 3-Sat in the undirected case. Finally, we consider games with networks of constant sizes and present a polynomial time algorithm deciding whether a Nash equilibrium exists.

2.4.1 Networks with Directed Edges

Theorem 2.9. *It is* NP*-complete to decide whether a player-specific network congestion game with* directed edges *and two players possesses a Nash equilibrium.*

Proof. Obviously, the decision problem belongs to NP as one can decide in polynomial time whether a state S of a player-specific network congestion game with directed edges and two players is a Nash equilibrium. In order to prove that the problem is complete, we present a polynomial time reduction from the Directed-Edge-Disjoint Path problem with two disjoint source-sink pairs. An instance of this problem consists of a directed graph $G = (V, E)$ and two disjoint node pairs, (s_1, t_1) and (s_2, t_2). Given such an instance, we like to decide whether there exist pairwise edge-disjoint paths between the two node pairs. This problem is known to be NP-complete [FHW80].

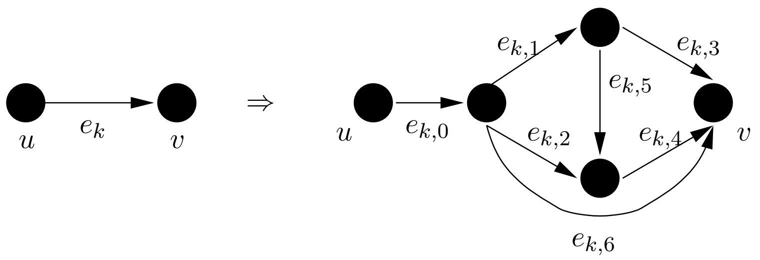


Figure 2.2: The gadget G_{e_k} by which an edge e_k is replaced.

Given an instance $(G,(s_1,t_1),(s_2,t_2))$ of the Directed-Edge-Disjoint Path problem with two source-sink pairs, we construct a player-specific network congestion game with two players as follows. We substitute every edge $e_k \in E$ by the gadget G_{e_k} presented in Figure 2.2 in order to obtain the network $G_\Gamma = (V_\Gamma, E_\Gamma)$ on which the game is played. Player $i \in \{1,2\}$ wants to allocate a path between the nodes s_i and t_i in G_Γ. Observe that this construction ensures a one-to-one correspondency between the paths in G and in G_Γ in the natural way if one ignores the precise subpaths through every gadget.

Let M be a sufficiently large number. The player-specific latency functions of the edges $e_{k,0}, \ldots e_{k,6}$ are defined as presented in Table 2.4.1. Observe that every gadget G_{e_k} implements a subgame that is played by the players if both want to allocate a path connecting u and v. If only one player wants to allocate such a path, then it allocates a player-specific shortest path from u to v. If we choose M sufficiently large such that the first player never allocates one of the edges $e_{k,2}$ and $e_{k,3}$ and such that the second player never allocates one of the edges $e_{k,5}$ or $e_{k,6}$, then the latencies of these shortest path are 3 and 5. Suppose now that the two players play such a subgame. In this case it is not difficult to verify that the subgame does not possess a Nash equilibrium.

	$e_{k,0}$		$e_{k,1}$		$e_{k,2}$		$e_{k,3}$		$e_{k,4}$		$e_{k,5}$		$e_{k,6}$	
congestion	1	2	1	2	1	2	1	2	1	2	1	2	1	2
player 1	0	M	1	1	M	M	M	M	1	20	1	1	5	5
player 2	0	M	1	20	1	1	4	5	5	5	M	M	M	M

Table 2.1: The player-specific latency functions of the edges $e_{k,0}, \ldots, e_{k,6}$.

Suppose now that we are given two node-disjoint paths P_1 and P_2 in G connecting s_1 and t_1, and s_2 and t_2. We map these paths to paths in G_Γ in the natural way and choose player-specific shortest paths through every gadget. Let $n(P_i)$ be the number of edges on path P_i. Thus, player 1 has latency $3 \cdot n(P_1)$ and player 2 has latency $5 \cdot n(P_2)$. If one of the two players had an incentive to change its strategy, then it will only choose a path in which it shares no gadget with the other player, as otherwise its latency would increase to at least M. This is true as in this case the players would share at least one edge $e_{k,0}$. This also implies that the latency of the other player does not increase due to the strategy change of the first player. Observe that this holds for any further best response. Thus, the players converge to an equilibrium after $O(n)$ best responses as the latency of a player decreases by at least the cost of the shortest path through a gadget.

Suppose now that we are given a Nash equilibrium of Γ. In this case the players do not share a gadget as otherwise the state is no Nash equilibrium. Hence, we can easily construct edge-disjoint paths in G connecting (s_1, t_1) and (s_2, t_2). □

2.4.2 Networks with Undirected Edges

Theorem 2.10. *It is* NP*-complete to decide whether a player-specific network congestion game with* undirected edges *and two players possesses a Nash equilibrium.*

Proof. Obviously, the decision problem belongs to NP as one can decide in polynomial time whether a state S of a player-specific network congestion game with undirected edges and two players is a Nash equilibrium. In order to prove that the problem is complete, we present a polynomial time reduction from 3-Sat. Let φ be a 3-Sat formula with n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$. We assume that every clause in φ contains exactly three pairwise disjoint literals. Given φ we construct a player-specific network congestion game with undirected edges and two players as follows. For the sake of simplicity we refer to the players as bit and clause player. Our construction satisfies the following three properties.

1. The bit player can choose between 2^n different paths each of them determining a unique assignment of the n variables.
2. The clause player can check for every clause separately whether there exists a variable satisfying that clause.
3. If and only if there exists an unsatisfied clause, then both players are forced to choose paths through a special gadget. This gadget implements a subgame which does not possess a Nash equilibrium if both players participate.

In the following we define three different kinds of gadgets called variable, clause and subgame gadget and describe how they are connected. The gadgets consist of bold, dashed and dotted edges. Bold edges appear in all three kinds of gadgets, whereas dotted edges do not appear in the variable gadgets, and dashed edges do not appear in the clause gadgets. The player-specific latency functions will be chosen in such a way that the bit player never chooses one of the dotted edges, and that the clause player never chooses one of the dashed edges.

For every variable x_i there is a variable gadget G_{x_i} as depicted in Figure 2.3. Without loss of generality let $\{C_1, \ldots, C_k\}$ be the set of clauses in which x_i appears as positive literal. For every such clause C_j there is a bold edge $e_{i,j}$ on the upper path in G_{x_i}. Additionally, let $\{C_{k+1}, \ldots, C_l\}$ be the set of clauses in which x_i appears as negative literal. For every such clause C_j there is a bold edge $\bar{e}_{i,j}$ on the lower path in G_{x_i}. Bold edges are connected by dashed edges as shown in the figure. The order of the

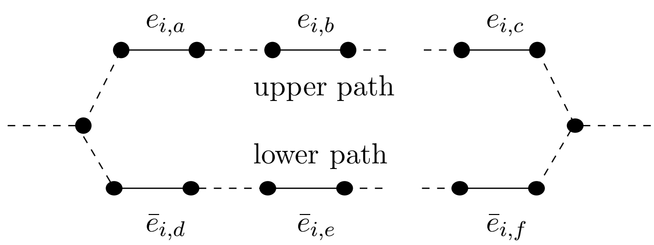


Figure 2.3: The bit gadget of variable x_i.

bold edges can be chosen arbitrarily. Additionally, the gadgets G_{x_i} are arranged one after the other starting with gadget G_{x_1} and finishing with gadget G_{x_n}. They are connected by dashed edges.

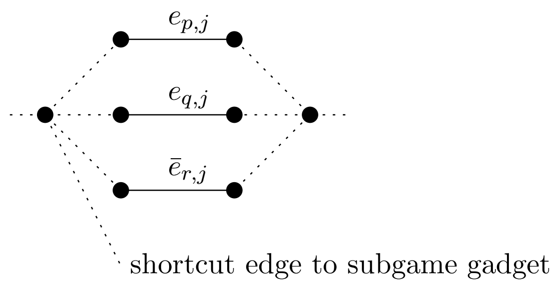


Figure 2.4: The clause gadget of clause $C_j = (x_p, x_q, \bar{x}_r)$.

For every clause C_j there is a clause gadget G_{C_j} as depicted in Figure 2.4. For every variable x_i which appears as positive literal in C_j there is a bold edge $e_{i,j}$. For every variable x_i which appears as negative literal in C_j there is a bold edge $\bar{e}_{i,j}$. Observe that edges $e_{i,j}$ and $\bar{e}_{i,j}$ coincide with the corresponding edges in the variable gadgets G_{x_i}. Bold edges are connected by dotted edges as shown in the figure. Additionally, there is a shortcut edge from the leftmost node from every clause gadget to the subgame gadget. The gadgets G_{C_j} are arranged one after the other starting with gadget G_{C_1} and finishing with gadget G_{C_m}. They are connected by dotted edges.

The subgame gadget is depicted in Figure 2.5. Basically it consists of three bold edges e_1, e_2, e_3 arranged as a triangle. The dotted shortcut edges from the clause

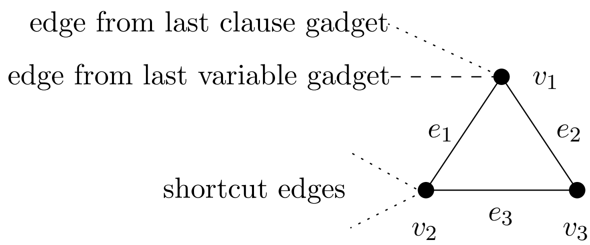


Figure 2.5: The subgame gadget

gadgets are connected to vertex v_2. Additionally, there is a dotted edge from the rightmost node of the clause gadget G_{C_m} to node v_1, and there is a dashed edge from the rightmost node of the subgame gadget G_{x_n} to node v_1. Note that this gadget is the same one as considered in Section 2.3.1 except that we exchange the source and the sink of the first player.

It remains to define the source and target nodes of the players and the latency functions of the edges. The bit player wants to allocate a path from the left most node of the variable gadget G_{x_1} to the node v_3 of the subgame gadget. The clause player, however, wants to allocate a path connecting the leftmost node from the clause gadget G_{C_1} with the node v_1 of the subgame gadget.

The player-specific latency functions are defined as follows. Let M be a sufficiently large number. The bit player always has latency 0 on bold and dashed edges, and latency M on every dotted edge. On bold edges the clause player has latency 0 if it does not share it with the variable player, otherwise it has latency M. On dotted edges the clause player always has latency 0. The player-specific latency functions of the edges $e_1, \ldots, e_3$ are depicted in Table 2.2. If we choose M sufficiently large, then the bit player never allocates a dotted edge, and the clause player never allocates a dashed edge. For the simplicity of presentation, we assume that both players always allocated cycle-free paths as best responses, that is, they never choose paths visiting a node twice. One can easily enforce this by a slightly modification of the latency of the dashed and dotted edges.

	e_1		e_2		e_3	
congestion	1	2	1	2	1	2
clause player	20	21	12	15	4	10
bit player	1	21	5	15	3	10

Table 2.2: The player-specific latency functions of the edges e_1, e_2, e_3.

Suppose now that there exists a satisfying assignment $\bar{x}$ of the given 3-SAT formula. In this case, we can construct a Nash equilibrium as follows. If $x_i = 0$, then the bit player chooses the upper path in gadget G_{x_i}, otherwise it chooses the lower path. Intuitively, it chooses a path that corresponds to the negation of $\bar{x}$. Additionally, it chooses the player-specific shortest path with respect to congestion 1 in the subgame gadget connecting v_1 and v_3. This path is simply the edge e_2. In this case, the bit player has latency 3. Observe that this is the globally shortest path of the bit player. The clause player chooses a path through every clause gadget along which it does not share a bold edge with the bit player. This is possible since $\bar{x}$ is a satisfying assignment and since the bit player chooses a path that corresponds to the negation of $\bar{x}$. In this case, the bit player enters the subgame gadget at its target node v_1. Observe that this path has latency 0, which is best possible. We conclude that we can construct a Nash equilibrium if we are given a satisfying assignment, since we can assign both players to globally shortest paths.

Suppose now that no satisfying assignment exists. In this case, there always exists an unsatisfied clause C_j, and thus the clause player cannot choose a path through that gadget along which both players do not share a bold edge. Due to the choice of M, the bit player always switches to the shortcut edge of G_{C_j} as best response and enters the subgame gadget at node v_2. Since the bit player always enters the subgame gadget at v_1, the players are forced to play the subgame defined by the subgame gadget. By the same arguments as in Section 2.3.1 this gadget does not possess a Nash equilibrium. We conclude that no Nash equilibrium exists if no satisfying assignment exists. □

2.4.3 Networks of Constant Size

Theorem 2.11. *One can decide in polynomial time whether a player-specific network congestion game* Γ *with a constant number of (un)directed edges possesses a Nash equilibrium.*

Proof. The algorithm we present generalizes a technique introduced by Chakrabarty et al. [CMN05] in order to compute a social optimal state of a player-specific singleton congestion game with a constant number of resources. Without loss of generality let $\mathcal{P} = \{P_1, \dots P_l\}$ be the set of all simple paths in the network. Note that $l \leq 2^m$ is constant. Given a state S of Γ we slightly abuse notion and denote by $x(S)$ the congestion vector $(x_1(S), \dots, x_l(S))$ where $x_i(S)$ equals the number of players choosing path P_i in S. Observe that there are at most n^l such vectors. In the following, we describe how to decide whether there exists an equilibrium S of Γ such that $\bar{x}(S)$ equals a given congestion vector $\bar{x} = (x_1, \dots, x_l)$.

Given a congestion vector $x = (x_1, \dots, x_l)$ we construct a directed graph $G_x = (\{s,t\} \cup \mathcal{N} \cup \mathcal{P}, E(x))$ with edge capacities as follows. For every player $i \in \mathcal{N}$ there is a vertex u_i which is connected to the vertex s. The capacity of such an edge equals 1. For every path $P_j \in \mathcal{P}$ there is a vertex v_j which is connected to the vertex t. The capacity of such an edge equals x_j. Furthermore, a vertex v_i is connected to a vertex w_j if the following conditions are satisfied:

1. Player i does not want to change its strategy if it would play strategy P_j.
2. The congestion on the edges is determined by the vector x.

Now we like to decide whether there exists a s-t-flow of capacity n. Observe that such a flow exists if and only if x_j units of flow can flow from x_j different player vertices u_i to path vertices v_j. Thus, if such a flow exists, and if we assign a player to that path to which the unit of flow originating in its vertex flows, we obtain a Nash equilibrium, since the construction ensures that the player is satisfied.

Finally, since there are polynomially many different vectors x, and since the construction and analysis of G_x can be done in polynomial time, we obtain a polynomial time algorithm. □

The running time of the algorithm is $O(poly(2^m) \cdot poly(n^{2^m}))$. An interesting open problem is to prove that the problem is *fixed parameter tractable*, that is, to develop an algorithm with running time $O(poly(2^m) \cdot poly(n))$. Finally, note that the above algorithm is not restricted to networks but applies to every player-specific congestion game with constant number of resources.

CHAPTER 3

Best Response Dynamics

In a Nash equilibrium each player has chosen the best strategy given fixed choices of the others, hence, none of them wants to change its strategy. At every non-equilibrium, however, at least one player wants to deviate from its current strategy and select a different one strictly decreasing its latency. This observation immediately motivates to study dynamics that arise when players sequentially play best responses.

In this chapter, we consider such *best response dynamics* and aim to upper and lower bound the number of steps until they terminate in a Nash equilibrium. We assume that the players have complete knowledge about the game and about its current state. This guarantees that players can compute best responses. Furthermore, we assume that two players do not change their strategies at the same time. In this case, a minimum of coordination among the players is needed which can be achieved, e. g. by the means of a *schedule* selecting the next player to act given the current state of the game and probably the history of the play. There are many different approaches to design such a schedule. For example:

- the *round robin schedule* selects the next player according to a fixed permutation of the players. If the selected player has no incentive to change its strategy, then the next player is selected.
- among all unsatisfied players the *random schedule* chooses a player uniformly at random.
- among all unsatisfied players the *largest improvement schedule* chooses that player which can decrease its latency most.
- among all unsatisfied players the *longest waiting time schedule* selects that player which has not been selected for the longest period of time.

Obviously, in case of potential games every best response dynamics terminates after a finite number of steps regardless of the chosen schedule. Monderer and Shapley [MS96] refer to this property by the term *finite improvement property.* In this chapter, we take a worst case perspective on best response dynamics in specific classes of potential games, and aim to upper and lower bound the maximum length of best response dynamics. In other words, we aim to upper and lower bound the maximum number of best responses until players reach a Nash equilibrium which holds regardless of the chosen schedule and the initial state. We refer to this quantity by the *convergence time of best response dynamics.*

In non-potential games, however, it can happen that sequential best response dynamics never reach a Nash equilibrium at all, since the dynamics can cycle. However, if from every state of a non-potential game a sequence of best response which leads to a Nash equilibrium exists, then the random schedule is guaranteed to terminate after a finite number of steps. Milchtaich [Mil96] calls games admitting this property *weakly acyclic.* In this chapter, we also consider the maximum expected convergence time of the random schedule in a class of weakly acyclic congestion games in which from every state a polynomially long sequence of best responses leading to a Nash equilibrium exists. We refer to this quantity by the *convergence time of random best response dynamics.*

The rest of this chapter is organized as follows. At first, we consider the convergence time of best response dynamics in standard congestion games with arbitrary strategy spaces and then proceed with weighted singleton congestion games. Recall that both classes of games are potential games. In case of standard games, we study conditions on the combinatorial structure of the strategy spaces of individual players that guarantee polynomial convergence time. Surprisingly, the maximal property is the same as in case of the existence of Nash equilibria in weighted and player-specific congestion games. Namely, the matroid property is the maximal property of the combinatorial structure of the strategy spaces of individual players that guarantees polynomial time convergence. In case of weighted congestion games, however, we present a super-polynomial lower bound on the convergence time even if the players' strategy spaces are singleton sets only. In contrast to previous such constructions the maximum latency and the maximum weight of a player are polynomially bounded in the number of players.

Additionally, we consider best response dynamics in player-specific matroid congestion games. At first, we prove that these games are weakly acyclic, that is, we prove that from every state of such a game there exists a polynomial long sequence of better responses leading to a Nash equilibrium. We proceed to consider the convergence time of random best response dynamics in player-specific singleton congestion games and prove polynomial upper bounds for two special cases. Additionally, we present empirical results supporting the following conjecture: There exists a super-polynomial lower bound on the convergence time of random best response dynamics in general player-specific singleton congestion games.

3.1 Standard Congestion Games

In this section, we consider the convergence time of best response dynamics in standard congestion games. Since Rosenthal's potential function yields a pseudo-polynomial upper bound only, we study conditions on the combinatorial structure of the strategy spaces of individual players that guarantee polynomial convergence time.

In general, we cannot hope for a polynomial upper bound on the convergence time since Fabrikant et al. [FPT04] show that there exist standard network congestion games with initial states such that every best response sequence starting from these states needs an exponential number of steps to reach a Nash equilibrium. On the other extreme, we find singleton standard congestion games for which Ieong et al. [IMN+05] show that best response dynamics always reach a Nash equilibrium after a polynomial number of steps. Hence, the assumption that each strategy space is a singleton set can be seen as one such condition. We show that the analysis of Ieong et al. [IMN+05] can be generalized towards *standard matroid congestion games*, that is, if the strategy space of each player consists of the bases of a matroid over the set of resources, then best response dynamics are guaranteed to terminate after a polynomial number of best responses. This result holds regardless of the global structure of the game and for any kind of latency functions, even for non-monotone ones.

We can also show that the matroid property is the maximal property of the combinatorial structure of the individual players' strategy spaces that guarantees polynomial convergence time of best response dynamics without taking into account how the strategy spaces of different players are interweaved. To this end, given a non-matroid set system we show how to construct a standard congestion game such that the players' strategy spaces are isomorphic to the given set system and such that there exists an exponentially long best response sequence.

Additionally, we show that this characterization holds for ε-greedy players, too. That is, even if players only deviate if their relative latency gain is sufficiently large, then the matroid property is the maximal property that guarantees polynomial time convergence. Finally, we consider better response dynamics in which players do not necessarily play best responses but also better responses. The result of Ieong et al. [IMN+05] about the convergence time of best response dynamics in singleton games even holds for better response dynamics. However, we observe that it cannot be extended towards matroid games.

3.1.1 Matroid Strategy Spaces

Ieong et al. [IMN+05] show that in standard singleton congestion games best response dynamics terminate after at most n^2m steps. Recall that singleton games are standard matroid congestion games with $\text{rk}(M_i) = 1$ for every player i. In the following, we extend their analysis to general matroid congestion games.

Theorem 3.1. *Let Γ be a standard matroid congestion game. Then players reach a Nash equilibrium after at most $n^2m \cdot \text{rk}(\Gamma)$ best responses. In the case of identical latency functions, players reach a Nash equilibrium after at most $n^2 \cdot \text{rk}(\Gamma)$ best responses.*

Proof. Consider a list of all latencies $\ell_r(i)$ with $r \in \mathcal{R}$ and $1 \leq i \leq n$ and assume that this list is sorted in a non-decreasing order. For each resource r, we define an alternative latency function $\tilde{\ell}_r : \mathbb{N} \to \mathbb{N}$ where, for each possible congestion i, $\tilde{\ell}_r(i)$ equals the rank of the latency $\ell_r(i)$ in the aforementioned list of all latencies. We assume that equal latencies receive the same rank.

Lemma 3.2. *Let S be a state of a standard matroid congestion game, and let $s_i^* \in \Sigma_i$ a best response of player i to S with respect to the latencies ℓ_r which strictly decreases the latency of i. Then s_i^* strictly decreases the latency of player i with respect to the latencies $\tilde{\ell}_r$.*

Proof. Consider the bipartite graph $G(s_i^* \Delta s_i)$ which contains a perfect matching P_M due to Corollary 1.3. Let $S^* = S \oplus s_i^*$ and observe that for every edge $\{r^*, r\} \in P_M$, with $r^* \in s_i^* \setminus s_i$ and $r \in s_i \setminus s_i^*$, $\ell_{r^*}(x_{r^*}(S^*)) \leq \ell_r(x_r(S^*) + 1) = \ell_r(x_r(S))$, since otherwise, s_i^* is not a best response with respect to the latencies ℓ_r. Additionally, there exists at least one edge with $\ell_{r^*}(x_{r^*}(S^*)) < \ell_r(x_r(S^*)+1) = \ell_r(x_r(S))$ since s_i^* strictly decreases the latency of player i. Finally, the same inequalities also hold for the latencies $\tilde{\ell}_r$ as they correspond to the ranks of the original latencies. Thus the claim follows. □

Now due to Lemma 3.2, whenever a player plays a best response with respect to the latencies ℓ_r, Rosenthal's potential decreases with respect to the latencies $\tilde{\ell}_r$. Since there are at most $n \cdot m$ different latencies, $\tilde{\ell}_r(x_r) \leq n \cdot m$ for all resources $r \in \mathcal{R}$ and for all possible congestion values x_r. Hence,

$$\tilde{\Phi}(S) = \sum_{r \in \mathcal{R}} \sum_{i=1}^{x_r(S)} \tilde{\ell}_r(i) \leq \sum_{r \in \mathcal{R}} \sum_{i=1}^{x_r(S)} nm \leq n^2 m \cdot \mathrm{rk}(\Gamma) \ ,$$

where the latter inequality holds as each of the n player occupies at most $\mathrm{rk}(\Gamma)$ resources. Since $\tilde{\Phi}(S)$ is lower bounded by 0 and decreases by at least one if a player plays a best response with respect to the latencies ℓ_r, the first part of the theorem follows. In the special case of identical latency functions, there are at most n different latencies instead of $n \cdot m$, and thus the second part of the theorem follows. □

Note that Theorem 3.1 is independent of any assumptions on the latency functions. In particular, we do not assume monotonicity nor that all latencies have the same sign. Even more, it is independent of any schedule which selects the next player to play a best response.

3.1.2 Non-Matroid Strategy Spaces

In the previous section, we showed that the matroid property guarantees fast convergence of best response dynamics in standard congestion games. In this section, we show that the matroid property is also the maximal property of the combinatorial structure of the strategy spaces of individual players that guarantees fast convergence to Nash equilibria.

Theorem 3.3. *Let Σ be a non-matroid set system and an antichain on a set of resources $\mathcal{R}$. Then, for every $n \in \mathbb{N}$, there exists a standard congestion game Γ with*

- $4n + 1$ *players each of which having a strategy space isomorphic to Σ, and*
- $O(n \cdot |\mathcal{R}|)$ *resources with positive and non-decreasing latency functions*

such that there exists a best response sequence of length 2^n.

Proof. A well-known technique for constructing local search problems with exponentially long improvement sequences is to construct instances that resemble the behavior of a binary counter (see, e. g., [ADK+04, Hak89, Orp97]). Here we construct a game that consists of n gadgets $G_0, \ldots, G_{n-1}$ that correspond to the bits of the counter. Each of these gadgets has a 0-state and a 1-state and for each gadget there exists a best response sequence from its 1-state to its 0-state when no other gadget interferes with it. A gadget which is in state 0 can be triggered by another gadget to change to state 1. The crucial property of our construction is that whenever a gadget G_i changes its state from 0 to 1, then it triggers gadget G_{i-1} twice. Hence, if G_{n-1} is triggered once, then every gadget G_i is triggered 2^{n-i-1} times. Thus the game possesses a best response sequence of length at least 2^n.

In the following, we denote by Σ_i a set system over a set of resources $\mathcal{R}_i$. We assume that Σ_i is isomorphic to Σ, and that Σ_i is the strategy space of some player i. Due to Lemma 1.10, we can for every player $i \in \mathcal{N}$, identify three resources a_i, b_i, and $c_i \in \mathcal{R}_i$ with the properties as in Definition 1.9. These are the only resources of player i that it shares with other players. Resources in the set $\mathcal{R}_i \setminus \{a_i, b_i, c_i\}$ are exclusively used by player i. We choose the latencies of the resources in $\mathcal{R}_i \setminus \{a_i, b_i, c_i\}$ in such a way that the $(1, 2)$-exchange property is satisfied for a_i, b_i, and c_i. Thus, k, the parameter from Definition 1.9, is chosen as upper bound on the maximum latency on one of these three resources. To simplify matters, we can assume without loss of generality that every player i is interested in only three resources, namely a_i, b_i, and c_i, and that it is only allowed to play either the strategy $\{a_i\}$ or the strategy $\{b_i, c_i\}$. Since we have made no restrictions on the global structure of the game, we can interweave the resources a_i, b_i, c_i of different players in an arbitrary manner.

Each gadget G_i consists of 6 resources $r_i^0, r_i^1, r_i^2, r_i^3, r_i^4, r_i^5$ and 4 players which we call Init_i-, Trigger_i-, P_i^1-, and P_i^2-player. Every player has two strategies, namely a 0-strategy and a 1-strategy. If all players of gadget G_i play their 0-strategies, then we say that gadget G_i is in its 0-state. Similarly, if all players play their 1-strategies, then we say that G_i is in its 1-state. If gadget G_i is in state 0, then Init_i is the player who is triggered by the player Trigger_{i+1} from gadget G_{i+1} and initiates a sequences of best responses resetting G_i to its 1-state.

For every player its 0-strategy consists of one resource and its 1-strategy consists of two resources. The strategy spaces of the players are defined as follows:

$$
\begin{aligned}
\Sigma_{\text{Init}_i} &= \{\{r_i^0\}, \{r_i^1, r_i^2\}\}, \\
\Sigma_{\text{Trigger}_i} &= \{\{r_i^1\}, \{r_i^3, r_{i-1}^0\}\}, \\
\Sigma_{\text{P}_i^1} &= \{\{r_i^2\}, \{r_i^3, r_i^4\}\}, \\
\Sigma_{\text{P}_i^2} &= \{\{r_i^4\}, \{r_i^1, r_i^5\}\}.
\end{aligned}
$$

Now we describe the aforementioned best response sequence of gadget G_i in detail. Assume that gadget G_i is in its 0-state, that is, every player plays its 0-strategy. If player Init_i is triggered by the player Trigger_{i+1} from gadget G_{i+1}, that is, if Trigger_{i+1} allocates the resource r_i^0, then the following sequence of strategy changes can take place in gadget G_i.

1. Init_i changes to its 1-strategy.
2. Trigger_i changes to its 1-strategy.
3. P_i^1 changes to its 1-strategy.
4. Trigger_i changes back to its 0-strategy.
5. P_i^2 changes to its 1-strategy.
6. Trigger_i changes to its 1-strategy again.

