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John T. Edwards and Neville J. Ford

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Abstract

This paper is concerned with the qualitative behaviour of solutions to difference equations. We focus on boundedness and stability of solutions and we present a unified theory that applies both to autonomous and non-autonomous equations and to nonlinear equations as well as linear equations. Our presentation brings together new, established, and hard-to-find results from the literature and provides a theory that is both memorable and easy to apply. We show how the theoretical results given here relate to some of those in the established literature and by means of simple examples we indicate how the use of Lipschitz constants in this way can provide useful insights into the qualitative behaviour of solutions to some nonlinear problems including those arising in numerical analysis.

1 Introduction

We consider the qualitative behaviour and stability of solutions to difference equations of fixed finite order k that take the form

$$x_{n+1} = f(n, x_n, x_{n-1}, ..., x_{n-k+1}). \tag{1.1}$$

The function $f : \mathbb{Z}^+ \times \mathbb{R}^k \to \mathbb{R}$. For a unique solution to an equation of the form (1.1) one needs to specify k initial values $x_0, x_1, \ldots, x_{k-1}$.

A more compact notation uses the vector form of the same equation:

$$y_{n+1} = f(n, y_n) \tag{1.2}$$

where here the vector $y_n = (x_n, x_{n-1}, \ldots, x_{n-k+1})^T$.

If $f(n, y_n) = A(n)y_n + b(n)$ then (1.2) is said to be linear. If $b(n) = 0$ for every n then the equation is homogeneous. Homogeneous linear equations always have the equilibrium solution $y_n = 0$. In the following discussion we shall work with the vector form of the equation unless otherwise stated.

Difference equations can arise in a number of ways. They may be the natural model of a discrete process (in combinatorics, for example) or they may be a discrete approximation of a continuous process. The wide literature on the subject reflects the particular standpoints of the authors.

Our interest in difference equations of this type is motivated by the fact that equations of the form (1.1) commonly arise when ordinary, or delay, differential equations (for example) are solved numerically. Correspondingly, Volterra difference equations of the form (1.3) arise in the numerical solution of Volterra integral or integro-differential equations by methods with a fixed step length.

$$x_{n+1} = f(n, x_n, x_{n-1}, ..., x_1, x_0). \tag{1.3}$$

These latter equations, with varying length history, present particular challenges to analysis and we shall not discuss them here. However we refer to a sequel to the present paper ([9]) in which Volterra-type problems are analysed.

One common theme of recent work in numerical analysis is the desire to model long term (qualitative) properties of the original problem in the numerical solution. In the long term, errors in numerical solutions grow, and it is not reasonable to demand that global errors in the numerical solution shall converge to zero with small step sizes h. However it is important that key features of the solution (boundedness, oscillations, periodic or closed orbits, stability) should be preserved. The aim is to identify *good* numerical methods, which are those that can be relied upon to reproduce faithfully the true qualitative behaviour of solutions to a class of problems.

The analysis of numerical methods applied to autonomous linear problems is well-developed. The direct analysis of nonlinear and non-autonomous problems is less well understood and is dependent on the availability of suitable general theorems on the behaviour of solutions to difference equations. In this paper we have chosen to concentrate on the properties of boundedness and stability of solutions. These properties are important, and they coincide for certain classes of simple equations that have been analysed previously. When one considers more general problems it becomes important to discriminate between the properties of boundedness and of stability.

2 Definitions of stability and related concepts

While the fundamental idea of stability is widely understood, there remains some latitude in definitions among authors and so for the sake of clarity, we give definitions of the key concepts here.

Definition 2.1 (Stability) Consider the difference scheme (1.2) and let $\{y_n\}$ be the solution with respect to initial condition $y_0 = \alpha \in \mathbb{R}^k$ and $\{z_n\}$ be the solution with respect to the initial condition $z_0 = \beta \in \mathbb{R}^k$. The solution $\{y_n\}$ is then said to be stable if, whenever $\epsilon > 0$ is given, there exists $\delta_\epsilon > 0$ for which $\|y_n - z_n\| < \epsilon$ whenever $\|\alpha - \beta\| < \delta_\epsilon$.

For homogeneous linear equations, it is quite easy to show that *every solution of (1.2) is stable if and only if every solution is bounded* and this motivates some authors to make their initial definition of stability in terms of boundedness (see below). It is important to remember that this equivalence of boundedness and stability (in the sense of definition 2.1) does not persist for nonlinear problems (or even for non-homogeneous linear problems).

Lakshmikantham and Trigiante ([15]) give the following definition of stability for the linear difference equation of the form:

$$y_{n+1} = A(n)y_n + g(n). \tag{2.1}$$

Definition 2.2 The solution $\{y_n\}$ of (2.1) is stable (some authors say *globally stable*) if and only if $\|y_n - z_n\|$ is bounded for *any other solution* z_n of (2.1).

It is routine to prove the following:

Proposition 2.1 Every solution $\{y_n\}$ of a *linear* equation of the form (2.1) is stable in the sense of definition 2.2 if and only if it is stable in the sense of definition 2.1.

