Towards Universal Logic: Gaggle Logics

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Guillaume Aucher
Univ Rennes, CNRS, IRISA
263, Avenue du Général Leclerc, 35042 Rennes Cedex, France
guillaume.aucher@irisa.fr

Abstract

A class of non–classical logics called gaggle logics is introduced, based on a Kripke–style relational semantics and inspired by Dunn's gaggle theory. These logics deal with connectives of arbitrary arity and we show that they capture a wide range of non–classical logics. In particular, we list the 96 binary connectives and 16 unary connectives of basic gaggle logic and relate their truth conditions to the non-classical logics of the literature. We establish connections between gaggle theory and group theory. We show that Dunn's abstract law of residuation corresponds to an action of transpositions of the symmetric group on the set of connectives of gaggle logics and that Dunn's families of connectives are orbits of the same action. Other operations on connectives, such as dual and Boolean negation, are also reformulated in terms of actions of groups and their combination is defined by means of free groups and free products. We show how notions of groups arise naturally from our gaggle logics and how gaggle logics can be canonically defined from given groups. Our other main contribution deals with the proof theory of gaggle logics. We show how sound and complete calculi can be systematically computed from any basic gaggle logic with or without Boolean connectives. These calculi are display calculi and we prove that the cut rule can be systematically eliminated from proofs. This allows us to prove that basic gaggle logics are decidable.

Keywords: substructural logics, residuation, gaggle theory, display calculus, group theory, action of group, free group and free product.

1 Introduction

A wide variety of non–classical logics have been introduced over the past decades, such as relevant logics, linear logics and Lambek calculi, to name just a few. On the one hand, this diversity is an asset since each logic has an interest for a specific purpose, and one can select, and resort to, some of them for reasoning about a given applicative issue [38]. In fact, many of these non–classical logics have been developed for solving concrete problems in computer science: for example, dynamic logics [24], Hoare and separation logics [25, 43] for reasoning about computer programs, and description logics [3] for formalizing ontologies of the semantic web. Acknowledging and dealing with this plurality and diversity of logics is in a sense at the origin of the development of a philosophical stance in logic called "logical pluralism" [5]. On the other hand, and from a theoretical point of view, this plurality can be felt as problematic because it threatens the unity and the unifying power of logic. Indeed, all logics already have in common the same terminology and notions, such as truth, validity, conservativity and interpolation, and this is also an asset. Nevertheless, one can argue that non–classical logics are still disorganized and scattered and somehow miss a common formal ground. As Gabbay summarised the state of play (vis-à-vis non-monotonic logics) in the early 1980s, "we have had a multitude of systems generally accepted as 'logics' without a unifying underlying theory and many had semantics without proof theory. Many had proof theory without semantics, though almost all of them were based on some sound intuitions of one form or another. Clearly there was the need for a general unifying framework." [15, p. 184].

In response to that situation, a number of efforts have been made by some logicians to provide a genuine unity to logic as witnessed for example by the development of abstract model theory and "institutions" [4, 33, 19], the introduction of "labelled deductive systems" by Gabbay [17] or the "basic logic" of Sambin & al. [45] (see [16] for details and more examples). This led to the rise of a research thread sometimes referred to (nowadays) as "Universal Logic". Many kinds of semantics, such as algebraic, categorial, topological, phase or relational semantics, have been introduced and developed, sometimes for the express purpose of tackling this issue [46]. Within that line of research, Dunn's gaggle theory [10, 11, 7] is one of the most well–known frameworks based on the relational Kripke-style semantics which itself deals with the aforementioned problem. Dunn's gaggle theory is an attempt to understand the Kripke semantics of non-classical logics in a disciplined, systematic way.[1]

We share the ideal and the objective of "Universal Logic", but, in our view, gaggle theory is only a first step. Indeed, this theory does not really introduce an actual logic or logical framework that can serve as a foundation for non–classical logics, in the same way as the Lambek calculus is sometimes presented as the foundational logic of the varied substructural logics [42]. However, as we will show, gaggle theory provides formal methods to define a generic logic. In fact, it allows us to define a

[1]Dunn "owe[s] the name "gaggle" to [his] colleague Paul Eisenberg (a historian of philosophy, not a logician), who supplied it at [his] request for a name like a "group", but which suggested a certain amount of complexity and disorder." [10, p. 31]

class of logics that can handle connectives of arbitrary arity. Building on (partial) gaggle theory, we will define a class of non-classical logics that we call gaggle logics and which generalize the Lambek calculus and other substructural logics in many directions.

In doing so, we will establish connections between gaggle theory and group theory. We will show that Dunn's abstract law of residuation corresponds to an action of transpositions of the symmetric group (the group of permutations) on the set of connectives of gaggle logics and that Dunn's families of connectives are orbits of the same action. Other operations on connectives, such as dual and Boolean negation, will also be reformulated in terms of actions of groups, and their combination will be defined by means of free groups and free products. We will also show how notions of groups arise naturally from our gaggle logics and how gaggle logics can be canonically defined from given groups.

Our other main contribution will deal with the proof theory of gaggle logics. We will show how sound and complete calculi can be systematically computed and defined for any basic gaggle logic given by its set of connectives. This generic result is in line with our 'universal' approach explained above and constitutes the main technical advance of the article. We will use a specific Henkin construction method to prove the strong completeness of our calculi. Our main objective is to obtain sound and complete proof calculi for basic gaggle logics without the Boolean connectives. However, we will need to add them anyway and proceed in two steps. Firstly, we will consider a language with the Boolean connectives and prove completeness with them (Section 7). Secondly, after proving the cut elimination (via the proof of conditions $(C1)$ – $(C8)$), we will obtain sound and complete calculi for basic gaggle logics without the Boolean connectives thanks to a proof-theoretical analysis of the calculi obtained (Sections 8 and 9, proof of Theorem 53). The cut elimination will also entail that basic gaggle logics are conservative extensions of each other and are decidable.

Organization of the article. In Section 2, we recall the basic results of (partial) gaggle theory. In Section 3, we recall the basics of group theory including the symmetric group (the group of permutations), free groups, free products and actions of groups. In Section 4, we introduce our gaggle logics and define our actions of groups on the gaggle connectives, in particular the residuation and the Boolean negation. In Section 5, we prove that Dunn's abstract laws of residuation are actions of transpositions of the symmetric group on the set of connectives and that Dunn's families of connectives are orbits of the action of the symmetric group. In Section 6, we relate our gaggle logics with the literature by listing the 96 binary connectives and the 16 unary connectives of basic gaggle logic while mentioning which connectives have already been introduced in a publication. We also mention two logics which cannot be embedded in gaggle logics. In Section 7, we introduce our display calculi. In Section 8 we prove that our calculi satisfy the display property and that the cut rule can be eliminated from any proof. Then, in Section 9, thanks to cut–elimination, we provide sound and strongly complete display calculi for gaggle logics without Boolean connectives. We also prove that basic gaggle logics are decidable. In Section 10, we show how notions of groups arise naturally from our gaggle logics and how gaggle logics can be canonically defined from given groups. We conclude in Section 11. Long proofs are in the Appendix.

2 The core of gaggle theory

We present the core ideas of (partial) gaggle theory [10, 11]. Partial gaggle first appeared in Dunn [11] as a generalization of a gaggle that has just an underlying poset, not necessarily a distributive lattice as required for a gaggle in Dunn [10]. For our purpose, the presentation of (partial) gaggle theory is slightly different from the usual presentation of this theory. The definitions are the same (although they are sometimes instantiated) but the results of this theory are differently presented. Our results can nevertheless easily be obtained from the original presentation [11].

In this section, we consider given an integer $n \in \mathbb{N}$ and a non-empty set W. $\mathcal{P}(W)$ is the set of subsets of W and if S is a set, S^n is the Cartesian product $S \times \ldots \times S$, n times. A *n–ary function f on $\mathcal{P}(W)$* is a function $f : \mathcal{P}(W)^n \to \mathcal{P}(W)$ and a *n–ary relation R over W* is a subset of W^n. We write $Rw_1 \ldots w_n$ for $(w_1, \ldots, w_n) \in R$. For all $m, n \in \mathbb{N}$, the expression $[\![m; n]\!]$ denotes the set $\{m, \ldots, n\}$ if $m \leq n$, and the empty set $\varnothing$ otherwise. In the sequel, we will resort to polarity groups, in particular to the *negation group* $P_{(+,-)}$ and later to the *anti-group* $P_{(+,\sim)}$.

Definition 1 (Polarity groups)**.** Let (x, y) be an ordered pair. The *polarity group associated to* (x, y) is $P_{(x,y)} \triangleq (\{x, y\}, \cdot)$ where the operation $\cdot : P_{(x,y)} \times P_{(x,y)} \to P_{(x,y)}$ is defined by $x \cdot y = y \cdot x = y$ and $x \cdot x = y \cdot y = x$. For all $\pm, \pm' \in \{x, y\}$, we write $\pm\pm'$ for $\pm \cdot \pm'$. □

Note that x is the neutral element of a polarity group.

Definition 2 (Trace, contrapositive trace)**.** A *(n–ary) trace* is a tuple $t = (\pm_1, \ldots, \pm_n, \pm) \in \{+, -\}^{n+1}$, often denoted $t = (\pm_1, \ldots, \pm_n) \mapsto \pm$. If $j \in [\![1; n]\!]$, then the *contrapositive trace of t with respect to its j^{th} argument* is the trace $t^j \triangleq (\pm_1, \ldots, -\pm, \ldots, \pm_n) \mapsto -\pm_j$. □

Note that the contrapositive operation on traces is symmetric: $\left(t^j\right)^j = t$.

Example 3. The 2–ary traces $(-,-) \mapsto -$ and $(-,+) \mapsto +$ are contrapositive with respect to (w.r.t.) their first argument.