Moreover, if all players play their 1-strategy and Init_i is not triggered by the player Trigger_{i+1}, then there exists a sequence of best responses such that all players of G_i change back to their 0-strategies. We assume that Init_i changes to its 0-strategy first, then P_i^1, P_i^2, and finally Trigger_i. Observe that this construction ensures the property that gadget G_{i+1} resets gadget G_i twice from state 0 to state 1 every time it changes its own state from 0 to 1. The first time gadget G_i triggers gadget G_{i-1} takes place after the first two strategy changes of the aforementioned sequence have been performed. In the last step of this sequence, gadget G_i triggers G_{i-1} for the second time.

Hence, this construction ensures the existence of best response sequences of length at least 2^n. Therefore, assume that initially every gadget is in its 0-state and that gadget G_{n-1} is triggered to change its state to 1. This can be accomplished by, e. g., introducing one additional player who allocates resource r_{n-1}^0. If all players act according to the aforementioned sequence of strategy changes, then every gadget G_i is reseted from its 0-state to its 1-state 2^{n-i-1} times.

Subsequently, let $*$ denote either 1 or 2. If the following inequalities are satisfied, then all six strategy changes in the aforementioned sequence of strategy changes are best responses. Recall that m denotes the number of resources. Moreover, adding m is due to Definition 1.9.

$$\begin{array}{lll}
1. & \ell_{r_i^0}(2) & > \ell_{r_i^1}(2) + \ell_{r_i^2}(2) + m, \\
2. & \ell_{r_i^1}(2) & > \ell_{r_i^3}(1) + \ell_{r_{i-1}^0}(*) + m, \\
3. & \ell_{r_i^2}(2) & > \ell_{r_i^4}(2) + \ell_{r_i^3}(2) + m, \\
4. & \ell_{r_i^1}(2) + m & < \ell_{r_i^3}(2) + \ell_{r_{i-1}^0}(*), \\
5. & \ell_{r_i^4}(2) & > \ell_{r_i^5}(1) + \ell_{r_i^1}(3) + m, \\
6. & \ell_{r_i^1}(3) & > \ell_{r_i^3}(2) + \ell_{r_{i-1}^0}(*) + m.
\end{array}$$

Let $\beta \geq 2$ be chosen arbitrarily, and for every gadget G_i, let $c_i = m \cdot \beta^{20i}$. We use c_i to scale the latencies of the resources in such a way that the best response of the player Trigger$_i$ is independent of the latency on the resource r_{i-1}^0.

Next we describe how to choose the latencies of the resources in order to achieve that the aforementioned best response sequences exist. We set $\ell_{r_i^j}(1) = c_i \cdot \beta^{2j}$ for every resource r_i^j and for every gadget G_i and furthermore $\ell_{r_i^0}(2) = c_i \cdot \beta^{20}, \ell_{r_i^1}(2) = c_i \cdot \beta^{8}, \ell_{r_i^1}(3) = c_i \cdot \beta^{14}, \ell_{r_i^2}(2) = c_i \cdot \beta^{18}, \ell_{r_i^3}(2) = c_i \cdot \beta^{10}, \ell_{r_i^4}(2) = c_i \cdot \beta^{16}$. One can easily verify that the aforementioned inequalities are all satisfied. Furthermore, observe that the second sequence of strategy changes in which G_i changes its state from 1 to 0 consists of best responses only since in this sequence every player changes to a resource that no other player allocates. □

The previous theorem shows that given a non-matroid antichain we can always construct a congestion game with an exponentially long best response sequence. Note that we are only interested in the combinatorial structure of the strategy spaces, and that we assume that the strategy spaces of different players can be interweaved arbitrarily. This matches the setting of our upper bound in Theorem 3.1 where we proved that in every standard matroid congestion game best response dynamics terminate after at most $n^2 m \cdot \mathrm{rk}(\Gamma)$ steps. The assumption that Σ is an antichain is natural when all latency functions are positive as, in this case, supersets are dominated by subsets so that supersets are never used as best responses. Hence, we can conclude the following corollary.

Corollary 3.4. *The matroid property is the maximal property of the combinatorial structure of the strategy spaces of individual players that guarantees polynomial time convergence of best response dynamics in standard congestion games with positive, non-decreasing latency functions.*

3.1.3 A Note on ε-greedy Players

Theorem 3.1 also holds if each player is ε-greedy. Moreover, the instances constructed in the proof of Theorem 3.3 possess the property that a player who decreases its latency even decreases it by a factor of at least β. Recall that we can choose $\beta \geq 2$ arbitrarily. Hence, with the same discussion as above we can conclude the following corollary.

Corollary 3.5. *The matroid property is the maximal property of the combinatorial structure of the strategy spaces of individual players that guarantees polynomial time convergence of best response dynamics in standard congestion games with positive, non-decreasing latency functions in which all players are ε-greedy.*

Note that Skopalik and Vöcking [SV08] strengthen this result in the sense that they show the existence of instances in which every sequence of best responses is exponentially long. Also note that the above corollary does not conflict with the result of Chien and Sinclair [CS07]. They prove that ε-greedy players reach an approximate Nash equilibrium after a polynomial number of best responses if two additional requirements are satisfied. At first, they require the game to be symmetric, i. e. all players have the same strategy space, and secondly that the latency functions satisfy the α-bounded jump condition. Especially the first requirement is not satisfied in our construction.

3.1.4 A Note on Better Response Dynamics

In singleton standard congestion games even the better response dynamics reaches a Nash equilibrium after at most n^2m better responses [IMN$^+$05]. In general standard matroid congestion games, however, even a single player can play exponentially many better responses until it reaches a best response, and hence a Nash equilibrium of the single-player game.

Theorem 3.6. *For every $m \in \mathbb{N}$ there exists a standard matroid congestion games with a single player, $\binom{m}{2}$ resources and initial state such that the player can play m^{m-2} better responses until it reaches a Nash equilibrium.*

Proof. Consider a standard spanning tree congestion game with a single player on the complete graph G_m with m vertices and $\binom{m}{2}$ edges $e_1, \dots, e_k$. Let 2^i be the latency the player observes if it allocates edge e_i. Since G has m^{m-2} different spanning tree [KV00], and since all of them have pairwise disjoint latencies, there exists a better response sequence of length m^{m-2}. □

3.2 Weighted Congestion Games

In this section, we consider best response dynamics in weighted matroid congestion games. In Theorem 2.1 we showed that players playing lazy best responses eventually reach a Nash equilibrium in every such game. From the proof we can conclude that every lazy best response dynamics terminates after at most

$$\min\left\{\left(\sum_{i=1}^{n}\omega_i\right)^m, \binom{m}{\mathrm{rk}(\Gamma)}^n\right\}$$

strategy changes. The first term is an upper bound on the maximum number of different vectors $\bar{z}(S)$ and the second one bounds the number of different states of a matroid congestion game Γ with n players, m resources and maximum rank $\text{rk}(\Gamma)$. As a first result, we present an example showing that arbitrary best responses do not necessarily lead to a Nash equilibrium.

Theorem 3.7. *There exists a weighted matroid congestion game in which arbitrary best response dynamics can cycle.*

Proof. Consider a weighted matroid congestion game with four resources $\{1, 2, 3, 4\}$ and two players with weights $\omega_1 = 1$ and $\omega_2 = 2$. We define the strategy spaces as follows:

$$\Sigma_1 = \{\{1\}, \{3\}\} \qquad \text{and} \qquad \Sigma_2 = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\} \ .$$

Observe that both strategy spaces are sets of bases of matroids on subsets of the resources. Additionally, we define non-decreasing latency functions. A dash denotes a value we do not have to care about.

	$x_r = 1$	$x_r = 2$	$x_r = 3$
$\ell_1(n_1)$	2	20	20
$\ell_2(n_2)$	-	9	-
$\ell_3(n_3)$	4	8	10
$\ell_4(n_4)$	-	20	-

Now consider the following cycle of states:

$$(\{3\}, \{1, 3\}) \to (\{3\}, \{2, 4\}) \to (\{1\}, \{2, 4\}) \to (\{1\}, \{1, 3\}) \to (\{3\}, \{1, 3\}) \ .$$

Each strategy change induces a set of inequalities in order to be a best response. One can easily verify that all these inequalities are satisfied by the above defined latency functions. Hence, players playing arbitrary best responses do not necessarily converge to a Nash equilibrium in weighted matroid congestion games. □

The latency functions in the previous example are non-decreasing but not strictly increasing. We leave open the question whether in arbitrary weighted matroid congestion games with strictly increasing latency functions players always converge to an equilibrium.

Next we focus on the convergence time of best response dynamics in weighted singleton congestion games. Note that in such games every best response is lazy, and hence, best response dynamics cannot cycle. We present a family of weighted singleton congestion games possessing super-polynomial long best response sequences although every player has either weight one or n and all latencies are polynomially bounded in the number of players and resources. This result improves the results of Even-Dar et al. [EDKM03] who already considered the convergence time in weighted singleton congestion games. In contrast to this thesis, they formulated their model in terms of scheduling selfish jobs on machines. They distinguish between different machine models, including identical, restricted, related and unrelated machines, and different kinds of weights, including integer weights, K distinct weights and identical weights. They prove upper and lower bounds on the convergence time depending on the schedule which selects the next player to deviate. Most related to our result is Theorem 9 in [EDKM03] which states that there exists a family of weighted singleton congestion games with symmetric players and identical machine with exponentially long best response sequences. However, in contrast to our result they use exponentially large weights.

Theorem 3.8. *There exists a constant $c > 0$ such that for every $n \in \mathbb{N}$, there exists a weighted singleton congestion game Γ with at most cn^2 players and at most cn resources that possesses a best response sequence of length 2^n. The players in Γ have either weight 1 or weight n, and the maximum latency is upper bounded by cn^3.*

Proof. A well known technique for constructing instances of local search problems with exponentially long best response sequences is to construct instances that resemble the behavior of a binary counter (see, e. g., [ADK$^+$04, Hak89, Orp97]). In the proof of Theorem 3.1 we already applied this technique to prove the existence of exponentially long best response sequence in non-matroid standard congestion games. Unfortunately, we cannot adopt the construction presented there to weighted singleton congestion games as players only allocate single resources. For that reason, we present a different construction below.

Let $n \in \mathbb{N}$ be chosen arbitrarily. We construct a weighted singleton congestion game with $O(n^2)$ players and $O(n)$ resources that resembles the behavior of a binary counter counting from 0 to $2^n - 1$. The instance consists of n gadgets $G_0, \ldots, G_{n-1}$ where gadget G_i represents the i-th bit of the counter; G_0 represents the least significant bit, G_{n-1} the most significant bit. For every gadget G_i, we define three main configurations, namely a 0-state, a 1-state and a reset state, with the following properties.

1. If gadget G_i is in its 0-state and no gadget G_j with $j > i$ is in its reset state, then there exists a best response sequence of gadget G_i such that G_i first changes to its reset state and then to its 1-state.

2. If gadget G_i is in its 1-state and at least one gadget G_j with $j > i$ is in its reset state, then there exists a best response sequence of gadget G_i such that G_i changes to its 0-state.

One can easily verify that these two properties ensure that there exists a best response sequence of all gadgets that resembles a binary counter counting from 0 to $2^n - 1$: Initially all gadgets are in their 0-state. First gadget G_0 changes to its 1-state, then gadget G_1. However, when gadget G_1 changes to its 1-state it passes its reset state, and therefore resets gadget G_0. Afterwards gadget G_0 may change back to its 1-state. We proceed with gadget G_2 that resets the gadgets G_0 and G_1 by changing to its 1-state. We may continue with gadget G_0 and so on.

Now we describe the gadgets $G_0, \ldots, G_{n-1}$ in detail. Gadget G_i consists of $i + 2$ players and 3 resources r_1^i, r_2^i and r_3^i. There are two main players, the *bit player* and the *reset player*, and i additional players, which we call *connection players.* The bit player and the reset player both have weight n, and each connection player has weight 1. Later, we will define latency functions and strategy spaces such that the best responses of the connection players are uniquely determined by the choice of the reset player. The purpose of the connection players is to propagate the decision of the reset player to the gadgets $G_0, \ldots, G_{i-1}$. The latency functions of the resources are defined as follows.

$$d_{r_1^i}(n_{r_1^i}) = \begin{cases} 3(n-i+1)+1 & \text{if } n_{r_1^i} \leq 2n-i-2 \\ 3n^2(i+1)+2 & \text{otherwise} \end{cases}$$

$$d_{r_2^i}(n_{r_2^i}) = \begin{cases} 3(n-i+1)+2 & \text{if } n_{r_2^i} \leq n \\ 3n^2(i+1)+1 & \text{otherwise} \end{cases}$$

$$d_{r_3^i}(n_{r_3^i}) = \begin{cases} 3(n-i+1)+3 & \text{if } n_{r_3^i} \leq i \\ 3n^2(i+1) & \text{otherwise} \end{cases}$$

We denote by Σ^i_{Bit} and Σ^i_{Reset} the strategy spaces of the bit and reset player, respectively, and by $\Sigma^i_{\text{Con}_j}$ the strategy space of the j-th connection player, with $0 \leq j \leq i-1$. Let the strategy spaces be defined as

$$\Sigma^i_{\text{Bit}} = \{\{r_1^i\}, \{r_2^i\}\} \qquad \Sigma^i_{\text{Reset}} = \{\{r_3^i\}, \{r_2^i\}\} \qquad \Sigma^i_{\text{Con}_j} = \{\{r_1^j\}, \{r_3^i\}\} \ .$$

For every player we name the first strategy according to the above given order, its 0-strategy and the second one its 1-strategy. Figure 3.1 illustrates our construction.

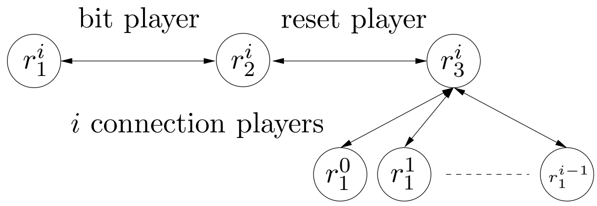


Figure 3.1: Illustration of gadget G_i. Nodes represent resources, edges represent players.

In the following, we describe the state of gadget G_i by a pair of bits (x, y), meaning that the bit player plays its x-strategy and that the reset player plays its y-strategy. When describing the state of a gadget by such a pair, we assume that the connection players have played their best responses according to strategy y. We denote by $(0,0)$ the 0-state of gadget G_i, by $(1,0)$ the 1-state, and by $(0,1)$ the reset state. We can then formulate the aforementioned properties of gadget G_i in terms of sequences of states (x, y).

1. If gadget G_i is in state (0,0) and every gadget G_j with $j > i$ is in state (0,0) or (1,0), then there exists a best response sequence of gadget G_i such that G_i first changes to its reset state (0,1) and then to the state (1,0).
2. If gadget G_i is in state (1,0) and at least one gadget G_j with $j > i$ is in state (0,1), then there exists a best response sequence of gadget G_i such that G_i changes to state (0,0).

It remains to be shown that the latency functions are chosen in the right way, that is, all strategy changes are best responses. We first show that the connection players of gadget G_i are solely controlled by the reset player of that gadget. Therefore, consider the following two cases.

Case 1: If the reset player plays its 0-strategy $\{r_3^i\}$, then the best response for every connection player is its 0-strategy. This is true since in this case the latency on resource r_3^i equals $3n^2(i+1)$ and the maximum latency on any resource r_1^j is at most $3n^2(j+1)+2$ which is less than $3n^2(i+1)$ because $j < i$.

Case 2: If the reset player plays its 1-strategy $\{r_2^i\}$, then the best response for every connection player is its 1-strategy. This is true since in this case the latency on r_3^i equals $3(n-i+1)+3$, and the minimum latency on any resource r_1^j is at least $3(n-j+1)+1$ which is larger than $3(n-i+1)+3$ because $j < i$.

In the following, we assume that immediately after each strategy change of the reset player, the connection players of the corresponding gadget change their strategies appropriately. Hence, when we say that the reset player of gadget G_i plays its x-strategy, $x \in \{0,1\}$, we implicitly assume that all connection players of that gadget play their x-strategies, too. Now we study the aforementioned best response sequences of the bit and reset players of a gadget G_i in detail.

1. Gadget G_i is in state $(0,0)$ and all reset players of the gadgets G_j with $j > i$ play their 0-strategy. In this case, the reset player can decrease its latency from $3n^2(i+1)$ to $3(n-i+1)+2$ by changing to its 1-strategy. After that, gadget G_i is in state $(0,1)$, and the bit player can decrease its latency from $3n^2(i+1)+2$ to $3n^2(i+1)+1$. After that, gadget G_i is in state $(1,1)$, and the reset player can decrease its latency from $3n^2(i+1)+1$ to $3n^2(i+1)$ by changing to its 0-strategy. After that, the gadget is in state $(1,0)$ and as long as no reset player of a gadget G_j with $j > i$ plays its 1-strategy it stays in this state.

2. Gadget G_i is in state $(1,0)$ and at least one reset player of a gadget G_j with $j > i$ plays its 1-strategy. In this case, the cumulative weight of all players allocating resource r_1^i is at most $n-i-2$. Hence, the bit player can decrease its latency from $(3n-i+1)+2$ to $(3n-i+1)+1$ by changing to its 0-strategy. After that, the gadget is in state $(0,0)$.

Altogether, this shows that the aforementioned sequence of strategy changes is a best response sequence and results in counting from 0 to 2^n-1. □

Let us briefly mention that our construction can even be implemented with players who have only weights 1 or 2. In order to achieve this, one has to introduce additional players that propagate the decision of the reset players to the connections players. Based on the observation that a player with weight 2 can displace two players of weight 1 from a resource, these players can be arranged in a binary tree with i leaves that propagate the decision to the connection players. As this construction is rather technical and does not give new insights, we do not present the details. Altogether, we conclude the following corollary which is in contrast to standard congestion games for which Rosenthal's potential functions implies pseudo-polynomial time convergence.

Corollary 3.9. *There exists no pseudo-polynomial upper bound on the convergence time of best response dynamics in weighted singleton congestion games.*

3.3 Player-Specific Congestion Games

In contrast to the games studied in the previous two sections, player-specific matroid congestion games are not potential games, since even in singleton games best response dynamics can cycle [Mil96]. On the positive side, Milchtaich [Mil96] observes that player-specific singleton congestion games are weakly acyclic, since from every state of such a game there exists a polynomial long sequence of best responses leading to a Nash equilibrium. As a first result, we prove that a similar property holds for player-specific matroid congestion games as well. Namely, we show that from every state of a player-specific matroid congestion game there exists a polynomial long sequence of *better* responses leading to a Nash equilibrium. We failed to prove the existence of best response sequences and leave it as an open question whether such sequences exist, too.

Additionally, we consider the convergence time of random best response dynamics in player-specific singleton congestion games. Currently, we are not able to analyze the convergence time in arbitrary player-specific singleton congestion games, but our theoretical and experimental results support the following conjecture.

Conjecture 3.10. *There exists a family of player-specific singleton congestion games with corresponding initial states such that the expected convergence time of random best response dynamics which start in the initial states is super-polynomial.*

In order to gain insights into random best response dynamics in player-specific singleton congestion games, we begin with very simple yet interesting classes of games. Namely, we consider games in which each player only chooses between two resources. These games can be represented as multi-graphs: each resource corresponds to a node and each player to an edge. In the following, we call games that can be represented as graphs with topology t player-specific congestion games on topology t. At first, we consider games on trees and on circles. We prove that player-specific congestion games on trees are potential games and deduce an upper bound of $O(n^2)$ on the convergence time of best response dynamics. The result bases on the observation that one can replace the player-specific latency functions by common latency functions without changing the players' preferences. Thus, we can apply the result of Ieong et al. [IMN$^+$05] to upper bound the convergence time. We proceed with player-specific congestion games on circles, and show that these games are the simplest games in which the best response dynamics can cycle. As we are only given four different latency values for each player, we characterize with respect to the order of these four values in which cases the best response dynamics can cycle. We observe that only one such case exists. Finally, we analyze the convergence time of the random best response dynamics in such games, and prove an upper bound of $O(n^2)$ on the expected convergence time. In order to prove this result, we introduce the notion of over- and underload tokens. An overload token indicates that a resource is shared by two players, an underload token indicates that it is unused. We observe that the number of tokens cannot increase, and that once in a while tokens get stuck or vanish.

Based on the insights gained from analyzing player-specific congestion games on circles we present a family of games and conjecture that there does not exist a polynomial upper bound on the expected convergence time of random best response dynamics. Obviously, this depends on the initial state, and so we implicitly assume that the initial configuration is chosen appropriately. Our conjecture is motivated and supported by a slightly different notion of over- and underload tokens. Their definition now depends on the fact that every resource has a fixed congestion that it takes at every Nash equilibrium. In contrast to games on circles we show that the number of over- and underload tokens can also increase if the initial configuration is chosen appropriately. Intuitively one may think of the number of tokens as a measure of derangement of order. In games on circles this measure can only decrease whereas it can also increase in general games. We fail to give a rigorous proof of a super-polynomial lower bound. However, we support our conjecture by empirical results obtained from simulations.

3.3.1 Short Better Response Sequences to Nash Equilibria

In this section, we show that from every state of a player-specific matroid congestion game there exists a polynomial long sequence of *better* responses leading to a Nash equilibrium. We leave it as an open question whether short sequences of *best* responses always exist, too.

Theorem 3.11. *Let Γ be a player-specific matroid congestion game with non-decreasing latency functions, and let S be an arbitrary state of Γ. Then there exists a sequence of better response of length at most $2n^2\, m\, \mathrm{rk}^2(\Gamma)$ which starts in state S and terminates in a Nash equilibrium.*

Proof. The proof uses similar arguments as the proof of Theorem 2.2, except that initially every player places all its tokens. After the first placement of the tokens, which corresponds to the given state S, we assume that all tokens are *deactivated*, i. e., players are not allowed to move them in order to decrease their latencies. We then consider a sequence of games $\Gamma_0, \dots, \Gamma_\tau$, where Γ_{k+1} is obtained from Γ_k by activating one more token. We can achieve that deactivated tokens are not moved by setting the latency of the corresponding player on the corresponding resource to 0. Then activating a token corresponds to restoring the latency function. Thus, each game Γ_k is a player-specific matroid congestion game. Given a Nash equilibrium S_k of Γ_k, we show that there exists a short improvement sequence in Γ_{k+1} from the former equilibrium S_k to a Nash equilibrium S_{k+1} of Γ_{k+1}. Obviously, by concatenating all these sequences we obtain a better response sequence from S to a Nash equilibrium of Γ.

Assume as induction hypothesis that k tokens have been activated so far and that the state S_k is a Nash equilibrium of Γ_k. Suppose now, that an additional token t_0 of player i_0 is activated, and that i_0 moves t_0 to a resource r_1 in order to decrease its latency. After that, we are in a situation similar to the one in the proof of Theorem 2.2, that is, the congestion on one resource r_1 is increased by one compared to the Nash equilibrium S_k of Γ_k. In contrast to the situation in the proof of Theorem 2.2, in which the congestion of the other resources remains unchanged, there exists a resource r_0 whose congestion is decreased by one compared to the congestion in Γ_k. Assume that we place a dummy token on resource r_0 which artificially increases the congestion by one. In this case, we can consider the same token migration process as in the proof of Theorem 2.2.

In contrast to the proof of Theorem 2.2, there are two different ways in which this process can terminate. If the process returns to r_0, i. e., if it moves a token onto r_0, we terminate the process and remove the dummy token from r_0. If the process does not return to r_0, then it is not affected by the dummy token and by the same arguments as in the proof of Theorem 2.2 it follows that it terminates after at most $n\, m\, \mathrm{rk}(\Gamma)$ moves of tokens.

In the first case, if at some point in time a player moves a token t_j from a resource r_{j-1} to the resource $r_j = r_0$, then after removing the dummy token from r_0 we have reached a Nash equilibrium of Γ_{k+1} due to Invariant 2.3. Since the resource r_0 is not involved in the previous moves of tokens, each of these movements reduces the latency of the corresponding player also in the game Γ_{k+1} without the dummy token. In the last step a player moves a token onto r_0 and improves its latency even if the dummy token is present. Hence, it also decreases its latency in Γ_{k+1} without the dummy token.

In the second case, we have almost reached a Nash equilibrium. That is, all players were satisfied if we would not remove the dummy token. Suppose now that we remove the dummy token. As the latency functions are non-decreasing, only players who can move tokens onto r_0 may have an incentive to change their strategies. From Corollary 1.6 we can conclude that those players who have an incentive to change their strategies, with respect to the tokens they are allowed to move, only need to move a token onto r_0 in order to obtain a best response.

Suppose now that player i'_0 moves a token t'_0 from resource r'_1 to r_0. Afterwards, the congestion on r_0 equals the congestion in the former equilibrium with respect to the dummy token, and the congestion on r'_1 is decreased by one. Again only players who can move a token onto r'_1 have an incentive to change their strategy. We can continue this process obtaining an additional token migration process in which a token t_{j+1} moves to the resource from which token t_j was removed. As before, we have to show that this token migration process is finite and terminates in a Nash equilibrium of Γ_{k+1}. The fact that it terminates in a Nash equilibrium is ensured by the following invariant which is a slight variation of Invariant 2.3.

Invariant 3.12. *For every $j \geq 0$, after player i'_j removes token t'_j from resource r'_{j+1},*

a) only players who can move a token onto r'_{j+1} may violate the Nash equilibrium condition, *b) the Nash equilibrium condition of all players would be satisfied if one ignores the missing token on r'_{j+1}, that is, if each player calculates the latency on r'_{j+1} as if there were one additional token on this resource.*

Invariant 3.12 can be proven analogously to Invariant 2.3. Therefore, its proof is omitted. It remains to show that the second token migration process is also finite. Again, the same arguments as in the proof of Theorem 2.2 show that this is true, and we conclude that the second process terminates after at most $nm\,\mathrm{rk}(\Gamma)$ moves of tokens in a Nash equilibrium of Γ_{k+1}.

Altogether, we have shown that there exists a better response sequence of length $2nm\,\mathrm{rk}(\Gamma)$ from S_k to a Nash equilibrium of Γ_{k+1}. As the number of tokens τ is upper bounded by $n\,\mathrm{rk}(\Gamma)$, the theorem follows. □

3.3.2 The Type of a Player

Before we present our results about the convergence time of random best response dynamics in player-specific singleton congestion games, we introduce the *type of a player*.

Ieong et al. [IMN+05] observe that in standard singleton congestion games one can always replace the latency values $\ell_r(x_r)$ with $r \in \mathcal{R}$ and $1 \leq x_r \leq n$ by their ranks in the sorted list of these values without affecting the players preferences in any state of the game. Recall that we already applied this approach in the proof of Theorem 3.1. Additionally, note that this approach is not restricted to standard singleton congestion games but also applies to player-specific singleton congestion games. That is, given a player-specific congestion game Γ, fix a player i and consider a list of all latencies $\ell^i_r(x_r)$ with $r \in \mathcal{R}$ and $1 \leq x_r \leq n$. Assume that this list is sorted in a non-decreasing order. For each resource r, we define an alternative player-specific latency function $\tilde{\ell}^i_r : \mathbb{N} \to \mathbb{N}$ where, for each possible congestion x_r, $\tilde{\ell}^i_r(x_r)$ equals the rank of the latency $\ell^i_r(x_r)$ in the aforementioned list of all latencies. Under the assumption that all latencies are *pairwise disjoint*, all ranks are unique. In the following, we stick to this natural assumption, and we define the *type of a player i* by the order of the player-specific latencies $\ell^i_r(1), \ldots, \ell^i_r(n)$ of the resources $r \in \Sigma_i$.

3.3.3 Games on Trees

In this section, we consider the convergence time of random best response dynamics in player-specific congestion games on trees. Note that in such games the number of resources equals the number of players plus one. We observe that one can always replace the player-specific latency functions by common latency functions such that the players' types are preserved. From this observation, we conclude the following theorem.

Theorem 3.13. *In every player-specific congestion game on a tree every best response dynamics terminates after at most $2n^2$ steps.*

Proof. Let Γ be a player-specific congestion game Γ on a tree. In the following, we describe how to replace the player-specific latency functions of Γ by common latency functions $\ell_r : \mathbb{N} \to \mathbb{N}$, $r \in \mathcal{R}$, with the following property: For every player i its type with respect to the player-specific latency functions equals its type with respect to the standard latency functions. Remember that the types completely describe the preferences of the players, and hence, best response dynamics in Γ are not affected by replacing the player-specific latency functions by common ones. Since the resulting game is a standard singleton congestion game, we can apply the result of Ieong et al. [IMN+05] to upper bound the convergence time. Obviously, the same bound holds in Γ. Thus, by applying the proof of the convergence time in standard singleton congestion game as presented in [IMN+05], we conclude that every best response dynamics for player-specific congestion games on trees terminates after at most $2n^2$ steps.