Remark 1ex

1. It follows that, for a linear equation, stability is a global property and stability of a solution implies that *all solutions are either bounded or unbounded.*

2. For nonlinear equations it is easy to construct examples where the properties of boundedness of all solutions and stability of a solution do not coincide.

   Example 2.1 The simple first order linear equation

$$y_{n+1} = y_n + 1 \tag{2.2}$$

   is an example of a difference equation where every solution is stable but unbounded.

Example 2.2 Consider the order 1 difference equation

$$y_{n+1} = y_n^{\frac{1}{3}}. \tag{2.3}$$

Here, for all non-zero initial values y_0 the solution $\{y_n\}$ satisfies $|y_n| \to 1$ whereas the solution $y_n = 0$ is unstable.

We now define two further types of stability: asymptotic stability and exponential stability.

Definition 2.3 (Asymptotic stability) Consider the difference scheme (1.2) and let $\{y_n\}$ be the solution with respect to initial condition $y_0 = \alpha$ and $\{z_n\}$ be the solution with respect to the initial condition $z_0 = \beta$. The solution $\{y_n\}$ is then said to be asymptotically stable if there exists $\delta > 0$ for which $\|y_n - z_n\| \to 0$ whenever $\|\alpha - \beta\| < \delta$.

Definition 2.4 (Exponential stability) Consider the difference scheme (1.2) and let $\{y_n\}$ be the solution with respect to initial condition $y_0 = \alpha$ and $\{z_n\}$ be the solution with respect to the initial condition $z_0 = \beta$. The solution $\{y_n\}$ is then said to be exponentially stable if there exist constants $a, \delta > 0, \eta \in (0, 1)$ for which $\|y_n - z_n\| < a\|\alpha - \beta\|\eta^n$ whenever $\|\alpha - \beta\| < \delta$.

For linear autonomous equations, it is straightforward to prove:

Proposition 2.2 Any solution $\{y_n\}$ of the linear autonomous equation

$$y_{n+1} = Ay_n + b \tag{2.4}$$

is exponentially stable if and only if it is asymptotically stable.

Note 2.1 In the definitions here, we have been careful to refer to stability concepts as they apply to a particular solution $\{y_n\}$ (of (1.2) for example) corresponding to a specific initial value. Frequently one may consider the stability of the zero solution of (1.2) (assuming $f(n, 0) = 0$, a condition not satisfied by (2.2)). Some authors refer to the stability of an equation, and by this they mean the stability (in the sense of our definitions) of *all solutions*. In many equations (see the examples in this paper) different solutions exhibit quite different stability properties.

3 Some insights from the numerical solution of differential equations

The classical stability analysis for simple numerical methods considers the homogeneous linear differential test equation

$$y'(t) = \lambda y(t), \qquad y(0) = y_0. \tag{3.1}$$

One can easily show that *all* the solutions to (3.1) are stable with respect to small perturbations in the initial condition y_0 if $Re\lambda \leq 0$ and asymptotically stable if $Re\lambda < 0$. Indeed, if $Re\lambda \leq 0$ then all the solutions to (3.1) are bounded.

One then considers conditions on numerical methods that ensure that the stability behaviour is reproduced in the numerical scheme (see, for example, [2, 12, 13, 16]). One finds, for example, that the solution to the approximate problem is stable for $Re\lambda \leq 0$ only for a restricted range of h (explicit Euler rule) or for all $h > 0$ (Trapezium rule or implicit Euler rule). On the other hand, for $Re\lambda \geq 0$ the solution is (correctly) unstable for all $h > 0$ when the explicit Euler rule is used but (surprisingly) stable for large $h > 0$ when the implicit rule is used.

Here we see a situation where additional constraints on the method (on the value of h) may need to be imposed (dependent on the value of λ) to ensure that the true qualitative behaviour is reproduced faithfully. Stable methods (those for which stability is preserved for every value $h > 0$) are highly regarded ([12, 16]).

One can explore, in a similar way, constraints that need to be imposed on the method in order that other qualitative properties of the true solution be preserved in the numerical scheme. For example in our recent paper [7] we consider in detail changes in the dynamical behaviour of solutions to a difference system for varying parameter values.

When equation (3.1) is approximated using a simple numerical scheme, one obtains a linear constant coefficient difference equation of fixed finite order r. The equation can be expressed in the matrix-vector form

$$y_n = Ay_{n-1}. \tag{3.2}$$

Here the vector $y_n = (x_n, x_{n-1}, ..., x_{n-(r-1)})^T$ and the matrix A (the *companion matrix*) contains the difference equation in the first row and shift operators in rows $2, ..., r$. The dynamical behaviour of solutions to (3.2) is determined by the eigenvalues of A. The situation is simplest when the eigenvalues, λ_i of A are distinct: all solutions are asymptotically stable when $\mathrm{Max}|\lambda_i| < 1$, stable when $\mathrm{Max}|\lambda_i| = 1$ and unstable when $\mathrm{Max}|\lambda_i| > 1$. In cases where a repeated eigenvalue of magnitude 1 is the largest eigenvalue, and the corresponding Jordan block is not diagonal, all solutions are unstable.