Definition 4 (Relation negation and permutation)**.** Let R be an arbitrary $n+1$–ary relation over W. Then, for all $j \in \{1,\ldots,n\}$, we define the $n+1$–ary relation $-R$ as follows: for all $w_1,\ldots,w_n,w \in W$,

$$-Rw_1\ldots w_n w \text{ iff } (w_1,\ldots,w_n,w) \notin R$$

$\mathfrak{S}_{n+1}$ denotes the set of permutations of the set $[\![1;n+1]\!]$ (see Section 3 for details). If $\sigma \in \mathfrak{S}_{n+1}$ is a permutation then its inverse permutation is denoted σ^-. We define the $n+1$–ary relation R^σ as follows: for all $w_1,\ldots,w_{n+1} \in W$,

$$R^\sigma w_1\ldots w_{n+1} \text{ iff } Rw_{\sigma^-(1)}\ldots w_{\sigma^-(n+1)}$$

We also define $+R \triangleq R$ and if $\pm \in \{+,-\}$ then $R^{\pm\sigma}$ denotes $\pm R^\sigma$. □

Definition 5 (Logical functions associated to a trace and a relation)**.** Let $t = (\pm_1,\ldots,\pm_n) \mapsto \pm$ be a n–ary trace and let R be a $n+1$–ary relation on W. The *n–ary function f on $\mathcal{P}(W)$ associated to t and R*, denoted f^t_R, is defined as follows:

- If $n = 0$, $f^t_R \triangleq R$;
- If $n > 0$, then for all $W_1,\ldots,W_n \in \mathcal{P}(W)$,

$$f^t_R(W_1,\ldots,W_n) \triangleq \left\{w \in W \mid \mathcal{C}^t_R(W_1,\ldots,W_n,w)\right\}$$

where $\mathcal{C}^t_R(W_1,\ldots,W_n,w)$ is called the *truth condition* of the function f^t_R and is defined as follows:

- if $\pm = +$: "for all $w_1,\ldots,w_n \in W$, we have $w_1 \pitchfork W_1$ or … or $w_n \pitchfork W_n$ or $Rw_1\ldots w_n w$";
- if $\pm = -$: "there are $w_1,\ldots,w_n \in W$ such that $w_1 \pitchfork W_1$ and … and $w_n \pitchfork W_n$ and $Rw_1\ldots w_n w$";

where, for all $j \in [\![1;n]\!]$, $w_j \pitchfork W_j \triangleq \begin{cases} w_j \in W_j & \text{if } \pm_j\pm = +; \\ w_j \notin W_j & \text{if } \pm_j\pm = -. \end{cases}$ □

Example 6. Let R be a 3–ary relation on W and let σ be the permutation $(2,3,1)$ on the set $[\![1;3]\!]$ (see Section 3 for details). Then, we have that $R^\sigma uvw$ if, and only if, $Rwuv$.

- If $t = (-,-) \mapsto -$ then the function $f_R^t : \mathcal{P}(W) \times \mathcal{P}(W) \to \mathcal{P}(W)$, whose truth condition is $\mathcal{C}_R^t(W_1, W_2, w) = \exists uv\,(u \in W_1 \wedge v \in W_2 \wedge Ruvw)$, defines the semantics of a connective, that we denote $\circ$, as follows: for all $w \in W$,

$$\begin{aligned} w \in [\![\varphi \circ \psi]\!] &\text{ iff } w \in f_R^t([\![\varphi]\!], [\![\psi]\!]) \\ &\text{ iff } \exists uv\,(u \in [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge Ruvw) \end{aligned}$$

- If $t = (-,+) \mapsto +$ then the function $f_{-R^\sigma}^t : \mathcal{P}(W) \times \mathcal{P}(W) \to \mathcal{P}(W)$, whose truth condition is $\mathcal{C}_{-R^\sigma}^t(W_1, W_2, w) = \forall vu\,(v \in W_1 \vee u \in W_2 \vee -R^\sigma vuw)$, defines the semantics of a connective that we denote $\backslash$, as follows: for all $w \in W$,

$$\begin{aligned} w \in [\![\varphi \backslash \psi]\!] &\text{ iff } w \in f_{-R^\sigma}^{t'}([\![\varphi]\!], [\![\psi]\!]) \\ &\text{ iff } \forall vu\,(v \notin [\![\varphi]\!] \vee u \in [\![\psi]\!] \vee -R^\sigma uvw) \\ &\text{ iff } \forall vu\,((Rwuv \wedge u \in [\![\varphi]\!]) \to v \in [\![\psi]\!])\,. \end{aligned}$$

Definition 7 (Isotonic and antitonic functions)**.** Let f be a n–ary function on $\mathcal{P}(W)$. We say that f is *isotonic (resp. antitonic) with respect to the j^{th} argument*, written $tn(f,j) = +$ (resp. $tn(f,j) = -$), when for all $W_1, \dots, W_{j-1}, W_{j+1}, \dots, W_n, X, Y \in \mathcal{P}(W)$,

$$\begin{aligned} &\text{if } X \subseteq Y \\ &\text{then } f(W_1, \dots, W_{j-1}, X, W_{j+1}, \dots, W_n) \subseteq f(W_1, \dots, W_{j-1}, Y, W_{j+1}, \dots, W_n) \\ &(\text{resp. } f(W_1, \dots, W_{j-1}, Y, W_{j+1}, \dots, W_n) \subseteq f(W_1, \dots, W_{j-1}, X, W_{j+1}, \dots, W_n))\,. \end{aligned}$$

□

Example 8. If $[\![\varphi]\!] \subseteq [\![\varphi']\!]$ then $[\![\varphi' \backslash \psi]\!] \subseteq [\![\varphi \backslash \psi]\!]$ because $tn(f_{-R^\sigma}^{t'}, 1) = -$, and $[\![\varphi \circ \psi]\!] \subseteq [\![\varphi' \circ \psi]\!]$ because $tn(f_R^t, 1) = +$.

Definition 9 (Relation transformations)**.** Let R be an arbitrary $n+1$–ary relation over W. Then, for all $j \in \{1, \dots, n\}$, we define the $n+1$–ary relation R^j as follows: for all $w_1, \dots, w_n, w \in W$,

$$R^j w_1 \dots w_n w \text{ iff } Rw_1 \dots w \dots w_n w_j$$

If $t = (\pm_1, \dots, \pm_n) \mapsto \pm$ and $t' = (\pm_1', \dots, \pm_n') \mapsto \pm'$ are two n–ary traces which are contrapositive w.r.t. their j^{th} argument, we define the $n+1$–ary relation $(t', t)(R)$ over W as follows:

$$(t', t)(R) \triangleq \begin{cases} R^j & \text{if } \pm = \pm'; \\ -R^j & \text{otherwise.} \end{cases}$$

□

Theorem 10. *Let R be a $n{+}1$–ary relation over W. Let $t = (\pm_1, \ldots, \pm_n) \mapsto \pm$ and $t' = (\pm'_1, \ldots, \pm'_n) \mapsto \pm'$ be two contrapositive n–ary traces w.r.t. their j^{th} argument. Let f (resp. f') be the n–ary function on $\mathcal{P}(W)$ associated to t and R (resp. associated to t' and $(t', t)(R)$). Then, if $n > 0$:*

1. *for all $j \in [\![1; n]\!]$, $tn(f, j) = \pm_j \pm$ (and thus $tn(f', j) = \pm'_j \pm'$ too);*
2. *f and f' satisfy the* abstract law of residuation w.r.t. their j^{th} argument*: for all $W_1, \ldots, W_n, X \in \mathcal{P}(W)$,*

$$S(f, W_1, \ldots, W_j, \ldots, W_n, X) \quad \textit{iff} \quad S(f', W_1, \ldots, X, \ldots, W_n, W_j).$$

where $S(f, W_1, \ldots, W_n, X) \triangleq \begin{cases} f(W_1, \ldots, W_n) \subseteq X & \textit{if } \pm = - \\ X \subseteq f(W_1, \ldots, W_n) & \textit{if } \pm = +. \end{cases}$

Example 11. Let us define $\varphi \Vdash \psi$ by for all $w \in W$, $w \in [\![\varphi]\!]$ implies that $w \in [\![\psi]\!]$. Then, the following holds:

- if $\psi \Vdash \psi'$ then $\varphi \circ \psi \Vdash \varphi \circ \psi'$ because $tn(f^t_R, 2) = +$, and if $\varphi \Vdash \varphi'$ then $\varphi' \backslash \psi \Vdash \varphi \backslash \psi$ because $tn(f^{t'}_{-R^\sigma}, 1) = -$. In other words, f^t_R is isotonic w.r.t. its second argument and $f^{t'}_{-R^\sigma}$ is antitonic w.r.t. its first argument.
- $\varphi \circ \psi \Vdash \chi$ iff $\varphi \Vdash \psi \backslash \chi$, because t and t' are contrapositive w.r.t. their first argument.

3 Group theory

We first recall some basics of group theory (see for instance [44] for more details).

Permutations and cycles. If X is a non-empty set, a *permutation* is a bijection $\sigma : X \to X$. We denote the set of all permutations of X by $\mathfrak{S}_X$. In the important special case when $X = \{1, \ldots, n\}$, we write $\mathfrak{S}_n$ instead of $\mathfrak{S}_X$. Note that $|\mathfrak{S}_n| = n!$, where $|Y|$ denotes the number of elements in a set Y. A permutation σ on the set $\{1, \ldots, n\}$ such that $\sigma(1) = x_1, \sigma(2) = x_2, \ldots, \sigma(n) = x_n$ is denoted $(x_1, x_2, \ldots, x_n)$. For example, $(1, 3, 2)$ is the permutation σ such that $\sigma(1) = 1, \sigma(2) = 3$ and $\sigma(3) = 2$.

If $x \in X$ and $\sigma \in \mathfrak{S}_X$, then σ *fixes* x if $\sigma(x) = x$ and σ *moves* x if $\sigma(x) \neq x$. Let $j_1, \ldots, j_r$ be distincts integers between 1 and n. If $\sigma \in \mathfrak{S}_n$ fixes the remaining $n - r$ integers and if $\sigma(j_1) = j_2, \sigma(j_2) = j_3, \ldots, \sigma(j_{r-1}) = j_r, \sigma(j_r) = j_1$ then σ is an r*–cycle*; one also says that σ is a cycle of *length* r. Denote σ by $(j_1\ j_2\ \ldots\ j_r)$. A 2–cycle which merely interchanges a pair of elements is called a *transposition*.