We prove the theorem by induction on the number of players and describe how to construct a sequence of player-specific congestion games $\Gamma_1, \ldots, \Gamma_n$ on trees with the following properties. Γ_1 is obtained from Γ by removing the players 2 to n from the game. The set of resources in Γ_0 is the set of the two resources the first player is interested in. Now Γ_i is obtained from Γ_{i-1} by adding one player and one resource to Γ_i. The player and the resource is chosen in such a way that Γ_i is a player-specific congestion game on a tree. That is, we choose a player i who is interested in a resource r of Γ_{i-1}, and add the additional resource r' the player is interested in to Γ_i.

Obviously Γ_1, the player-specific congestion game with a single player and two resources, is a standard congestion game. Assume as induction hypothesis that we already replaced the player-specific latency functions in Γ_{i-1} by common ones without affecting the players' types. For ease of notation let Γ^*_{i-1} be this game. In the following, we assume that for every resource r in Γ^*_i its latency functions is defined for all possible congestion values x_r between 1 and n and not only for the maximum number of players that are interested in r in Γ^*_i.

Given Γ^*_{i-1}, we now describe how to choose the latency functions ℓ_r of the resources in Γ^*_i such that the players in Γ^*_i and Γ_i have the same types. The latency functions of the resources r that belong to Γ^*_{i-1} are the same as in Γ^*_{i-1}. Additionally, we assume that for every such resource r and every congestion $1 < x_r \leq n$, $\ell_r(x_r) - \ell_r(x_r - 1) \geq n$. If this is not the case, then due to our assumption that the latency functions are strictly increasing, we can scale all latencies by a factor of n in order to achieve the desired goal. Thus, it remains to choose a latency function of the additional resource r' that does not belong to Γ^*_{i-1}. Since the gap between consecutive values of the latency function ℓ_r is large enough, we can realize every type for the additional player by choosing the latency function $\ell_{r'}$ appropriately. □

3.3.4 Games on Circles

In this section, we consider the convergence time of random best response dynamics in player-specific congestion games on circles. Without loss of generality, we assume that the resources are enumerated from $0, \ldots, n-1$, and that they are arranged in increasing order clockwise. Furthermore, we assume without loss of generality that for every player i, $\Sigma_i = \{r_i, r_{i+1 \bmod n}\}$. In the following, we call r_i the 0- and $r_{i+1 \bmod n}$ the 1-strategy of player i. Furthermore, we drop the mod n terms and assume that all indices are computed modulo n. Due to our assumptions on the latency functions, there are six different types of players in such games:

$$\begin{array}{lllllllll} \ell^i_{r_i}(1) & < & \ell^i_{r_i}(2) & < & \ell^i_{r_{i+1}}(1) & < & \ell^i_{r_{i+1}}(2) & & \text{type 1} \\ \ell^i_{r_i}(1) & < & \ell^i_{r_{i+1}}(1) & < & \ell^i_{r_i}(2) & < & \ell^i_{r_{i+1}}(2) & & \text{type 2} \\ \ell^i_{r_i}(1) & < & \ell^i_{r_{i+1}}(1) & < & \ell^i_{r_{i+1}}(2) & < & \ell^i_{r_i}(2) & & \text{type 3} \end{array}$$

We name the three other types, which can be obtained by exchanging the identities of the resources r_i and r_{i+1} in the above inequalities, type $1'$, type $2'$, and type $3'$. Furthermore, we call two players i and j *consecutive*, if they share a resource, that is, if $j = i+1$ or $i = j+1$. Given a state S, we call two consecutive players *synchronized*, if both play the same strategy, that is, if both either play their 0- or their 1-strategy. Moreover, we call a set of consecutive players $i, \ldots, j$ synchronized if all players play the same strategy.

Best Response Dynamics can Cycle

Next we present an infinite family of games in which best response dynamics can cycle. From this construction we derive a lower bound of $\Omega(n^2)$ on the convergence time of random best response dynamics in player-specific congestion games on circles.

Consider a game on a circle with n players which are all of type 3. It is not difficult to verify that this game possesses only two Nash equilibria: either all players play their 0-strategy or their 1-strategy. Consider now a state S with the following properties: In S we can partition the players into two non-empty sets $\mathcal{S}_0$ and $\mathcal{S}_1$ of synchronized players. Players in $\mathcal{S}_0$ all play their 0-strategy, whereas players in $\mathcal{S}_1$ all play their 1-strategy. Again, it is not difficult to verify that in every such state there are exactly two players who have an incentive to change their strategies. From both sets only the first player clockwise has an incentive to change its strategy. Thus, best response dynamics can cycle. We obtain such a cycle by selecting players from the two sets alternately and letting them play best responses.

In order to prove a lower bound on the convergence time of random best response dynamics, observe that with probability 1/2 the total number of players playing their 0-strategy increases or decreases by one whenever a player is selected uniformly at random. After the strategy change, either all players are synchronized, and therefore random best response dynamics terminates, or again we are in a state S' with two sets of synchronized players. Observe now that this process is isomorphic to a random walk on a line with nodes $v_0, \ldots, v_n$. The node v_i corresponds to the fact that i players play their 0-strategy. As the expected time of a random walk on a line with $n+1$ nodes to reach one of the two ends of the line is $\Theta(n^2)$ if the walk starts in the middle of the line [Lov96], we obtain a lower bound of $\Omega(n^2)$.

Corollary 3.14. *There exists a family of instances of player-specific congestion games on circles with corresponding initial states such that the number of steps until random best response dynamics terminates is lower bounded by* $\Omega(n^2)$.

An Upper Bound

In this paragraph, we present an upper bound on the convergence time of random best response dynamics in player-specific congestion games on circles. We prove the following theorem which matches the lower bound presented in the above corollary.

Theorem 3.15. *In every player-specific congestion game on a circle random best response dynamics terminates after $O(n^2)$ steps in expectation.*

The remainder of this paragraph is organized as follows. We characterize with respect to the types of the players in which cases there are cycles in the transition graphs of such games. We show that cycles only exist if all players are of type 3 or type $3'$. We analyze the convergence time of deterministic best response dynamics in games in which best response dynamics cannot cycle by developing a general framework that allows to derive potential functions from which one can easily derive upper bounds. Finally, we analyze the convergence time of random best response dynamics in the case of games with players of type 3 or type $3'$.

The Impact of Type 1 Players Firstly, we investigate the impact of type 1 players on the existence of cycles in the transition graphs and on the convergence time of best response dynamics. We claim that best response dynamics in games with at least one player of type 1 cannot cycle. Intuitively, this is true since every player of type 1 changes its strategy at most once, whereas in a cycle every player changes its strategy at least twice.

Lemma 3.16. *Let Γ be a player-specific congestion game on a circle. If there exists at least one player of type 1 or $1'$, then best response dynamics cannot cycle. Moreover, every best response dynamics terminates after at most $4n^2$ steps.*

In order to prove Lemma 3.16, we first prove the following observation.

Observation 3.17. *Let Γ be a player-specific congestion game on a circle in which best response dynamics can cycle. Then* every *player changes its strategy at least twice in every cycle of $TG(\Gamma)$.*

Proof. The fact that players being involved in the cycle change their strategy an even number of times is obvious. Thus, it remains to show that *every* player changes its strategy. For contradiction, assume that there exists a player i and a cycle such that player i does not change its strategy on that cycle. In this case, we could remove the player from the game, and artificially increase the congestion on the resource the player allocates by one. We would then obtain a player-specific congestion game on a tree in which best response dynamics can cycle. This is a contradiction to Theorem 3.13. □

Next we prove Lemma 3.16 for type 1 players. The proof for type $1'$ players is essentially the same.

Proof of Lemma 3.16. Without loss of generality, let player 0 be of type 1. Then observe that player 0 will never play its 1-strategy again, once it played its 0-strategy. Thus, by Lemma 3.17, best response dynamics cannot cycle.

In order to prove the convergence time, observe that if we fix player 0 to one of its strategies, then we obtain a player-specific congestion game on a tree. Due to Theorem 3.13, the convergence time in such games is upper bounded by $2n^2$. Since player 0 changes its strategy at most once we obtain the upper bound on the convergence time as stated in the lemma. □

In the following, we will assume that there exists no player of type 1 or $1'$, as otherwise we could apply Lemma 3.16.

A Framework to Analyze the Convergence Time In this paragraph, we present a framework to analyze the convergence time of best response dynamics in player-specific congestion games on circles. Let Γ be a game such that there is no player of type 1 or $1'$. At first we investigate whether there is a sufficient condition such that player i does not want to change its strategy in a state S of Γ.

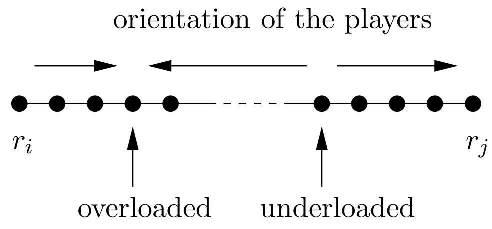
Observation 3.18. *Suppose that player i is not of type 1 or $1'$. Then if it is synchronized with the players $i-1$ and $i+1$ in S, it has no incentive to change its strategy.*

In the following, we call a resource r *overloaded* in state S if two players share r. Additionally, we call a resource r' *underloaded* in state S if no player allocates r'. Obviously in every state of Γ, the total number of overloaded resources equals the total number of underloaded resources. From Observation 3.18, we conclude that in every state S only players who allocate a resource that is currently overloaded or who could allocate a resource that is currently underloaded might have an incentive to change their strategy.

Based on this observation, we now present a general framework to analyze the convergence time of best response dynamics. At first, we introduce the notion of *over- and underload tokens*. Given an arbitrary state S of Γ, we place an *overload token* on every overloaded resource. Additionally, we place an *underloaded token* on every underloaded resource. Obviously, over- and underload tokens alternate on the circle. Furthermore, note that a legal placement of tokens uniquely determines the strategies the players play. A placement of tokens is legal if no two tokens share a resource, and if the tokens alternate on the circle.

In the following, we investigate in which directions tokens move if players play best responses. Consider first a sequence of resources $r_i, \ldots, r_j$ and assume that players $i, \ldots, j-1$ are of the same type t. Additionally, assume that an overload token is placed on resource r_k, and that an underload token is placed on resource r_l with $i < k < l < j$. The scenario we consider is depicted in Figure 3.2.

Assume at first, that the distance (number of edges) between the two tokens is at least two, i.e., $|l-k| \geq 2$. In this case, each token can only move in one direction. The directions are uniquely determined by the type of the players. They can be derived



	overload	underload
type 2	anticlockwise	clockwise
type 2′	clockwise	anticlockwise
type 3	clockwise	clockwise
type 3′	anticlockwise	anticlockwise

Figure 3.2: In which directions do the tokens move?

from investigating, with respect to the players' type t, which players have incentives to change their strategy. The directions are stated in Figure 3.2, too. Assume now that the distance between the two tokens is one. That is, $k = l - 1$. Then there exists a player who is interested in the over- and underloaded resource, and who currently allocates the overloaded one. It is not difficult to verify that this player always has an incentive to change its strategy. Note that this holds regardless of the player's type since we assumed that there are no players of type 1 and $1'$. Observe that after the strategy change of this player all players $i, \ldots, j - 1$ are synchronized and therefore there exist no over- and underloaded resources anymore. In the following, we call such an event a *collision of tokens.*

So far, we considered sequences of players of the same type and observed that there is a unique direction in which tokens of the same kind move. In sequences with multiple types of players such unique directions do not exist any longer, i.e., overload as well as underload tokens can move in both directions. However, if two players of different types share a resource and if due to best responses of both players an over- or underload token moves onto this resource, then the token could stop there. In the following, we formalize this observation with respect to overload tokens and introduce the notion of *termination points.*

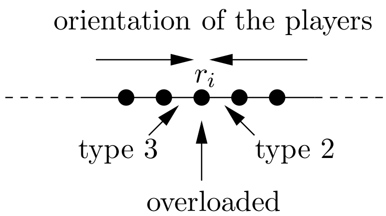
Definition 3.19. *We call a resource r_i a* termination point *of an overload token if the following conditions are fulfilled.*

1. *The players $i - 1$ and i have different types. Let these types be t_{i-1} and t_i.*
2. *In sets of consecutive players of type t_{i-1} overload tokens move clockwise, whereas they move anticlockwise in sets of consecutive players of type t_i.*

We illustrate the definition in Figure 3.3a. Let player $i-1$ be of type 3, and let player i be of type 2. In this case, the requirements of the definition are satisfied. Assume that player $i - 1$ plays its 1-strategy and that it is synchronized with player $i - 2$.

Additionally, assume that player i plays its 0-strategy and that it is synchronized with player $i+1$. Observe now that the token cannot move as neither player $i-1$ nor player i has an incentive to change its strategy. Suppose now that initially all players along the path play their 0-strategy. Then an overload token that moves from the left to the right along the path stops at r_i. The token may only move on if one of the two players is not synchronized with its neighbor any longer. In this case, this player always has an incentive to change its strategy as it can allocate a resource that is currently underloaded. Thus, an underload and an overload token collide. Additionally, if initially all players play their 1-strategy and an overload token moves from the right to the left along the path, we observe the same phenomenon. The token cannot pass the resource r_i unless it collides with an underload token.

Note that the definition of a termination point can easily be adopted to underload tokens. A list of all termination points is given in Figure 3.3b. In the left column we present all termination points for overload tokens, in the right one for underload tokens.



(a) An Example of a termination point

$\rightarrow\rightarrow\leftarrow\leftarrow$	$\leftarrow\leftarrow\rightarrow\rightarrow$
$2'$ 2	2 $2'$
3 $3'$	3 $3'$
3 2	-
-	2 $3'$
-	3 $2'$
$2'$ $3'$	-

(b) A List of all termination points

Figure 3.3: Termination Points.

Analyzing the Convergence Time In this paragraph, we analyze the convergence time in player-specific congestion games on circles. We distinguish between the following four cases.

Case 1: For both kinds of tokens there exists at least one termination point.

Case 2: Only for one kind of token there exists at least one termination point.

Case 3: There exist no termination points but over- and underload tokens move in opposite directions.

Case 4: There exist no termination points and over- and underload tokens move in the same direction.

In the first two cases, we present potential functions from which we derive that every best response dynamics terminates after $O(n^2)$ steps. In the third case, we can do slightly better and prove an upper bound of $O(n)$ on the convergence time. In all cases one can easily construct matching lower bounds. Only in the fourth case, best response dynamics can cycle, however, we prove that random best response dynamics terminate after $O(n^2)$ steps in expectation.

Before we take a closer look at the different cases, we discuss which games with respect to their players' types belong to which case. Games with players of type 2 and $2'$ only or with players of type 3 and $3'$ only belong to the first case. Additionally, some games with more than two types of players belong to this case. The second case covers all games with at least three different kinds of players which do not belong to the first case. Furthermore, it covers games with type 2 and type 3 players, with type $2'$ and type $3'$ players, type $2'$ and type 3 players, and with type 2 and type $3'$ players. Games with type 2 players only, or games with type $2'$ players only belong to the third case. Finally, games with type 3 players only and games with type $3'$ players only belong to the fourth case. These observations can easily be derived from Figure 3.3b.

Lemma 3.20 (Case 1). *Let Γ be a player-specific congestion game on a circle with termination points for both kinds of tokens. Then Γ is a potential game, and every best response dynamics terminates after $O(n^2)$ steps.*

Proof. Let S be a state of Γ and consider the mapping that maps every token in S to the next termination point lying in the direction in which the token moves. In the following, we define $d(t, S)$ as the distance of a token t in state S to its corresponding termination point. Obviously $d(t, S) \leq n$. Consider now the potential function $\Phi(S) = \sum_{\text{token } t} d(t, S)$ and suppose that a player plays a best response. Then either one token moves closer to its termination point or two tokens collide. In both cases $\Phi(S)$ decreases by at least 1. Thus, $\Phi(S)$ strictly decreases if a player plays a best response and therefore, Γ is a potential game. Moreover, as $\Phi(S)$ is upper bounded by $O(n^2)$, every best response dynamics terminates after $O(n^2)$ steps. □

Lemma 3.21 (Case 2). *Let Γ be a player-specific congestion game on a circle with termination points only for one kind of token. Then Γ is a potential game, and every best response dynamics terminates after $O(n^2)$ steps.*

Proof. Without loss of generality, assume that termination points only exist for overload tokens. In this case, we define $d(t_o, S)$ for every overload token t_o as in the proof of Lemma 3.20. For every underload token t_u we define $d(t_u, S)$ as follows. Let t_o be the first overload token lying in the same direction as t_u moves.

1. If t_o moves in the opposite direction than t_u, we define $d(t_u, S)$ as the distance between the two tokens, where the distance of two tokens moving in opposite directions is defined as the maximum number of moves of these tokens until they collide.

2. If t_o moves in the same directions as t_u, then we define $d(t_u, S)$ as the distance between t_u and t_o plus the distance between t_o and the first termination point at which t_o has to stop. Thus, $d(t_u, S)$ equals the maximum number of moves of these two tokens until they collide.

Observe, that for every underload token t_u, $d(t_u, S) \leq 2n$. Consider the potential function $\Phi\colon \Sigma_1 \times \ldots \times \Sigma_n \to \mathbb{N} \times \mathbb{N}$ with $\Phi(S) = (\Phi_1(S), \Phi_2(S))$, where $\Phi_1(S)$ equals the total number of overload tokens in S and $\Phi_2(S)$ equals the sum of all $d(t, S)$ for all under- and overload tokens. Suppose now that a player plays a best response. Obviously if two tokens collide, then $\Phi_1(S)$ decrease by one. Moreover, if there is no collision, then $\Phi_2(S)$ decreases. Note that in the first case Φ_2 may increase. This may happen if, due to the collision, $d(t_u, S)$ of a remaining underload token t_u has to be recomputed as its associated overload token has been removed. The new value is upper bounded by the sum of the old values of t_u and the collided underload token plus 1. Now consider the lexicographic order $<_\Phi$ of the states of Γ with respect to Φ. Let S and S' be two states of Γ. Then

$$S <_\Phi S' \quad \Leftrightarrow \quad \begin{cases} \Phi_1(S) < \Phi_1(S') & \text{or} \\ \Phi_1(S) = \Phi_1(S') \quad \text{and} \quad \Phi_2(S) < \Phi_2(S') & . \end{cases}$$

Observe that Φ strictly decreases if a player plays a best response. Thus, Γ is a potential game. Additionally, observe that Φ_1 is upper bounded by n, and that Φ_2 is upper bounded by n^2. However, as Φ_2 only increases by one when Φ_1 decreases, we conclude that every best response dynamics terminates after $O(n^2)$ steps. □

Lemma 3.22 (Case 3). *Let Γ be a player-specific congestion game on a circle with no termination points in which over- and underload tokens move in opposite directions. Then Γ is a potential game, and every best response dynamics terminates after $O(n)$ steps.*

Proof. Let S be a state of Γ and consider the one-to-one mapping that maps every overload token to the next underload token lying in the direction in which the token moves. We define the distance of such a pair of tokens as the maximum number of moves of these two tokens until they collide.

Suppose now that a player plays a best response. Then either the number of overload tokens or the distance between one pair of tokens decreases by one. Consider now the potential function $\Phi\colon \Sigma \to \mathbb{N} \times \mathbb{N}$ with $\Phi(S) = (\Phi_1(S), \Phi_2(S))$, where $\Phi_1(S)$ equals the number of overload tokens in S, and $\Phi_2(S)$ equals the sum of all distances of pairs of tokens. Observe now that in the case of a best response, Φ_1 either decreases by 1 or remains unchanged. In the first case, Φ_2 may increase by 1. This is true as tokens from different pairs may collide. However, this can happen at most n times. If this happens, the remaining two tokens form a new pair whose distance equals the sum of the distances of the previous pairs plus 1. In the second case, Φ_2 decreases by 1. Then by similar arguments as in the proof of Lemma 3.21, we conclude that Γ is a potential game. Finally, observe that Φ_1 is upper bounded by n. Moreover, Φ_2 is upper bounded by n, too. As Φ_2 only increases by one when Φ_1 decreases, we conclude that every best response dynamics terminates after $O(n)$ steps. □

In the following, we present a proof of the fourth case for players of type 3. By symmetry of the types 3 and $3'$, the same result holds for games with players of type $3'$, too.

Lemma 3.23 (Case 4)**.** *Let* Γ *be a player-specific congestion game on a circle in which all players are of type 3. Then random best response dynamics terminates after* $O(n^2)$ *steps in expectation.*

Proof. In order to prove the lemma, we first prove the following observation.

Observation 3.24. *In every state* S *of* Γ *the number of players which want to change from their 0- to their 1-strategy equals the number of players which want to change from their 1- to their 0-strategy.*

Proof. In the following, we call a synchronized set of consecutive players *maximal* if the next players to both ends of the set play different strategies than the synchronized players. Obviously, in every state S of Γ which is not an equilibrium the number of maximal synchronized sets of players playing their 0-strategy equals the number of maximal synchronized sets of players playing their 1-strategy.

We now prove that in every maximal synchronized set of consecutive players only the first player clockwise has an incentive to change its strategy. Thus, in every maximal set, there is only a single player who wants to change its strategy. Note that this suffices to prove the lemma.

Firstly, consider a maximal, synchronized subset of consecutive players $\mathcal{N}' = \{i, \dots j\}$ which all play their 0-strategy. Then player $i-1$ plays its 1-strategy, and therefore the players $i-1$ and i share resource r_i. In this case, player i can decrease its latency by changing to its 1-strategy. Other players $k \in \mathcal{N}'$, $k \neq i$, do not have an incentive to change their strategy as this would increase their latency.

Secondly, consider a maximal synchronized subset of consecutive players $\mathcal{N}' = \{i, \dots j\}$ which all play their 1-strategy. Then player $i-1$ plays its 0-strategy and therefore no player currently allocates resource r_i. Observe now that player i may decrease its latency by changing to its 0-strategy. Again, all other players $k \in \mathcal{N}' \setminus \{i\}$ do not have an incentive to change their strategy as this would increase their latency. This is especially true for the last player, who currently allocates an overloaded resource. □

Consider random best response dynamics in which an unsatisfied player is selected uniformly at random. From Observation 3.24 we conclude that the total number of players playing their 0-strategy increases or decreases by 1 with probability 1/2. Combining this with the observation that at a Nash equilibrium all players play the same strategy, we conclude that every random best response dynamics is isomorphic to a random walk on a line with $n+1$ vertices. Vertex v_i corresponds to the fact that i players play their 0-strategy. As the time of such a random walk to reach one of the two ends of the line is $O(n^2)$, the lemma follows. □

3.3.5 Games on General Graphs

In this section, we consider player-specific congestion games on general graphs and present evidence supporting Conjecture 3.10 by constructing a family of instances for which experimental results clearly show a super-polynomial convergence time.

Our analysis of player-specific congestion games on circles is based on the notion of over- and underload tokens, and there is no straightforward extension of this notion to player-specific singleton congestion games on general graphs. The instances we construct have, however, the property that every resource has a fixed congestion that is taken at every Nash equilibrium, and we can define tokens with respect to these congestion values. To be precise, if the congestion on a resource deviates by Δx from the equilibrium congestion, we place Δx overload tokens in the case $\Delta x > 0$ and we place $-\Delta x$ underload tokens in the case $\Delta x < 0$. Note that for circles with type 3 players this definition coincides with the former definition of tokens.

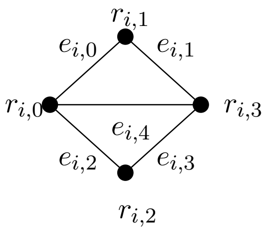
The crucial property of games on circles with type 3 players leading to polynomial convergence is that the number of tokens cannot increase. The instances we construct in this paragraph are in some sense similar to circles with type 3 players, but we attach additional gadgets to the nodes which can occasionally increase the number of tokens. We start with a circle with n type 3 players and replace each edge by n parallel edges. This modification allows each node to store more than one token of the same kind if the preferences of the players are adjusted accordingly. Other properties are not affected by this modification, that is, over- and underload tokens still move in the same direction with approximately the same speed and if an overload and an underload token meet, they both vanish. Each time a node contains at least two tokens of the same kind, the gadget attached to the node is triggered with constant probability. If a gadget is triggered, it can emit a new pair of overload and underload token into the circle. Usually, this new pair is stored in the gadget and only emitted after the triggering tokens have moved on a linear number of steps. The new tokens are not emitted simultaneously but the second one is usually only emitted after the first one has moved on a linear number of steps in order to prevent the new tokens from canceling each other out immediately.

Initially, we introduce two overload tokens at node 0 and two underload tokens at node $n/2$. The two overload tokens move independently through the circle starting at the same node. Typically, they meet a couple of times before they meet the underload tokens and vanish. The same is true for the underload tokens as well, meaning that typically a couple of gadgets get triggered before the initial tokens vanish. Hence, the number of tokens has a tendency to increase. Since the triggered gadgets emit the stored tokens in a random order, the random process soon becomes unwieldy and we fail to rigorously prove that it takes super-polynomial time in expectation until all tokens vanish. This conjecture is, however, strongly supported by simulations.

Our Construction

Given $n \in \mathbb{N}$, we construct a player-specific congestion game Γ_n consisting of n gadgets $G_0, \dots, G_{n-1}$ as follows. A single gadget G_i is depicted in Figure 3.4a. It consists of 4 resources $r_{i,0}, \dots, r_{i,3}$ and $5n$ players. Each edge in the figure represents n of them. The gadgets are arranged on a circle, such that for every i the resources $r_{i,3}$ and $r_{i+1,0}$ coincide. Thus, for every i, $6n$ players are interested in $r_{i,0}$ and $r_{i,3}$, and $2n$ players are interested in $r_{i,1}$ and $r_{i,2}$.

For every player which chooses between the two resources $r_{i,k}$ and $r_{i,l}$ with $l < k$ we call $r_{i,l}$ the 0-strategy and $r_{i,k}$ the 1-strategy of that player. In the following, we refer to a player represented by an edge $e_{i,j}$ by the term *type j player*. The player-specific latency functions are defined as follows. All players of the same type j have the same functions for the two resources they choose between. We define these functions in terms of a threshold t for their 0-strategies, meaning that the 0-strategy is a best response if and only if the total number of *other* players allocating the 0-strategy resource is less or equal to the threshold t. Otherwise the 1-strategy is best response. The thresholds are defined as depicted in Figure 3.4b.



(a) Gadget G_i

type 0	$t_0 = 3n$
type 1	$t_1 = n - 1$
type 2	$t_2 = 3n - 2$
type 3	$t_3 = n - 1$
type 4	$t_4 = 3n - 1$

(b) The player-specific latency functions

Figure 3.4: The lower bound construction.

In the next paragraphs, we prove that every resource has the same congestion at every Nash equilibrium. We proceed with a description of how gadgets can generate new tokens. Finally, we present results obtained from simulations.

Properties of Nash Equilibria

In order to simplify our proceeding discussion, we introduce the term $c^b_{i,j}(S) \in \mathbb{N}$, $b \in \{0,1\}$, to denote the number of type j players in gadget i who play their b-strategy in state S. Furthermore, we define $x_{i,j}(S) = x_{r_{i,j}}(S)$. In the following, let S^* be a Nash equilibrium of Γ_n. For ease of notation, we use $c^b_{i,j} = c^b_{i,j}(S^*)$ and $x_{i,j} = x_{i,j}(S^*)$. The following observation is true because S^* is a Nash equilibrium.

Observation 3.25. *Let $j \in \{1,3\}$ and $b \in \{0,1\}$. Then for every $0 \leq i < n$ the number of type j-players playing their b-strategy in gadget G_i in S^* is uniquely determined by the number of type $j-1$ players playing their b-strategy in gadget G_i in S^*, i.e., $c^b_{i,j-1} = c^b_{i,j}$.*

Next we prove that every resource has the same congestion at every Nash equilibrium.

Lemma 3.26. *For every Nash equilibrium S^* of Γ_n and every $0 \leq i < n$,*

$$x_{i,0} = 3 \cdot n \quad and \quad x_{i,1} = x_{i,2} = n \ .$$

Proof. At first, observe that for every gadget G_i, it holds

$$c^0_{i,0} \geq c^0_{i,4} \geq c^0_{i,2} \ .$$

If the first inequality were not true, then there exist type 0 players in G_i playing their 1-strategy and type 4 players playing their 0-strategy. However, since S^* is a Nash equilibrium, all type 4 players in G_i who play their 0-strategy are satisfied and thus $x_{i,0} \leq 3n$. We observe that all type 0 players currently playing their 1-strategy have an incentive to change their strategy. A similar argument proves the second inequality. Essentially, the same arguments prove the following implications:

$$\begin{aligned} c^0_{i,0} < n \quad &\Rightarrow \quad c^0_{i,4} = 0, \\ c^0_{i,4} < n \quad &\Rightarrow \quad c^0_{i,2} = 0. \end{aligned}$$

Now consider an arbitrary gadget G_i and let $3n - k_{i-1}$ be the number of players from gadget G_{i-1} allocating resource $r_{i,0}$. In the following, we discuss how the parameter k_{i-1} affects the choices of the players in gadget G_i at the Nash equilibrium S^*. We prove that the best responses of the players in G_i are uniquely determined by the parameter k_{i-1}. In order to do so, we distinguish 6 cases.