When one writes down an expression for the solution of (3.2), the close relationship between boundedness of all solutions and stability is clearly to be seen. We make the following remarks:

1. If all $|\lambda_i| < 1$ then all solutions are bounded and asymptotically stable.

2. If all $|\lambda_i| \leq 1$ and only simple roots (of multiplicity one) satisfy $|\lambda_i| = 1$ then all solutions are bounded and stable.

3. If all λ_i are simple and there are $k(< r)$ such roots satisfying $|\lambda_i| \leq 1$, then there is a k-dimensional set of initial values leading to bounded solutions, all of which are unstable. In fact, unless exact arithmetic is used, even the initial conditions that would be expected to yield bounded solutions may yield unbounded solutions because of the errors introduced by the inexact arithmetic.

In other words stability corresponds to boundedness of all the solutions and instability arises as soon as any unbounded solution exists. It is also worth remarking that even in the case where all the λ_i satisfy $|\lambda_i| > 1$, there still exists a (unique) bounded solution $y = 0$ corresponding to a zero initial value.

Some authors have focused on the question of whether unbounded solutions to a difference scheme exist (see, for example, [8, 23]). For homogeneous linear problems this analysis relates directly to the question of stability, but the link for nonlinear problems is less clear.

The conventional approach to the stability analysis of difference equations of the type we discuss is to begin with a detailed analysis of autonomous linear equations of the form

$$y_{n+1} = Ay_n + b. \tag{3.3}$$

One can then extend the analysis to cover non-autonomous linear equations of the form

$$y_{n+1} = A(n)y_n + b(n). \tag{3.4}$$

To investigate the stability and boundedness of solutions to nonlinear equations, the usual approach is to concentrate on equations that are related in some way to the linear equations whose behaviour is known to be covered by the existing theory. Thus, results are presented that consider equations of the form

$$y_{n+1} = Ay_n + f(y_n) \tag{3.5}$$

or

$$y_{n+1} = A(n)y_n + f(n, y_n) \tag{3.6}$$

where the perturbation f is assumed to be small in some sense that is made precise in statements of the various theorems (see, for example [8, 5, 23]). The established theory treats wide classes of *almost* *linear* problems. One may also be tempted to apply a linearisation method to yield corresponding insight into numerical solutions to nonlinear differential equations of the form

$$y'(t) = f(t, y(t)). \tag{3.7}$$

This is the basic idea behind the use of linear test equations to analyse the stability of numerical methods. Lambert([16]) draws attention to the danger of making errors through this approach. Direct nonlinear analysis of stability is generally to be preferred to a (possibly risky) application of a linearisation method. Mazzia and Trigiante ([21]) draw attention to the fact that linearisation is performed close to an equilibrium solution and that therefore a linearisation method is ineffective in analysing behaviour of numerical approximations of (for example) chaotic solutions. In fact, so many authors use a linearisation method to analyse stability that it is not always made clear exactly what assumptions are being made about the problem under analysis.

In this paper we have tried to avoid making any assumptions about the form of the difference equation under analysis. In particular we have avoided results that can be obtained through considering small perturbations of linear problems. Instead we have tried to present results in as general a form as possible and then to relate the results we have obtained to those that are already established. It is our aim to provide theoretical results that are more easily remembered and more readily applied in a wider range of problems than previously.

Remark 1ex An alternative approach to direct nonlinear stability analysis is the use of Lyapunov's direct method. Lyapunov analysis will not be considered explicitly in the current work although a suitable Lyapunov function could be defined in terms of the vector norm.

4 A unified stability theory

In this section we provide a unified theory for the analysis of stability and boundedness of solutions of autonomous and non-autonomous difference equations, both linear and nonlinear. As we remarked in the previous section, it is usual for the theoretical results to be presented separately for these different classes of problem but it is our aim to provide a sequence of theoretical results that are useful in providing insight across all these problem classes. It would appear attractive to present a single theorem that encapsulates all our results (and this is indeed possible (see Theorem 4.4) but we consider it beneficial to present several theorems. We provide results that are transparent at the cost of poorer generality and more general results that suffer from somewhat poorer transparency. Our tool in the analysis is the Lipschitz condition, familar from several other areas of mathematics. In this situation (as elsewhere) the use of the Lipschitz condition enables us to develop a direct theory that applies both to linear and nonlinear problems.

The concept of *Lipschitz Stability* for differential equations was introduced in [3, 4]. Its application to difference equations is mentioned, for example, in [22]. The book by Agarwal ([1]) published in 1992 provides an excellent survey of the state of the art at that time and focuses on other approaches to stability analysis. Many recent papers continue to concentrate on linear (or linearised) stability theory. We refer the interested reader to the works [6, 18, 20] for further reading, however we have not found elsewhere the unified approach that we introduce here.

