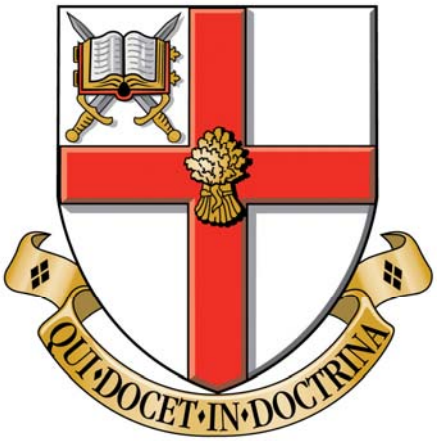

University of Chester

Author(s): John T Edwards; Neville J Ford; A Charles Simpson

Title: The numerical solution of linear multi-term fractional differential equations: Systems of equations

Date: 2002

Originally published in: Journal of Computational and Applied Mathematics

Example citation: Edwards, J. T., Ford, N. J., & Simpson, A. C. (2002). The numerical solution of linear multi-term fractional differential equations: Systems of equations. *Journal of Computational and Applied Mathematics*, 148(2), 401-418.

Version of item: Author's preprint

Available at: http://hdl.handle.net/10034/13242

The Numerical Solution of Linear Multi-term Fractional Differential Equations: Systems of Equations

John T. Edwards, Neville J. Ford [1] and A. Charles Simpson

Mathematics Department, Chester College, Parkgate Road, Chester, CH1 4BJ, UK.

John Edwards and Neville Ford are members of the Manchester Centre for Computational Mathematics.

Abstract

In this paper we show how the numerical approximation of the solution of a linear multi-term fractional differential equation can be calculated by reduction of the problem to a system of ordinary and fractional differential equations each of order at most unity. We begin by showing how our method applies to a simple class of problems and we give a convergence result. We solve the Bagley Torvik equation as an example. We show how the method can be applied to a general linear multi-term equation and give two further examples.

Keywords: Fractional differential equations, multi-term equations, numerical methods.

AMS subject classification: 65L05, 65L06

1 Introduction

The use of fractional differential and integral operators in mathematical models has become increasingly widespread in recent years. Several forms of fractional differential equations have been proposed in standard models, and there has been significant interest in developing numerical schemes for their solution. However, much of the work published to date has been concerned with linear *single term* equations and, of these, equations of order less than unity have been most often investigated. In this paper we focus on providing numerical solutions to linear *multi-term equations* where the highest order derivative may be greater than one. These problems arise, for instance, in the Basset equation [16] and the Bagley Torvik equation [20]. We are aware that one constraint on their more

[1] corresponding author email: njford@chester.ac.uk

widespread use in models is the lack of good quality numerical schemes for their solution and this matter has been partially addressed in the recent papers [7,9].

Our approach is to investigate how one can solve multi-term and high order linear fractional differential equations, of the form given by the equation

$$\sum_{i=0}^{n} b_i D^{\alpha_i} y(t) = g(t),\ b_i \in \mathbb{R},\ b_n \neq 0,\ \alpha_i \geq 0, \tag{1}$$

as a system of mixed fractional and ordinary differential equations of order ≤ 1. We will always assume, for convenience, that $i > j \Longrightarrow \alpha_i > \alpha_j$.

2 Background

Fractional Derivatives We follow the developing convention in applications of using the Caputo version of the fractional derivative in preference to the Riemann-Liouville version found in the current generation of texts.

Thus, in our work, D_*^q denotes the fractional differential operator of order $q \notin \mathbb{N}$ denoted and defined by (cf., e.g., [11, eq. (1.17)])

$$D_*^q y(t) := J^{m-q} y^{(m)}(t),$$

where m is the integer defined by the relation $m - 1 < q < m$ and J^μ is the fractional integral operator,

$$J^\mu g(t) := \frac{1}{\Gamma(\mu)} \int_0^t (t-u)^{\mu-1} g(u) du.$$

The reason for this preference is as follows: when we interpret the fractional differential operators in (1) as Caputo fractional derivatives then, with suitable conditions on the forcing function g and with the initial values $y^{(i)}(0) = y_0^i,\ i = 0, \ldots, m-1$ specified, the system has a unique solution. If we were to interpret the fractional differential operators as Riemann-Liouville fractional derivatives we would have to specify our initial conditions in terms of fractional integrals and their derivatives. The initial conditions required by the Caputo definition coincide with identifiable physical states, and this leads to the preference, amongst modellers, for the Caputo definition.[2] It also proves more convenient in our analysis to use the Caputo derivative.

The basic analytical results on existence and uniqueness of solutions to fractional differential equations are given in Samko et al [19] and Podlubny [18]. For equations defined in terms of the Caputo fractional differential operator further discussion of these matters is contained (for example) in the recent papers by Diethelm and Ford [6–8].

[2] For multiterm equations such as those we consider, the Riemann Liouville definition of the fractional derivative would require initial conditions corresponding to *each fractional order of derivative* that appears in the equations. The Caputo version merely requires us to specify the initial conditions for integer order derivatives.

Established Methods (Analytical and Numerical) In [15] Miller and Ross gave a method for calculating the analytic solution to a multi-term fractional differential equation of the form

$$[D^{n\nu} + a_1 D^{(n-1)\nu} + \cdots + a_{n-1} D^{\nu} + a_n D^0] y(t) = 0,\ \nu = \frac{1}{q},\ q \in \mathbb{N}. \tag{2}$$

They interpret the fractional operator $D^{i\nu}$ as D^{ν} applied i-times. Under this interpretation they describe the multi-term fractional differential equation as a sequential fractional differential equation. In [7,8] Diethelm and Ford showed that this intepretation arises naturally when using the Caputo fractional derivative, and thus developed a numerical algorithm. Given the density of the rational numbers any fractional differential equation can be approximated arbitrarily well by this method since the finite precision arithmetic of the computer means that all fractional derivatives must, in any case, be approximated to machine precision in any numerical scheme. This approach may result in a very large system of equations.

In [9] Luchko and Diethelm gave an algorithm for calculating the approximate numerical solution of multi-term linear fractional differential equations based on [13]. This method may require a large amount of computational effort to calculate its weights.

In this paper we give a numerical method for solving fractional differential equations based on treating the fractional differential equation as a system of equations. Our approach is to use a system where the equations do not all have the *same* order. This avoids the problems of order blow up (seen, for example, in [7,8]), and also some of the costs of computation of quadrature weights. We draw attention to the fact that solving fractional differential equations may, in general, be a very computationally intensive problem and we refer the interested reader to our recent paper [10] where this matter is discussed in rather greater detail.