1. **Case** $k_{i-1} = 0$**:** All type 1, type 3, and type 4 players in gadget G_{i-1} play their 1-strategy. Due to Observation 3.25 we conclude that all type 0 and type 2 players in G_{i-1} play their 1-strategy as well, and therefore the congestion on $r_{i-1,0}$ is at most $3n$. In this case, however, all type 0 players in G_{i-1} have an incentive to play a best response. We conclude that this case does not appear in a Nash equilibrium.
2. **Case** $1 \leq k_{i-1} < n$**:** $k_{i-1}+1$ type 0 and $k_{i-1}+1$ type 1 players in G_i play their 0-strategy. The remaining players in G_i play their 1-strategy. Thus $k_i = k_{i-1}+1$.
3. **Case** $k_{i-1} = n$**:** All type 0 and all type 1 players in G_i play their 0-strategy; all other players in G_i play their 1-strategy. Thus $k_i = k_{i-1}$.
4. **Case** $n < k_{i-1} \leq 2n$**:** All type 0 and all type 1 players in G_i play their 0-strategy. Additionally, $k_{i-1} - n$ type 4 players in G_i play their 0-strategy. The remaining players in G_i play their 1-strategy. Thus $k_i = k_{i-1}$.
5. **Case** $2n < k_{i-1} < 3n$**:** All type 0, all type 1 and all type 4 players in G_i play their 1-strategy. Additionally, $k_{i-1} - 2n - 1$ type 3 and $k_{i-1} - 2n - 1$ type 4 players in G_i play their 0-strategy. The remaining players in G_i play their 1-strategy. Thus $k_i = k_{i-1} - 1$.
6. **Case** $k_{i-1} = 3n$**:** Similar arguments as in the first case show that this case does not appear in a Nash equilibrium.

As an intermediate observation we conclude that the lemma is true if at least one gadget G_i exists for which $n \leq k_i \leq 2n$ holds. In this case, $k_{i-1} = k_i$ for every $1 \leq i < n$ and the players play the strategies as described above.

Next we take a closer look at the second and fifth case. We begin with the second one in which $1 \leq k_{i-1} < n$ implies $k_i = k_{i-1} + 1$ which implies $k_{i+1} = k_{i-1} + 2$ and so on until $k_j = n$. In this case we enter the third case which implies $k_{j+1} = n$ and so on. Obviously, this leads to a contradiction since $k_{i-1} < n$. Thus, whenever there exists a gadget for which $k_{i-1} < n$ holds, S^* is not a Nash equilibrium. Similar arguments show that the fifth case does not appear in a Nash equilibrium either. □

Generating New Tokens

Motivated by Lemma 3.26 we are now ready to introduce a new notion of tokens.

Definition 3.27. *Let S be an arbitrary state of Γ_n and let x_r^* be the congestion on a resource r at every Nash equilibrium. Then we place over- and underload on the resources according to the following rules.*

1. *If $x_r(S) = x_r^* + k$, $k \in \mathbb{N}$, then we place k overload tokens on r.*
2. *If $x_r(S) = x_r^* - k$, $k \in \mathbb{N}$, then we place k underload tokens on r.*

Next we describe how the number of overload and underload tokens can increase. This can happen if there are either at least two overload or at least two underload tokens on $r_{i,0}$. In the following, we discuss the first case in detail. The second case in only depicted in Figure 3.7.

Consider a single gadget G_i as depicted in Figure 3.6a. Numbers attached to resources correspond to the number of tokens lying on them. Positive numbers indicate that overload tokens are present, negative numbers indicate that underload tokens are present. Numbers a attached to edges indicate that a players represented by that edge play their 0-strategy, whereas $n-a$ players play their 1-strategy.

Configuration 3.6a: Initially, there are two overload tokens on $r_{i,0}$. In this case, all type 0 and all type 4 players have an incentive to change to their 1-strategies. All other players are satisfied. With probability 2/3, given that a player from G_i is selected, a type 0 player is selected and Configuration 3.6b is reached, in which there is one overload token on $r_{i,0}$ and one on $r_{i,1}$.

Configuration 3.6b: All type 1 and all type 4 players have an incentive to change to their 1-strategy. With probability 2/3 Configuration 3.6c is reached in which there is one overload token on $r_{i,0}$ and one on $r_{i,3}$.

Configuration 3.6c: Still all type 4 players have an incentive to change to their 1-strategy. However, we assume that the overload token which currently lies on $r_{i,0}$ moves on due to a best response of a player in gadget G_{i+1}. In this case, Configuration 3.6d is reached in which there is still one overload token on $r_{i,0}$. Additionally, one overload token is in gadget G_{i+1}.

Configuration 3.6d: Again, all type 4 players have an incentive to change to their 1-strategy. In this case, we select one of these players and Configuration 3.6e is reached in which there is one overload token on $r_{i,4}$.

Configuration 3.6e: In this configuration, the overload token on $r_{i,4}$ can move to the next gadget. Observe that this event is much more likely than the next one, in which the only type 0 player playing its 1-strategy switches back to its 0-strategy. All other players are satisfied. If both events take place Configuration 3.6f is reached. Note, that in this case additional tokens are generated. There is a new underload token on $r_{i,1}$ and a new overload token on $r_{i,0}$.

Configuration 3.6f: Finally, all $n-1$ type 4 players playing their 0-strategy have an incentive to change to their 1-strategy. Additionally, the only type 1 player playing its 1-strategy wants to change back to its its 0-strategy.

Simulations

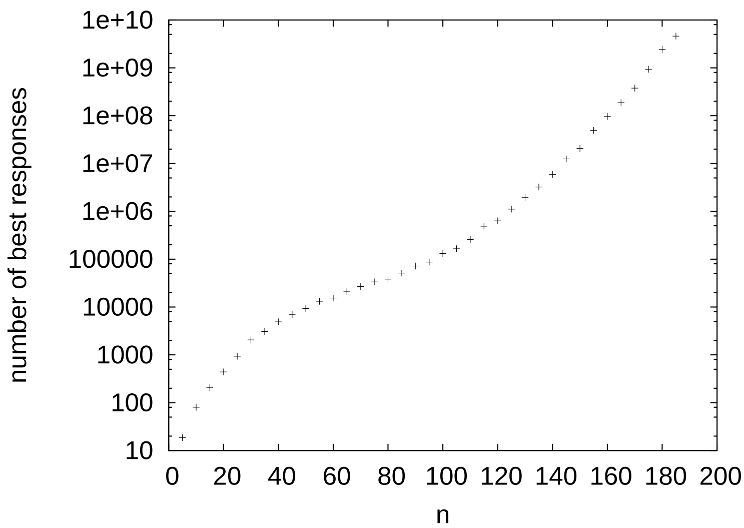
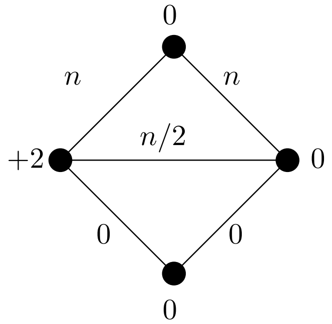


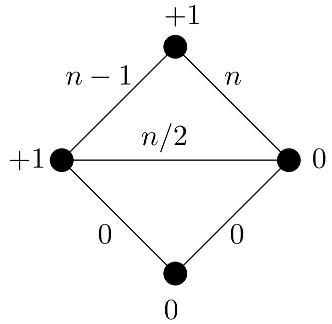
Figure 3.5: Average number of best responses

We simulated random best response dynamics in games Γ_n and obtained the results shown in Figure 3.5. On the x-axis we plotted the parameter n, on the y-axis the average number of best responses until random best response dynamics terminated. Observe that the y-axis is plotted in log-scale. For every $n \in \{5, 10, \ldots, 180, 185\}$ we started random best response dynamics from the following initial configuration: all type 0 and all type 1 players play their 0-strategies; all type 2 and all type 3 players play their 1-strategies. Additionally, $n/2$ type 4 players in the gadgets $G_0, \ldots, G_{n/2-1}$ and $n/2+2$ type 4 players in the gadgets $G_{n/2}, \ldots, G_{n-1}$ play their 1-strategy. All other type 4 players play their 0-strategy. This initial configuration corresponds to placing two overload tokens on $r_{0,0}$ and two underload tokens on $r_{n/2,0}$. For $n \leq 160$ we took the average over 400 runs, and for larger n we took the average over 100 runs.

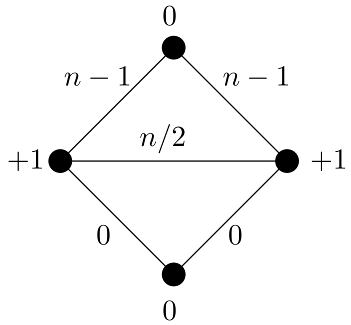
Unfortunately, it does not seem feasible to simulate best response dynamics for much larger values of n. We believe, however, that the results in Figure 3.5 are a clear indication for a super-polynomial, maybe even exponential, convergence time.



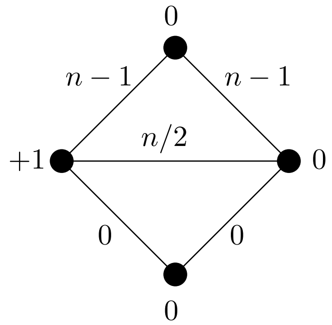
(a) Initial configuration.



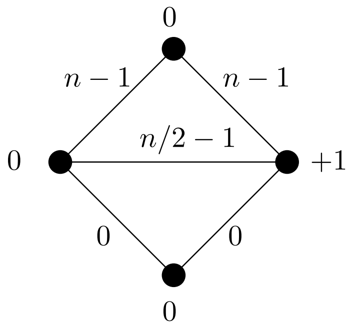
(b) One overload token detours to the upper path.



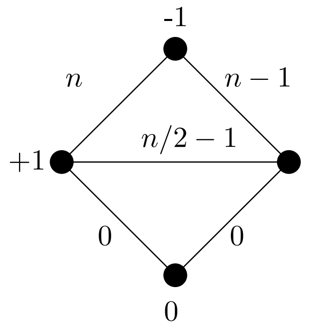
(c) It continues on the upper path



(d) ... and moves to the next gadget.

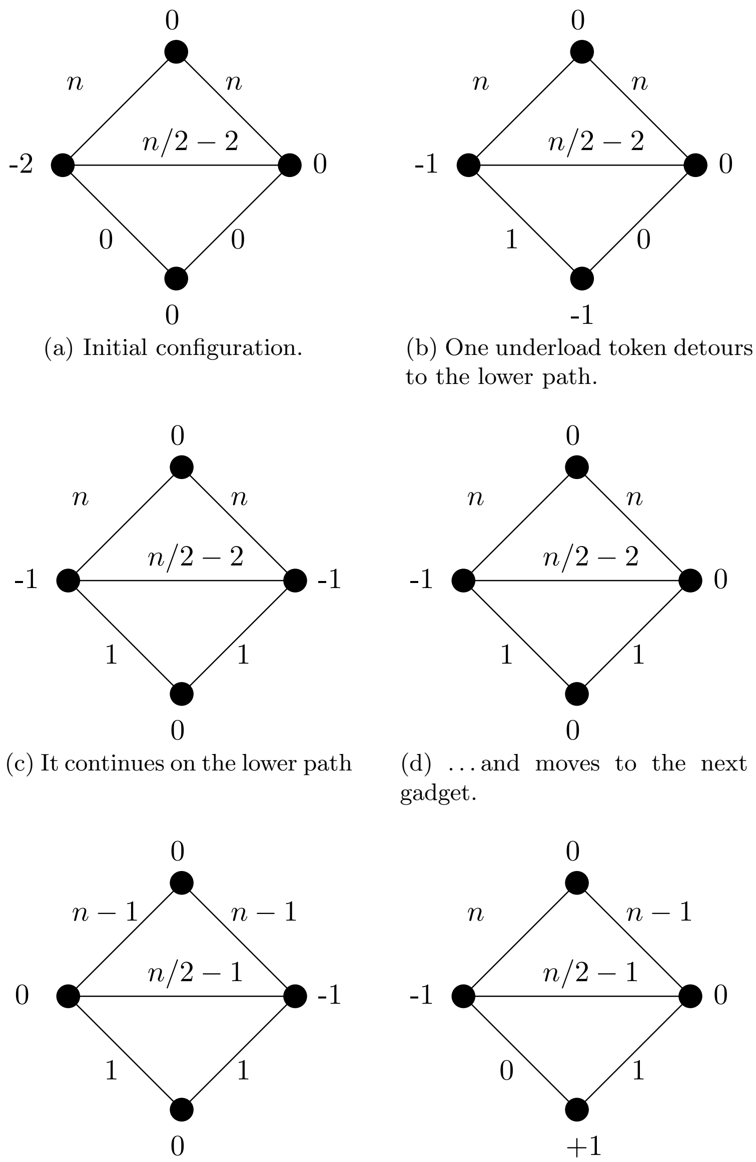


(e) The second overload token moves.



(f) New tokens are generated.

Figure 3.6: The number of tokens increases along the upper path.



(a) Initial configuration.

(b) One underload token detours to the lower path.

(c) It continues on the lower path

(d) . . . and moves to the next gadget.

(e) The second overload token moves.

(f) New tokens are generated.

Figure 3.7: The number of tokens increases along the lower path.

CHAPTER 4

Imitation Dynamics

Recall that the notion of Nash equilibria in congestion games is based on two assumptions. At first, it is assumed that players have complete knowledge about the game, i. e., about their strategy spaces, the set of resources including their latency functions, and in every state of the game about the choices of the other players. Secondly, it is assumed that players act rationally, i. e., they switch to other strategies if and only if this would decrease their latencies. In the previous chapters we have complied to these assumptions and investigated the existence of Nash equilibria and the convergence time of sequential best response dynamics in different classes of congestion games. In this chapter, however, we relax the first assumption in the following way. We consider scenarios for which players have only little or no experience at all upon the available options, that is, we assume that the players do not know their entire strategy spaces. In such scenarios a good strategy to follow is to *imitate* others coping successfully with the situation. It is widely accepted that this is a natural behavior, and thus, imitating behavior has been studied intensively in economics and game theory [HS98, Wei95].

In this chapter, we focus on *imitation dynamics* in symmetric standard congestion games[1] that emerge if myopic players imitate each other in order to improve their own situation. For the sake of mathematical tractability we make the following assumptions. We assume that in every state of the game each player knows its current strategy and its latency. Furthermore, we assume that each player has the capability of observing or sampling other players in order to get to know their current strategies. Additionally, we assume that a player can compute its latency if it were the only player to switch to a different strategy. Hence, a player can decide if imitating someone else is profitable. However, we also assume that players are oblivious, that is, once they switch to other strategies they forget about their previous choices.

[1]Throughout this chapter, we consider symmetric standard congestion games only. Hence, we omit the terms symmetric and standard and refer by the expression congestion game to such a game.

Under these assumptions we consider dynamics that emerge if players *sequentially* or *concurrently* imitate each other in order to decrease their latencies. It is quite obvious that there exist stable states of such dynamics, i. e., states in which none of the players can successfully imitate a different player. For example, every Nash equilibrium or every state in which all players choose the same strategy is such a state. Moreover, sequential imitation dynamics converge to stable states as Rosenthal's potential decreases in case of a profitable imitation. Also note that in the worst case it may take an exponential number of steps until sequential best response dynamics terminate. For that reasons we focus on concurrent imitation dynamics in which players concurrently imitate each other in a round based fashion and investigate under which conditions we can benefit from concurrency.

The major drawback of concurrent imitation dynamics is that several players might want to imitate the same player at the same time. However, if all of them act that way their latencies might be larger than before the migration. In this case undesirable *overshooting effects* occur. In order to circumvent these drawbacks we propose a protocol which we call the Imitation Protocol and analyze the convergence properties of imitation dynamics emerging if players concurrently apply this protocol. The Imitation Protocol consists of a sampling and a migration step. Initially, each player samples another player uniformly at random. Then it considers the latency gain that it would have by adopting the strategy of the sampled player, under the assumption that no one else changes its strategy. If this latency gain is not too small the player adopts the sampled strategy with a *migration probability* mainly depending on the anticipated latency gain.

This Imitation Protocol has several appealing properties: it is simple, stateless, based on local information, and is compatible with the selfish incentives of the players. Moreover, the Imitation Protocol avoids overshooting effects as the migration probabilities, according to which the players decide whether they migrate or not, are defined in a suitable manner without sacrificing the benefit of concurrency. We show that it suffices to scale the migration probabilities by the *elasticity* of the latency functions in order to avoid overshooting effects. The elasticity is a well-known parameter in economics which describes characteristic properties of markets, e. g. the ratio of the percent change of the demand of a good to the percent change of its price. For example, in case of polynomial latency functions with positive coefficients and maximum degree d the elasticity is bounded from above by d. As we already discussed above, a natural solution concept to imitation dynamics is *imitation stability*. Subsequently, we call a state *imitation-stable* if no more improvements are possible based on the Imitation Protocol.

As our first result we prove that the Imitation Protocol succeeds in avoiding overshooting effects and converges in a monotonic fashion to imitation-stable states. More precisely, we show that Rosenthal's potential function decreases on expectation as long as the system is not yet stable. Thus, the potential is a *super-martingale* and eventually reaches a local minimum, corresponding to an imitation-stable state. Hence, as a corollary, we see that an imitation-stable state is reached in pseudo-polynomial time.

Our main result, however, is a much stronger bound on the time to reach approximate imitation-stable states. What is a natural definition of approximate stability in our setting? By repeatedly sampling other players, a player gets to know the average latency of the system. It is approximately satisfied if it does not sustain a latency much larger than the average. Hence, we say that a state is approximately stable if almost all players are almost satisfied. More precisely, we consider states in which at most a δ-fraction of the players deviates by more than an ε-fraction (in any direction) from the average latency. We show that the expected time to reach such a state is polynomial in the inverse of the approximation parameters δ and ε as well as in the maximum elasticity of the latency functions, and logarithmic in the ratio between maximum and minimum potential. Hence, if the maximum latency of a path is fixed, the convergence time is only logarithmic in the number of players and independent of the size of the strategy space and the number of resources.

We complement these results by various lower bounds. At first, it is clear that pseudo-polynomial time is required to reach exact imitation-stable states. This follows from the fact that there exist states in which all latency improvements are arbitrarily small, resulting in arbitrarily small migration probabilities. Hence, already a single step may take pseudo-polynomially long. As a concept of approximately stable states one could have required *all* players to be approximately satisfied, rather than only all but a δ-fraction. This, however, would require to wait a polynomial number of rounds for the last player to become approximately satisfied, as opposed to our logarithmic bound. Finally, we consider sequential imitation processes in which only one player may move at a time. We extend a construction from [ARV08] to show that there exist instances in which the shortest sequence of imitations that leads to an imitation-stable state is exponentially long. In this construction, however, a player deviates to a different strategy regardless of how big its latency gain is. This is in contrast to the Imitation Protocol in which players only deviate if the anticipated latency gain exceeds a small threshold.

Note that there is a fundamental limitation to imitation dynamics: they are not innovative, i. e., they cannot explore new strategies. Even worse, due to our assumption that players are oblivious the knowledge about a strategy gets lost once no player uses the strategy any longer. In case of the Imitation Protocol the latter drawback reads as follows. It might happen with small but non-zero probability that all players currently using the same strategy s migrate towards other strategies and no other player migrates towards s. For singleton games, i. e., games in which each strategy is a singleton set, in which empty links have latency zero, we show that the probability of this event occurring in a polynomial number of rounds is negligible. This also has an important consequence: The sum of the players' latencies of a state to which the Imitation Protocol converges is, on expectation, not much worse than this sum at a Nash equilibrium. More precisely, we show for the case of linear latency functions that the expected sum of the players' latencies of a state to which the Imitation Protocol converges is within a constant factor of the optimal solution.

We conclude this chapter with a discussion of a possible extension of the Imitation Protocol. In cases in which convergence to a Nash equilibrium is required it is possible to adjust the dynamics and occasionally let players use an Exploration Protocol. Using such a protocol, players sample other strategies directly instead of sampling them by looking at other players. We show that a suitable definition of such a protocol and a suitable combination with the Imitation Protocol guarantee convergence to Nash equilibria in the long run.

4.1 Additional Notations and Useful Facts

In this chapter, we slightly change and extend the notions as introduced in Section 1.1.1. For ease of presentation, we formulate our results in the terms of symmetric network congestion games. Recall that in such a game all players strive to select a shortest path from the same source to the same sink. Subsequently, we refer by $\mathcal{P}$ to the set of paths in a given network connecting a particular source-sink pair, and by p to their number, i. e., $p = |\mathcal{P}|$. Furthermore, we use the terms edge e and path P instead of the terms resource r and strategy s. Additionally, we assume that the latency functions $\ell_e \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are non-decreasing and *differentiable*, and that $\ell_e(x) > 0$ for all $x > 0$.

Previously, we referred to a state by S and denoted by $x_e(S)$ the congestion on edge e. In this chapter, we refer to a state by x as we are more interested in the number of players allocating a particular resource or selecting a particular strategy rather than which player plays what strategy. Hence, given a state x, we denote by x_e, $e \in E$ the number of players allocation edge e, and by $x_P, P \in \mathcal{P}$, the number of players utilizing path P. Furthermore, we denote by $\ell_P(x) = \sum_{e \in P} \ell_e(x_e)$ the latency of path $P \in \mathcal{P}$ in state x. Hence, the latency of a player is the latency of the path it chooses.

For brevity, for all paths $P \in \mathcal{P}$, let 1_P denote the p-dimensional unit vector with the one in position P. Thus, in state x a player would switch from path P to path Q if this strictly decreases its latency, i. e., if $\ell_P(x) > \ell_Q(x + 1_Q - 1_P)$. For every path $P \in \mathcal{P}$ let

$$\ell_P^+(x) = \ell_P(x + 1_P) \ .$$

Note that $\ell_P^+(x) \geq \ell_P(x + 1_P - 1_Q)$ for every path $Q \in \mathcal{P}$. Additionally, let

$$L_{\text{av}}(x) = \sum_{P \in \mathcal{P}} \frac{x_P}{n} \ell_P(x)$$

denote the average latency of the paths in state x, and let

$$L_{\text{av}}^+(x) = \sum_{P \in \mathcal{P}} \frac{x_P}{n} \ell_P(x + 1_P) \ .$$

Finally, let

$$\ell_{\max} = \max_x \max_{P \in \mathcal{P}} \ell_P(x)$$

denote the maximum latency of any path. Throughout this chapter, whenever we consider a fixed state x we simply drop the argument (x) from Φ, ℓ_P, ℓ_P^+, L_{av}, and L_{av}^+.

Finally, given a fixed game let $\Phi^* = \min_x \Phi(x)$ be the minimum of Rosenthal's potential function. Due to our definition of the latency functions $\Phi^* > 0$ holds.

4.1.1 The Elasticity and the Slope of Latency Functions

In order to give a precise definition of the Imitation Protocol we need to introduce two additional quantities.

The Elasticity of Latency Functions

In order to bound the steepness of the latency functions and the effect that overshooting effects may have, we propose to scale the migration probabilities of the players by the elasticity of the latency functions. Formally, it is defined as follows. Recall that n denotes the number of player.

Definition 4.1. *Consider a differentiable latency function $\ell \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Then the elasticity of ℓ at point $x \in \,]0, n]$ is defined as*

$$d(x) = \frac{\ell'(x) \cdot x}{\ell(x)} \quad .$$

The function ℓ has elasticity d if $d(x) \leq d$ for every $x \in \mathbb{R}_{\geq 0}$.

Graphically, the elasticity at some point x can be interpreted as follows. The slope of $\ell(\cdot)$ at x can be estimated by the slope of the secant through the origin and the point $(x, \ell(x))$. Then the elasticity of $\ell(\cdot)$ at x is the factor by which this estimate is wrong. As an example, consider the polynomial function $\ell(x) = a \cdot x^d$ which has elasticity d for every $x \in \mathbb{R}_{\geq 0}$. Throughout this chapter, we frequently apply the following fact.

Fact 4.2. *Given a latency function with elasticity d, it holds that for any $x_0 \in \mathbb{R}_{>0}$ and $\alpha \geq 1$, $\ell_e(\alpha\, x_0) \leq \ell_e(x_0) \cdot \alpha^d$ and for $0 \leq \alpha < 1$, $\ell_e(\alpha\, x_0) \geq \ell_e(x_0) \cdot \alpha^d$.*

Proof. We first consider the case $\alpha \geq 1$. Hence, given $\ell(x_0)$ we like to derive an upper bound on $\ell(\alpha\, x_0)$. Since $\ell(\cdot)$ has elasticity d

$$\frac{\ell'(x) \cdot x}{\ell(x)} \quad \leq \quad d$$

holds for every $x \geq 0$. Note that the functional equality $(\ell'(x) \cdot x)/\ell(x) = d$ together with our boundary condition on $\ell(x_0)$ is solved by the function $\ell^*(x) = a \cdot x^d$ for $a = \ell(x_0)/x_0^d$. Hence, for $x \geq x_0$, $\ell(x) \leq a\, x^d \leq \ell(x_0)/x_0^d \cdot x^d$. For $x = \alpha x_0$, this proves the claim. The case $0 \leq \alpha < 1$ is treated similarly. □

The Slope of Latency Functions

As we already mentioned in the introduction, players applying the Imitation Protocol probably switch to different strategies if the anticipated latency gain is sufficiently large. As we will see below, the gain should be larger than the maximum slope on almost empty edges. Let ν_e denote the maximum slope on almost empty edges, i. e.,

$$\nu_e = \max_{x \in \{1,\ldots,d\}} \{\ell_e(x) - \ell_e(x-1)\} \quad .$$

Furthermore, for $P \in \mathcal{P}$, let $\nu_P = \sum_{e \in P} \nu_e$. Subsequently, we choose ν such that $\nu \geq \max_{P \in \mathcal{P}} \nu_P$.

As an example, consider monomial latency functions of the form $a \cdot x^d$, $a \geq 0$. Then a very rough upper bound of v is $m \cdot a \cdot d^d$.

4.1.2 Useful Facts

Next, we present some facts which we frequently apply in this chapter.

Fact 4.3 (Chernoff Bounds, see [HR90])**.** *Let X be a sum of Bernoulli variables. Then*

$$\mathbb{P}\left[X \geq k \cdot \mathbb{E}\left[X\right]\right] \quad \leq \quad \mathrm{e}^{-\mathbb{E}[X]\, k \cdot (\ln k - 1)} \quad ,$$

and, for $k \geq 4 > \mathrm{e}^{4/3}$,

$$\mathbb{P}\left[X \geq k \cdot \mathbb{E}\left[X\right]\right] \quad \leq \quad \mathrm{e}^{-\frac{1}{4}\, \mathbb{E}[X]\, k\, \ln k} \quad .$$

Equivalently, for $k \geq 4\, \mathbb{E}\left[X\right]$,

$$\mathbb{P}\left[X \geq k\right] \leq \mathrm{e}^{-\frac{1}{4}\, k\, \ln(k/\mathbb{E}[X])} \quad .$$

Fact 4.4. *For any $r > 0$ and $x \in [0, r]$, it holds that $(\mathrm{e}^x - 1) \leq x \cdot \frac{\mathrm{e}^r - 1}{r}$.*

Fact 4.5 (Geometric Series)**.** *For every $c \in]0, 1[$ it holds*

$$\sum_{k=l}^{\infty} c^k \;=\; \frac{c^l}{1 - c} \quad .$$

Fact 4.6 (Jensen's Inequality)**.** *Let $f \colon \mathbb{R} \to \mathbb{R}$ be a convex function, and let $a_1, \ldots, a_k$, $x_1, \ldots, x_k \in \mathbb{R}$. Then*

$$f\left(\frac{\sum_{i=1}^{k} a_i x_i}{\sum_{i=1}^{k} a_i}\right) \quad \leq \quad \frac{\sum_{i=1}^{k} a_i f(x_i)}{\sum_{i=1}^{k} a_i} \quad .$$

If $f(x) = x^2$, then

$$\left(\frac{\sum_{i=1}^{k} a_i x_i}{\sum_{i=1}^{k} a_i}\right)^2 \quad \leq \quad \frac{\sum_{i=1}^{k} a_i (x_i)^2}{\sum_{i=1}^{k} a_i}$$

$$\Leftrightarrow \quad \frac{1}{\sum_{i=1}^{k} a_i} \cdot \left(\sum_{i=1}^{k} a_i x_i\right)^2 \quad \leq \quad \sum_{i=1}^{k} a_i f(x_i) \quad .$$

Fact 4.7 ([FOV08b]). *Let $X_0, X_1, \ldots$ denote a sequence of non-negative random variables and assume that for all $i \geq 0$*

$$\mathbb{E}\left[X_i \mid X_{i-1} = x_{i-1}\right] \leq x_{i-1} - 1$$

and let τ denote the first time t such that $X_t = 0$. Then

$$\mathbb{E}\left[\tau \mid X_0 = x_0\right] \leq x_0 \ .$$

Fact 4.8 ([FOV08b]). *Let $X_0, X_1, \ldots$ denote a sequence of non-negative random variables and assume that for all $i \geq 0$ $\mathbb{E}\left[X_i \mid X_{i-1} = x_{i-1}\right] \leq x_{i-1} \cdot \alpha$ for some constant $\alpha \in (0, 1)$. Furthermore, fix some constant $x^* \in (0, x_0]$ and let τ be the random variable that describes the smallest t such that $X_t \leq x^*$. Then*

$$\mathbb{E}\left[\tau \mid X_0 = x_0\right] \leq \frac{2}{\log(1/\alpha)} \cdot \log\left(\frac{x_0}{x^*}\right) \ .$$

4.2 The Imitation Protocol

The Imitation Protocol (Protocol 1) proceeds in two steps. At first, each player samples another player uniformly at random. Then it considers the latency gain that it would have by adopting the strategy of the sampled player, under the assumption that no one else changes its strategy. If this latency gain is not too small the player adopts the sampled strategy with a *migration probability* mainly depending on the anticipated latency gain $(\ell_P(x) - \ell_Q(x + 1_Q - 1_P))/\ell_P(x)$ and on the elasticity of the latency functions given that the absolute latency exceeds the threshold ν.