Two permutations $\sigma, \tau \in \mathfrak{S}_X$ are *disjoint* if every x moved by one is fixed by the other. A family of permutations $\sigma_1, \sigma_2, \dots, \sigma_n$ is *disjoint* if each pair of them is disjoint. Every permutation $\sigma \in \mathfrak{S}_n$ is either a cycle or a product of disjoint cycles. Moreover, this factorization is unique except for the order in which the factors occur.

Groups. A *group* $(G, \circ)$ is a non–empty set G equipped with an associative operation $\circ : G \times G \to G$ and containing an element denoted 1_G called the *neutral element* such that:

- $1_G \circ a = a = a \circ 1_G$ for all $a \in G$;
- for every $a \in G$, there is an element $b \in G$ such that $a \circ b = 1_G = b \circ a$.

This element b is unique and called the *inverse* of a, denoted a^{-1}. The set $\mathfrak{S}_n$ with the composition operation is a group called the *symmetric group on n letters*.

A non–empty subset S of a group G is a *subgroup* of G if $s \in S$ implies $s^{-1} \in S$ and $s, t \in S$ imply $s \circ t \in S$. In that case, S is also a group in its own right.

If X is a subset of a group G, then the smallest subgroup of G containing X, denoted by $\langle X \rangle$, is called the *subgroup generated by X*. For example, $\mathfrak{S}_n = \langle (1\,2), (2\,3), \dots, (i\;i+1), \dots, (n-1\;n) \rangle = \langle (n\,1), (n\,2), \dots, (n\;n-1) \rangle = \langle (n-1\;n), (1\,2\,\dots\,n) \rangle$. $\mathfrak{S}_n$ is also generated by $(1\;2)$ and 3–cycles. For $n \geq 3$, the *alternating group* $\mathfrak{A}_n$ is the subgroup of $\mathfrak{S}_n$ generated by the n–cycles of $\mathfrak{S}_n$.

In fact, if X is non–empty, then $\langle X \rangle$ is the set of all the words on X, that is, elements of G of the form $x_1^{\pm_1} x_2^{\pm_2} \dots x_n^{\pm_n}$ where $x_1, \dots, x_n \in X$ and $\pm_1, \dots, \pm_n$ are either -1 or empty.

Free groups and free products. If X is a subset of a group F, then F is a *free group* with *basis* X if, for every group G and every function $f : X \to G$, there exists a unique homomorphism $\varphi : F \to G$ extending f. One can prove that a free group with basis X always exists and that X generates F. We therefore use the notation $F = \langle X \rangle$ also for free groups.

If G and H are groups, the *free product* of G and H is a group P and homomorphisms j_G and j_H such that, for every group Q and all homomorphisms $f_G : G \to Q$ and $f_H : H \to Q$, there exists a unique homomorphism $\varphi : P \to Q$ with $\varphi j_G = f_G$ and $\varphi j_H = f_H$. Such a group always exists and it is unique modulo isomorphism, we denote it $G * H$. This definition can be generalized canonically to the case of a finite number of groups $G_1, \dots, G_n$, yielding the free product $G_1 * \dots * G_n$.

Group actions. If X is a set and G a group, an *action of G on X* is a function $\alpha : G \times X \to X$ given by $(g, x) \mapsto gx$ such that:

- $1x = x$ for all $x \in X$;
- $(g_1 g_2)x = g_1(g_2 x)$ for all $x \in X$ and all $g_1, g_2 \in G$.

An action of G on X is *transitive* if for every $x, y \in X$, there exists $g \in G$ such that $y = gx$; it is *faithful* if for $gx = x$ for all $x \in X$ implies that $g = 1$.

If $x \in X$ and α an action of a group G on X, then the *orbit* of x under α is $\mathcal{O}_\alpha(x) \triangleq \{\alpha(g,x) \mid g \in G\}$. The orbits form a partition of X. The *stabilizer* of x, denoted by G_x, is the subgroup $G_x \triangleq \{g \in G \mid gx = x\}$ of G. If G is finite, then we have that $|\mathcal{O}_\alpha(x)| = \frac{|G|}{|G_x|}$. Moreover, if X and G are finite then the number N of orbits of X is $N = \frac{1}{|G|}\sum_{\tau \in G} F(\tau)$ where, for $\tau \in G$, $F(\tau)$ is the number of $x \in X$ fixed by τ (Burnside's lemma). Finally, if $X' \subseteq X$ then $\mathcal{O}_\alpha(X')$ denotes $\bigcup\limits_{x' \in X'} \mathcal{O}_\alpha(x')$.

Fact 12. *If α is an action of G on a set X and H is a subgroup of G, then the restriction of α to H, denoted α_H, is also an action of H on the set X.*

Definition 13. Let G and H be two groups. If α and β are actions of G and H on a set X, then the *free action* $\alpha * \beta$ is the mapping $\alpha * \beta : G * H \times X \to X$ given by $\alpha * \beta(g,x) \triangleq \alpha(g_1, \beta(h_1, \ldots, \alpha(g_n, \beta(h_n, x))))$, where $g = g_1 h_1 \ldots g_n h_n$ is the factorization of g in the free group $G * H$. □

This definition can be generalized canonically to the case of a finite number of actions $\alpha_1, \ldots, \alpha_n$, yielding the mapping $\alpha_1 * \ldots * \alpha_n$.

Proposition 14. *If $\alpha_1, \ldots, \alpha_n$ are actions of $G_1, \ldots, G_n$ on a set X respectively, then the mapping $\alpha_1 * \ldots * \alpha_n$ is an action of the (free) group $G_1 * \ldots * G_n$ on X.*

4 From gaggle theory to gaggle logics

The introduction of the formal concepts of gaggle theory are motivated by some heuristic and logical reasons (see for example [41] for informal explanations). We are going to reformulate these formal concepts of gaggle theory because we want to make more clear the connection between traces and the relational Kripke–style semantics that they induce. Thereby, we replace the notion of trace by our notion of 'signature' which highlights and distinguishes in a more immediate way the different semantic ingredients that compose gaggle theory. More specifically, the output of a trace (+ or −) is replaced by a quantification signature (∀ or ∃). Doing so, our reformulation will capture and represent more directly and faithfully the tonicity of the connective defined by a given trace/signature and the formulation of its truth condition (even if, as we said, the notion of trace output was introduced for different heuristic reasons [41]).

In this section, we show how gaggle theory, and in particular Definition 5, leads to the definition of finite families of connectives of arbitrary arities which are related to each other by the abstract law of residuation of Theorem 10.

4.1 From traces to gaggle connectives

Informally, $\forall$ is associated with $+$ and $\exists$ is associated with $-$. We formalize this association with the function $\pm : \{\forall, \exists\} \to \{+, -\}$ defined by $\pm(\forall) \triangleq +, \pm(\exists) \triangleq -$ and the inverse function $Æ : \{+, -\} \to \{\forall, \exists\}$ defined by $Æ(+) \triangleq \forall, Æ(-) \triangleq \exists$. Also, we define the function $+ : \{\forall, \exists\} \to \{\forall, \exists\}$ by $+(\forall) \triangleq \forall$ and $+(\exists) \triangleq \exists$ and the function $- : \{\forall, \exists\} \to \{\forall, \exists\}$ by $-(\forall) \triangleq \exists$ and $-(\exists) \triangleq (\forall)$. For better readability, we write $+\forall, +\exists, -\forall, -\exists$ instead of $-(\forall), +(\exists), -(\forall), -(\exists)$.

Definition 15 (Signatures versus traces)**.** A *(n–ary) signature* s is a tuple $s = (Æ, (\pm_1, \dots, \pm_n)) \in \{\forall, \exists\} \times \{+, -\}^n$. If $s = (Æ, (\pm_1, \dots, \pm_n))$ is a n–ary signature and $t = (\pm_1, \dots, \pm_n, \pm)$ a n–ary trace, then

- The *trace* $T(s)$ *equivalent to* s is the trace $(\pm'_1, \dots, \pm'_n) \mapsto \pm$ where $\pm \triangleq \pm(Æ)$ and $\pm'_j \triangleq \pm\pm_j$ for all $j \in [\![1; n]\!]$.
- The *signature* $S(t)$ *equivalent to* t is the signature $(Æ, (\pm'_1, \dots, \pm'_n))$ where $Æ \triangleq Æ(\pm)$ and $\pm'_j \triangleq \pm\pm_j$ for all $j \in [\![1; n]\!]$. □

Note that the derived notion of tonicity $tn(f, j)$ determined in Theorem 10 is now taken as primitive with our notion of signature. Then, we can easily prove the following:

$$s = S(T(s)) \qquad\qquad t = T(S(t))$$

We also reformulate the definition of contrapositive trace in terms of signature as follows. If $s = (Æ, (\pm_1, \dots, \pm_n))$ is a n–ary signature and $r_j = (n+1\ j)$ a transposition with $j \in [\![1; n]\!]$, then we define

$$r_j s \triangleq (-\pm_j Æ, (-\pm_j \pm_1, \dots, \pm_j, \dots, -\pm_j \pm_n)). \tag{1}$$

Then, we can easily prove the following: for all n–ary traces t and n–ary signatures s,

$$r_j s = S\left(T(s)^j\right) \qquad\qquad t^j = T\left(r_j S(t)\right)$$

Moreover, for every cycle c fixing $n+1$, we define

$$cs \triangleq \left(Æ, \left(\pm_{c(1)}, \pm_{c(2)}, \ldots, \pm_{c(n)}\right)\right). \tag{2}$$

This definition is coherent with Expression (1). Indeed, the transpositions $(n+1\ 1), (n+1\ 2), \ldots, (n+1\ n)$ generate $\mathfrak{S}_{n+1}$ and every cycle fixing $n+1$ can be factorized into a sequence of transpositions of the form $(n+1\ j)$ so that, applying iteratively Expression (1), we obtain Expression (2).