3 Discretisation of Derivatives

There are of course many variants on how to discretise derivatives, both of integer and of fractional order, see for example Lambert [12] and Lubich [13]. To faciliate our exposition we have chosen to use simple discretisations so that the important points introduced here are not confused with other mathematical details. We will need to approximate derivatives of both integer and fractional order.

To solve a first order differential equation we will use the trapezium rule:

$$Dy = f \Rightarrow y_i = y_{i-1} + \frac{1}{2} h (f_i + f_{i-1}).$$

To discretise the fractional derivative we will use Diethelm's method:

$$D^\alpha y = \frac{1}{{}^\alpha\gamma_i}\left(\sum_{k=0}^{i} {}^\alpha\omega_{k,i} y_{i-k} + \frac{y_0}{\alpha}\right), \tag{3}$$

where ${}^\alpha\gamma_i = (ih)^\alpha\Gamma(-\alpha)$ and ${}^\alpha\omega_{k,0}\ \ldots, {}^\alpha\omega_{k,i}$ are convolution weights derived from the fact that the fractional operator is defined in terms of a convolution integral. For this method the weights are easily calculated (see Diethelm [3]).

It should be noted that D and D^α are operators and so their appearance in matrices is a notational device.

Systems of equations We recall that to solve the integer-order equation

$$\begin{aligned} &D^n y + \cdots + b_1 y = f, \\ &y^{(i)}(0) = y_0^{(i)},\ i = 0, \ldots, n-1, \end{aligned}$$

as a system of first order equations we let

$$\begin{aligned} {}^1Y &= y \\ {}^{i+1}Y &= D^i y,\ i = 1, \ldots, n. \end{aligned}$$

Expressed in matrix notation this gives

$$\begin{pmatrix} D & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & D \end{pmatrix} \begin{pmatrix} {}^1Y \\ \vdots \\ {}^nY \end{pmatrix} = \begin{pmatrix} {}^2Y \\ \vdots \\ -\sum_{i=1}^n b_i\, {}^iY + f \end{pmatrix}.$$

One may then proceed to discretise the derivative and solve the resulting matrix system to obtain an approximation to the solution y.

Suppose now that we wish to to discretise the equation

$$Dy + a_1 D^{\alpha_2} y + a_2 D^{\alpha_1} + a_3 y = f,\ 1 > \alpha_2 > \alpha_1 > 0, \tag{4}$$

then we can identify two natural alternative ways to write this (formally) as a system after setting ${}^1Y = y$. We could write

$$\begin{aligned} {}^2Y &= D^{\alpha_1}\, {}^1Y \\ {}^3Y &= D^{\alpha_2}\, {}^1Y \\ {}^4Y &= D\, {}^1Y \end{aligned}$$

or

$$
\begin{aligned}
{}^2Y &= D^{\alpha_1}\,{}^1Y \\
{}^3Y &= D^{\alpha_2-\alpha_1}\,{}^2Y \\
{}^4Y &= D^{1-\alpha_2}\,{}^3Y.
\end{aligned}
$$

We choose the former since it requires the weights for one fewer fractional derivative to be calculated (and therefore one fewer convolution sum per iteration) and has less propagation of rounding error. We consider the mathematical justification for this approach briefly in the next section (see also [7]).

4 Our numerical scheme in practice

Later we shall solve an arbitrary multi-term linear fractional differential equation but to make our method clear and to avoid complicated notation, for the moment we will consider a smaller class of problems. We consider equations (such as the Bagley-Torvik equation) where the highest order of derivative is an integer and where there is at most one non-integer order derivative between successive integer orders.

Consider, for example, the equation

$$
D^2y + b_3D^{\alpha_3}y + b_2Dy + b_1D^{\alpha_1}y + b_0y = f,\ \alpha_3 = 1+\beta, \alpha_1 = \alpha,\ \alpha,\beta \in (0,1),\ b_i \in \mathbb{R}. \tag{5}
$$

We write equation (5) as the system

$$
\begin{aligned}
{}^1Y &= y \\
{}^2Y &= D^{\alpha}y = D^{\alpha}\,{}^1Y \\
{}^3Y &= Dy = D^{1}\,{}^1Y \\
{}^4Y &= D^{\beta}\,Dy = D^{\beta}\,{}^3Y \\
{}^5Y &= D\,{}^2y = D^{1}\,{}^3Y
\end{aligned} \tag{6}
$$

which in matrix form is

$$
\begin{pmatrix} D^{\alpha} & 0 & 0 & 0 \\ D & 0 & 0 & 0 \\ 0 & 0 & D^{\beta} & 0 \\ 0 & 0 & D & 0 \end{pmatrix}
\begin{pmatrix} {}^1Y \\ {}^2Y \\ {}^3Y \\ {}^4Y \end{pmatrix}
=
\begin{pmatrix} {}^2Y \\ {}^3Y \\ {}^4Y \\ f - \sum_{j=1}^{4} b_{j-1}\,{}^jY \end{pmatrix}.
$$

In the paper [7] a formal proof is given that equation (5) is equivalent to (6) in the sense that a function y satisfies (5) if and only if it is the first component of a vector solution of (6).

Consider the equation $D^{\alpha\,1}Y = {}^2Y$. We set our approximations ${}^iY_j \approx {}^iY(jh)$ for some fixed $h > 0$ and we discretise this first component as

$$\frac{1}{{}^\alpha\gamma_i}(\sum_{k=0}^{i} {}^\alpha\omega_{k,i}\,{}^1Y_{i-k} + \frac{{}^1Y_0}{\alpha}) = {}^2Y_i, \tag{7}$$

and put

$${}^1S_i = \sum_{k=1}^{i} {}^\alpha\omega_{k,i}\,{}^1Y_{i-k} + \frac{{}^1Y_0}{\alpha}, \tag{8}$$

so we have

$${}^\alpha\omega_{0,i}\,{}^1Y_i + {}^1S_i = {}^\alpha\gamma_i\,{}^2Y_i, \tag{9}$$

and it follows that

$$-{}^\alpha\omega_{0,i}\,{}^1Y_i + {}^\alpha\gamma_i\,{}^2Y_i = {}^1S_i. \tag{10}$$