We recall the definition:

Definition 4.1 (Uniform Lipschitz condition) Let $f(x, y)$ be a function defined for $x \in X, y \in Y$ where X is some arbitrary set and Y is a normed space, then f satisfies a uniform Lipschitz condition on $X \times Y$ with respect to its second argument if there exists a constant L such that

$$\|f(x, y) - f(x, z)\| \leq L\|y - z\| \tag{4.1}$$

for every $x \in X$ and for all choices of $y, z \in Y$.

Remark 1ex

1. Note that this property could be true in some particular norm but not in another.
2. In the analysis of differential equations of the form (3.7), one often encounters the weaker *one-sided Lipschitz condition*

$$\frac{1}{2}\frac{d}{dt}\|y-z\|^2 =< f(x,y)-f(x,z), y-z >\leq c\|y-z\|^2$$

which provides a bound on the derivative of $\|y-z\|^2$ in terms of $\|y-z\|^2$. There seems to be no useful corresponding weaker Lipschitz condition for the difference equation. The corresponding idea would be to seek a bound on

$$\|y_{n+1}-z_{n+1}\|^2 - \|y_n - z_n\|^2$$

in terms of $\|y_n - z_n\|^2$ which is equivalent to requiring a Lipschitz condition.

For our purposes here, it will often be sufficient to consider a *local Lipschitz condition* defined as follows:

Definition 4.2 (Local Lipschitz condition) Let $f(x, y)$ be a function defined for $x \in X, y \in Y$, then f satisfies a local Lipschitz condition with respect to its second argument in a neighbourhood D of some point w if there exists a constant L^D such that

$$\|f(x,y)-f(x,z)\| \leq L^D\|y-z\| \tag{4.2}$$

for every $x \in X$ and for all choices of $y, z \in D$.

We consider now the general difference equation of the form

$$y_{n+1} = f(n, y_n). \tag{4.3}$$

Here the sequence $\{y_n\}$ may consist of $k-$vectors and the function f may be quite general. When the function f varies according to the value of its first argument, equation (4.3) is called a non-autonomous equation, otherwise it is autonomous. It is usual for authors to consider the cases of autonomous and non-autonomous equations separately and we will adopt this approach at first to provide greater transparency in the discussion. It is also usual for authors to consider first the case where the function f is linear in its second argument. However here we provide a unified analysis.

Equation (4.3) is usually regarded as an initial value problem and the sequence $\{y_n\}$ generated is easily shown to be unique given a starting value y_0. We can write the solution map $\Phi(n, y_0)$ to represent the propagation of the sequence $\{y_n\}$ from the initial vector y_0 in the following form:

$$y_n = \Phi(n, y_0). \tag{4.4}$$

We are now in a position to give theorems on the stability and boundedness of solutions of difference equations.

Remark 1ex The approach we describe, based on the use of the solution map, is analogous to the methods described, for example, in [25] based on the use of dynamical systems theory. However, for non-autonomous problems, the problem does not define a dynamical system as defined there (it is described as a *discrete process* in [17]) and our discussion is more general.

4.1 Simple theorems based on uniform Lipschitz conditions

We begin with a simple theorem that indicates the type of result that is possible for autonomous problems.

Theorem 4.1 (Basic Theorem) Let the sequence $\{y_n\}$ satisfy an autonomous difference equation of the form

$$y_{n+1} = f(y_n) \tag{4.5}$$

where the function f satisfies a uniform Lipschitz condition with Lipschitz constant $L < 1$ then every solution to (4.5) is asymptotically stable. If, instead, f satisfies a uniform Lipschitz condition with Lipschitz constant $L = 1$ then every solution to (4.5) is stable. Further, either all solutions are bounded or all solutions are unbounded.

The proof of this result is straightforward and it provides a very simple criterion for boundedness and asymptotic stability of solutions to equations of the type (4.5). It can be strengthened to include a much wider range of nonlinear problems by allowing a local, rather than global, Lipschitz condition (see the next subsection).

We can give a corresponding simple result for non-autonomous problems:

Theorem 4.2 (Basic theorem for non-autonomous problems) Let the sequence $\{y_n\}$ satisfy a difference equation of the form

$$y_{n+1} = f(n, y_n) \tag{4.6}$$

where the function f satisfies, for each value n, a uniform Lipschitz condition with respect to its second argument, with Lipschitz constant $L_n \le M < 1$ then every solution to (4.6) is asymptotically stable. Further all solutions are bounded or all are unbounded.

Remark 1ex

1. In fact the condition $L_n \le M < 1$ can be relaxed somewhat. The conclusions of Theorem 4.2 hold whenever $\prod_{i=0}^{k} L_n \to 0$ as $k \to \infty$.

2. As in the previous theorem, if the Lipschitz constants satisfy $L_n \le 1$ (or the weaker condition $\prod_{i=0}^{k} L_n$ bounded as $k \to \infty$) then the conclusion is that every solution is bounded and stable rather than asymptotically stable.

3. The strength of these theorems lies in the fact that the conditions are very easy to check and that the theorems give results both for linear and for nonlinear equations.