Definition 16 (Gaggle connectives)**.** The set of *atoms* $\mathbb{P}$ and *connectives* $\mathbb{C}$ are:

$$\mathbb{P} \triangleq \mathfrak{S}_1 \times \{+,-\} \times \{\forall,\exists\} \qquad \mathbb{C} \triangleq \mathbb{P} \cup \bigcup_{n\in\mathbb{N}^*} \mathfrak{S}_{n+1} \times \{+,-\} \times \{\{\forall,\exists\} \times \{+,-\}^n\}.$$

Both atoms and connectives can be represented by triples $p = (1, \pm, Æ)$ (for atoms) and $\otimes = (\sigma, \pm, (Æ, (\pm_1, \ldots, \pm_n)))$ (for connectives) where $\sigma \in \mathfrak{S}_{n+1}$, $\pm \in \{+,-\}$ and $(Æ, (\pm_1, \ldots, \pm_n)) \in \{\forall,\exists\} \times \{+,-\}^n$. The *arity* of an atom is 0, the *arity* of a connective $\otimes = (\sigma, \pm, (Æ, (\pm_1, \ldots, \pm_n))) \in \mathbb{C}$, denoted $a(\otimes)$, is n, its *signature* is $(Æ, (\pm_1, \ldots, \pm_n))$, its *quantification signature* is Æ and its *tonicity signature* is $(\pm_1, \ldots, \pm_n)$. For all $j \in [\![1;n]\!]$, $tn(\otimes, j)$ denotes $\pm_j$. Atoms are denoted p, p_1, p_2, *etc.* and connectives are denoted $\otimes, \otimes_1, \otimes_2$, *etc.* The set of n–ary connectives, for $n > 0$, is denoted $\mathbb{C}_n$. □

Fact 17. *The number of n–ary gaggle connectives is $(n+1)! \cdot 2^{n+2}$.*

Proof: It follows from the very definition of connectives. □

4.2 Actions of groups on gaggle connectives

In this section, we introduce actions on the set of gaggle connectives. In the next sections, we will show that they generalize standard notions of residuations, duals and Boolean negation.

Definition 18 (Action of the symmetric group)**.** Let $n \in \mathbb{N}^*$. We define the function $\alpha_n : \mathfrak{S}_{n+1} \times \mathbb{C}_n \to \mathbb{C}_n, (\tau, \otimes) \mapsto \tau\otimes$ inductively as follows. Let $\otimes = (\sigma, \pm, (Æ, (\pm_1, \ldots, \pm_n))) \in \mathbb{C}_n$ and let $c \in \mathfrak{S}_{n+1}$.

- If c is the transposition $r_j = (j\ n+1)$, then $r_j\otimes \triangleq (r_j \circ \sigma, -\pm_j \pm, r_j s)$, *i.e.*:

$$r_j\otimes \triangleq ((j\ n+1) \circ \sigma, -\pm_j \pm, (-\pm_j Æ, (-\pm_j \pm_1, \ldots, \pm_j, \ldots, -\pm_j \pm_n)).$$

  The connective r_j is called the *residual of $\otimes$ w.r.t. its j^{th} argument.*

Permutations of $\mathfrak{S}_2$	1–ary signatures
$\tau_1 = (1,2)$	$t_1 = (\exists, +)$
$\tau_2 = (2,1)$	$t_2 = (\forall, +)$
	$t_3 = (\forall, -)$
	$t_4 = (\exists, -)$

Permutations of $\mathfrak{S}_3$	2–ary signatures
$\sigma_1 = (1,2,3)$	$s_1 = (\exists, (+,+))$
$\sigma_2 = (3,2,1)$	$s_2 = (\forall, (+,-))$
$\sigma_3 = (3,1,2)$	$s_3 = (\forall, (-,+))$
$\sigma_4 = (2,1,3)$	$s_4 = (\forall, (+,+))$
$\sigma_5 = (2,3,1)$	$s_5 = (\exists, (+,-))$
$\sigma_6 = (1,3,2)$	$s_6 = (\exists, (-,+))$
	$s_7 = (\exists, (-,-))$
	$s_8 = (\forall, (-,-))$

Figure 1: Permutations of $\mathfrak{S}_2$ and $\mathfrak{S}_3$ and 'families' of 1–ary and 2–ary signatures

- If c is the cycle $(j_1 \ j_2 \ \dots \ j_k \ n+1)$, then $c\circledast \triangleq r_{j_1}(r_{j_2} \dots (r_{j_k}\circledast))$, where $r_j \triangleq (j \ n+1)$ for all j.
- If c is a cycle fixing $n+1$, then $c\circledast \triangleq (c \circ \sigma, \pm, cs)$, *i.e.*:

$$c\circledast \triangleq \left(c \circ \sigma, \pm, \left(\text{Æ}, \left(\pm_{c(1)}, \pm_{c(2)}, \dots, \pm_{c(n)}\right)\right)\right).$$

Finally, if τ is an arbitrary permutation of $\mathfrak{S}_{n+1}$, it can be factorized into a product of disjoint cycles $\tau = c_1 c_2 \dots c_k$ and this factorization is unique (modulo its order) [44]. So, we define $\tau\circledast \triangleq c_1(c_2 \dots (c_k\circledast))$. □

The mapping α_n is well-defined because one can easily prove that any other ordering of the disjoint cycles $c_1, \dots, c_k$ of τ yields the same outcome for $\tau\circledast$. Our definition is based on cycles and not on transpositions because the decomposition of any permutation into disjoint cycles is unique (modulo its order), unlike its decomposition into transpositions.

Proposition 19. *For all $n \in \mathbb{N}^*$, the mapping $\alpha_n : \mathfrak{S}_{n+1} \times \mathbb{C}_n \to \mathbb{C}_n$ is a group action of $\mathfrak{S}_{n+1}$ on $\mathbb{C}_n$. For all $n \in \mathbb{N}^*$, the group actions α_n (and all their restrictions to subgroups G) are not transitive, the cardinality of each orbit is $|\mathfrak{S}_{n+1}|$ (resp. $|G|$) and the number of orbits is $4 \cdot 2^n$ (resp. $\frac{|\mathbb{C}_n|}{|G|}$).*

Proof: (sketch) The condition $(\tau_1 \circ \tau_2)\circledast = \tau_1(\tau_2 \circledast)$ of the definition of group actions is proved by induction on τ_1. The other results follow from group theory because for all $x \in \mathbb{C}_n$, $G_x = \{1\}$. □

Definition 20 (Actions of the negation group and the anti-group)**.** Let $n \in \mathbb{N}^*$. We define the functions $\beta_n : P_{(+,-)} \times \mathbb{C}_n \to \mathbb{C}_n, (\pm, \circledast) \mapsto \pm\circledast$ and $\gamma_n : P_{(+,\sim)} \times \mathbb{C}_n \to \mathbb{C}_n, (\pm, \circledast) \mapsto \pm\circledast$ as follows: if $\circledast = (\sigma, \pm, (Æ, (\pm_1, \dots, \pm_n))) \in \mathbb{C}_n$, then

- $+\circledast \triangleq \circledast$
- $\sim \circledast \triangleq (\sigma, -\pm, (Æ, (\pm_1, \dots, \pm_n)))$
- $-\circledast \triangleq (\sigma, -\pm, (-Æ, (-\pm_1, \dots, -\pm_n)))$.

$-\circledast$ and $\sim \circledast$ are called the *Boolean negation* and the *symmetry* of $\circledast$ respectively. Moreover, if $\circledast$ is an atom $p = (1, \pm, Æ)$, then we also define $-p \triangleq (1, -\pm, -Æ)$. □

As we will see in Proposition 29, our definition of Boolean negation does correspond to the intended (Boolean) negation.

Proposition 21. *For all $n \in \mathbb{N}^*$, the functions β_n and γ_n are non–transitive actions. For both actions, the cardinality of each orbit is 2 and the number of orbits is $\frac{|\mathbb{C}_n|}{2}$.*

Proof: It follows from the application of Burnside Lemma. Only + fixes connectives of $\mathbb{C}_n$ and it fixes all of them. − and ~ do not fix any element of $\mathbb{C}_n$. □

4.3 Gaggle logics

Our introduction of 'gaggle logics', like many semantic-based logics, is made in three parts: first, we define their language (Definition 22), then their class of models (Definition 24) and finally their satisfaction relation (Definition 25).

Definition 22 ((Boolean) gaggle language)**.** The *gaggle language* $\mathcal{L}^0$ is the smallest set that contains the propositional letters and that is closed under the gaggle connectives. That is,

- $\mathbb{P} \subseteq \mathcal{L}^0$;
- for all $\circledast \in \mathbb{C}$ of arity $n > 0$ and for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}^0$, we have $\circledast(\varphi_1, \dots, \varphi_n) \in \mathcal{L}^0$.

The *Boolean gaggle language* $\mathcal{L}$ is the smallest set that contains the propositional letters and that is closed under the gaggle connectives as well as the Boolean connectives $\wedge, \vee$ and $\neg$.

Elements of $\mathcal{L}$ are called *formulas* and are denoted $\varphi, \psi, \alpha, \ldots$ For all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}$, $\varphi_1 \wedge \ldots \wedge \varphi_n$ and $\varphi_1 \vee \ldots \vee \varphi_n$ stand for $((\varphi_1 \wedge \varphi_2) \wedge \ldots \wedge \varphi_n)$ and $((\varphi_1 \vee \varphi_2) \vee \ldots \vee \varphi_n)$ respectively.

If $\mathsf{C} \subseteq \mathbb{C} \cup \{\wedge, \vee, \neg\}$ is such that $\mathsf{C} \cap \mathbb{P} \neq \emptyset$, then an element of $\mathcal{L}_{\mathsf{C}}$ is an element of $\mathcal{L}$ that contains only connectives and atoms of C. *In the sequel, we assume that all the sets of atoms and connectives $\mathsf{C} \subseteq \mathbb{C} \cup \{\wedge, \vee, \neg\}$ are such that $\mathsf{C} \cap \mathbb{P} \neq \emptyset$.* □

Remark 23. We could consider a countable number of copies of the atoms and connectives: $\mathbb{P}' \triangleq \bigcup_{i \in \mathbb{N}} \{\circledast_i \mid \circledast \in \mathbb{P}\}$, $\mathbb{C}' \triangleq \bigcup_{i \in \mathbb{N}} \{\circledast_i \mid \circledast \in \mathbb{C}\}$. Indeed, in general we need a countable number of atoms or, like in some modal logics, we need multiple modalities of the same (similarity) type. All the results that follow would still hold in this extended language.