We proceed similarly to obtain the third row in the discrete scheme. Now for the final row, we put

$${}^3Y_i = {}^3Y_{i-1} + \frac{h}{2}(f_i + f_{i-1} - \sum_{j=1}^{4} b_{j-1}({}^jY_i + {}^jY_{i-1})) \tag{11}$$

so with $F_i = (f_i + f_{i-1})h/2$ and

$${}^4S_i = {}^3Y_{i-1} + F_i - \frac{h}{2}\sum_{j=1}^{4} b_{j-1}\,{}^jY_{i-1}, \tag{12}$$

we have

$$\frac{h}{2}b_0\,{}^1Y_i + \frac{h}{2}b_1\,{}^2Y_i + (1 + \frac{h}{2}b_2)\,{}^3Y_i + \frac{h}{2}b_3\,{}^4Y_i = {}^4S_i. \tag{13}$$

This leads to the matrix form:

$$\begin{pmatrix} -{}^\alpha\omega_{0,i} & {}^\alpha\gamma_i & 0 & 0 \\ 1 & 0 & -\frac{h}{2} & 0 \\ 0 & 0 & -{}^\beta\omega_{o,i} & {}^\beta\gamma_i \\ \frac{hb_0}{2} & \frac{hb_1}{2} & 1 + \frac{hb_2}{2} & \frac{hb_3}{2} \end{pmatrix} \begin{pmatrix} {}^1Y_i \\ {}^2Y_i \\ {}^3Y_i \\ {}^4Y_i \end{pmatrix} = \begin{pmatrix} {}^1S_i \\ {}^2S_i \\ {}^3S_i \\ {}^4S_i \end{pmatrix}, \tag{14}$$

where

$$
\begin{aligned}
{}^1S_i &= \sum_{k=1}^{i} {}^{\alpha}\omega_{k,i}\ {}^1Y_{i-k} + {}^1Y_0/\alpha \\
{}^2S_i &= {}^1Y_{i-1} + \frac{h}{2}\,{}^3Y_{i-1}, \\
{}^3S_i &= \sum_{k=1}^{i} {}^{\beta}\omega_{k,i}\ {}^3Y_{i-k} + {}^3Y_0/\beta, \\
{}^4S_i &= {}^3Y_{i-1} + F_i - \frac{h}{2}\sum_{j=1}^{4} b_{j-1}\,{}^jY_{i-1}.
\end{aligned}
\tag{15}
$$

and $F_i = h(f_i + f_{i-1})/2$, $f_i = f(ih)$.

It is quite straightforward to show:

Theorem 4.1 *The order of convergence of the resulting method is equal to the lowest order of the methods that it comprises.*

Proof: First we remark that each of the matrices on the left hand side of the system (14) has determinant that is $\mathcal{O}(1)$ as $h \to 0$. It follows that the entries in the inverse matrices are also $\mathcal{O}(1)$ as $h \to 0$.

Now assume that, for each i, j the values ${}^i\epsilon_j$ are defined by ${}^iY(jh) = {}^iY_j + {}^i\epsilon_j$. Since we start with the exact initial conditions, it follows that each value ${}^i\epsilon_0 = 0$.

By linearity of the numerical method in (14) and since the causal sums in (15) are bounded (see [3], p5) it follows that each ${}^iY_{j+1}$ satisfies

$$
{}^iY_{j+1} = {}^iY((j+1)h) + \mathcal{O}(h^{s_i}) + L\max_{i=1,\ldots,4,k=1,\ldots,j}|{}^i\epsilon_k| \tag{16}
$$

Here h^{s_i} is the order of the numerical method applied in row i of the scheme (14) when the right hand side of (15) is evaluated using *exact* function values ${}^iY(kh)$. The final term in (16) takes account of the cumulative effect of the errors ${}^i\epsilon_k, i = 1, \ldots, 4, k = 1, \ldots, j$. The (finite) constant L is determined by taking the largest element of the inverses of the matrices in (14) and multiplying by a bound on the coefficients on the right hand side of (15).

Now we can complete the proof by induction. As we remarked earlier, the values ${}^i\epsilon_0$ are zero. Therefore the values ${}^i\epsilon_1$ given by (16) are $\mathcal{O}(h_i^s)$. Now let $s = \min(s_1, s_2, s_3, s_4)$ and it is easy to see that for $j \geq 2$ the last term in (16) dominates and ${}^i\epsilon_j = \mathcal{O}(h^s)$. □

To apply the method in practice we need initial conditions for equation (6) which involves giving initial conditions for each of the components ${}^iY$. As described in the recent paper [7], since we are using the Caputo definition for the fractional derivative, one simply puts the non-integer order initial conditions as 0 and uses the integer order initial conditions as given in the original problem.

We have calculated the solutions to the equations:

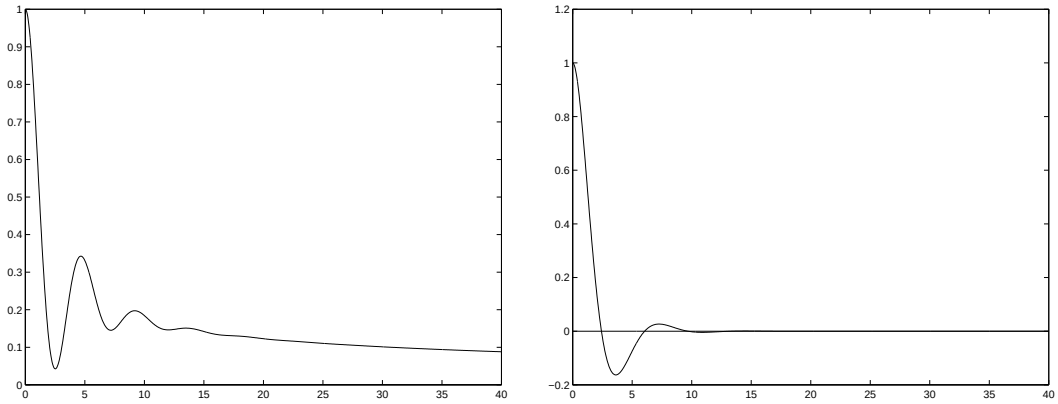

Fig. 1. $D^2y + D^{0.5}y + y = 0, D^2y + Dy + y = 0, y_0 = 1, y_0' = 0$

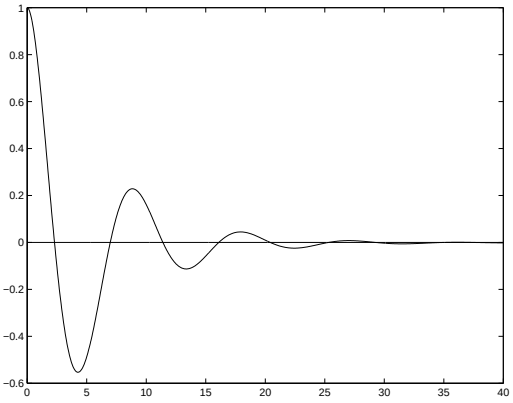

Fig. 2. $D^2y + D^{1.5}y + y = 0, y_0 = 1, y_0' = 0$

$$D^2y + D^{0.5}y + y = 0, \tag{17}$$

$$D^2y + Dy + y = 0, \tag{18}$$

$$D^2y + D^{1.5}y + y = 0, \tag{19}$$

subject to the initial conditions $y(0) = 1$, $y'(0) = 0$. In these and later examples we use step size $h = 0.01$. We show the approximate solutions in Figures 1 and 2.

The figures are consistent with those we have seen elsewhere depicting fractional generalisations of velocity damping.

5 Solution of the Bagley Torvik equation

As an example that arises in applications, we solve the Bagley Torvik equation which arises, for instance, in modelling the motion of a rigid plate immersed in a Newtonian fluid. It is a particularly interesting example for us to consider because it was solved in the paper [7].

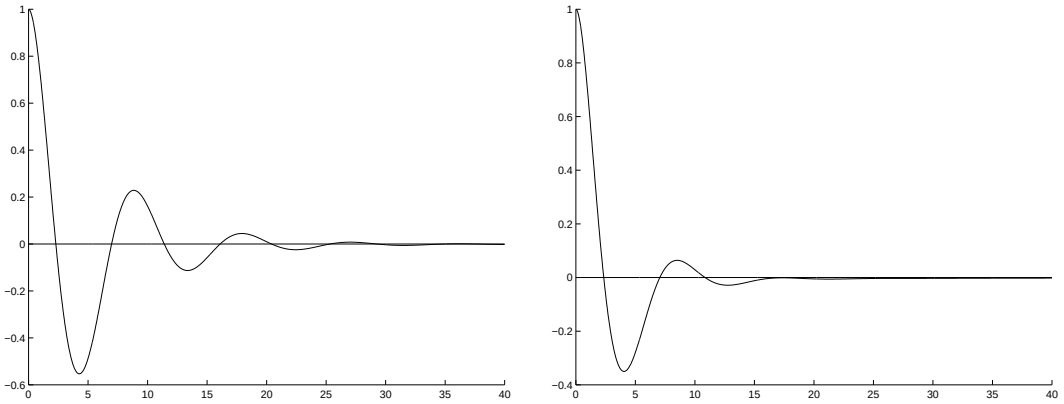

Fig. 3. $\alpha = 0.5$, $\alpha = 0.25, y_0 = 1, y_0' = 0$

$$D^2 y + b_2 D^{\alpha_1} y + b_1 y = f, \tag{20}$$
$$y(0) = y_0,\ y'(0) = y_0',\ \alpha_1 = 1 + \alpha, \alpha \in (0,1).$$

D and D^α are linear operators so, as before, we can rewrite this as

$$\begin{aligned} {}^1Y &= y, \qquad {}^2Y = Dy, \\ {}^3Y &= D^\alpha Dy,\ {}^4Y = D^2 y, \end{aligned}$$

which written in matrix notation is

$$\begin{pmatrix} D & 0 & 0 \\ 0 & D^\alpha & 0 \\ 0 & D & 0 \end{pmatrix} \begin{pmatrix} {}^1Y \\ {}^2Y \\ {}^3Y \end{pmatrix} = \begin{pmatrix} {}^2Y \\ {}^3Y \\ f - b_2\,{}^3Y - b_1\,{}^1Y \end{pmatrix}.$$

Discretising as above this results in the matrix system

$$\begin{pmatrix} 1 & -\frac{h}{2} & 0 \\ 0 & -\omega_{0,i} & \gamma_i \\ \tau_1 & 1 & \tau_2 \end{pmatrix} \begin{pmatrix} {}^1Y_i \\ {}^2Y_i \\ {}^3Y_i \end{pmatrix} = \begin{pmatrix} {}^1Y_{i-1} + \frac{h}{2}\,{}^2Y_{i-1} \\ {}^2S_i \\ F_i - \tau_1\,{}^1Y_{i-1} + {}^2Y_{i-1} - \tau_2\,{}^3Y_{i-1} \end{pmatrix}.$$

where $F_i = h(f_i + f_{i-1})/2$, $\tau_1 = hb_1/2$, and $\tau_2 = hb_2/2$.

We have calculated the solutions in Figures 3 and 4 to the example Bagley Torvik equation:

$$D^2 y + D^{1+\alpha} y + y = 0,\ y(0) = 1,\ y'(0) = 0 \tag{21}$$

for $\alpha = 0.5, 0.25, 0.75$ respectively. Notice that, when $\alpha = 0.5$, (19) coincides with (21).

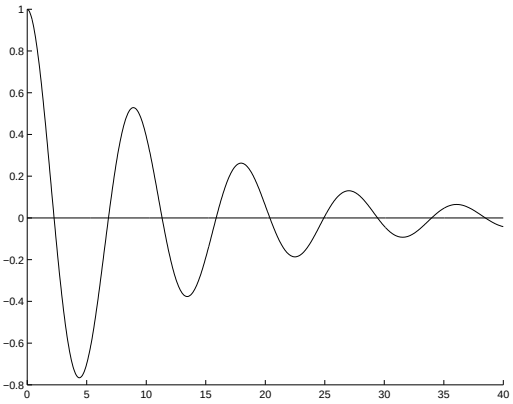

Fig. 4. $\alpha = 0.75, y_0 = 1, y_0' = 0$

6 Numerical solution of general multi-term linear equations

For a general linear multi-term fractional equation we consider

$$\sum_{s=0}^{p} c_s D^{\beta_s} y = f, \tag{22}$$

where $0 \leq \beta_0 < \beta_r < \beta_{r+1} < \beta_p, c_p = 1, c_s \in \mathbb{R}$. Note that the highest order β_p need not be an integer. However we need to regard every whole number derivative as being present in our equation, (if necessary with coefficient zero) so as to use all the initial conditions. To emphasise how such an equation needs to be solved we suggest that one might use a notation that emphasises this: we collect all the orders within each interval $(j, j+1], j \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ and so we shall use the form

$$b_{0,0} y + \sum_{j=0}^{m} \sum_{r=1}^{n_j} b_{r,j} D^{j+\alpha_{r,j}} y = f, \tag{23}$$

where, for $j = 1, 2, \ldots, m-1$, we have $0 < \alpha_{1,j} < \alpha_{2,j} < \ldots < \alpha_{n_j,j} = 1$ and $0 < \alpha_{1.m} < \alpha_{2,m} < \ldots, < \alpha_{n_m,m} \leq 1$ and $b_{i,j} \in \mathbb{R}$. Also, $n_0 + \cdots + n_m = p$ and we let $p_k = \sum_{j=0}^{k-1} n_j$.