4. These theorems provide sufficient (but not necessary) conditions for stability.

5. In the next subsections, we present a more general theory:

   (a) We consider (in section 4.2) the situation where the Lipschitz conditions hold locally rather than globally (particularly important for nonlinear problems).

   (b) We show (in section 4.3) that the conditions $L < 1$ (respectively $L_n < 1$) can be relaxed somewhat.

Example 4.1 Consider the nonlinear difference equation of the form

$$x_{n+1} = \alpha(n, x_n) + \beta(n, x_{n+1}) \tag{4.7}$$

where $\alpha(n, y), \beta(n, y)$ satisfy Lipschitz conditions with Lipschitz constants $L_n^{\alpha}, L_n^{\beta}$ respectively, where $L_n^{\alpha} + L_n^{\beta} < 1$. Then every solution of (4.7) is asymptotically stable.

4.2 Theorems based on local Lipschitz conditions

In essence, the results of the previous subsection are based on the fact that the existence of a global Lipschitz constant $L < 1$ implies that the difference equations give iterative schemes $y_{n+1} = f(y_n)$ that are global contraction mappings. As is well known, such schemes always have a unique fixed point (see, for example, [24]) that is the unique limit of the sequence $\{y_n\}$ (independent of the initial value y_0). In practice we may meet difference equation schemes with more than one equilibrium solution and therefore it is appropriate to consider the behaviour of solutions *in a neighbourhood of some equilibrium.* Depending on the initial value y_0, the long-term behaviour of the solution will vary. In this case, it is not possible for the difference equation to satisfy a global Lipschitz condition with $L < 1$ but we can give (instead) sufficient conditions for asymptotic stability of the equilibrium solution in terms of a local Lipschitz condition.

Theorem 4.3 (Local Lipschitz uniqueness and stability theorem) Let the sequence $\{y_n\}$ satisfy a difference equation of the form

$$y_{n+1} = f(n, y_n) \tag{4.8}$$

and let the value d be *an equilibrium solution* of (4.8). (In other words, we assume $f(n, d) = d$ for all $n \in \mathbb{N}$.) Assume that for some sphere D with d as its centre, the function f satisfies, for each value n, a local Lipschitz condition with respect to its second argument, with Lipschitz constant $L_n^D \le M < 1$ then the equilibrium solution $y_n = d$ to (4.8) is asymptotically stable.

Remark 1ex

1. Note that the conclusions to Theorem 4.3 do not include any assertion about boundedness or unboundedness of the solutions. It is important to realise that for nonlinear equations (where Lipschitz conditions may be local rather than global) the properties of boundedness and stability are really quite different. We draw attention to examples

   (a) of equations all of whose solutions are bounded and whose equilibrium solutions may be unstable (see Example 2.2)

   (b) of equations whose equilibrium solutions are stable but which have unbounded solutions (see Example 6.1).

   These situations cannot arise for linear equations.

2. With care, one can relax the condition $L_n^D \le M < 1$ and consider, instead, a condition of the form $\prod_{i=0}^{k} L_n^D \to 0$ as in the previous result.

4.3 A more general theorem

The previous two subsections gave theorems that are useful because they cover both linear and nonlinear problems and are very easy to apply. They give sufficient conditions for the stability of equilibrium solutions (for example) but the conditions they give are not necessary. Indeed one can give (see Example 4.2 below) some quite elementary difference equations whose solutions are all bounded and asymptotically stable but which do not satisfy the conditions we have given in Theorems 4.1, 4.2, 4.3.

Example 4.2 The difference equations

$$y_{n+1} = f(n)y_n, \quad f(n) = 2 \text{ when } n \text{ is even}, \ f(n) = \frac{1}{4} \text{ when } n \text{ is odd} \tag{4.9}$$

has solutions all of which tend to zero as $n \to \infty$. However, when n is even, any Lipschitz constant for f satisfies $L_n \ge 2$.

To give a theorem that is more generally applicable we consider Lipschitz conditions imposed not on the function f in the difference equation (at each step) but instead on the solution map Φ that we introduced in (4.4). We can give the following theorem:

Theorem 4.4 Consider the equation

$$y_{n+1} = f(n, y_n) \tag{4.10}$$

with solution map operator

$$y_n = \Phi(n, y_0). \tag{4.11}$$

1. Assume that, for each n, $\Phi(n, y_0)$ satisfies a uniform Lipschitz condition (with Lipschitz constant L_n) with respect to its second argument and the values $L_n \leq M < \infty$. Then every solution of (4.10) is stable (but need not be bounded). If, further, $L_n \to 0$ as $n \to \infty$ then there exists a unique equilibrium solution to (4.10) and it is asymptotically stable. If, additionally, for every $n \in \mathcal{N}$, $L_n < \zeta^n$ for some $|\zeta| < 1$ then the unique equilibrium solution is exponentially stable.