Definition 24 (C–models and C–frames)**.** Let $\mathsf{C} \subseteq \mathbb{C}$. A *C–model* is a tuple $M = (W, \mathcal{R})$ where W is a non-empty set and $\mathcal{R}$ is a set of relations over W. Each n–ary connective $\circledast \in \mathsf{C}$ is associated to a $n+1$–ary relation $R_\circledast$ such that for all connectives $\circledast_1, \circledast_2 \in \mathsf{C}$, we have that $R_{\circledast_1} = R_{\circledast_2}$ iff $\mathcal{O}_{\alpha_n * \beta_n}(\circledast_1) = \mathcal{O}_{\alpha_n * \beta_n}(\circledast_2)$.

We abusively write $w \in M$ for $w \in W$. A *pointed C–model* (M, w) is a C–model M together with a state $w \in M$. The class of all pointed C–models is denoted $\mathcal{M}_{\mathsf{C}}$ and simply $\mathcal{M}$ when $\mathsf{C} = \mathbb{C}$. A *C–frame* is a $\mathsf{C} \backslash \mathbb{P}$–model. The class of all pointed C–frames is denoted $\mathcal{F}_{\mathsf{C}}$ and simply $\mathcal{F}$ when $\mathsf{C} = \mathbb{C}$. □

Definition 25 (Gaggle logics)**.** Let $\mathsf{C} \subseteq \mathbb{C}$ and let $M = (W, \mathcal{R})$ be a C–model. We define the *interpretation function of $\mathcal{L}_{\mathsf{C}}$ in M*, denoted $[\![\cdot]\!]^M : \mathcal{L}_{\mathsf{C}} \to \mathcal{P}(W)$, inductively as follows: for all $p \in \mathsf{C} \cap \mathbb{P}$ and all $\circledast \in \mathsf{C}$ of arity $n > 0$ and signature denoted $(\sigma, \pm, s)$, for all $\varphi, \psi, \varphi_1, \ldots, \varphi_n \in \mathcal{L}_{\mathsf{C}}$,

$$\begin{array}{rcl}
[\![p]\!]^M & \triangleq & \pm R_p \\
[\![\neg\varphi]\!]^M & \triangleq & W - [\![\varphi]\!]^M \\
[\![(\varphi \wedge \psi)]\!]^M & \triangleq & [\![\varphi]\!]^M \cap [\![\psi]\!]^M \\
[\![(\varphi \vee \psi)]\!]^M & \triangleq & [\![\varphi]\!]^M \cup [\![\psi]\!]^M \\
[\![\circledast(\varphi_1, \ldots, \varphi_n)]\!]^M & \triangleq & f_\circledast([\![\varphi_1]\!]^M, \ldots, [\![\varphi_n]\!]^M)
\end{array}$$

where the function $f_\circledast = f^t_{R^{\pm\sigma}_\circledast}$ with $t = T(s)$ defined in Section 4.1 and $f^t_{R^{\pm\sigma}_\circledast}$ in Definition 5. That is, $f_\circledast$ is defined as follows: for all $W_1, \ldots, W_n \in \mathcal{P}(W)$, $f_\circledast(W_1, \ldots, W_n) \triangleq \{w \in W \mid \mathcal{C}^\circledast(W_1, \ldots, W_n, w)\}$ where $\mathcal{C}^\circledast(W_1, \ldots, W_n, w)$ is called the *truth condition* of $\circledast$ and is:

- if Æ = ∀: "$\forall w_1, \ldots, w_n \in W\ (w_1 \pitchfork W_1 \vee \ldots \vee w_n \pitchfork W_n \vee R_{\circledast}^{\pm\sigma} w_1 \ldots w_n w)$";
- if Æ = ∃: "$\exists w_1, \ldots, w_n \in W\ (w_1 \pitchfork W_1 \wedge \ldots \wedge w_n \pitchfork W_n \wedge R_{\circledast}^{\pm\sigma} w_1 \ldots w_n w)$";

where, for all $j \in [\![1;n]\!]$, $w_j \pitchfork W_j \triangleq w_j \in W_j$ if $\pm_j = +$ and $w_j \pitchfork W_j \triangleq w_j \notin W_j$ if $\pm_j = -$ and $R_{\circledast}^{\pm\sigma} w_1 \ldots w_{n+1}$ iff $\pm R_{\circledast} w_{\sigma^-(1)} \ldots w_{\sigma^-(n+1)}$ (we recall that $+R_{\circledast} \triangleq R_{\circledast}$ and $-R_{\circledast} \triangleq W^{n+1} - R_{\circledast}$).

We extend the definition of the interpretation function $[\![\cdot]\!]^M$ to C–frames as follows: for all $\varphi \in \mathcal{L}_{\mathsf{C}}$ and all C–frames F,

$$[\![\varphi]\!]^F \triangleq \bigcap \left\{ [\![\varphi]\!]^{(F,\mathcal{P})} \mid \mathcal{P} \text{ a set of } n\text{–ary relations over } W \text{such that } (F,\mathcal{P}) \text{ is a C–model} \right\}$$

If $\mathcal{E}_{\mathsf{C}}$ is a class of pointed C–models or C–frames, the *satisfaction relation* $\Vdash \subseteq \mathcal{E}_{\mathsf{C}} \times \mathcal{L}_{\mathsf{C}}$ is defined as follows: for all $\varphi \in \mathcal{L}_{\mathsf{C}}$ and all $(M,w) \in \mathcal{E}_{\mathsf{C}}$, $((M,w),\varphi) \in \Vdash$ iff $w \in [\![\varphi]\!]^M$. We usually write $(M,w) \Vdash \varphi$ instead of $((M,w),\varphi) \in \Vdash$. The triple $(\mathcal{L}_{\mathsf{C}}, \mathcal{E}_{\mathsf{C}}, \Vdash)$ is a logic called the *gaggle logic associated to $\mathcal{E}_C$ and C*. The logics of the form $(\mathcal{L}_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$ are called *basic gaggle logics*. We call them *Boolean (basic) gaggle logics* when their language includes the Boolean connectives $\wedge, \vee, \neg$. □

The truth conditions of the above definitions have been introduced in a different formal approach by Bimbó & Dunn [7] and for some particular cases by Dunn [10] and Dunn & Hardegree [13]. However, it is the first time that they are spelled out systematically and in a comprehensive manner.

Example 26 (Lambek calculus, modal logic)**.** The Lambek calculus $(\mathcal{L}_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$ where $\mathsf{C} = \{p, \circ, \backslash, /\}$ defined in Section 2 is an example of basic gaggle logic. Here $\circ, \backslash, /$ are the connectives $(\sigma_1, +, s_1), (\sigma_5, -, s_3), (\sigma_3, -, s_2)$. Another example of gaggle logic is modal logic $(\mathcal{L}_{\mathsf{C}}, \mathcal{E}_{\mathsf{C}}, \Vdash)$ where $\mathsf{C} = \{p, \top, \bot, \wedge, \vee, \Diamond, \Box\}$ is such that

- $\top, \bot$ are the connectives $(1, +, \exists)$ and $(1, -, \forall)$ respectively;
- $\wedge, \vee, \Diamond, \Box$ are the connectives $(\sigma_1, +, s_1), (\sigma_1, -, s_4), (\tau_2, +, s_1), (\tau_2, -, s_2)$ respectively;
- the C-models $M = (W, \mathcal{R}) \in \mathcal{E}_{\mathsf{C}}$ are such that $R_\wedge = R_\vee = \{(w,w,w) \mid w \in W\}$, $R_\Diamond = R_\Box$, $R_\top = R_\bot = W$.

Indeed, one can easily show that, with these conditions on the C–models of $\mathcal{E}_{\mathsf{C}}$, we have that for all $(M,w) \in \mathcal{E}_{\mathsf{C}}$, $(M,w) \Vdash (\sigma_1, +, s_1)(\varphi,\psi)$ iff $(M,w) \Vdash \varphi$ and $(M,w) \Vdash \psi$, and $(M,w) \Vdash (\sigma_1, -, s_4)(\varphi,\psi)$ iff $(M,w) \Vdash \varphi$ or $(M,w) \Vdash \psi$. Note that the Boolean conjunction and disjunction $\wedge$ and $\vee$ are defined using the connectives of $\mathbb{C}$ by means of special relations $R_\wedge$ and $R_\vee$. They could obviously be defined directly. Many more examples will be given in Section 6.

5 Residual, Boolean negation, dual and switch

The action of specific permutations on the set of connectives corresponds to well–known operations used in proof theory. For example, the action of a transposition $(j\ n+1)$ corresponds to the abstract law of residuation for the j^{th} argument. This operation of residuation turns out to be central since every permutation can be decomposed into a composition of transpositions. Yet, we argue that the actions of cycles is more central because every permutation can be decomposed *uniquely* into disjoint cycles. Moreover, the symmetric group $\mathfrak{S}_{n+1}$ is also generated by the cycles $(1\ \dots\ n+1)$ and $(n\ n+1)$ and the alternation group is generated by the $n+1$-cycles of $\mathfrak{S}_{n+1}$. This confirms an observation already made in [2] which highlighted the role of 3-cycles for substructural and update logics in the formal connections that exist between connectives.

Proposition 27. *Let t be a n-ary trace, R a $n+1$-ary relation over W and $\sigma \in \mathfrak{S}_{n+1}$. Then, $f^t_{R^{\pm\sigma}} = f_\circledast$ where $\circledast = (\sigma, \pm, S(t)) = (\sigma, \pm, (Æ, (\pm_1, \dots, \pm_n)))$. Moreover, if $j \in [\![1;n]\!]$, then the n-ary function associated to t^j and $(t^j, t)(R)$ of Definition 5 is $f_{r_j\circledast}$ where $r_j\circledast$, the* residual of $\circledast$ w.r.t. its j^{th} argument*, was defined in Definition 18:*

$$r_j\circledast \triangleq \left((j\ n+1)\circ\sigma, -\pm_j \pm, \left(-\pm_j Æ, \left(-\pm_j \pm_1, \dots, \pm_j, \dots, -\pm_j \pm_n\right)\right)\right).$$

Therefore, we have the following property: for all $\varphi_1, \dots, \varphi_j, \dots, \varphi_n, \varphi \in \mathcal{L}$,

$$S\left[\circledast, \varphi_1, \dots, \varphi_j, \dots, \varphi_n, \varphi\right] \qquad \textit{iff} \qquad S\left[r_j\circledast, \varphi_1, \dots, \varphi, \dots, \varphi_n, \varphi_j\right] \tag{3}$$

where $S\left[\circledast, \varphi_1, \dots, \varphi_n, \varphi\right] \triangleq \begin{cases} \circledast(\varphi_1, \dots, \varphi_n) \Vdash \varphi & \textit{if } Æ = \exists \\ \varphi \Vdash \circledast(\varphi_1, \dots, \varphi_n) & \textit{if } Æ = \forall \end{cases}.$

Proof: It follows straightforwardly from our definitions. Expression (3) follows from Theorem 10 (item 2). □

Hence, $r_j\circledast$ does correspond to the residual connective of $\circledast$ w.r.t. its j^{th} argument as it is usually defined in Dunn's theory.