We write equation (23) as the system

$$\begin{aligned}
{}^{1}Y &= y \\
{}^{2}Y &= D^{\alpha_{1,0}} y \\
&\vdots \\
{}^{1+n_0}Y &= Dy \\
{}^{2+n_0}Y &= D^{\alpha_{1,1}} Dy \\
&\vdots \\
{}^{p+1}Y &= D^{\alpha_{n_m,m}} D^{m-1} y
\end{aligned}$$

which in matrix form is

$$\mathcal{D}Y = \begin{pmatrix} \mathcal{D}_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \mathcal{D}_m \end{pmatrix} \begin{pmatrix} {}^1Y \\ \vdots \\ {}^{1+n_0}Y \\ \vdots \\ {}^pY \end{pmatrix} = \begin{pmatrix} {}^2Y \\ \vdots \\ {}^{2+n_0}Y \\ \vdots \\ f - \sum_{k=0}^{p-1} c_k \, {}^kY \end{pmatrix}$$

where for $k = 1, \dots, m-1$, $\mathcal{D}_k$ is an n_k by n_k matrix of differential operators of the form

$$\begin{pmatrix} D^{\alpha_{1,k}} & 0 & \ddots & 0 \\ D^{\alpha_{2,k}} & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ D & 0 & \dots & 0 \end{pmatrix}$$

The corresponding expression holds for $k = m$ if $\alpha_{n_m,m} = 1$, which we shall call case 1. If $\alpha_{n_m,m} \neq 1$ which we shall call case 2, we have

$$\mathcal{D}_m = \begin{pmatrix} D^{\alpha_{1,m}} & 0 & \ddots & 0 \\ D^{\alpha_{2,m}} & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ D^{\alpha_{n_m,m}} & 0 & \dots & 0 \end{pmatrix}$$

We discretise the derivatives in $\mathcal{D}$ to produce $\bar{\mathcal{D}}$. The matrix $\bar{\mathcal{D}}$ will consist of square matrix blocks $\bar{\mathcal{D}}_k$ along the diagonal which are the discrete analogues of $\mathcal{D}_k$ for $k = 1, \dots, m-1$. The bottom horizontal band of the matrix consists of the matrices $\bar{\mathcal{D}}_{m,1}, \dots, \bar{\mathcal{D}}_{m,m}$.

This means we will write the discretised system as

$$\bar{\mathcal{D}}Y_i = S_i \tag{24}$$

where

$$\bar{\mathcal{D}} = \begin{pmatrix} \bar{\mathcal{D}}_1 & 0 & \dots & \dots & \dots & 0 \\ 0 & \bar{\mathcal{D}}_2 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 & \bar{\mathcal{D}}_{m-1} & 0 \\ \bar{\mathcal{D}}_{m,1} & \dots & \dots & \dots & \dots & \bar{\mathcal{D}}_{m,m} \end{pmatrix} \tag{25}$$

and

$$\bar{\mathcal{D}}_k = \begin{pmatrix} -^{1,k}\omega_{0,i} & {}^{1,k}\gamma_i & 0 & \dots & \dots & 0 \\ -^{2,k}\omega_{0,i} & 0 & {}^{2,k}\gamma_i & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -^{n_k-1,k}\omega_{0,i} & 0 & \ddots & 0 & {}^{n_k-1,k}\gamma_i & 0 \\ 1 & 0 & \dots & \dots & 0 & -\frac{h}{2} \end{pmatrix}$$

and consequently by implication we also have

$$^k\bar{Y}_i = \begin{pmatrix} {}^{1+p_k}Y_i \\ \vdots \\ \vdots \\ {}^{p_{k+1}}Y_i \end{pmatrix}$$

and

$$^k\bar{S}_i = \begin{pmatrix} {}^{1+p_k}S_i \\ \vdots \\ \vdots \\ {}^{p_{k+1}}S_i \end{pmatrix},$$

where

$$
\begin{aligned}
{}^{j+p_k}S_i &= \sum_{r=0}^{i-1} {}^{j+p_k}\omega_{i-r,j+p_k\beta-(k-1)}\ {}^{1+p_k}Y_r + \frac{{}^{1+p_k}Y_0}{{}^{j+p_k}\beta}, \quad j = 1,\ldots,n_k-1 \\
{}^{p_{k+1}}S_i &= {}^{1+p_k}Y_{i-1} + \frac{h}{2}\,{}^{p_{k+1}}Y_{i-1}.
\end{aligned}
$$

Discretising for $k = m$ and collecting terms together gives the first $n_m - 1$ rows of $\bar{\mathcal{D}}_{m,m}$ as

$$
\begin{pmatrix}
-{}^{1,m}\omega_{0,i} & {}^{1,m}\gamma_i & 0 & \ldots & 0 \\
-{}^{2,m}\omega_{0,i} & 0 & {}^{2,m}\gamma_i & 0 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
-{}^{n_m-1,m}\omega_{0,i} & 0 & \ldots & {}^{n_m-1,m}\gamma_i & 0
\end{pmatrix}.
$$

For the last row, analytically, we have (for $\alpha_{n_m,m} \neq 1$)

$$
D^{\alpha_{n_m,m}}\,{}^{1+p_m}Y_i = f - \sum_{s=1}^{p} c_{s-1}\,{}^{s}Y_i \tag{26}
$$

which on discretisation becomes

$$
\frac{1}{{}^{n_m,m}\gamma_i}\Big(\sum_{k=0}^{i} {}^{n_m,m}\omega_{i,k}\,{}^{1+p_m}Y_{i-k} + \frac{{}^{1+p_m}Y_0}{\alpha_{n_m,m}}\Big) = f_i - \sum_{s=1}^{p} c_{s-1}\,{}^{s}Y_i. \tag{27}
$$