Protocol 1 The Imitation Protocol, repeatedly executed by all players in parallel.

Let P denote the path of the player in state x
Sample another player uniformly at random. Let Q denote its path.
if $\ell_P(x) > \ell_Q(x + 1_Q - 1_P) + \nu$ **then**
  With probability

$$\mu_{PQ} = \frac{\lambda}{d} \cdot \frac{\ell_P(x) - \ell_Q(x + 1_Q - 1_P)}{\ell_P(x)}$$

  migrate from path P to path Q
end if

Our analysis concentrates on dynamics that result from the protocol being executed by the players in parallel in a round-based fashion. These dynamics generate a sequence of states $x(0), x(1), \ldots$. The resulting dynamics converge to a state that is stable in the sense that imitation cannot produce further progress, i. e., $x(t+1) = x(t)$ with probability 1. Such a state is called an *imitation-stable state*. In other words, a state is imitation-stable if it is ε-Nash with $\varepsilon = \nu$ with respect to the strategy space restricted to the current support. Here ε-Nash means that no player can improve its own latency by more than ε.

As discussed in the introduction, the main difficulty in the design of the protocol is to bound overshooting effects. To get an intuition of this problem, consider two parallel resources of which the first has the constant latency function $\ell_1(x) = c$ and the second has the latency function $\ell_2(x) = x^d$. Recall that the elasticity of ℓ_2 is d. Furthermore, assume that only a small number of players x_2 utilizes resource 2 whereas the majority of $n - x_2$ users utilizes resource 1. Let $b = c - x_2^d > 0$ denote the latency difference between the two resources. A simple calculation shows that using the protocol without the damping factor $1/d$, the expected latency increase on resource 2 would be $\Theta(b \cdot d)$, overshooting the balanced state by a factor d. For this reason, we reduce the migration probability accordingly. The constant λ will be determined later.

Note that the arguments in the last paragraph hold for the *expected* load changes. Our protocol, however, has to take care of probabilistic effects, i. e., the realized migration vector may differ from its expectation. Typically, we can use the elasticity to bound the impact of this effect. However, the elasticity does not give a good estimate of the latency increase if the congestion on an edge is very small, i. e., less than d. In this case, the number of joining players is not concentrated sharply enough around its expectation. Therefore, we add an additional requirement that players only migrate if the anticipated latency gain is at least ν and use this to bound probabilistic effects if the congestion of the edge is less than d. Let us remark that we will see below (Theorem 4.18) that for a large class of singleton games it is very unlikely, that an edge will ever have a load of d or less, so the protocol will behave in the same way with high probability for a polynomial number of rounds even if this additional requirement is dropped.

4.2.1 Convergence to Imitation-Stable States

In this section, we prove *that* imitation dynamics generated by the Imitation Protocol converge to imitation stable states since in each round Rosenthal's potential function $\Phi(x)$ decreases in expectation. From this result we can derive a pseudo-polynomial upper bound on the convergence time to imitation stable states.

Consider two states x and x' as well as a *migration vector* $\Delta x = (\Delta x_P)_{P \in \mathcal{P}}$ such that $x' = x + \Delta x$. We may imagine Δx as the result of one round of the Imitation Protocol although the following lemma is independent of how Δx is constructed. Furthermore, we consider Δx to be composed of a set of migrations of players between pairs of paths, i. e., Δx_{PQ} denotes the number of players who switch from path P to path Q, and Δx_P denotes the total increase or decrease of the number of players utilizing path P, that is,

$$\Delta x_P = \sum_{Q \in \mathcal{P}} (x_{QP} - x_{PQ}) \ .$$

Also, let $\Delta x_e = \sum_{P \ni e} \Delta x_P$ denote the induced change of the number of players utilizing edge $e \in E$. In order to prove convergence, we define the *virtual potential gain*

$$V_{PQ}(x, \Delta x) = x_{PQ} \cdot (\ell_Q(x + 1_Q - 1_P) - \ell_P(x))$$

which is the sum of the potential gains each player migrating from path P to path Q would contribute to $\Delta\Phi$ if each of them was the only migrating player. Note that if a player improves the latency of its path, the potential gain is negative. The sum of all virtual potential gains is a very rough lower bound on the true potential gain $\Delta\Phi(x, \Delta x) = \Phi(x + \Delta x) - \Phi(x)$. In order to compensate for the fact that players concurrently change their strategies, consider the *error term* on an edge $e \in E$:

$$F_e(x, \Delta x) = \begin{cases} \sum_{u=x_e+1}^{x_e+\Delta x_e} \ell_e(u) - \ell_e(x_e+1) & \text{if } \Delta x_e > 0 \\ \sum_{u=x_e+\Delta x_e+1}^{x_e} \ell_e(x_e) - \ell_e(u) & \text{if } \Delta x_e < 0 \\ 0 & \text{if } \Delta x_e = 0 \end{cases}$$

Subsequently, we show that the sum of the virtual potential gains and the error terms is indeed an upper bound on the true potential gain $\Delta\Phi(x, \Delta x)$. A similar result is shown in [FV08] for a continuous model.

Lemma 4.9. *For any assignment x and migration vector Δx it holds that*

$$\Delta\Phi(x, \Delta x) \le \sum_{P,Q \in \mathcal{P}} V_{PQ}(x, \Delta x) + \sum_{e \in E} F_e(x, \Delta x) \ .$$

Proof. We first express the virtual potential gain in terms of latencies on the edges. Clearly,

$$\begin{aligned} \sum_{P,Q \in \mathcal{P}} V_{PQ}(x, \Delta x) &= \sum_{P,Q \in \mathcal{P}} x_{PQ} \cdot (\ell_Q(x + 1_Q - 1_P) - \ell_P(x)) \\ &\le \sum_{P,Q \in \mathcal{P}} x_{PQ} \cdot \left(\sum_{e \in Q} \ell_e(x_e + 1) - \sum_{e \in P} \ell_e(x_e) \right) \\ &\le \sum_{e:\Delta x_e > 0} \Delta x_e \cdot \ell_e(x_e + 1) + \sum_{e:\Delta x_e < 0} \Delta x_e \cdot \ell_e(x_e) \ . \end{aligned} \tag{4.1}$$

The true potential gain, however, is

$$\begin{aligned} \Delta\Phi(x, \Delta x) &= \sum_{e:\Delta x_e > 0} \sum_{u=x_e+1}^{x_e+\Delta x_e} \ell_e(u) - \sum_{e:\Delta x_e < 0} \sum_{u=x_e-\Delta x_e+1}^{x_e} \ell_e(u) \\ &= \sum_{e:\Delta x_e > 0} \left(\Delta x_e \cdot \ell_e(x_e + 1) + \sum_{u=x_e+1}^{x_e+\Delta x_e} (\ell_e(u) - \ell_e(x_e + 1)) \right) \\ &\quad + \sum_{e:\Delta x_e < 0} \left(\Delta x_e \cdot \ell_e(x_e) + \sum_{u=x_e-\Delta x_e+1}^{x_e} (\ell_e(x_e) - \ell_e(u)) \right) \ . \end{aligned}$$

Substituting Equation (4.1) for the left term of each sum and the definition of F_e for the right term of each sum, we obtain the claim of the Lemma. □

In the following, we consider Δx to be a migration vector generated by the IMITATION PROTOCOL rather than an arbitrary vector. In this case, Δx is a random variable and all probabilities and expectations are taken with respect to the IMITATION PROTOCOL. In order to prove that the potential decreases in expectation, we derive a bound on the size of the error terms. We show that the error terms reduce the virtual potential gain by at most a factor of two, or, put a different way, that the true potential gain is at least half of the virtual potential gain.

Lemma 4.10. *Let x denote a state and let the random variable Δx denote a migration vector generated by the* IMITATION PROTOCOL. *Then,*

$$\mathbb{E}\left[\Delta\Phi(x, \Delta x)\right] \quad \leq \quad \frac{1}{2} \sum_{P,Q \in \mathcal{P}} \mathbb{E}\left[V_{PQ}(x, \Delta x)\right] \quad .$$

Proof. For any given round, each term in V_{PQ}, $P, Q \in \mathcal{P}$ and F_e, $e \in E$ can be associated with an player. Fix an player i migrating from, say, P to Q. Its contribution to the $V_{PQ}(x, \Delta x)$ is $\ell_Q(x + 1_Q - 1_P) - \ell_P(x)$ (this is the same for all players moving from P to Q). It may also contribute to F_e, $e \in P \cup Q$. However, this contribution depends on Δx_e and whether i migrates towards or away from e. Subsequently, we describe how to derive upper and lower bounds on these contributions depending on whether i migrates towards e or away from e.

Below, we consider subsets $\mathcal{N}' \subset \mathcal{N}$ of the players and assume that they are ordered with respect to ascending migration probabilities $\mu_{P_j Q_j}$, where P_j and Q_j denote the origin and destination path of player $j \in \mathcal{N}'$. Ties are broken arbitrarily.

Fix an edge $e \in Q \setminus P$ and let $A^+(e)$ denote the set of players migrating to $e \in Q \setminus P$. Let $\Delta\tilde{x}_e$ denote the number of players in $A^+(e)$ which occur in our ordering with respect to μ_{PQ} before player i.

Player i's contribution to $F_e(x, \Delta x)$, $e \in Q \setminus P$, is upper bounded by $\Delta\tilde{\ell}_e(\Delta\tilde{x}_e)$ where we define the error function $\Delta\tilde{\ell}_e(\delta) = \ell_e(x_e + 1 + \delta) - \ell_e(x_e + 1)$. In this case, we forgot about the positive effects players departing from e might have. For an illustration, see Figure 4.1. For brevity, let us write $\ell_e = \ell_e(x_e)$ and $\ell_e^+ = \ell_e(x_e + 1)$ as well as $\ell_P = \ell_P(x)$ and $\ell_Q^+ = \ell_P(x_e + 1_Q - 1_P)$. For $e \in Q \setminus P$ we show that

$$\mathbb{E}\left[\Delta\tilde{\ell}_e\left(\Delta\tilde{x}_e\right)\right] \quad \leq \quad \frac{1}{8} \cdot (\ell_P - \ell_Q^+) \cdot \left(\frac{\ell_e^+}{\ell_Q^+} + \frac{\nu_e}{\nu_Q}\right) \quad . \tag{4.2}$$

Now, fix an edge $e \in P \setminus Q$ and let $A^-(e)$ denote the set of players migrating away from $e \in P \setminus Q$. Let $\Delta\tilde{x}_e$ denote the number of players in $A^-(e)$ which occur in our ordering with respect to μ_{PQ} before player i. Player i's contribution to $F_e(x, \Delta x)$, $e \in P \setminus Q$ is lower bounded by $\Delta\tilde{\ell}_e(\Delta\tilde{x}_e)$ where $\Delta\tilde{\ell}_e(\delta)$ is defined as above. Hence, we forgot about the positive effects players migrating towards e might have. For $e \in P \setminus Q$ we show that

$$\mathbb{E}\left[\Delta\tilde{\ell}_e\left(\Delta\tilde{x}_e\right)\right] \quad \leq \quad \frac{1}{8} \cdot (\ell_P - \ell_Q^+) \cdot \left(\frac{\ell_e}{\ell_P} + \frac{\nu_e}{\nu_P}\right) \quad . \tag{4.3}$$

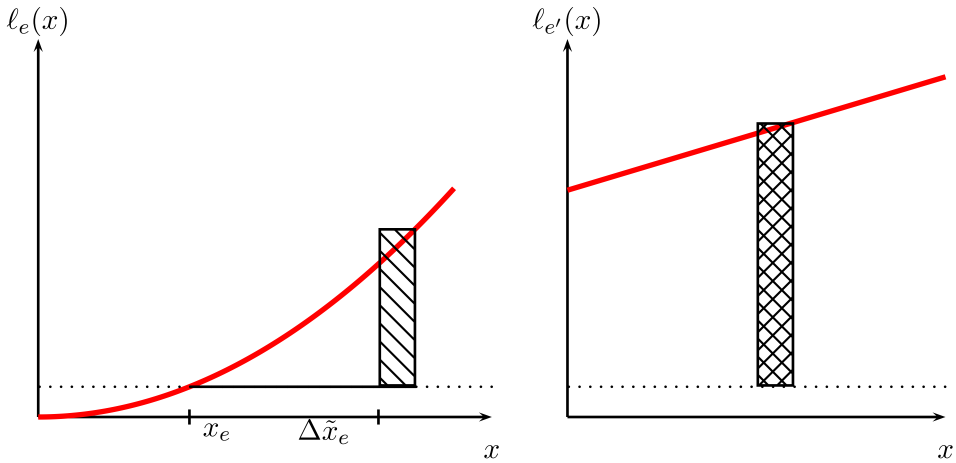


Figure 4.1: Potential gain of a player migrating from edge e' towards edge e. The hatched area is the player's virtual potential gain. The shaded area on the left is this player's contribution to the error term, caused by the $\Delta\tilde{x}_e$ players ranking before the player under consideration (with respect to μ_{PQ}).

Thus, the expected sum of the error terms of an player migrating from P to Q is at most

$$\frac{\ell_P - \ell_Q^+}{8}\left(\sum_{e\in P\setminus Q}\left(\frac{\ell_e}{\ell_P}+\frac{\nu_e}{\nu_P}\right)+\left(\sum_{e\in Q\setminus P}\frac{\ell_e^+}{\ell_Q^+}+\frac{\nu_e}{\nu_Q}\right)\right) \quad\le\quad \frac{1}{2}(\ell_P-\ell_Q^+) \ ,$$

i. e., half of its virtual potential gain, which proves the lemma. First, consider the case that $e \in Q \setminus P$ where Q denotes the destination path of agent i.

For brevity, let us write $I_{PQ} = (\ell_P - \ell_Q^+)/\ell_P$ for the incentive to migrate from P to Q. Again, consider the case that $e \in Q$ where Q denotes the destination path of player i. Then, due to our order of the players,

$$\mathbb{E}\left[\Delta\tilde{x}_e\right] \le n\cdot\frac{x_e}{n}\cdot\mu_{PQ} \le \frac{\lambda\cdot x_e\cdot I_{PQ}}{d} \tag{4.4}$$

implying

$$x_e \ge \frac{\mathbb{E}\left[\Delta\tilde{x}_e\right]\cdot d}{\lambda\cdot I_{PQ}} \ . \tag{4.5}$$

Furthermore, due to the elasticity of ℓ_e, and using $(1+1/x)^x \le \exp(1)$, we obtain

$$\begin{aligned}\Delta\tilde{\ell}_e(\delta) \quad&\le\quad \ell_e^+\cdot\left(\frac{x_e+1+\delta}{x_e+1}\right)^d - \ell_e^+\\ &\le\quad \ell_e^+\cdot\left(1+\frac{\delta}{x_e}\right)^d - \ell_e^+\\ &\le\quad \ell_e^+\cdot\left(\mathrm{e}^{\frac{d\,\delta}{x_e}}-1\right) \ .\end{aligned} \tag{4.6}$$

Subsequently, we consider two cases.

Case 1: $\mathbb{E}\left[\Delta\tilde{x}_e\right] \geq \frac{1}{64}$**.** Substituting Inequality (4.5) into Inequality (4.6), we obtain for every $\kappa \in \mathbb{R}_{\geq 0}$

$$\Delta\tilde{\ell}_e\left(\kappa\,\mathbb{E}\left[\Delta\tilde{x}_e\right]\right) \quad\leq\quad \ell_e^+ \cdot \left(\mathrm{e}^{\kappa\,\lambda\,I_{PQ}} - 1\right) \quad .$$

Now, note that for every $k \in \mathbb{N}$ and $\kappa \in [k, k+1]$

$$\begin{aligned} \mathbb{P}\left[\Delta\tilde{x}_e \geq \kappa\,\mathbb{E}\left[\Delta\tilde{x}_e\right]\right] &\leq \mathbb{P}\left[\Delta\tilde{x}_e \geq k\,\mathbb{E}\left[\Delta\tilde{x}_e\right]\right] \quad \text{and} \\ \Delta\tilde{\ell}_e(\kappa\,\mathbb{E}\left[\Delta\tilde{x}_e\right]) &\leq \Delta\tilde{\ell}_e((k+1)\,\mathbb{E}\left[\Delta\tilde{x}_e\right]) \end{aligned}$$

hold. Applying a Chernoff bound (Fact 4.3), we obtain an upper bound for the expectation of $\mathbb{E}\left[\Delta\tilde{\ell}_e\left(\Delta\tilde{x}_e\right)\right]$ as follows.

$$\begin{aligned} &\mathbb{E}\left[\Delta\tilde{\ell}_e\left(\Delta\tilde{x}_e\right)\right] \\ \leq\ & \sum_{k=1}^{\infty} \mathbb{P}\left[\Delta\tilde{x}_e \geq k\,\mathbb{E}\left[\Delta\tilde{x}_e\right]\right] \cdot \Delta\tilde{\ell}_e((k+1)\,\mathbb{E}\left[\Delta\tilde{x}_e\right]) \\ \leq\ & \Delta\tilde{\ell}_e^+\left(5\,\mathbb{E}\left[\Delta\tilde{x}_e\right]\right) + \sum_{k=5}^{\infty} \mathbb{P}\left[\Delta\tilde{x}_e \geq k\,\mathbb{E}\left[\Delta\tilde{x}_e\right]\right] \cdot \Delta\tilde{\ell}_e((k+1)\,\mathbb{E}\left[\Delta\tilde{x}_e\right]) \\ \leq\ & \ell_e^+ \cdot \left(\mathrm{e}^{5\,\lambda\,I_{PQ}} - 1\right) + \sum_{k=5}^{\infty} \mathrm{e}^{-\frac{1}{4}\,\mathbb{E}[\Delta\tilde{x}_e]\,k\,\ln k} \cdot \ell_e^+ \cdot \left(\mathrm{e}^{(k+1)\,\lambda\,I_{PQ}} - 1\right) \\ \leq\ & \ell_e^+ \cdot \left(\mathrm{e}^{5\,\lambda\,I_{PQ}} - 1\right) + \sum_{k=5}^{\infty} \mathrm{e}^{-\frac{1}{4}\,\mathbb{E}[\Delta\tilde{x}_e]\,k} \cdot \ell_e^+ \cdot \left(\mathrm{e}^{2\,k\,\lambda\,I_{PQ}} - 1\right) \\ \leq\ & \ell_e^+ \cdot \left(\mathrm{e}^{5\,\lambda\,I_{PQ}} - 1\right) + \int_4^{\infty} \mathrm{e}^{-\frac{1}{4}\,\mathbb{E}[\Delta\tilde{x}_e]\,u} \cdot \ell_e^+ \cdot \left(\mathrm{e}^{2\,u\,\lambda\,I_{PQ}} - 1\right)\,du \\ =\ & \ell_e^+ \cdot \left(\mathrm{e}^{5\lambda I_{PQ}} - 1 + \mathrm{e}^{-\mathbb{E}[\Delta\tilde{x}_e]} \frac{\mathrm{e}^{8\,\lambda\,I_{PQ}} - 1 + \frac{8\,\lambda\,I_{PQ}}{\mathbb{E}[\Delta\tilde{x}_e]}}{\frac{1}{4}\mathbb{E}\left[\Delta\tilde{x}_e\right] - 2\,\lambda\,I_{PQ}}\right) \quad . \end{aligned}$$

Now, due to Fact 4.4 with $r = 1$ and our assumption that $\mathbb{E}\left[\Delta\tilde{x}_e\right] \geq 1/64$, we obtain

$$\begin{aligned} \mathbb{E}\left[\Delta\tilde{\ell}_e\left(\Delta\tilde{x}_e\right)\right] &\leq \lambda \cdot \ell_e^+ \cdot I_{PQ} \cdot \left(5\,(\mathrm{e}-1) + \frac{8\,(\mathrm{e}-1) + 8 \cdot 64}{\frac{1}{4 \cdot 64} - 2\,\lambda}\right) \\ &\leq c \cdot \lambda \cdot \ell_e^+ \cdot \frac{\ell_P - \ell_Q^+}{\ell_P} \\ &\leq c \cdot \lambda \cdot \ell_e^+ \cdot \frac{\ell_P - \ell_Q^+}{\ell_Q^+} \end{aligned}$$

for some constant c. The first inequality holds if $\lambda < 1/512$, proving Equation (4.2) if λ is chosen small enough.

Case 2: $\mathbb{E}\left[\Delta\tilde{x}_e\right] < \frac{1}{64}$**.** Again, in this case we can apply a Chernoff bound (Fact 4.3) to upper bound $\mathbb{E}\left[\Delta\tilde{\ell}_e\left(\Delta\tilde{x}_e\right)\right]$.

$$\begin{aligned}
\mathbb{E}\left[\Delta\tilde{\ell}_e\left(\Delta\tilde{x}_e\right)\right] &\leq \sum_{k=1}^{n} \mathbb{P}\left[\Delta\tilde{x}_e = k\right] \cdot \Delta\tilde{\ell}_e(k) \\
&\leq \sum_{k=1}^{n} \mathbb{P}\left[\Delta\tilde{x}_e \geq \frac{k}{\mathbb{E}\left[\Delta\tilde{x}_e\right]}\,\mathbb{E}\left[\Delta\tilde{x}_e\right]\right] \cdot \Delta\tilde{\ell}_e(k) \\
&\leq \sum_{k=1}^{n} \mathrm{e}^{-k\,(\ln(k/\mathbb{E}[\Delta\tilde{x}_e])-1)} \cdot \Delta\tilde{\ell}_e(k)
\end{aligned}$$

There are two subcases:

Case 2a: $x_e > d$**.** In order to bound the expected latency increase, we apply the elasticity bound on ℓ_e:

$$\begin{aligned}
&\mathbb{E}\left[\Delta\tilde{\ell}_e(\Delta\tilde{x}_e)\right] \\
\leq\ & \sum_{k=1}^{n} \mathrm{e}^{-k\,(\ln(k/\mathbb{E}[\Delta\tilde{x}_e])-1)} \cdot \ell_e^+ \cdot \left(\mathrm{e}^{\frac{k\,d}{x_e}} - 1\right) \\
\leq\ & \ell_e^+ \cdot \sum_{k=1}^{n} \mathrm{e}^{-k\,(\ln(k)-\ln(\mathbb{E}[\Delta\tilde{x}_e])-1)} \cdot \left(\mathrm{e}^{\frac{k\,d}{x_e}} - 1\right) \\
\leq\ & \ell_e^+ \cdot \sum_{k=1}^{n} \left(\mathbb{E}\left[\Delta\tilde{x}_e\right]\,(\mathrm{e}^k\,\mathbb{E}\left[\Delta\tilde{x}_e\right]^{k-1})\right)\,\mathrm{e}^{-k\,(\ln k)} \cdot \left(\mathrm{e}^{\frac{k\,d}{x_e}} - 1\right) \\
\leq\ & \ell_e^+ \cdot \mathbb{E}\left[\Delta\tilde{x}_e\right] \cdot \sum_{k=1}^{n} \mathrm{e}^{-k\,(\ln k)} \cdot \left(\mathrm{e}^{\frac{k\,d}{x_e}} - 1\right) \quad .
\end{aligned}$$

Now, splitting up the sum, we define

$$\begin{aligned}
L_1 &= \mathbb{E}\left[\Delta\tilde{x}_e\right] \sum_{k=1}^{\left\lfloor\frac{8\,x_e}{d}\right\rfloor} \mathrm{e}^{-k\,(\ln k)} \cdot \left(\mathrm{e}^{\frac{k\,d}{x_e}} - 1\right) \\
&\leq \mathbb{E}\left[\Delta\tilde{x}_e\right] \frac{(\mathrm{e}^8 - 1)\,d}{8\,x_e} \sum_{k=1}^{\left\lfloor\frac{8\,x_e}{d}\right\rfloor} \mathrm{e}^{-k\,(\ln k)} \cdot k \\
&\leq \frac{\mathrm{e}^8}{4} \cdot \mathbb{E}\left[\Delta\tilde{x}_e\right] \frac{d}{x_e} \\
&\leq \frac{\mathrm{e}^8}{4} \cdot \lambda\, I_{PQ} \quad ,
\end{aligned}$$

where the first inequality uses the observation that $\mathrm{e}^{\frac{k\,d}{x_e}} \leq \mathrm{e}^8$ since $k \leq \lfloor 8x_e/d \rfloor$, and Fact 4.4 (with $r = 8$). Additionally, where the third inequality uses the observation that $\sum_{k=1}^{\infty} \mathrm{e}^{-k\,(\ln k)} \cdot k \leq 2$, and finally where the last inequality uses Inequality (4.4).

For the second part of the sum, let

$$\begin{aligned} L_2 &= \mathbb{E}\left[\Delta\tilde{x}_e\right] \sum_{k=\lceil\frac{8\,x_e}{d}\rceil}^{\infty} \mathrm{e}^{-k\,(\ln k)} \cdot \left(\mathrm{e}^{\frac{k\,d}{x_e}} - 1\right) \\ &\le \mathbb{E}\left[\Delta\tilde{x}_e\right] \sum_{k=\lceil\frac{8\,x_e}{d}\rceil}^{\infty} \mathrm{e}^{-k\,(\ln k)+\frac{k\,d}{x_e}} \\ &= \mathbb{E}\left[\Delta\tilde{x}_e\right] \sum_{k=\lceil\frac{8\,x_e}{d}\rceil}^{\infty} \mathrm{e}^{-k\,(\ln k-1)} && \text{(since } x_e > d\text{)} \\ &\le \mathbb{E}\left[\Delta\tilde{x}_e\right] \sum_{k=\lceil\frac{8\,x_e}{d}\rceil}^{\infty} \mathrm{e}^{-\frac{1}{2}\,k\,\ln k} && \text{(since } k \ge \lceil\tfrac{8\,x_e}{d}\rceil \ge 8\text{)} \\ &\le \mathbb{E}\left[\Delta\tilde{x}_e\right] \sum_{k=\lceil\frac{8\,x_e}{d}\rceil}^{\infty} \left(\frac{d}{8\,x_e}\right)^{\frac{1}{2}k} \quad . \end{aligned}$$

Due to Fact 4.5 and since $x_e > d$

$$\begin{aligned} L_2 &= \mathbb{E}\left[\Delta\tilde{x}_e\right] \frac{\left(\frac{d}{8\,x_e}\right)^{8/2}}{1 - \sqrt{\frac{d}{8\,x_e}}} \\ &\le \mathbb{E}\left[\Delta\tilde{x}_e\right] \frac{d}{x_e} \\ &\le \lambda\, I_{PQ} \quad . \end{aligned}$$

Reassembling the sum, we obtain

$$\begin{aligned} \mathbb{E}\left[\Delta\tilde{\ell}_e(\Delta\tilde{x}_e)\right] &\le \ell_e^+ \cdot (L_1 + L_2) \\ &\le \ell_e^+ \cdot \left(\frac{\mathrm{e}^8}{4} + 1\right) \lambda\, I_{PQ} \quad . \end{aligned}$$

Again, by the same arguments as at the end of Case 1 this proves Equation (4.2) if λ is less than $1/(2\mathrm{e}^8 + 8)$.