2. Let D be a sphere around an equilibrium point d of (4.10). Assume further that, for each n, $\Phi(n, y_0)$ satisfies a local Lipschitz condition (with Lipschitz constant L_n^D) with respect to its second argument and the values $L_n \leq M < \infty$. Then the equilibrium solution of (4.10) is stable. If, further, $L_n \to 0$ as $n \to \infty$ then the equilibrium solution to (4.10) is asymptotically stable. If, additionally, for every $n \in \mathcal{N}$, $L_n < \zeta^n$ for some $|\zeta| < 1$ then the equilibrium solution is exponentially stable.

Proof: The proof of the first part of the theorem proceeds as follows. Let y_n and z_n be solutions of (4.10) with initial conditions y_0 and z_0 respectively. Let $\epsilon > 0$ be given. Set $\delta = \frac{\epsilon}{M}$. Then provided $\|y_0 - z_0\| < \delta$ it follows, by the fact that $L_n \leq M$, that

$$\|y_n - z_n\| = \|\Phi(n, y_0) - \Phi(n, z_0)\| < L_n \|y_0 - z_0\| \leq M \|y_0 - z_0\| = \epsilon \tag{4.12}$$

so every solution is stable.

Now assume $L_n \to 0$ as $n \to \infty$. Let $\epsilon > 0$ be given, and fix initial values y_0, z_0. Put $\delta = \|y_0 - z_0\|$ and choose N to satisfy $L_n < \frac{\epsilon}{\delta}$ for all $n > N$. It follows that, for $n > N$, $\|y_n - z_n\| < \epsilon$. But ϵ was arbitrary so $\|y_n - z_n\| \to 0$ as $n \to \infty$. It is easy to see that $y = \lim(y_n)$ is an equilibrium solution and it is unique.

The final conclusion of the first part of the theorem follows by substituting the expression for L_n into equation (4.12).

The second part of the theorem follows in exactly the same way as the first part, but using local Lipschitz conditions on D in place of global conditions.

Remark 1ex Note that Theorem 4.4 gives sufficient conditions for stability (respectively asymptotic stability, exponentional stability) of (4.10). The question then arises as to whether the conditions are also necessary. We make the following observations:

1. For linear equations, the existence of a sequence of Lipschitz constants L_n with a finite bound M is necessary for stability and the existence of a sequence L_n satisfying the additional condition $L_n \to 0$ as $n \to \infty$ is necessary for asymptotic stability.

2. For nonlinear equations the situation is not so clear. For example, we could define a sequence of functions $\Phi(n, y)$ in the following way.

$$\Phi(n, 0) = 0 \tag{4.13}$$

$$\Phi(n, y) = 1, \quad 0 < \|y\| < \frac{1}{n} \tag{4.14}$$

$$\Phi(n, y) = 0, \quad \|y\| \geq \frac{1}{n}. \tag{4.15}$$

Using this definition of Φ, the solution $y_n = 0$ is asymptotically stable (but *not uniformly so*) and no bounded set of Lipschitz constants exists for $\Phi(n, y)$ in a neighbourhood of $y = 0$. However, if one makes additonal assumptions on the behaviour of $\Phi(n, y)$ it is possible to give necessary conditions for (asymptotic) stability. For example, if we assume that for each n, $\frac{\Phi(n,y)}{y}$ is continuous in a neighbourhood of $y = 0$, or that $\Phi(n, y)$ has bounded derivative in a neighbourhood of $y = 0$ such a result is possible.

5 Relationship to existing theory

As we remarked previously, the conventional approach to the stability analysis of difference equations begins with consideration of the autonomous linear problem. The fundamental theorem is as follows (see [15] Theorems 4.3.1 and 4.3.2)

Theorem 5.1 For the equation

$$y_{n+1} = Ay_n. \tag{5.1}$$

1. The zero solution is asymptotically stable if and only if the eigenvalues of the matrix A are all inside the unit disk.

2. The zero solution is stable if and only if the eigenvalues of the matrix A have modulus less than one and those of modulus one are semi-simple (that is, the corresponding Jordan block is diagonal).

In our treatment, we do not deal directly with the eigenvalues of the matrix A but instead we consider a Lipschitz condition that arises for the matrix A or for the solution operator Φ that it induces. In this (linear) equation, any such Lipschitz condition will apply globally and will therefore imply both boundedness of all solutions as well as (asymptotic) stability. We remark that the matrix 2-norm provides a suitable Lipschitz constant for use in Theorem 4.1 or Theorem 4.4 and that the 2-norm of a matrix (it largest singular value) is bounded below by the largest eigenvalue. It is easy to show that some matrices A do not satisfy the conditions of Theorem 4.1 even though they do satisfy Theorem 5.1. However it is simple to show that in case 1 of Theorem 5.1 $\Phi(n, y) = A^n y$ satisfies a Lipschitz condition with $L_n \to 0$ and in case 2 of Theorem 5.1 $\Phi(n, y) = A^n y$ satisfies a Lipschitz condition with $L_n \le M < \infty$. It follows that Theorem 4.4 gives the same conditions for stability and boundedness of solutions to (5.1) as given in Theorem 5.1.