Rearranging this gives

$$
-{}^{n_m,m}\omega_{i,0}\,{}^{1+p_m}Y_i + {}^{n_m,m}\gamma_i \sum_{s=1}^{p} c_{s-1}\,{}^{s}Y_i = {}^{n_m,m}S_i, \tag{28}
$$

where

$$
{}^{n_m,m}S_i = {}^{n_m,m}\gamma_i f_i - \sum_{k=0}^{i-1} {}^{n_m,m}\omega_{i,k}\,{}^{1+p_m}Y_{i-k} - \frac{{}^{1+p_m}Y_0}{\alpha_{n_m,m}}. \tag{29}
$$

Then for $k = 1,\ldots,m-1$, $\bar{\mathcal{D}}_{m,k}$ has all zero entries for the first $n_m - 1$ rows and the last row is given by, for $\bar{\mathcal{D}}_{m,1}$

$$
({}^{n_m,m}\gamma_i c_0 \ \ldots \ {}^{n_m,m}\gamma_i c_{n_0}), \tag{30}
$$

and for $\bar{\mathcal{D}}_{m,k}$ (for $k = 2, \ldots, m-1$)

$$({}^{n_m,m}\gamma_i c_{p_k+1} \ \ldots \ {}^{n_m,m}\gamma_i c_{p_{k+1}}), \tag{31}$$

and the last row of $\bar{\mathcal{D}}_{m,m}$ is given by

$$({}^{n_m,m}\gamma_i c_{p_m+1} - {}^{n_m,m}\omega_{i,0} \ \ldots \ {}^{n_m,m}\gamma_i c_p). \tag{32}$$

In the case where $\alpha_{n_m,m} = 1$ we have

$$D^{\,1+p_m} Y = f - \sum_{s=1}^{p} c_{s-1}\, {}^{s}Y, \tag{33}$$

which on discretisation yields

$${}^{1+p_m}Y_i = {}^{1+p_m}Y_{i-1} + \frac{h}{2}(f_i + f_{i-1} - \sum_{s=1}^{p} c_{s-1}\, {}^{s}(Y_i + Y_{i-1})) \tag{34}$$

and on rearranging gives

$${}^{1+p_m}Y_i + \frac{h}{2}\sum_{s=1}^{p} c_{s-1}\, {}^{s}Y_i, = {}^{n_m,m}S_i. \tag{35}$$

where

$${}^{n_m,m}S_i = {}^{1+p_m}Y_{i-1} + F_i - \frac{h}{2}\sum_{s=1}^{p} c_{s-1}\, {}^{s}Y_{i-1}. \tag{36}$$

Then for $k = 1, \ldots, m-1$ $\bar{\mathcal{D}}_{m,k}$ has all zero entries for the first $n_m - 1$ rows and the last row is given by, for $\bar{\mathcal{D}}_{m,1}$

$$(\tfrac{h}{2}c_0 \ \ldots \ \tfrac{h}{2}c_{n_0}) \tag{37}$$

and for $\bar{\mathcal{D}}_{m,k}$ (for $k = 2, \ldots, m-1$)

$$(\tfrac{h}{2}c_{p_k+1} \ \ldots \ \tfrac{h}{2}c_{p_{k+1}}) \tag{38}$$

and the last row of $\bar{\mathcal{D}}_{m,m}$ is given by

$$(1 + \tfrac{h}{2}c_{p_m+1} \ \tfrac{h}{2}c_{p_m+2} \ldots \ \tfrac{h}{2}c_p). \tag{39}$$

In the next two sections we consider the solution of two particular types of example. We considered before the Bagley Torvik equation. Here we had a highest order of derivative

equal to 2 (an integer) and fractional derivative greater than unity. The fractional oscillation equation has highest order of derivative $1+\alpha$ which is non-integral. The Basset equation has highest order derivative unity and a non-integer derivative of lower order. Thus each of these equations represents a distinctive class of problems.

7 Solution of the Fractional Oscillation Equation

By way of example, we apply the above techniques to the fractional oscillation equation

$$D^{1+\alpha}y(t) + by(t) = f(t),\ y(0) = y_0,\ y'(0) = y_0', \tag{40}$$

where $\alpha \in (0,1)$, $t \in [0,\infty)$, and f satisfies a Lipschitz condition in t.

We can represent (40), since D and D^α are linear operators, as the system

$$\begin{aligned} {}^1Y &= y \\ {}^2Y &= Dy \\ {}^3Y &= D^\alpha Dy \end{aligned}$$

which in matrix form is

$$\begin{pmatrix} D & 0 \\ 0 & D^\alpha \end{pmatrix} \begin{pmatrix} {}^1Y \\ {}^2Y \end{pmatrix} = \begin{pmatrix} {}^2Y \\ -b\,{}^1Y + f(t) \end{pmatrix}.$$

We then discretise the derivatives by the schemes previously described. For the first equation, using the trapezium method, we have

$${}^1Y_i = {}^1Y_{i-1} + ({}^2Y_i + {}^2Y_{i-1})h/2.$$

For the second equation by Diethelm's method we have

$$-b\,{}^1Y_i + f_i = \frac{1}{\gamma_i}(\omega_{0,i}\,{}^2Y_i + {}^2S_i),$$

where ${}^2S_i = \sum_{k=1}^{i} \omega_{k,i}\,{}^2Y_{i-k} + {}^2Y_0/\alpha$.