Case 2b: $x_e \le d$**.** In this case we separate the upper bound on $\mathbb{E}\left[\Delta\tilde{\ell}_e(\Delta\tilde{x}_e)\right]$ into the section up to d and above d. For the first section we use the fact that each additional player on resource e causes a latency increase of at most ν_e as long as the load is at most d. We define the contribution to the expected latency increase by the events that up to $d - x_e$ join resource e, i. e., afterwards the congestion is still at most d. In this case, we may use

ν_e to bound the contribution of each player:

$$\begin{aligned} L_1 &\leq \sum_{k=1}^{d-x_e} \mathrm{e}^{-k\,(\ln(k/\mathbb{E}[\Delta\tilde{x}_e])-1)} \cdot k\,\nu_e \\ &\leq \mathrm{e}\,\nu_e\,\mathbb{E}\left[\Delta\tilde{x}_e\right] + \nu_e\,\mathbb{E}\left[\Delta\tilde{x}_e\right]^2 \sum_{k=2}^{d-x_e} \mathrm{e}^{-k\,(\ln(k)-1)} \cdot k \\ &\leq \mathrm{e}\,\nu_e\,\mathbb{E}\left[\Delta\tilde{x}_e\right] \cdot (1 + 8\,\mathbb{E}\left[\Delta\tilde{x}_e\right]/\mathrm{e}) \\ &\leq 3\,\nu_e\,\mathbb{E}\left[\Delta\tilde{x}_e\right] \quad , \end{aligned}$$

where the third inequality holds since $\sum_{k=2}^{d-x_e} \mathrm{e}^{-k\,(\ln(k)-1)} \cdot k \leq 8$, and where the last inequality holds since $\mathbb{E}\left[\Delta\tilde{x}_e\right] < 1/64$.
For the contribution of the players increasing the load on resource e to above d we use the elasticity constraint again. This time, we do not consider the latency increase with respect to $\ell_e^+(x_e)$ but with respect to $\ell_e(d)$:

$$L_2 = \sum_{k=d-x_e+1}^{n} \mathrm{e}^{-k\cdot(\ln(k)/\mathbb{E}[\Delta\tilde{x}_e])-1)} \cdot \ell_e(d) \cdot \left(\mathrm{e}^{\frac{d\,(k-(d-x_e))}{d}} - 1\right) \quad .$$

By the same arguments as in case (2a),

$$\begin{aligned} & L_2 \\ \leq\; & \ell_e(d) \cdot \mathbb{E}\left[\Delta\tilde{x}_e\right] \cdot \sum_{k=d-x_e+1}^{\infty} \mathrm{e}^{-k\,\ln k + k - (d-x_e)} \\ =\; & \ell_e(d) \cdot \mathbb{E}\left[\Delta\tilde{x}_e\right] \cdot \sum_{k=1}^{\infty} \mathrm{e}^{-(k+(d-x_e))\,\ln(k+(d-x_e))+k} \\ =\; & \ell_e(d) \cdot \mathbb{E}\left[\Delta\tilde{x}_e\right] \cdot \mathrm{e}^{-(d-x_e)} \cdot \sum_{k=1}^{\infty} \mathrm{e}^{-(k+(d-x_e))\,\ln(k+(d-x_e))+k+d-x_e} \quad . \end{aligned}$$

Consider the series in the above expression as a function of $u = (d - x_e)$ and denote it by $S(u)$. Note that $S(u)$ converges for every $u \geq 0$ and $S(u) \to 0$ as $u \to \infty$. In particular, $S(u) < 8$ for any $u \geq 0$, so

$$\begin{aligned} L_2 &\leq 8\,\ell_e(d) \cdot \mathbb{E}\left[\Delta\tilde{x}_e\right] \cdot \mathrm{e}^{-(d-x_e)} \\ &\leq 8\,(\ell_e(x_e) + (d - x_e)\,\nu_e) \cdot \mathbb{E}\left[\Delta\tilde{x}_e\right] \cdot \mathrm{e}^{-(d-x_e)} \quad . \end{aligned}$$

Since $(d - x_e) \cdot \mathrm{e}^{-(d-x_e)} < 1/2$,

$$L_2 \leq 4\,(\ell_e(x_e) + \nu_e) \cdot \mathbb{E}\left[\Delta\tilde{x}_e\right] \quad .$$

Altogether,

$$\begin{aligned} \mathbb{E}\left[\Delta\tilde{\ell}_e(\Delta\tilde{x}_e)\right] &\leq L_1 + L_2 \\ &\leq 7\,\nu_e\,\mathbb{E}\left[\Delta\tilde{x}_e\right] + 4\,\ell_e(x_e)\,\mathbb{E}\left[\Delta\tilde{x}_e\right] \\ &\leq 7\,\nu_e\,\mathbb{E}\left[\Delta\tilde{x}_e\right] + 4\,\frac{\lambda\,x_e\,I_{PQ}}{d}\ell_e(x_e) \\ &\leq \frac{7}{64}\,\nu\,\frac{\nu_e}{\nu_Q} + \frac{4\,\lambda\,x_e\,I_{PQ}}{d}\ell_e(x_e) \end{aligned}$$

where we have used Equation (4.4) for the third inequality, and the inequalities $\mathbb{E}[\Delta\tilde{x}_e] < 1/64$ and $\nu \geq \nu_Q$ for the last step. Since $x_e \leq d$ and $\ell_P - \ell_Q^+ \geq \nu$,

$$\mathbb{E}\left[\Delta\tilde{\ell}_e(\Delta\tilde{x}_e)\right] \leq \frac{1}{8}(\ell_P - \ell_Q^+)\frac{\nu_e}{\nu_Q} + \frac{4\,\lambda\,(\ell_P - \ell_Q^+)}{\ell_P}\ell_e(x_e)$$

again proving Equation (4.2) if $\lambda \leq 1/32$.

Finally, the case $e \in P$ is very similar. □

Note that all migrating players add a negative contribution to the virtual potential gain since they migrate only from paths with currently higher latency to paths with lower latency. Hence, together with Lemma 4.10, we can derive the next corollary.

Corollary 4.11. *Consider a symmetric network congestion game Γ and let x and x' denote states of Γ such that x' is a random state generated after one round of executing the* Imitation Protocol. *Then*

$$\mathbb{E}\left[\Phi(x')\right] \leq \Phi(x)$$

with strict inequality as long as x is not imitation-stable. Thus, Φ is a supermartingale.

It is obvious *that* the sequence of states generated by the Imitation Protocol terminates at an imitation-stable state. From Lemma 4.10 we can immediately derive an upper bound on the time to reach such a state. However, since for arbitrary latency functions the minimum possible latency gain may be very small, this bound can clearly be only pseudo-polynomial. To see this, consider a state in which only one player can make an improvement. Then the expected time until the player moves is inversely proportional to its latency gain.

Theorem 4.12. *Consider a symmetric network congestion game in which all players use the* Imitation Protocol. *Let x denote the initial state of the dynamics. Then the dynamics converge to an imitation-stable state in expected time*

$$O\left(\frac{d\,n\,\ell_{\max}\,\Phi(x)}{\nu^2}\right) .$$

Proof. By definition of the Imitation Protocol, the expected virtual potential gain in any state x' which is not yet imitation-stable is at least

$$\mathbb{E}\left[\sum_{P,Q\in\mathcal{P}} V_{PQ}(x',\Delta x')\right] \leq -\nu\cdot\frac{\lambda}{d\,n}\cdot\frac{\nu}{\ell_{\max}} .$$

Hence, also the expected potential gain $\mathbb{E}[\Delta\Phi(x')]$ in every intermediate state x' of the dynamics is bounded from above by at least half of the above value. From this it follows that the expected time until the potential drops from at most $\Phi(x)$ to the minimum potential Φ^* is at most

$$\frac{d\,n\,\ell_{\max}(\Phi(x)-\Phi^*)}{\lambda\,\nu^2} \ .$$

Formally, this is a consequence of Fact 4.7. ☐

It is obvious that this result cannot be significantly improved since we can easily construct an instance and a state such that the only possible improvement that can be made is ν. Hence, already a single step takes pseudo-polynomially long. In case of polynomial latency functions Theorem 4.12 reads as follows.

Corollary 4.13. *Consider a symmetric network congestion game with polynomial latency functions with maximum degree d and minimum and maximum coefficients a_{min} and a_{max}, respectively. Then the dynamics converges to an imitation-stable state in expected time*

$$O\left(d^3 m^2 n^{2d+2} \cdot \left(\frac{a_{max}}{a_{min}}\right)^2\right) \ .$$

In case of a singleton congestion game with monimial latency function we can improve the corollary as follows.

Corollary 4.14. *Consider a symmetric single congestion game with monomial latency functions with maximum degree d. Then the dynamics converges to an imitation-stable state in expected time $O\left(n^{2d+2}\right)$.*

Let us remark that all proofs in this section do not rely on the assumption that the underlying congestion game is symmetric. In fact, the lemmas also hold for asymmetric congestion games in which each player samples only among players that have the same strategy space.

4.2.2 Convergence to Approximate Equilibria

Theorem 4.12 guarantees convergence of concurrent imitation dynamics generated by the Imitation Protocol to an imitation-stable state in the long run. However, it does not give a reasonable bound on the time due to the small progress that can be made. Hence, as our main result of this chapter, we present bounds on the time to reach an *approximate equilibrium*. Here we relax the definition of an imitation-stable state in two aspects: We allow only a small minority of players to deviate by more than a small amount from the average latency. Our notion of an approximate equilibrium is similar to the notion used in [BEDL06, FRV06, FKS08a]. It is motivated by the following observation. When sampling other players each player comes to know its latency if it would adopt that players' strategy. Hence to some extend each player can compute the average latency L^+_{av} and determine if its own latency is above or below that average.

Definition 4.15 ((δ,ε,ν)-equilibrium)**.** *Given a state x, let the set of* expensive *paths be $\mathcal{P}^+_{\varepsilon,\nu} = \{P \in \mathcal{P} : \ell_P(x) > (1+\varepsilon)\, L^+_{av} + \nu\}$ and let the set of* cheap *paths be $\mathcal{P}^-_{\varepsilon,\nu} = \{P \in \mathcal{P} : \ell_P(x) < (1-\varepsilon)\, L_{av} - \nu\}$. Let $\mathcal{P}_{\varepsilon,\nu} = \mathcal{P}^+_{\varepsilon,\nu} \cup \mathcal{P}^-_{\varepsilon,\nu}$. A configuration x is at a* (δ,ε,ν)-equilibrium *if and only if it holds that $\sum_{P \in \mathcal{P}^{\varepsilon,\nu}} x_P \leq \delta \cdot n$.*

Intuitively, a state at (δ,ε,ν)-equilibrium is a state in which almost all players are almost satisfied when comparing their own situation with the situation of other players. One may hope that it is possible to reach a state in which *all* players are almost satisfied quickly. This relaxation would conceptually be the same as a Nash equilibrium. We will argue below, however, that there is no rapid convergence to such states.

Theorem 4.16. *For an arbitrary initial assignment x_0, let τ denote the first round in which the* Imitation Protocol *reaches a (δ,ε,ν)-equilibrium. Then*

$$\mathbb{E}\left[\tau\right] = O\left(\frac{d}{\varepsilon^2\,\delta}\log\left(\frac{\Phi(x_0)}{\Phi^*}\right)\right) \quad .$$

Proof. We consider a state $x(t)$ that is not at a (δ,ε,ν)-equilibrium and derive a lower bound on the expected potential gain. There are two cases. Either at least half of the players utilizing paths in $\mathcal{P}_{\varepsilon,\nu}$ utilize paths in $\mathcal{P}^+_{\varepsilon,\nu}$ or at least half of them utilize paths in $\mathcal{P}^-_{\varepsilon,\nu}$.

Case 1: Many players use expensive paths, i. e., $\sum_{P \in \mathcal{P}^+_{\varepsilon,\nu}} x_P \geq \delta\, n/2$. Let us define the volume T and the average ex-post latency C of potential destination paths, i. e., paths with ex-post latency at most $(1+\varepsilon)L^+_{\mathrm{av}}$, by

$$T = \sum_{Q:\ell^+_Q \leq (1+\varepsilon)L^+_{\mathrm{av}}} \frac{x_Q}{n} \quad \text{and} \quad C = \frac{1}{T}\sum_{Q:\ell^+_Q \leq (1+\varepsilon)L^+_{\mathrm{av}}} \frac{x_Q}{n}\ell^+_Q \ .$$

Clearly,

$$L^+_{\mathrm{av}} = \sum_P \frac{x_P}{n}\ell^+_P \geq T \cdot C + (1-T)\cdot(1+\varepsilon)\,L^+_{\mathrm{av}} \ ,$$

and solving for T yields

$$T \geq \frac{\varepsilon\, L^+_{\mathrm{av}}}{(1+\varepsilon)\,L^+_{\mathrm{av}} - C} \quad . \tag{4.7}$$

We now give a lower bound on the expected virtual potential gain given that the current state is not at a (δ,ε,ν)-equilibrium. We consider only the contribution of players utilizing paths in $\mathcal{P}^+_{\varepsilon,\nu}$ and sampling paths with ex-post latency below $(1+\varepsilon)\,L^+_{\mathrm{av}}$. Then

$$\begin{aligned}
&\mathbb{E}\left[\sum_{P,Q} V_{PQ}\right] \\
\leq\ & -\frac{\lambda}{d}\sum_{P \in \mathcal{P}^+_{\varepsilon,\nu}} x_P \sum_{Q:\ell^+ \leq (1+\varepsilon)L^+_{\mathrm{av}}} \frac{x_Q}{n}\cdot\frac{(\ell_P - \ell_Q(x+1_Q-1_P))^2}{\ell_P} \\
\leq\ & -\frac{\lambda}{d}\sum_{P \in \mathcal{P}^+_{\varepsilon,\nu}} x_P\ell_P \sum_{Q:\ell^+ \leq (1+\varepsilon)L^+_{\mathrm{av}}} \frac{x_Q}{n}\cdot\left(\frac{\ell_P - \ell^+_Q}{\ell_P}\right)^2 \quad .
\end{aligned}$$

Using Jensen's inequality (Fact 4.6) and substituting $\ell_P \geq L_{\text{av}}^+$ yields

$$\mathbb{E}\left[\sum_{P,Q} V_{PQ}\right] \quad \leq \quad \frac{-\frac{\lambda}{d} L_{\text{av}}^+ \sum_{P\in\mathcal{P}_{\varepsilon,\nu}^+} x_P \left(\sum_{Q:\ell^+\leq(1+\varepsilon)L_{\text{av}}^+} \frac{x_Q}{n} \cdot \frac{\ell_P - \ell_Q^+}{\ell_P}\right)^2}{\sum_{Q:\ell_Q^+\leq(1+\varepsilon)L_{\text{av}}^+} \frac{x_Q}{n}} \ .$$

Now we substitute $\ell_P \geq (1+\varepsilon)\, L_{\text{av}}^+$ and use the fact that the squared expression is monotone in ℓ_P. Furthermore, we substitute the definition of T and C to obtain

$$\begin{aligned}
&\mathbb{E}\left[\sum_{P,Q} V_{PQ}\right] \\
\leq\ & -\frac{\lambda}{d} L_{\text{av}}^+ \sum_{P\in\mathcal{P}_{\varepsilon,\nu}^+} x_P \left(\frac{T\,(1+\varepsilon)L_{\text{av}}^+ - \sum_{Q:\ell^+\leq(1+\varepsilon)L_{\text{av}}^+} \frac{x_Q\,\ell_Q^+}{n}}{(1+\varepsilon)L_{\text{av}}^+}\right)^2 \cdot \frac{1}{T} \\
\leq\ & -\frac{\lambda}{d} L_{\text{av}}^+ \sum_{P\in\mathcal{P}_{\varepsilon,\nu}^+} x_P \left(\frac{T\,(1+\varepsilon)L_{\text{av}}^+ - T\,C}{(1+\varepsilon)L_{\text{av}}^+}\right)^2 \cdot \frac{1}{T} \\
=\ & -\frac{\lambda}{d} L_{\text{av}}^+ \cdot \left(\frac{(1+\varepsilon)L_{\text{av}}^+ - C}{(1+\varepsilon)L_{\text{av}}^+}\right)^2 \cdot T \cdot \sum_{P\in\mathcal{P}_{\varepsilon,\nu}^+} x_P \ .
\end{aligned}$$

We can now use the tradeoff shown in Equation (4.7), $C \leq L_{\text{av}}^+$, and

$$\sum_{P\in\mathcal{P}_{\varepsilon,\nu}^+} x_P > \delta\, n/2$$

to obtain

$$\begin{aligned}
\mathbb{E}\left[\sum_{P,Q} V_{PQ}\right] \quad &\leq \quad -\frac{\lambda}{d} \cdot L_{\text{av}}^+ \cdot \frac{(1+\varepsilon)L_{\text{av}}^+ - C}{((1+\varepsilon)L_{\text{av}}^+)^2} \cdot \varepsilon\, L_{\text{av}}^+ \cdot \sum_{P\in\mathcal{P}_{\varepsilon,\nu}^+} x_P \\
&\leq \quad -\frac{\lambda}{d} \cdot \varepsilon \cdot \frac{\varepsilon\, L_{\text{av}}^+}{(1+\varepsilon)^2} \cdot \frac{\delta\, n}{2} \\
&\leq \quad -\Omega\left(\frac{\varepsilon^2 \cdot \delta}{d} \cdot n\, L_{\text{av}}^+\right) \ .
\end{aligned}$$

Since $nL_{\text{av}}^+ \geq \Phi$, we have by Lemma 4.10

$$\mathbb{E}\left[\Phi(x(t+1))\right] \leq \Phi(x(t)) - \frac{1}{2}\mathbb{E}\left[\sum_{P,Q} V_{PQ}\right] \leq \Phi(x(t))\left(1 - \Omega\left(\frac{\varepsilon^2 \cdot \delta}{d}\right)\right) \ .$$

Case 2: Many players use cheap paths, i. e., $\sum_{P\in\mathcal{P}_{\varepsilon,\nu}^-} x_P \geq \delta\, n/2$. This time, we define the volume T and average latency C of paths which are potential origins of players migrating towards $\mathcal{P}_{\varepsilon,\nu}^-$.

$$T = \sum_{Q:\ell_Q\geq(1-\varepsilon)L_{\text{av}}} \frac{x_Q}{n} \quad \text{and} \quad C = \frac{1}{T} \sum_{Q:\ell_Q\geq(1-\varepsilon)L_{\text{av}}} \frac{x_Q}{n}\ell_Q \ .$$

This time,

$$L_{\text{av}} \le T \cdot C + (1-T) \cdot (1-\varepsilon)\, L_{\text{av}}$$

implying

$$T \ge \frac{\varepsilon\, L_{\text{av}}}{C - (1-\varepsilon)\, L_{\text{av}}} \quad . \tag{4.8}$$

Similar to Case 1 we now give a lower bound on the contribution to the virtual potential gain caused by players with latency at least $(1-\varepsilon)L_{\text{av}}$ sampling players in $\mathcal{P}^-_{\varepsilon,\nu}$.

$$\mathbb{E}\left[\sum_{P,Q} V_{PQ}\right] \;\le\; -\frac{\lambda}{d} \sum_{Q:\ell_Q \ge (1-\varepsilon)L_{\text{av}}} x_Q\, \ell_Q \sum_{P \in \mathcal{P}^-_{\varepsilon,\nu}} \frac{x_P}{n} \cdot \left(\frac{\ell_Q - \ell_P^+}{\ell_Q}\right)^2 \quad .$$

we rearrange the sum, apply Jensen's inequality (Fact 4.6) to obtain

$$\begin{aligned}
\mathbb{E}\left[\sum_{P,Q} V_{PQ}\right] &\le -\frac{\lambda}{d} \sum_{P \in \mathcal{P}^-_{\varepsilon,\nu}} x_P \sum_{Q:\ell_Q \ge (1-\varepsilon)L_{\text{av}}} \frac{x_Q\, \ell_Q}{n} \cdot \left(\frac{\ell_Q - \ell_P^+}{\ell_Q}\right)^2 \\
&\le \frac{-\frac{\lambda}{d} \sum_{P \in \mathcal{P}^-_{\varepsilon,\nu}} x_P \left(\sum_{Q:\ell_Q \ge (1-\varepsilon)L_{\text{av}}} \frac{x_Q\, \ell_Q}{n} \cdot \frac{\ell_Q - \ell_P^+}{\ell_Q}\right)^2}{\sum_{Q:\ell_Q \ge (1-\varepsilon)L_{\text{av}}} \frac{x_Q\, \ell_Q}{n}} \\
&= -\frac{\lambda}{d} \sum_{P \in \mathcal{P}^-_{\varepsilon,\nu}} x_P \left(\sum_{Q:\ell_Q \ge (1-\varepsilon)L_{\text{av}}} \frac{x_Q}{n} \cdot (\ell_Q - \ell_P^+)\right)^2 \cdot \frac{1}{C\,T} \\
&= -\frac{\lambda}{d} \sum_{P \in \mathcal{P}^-_{\varepsilon,\nu}} x_P \left(T \cdot (C - \ell_P^+)\right)^2 \cdot \frac{1}{C\,T} \\
&\le -\frac{\lambda}{d} \left(T \cdot (C - (1-\varepsilon)\, L_{\text{av}})\right)^2 \cdot \frac{1}{C\,T} \cdot \sum_{P \in \mathcal{P}^-_{\varepsilon,\nu}} x_P \quad .
\end{aligned}$$

Finally, using Equation (4.8) and $C\,T \le L_{\text{av}}$,

$$\begin{aligned}
\mathbb{E}\left[\sum_{P,Q} V_{PQ}\right] &\le -\frac{\lambda}{d} (\varepsilon\, L_{\text{av}})^2 \cdot \frac{1}{C\,T} \cdot \sum_{P \in \mathcal{P}^-_{\varepsilon,\nu}} x_P \\
&\le -\frac{\lambda\, \varepsilon^2\, L_{\text{av}}}{d} \delta n \\
&\le -\Omega\left(\frac{\delta\, \varepsilon^2\, \Phi}{d}\right) \quad .
\end{aligned}$$

In both cases, the potential decreases by at least a factor of $(1 - \Omega(\varepsilon^2\, \delta/d))$ in expectation, which, by Lemma 4.7, implies that the expected time to reach a state with $\Phi(x(t)) \le \Phi^*$ is at most the time stated in the theorem. □

From Theorem 4.16 we can immediately derive the next corollary.

Corollary 4.17. *Consider a symmetric network congestion game with polynomial latency functions of maximum degree d and with minimum and maximum coefficients* $a_{\max}$ *and* $a_{\min}$*, respectively. If all players use the* Imitation Protocol*, then the expected convergence time of imitation dynamics to an* (δ,ε,ν)*-equilibrium is upper bounded by*

$$O\left(\frac{d^2}{\varepsilon^2\,\delta}\cdot\log\left(n\,m\frac{a_{\max}}{a_{\min}}\right)\right) \quad .$$

Let us remark, that (δ,ε,ν)-equilibria are transient, i. e., they can be left again once they are reached, for example, if the average latency decreases or if players migrate towards low-latency paths. However, our proofs actually do not only bound the time until a (δ,ε,ν)-equilibrium is reached for the first time, but rather the expected total number of rounds in which the system is not at a (δ,ε,ν)-equilibrium.

Note that in the definition of (δ,ε,ν)-equilibria we require the majority of players to deviate by no more than a small amount from L_{av}^{+}. This is because the expected latency of a path sampled by a player is L_{av}, but the latency of the destination path becomes larger if the player migrates. We use L_{av}^{+} as an upper bound in our proof, although we could use a slightly smaller quantity in cases where the origin Q and the destination P intersect, namely $\ell_P(x + 1_P - 1_Q)$. Using an average over P and Q of this quantity rather than L_{av}^{+} would result in a slightly stronger definition of (δ,ε,ν)-equilibria. However, we go with the definition as presented above for the sake of clarity of presentation.

Let us conclude this section by showing that there are fundamental limitations to fast convergence. One could hope to show fast convergence towards a state in which *all* players are approximately satisfied, i. e., $\delta = 0$. However, any protocol that proceeds by sampling either a strategy or a player and then possibly migrates, takes at least expected time $\Omega(n)$ to reach a state in which all players sustain a latency that is within a constant factor of L_{av}^{+}. To see this, consider an instance with $n = 2\,m$ players and identical linear latency functions. Now let $x_1 = 3$, $x_2 = 1$ and $x_i = 2$ for $3 \leq i \leq n$. Then the probability that one of the players currently using resource 1 samples resource 2 is at most $O(1/m) = O(1/n)$. Since this is the only possible improvement step, this yields the desired bound.

4.3 Imitation Dynamics in Singleton Games

In this section, we improve on the results presented in the previous section and consider imitation dynamics in the special case of singleton congestion games. A major drawback of the Imitation Protocol is that players which rely on this protocol cannot explore the complete set of edges if the dynamics starts in a state in which some edges are unused. Even worse, the event that an edge becomes unused in later states, although it has been used in the initial state, is not impossible. It is clear, however, that when starting from a random initial distribution of players among the edges, the probability of emptying an edge becomes increasingly unlikely *as the number of players increases.* In the first part of this section, we formalize this intuitive observation and show that under some mild assumptions the probability of loosing an edge is exponentially small.

Furthermore, we consider the loss in the global performance due to the selfish behavior of the players. To this end, we introduce the *Price of Imitation* which is defined as the ratio between the expected social cost of the state to which the IMITATION PROTOCOL converges and the optimum social cost.

4.3.1 The Unlikely Event of Loosing an Edge

In this section, we formalize the intuitive observation that the probability of emptying an edge in a singleton congestion game becomes increasingly unlikely as the number of players increases.

Consider a family of singleton congestion games over the *same* set of edges with latency functions without offsets. Then the probability that an edge becomes unused is exponentially small in the number of players. To this end, consider a vector of continuous latency functions $\mathcal{L} = (\ell_e)_{i\in[m]}$ with $\ell_e : [0,1] \to \mathbb{R}_{\geq 0}$. To use these functions for games with a finite number of players, we have to normalize them appropriately. For any such function $\ell \in \mathcal{L}$, let ℓ^n with $\ell^n(x) = \ell(x/n)$ denote the respective scaled function. We may think of this as having n players with weight $1/n$ each. Note that this transformation leaves the elasticity unchanged, whereas the step size ν decreases as n increases. For a vector of latency functions $\mathcal{L} = (\ell_e)_{i\in[m]}$, let $\mathcal{L}^n = (\ell_e^n)_{i\in[m]}$.

Theorem 4.18. *Fix a vector of latency functions $\mathcal{L}$ with $\ell_e(0) = 0$ for all $i \in [m]$. For the singleton congestion game over $\mathcal{L}^n$ with n players, the probability that the* IMITATION PROTOCOL *with random initialization generates a state with $x_e = 0$ for some $i \in [m]$ within* $\text{poly}(n)$ *rounds is bounded by $2^{-\Omega(n)}$.*

Proof. Let d denote an upper bound on the elasticity of the functions in $\mathcal{L}$, and let $\text{opt}_{\mathcal{L}} = \min_y\{L_{\text{av}}(y)\}$ where the minimum is taken over all $y \in \{y' \in \mathbb{R}^m_{\geq 0} \mid \sum_e y'_e = 1\}$. In other words, $\text{opt}_{\mathcal{L}}$ corresponds to the minimum average latency achievable in a fractional solution. For any $e \in [m]$, by continuity and monotonicity, there exists an $y_e > 0$ such that $\ell_e(y_e) < \text{opt}_{\mathcal{L}} / 4^d$ and $y_e < 1/m$.

Consider the congestion game with n players and fix an arbitrary edge $e \in [m]$. In the following, we upper bound the probability that the congestion on edge e falls below $n\, y_e/2$. At first consider the random initialization in which each resource receives an expected number of n/m players. The probability that $x_e < n\, y_e/2 \leq n/(2\, m)$ is at most $2^{-\Omega(n\, y_e)}$. Now, consider any assignment x with $x_j > n\, y_j/2$ for all $e \in [m]$. There are two cases.

Case 1: $x_e > y_e\, n$**.** Since in expectation our policy removes at most a λ/d fraction of the players from edge e, the expected load in the subsequent round is at least $(1 - \lambda/d)\, x_e$. Since for sufficiently small λ it holds that $1 - \lambda/d \geq 3/4$, we can apply a Chernoff bound (Fact 4.3) in order to obtain an upper bound of $2^{-\Omega(x_e)}$ for the probability that the congestion on e decreases to below $x_e/2 \geq y_e\, n/2$.

Case 2: $y_e n/2 < x_e \le y_e n$. Hence, $\ell_e^n(x_e) \le \mathrm{opt}_{\mathcal{L}}/4^d$. In the following, let n^- denote the number of players on edges r with $\ell_r^n(x_r+1) < \ell_e^n(x_e)$, and let n^+ denote the number of players utilizing edges with latency above $\mathrm{opt}_{\mathcal{L}}$. There are two subcases:

Case 2a: $n^- = 0$. Then the probability that an player leaves edge e is 0.