The non-autonomous linear equation

$$y_{n+1} = A(n)y_n \tag{5.2}$$

has presented some difficulties in the past for the conventional analysis. It turns out that the natural condition on the eigenvalues of each $A(n)$ (that each $A(n)$ has eigenvalues within the unit disk) is *not* sufficient to guarantee stability of solutions to (5.2). For example the authors of [15] give the example where

$$A(n) = \frac{1}{8}\begin{pmatrix} 0 & 9 + (-1)^n 7 \\ 9 - (-1)^n 7 & 0 \end{pmatrix} \tag{5.3}$$

which have eigenvalues $\pm 2^{-1/2}$ but for which all solutions (apart from the zero solution) are unbounded as $n \to \infty$. Theorem 4.4 gives a direct way of checking, because the largest singular value of each matrix $A(n)$ is 4 and the Lipschitz constant for each $\Phi(n, y)$ is $4^n \to \infty$. It follows that, in this case, (5.2) does not satisfy the conditions of Theorem 4.4. We note that the recent works ([10, 11]) develop methods for analysing asymptotic properties of infinite products of a family of matrices.

The next most favoured approach in the conventional analysis seems to be to consider *Stability by the first approximation.* In other words, the equation

$$y_{n+1} = A(n)y_n + f(n, y_n) \tag{5.4}$$

is considered. The usual idea is to impose conditions which make the linear equation

$$y_{n+1} = A(n)y_n \tag{5.5}$$

well-behaved and then impose conditions on the function f that ensure that the perturbation introduced is *small*. Typically, this smallness is described in terms of a bound on $\|f(n, y_n)\|$. It is then quite simple to prove results of this type through an appeal to Theorem 4.4. The approach is as follows:

Write the solution map $\Phi(n, y)$ in the form

$$\Phi(n, y_0) = \prod_{k=1}^{n} A(k)y_0 + \sum_{j=1}^{n}\prod_{k=j}^{n} A(k)f_j \tag{5.6}$$

where the f_j are values of the function $f(j, y_j)$.

By the hypotheses on the linear equation, we deduce properties of the Lipschitz constants for the products $\prod_{k=j}^{n} A(k)$. We combine this with knowledge of the behaviour of $\{f_j\}$ to prove that the required properties on $\Phi(n, y_0)$ hold.

Remark 1ex

1. Theorems 4.7.1, 4.7.2 and 4.7.3 from [15] can all be proved in this way.
2. Elsewhere in the literature one can find stability theorems for equations of the form

   $$y_{n+1} = y_n g(n, y_n) \tag{5.7}$$

   The stability of many such equations can be analysed by appealing to Theorem 4.4.
3. The principal Theorem (Theorem 1) of the recent paper [14] may be deduced from Theorem 4.4.

6 Some examples

In this section we consider some simple examples that illustrate different types of qualitative behaviour of solutions to simple nonlinear difference equations.

Example 6.1 Consider the first order equation:

$$y_{n+1} = y_n^2. \tag{6.1}$$

It is a simple matter to see that 0 and 1 are both steady state equilibrium solutions of (6.1). In a neighbourhood (a disk of radius $\frac{1}{2}$) of the equilibrium point 0 the Lipschitz constant is less than unity in magnitude. By Theorem 4.4 it follows that $y_n = 0$ is an asymptotically stable solution. Close to the equilibrium point 1 the Lipschitz constant is always larger than unity and one can infer that this equilibrium value is unstable. Indeed, for initial values $y_0 > 1$ equation (6.1) has unbounded solutions.

Example 6.2 As another very simple first order example we consider the equation

$$y_{n+1} = y_n^{\frac{1}{3}}. \tag{6.2}$$

Here we adopt the convention that the real cube root is always chosen. There are equilibrium values of -1, 0 and 1. There is no finite Lipschitz constant available in a region of the origin. However there are Lipschitz constants of magnitude less than unity in a small neighbourhood of each of the solutions $y_n = \pm 1$. It follows that the solutions $y_n = 1$ and $y_n = -1$ are each asymptotically stable. The solution $y_n = 0$ is unstable. In fact, one can see quite easily that any non-constant solution to (6.2) with initial value $|y_0| > \sqrt{\frac{1}{27}}$ is asymptotically stable and has limiting value $y_n \to \text{sign}(y_0)$. For this equation, all the solutions are bounded and there exist both stable and unstable equilibria.

Finally we turn to two examples that illustrate how our ideas may be applied to numerical methods for differential equations.

Example 6.3 For the ordinary differential equation

$$y'(t) = -y^{2\ell+1}(t), \quad y(0) = a, \quad \ell \in \mathbb{N} \tag{6.3}$$

it is easy to show that $y(t) = 0$ is an asymptotically stable equilibrium solution. When the simplest possible numerical method (the explicit Euler rule) is applied to (6.3) we obtain the difference scheme

$$y_{n+1} = y_n - hy_n^{2\ell+1}. \tag{6.4}$$

One can calculate a Lipschitz constant for the solution map of equation (6.4) which is bounded if $0 \le (2\ell+1)hy_0^{2\ell} \le 2$ and tends to zero if $0 < (2\ell+1)hy_0^{2\ell} < 2$. Thus, for fixed y_0 and sufficiently small $h > 0$ the zero solution to the discrete scheme is asymptotically stable.