Definition 28 (Dual and switch operations)**.** Let $\circledast = (\sigma, \pm, (Æ, (\pm_1, \dots, \pm_n))) \in \mathbb{C}$ be a n–ary connective and let $j \in [\![1;n]\!]$.

- The *switch of $\circledast$ w.r.t. its j^{th} argument* is the n-ary connective

$$s_j\circledast \triangleq (\sigma, \pm, (Æ, (\pm_1, \dots, -\pm_j, \dots, \pm_n))).$$

- The *dual of* $\circledast$ *w.r.t. its* j^{th} *argument* is the n–ary connective

$$d_j\circledast \triangleq (\sigma, -\pm, (-Æ, (-\pm_1, \ldots, \pm_j, \ldots, -\pm_n))).$$

- The *dual of* $\circledast$ is the n-ary connective

$$d\circledast \triangleq (\sigma, -\pm, (-Æ, (\pm_1, \ldots, \pm_n))).$$

□

The following proposition shows that our terminology for "Boolean negation" and "dual" is appropriate and does correspond to the standard intuitive meaning (see Blackburn & Al. [8, Def 1.13] for example).

Proposition 29. *Let* $\circledast \in \mathbb{C}$ *be a* n*–ary connective and let* $\varphi_1, \ldots, \varphi_n \in \mathcal{L}$*. Then, for all (appropriate) pointed models* (M, w),

$$\begin{array}{lll}
(M,w) \Vdash -\circledast(\varphi_1,\ldots,\varphi_n) & \textit{iff} & (M,w) \Vdash \circledast(\varphi_1,\ldots,\varphi_n) \textit{ does not hold} \\
(M,w) \Vdash s_j \circledast(\varphi_1,\ldots,\varphi_n) & \textit{iff} & (M,w) \Vdash \circledast(\varphi_1,\ldots,\neg\varphi_j,\ldots,\varphi_n) \\
(M,w) \Vdash d_j \circledast(\varphi_1,\ldots,\varphi_n) & \textit{iff} & (M,w) \Vdash -\circledast(\varphi_1,\ldots,\neg\varphi_j,\ldots,\varphi_n) \\
(M,w) \Vdash d \circledast(\varphi_1,\ldots,\varphi_n) & \textit{iff} & (M,w) \Vdash -\circledast(\neg\varphi_1,\ldots,\neg\varphi_n)
\end{array}$$

The following proposition shows that the switch as well as the dual operations are definable in terms of residuations and Boolean negation.

Proposition 30. *If* $\circledast \in \mathbb{C}_n$ *is a* n*–ary connective, then for all* $j \in [\![1; n]\!]$,

- $s_j\circledast = r_j - r_j\circledast$
- $d_j\circledast = r_j - r_j - \circledast$
- $d\circledast = s_1 \ldots s_n - \circledast$.

Proof: See the Appendix, Section A. □

Proposition 31. *Dunn's (complete) families of* n*–ary connectives are orbits* $\mathcal{O}_{\alpha_n}(\circledast)$ *of the group action* α_n*. These families/orbits form a partition of the set of* n*–ary connectives.*

Proof: It follows easily from Dunn's and our definitions. □

Dunn's families of n–ary connectives are called "complete families" of operations by Bimbó & Dunn [7]. Likewise, two n–ary connectives $\circledast, \circledast \in \mathbb{C}_n$ are "colligated" in the sense of Bimbó & Dunn [7] when they belong to the same orbit $\mathcal{O}_{\alpha_n}(\circledast)$.

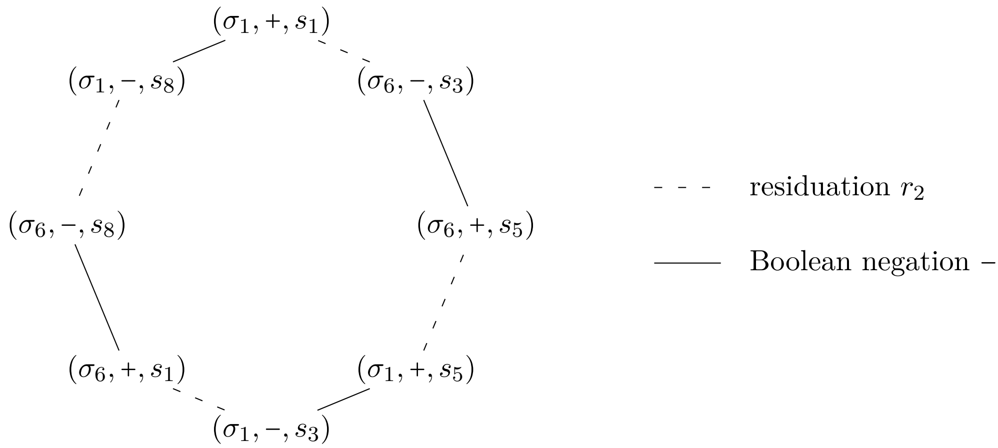

Figure 2: The 8 connectives of the orbit $\mathcal{O}_{\alpha_{G_2}}((\sigma_1, +, s_1))$

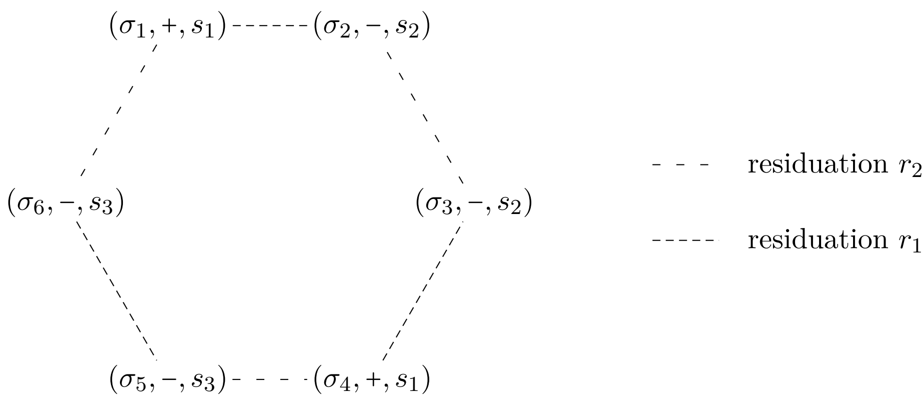

Figure 3: The 6 connectives of the orbit $\mathcal{O}_{\alpha_2}((\sigma_1, +, s_1))$

Proposition 32. *Let $n \in \mathbb{N}^*$, $j \in [\![1; n]\!]$ and let us define $G_j = \langle r_j \rangle * P_{(+,-)}$. Since G_j is a subgroup of $\mathfrak{S}_{n+1} * P_{(+,-)}$, let us denote by α_{G_j} the action of G_j on $\mathbb{C}_n$ induced by the free action $\alpha_n * \beta_n$. Then, for all connectives $\circledast$ of arity n,*

1. *$\mathcal{O}_{\alpha_{G_j}}(\circledast)$ is isomorphic to a cyclic group of order 8.*

2. *$\{\mathcal{O}_{\alpha_n * \beta_n}(\circledast), \mathcal{O}_{\alpha_n * \beta_n}(\sim \circledast)\}$ forms a partition of the set $\mathbb{C}_n$ of connectives of arity n. Moreover, the mapping $\tilde{\cdot} : \mathcal{O}_{\alpha_n * \beta_n}(\circledast) \to \mathcal{O}_{\alpha_n * \beta_n}(\sim \circledast)$, $x \mapsto \sim x$ is involutive.*