Rearranging gives

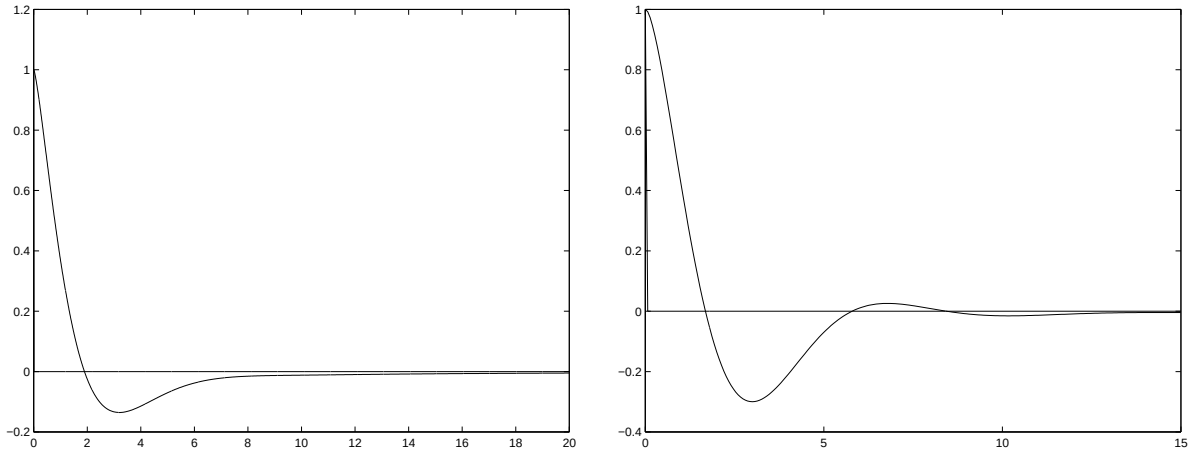

Fig. 5. $D^{1.3}y = -y,\ y_0 = 1, D^{1.5}y = -y,\ y_0 = 1, y_0' = 0$

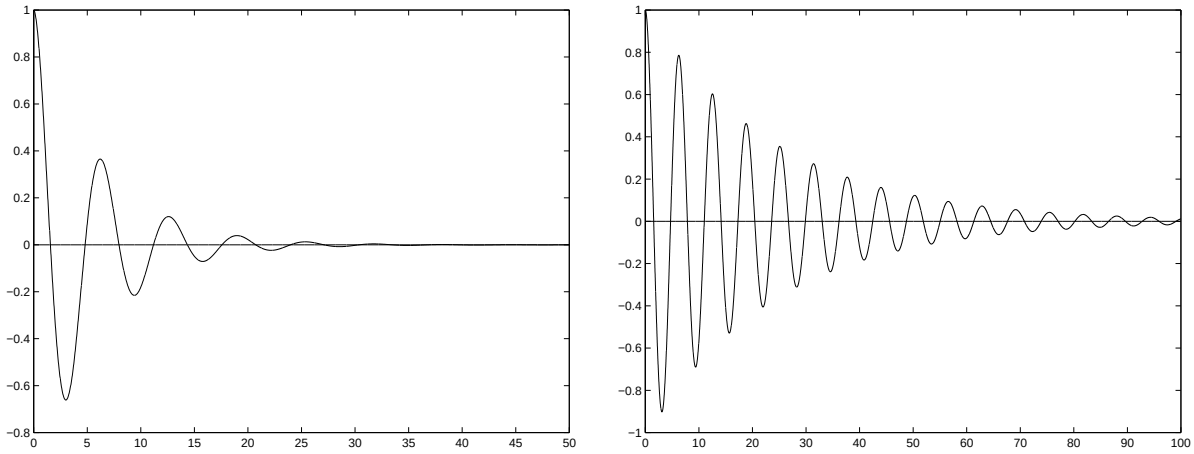

Fig. 6. $D^{1.8}y = -y,\ y_0 = 1, y_0' = 0, D^{1.95}y = -y,\ y_0 = 1, y_0' = 0$

$$\begin{pmatrix} 1 & -\frac{h}{2} \\ \gamma_i b\ \omega_{0,i} & \end{pmatrix} \begin{pmatrix} {}^1Y_i \\ {}^2Y_i \end{pmatrix} = \begin{pmatrix} {}^1Y_{i-1} + \frac{h\,{}^2Y_{i-1}}{2} \\ \gamma_i f_i - {}^2S_i \end{pmatrix}.$$

Remark 7.1 *By exactly the same argument as before, we can deduce that the order of our method is the lowest order* $(2-\alpha)$ *of the numerical methods which are used.*

We give diagrams of approximate solutions for some example fractional oscillation equations in Figures 5 and 6 (with $h = 0.1$). These equations were previously studied by Blank in [1] and our figures show that the method we have used gives solutions consistent with those found by collocation techniques.

8 Solution of the Basset Equation

Consider the linear fractional differential equation (41) which arises, for instance, in the study of the generalised Basset force occuring when a sphere sinks in a (relatively less dense) viscous fluid (see Mainardi [16]),

$$\begin{aligned} &Dy + b_2 D^\alpha y + b_1 y = f, \\ &\quad y(0) = y_0,\ \alpha \in (0,1). \end{aligned} \tag{41}$$

We can rewrite this as

$$\begin{aligned} {}^1Y &= y \\ {}^2Y &= D^\alpha y \\ {}^3Y &= Dy \end{aligned}$$

which expressed in matrix notation gives

$$\begin{pmatrix} D^\alpha & 0 \\ D & 0 \end{pmatrix} \begin{pmatrix} {}^1Y \\ {}^2Y \end{pmatrix} = \begin{pmatrix} {}^2Y \\ f - b_2\,{}^2Y - b_1\,{}^1Y \end{pmatrix}.$$

On discretisation this results in the system

$$\begin{pmatrix} -\omega_{0,i} & \gamma_i \\ \tau_1 & \tau_3 \end{pmatrix} \begin{pmatrix} {}^1Y_i \\ {}^2Y_i \end{pmatrix} = \begin{pmatrix} {}^1S_i \\ F_i + \tau_2\,{}^1Y_{i-1} - \tau_3\,{}^2Y_{i-1} \end{pmatrix},$$

where $F_i = \frac{h}{2}(f_i + f_{i-1})$, $\tau_1 = (1 + \frac{hb_1}{2})$, $\tau_2 = (1 - \frac{hb_1}{2})$, $\tau_3 = (\frac{hb_2}{2})$, and ${}^1S_i = \sum_{k=1}^{i} \omega_{k,i}\,{}^1Y_{i-k} + {}^1Y_0/\alpha$.

Using the formulation of the problem given by Mainardi in [16] we let $\beta = 9/(1+2\chi)$, and $\chi = 0.5,\ 2,\ 10,\ 100$. Our calculated approximate solutions (with $h = 0.1$) to the Basset equation are given in Figures 7 and 8.

As before, we have analysed the convergence of the method, which equals that of the lowest order method $(2 - \alpha)$ employed here. We have compared our plots with those given by Mainardi [16] (obtained by an unspecified method) and there is a strong level of agreement in our solutions.