Case 2b: $n^- \ge 1$. We first show that $n^+ \ge 4\max\{n^-, x_e\}$. For the sake of contradiction, assume that $n^+ < 4\,n^-$. Now consider an assignment where all of these players are shifted to edges r with latency $\ell_r^n(x_r) < \ell_e^n(x_e) \le \mathrm{opt}_{\mathcal{L}}/4^d$, where edge r receives $n^+ \cdot x_r/n^-$ (fractional) players. In this assignment, the congestion on all edges is increased by no more than a factor of $n^+/n^- < 4$. Hence, due to the limited elasticity, this increases the latency by strictly less than a factor of 4^d. Then all edges have a latency of less than $\mathrm{opt}_{\mathcal{L}}/4 \cdot 4 = \mathrm{opt}_L$ and some have latency strictly less than opt_L, a contradiction. The same argument also holds if we consider only resource e rather than all resources r considered above. Hence, also $n^+ \ge 4\,x_e$.

Now consider the number of players leaving edge e. Clearly,

$$\mathbb{E}\left[\Delta X_e^-\right] \le x_e \cdot \frac{\lambda}{d} \sum_{r:\ell_r^n(x_r+1)<\ell_e^n(x_e)} \frac{x_r}{n} = x_e \cdot \frac{\lambda\, n^-}{d\, n} \quad .$$

All players with current latency at least $\mathrm{opt}_{\mathcal{L}}$ can migrate to resource e since the anticipated latency gain is larger than ν. Hence, the number of players migrating towards e, is at least

$$\begin{aligned}
\mathbb{E}\left[\Delta X_e^+\right] &\ge \sum_{r:\ell_r^n(x_r)\ge \mathrm{opt}_{\mathcal{L}}} x_r \cdot \frac{\lambda\, x_e \cdot (\ell_r^n(x_r) - \ell_e^n(x_e+1))}{n\, d\, \ell_r^n(x_r)} \\
&\ge \frac{\lambda\, x_e}{n\, d} \cdot \sum_{r:\ell_r^n(x_r)\ge \mathrm{opt}_{\mathcal{L}}} x_r \cdot \frac{\ell_r^n(x_r) - 2^d \cdot \ell_e^n(x_e)}{\ell_r^n(x_r)} \\
&\ge \frac{\lambda\, x_e}{n\, d} \cdot \left(1 - \frac{1}{2^d}\right) \cdot n^+ \\
&\ge 2 \cdot x_e \cdot \frac{\lambda}{d\, n} \max\{n^-, x_e\} \quad .
\end{aligned}$$

The third inequality holds since $\ell_r^n \ge \mathrm{opt}_{\mathcal{L}}$ and $\ell_e^n \le \mathrm{opt}_{\mathcal{L}}/4^d$ and the last inequality holds since $d \ge 1$. For any $T \ge 0$ it holds that

$$\begin{aligned}
\mathbb{P}\left[\Delta X_e \ge 0\right] &\ge \mathbb{P}\left[(\Delta X_e^+ \ge T) \wedge (\Delta X_e^- \le T)\right] \\
&\ge \left(1 - \mathbb{P}\left[\Delta X_e^+ < T\right]\right) \cdot \left(1 - \mathbb{P}\left[\Delta X_e^- > T\right]\right) \quad .
\end{aligned}$$

Due to our lower bounds on $\mathbb{E}[\Delta X_e^+]$ and $\mathbb{E}[\Delta X_e^-]$ we can apply a Chernoff bound (Fact 4.3) on these probabilities. We set

$$T = 1.5\,\lambda\, \max\{x_e, n^-\}\, x_e/(d\, n)$$

which is an upper bound on $\mathbb{E}[\Delta X_e^-]$ and a lower bound on $\mathbb{E}[\Delta X_e^+]$, so

$$\begin{aligned}\mathbb{P}\left[\Delta X_e^+ < T\right] &\leq 2^{-\Omega(T)} \leq 2^{-\Omega(\lambda\, x_e^2/(d\, n))} \quad \text{and} \\ \mathbb{P}\left[\Delta X_e^- > T\right] &\leq 2^{-\Omega(T)} \leq 2^{-\Omega(\lambda\, x_e^2/(d\, n))} \ .\end{aligned}$$

Altogether,

$$\begin{aligned}\mathbb{P}\left[\Delta X_e \geq 0\right] &\geq \left(1 - 2^{-\Omega\left(\frac{\lambda\, x_e^2}{d\, n}\right)}\right) \cdot \left(1 - 2^{-\Omega\left(\frac{\lambda\, x_e^2}{d\, n}\right)}\right) \\ &= 1 - 2^{-\Omega\left(\frac{\lambda\, x_e^2}{d\, n}\right)} \ .\end{aligned}$$

Finally, since $x_e \geq n\, y_e/2$, $\mathbb{P}\left[\Delta X_e < 0\right] \leq 2^{-\Omega(\lambda\, n\, y_e^2/d)} = 2^{-\Omega(x_e)}$.

In all cases, the probability that the edge becomes unused is bounded by $2^{-\Omega(x_e)} = 2^{-\Omega(n)}$. Hence, the same holds also for $m = \text{poly}(n)$ edges and $\text{poly}(n)$ rounds. □

The proof does not only show that edges do not become empty with high probability, but also that the congestion does not fall below any constant congestion value. In particular, for the constant d this implies that with high probability the dynamics never reach case 2b of the proof of Lemma 4.10. This is the only place where our analysis relies on the parameter ν. Hence, for a large number of players we can remove it from the protocol and the dynamics converge to an exact Nash equilibrium.

4.3.2 The Price of Imitation

In the preceding section we have seen that it is unlikely that edges become unused when the granularity of the players decreases. If the instance, i. e., the latency functions and the number of users, is fixed, it is an interesting question, how much the performance can suffer from the fact that the Imitation Protocol is not innovative. We measure this degradation of performance by introducing the *Price of Imitation* which is defined as the ratio between the expected social cost of the state to which the Imitation Protocol converges, denoted I_Γ, and the optimum social cost. The expectation is taken with respect to the random choices of the Imitation Protocol, including random initialization.

We answer this question here for the case of linear latency functions of the form $\ell_e(x) = a_e\, x$. Then $d = 1$ is an upper bound on the elasticity and $\nu = a_{\max} = \max_{e \in E}\{a_e\}$. Choosing the average latency $SC(x) = \sum_{e \in E}(x_e/n) \cdot \ell_e(x_e)$ as the social cost measure, we show that the Price of Imitation is bounded by a constant. It is, however, obvious that the same also holds if we consider the makespan, i. e., the maximum latency, as social cost function.

The performance of the dynamics can be artificially degraded by introducing an extremely slow edge. Thus, $a_{\max}$ can be chosen extremely large such that any state is imitation-stable. However, such an edge can be removed from the instance without harming the optimal solution at all since it would not be used anyhow. We will call such edges *useless* and make this notion precise below.

Let us first define some quantities used in the proof. For a set of edges $\mathcal{R}' \subset \mathcal{R}$, let $A_{\mathcal{R}'} = \sum_{e \in \mathcal{R}'} \frac{1}{a_e}$ and let $A_\Gamma = A_\mathcal{R}$. For $\mathcal{R}' \subseteq \mathcal{R}$ let $\Gamma \setminus \mathcal{R}'$ denote the instance obtained from Γ by removing all edges in $\mathcal{R}'$. In the proof, we do not compare the outcome of the IMITATION PROTOCOL to the optimum solution, but rather to a lower bound, namely the optimal fractional solution. The optimal fractional solution $\tilde{x}_e$ can be computed as $\tilde{x}_e = n/(A_\Gamma \, a_e)$. For this solution, the latency of all edges is $a_e \cdot \tilde{x}_e = n/A_\Gamma$. An edge is *useless* if $\tilde{x}_e < 1$. In the following, we assume that there are no useless edges. Then we can show that the social cost of an imitation-stable state in which all edges are used does not differ by more than a small constant from the optimal social cost (Lemma 4.20) and that the Price of Imitation is small. In fact, whereas $\tilde{x}_e \geq 1$ is required for Lemma 4.20, we here need a slightly stronger assumption, namely that $\tilde{x}_e = \Omega(\log n)$.

Theorem 4.19. *Assume that for the optimal fractional solution, $\tilde{x}_e = \Omega(\log n)$ is large enough. The price of imitation is at most $(3+o(1))$. In particular, for $\delta > 0$, and any $n \geq n_0(\delta)$ for a large enough value $n_0(\delta)$ (which is independent of the instance),*

$$I_\Gamma \leq (3+\delta) \cdot \frac{n}{A_\Gamma} \quad .$$

We start by proving two lemmas.

Lemma 4.20. *Let x be a state in which no player can gain more than $a_{\max}$. Then*

$$\frac{n}{A_\Gamma} \leq SC(x) \leq 3 \frac{n}{A_\Gamma} \quad .$$

Proof. The lower bound has been proven above since n/A_Γ is the social cost of an optimal fractional solution. Also note that since there are no useless edges, $\tilde{x}_e \geq 1$ and hence $n/A_\Gamma \geq a_{\max}$.

For the upper bound, consider a state x in which no player can gain more than $a_{\max}$. For the sake of contradiction assume that there exists an edge $e \in [m]$ with $\ell_e(x_e) > 3\,n/A_\Gamma$. Since $x \neq \tilde{x}$ there exists a edge $f \neq e$ with $x_f < \tilde{x}_f$. In particular, $\ell_f(x_f+1) < n/A_\Gamma + a_{\max} \leq 2\,n/A_\Gamma \leq \ell_e(x_e) - a_{\max}$. The last inequality holds due to our assumption on $\ell_e(x_e)$ and since $n/A_\Gamma \geq a_{\max}$. Hence, any player on edge e can improve by $a_{\max}$ by migrating to f, a contradiction. □

The next lemma follows easily from Corollary 4.14.

Lemma 4.21. *The* IMITATION PROTOCOL *converges towards an imitation-stable state in time $O\left(n^4\right)$.*

Based upon the proof of Theorem 4.18 we can now bound the probability that a edge becomes empty for the case of linear latency functions more specifically.

Lemma 4.22. *The probability that all edges of the subset $\mathcal{R}' \subseteq \mathcal{R}$ become empty simultaneously in one round is bounded from above by*

$$\prod_{e \in \mathcal{R}} 2^{-\Omega\left(\frac{n}{A_\Gamma a_e}\right)} \quad .$$

Proof. Recall the bounds on the probability that an edge $e \in \mathcal{R}$ becomes empty in the proof of Theorem 4.18. Since we now consider linear latency functions, we may explicitly compute the value of $y_e = 1/(A_\Gamma\, a_e)$. Recall the two cases and the failure probability in the initialization:

Initialization: Here the error probability was at most

$$2^{-\Omega(n\, y_e)} = 2^{-\Omega(n/(A_\Gamma\, a_e)} \quad .$$

Case 1: $x_e > y_e\, n$**.** Here the error probability was at most

$$2^{-\Omega(x_e)} = 2^{-\Omega(n/(A_\Gamma\, a_e))} \quad .$$

Case 2: $y_e\, n/2 < x_e \leq y_e\, n$**.** Here the error probability was at most

$$2^{-\Omega(x_e^2/n)} = 2^{-\Omega(n/(A_\Gamma\, a_e)^2)} \quad .$$

In all cases, the probability that edge i becomes empty is at most $2^{-\Omega\left(\frac{n}{A_\Gamma a_e}\right)}$.

Furthermore, consider edges e and e' and let E and E' denote the events that e and e' become empty, respectively. It holds that $\mathbb{P}[E' \mid E] \leq \mathbb{P}[E']$. Therefore, $\mathbb{P}[E \cap E'] = \mathbb{P}[E] \cdot \mathbb{P}[E' \mid E] \leq \mathbb{P}[E] \cdot \mathbb{P}[E']$. Extending this argument to several edges yields the statement of the lemma. □

Using the above two lemmas, we can now prove the main theorem of this section.

Proof of Theorem 4.19. The proof is by induction on the number of edges m. Clearly, the statement holds for $m = 1$, in which case there is only one assignment. In the following we divide the sequence of state generated by the Imitation Protocol into *phases* consisting of several rounds. The phase is terminated by one of the following events, whatever happens first:

1. A subset of edges $\mathcal{R}'$ becomes empty.
2. The Imitation Protocol reaches an imitation-stable state.
3. The protocol enters round $\Theta(n^5)$.

If a phase ends because Event 1 occurs, we start a new phase for the instance $\Gamma \setminus \mathcal{R}'$. If it ends because of Event 3, we start a new phase for the original instance.

The probability for Event 1 is bounded by Lemma 4.22. Note that the probability is also bounded for up to poly(n) many rounds. If a phase ends with Event 2 we have $I_\Gamma \le 3\,\frac{n}{A_\Gamma}$ (Lemma 4.20). We bound the probability of this event by 1, which is trivially true. Event 3 happens with a probability at most $O(1/n)$. This can be shown using Lemma 4.21 and Markov's inequality. Note that the expected social cost is still at most I_Γ. Summing up over all three events, we obtain the following recurrence:

$$I_\Gamma \le \sum_{\mathcal{R}'\subset\mathcal{R}} \prod_{e\in\mathcal{R}'} 2^{-\Omega(n/(A_\Gamma\, a_e))} \cdot I_{\Gamma\setminus\mathcal{R}} + 3\cdot\frac{n}{A_\Gamma} + O\left(\frac{1}{n}\right)\cdot I_\Gamma$$

implying

$$I_\Gamma \cdot \left(1 - O\left(\frac{1}{n}\right)\right) \le 3\cdot\frac{n}{A_\Gamma} + \sum_{\mathcal{R}'\subset\mathcal{R}} \prod_{e\in\mathcal{R}'} 2^{-\Omega(\frac{n}{A_\Gamma\, a_e})} \cdot I_{\Gamma\setminus\mathcal{R}'} \ .$$

Substituting the induction hypothesis for $I_{\Gamma\setminus\mathcal{R}'}$, and introducing a constant c for the constant in the $\Omega()$,

$$\begin{aligned} I_\Gamma \cdot \left(1 - O\left(\frac{1}{n}\right)\right) &\le 3\cdot\frac{n}{A_\Gamma} + \sum_{\mathcal{R}'\subset\mathcal{R}} \prod_{e\in\mathcal{R}'} 2^{\frac{-c\,n}{(A_\Gamma\, a_e)}} \cdot 4\,\frac{n}{A_{\Gamma\setminus\mathcal{R}'}} \\ &= 3\cdot\frac{n}{A_\Gamma} + 4\,\frac{n}{A_\Gamma}\sum_{\mathcal{R}'\subset\mathcal{R}} 2^{\frac{-c\,n\,A_{\mathcal{R}'}}{A_\Gamma}} \cdot \frac{A_\Gamma}{A_{\Gamma\setminus\mathcal{R}'}} \ . \end{aligned}$$

Now, by our assumption that for all $e \in \mathcal{R}'$, $\tilde{x}_e = n/(A_\Gamma \cdot a_e) \ge \Omega(\log n)$, we know that for all e, $1/a_e \ge c'\,A_\Gamma \cdot \log n/n$ for a constant c' which we may choose appropriately. In particular, $A_{\mathcal{R}'} \ge |\mathcal{R}'|c'\,A_\Gamma \cdot \log n/n$ and $A_{\Gamma\setminus\mathcal{R}'} \ge c'\,A_\Gamma \cdot \log n/n$. Altogether,

$$\begin{aligned} I_\Gamma \cdot \left(1 - O\left(\frac{1}{n}\right)\right) &\le \frac{n}{A_\Gamma}\left(3 + 4\sum_{\mathcal{R}'\subset\mathcal{R}} 2^{-c\,c'\,|\mathcal{R}'|\log n} \cdot \frac{n}{c'\,\log n}\right) \\ &= \frac{n}{A_\Gamma}\left(3 + 4\sum_{k=1}^{m-1}\binom{m}{k} 2^{-c\,c'\,k\log n} \cdot \frac{n}{c'\,\log n}\right) \\ &\le \frac{n}{A_\Gamma}\left(3 + 4\sum_{k=1}^{m-1} n^k \cdot 2^{-c\,c'\,k\log n} \cdot \frac{n}{c'\,\log n}\right) \\ &\le \frac{n}{A_\Gamma}\left(3 + 4\sum_{k=1}^{m-1} 2^{-(c\,c'-1)\,k\log n} \cdot \frac{n}{c'\,\log n}\right) \\ &\le \frac{n}{A_\Gamma}\left(3 + 4\sum_{k=1}^{m-1} \frac{n^{-(c\,c'-1)\,k+1}}{c'\,\log n}\right) \\ &\le (3 + o(1))\,\frac{n}{A_\Gamma} \ , \end{aligned}$$

since the last sum is bounded by $o(n)$. This implies our claim. □

4.4 A Note on Sequential Imitation Dynamics

In Section 4.2.1, we proved that players concurrently applying the Imitation Protocol reach an imitation-stable state after a pseudo-polynomial number of rounds.

Recall that in this case each player decreases its latency by at least ν if it were the only player to change its strategy. In this section, we consider sequential imitation dynamics such that in each round a single player is permitted to imitate someone else. Furthermore, we assume that each player changes its path regardless of the anticipated latency gain. Now it is obvious that sequential imitation dynamics converge towards imitation-stable states as the potential Φ strictly decreases after every strategy change. Hence, we focus on the convergence time of such dynamics.

For such sequential imitation dynamics we prove an exponential lower bound on the number of rounds to reach an imitation-stable state. To be precise, we present a family of symmetric network congestion games with corresponding initial states such that every sequence of imitation leading to an imitation-stable state is exponentially long. To some extent this result complements Theorem 4.12 as it presents an exponential lower bound in a slightly different model. However, in this lower bound ν is arbitrary large and almost every state is imitation-state with respect to the Imitation Protocol.

Theorem 4.23. *For every $n \in \mathbb{N}$, there exists a symmetric network congestion game with n players, initial state S^{init}, polynomial bounded network size, and linear latency functions such that* every *sequential sequence of imitation that start in S^{init} is exponentially long.*

We do not give a complete proof of the theorem but we discuss how to adapt a series of constructions as presented in [ARV08] which shows that there exists a family of symmetric network congestion games with the same properties as stated in the above theorem such that *every best response dynamics* starting in S^{init} is exponentially long. To be precise, they prove that in every intermediate state of the best response dynamics *exactly* one player can play a best response. Recall that in best response dynamics players know the entire strategy space and that in each round one player is permitted the switch to the best available path.

In the following, we summarize the constructions presented in [ARV08]. At first, a PLS-reduction from the local search variant of MaxCut to threshold games is presented. In a threshold game, each player either allocates a single resource on its own or shares a bunch of resources with other players. Hence, in a threshold game each player chooses between two strategies only. The precise definition of these games is given below. Then a PLS reduction from threshold games to asymmetric network congestion games is presented. Finally, the authors of [ARV08] show how to transform an asymmetric network congestion game into a symmetric one such that the desired properties of best response dynamics are preserved. All PLS reductions are embedding, and there exists a family of instances of MaxCut with corresponding initial configurations such that in every intermediate configuration generated by a local search algorithm exactly one node can be moved to the other side of the cut. Therefore, there exists a family of symmetric network congestion games with the properties as stated above.

A naive approach to prove a lower bound on the convergence time of imitation dynamics in symmetric network congestion games is as follows. Building upon the lower bound of the convergence time of best responses dynamics, a player for every path is added to the game. Then the latency functions are adopted accordingly. However, in this case we would introduce an exponential number of additional players. In threshold games, however, the players' strategy spaces have size two only. Hence, we could apply this approach to threshold games. In the following, we present the details of this approach. It is then not difficult to verify that the PLS reductions mentioned above can be reworked in order to prove Theorem 4.23. However, note that this does not imply that computing an imitation-stable state is PLS-complete since one can always assign all players to the same strategy which obviously is an imitation-stable state.

Threshold games are a special class of congestion games in which the set of resources $\mathcal{R}$ can be divided into two disjoint sets $\mathcal{R}_{\text{in}}$ and $\mathcal{R}_{\text{out}}$. The set $\mathcal{R}_{\text{out}}$ contains exactly one resource r_i for every player $i \in \mathcal{N}$. This resource has a fixed latency T_i called the *threshold* of player i. Each player i has only two strategies, namely a strategy $S_i^{\text{out}} = \{r_i\}$ with $r_i \in \mathcal{R}_{\text{out}}$, and a strategy $S_i^{\text{in}} \subseteq \mathcal{R}_{\text{in}}$. The preferences of player i can be described in a simple and intuitive way: Player i prefers strategy S_i^{in} to strategy S_i^{out} if the latency of S_i^{in} is smaller than the threshold T_i. *Quadratic threshold games* are a subclass of threshold games in which the set $\mathcal{R}_{\text{in}}$ contains exactly one resource r_{ij} for every unordered pair of players $\{i, j\} \subseteq \mathcal{N}$. Additionally, for every player $i \in \mathcal{N}$ of a quadratic threshold game, $S_i^{\text{in}} = \{r_{ij} \mid j \in \mathcal{N}, j \neq i\}$. Moreover, for every resource $r_{ij} \in \mathcal{R}_{\text{in}}$: $\ell_{r_{ij}}(x) = a_{i,j} \cdot x$ with $a_{ij} \in \mathbb{N}$, and for every resource r_i: $\ell_{r_i}(x) = 1/2 \sum_{j \neq i} a_{ij} \cdot x$ to r_i.

Let Γ be a quadratic threshold game that has an initial state S^{init}, such that every best response dynamics which start in this state is exponentially long, and every intermediate state has a unique player which can improve its latency. Suppose now that we replace every player i in Γ by three players i_1, i_2 and i_3 which all have the same strategy spaces as player i has. Additionally, suppose that we choose new latency functions ℓ' for every resource r_i as follows: $\ell'_{r_i}(x) = 1/2 \sum_{j \neq i} a_{ij} \cdot x + 3/2 \sum_{j \neq i} a_{ij}$. Hence, we add an additional offset of $3/2 \sum_{j \neq i} a_{ij}$.

Suppose now that we assign every player i_1 to S_i^{out}, and every player i_2 to S_i^{in}. For every possible strategy that the i_3 players can use, their latency increases by $2 \sum_{j \neq i} a_{ij}$, compared to the equivalent state in the original game, in which every player i chooses the same strategy as player i_3 does. Hence, if we assign every player i_3 to the strategy chosen by player i in S^{init} and if the players i_1 and i_2 were not permitted to change their strategies, then we would obtain the desired lower bound on the convergence time of imitation dynamics in threshold games. However, since also i_1 and i_2 are permitted to imitate, it remains to show that whenever player i_3 has changed its strategy, then both i_1 and i_2 do not want to change their strategies anymore.

First, suppose that player i_3 switches from the strategy of player i_2 to the strategy of player i_1. Obviously, player i_1 does not want to change its strategy as otherwise i_3 would not have imitated i_1. Suppose now that i_2, whose strategy is dropped by i_3, also wants to imitate i_1. In this case, all three players would allocate S_i^{out}, and hence have latency $3 \sum_{r \in j \neq i} a_{ij}$. However, if player i_2 would stay with strategy S^{in} then its latency is upper bounded by $2 \sum_{r \in S_i^{\text{in}}} a_{ij}$. Hence, players i_1, i_2, i_3 will never select S^{out} at the same time.

Second, suppose that player i_3 switches from the strategy of player i_1 to the strategy of player i_2. Now, player i_2 does not want to change its strategy as otherwise i_3 would not have imitated i_2. Suppose now that i_1, whose strategy is dropped by i_3, also wants to imitate i_3. In this case, their latency would increase to at least $3\sum_{r\in j\neq i} a_{ij}$, whereas player i_1 would have latency $2\sum_{r\in j\neq i} a_{ij}$ if it would stay with strategy S^{out}. Hence, players i_1, i_2, i_3 will never select S^{in} at the same time.

Hence, by applying the above observations that the three players never allocate the same strategy at the same point in time we can conclude our claim.

4.5 Extensions Guaranteeing Convergence to Nash Equilibria

In Section 4.2.1 we have seen that, in the long run, the dynamics resulting from the Imitation Protocol converges to an imitation-stable state in pseudo-polynomial time. The Imitation Protocol and the concept of an imitation-stable state have the drawback that the dynamics can stabilize in quite a disadvantageous situation, e.g. when all players play the same expensive strategy. This is due to the fact that the strategy space is essentially restricted to the current strategy choices of the players. Strategies that might be attractive and offer a large latency gain are "lost" once no player uses them anymore.

A stronger result would be convergence towards a Nash equilibrium. In the literature, several other protocols are discussed. For all of the protocols we are aware of, the probability to migrate from one strategy to another depends in some continuous, non-decreasing fashion on the anticipated latency gain, and it becomes zero for zero gain. Hence, in a setting with arbitrary latency functions which we consider here there always exist simple instances and states that are not at equilibrium and in which only one improvement step is possible which has an arbitrarily small latency gain. Thus, it takes pseudo-polynomially long, until an exact Nash equilibrium is reached. Still, it might be desirable to design a protocol which reaches a Nash equilibrium in the long run. There are several ways to achieve this goal. We will discuss three of them here.

Theorem 4.18 states the following for a particular class of singleton congestion games. With an increasing number of players it becomes increasingly unlikely that useful strategies are lost. This allows to omit the parameter ν from the protocol. If no strategies are lost for a long period of time, the dynamics will converge towards an exact Nash equilibrium.

Secondly, we may add an additional "virtual player" to every strategy, such that the probability to sample a strategy never becomes zero. This has two implications on our analysis. On the one hand, there is a certain base load on all resources, denoted by x_e^0. We then need to have an upper bound on the elasticity of $\ell_e(x - x_e^0)$ which may be larger than the elasticity of $\ell_e(x)$ itself. Furthermore, we have to add $|\mathcal{P}|$ virtual players, which leaves the analysis of the time of convergence unchanged only if $n = \Omega(|\mathcal{P}|)$.

As a third alternative, we can add an exploration component to the protocol. With a probability of 1/2, the players can sample another path uniformly at random rather than another player. In this case, however, the elasticity d cannot be used as a damping factor anymore, since the expected increase of congestion may be much larger than the current load. Rather, we have to reduce the migration probability by a factor $\min\left\{1, \frac{|\mathcal{P}|\,\ell_{\min}}{\beta\, n}\right\}$ where β is an upper bound on the maximum slope and $\ell_{\min} = \min_{e\in E}\ell_e(1)$ is the minimum latency of an empty resource.

Protocol 2 Exploration Protocol, repeatedly executed by all players in parallel.

Let P denote the path of the player in state x.
Sample another path $Q \in \mathcal{P}$ uniformly at random.
if $\ell_P(x) > \ell_Q(x + 1_Q - 1_P)$ **then**
with probability

$$\mu_{PQ} = \min\left\{1, \lambda \cdot \frac{|\mathcal{P}|\,\ell_{\min}}{\beta\, n} \cdot \frac{\ell_P(x) - \ell_Q(x + 1_Q - 1_P)}{\ell_P(x)}\right\}$$

migrate from path P to bin Q.
end if

Lemma 4.24. *Let x denote a state and let Δx denote a random migration vector generated by the* Exploration Protocol*. Then,*

$$\mathbb{E}\left[\Delta\Phi(x, \Delta x)\right] \;\leq\; \frac{1}{2}\sum_{P,Q\in\mathcal{P}} \mathbb{E}\left[V_{PQ}(x, \Delta x)\right] \;.$$

Proof. Recall that Lemma 4.9 states the following for every state x and every migration vector Δx

$$\Delta\Phi(x, \Delta x) \;\leq\; \sum_{P,Q\in\mathcal{P}} V_{PQ}(x, \Delta x) + \sum_{e\in E} F_e(x, \Delta x) \;.$$

Now in order to proof Lemma 4.24, we apply the same approach as in the proof of Lemma 4.10. Hence, it remains to adapt the upper bound on $\mathbb{E}\left[\Delta\tilde{\ell}_e(\Delta\tilde{x}_e)\right]$ to the Exploration Protocol. Note that this is quite simple, since due to the linearity of expectation,

$$\begin{aligned}
\mathbb{E}\left[\Delta\tilde{\ell}_e(\Delta\tilde{x}_e)\right] \;&\leq\; \beta\,\mathbb{E}\left[\Delta\tilde{x}_e\right] \\
&\leq\; \beta\, n \cdot \lambda \cdot \frac{\ell_{\min}\,|\mathcal{P}|}{\beta\, n} \cdot \frac{1}{|\mathcal{P}|} \cdot \frac{\ell_P - \ell_Q^+}{\ell_P} \\
&\leq\; \lambda \cdot \frac{\ell_e^+}{\ell_Q^+} \cdot (\ell_P - \ell_Q^+) \;,
\end{aligned}$$

where we have substituted the migration probability of the protocol and the fact that there are at most n players that may sample a path containing e. This proves Equation (4.2) if λ is chosen small enough. With opposite signs, the same argument holds if $e \in P$. □

Since we have omitted the parameter ν from the protocol, we now need a lower bound on the minimum improvement that is possible when the system is not yet at an imitation-stable state in order to give an upper bound on the convergence time. Formally, let

$$\kappa = \min_{x} \min_{\substack{P,Q \in \mathcal{P} \\ \ell_p(x) > \ell_Q(x+1_Q-1_P)}} \{\ell_P(x) - \ell_Q(x+1_Q-1_P)\} \ .$$

Theorem 4.25. *Consider a symmetric network congestion game in which all players use the* EXPLORATION PROTOCOL*. Let x denote the initial state of the dynamics. Then the dynamics converge to a Nash equilibrium in expected time*

$$O\left(\frac{\Phi(x)\,\beta\,n\,\ell_{\max}}{\ell_{\min}\,\kappa^2}\right) \ .$$

Proof. In every state which is not a Nash equilibrium there exists a player currently utilizing path $P \in \mathcal{P}$ and a path $Q \in \mathcal{P}$ such that $\ell_Q \leq \ell_P - \kappa$. Hence, the expected virtual potential gain is at least

$$\mathbb{E}\left[V_{PQ}\right] \leq -\frac{1}{|\mathcal{P}|} \cdot \frac{\lambda\,|\mathcal{P}|\,\ell_{\min}}{\beta\,n} \cdot \frac{\kappa}{\ell_P} \cdot \kappa \leq -\frac{\lambda\,\ell_{\min}}{\beta\,n} \cdot \frac{\kappa^2}{\ell_{\max}} \ ,$$

and the true potential gain is at least half of this. Again, Lemma 4.7 yields the expected time until the potential decreases from at most Φ to $\Phi^* \geq 0$. □

It is obvious that an analogue of Lemmas 4.10 and 4.24 also holds for any protocol that is a combination of the IMITATION PROTOCOL and the EXPLORATION PROTOCOL, e. g. a protocol in which in every round every player executes the one or the other with probability one half. Then in order to bound the value of $\mathbb{E}\left[\Delta\tilde{\ell}_e(\Delta\tilde{x}_e)\right]$, we must make a case differentiation based on whether proportional or uniform sampling dominates the probability that other players migrate towards resource e. Such a protocol combines the advantages of the IMITATION PROTOCOL and the EXPLORATION PROTOCOL: In the long run, it converges to a Nash equilibrium, and reaches an approximate equilibrium as quickly as stated by Theorem 4.16 (up to a factor of 2).