Example 6.4 For the ordinary differential equation

$$y'(t) = -y^{2\ell}(t), \quad y(0) = a, \quad \ell \in \mathbb{N} \tag{6.5}$$

it is easy to show that $y(t) = 0$ is an unstable equilibrium solution (for $y(0) < 0, y(t) \not\to 0$). The explicit Euler rule applied to (6.5) we obtain the difference scheme

$$y_{n+1} = y_n - hy_n^{2\ell} \tag{6.6}$$

One can calculate a Lipschitz constant for the solution map of equation (6.6) which is bounded if $0 \le (2\ell)hy_0^{2\ell-1} \le 2$ and tends to zero if $0 < (2\ell+1)hy_0^{2\ell} < 2$. Thus, for fixed $y_0 > 0$ and sufficiently small $h > 0$ the solution tends to zero, but by choosing $y_0 < 0$ we can see that no bounded Lipschitz constant exists for the solution map and the zero solution to (6.6) is not asymptotically stable, regardless of the value of $h > 0$ chosen.

7 Further Work and Acknowledgements

The approach we describe here will be extended in our paper [9] to the treatment of difference equations of Volterra type.

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References

[1] R.P. Agarwal, *Difference equations and inequalities*, Marcel Dekker, New York, 1992.

[2] C.T.H. Baker and N.J. Ford, Some applications of the boundary locus method and the method of D-partitions, IMA J. Numer. Anal., 11, 1991, 143–158.

[3] F.M. Dannan and S. Elaydi, Lipschitz stability of nonlinear systems of differential equations, J. Math. Anal. Appl. 113, 1986, 562–577.

[4] F.M. Dannan and S. Elaydi, Lipschitz stability of nonlinear systems of differential equations. II Liapunov functions, J. Math. Anal. Appl. 143, 1989, 517–529.

[5] S. Elaydi, *An introduction to difference equations*, Springer, New York, 1996.

[6] S. Elaydi & A. Paterson, Stability of difference equations, *Differential equations and applications*, Vol I/II 235–238, Columbus, Ohio, 1989.

[7] J.T. Edwards, J.A. Roberts & N.J. Ford, The numerical simulation of an integrodifferential equation close to bifurcation points, Proceedings of IMACS World Congress 2000, Lausanne, 213-3.

[8] J. Entwistle, Existence of bounded solutions of linear and nonlinear difference equations, MSc Dissertation, Chester College, 1997.

[9] N.J. Ford & J.T. Edwards, Boundedness and stability of difference equations of Volterra type, Preprint.

[10] N. Guglielmi and M. Zennaro, On the asymptotic properties of a family of matricies, Preprint.

[11] N. Guglielmi and M. Zennaro, On the zero-stability of variable stepsize multistep methods: the spectral radius approach, Preprint.

[12] E. Hairer, S.P. Nørsett and G. Wanner, *Solving Ordinary Differential Equations I: Nonstiff Problems*, Springer, Berlin, 1993.

[13] A. Iserles, *A first course in the numerical analysis of differential equations*, Cambridge University Press, Cambridge, 1996.

[14] N. Kruse & T. Nesemann, Global asymptotic stability in some discrete dynamical systems, J. Math. Anal. Appl. 235, 1999, 151–158.

[15] V. Lakshmikantham and D. Trigiante, *Theory of difference equations: numerical methods and applications*, Academic Press, San Diego, 1988.

[16] J.D. Lambert, *Numerical methods for ordinary differential systems*, Wiley, Chichester, 1991.

[17] J.P. LaSalle, *The Stability of Dynamical Systems*, SIAM Publications, Philadelphia, 1976.

[18] J.P. LaSalle, *The stability and control of discrete processes*, Springer, Berlin, 1986.

[19] A.V. Lasunskii, Growth estimates for solutions to linear systems of difference equations in terms of coefficients, and their applications to stability problems, Differential Equations, 31.11, 1995, 1903–1904.

[20] J.W. Luo, Lipschitz stability of nonlinear systems of difference equations, Math. Theory Appl., 19, 1999, 85–87.

[21] F. Mazzia and D. Trigiante, The role of difference equations in numerical analysis, *Computers Math. Applic.* 28, 1994, 209–217.

[22] R. Medina, Stability and asymptotic behaviour of difference equations, JCAM, 80, 1997, 17–30.

[23] M.H. Moadab, Existence of bounded solutions of nonlinear difference equations, Libertas Mathematica, IX, 1989, 127–132.

[24] W. Rudin, *Principles of mathematical analysis*, McGraw Hill, 1964.

[25] A.M. Stuart & A.R. Humphries, *Dynamical systems and numerical analysis*, Cambridge University Press, Cambridge, 1996.