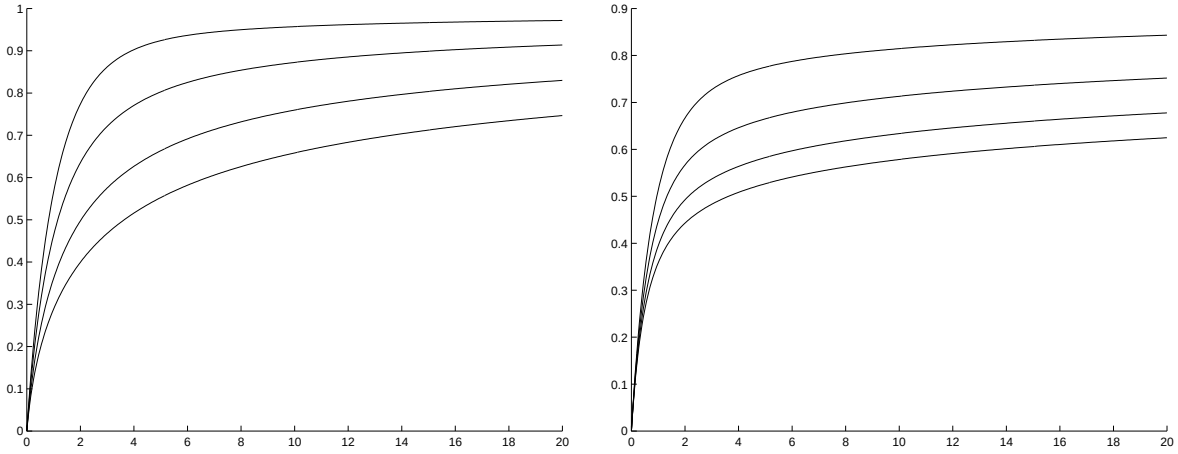

Fig. 7. $\alpha = 0.5, \alpha = 0.25, y_0 = 0$

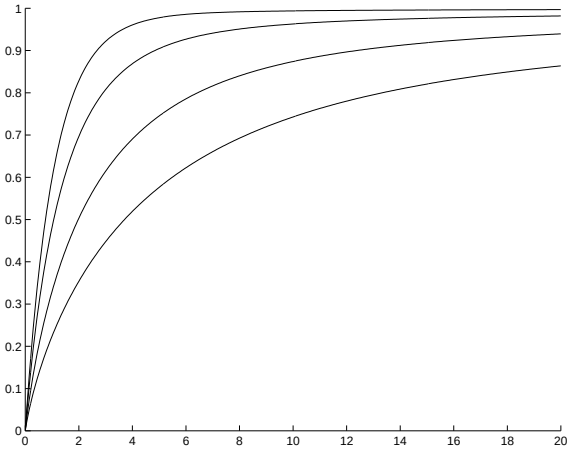

Fig. 8. $\alpha = 0.75, y_0 = 0$

9 Conclusion

We have shown that the use of systems of fractional differential equations of different degrees can provide a convergent method for the solution of multi-term equations. Our method can lead to numerical schemes of arbitrarily high order: we can apply, for example, high order fractional linear multistep methods, subject to the overhead of calculating the starting weights. We could also apply collocation methods successfully (see [1]). However the key feature of the approach we have introduced in this paper is the fact that the computational complexity of the calculation is kept under control. Our approach avoids the construction of excessively large systems of equations which can be needed in the methods described in [7,8].

References

[1] L.Blank, *Numerical treatment of differential equations of fractional order*, Numerical Analysis Report 287. Manchester Centre for Numerical Computational Mathematics, 1996.

[2] M.Caputo, *Linear models of dissipation whose Q is almost frequency independent II*, Geophys. J. Royal Astronomical Society, 13, 529–539, 1967.

[3] K.Diethelm, *An algorithm for the numerical solution of differential equations of fractional order*, Electronic Transactions on Numerical Analysis, Kent State University, 5, 1-6, 1997.

[4] K.Diethelm, *Numerical approximation of finite-part integrals with generalised compound quadrature formulae*, IMA J. Numer. Anal., 17, 479–493, 1997.

[5] K. Diethelm and A. Freed, *On the solution of nonlinear fractional order differential equations used in the modelling of viscoplasticity*, Scientific Computing in Chemical Engineering II — Computational Fluid Dynamics, Reaction Engineering, and Molecular Properties, F. Keil, W. Mackens, H. Voß and J. Werther (eds), 217–224, Springer, Heidelberg, 1999.

[6] K.Diethelm and N.J.Ford, *Analysis of fractional differential equations*, J. Math. Anal. Appl., 265, 229–248, 2002.

[7] K.Diethelm and N.J.Ford, *Numerical solution of the Bagley-Torvik equation*, BIT to appear.

[8] K.Diethelm and N.J.Ford, *The numerical solution of linear and nonlinear fractional differential equations involving fractional derivatives of several orders*, to appear.

[9] K. Diethelm and Y. Luchko, *Numerical solution of linear multi-term differential equations of fractional order*, J Comput. Anal. Appl., to appear.

[10] N.J. Ford and A.C. Simpson *The numerical solution of fractional differential equations: speed versus accuracy* Numerical Algorithms, 26.4, 333–346, 2001.

[11] R. Gorenflo and F. Mainardi, *Fractional Calculus: Integral and Differential Equations of Fractional Order*, in A. Carpinteri and F. Mainardi (Eds), Fractals and Fractional Calculus in Continuum Mechanics, Springer, 223–276, 1997, Wien.

[12] J.D.Lambert, *Numerical methods for ordinary differential systems*, Wiley, 1991.

[13] C.Lubich, *Discretized fractional calculus*, SIAM Journal of Mathematical Analysis, Volume 17, No. 3, 704–719, 1986.

[14] C.Lubich, *Convolution quadrature and discretized operational calculus. I*, Numerische Mathematik, 52, 129–145, 1988.

[15] K.S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, 209–217, John Wiley, 1993.

[16] F. Mainardi, *Some basic problems in continuum and statistical mechanics*, in A. Carpinteri and F. Mainardi (Eds), Fractals and Fractional Calculus in Continuum Mechanics, Springer, 291–348, 1997, Wien.

[17] K.B.Oldham and J.Spanier, *The fractional calculus*, Academic Press, 1974.

[18] I.Podlubny, *Fractional differential equations*, Academic Press, 1999.

[19] S.G.Samko, A.A.Kilbas, O.I. Marichev, *Fractional integrals and derivatives*, Gordon and Breach Science Publishers, 1993.

[20] P.J.Torvik and R.L.Bagley, *On the appearance of the fractional derivative in the behavior of real materials*, J. Appl. Mech., 51, 294–298, 1984.