CHAPTER 5

Congestion Games with Priorities

One drawback of the classical notion of congestion games as studied in the previous chapters is that all players are treated as equal, meaning each player has access to every resource regardless of the other players who already allocate it. In many scenarios, however, some players are more “important” than others, and hence resources are likely to slow down or stop those players with lower priorities.

In this chapter, we introduce *congestion games with priorities* to model scenarios in which players with higher priorities can prevent players with lower priorities from obtaining access to the same resources as they allocate. In our model, each resource can partition the set of players into classes of different priorities. As long as a resource is only allocated by players with the same priority, these players incur a latency depending on the congestion, as in the classical notion of congestion games. But if players with different priorities allocate a resource, only players with the highest priority incur a latency depending on the number of players with this priority, and players with lower priorities incur an infinite latency. Intuitively, they are displaced by the players with the highest priority.

Obviously, every congestion game can be enhanced with priorities. However, in this chapter we consider such singleton congestion games. To be precise, we consider standard and player-specific singleton congestion games with priorities and prove that every such game possesses a Nash equilibrium. We restrict ourselves to singleton congestion games, as our model of player-specific singleton congestion games with priorities does not only extend the notion of congestion games but also the well-known model of *two-sided matching markets*. This model was introduced by Gale and Shapely [GS62] to model markets on which different kinds of agents are matched to one another, for example men and women, students and colleges [GS62], interns and hospitals [Rot84], and firms and workers. Using the same terms as for congestion games, we say that the goal of a two-sided matching market is to match players and resources (or markets). In contrast to congestion games, each resource can only be matched to one player. With each pair of player and resource a payoff is associated, and players are interested in maximizing their payoffs. Hence, the payoffs implicitly define a preference list of the resources for each player. Additionally, each resource has a preference list of the players that is independent of the payoffs and that assigns a unique rank to each player. Every player can *propose* to one resource and if several players propose to a resource, only the most preferred player is *assigned* to that resource and receives the corresponding payoff. This way, every set of proposals corresponds to a bipartite matching between players and resources. A matching is *stable* if no player can be assigned to a resource from which it receives a higher payoff than from its current resource given the proposals of the other players. Gale and Shapely [GS62] show that stable matchings always exist and can be found in polynomial time.

In many congestion gamearguments thes it is unrealistic to assume that the resources have no preferences over the players. In the same way it is unrealistic to assume that the markets in a two-sided matching market have strict preference lists. Our model of player-specific congestion games with priorities can also be seen as a model of *two-sided matching markets with ties* in which several players can be assigned to one resource. If different players propose to a resource, only the most preferred ones are assigned to it. If the most preferred player is not unique, several players share the payoff of the resource. Such two-sided matching markets correspond to our model of congestion games with priorities, except that players are now interested in maximizing their payoffs instead of minimizing their latencies, which does not affect our results. Two-sided matching markets with ties have been extensively studied in the literature [GI89, IMMM99]. In these models, ties are somehow broken, i. e., despite ties in the preference lists, every resource can be assigned to at most one player. Hence, these models differ significantly from our model. One application of our model are markets into which different companies can invest: as long as the investing companies are of comparable size, they share the payoff of the market, but large companies can utilize their market power to eliminate smaller companies completely from the market. Player-specific congestion games and two-sided matching markets are the special cases of our model in which all players have the same priority or distinct priorities, respectively. In the following, we use the terms *two-sided matching markets with ties* and *player-specific congestion games with priorities* interchangeably.

We also consider a special case of *correlated two-sided matching markets with ties* in which the payoffs of the players and the preference lists of the resources are correlated. In this model, every resource prefers to be assigned to players which receive the highest payoff when assigned to it. We show that this special case is a potential game. Variants of correlated two-sided matching markets without ties have been studied in the context of content distribution in networks and distributed caching problems [FGMS06, GMV05, Mir05]. These markets have also been considered for discovering stable geometric configurations with applications in VLSI design [HHP06].

5.1 Formal Definition

Next we formally define standard and player-specific singleton congestion games with priorities and two sided-markets with ties.

Congestion Games with Priorities A singleton congestion game with priorities is a singleton congestion game in which additionally each resource $r \in \mathcal{R}$ assigns a *rank* $\text{rk}_r(i)$ to every player $i \in \mathcal{N}$. For a state S, let $\text{rk}_r(S) = \max_{i:r \in S_i} \text{rk}_r(i)$. We say that player i *allocates* resource r if $r \in s_i$, and we say that player i is *assigned* to resource r if $r \in s_i$ and $\text{rk}_r(i) = \text{rk}_r(S)$. We define $x^*_r(S)$ to be the number of players that are assigned to resource r, that is, the number of players i with $\{r\} = s_i$ and $\text{rk}_r(i) = \text{rk}_r(S)$. The latency that player i being assigned to r incurs on r is $\ell_r(x^*_r(S))$ or $\ell^i_r(x^*_r(S))$ depending on if the congestion game is a standard game or a player-specific game. Players which allocate a resource r but are not assigned to it incur an infinite latency on it.

Two-Sided Matching Markets with Ties A *two-sided matching market* consists of two disjoint sets $\mathcal{N} = \{1, \dots, n\}$ and $\mathcal{R} = \{r_1, \dots, r_m\}$. We use the terms *players* to denote elements from $\mathcal{N}$, and we use the terms *resources* and *markets* to denote elements from $\mathcal{R}$. In a two-sided matching market, every player can be *matched* to one resource, and every resource can be matched to one player. We assume that with every pair $(i, r) \in \mathcal{N} \times \mathcal{R}$, a payoff $p_{i,r}$ is associated and that player i receives payoff $p_{i,r}$ if it is matched to resource r. Hence, for each player the payoffs describe implicitly a preference list of the resource. Additionally, we assume that every resource has a strict preference list of the players, which is independent of the payoffs. Each player $i \in \mathcal{N}$ can *propose* to a resource $r_i \in \mathcal{R}$. Given a *state* $S = (r_1, \dots, r_n)$, each resource $r \in \mathcal{R}$ is matched to the *winner of* r, which is the player that r ranks highest among all players $i \in \mathcal{N}$ with $r = r_i$. If i is the winner of r, it receives a payoff of $p_{i,r}$. If a player proposes to a resource won by another player, it receives no payoff at all. We say that S is a *stable matching* if none of the players can unilaterally increase its payoff by changing its proposal given the proposals of the other players. That is, for each player i who is assigned to a resource r_i, each resource r from which it receives a higher payoff than from r_i is matched to a player that r prefers to i.

We define a *Two-Sided Matching Market with ties* to be a two-sided matching market in which the preference lists of the resources can have ties. Given a vector of proposals $S = (r_1, \dots, r_n)$, we say that a player $i \in \mathcal{N}$ is matched to resource $r \in \mathcal{R}$ if $r = r_i$ and if there is no player $j \in \mathcal{N}$ such that $r = r_j$ and j is strictly preferred to i by r. For a resource r, we denote by $x_r(S)$ the number of players proposing to r and by $x^*_r(S)$ the number of players that are matched to r. We assume that every player i has a non-increasing payoff function $p^i_r \colon \mathbb{N} \to \mathbb{N}$ for every resource r. A player i who is matched to resource r receives a payoff of $p^i_r(x^*_r(S))$. Also for two-sided matching markets with ties, we call a state S a stable matching if none of the players can increase its payoff given the proposals of the other players.

In *correlated two-sided matching markets with ties*, the preferences of players and resources are correlated. We assume that also the preference lists of the resources are chosen according to the payoffs that are associated with the pairs from $\mathcal{N} \times \mathcal{R}$. That is, a player $i \in \mathcal{N}$ is preferred to a player $j \in \mathcal{N}$ by resource $r \in \mathcal{R}$ if and only if $p_{i,r} > p_{j,r}$. Due to this construction, if two players i and j are both matched to a resource r, the payoffs $p_{i,r}$ and $p_{j,r}$ must be the same. We denote this payoff by $p_r(S)$, and we assume that it is split among the players that are matched to r. The payoff that a player receives that is matched to r is specified by a function $q_r(p_r(S), x_r^*(S))$ with $q_r(p_r(S), 1) = p_r(S)$ that is non-increasing in the number of players matched to r.

5.2 Existence of Nash Equilibria

In this section, we prove that every standard and every player-specific singleton congestion game with priorities possesses a Nash equilibrium. From our existence proofs we conclude that Nash equilibria can be computed efficiently.

Theorem 5.1. *Every standard singleton congestion game with priorities possesses a Nash equilibrium. Moreover, every such game is a potential game.*

Proof. Let $\mathcal{D} = (\mathbb{N} \cup \{\infty\}) \times \mathbb{N}$. For elements $a = (a_1, a_2) \in \mathcal{D}$ and $b = (b_1, b_2) \in \mathcal{D}$ we denote by "$<$" the lexicographic order on $\mathcal{D}$ in which the first component is to be minimized and the second component is to be maximized, i. e., we define $a < b$ if and only if $a_1 < b_1$ or if $a_1 = b_1$ and $a_2 > b_2$. We construct a potential function $\Phi \colon \Sigma_1 \times \cdots \times \Sigma_n \to \mathcal{D}^n$ that maps every state $S = (r_1, \ldots, r_n)$ to a vector of values from $\mathcal{D}$. In state S, every resource $r \in \mathcal{R}$ contributes $x_r(S)$ values to the vector $\Phi(S)$ and $\Phi(S)$ is obtained by sorting all values contributed by the resources in non-decreasing order according to the lexicographic order defined above. Resource r contributes the values $(\ell_r(1), \mathrm{rk}_r(S)), \ldots, (\ell_r(x_r^*(S)), \mathrm{rk}_r(S))$ to the vector $\Phi(S)$ and $x_r(S) - x_r^*(S)$ times the value $(\infty, 0)$. We claim that if state S' is obtained from S by letting one player play a better response, then $\Phi(S')$ is lexicographically smaller than $\Phi(S)$, i.e., there is a k with $\Phi_j(S) = \Phi_j(S')$ for all $j < k$ and $\Phi_k(S') < \Phi_k(S)$.

Assume that in state S player i plays a better response by changing its allocation from resource r_i to resource r_i'. We compare the two vectors $\Phi(S)$ and $\Phi(S')$, and we show that the smallest element added to the potential vector is smaller than the smallest element removed from the potential vector, showing that the potential decreases lexicographically. Due to the strategy change of player i, either the value $(\ell_{r_i}(x_{r_i}^*(S)), \mathrm{rk}_{r_i}(S))$ or the value $(\infty, 0)$ is replaced by the value $(\ell_{r_i'}(x_{r_i'}^*(S')), \mathrm{rk}_{r_i'}(S'))$. Since player i plays a better response, $\ell_{r_i'}(x_{r_i'}^*(S')) < \ell_{r_i}(x_{r_i}^*(S))$ or $\ell_{r_i'}(x_{r_i'}^*(S')) < \infty$, respectively, and hence, the term added to the potential is smaller than the term removed from the potential. In the following we show that all values that are contained in $\Phi(S)$ but not in $\Phi(S')$ are larger than $(\ell_{r_i'}(x_{r_i'}^*(S')), \mathrm{rk}_{r_i'}(S'))$. Clearly, only terms for the resources r_i and r_i' change and we can restrict our considerations to these two resources.

Let us consider resource r_i first. If the rank of r_i does not decrease by the strategy change of player i or if no player allocates resource r_i in state S', then only the term $(\ell_{r_i}(x_{r_i}^*(S)), \mathrm{rk}_{r_i}(S))$ or $(\infty, 0)$ is not contained in the vector $\Phi(S')$ anymore. All other terms contributed by resource r_i do not change. If the rank of resource r_i is decreased by the strategy change of player i, then additionally some terms $(\infty, 0)$ in the potential are replaced by other terms. Obviously, the removed terms $(\infty, 0)$ are larger than $(\ell_{r'_i}(x^*_{r'_i}(S')), \mathrm{rk}_{r'_i}(S'))$.

Now we consider resource r'_i. If the rank of r'_i does not increase by the strategy change of player i or if no player allocates r'_i in state S, then only the term $(\ell_{r'_i}(x^*_{r'_i}(S')), \mathrm{rk}_{r'_i}(S'))$ is added to the potential. All other terms contributed by r'_i do not change. If the rank of r'_i is increased by the strategy change of player i, then additionally the terms $(\ell_{r'_i}(1), \mathrm{rk}_{r'_i}(S)), \ldots, (\ell_{r'_i}(x^*_{r'_i}(S)), \mathrm{rk}_{r'_i}(S))$ are replaced by $x^*_{r'_i}(S)$ terms $(\infty, 0)$. In this case, $x^*_{r'_i}(S') = 1$ and the smallest removed term, $(\ell_{r'_i}(1), \mathrm{rk}_{r'_i}(S))$, is larger than

$$(\ell_{r'_i}(1), \mathrm{rk}_{r'_i}(S')) = (\ell_{r'_i}(x^*_{r'_i}(S')), \mathrm{rk}_{r'_i}(S'))$$

because $\mathrm{rk}_{r'_i}(S') > \mathrm{rk}_{r'_i}(S)$. □

Next we consider player-specific singleton congestion game with priorities and prove that every such games possesses a Nash equilibrium. The proof we present follows similar arguments as Milchtaich's proof showing that every player-specific singleton congestion game without priorities possesses a Nash equilibrium [Mil96]. Note that we extended this proof towards player-specific matroid congestion games.

Theorem 5.2. *Every player-specific singleton congestion game with priorities possesses a Nash equilibrium.*

Proof. In order to prove the existence of Nash equilibria, we compute a sequence of states $S^0, \ldots, S^k$ such that S^0 is the state in which no player allocates a resource and S^k is a state in which every player allocates a resource. Remember that we distinguish between allocating a resource and being assigned to it. Our construction ensures the invariant that in each state S^t in this sequence, every player who allocates a resource has no incentive to change its strategy. Clearly, this invariant is true for S^0 and it implies that S^k is a pure Nash equilibrium.

In state S^t we pick an arbitrary player i who is allocating no resource and we let it play its best response. If in state S^t there is no resource to which i can be assigned, then i can allocate an arbitrary resource without affecting the players who are already allocating a resource and hence without affecting the invariant. We still have to consider the case that after its best response, player i is assigned to a resource r. If we leave the strategies of the other players unchanged, the invariant may not be true anymore after the strategy change of player i. The invariant can, however, only be false for players who are assigned to resource r in state S^t. We distinguish between two cases in order to describe how the strategies of these players are modified in order to maintain the invariant.

Firstly, we consider the case that the rank of resource r does not change by the strategy change of player i. If there is a player j who is assigned to resource r in S^t and who can improve its strategy after i is also assigned to r, we change the strategy of j to the empty set, i. e., in state S^{t+1} player j belongs to the set of players which do not allocate any resource. Besides this, no further modifications of the strategies are necessary because all other players are not affected by the replacement of j by i on resource r. In the case that the rank of resource r increases by the strategy change of player i, all players which are assigned to resource r in state S^t are set to their empty strategy in S^{t+1}.

All that remains to be shown is that the described process terminates after a polynomial number of strategy changes in a stable state. We prove this by a potential function that is the lexicographic order of two components. The most important component is the sum of the ranks of the resources, i.e., $\sum_{r \in \mathcal{R}} \mathrm{rk}_r(S^t)$, which is to be maximized. Observe that this sum does not decrease in any of the two aforementioned cases, and that it increases strictly in the second case. Thus we need to show that after a polynomial number of consecutive occurrences of the first case, the second case must occur. Therefore, we need a second and less important component in our potential function. In order to define this component, we associate with every pair $(i, r) \in \mathcal{N} \times \mathcal{R}$ for which i is assigned to r in state S^t a *tolerance* $\mathrm{tol}_t(i, r)$ that describes how many players (including i) can be assigned to r without changing the property that r is an optimal strategy for i, i.e.,

$$\min\{\max\{b \mid \text{in } S^t,\ r \text{ is best resp. for } i \text{ if } i \text{ shares } r \text{ with } b-1 \text{ players}\}, n\} \ .$$

The second component of the potential function is the sum of the tolerances of the assigned pairs in S^t, which is to be maximized. We denote the set of assignments in state S^t by $E^t \subseteq \mathcal{N} \times \mathcal{R}$ and define the potential function as

$$\Phi(S^t) = \left(\sum_{r \in \mathcal{R}} \mathrm{rk}_r(S^t),\ \sum_{(i,r) \in E^t} \mathrm{tol}_a(i, r) \right) \ .$$

In every occurrence of the first case, the second component increases by at least 1. Since the values of the components are bounded from above by $m\,n$ and n^2 and bounded below from 0, the potential function implies that there can be at most $m\,n^3$ strategy changes before an equilibrium is reached. This does not include the last strategy change of players which are not assigned to any resource in the final state. In their last strategy change, these players allocate an arbitrary resource, which does not affect the potential. However, there are less than n such strategy changes. ☐

Let us remark that the potential function does not imply that the considered games are potential games because it increases only if the strategy changes are made according to the above described policy. Additionally, observe that the proof implicitly describes an efficient algorithm to compute a Nash equilibrium with at most $O(m\,n^3)$ strategy changes.

Corollary 5.3. *There exists a polynomial time algorithm to compute a Nash equilibrium of a player-specific singleton congestion game with priorities and non-decreasing player-specific latency functions.*

Finally, we consider correlated two-sided matching markets with ties and we show that these games are potential games.

Theorem 5.4. *Correlated two-sided matching markets with ties are potential games.*

Proof. We define a potential function $\Phi\colon \Sigma_1 \times \cdots \times \Sigma_n \to \mathbb{N}^n$ that is similar to the one used in the proof of Theorem 5.1 and we show that it increases strictly with every better response that is played. Again, each resource r contributes $x_r(S)$ values to the potential, namely the values $q_r(p_r(S), 1), \ldots, q_r(p_r(S), x_r^*(S))$ and $x_r(S) - x_r^*(S)$ times the value 0. In the potential vector $\Phi(S)$, all these values are sorted in non-increasing order. A state S' has a higher potential than a state S if $\Phi(S')$ is lexicographically larger than $\Phi(S)$, i.e., if there exists an index k such that $\Phi_j(S) = \Phi_j(S')$ for all $j < k$ and $\Phi_k(S) < \Phi_k(S')$.

Let S denote the current state and assume that there exists one player $i \in \mathcal{N}$ who plays a better response, leading to state S'. We show that $\Phi(S')$ is lexicographically larger than $\Phi(S)$. Assume that i changes its proposal from r_i to r_i'. Since i plays a better response, it must be matched to r_i' in state S'. That is, the value $q_{r_i'}(p_{i,r_i'}, x_{r_i'}^*(S'))$ is added to the potential. We show that only smaller values are removed from the potential, implying that the potential must increase lexicographically. If i is matched to r_i in state S, then only the value $q_{r_i}(p_{r_i}(S), x_{r_i}^*(S))$ is removed from the vector and maybe, if $x_{r_i}^*(S) = 1$, some 0 values are replaced by larger values. Since player i plays a better response, $q_{r_i}(p_{r_i}(S), x_{r_i}^*(S)) < q_{r_i'}(p_{i,r_i'}, x_{r_i'}^*(S'))$. If $x_{r_i'}^*(S') = 1$ and there are players assigned to r_i' in state S, then also the values $q_{r_i'}(p_{r_i'}(S), 1), \ldots, q_{r_i'}(p_{r_i'}(S), x_{r_i'}^*(S))$ are removed from the potential vector. In this case, player i displaces the previously assigned players from resource r_i', which implies $q_{r_i'}(p_{i,r_i'}, x_{r_i'}^*(S')) = q_{r_i'}(p_{i,r_i'}, 1) > q_{r_i'}(p_{r_i'}(S), 1)$, as desired. □

CHAPTER 6

Conclusion and Open Problems

In this thesis we presented new results about the existence of Nash equilibria in congestion games and on the time until players playing a congestion game reach a Nash equilibrium. In particular, we extended our insights on the impact of the combinatorial structures of the players' strategy spaces on these problems. Building upon previous work, we proved that every player-specific and every weighted congestion game possesses a Nash equilibrium if the players' strategy spaces correspond to the set of bases of matroids. Furthermore, we proved that best response dynamics in standard matroid congestion games are guaranteed to terminate quickly. Note that both results hold without further assumptions on the global structure of the game and on the latency functions except monotonicity. In order to prove these results we applied results from matroid theory, namely we took advantages of the *greedy property* and the *exchange property* of matroids.

We also showed that our results cannot be extended to larger classes of games if we solely consider the combinatorial structure of the strategy spaces of individual players. For these purposes, we provided a new characterization of non-matroid set systems which is complementary to the exchange properties of matroids in the following way. The exchange property of matroids says that it suffices to consider exchanges on the bases of pairs of resources in order to obtain a new base again. Our characterization of non-matroid, however, say that in case of a non-matroid set system there exists at least one resource that needs to be exchanged with two others. In other words, we showed that $(1, 2)$-exchanges cause the trouble.

Our characterization does not rule out the possibility that there exist other classes of player-specific or weighted congestion games which always possess Nash equilibria or classes of standard congestion games in which best response dynamics are guaranteed to terminate quickly. However, from our characterizations we can conclude that in this case additional properties on the latency functions or how the players' strategy spaces can be interweaved have to be taken into account. As an example for the first case consider the result presented by Fotakis et al. [FKS05] which shows that every weighted congestion game possesses a Nash equilibrium if the latency functions are linear. As an example for the second case consider the recent result of Fotakis [Fot08] which shows that best response dynamics in standard network congestion games are guaranteed to terminate quickly if the paths are linearly independent.

At this point we also like to point out that one can check in polynomial time whether a given strategy space is the set of bases of a matroid. Given an explicit description of the players' strategy spaces this follows easily from Corollary 1.2.

Let us compare our results on the impact of the combinatorial structures of the players' strategy spaces on the convergence time of best response dynamics in standard congestion games with the PLS-completeness results as presented in [ARV08]. There it is shown that computing a Nash equilibrium of a threshold game is PLS-complete. Recall that in such games players perform $(1, n)$-exchanges, i. e., either a player allocates a private resource on its own or share a bunch of n resources with the other players. Their analysis shows that $(1, n)$-exchanges cause the trouble in the complexity of computing Nash equilibria, whereas our results show that $(1, 2)$-exchanges cause the trouble in the convergence time. We believe that it is of particular importance to narrow the gap between $(1, 2)$- and $(1, n)$-exchanges in order to gain more insights in the complexity of computing Nash equilibria as threshold congestion games are the building block to prove other completeness results. To this end, one might first want to consider games with a constant number of players but with increasing number of resources.

Also note that we took a worst case perspective on best response dynamics in standard congestion games. In order to circumvent this pessimistic assumptions one might want to consider (semi-)random instances. Such an approach can explain why real world systems stabilize quite quickly. A similar approach has been proposed and considered in [Röe08]. There, (semi-)random traveling salesperson instances are considered with respect to the convergence time of the well-known 2-opt local search heuristic. It is shown that in (semi-)random instances the 2-opt local search heuristic terminates quickly, whereas in the worst case it can take exponentially long.

In Chapter 4 we also proposed and analyzed a natural protocol based on imitating profitable strategies in symmetric standard congestion games. If players concurrently use our Imitation Protocol, then the resulting dynamics converge rapidly to approximate equilibria, in which only a small fraction of players have latency significantly larger than average. In addition, in finite time the dynamics converges to an imitation-stable state, in which no player can improve its latency by more than ν by imitating a different player. To the best of our knowledge, this is the first protocol that applies to general strategy spaces. Previous work only considers singleton games [BFG$^+$06, BFHH07, EDM05, FKS08a].

As the Imitation Protocol has the disadvantage that attractive strategies offering large latency gain can get "lost" once no player uses them anymore we also analyzed the probability of this event in singleton congestion games. We showed that this event becomes unlikely to occur as the number of players increases. Hence, by removing the parameter ν from the protocol, imitation dynamics become likely to converge to Nash equilibria. We also showed that in singleton congestion games with linear latency function imitation dynamics terminated at a state which social cost is on expectation not much worse than that of a socially optimal state. In order to prove this result, we required that in a social optimum the number of players allocating a resource is lower bounded by $\Omega(\log n)$. Currently, we are not sure whether this result also holds without this assumption. Hence, we leave it as an open problem to prove or falsify it. Furthermore, we leave it as an open problem to extend this approach to games with general strategy spaces.

In Section 3.3 we considered the convergence time of random best response dynamics in player-specific singleton congestion games. We failed to prove an exponential lower bound on the convergence time of such dynamics, however, we supported this conjecture by the means of simulations. Our simulations are motivated by a careful analysis of games in which the derangement of order - measured in terms of under- and overload tokens - cannot increase. In our experimental lower bound, we carefully construct games in which their number can also increase. One might want to compare this conjecture with the recent exponential lower bound on the convergence time of random better and best response dynamics in two-sided matching markets [AGM$^+$08]. Chapter 5 presents close relationships between the two models, however, the lower bounds cannot be transferred immediately. In a two-sided matching market a player can be banished from a resource by someone else, however, this is impossible in a player-specific singleton congestion game. In the latter class of games, the player still has selected a feasible strategy except that its latency has increased.

In order to gain more insights into random best response dynamics in player-specific singleton congestion games one might want to consider the following intermediate problem: Consider a Nash equilibrium of a player-specific singleton congestion game and suppose that for some reason a single player changes its latency functions such that its current strategy is no longer its best choice. How long does it take until subsequent best response dynamics are guaranteed to terminate? It can easily be verified that such dynamics can cycle, too. However, one can also verify that the number of over- and underload tokens cannot increase. Hence, intuitively such random best response dynamics cannot last long. Furthermore, recall that player-specific singleton congestion games with linear latency functions are potential games [GMT06]. The convergence time of sequential best response dynamics in such games is an open problem, too.

Besides the classical assumptions that all players allocating a resource are treated as equal we introduced player-specific singleton congestion games with priorities in which a resource can foster certain players by assigning a higher priority to them. In our model players with less priority observe infinite latency if players with higher priorities allocate the same resource. We showed that every such game possesses a Nash equilibrium and observed interesting relationships to two-sided matching markets. However, we think that our model is not yet satisfying from the perspective of real world applications. In real world applications, it is more likely that certain players are slowed down or stopped if players with higher priorities are present. Moreover, resources will continue to process these players if the ones with higher priorities have been finished. Such models are considered by Farzad et al. [FOV08a] and by Immorlica et al. [ILMS05]. There, each resource comes along with a priority system/mechanism according to which the order in which the players are processed is determined. Note that in case of congestion games as considered in this thesis one can say that each resource applies a round-robin schedule with infinitesimal small time slots. We believe that more effort should be spend on defining more realistic models.

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