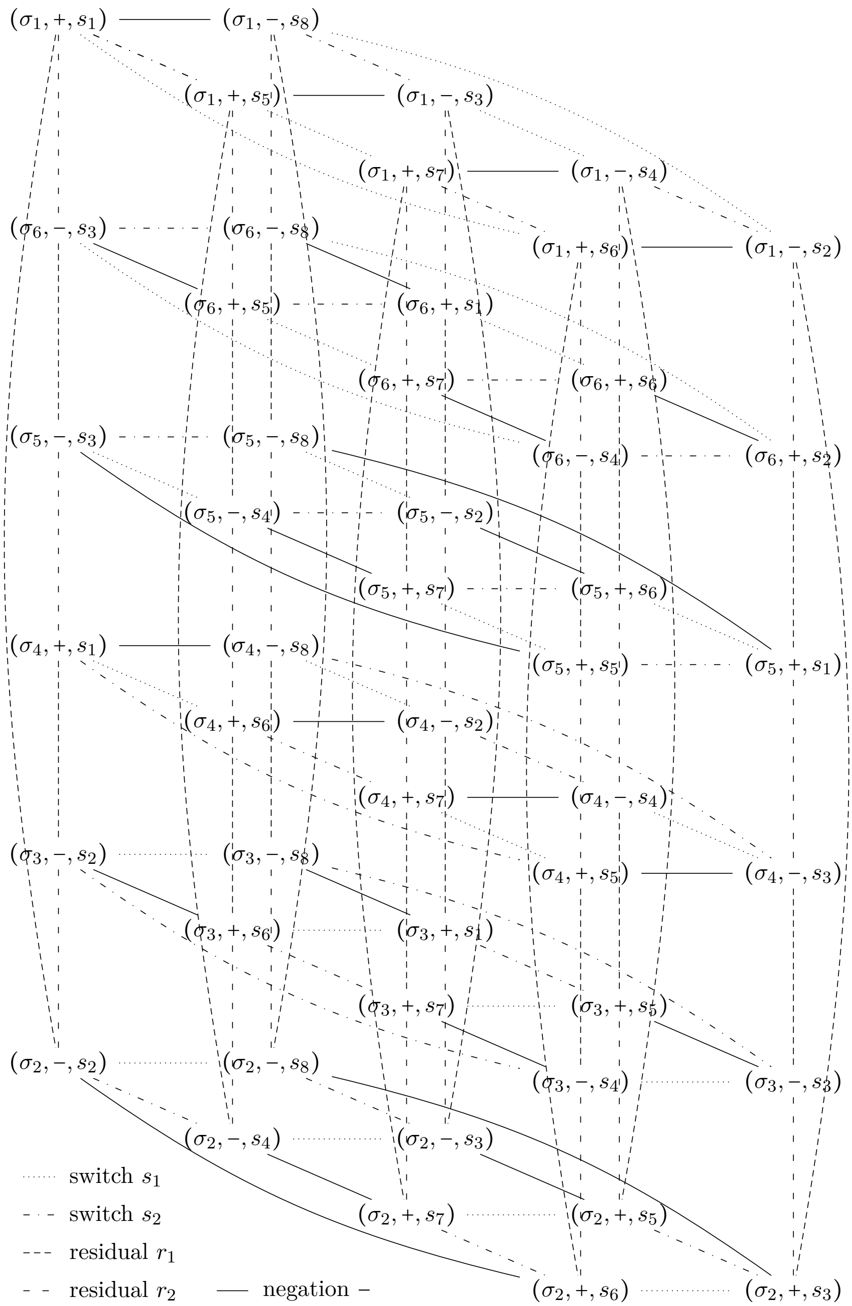

Figure 4: The 48 connectives of the orbit $\mathcal{O}_{\alpha_2 * \beta_2}((\sigma_1,+,s_1))$ related to each other by residual, negation and switch operations.

3. *For all $n \in \mathbb{N}^*$, the free action $\alpha_n * \beta_n * \gamma_n$ on the set of connectives $\mathbb{C}_n$ is transitive.*

Proof: See the Appendix, Section A. □

So, for every pair of connectives $(\circledast, \circledast')$, there exists a sequence of residuation(s), negation(s) and symmetry which transforms $\circledast$ into $\circledast'$. In other words, every gaggle connective $\circledast \in \mathbb{C}_n$ can be obtained from another connective $\circledast' \in \mathbb{C}_n$ with a suitable choice of element in the free groups $\mathfrak{S}_{n+1} * P_{(+,-)} * P_{(+,\sim)}$: for all $\circledast, \circledast' \in \mathbb{C}_n$, there is $g \in \mathfrak{S}_{n+1} * P_{(+,-)} * P_{(+,\sim)}$ such that $\circledast' = \alpha_n * \beta_n * \gamma_n(g, \circledast)$.

Example 33. In Figure 2, we represent the orbit $\mathcal{O}_{\alpha_{G_2}}((\sigma_1, +, s_1))$. It is isomorphic to a group of order 8 according to the first item of Proposition 32. In Figure 4, we represent the orbit $\mathcal{O}_{\alpha_2 * \beta_2}((\sigma_1, +, s_1))$ where the 48 binary connectives are related to each other by means of residuation, switch or Boolean negation. The other 48 binary connectives of the orbit $\mathcal{O}_{\alpha_2 * \beta_2}(\sim (\sigma_1, +, s_1))$ are obtained symmetrically by switching everywhere $-$ to $+$ and $+$ to $-$. These two orbits form a partition of $\mathbb{C}_2$ according to the second item of Proposition 32. The orbits $\mathcal{O}_{\alpha_2}(\circledast)$ of the binary connectives $\circledast$ of $\mathbb{C}_2$ are given in Figures 7, 8, 9, 10, 11 and 12. Every orbit $\mathcal{O}_{\alpha_2}(\circledast)$ is of cardinality $6 = |\mathfrak{S}_3|$. In order to follow common notations, binary connectives are denoted $\varphi \circledast \psi$ instead of $\circledast(\varphi, \psi)$. Finally, the orbit of $\mathcal{O}_{\alpha_2}((\sigma_1, +, s_1))$ is represented graphically in Figure 3, it corresponds to the outermost left vertical line of Figure 4.

6 Gaggle logics in the literature

In this section, we provide formal connections between our gaggle logics and substructural and non-classical logics. The last columns of our tables indicate the relevant publication where the gaggle logic connective was introduced for the first time. A logic close to our approach with connectives of arbitrary arity is the Generalized Lambek Calculus of Kolowska-Gawiejnovicz [26]. It is in fact the basic gaggle logic $(\mathcal{L}_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$ where $\mathsf{C} = \bigcup_{n \in \mathbb{N}^*} \{\circledast_n, r_i \circledast_n \mid, i = 1, \ldots, n\}$ with $\circledast_n$ the n-ary connnective $(1, +, (\exists, (+, \ldots, +)))$. ($\circledast_n$ and $r_i \circledast_n$ are denoted f and f/i in [26].)

6.1 Binary and unary connectives of basic gaggle logic

The truth conditions of the 16 unary gaggle connectives of gaggle logic are given in Figure 6 and those of the 96 binary gaggle connectives of gaggle logic in Figures 7, 8, 9, 10, 11 and 12. Many of these unary and binary connectives have already been introduced in the literature [30, 23, 28, 29, 40, 31, 22, 42, 2]. For example, the binary connectives $(\sigma_1, +, s_1)$, $(\sigma_5, s_3, -)$ and $(\sigma_3, s_2, -)$ are the fusion $\circ$, implication $\backslash$ and co-implication $/$ connectives of the Lambek calculus [30] used to illustrate our examples in Section 2. They are also denoted $\otimes_3$, $\supset_1$ and $\subset_2$ in update logic [2].[2] In the third column of the tables, we provide the bibliographical references where the connectives were first introduced. Note that each binary connective $\circledast$ has a commutative version $\circledast'$ which belongs to the same orbit/family so that for all formulas φ, ψ we have that $\varphi \circledast \psi = \psi \circledast' \varphi$. So, instead of 6 different connectives for each 2–ary orbit, we genuinely have 3 different connectives. This is in line with a result about colligated operations of Bimbó & Dunn [7]. For each orbit, one goes from one connective to the next by alternating residuations w.r.t. the first or the second argument, like in Figure 3. For example, $(\sigma_1, +, s_1) = r_1\ (\sigma_2, -, s_2) = r_1 r_2\ (\sigma_3, -, s_2) = r_1 r_2 r_1\ (\sigma_4, +, s_1) = r_1 r_2 r_1 r_2\ (\sigma_5, -, s_3) = r_1 r_2 r_1 r_2 r_1\ (\sigma_6, -, s_3)$.

To each family/orbit of connectives corresponds a series of laws of residuation. These laws are all instances of the same abstract law of residuation of Definition 10 and correspond to the action of transpositions of the form $(j\ n+1)$ on the set of connectives. They are of different types depending on the family/orbit to which they belong. These types were denoted in the literature: residuation connection, dual residuation connection, Galois connection and dual Galois connection (denoted *rp*, *drp*, *gc* and *dgc* by Goré [22]). These different 'types' of instance of the same abstract law of residuation for binary and unary connectives are given in Figure 5. In particular, note that the notion of dual residuation is the same as our definition of dual w.r.t. the j^{th} argument (Definition 28 and Proposition 29).

6.2 Non gaggle logics

Some connectives of non–classical logics are not connectives of gaggle logics. We mention two of them here. First, the standard modal connective interpreted over a neighborhood semantics [34, 35, 47]. It cannot be expressed by a combination of gaggle logic connectives, because its reformulation with a ternary relation contains an alternation of quantifiers that cannot occur in any function of Definition 5:

$$w \in [\![\Box\varphi]\!] \text{ iff } \exists u \forall v\, (Rwuv \leftrightarrow v \in [\![\varphi]\!]).$$

[2]There is a number of important typographical mistakes about *dual* update logic in [2]. In particular, in Definition 20 (dual update logic) of [2], y and z should be swapped in the truth conditions of $\prec_i$ and $\succ_i$. There are also some errors in the case study of Section 8 about bi-intuitionistic logic. A fully corrected version of [2] is available at `https://hal.inria.fr/hal-01476234v2/document`.

'Type' of the abstract law	Binary connectives	Unary connectives
Residuation	$\begin{array}{c} \varphi \otimes_{\mathrm{i}} \psi \Vdash \chi \\ \hline\hline \varphi \Vdash \psi \supset_{\mathrm{j}} \chi \\ \hline\hline \psi \Vdash \chi \subset_{\mathrm{k}} \varphi \end{array} \begin{array}{c} 1 \\ 2 \end{array}$	$\begin{array}{c} \Diamond^{-} \varphi \Vdash \psi \\ \hline\hline \varphi \Vdash \Box \psi \end{array} \quad \begin{array}{c} \Diamond \varphi \Vdash \psi \\ \hline\hline \varphi \Vdash \Box^{-} \psi \end{array}$
Dual residuation	$\begin{array}{c} \chi \Vdash \varphi \oplus_{\mathrm{i}} \psi \\ \hline\hline \psi \succ_{\mathrm{j}} \chi \Vdash \varphi \\ \hline\hline \chi \prec_{\mathrm{k}} \varphi \Vdash \psi \end{array}$	
Galois	$\begin{array}{c} \varphi \mid_{\mathrm{i}} \psi \Vdash \chi \\ \hline\hline \varphi \mid_{\mathrm{j}} \chi \Vdash \psi \\ \hline\hline \psi \mid_{\mathrm{k}} \chi \Vdash \varphi \end{array}$	$\begin{array}{c} \varphi^{\mathbf{1}} \Vdash \psi \\ \hline\hline {}^{\mathbf{1}}\psi \Vdash \varphi \end{array}$
Dual Galois	$\begin{array}{c} \chi \Vdash \varphi \downarrow_{\mathrm{i}} \psi \\ \hline\hline \psi \Vdash \varphi \downarrow_{\mathrm{j}} \chi \\ \hline\hline \varphi \Vdash \psi \downarrow_{\mathrm{k}} \chi \end{array}$	$\begin{array}{c} \psi \Vdash \varphi^{\mathbf{0}} \\ \hline\hline \varphi \Vdash {}^{\mathbf{0}}\psi \end{array}$

Figure 5: Instances of the abstract law of residuation
$(i, j, k) \in \{(3, 1, 2), (2, 3, 1), (1, 2, 3)\}$

Second, the disjunction of connexive logics interpreted over the ternary semantics of relevant logics [37]. It cannot be expressed in basic gaggle logic either, because its formulation contains a pattern of Boolean connectives absent from the functions of Definition 5:

$$w \in [\![\varphi \vee \psi]\!] \text{ iff } \exists uv \left(Rwuv \wedge (u \in [\![\varphi]\!] \vee v \in [\![\psi]\!]) \right).$$

7 Calculi for Boolean gaggle logics

After some general definitions in Section 7.1 and definitions of structures and consecutions for gaggle logics in Definition 40, we introduce in Section 7.3 our calculus for *Boolean* basic gaggle logics. The calculus is a display calculus.

7.1 Preliminary definitions

These definitions are very general and apply to any kind of formalism.

Gaggle connective	Truth condition	Substructural connective
The existentially positive orbit: residuations		
$(\tau_1, +, t_1)\ \varphi$	$\exists v\,(v \in [\![\varphi]\!] \wedge Rvw)$	$\Diamond^- \varphi$ [40] $\Diamond_\downarrow$ [10]
$(\tau_2, -, t_2)\ \varphi$	$\forall v\,(v \in [\![\varphi]\!] \vee -Rwv)$	$\Box\varphi$ [28]
The universally positive orbit: residuations		
$(\tau_1, +, t_2)\ \varphi$	$\forall v\,(v \in [\![\varphi]\!] \vee Rvw)$	$+_\downarrow\varphi$ [10] [13, p. 401]
$(\tau_2, -, t_1)\ \varphi$	$\exists v\,(v \in [\![\varphi]\!] \wedge -Rwv)$	[10]
The existentially negative orbit: Galois connections		
$(\tau_1, +, t_4)\ \varphi$	$\exists v\,(v \notin [\![\varphi]\!] \wedge Rvw)$	$?\varphi$ [10][13, p. 402] $\boxminus_1\varphi$ [10][7, Def. 10.7.7]
$(\tau_2, +, t_4)\ \varphi$	$\exists v\,(v \notin [\![\varphi]\!] \wedge Rwv)$	$?_\downarrow\varphi$ [10][14] [13, p. 402] $\boxminus_2\varphi$ [7, Def. 10.7.7]
The universally negative orbit: dual Galois connections		
$(\tau_1, +, t_3)\ \varphi$	$\forall v\,(v \notin [\![\varphi]\!] \vee Rvw)$	$\varphi^\perp$ [10, 12] $\varphi^{\mathbf{o}}$ [22] $\diamondminus_1\varphi$ [7, Def. 10.7.2]
$(\tau_2, +, t_3)\ \varphi$	$\forall v\,(v \notin [\![\varphi]\!] \vee Rwv)$	$\sim\varphi$ [20] $^\perp\varphi$ [10, 12] $^{\mathbf{o}}\varphi$ [22] $\diamondminus_2\varphi$ [7, Def. 10.7.2]
The symmetrical existentially positive orbit: residuations		
$(\tau_1, -, t_1)\ \varphi$	$\exists v\,(v \in [\![\varphi]\!] \wedge -Rvw)$	[10]
$(\tau_2, +, t_2)\ \varphi$	$\forall v\,(v \in [\![\varphi]\!] \vee Rwv)$	$+\varphi$ [10] [13, p. 402] φ^* [7, Def. 7.1.19]
The symmetrical universally positive orbit: residuations		
$(\tau_1, -, t_2)\ \varphi$	$\forall v\,(v \in [\![\varphi]\!] \vee -Rvw)$	$\Box^-\varphi$ [40] $\Box_\downarrow$ [10]
$(\tau_2, +, t_1)\ \varphi$	$\exists v\,(v \in [\![\varphi]\!] \wedge Rwv)$	$\Diamond\varphi$ [28]
The symmetrical existentially negative orbit: Galois connections		
$(\tau_1, -, t_4)\ \varphi$	$\exists v\,(v \notin [\![\varphi]\!] \wedge -Rvw)$	$?\varphi$ [10][7, Ex. 1.4.5] $\varphi^{\mathbf{1}}$ [22]
$(\tau_2, -, t_4)\ \varphi$	$\exists v\,(v \notin [\![\varphi]\!] \wedge -Rwv)$	$?_\downarrow\varphi$ [10] [7, Ex. 1.4.5] $^{\mathbf{1}}\varphi$ [22]
The symmetrical universally negative orbit: dual Galois connections		
$(\tau_1, -, t_3)\ \varphi$	$\forall v\,(v \notin [\![\varphi]\!] \vee -Rvw)$	[10]
$(\tau_2, -, t_3)\ \varphi$	$\forall v\,(v \notin [\![\varphi]\!] \vee -Rwv)$	$\neg_h\varphi$ [29, 42] $\perp\varphi$ [14]

Figure 6: The 1–ary gaggle connectives

Gaggle connective	Truth condition	Substructural connective
The conjunction orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1,+,s_1)\ )$: residuations		
$\varphi\ (\sigma_1,+,s_1)\ \psi$	$\exists uv\,(u \in \llbracket\varphi\rrbracket \wedge v \in \llbracket\psi\rrbracket \wedge Ruvw)$	$\varphi \circ \psi$ [30], $\varphi \otimes_3 \psi$ [2]
$\varphi\ (\sigma_2,-,s_2)\ \psi$	$\forall uv\,(u \in \llbracket\varphi\rrbracket \vee v \notin \llbracket\psi\rrbracket \vee -Rwvu)$	
$\varphi\ (\sigma_3,-,s_2)\ \psi$	$\forall uv\,(u \in \llbracket\varphi\rrbracket \vee v \notin \llbracket\psi\rrbracket \vee -Rvwu)$	$/$ [30], $\varphi \subset_2 \psi$ [2]
$\varphi\ (\sigma_4,+,s_1)\ \psi$ $= \psi\ (\sigma_1,+,s_1)\ \varphi$	$\exists uv\,(u \in \llbracket\varphi\rrbracket \wedge v \in \llbracket\psi\rrbracket \wedge Rvuw)$	
$\varphi\ (\sigma_5,-,s_3)\ \psi$ $= \psi\ (\sigma_2,-,s_2)\ \varphi$	$\forall uv\,(u \notin \llbracket\varphi\rrbracket \vee v \in \llbracket\psi\rrbracket \vee -Rwuv)$	$\backslash$ [30], $\varphi \supset_1 \psi$ [2]
$\varphi\ (\sigma_6,-,s_3)\ \psi$ $= \psi\ (\sigma_3,-,s_2)\ \varphi$	$\forall uv\,(u \notin \llbracket\varphi\rrbracket \vee v \in \llbracket\psi\rrbracket \vee -Ruwv)$	
The not–but orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1,+,s_6)\ )$: residuations		
$\varphi\ (\sigma_1,+,s_6)\ \psi$	$\exists uv\,(u \notin \llbracket\varphi\rrbracket \wedge v \in \llbracket\psi\rrbracket \wedge Ruvw)$	$\varphi \succ_3 \psi$ [2]
$\varphi\ (\sigma_2,+,s_6)\ \psi$	$\exists uv\,(u \notin \llbracket\varphi\rrbracket \wedge v \in \llbracket\psi\rrbracket \wedge Rwvu)$	
$\varphi\ (\sigma_3,-,s_4)\ \psi$	$\forall uv\,(u \in \llbracket\varphi\rrbracket \vee v \in \llbracket\psi\rrbracket \vee -Rvwu)$	$\varphi \oplus_2 \psi$ [2]
$\varphi\ (\sigma_4,+,s_5)\ \psi$ $= \psi\ (\sigma_1,+,s_6)\ \varphi$	$\exists uv\,(u \in \llbracket\varphi\rrbracket \wedge v \notin \llbracket\psi\rrbracket \wedge Rvuw)$	
$\varphi\ (\sigma_5,+,s_5)\ \psi$ $= \psi\ (\sigma_2,+,s_6)\ \varphi$	$\exists uv\,(u \in \llbracket\varphi\rrbracket \wedge v \notin \llbracket\psi\rrbracket \wedge Rwuv)$	$\varphi \prec_1 \psi$ [2]
$\varphi\ (\sigma_6,-,s_4)\ \psi$ $= \psi\ (\sigma_3,-,s_4)\ \varphi$	$\forall uv\,(u \in \llbracket\varphi\rrbracket \vee v \in \llbracket\psi\rrbracket \vee -Ruwv)$	
The but–not orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1,+,s_5)\ )$: residuations		
$\varphi\ (\sigma_1,+,s_5)\ \psi$	$\exists uv\,(u \in \llbracket\varphi\rrbracket \wedge v \notin \llbracket\psi\rrbracket \wedge Ruvw)$	$\varphi \prec_3 \psi$ [2]
$\varphi\ (\sigma_2,-,s_4)\ \psi$	$\forall uv\,(u \in \llbracket\varphi\rrbracket \vee v \in \llbracket\psi\rrbracket \vee -Rwvu)$	
$\varphi\ (\sigma_3,+,s_6)\ \psi$	$\exists uv\,(u \notin \llbracket\varphi\rrbracket \wedge v \in \llbracket\psi\rrbracket \wedge Rvwu)$	$\varphi \succ_2 \psi$ [2]
$\varphi\ (\sigma_4,+,s_6)\ \psi$ $= \psi\ (\sigma_1,+,s_5)\ \varphi$	$\exists uv\,(u \notin \llbracket\varphi\rrbracket \wedge v \in \llbracket\psi\rrbracket \wedge Rvuw)$	$\varphi \obslash \psi$ [23, 36]
$\varphi\ (\sigma_5,-,s_4)\ \psi$ $= \psi\ (\sigma_2,-,s_4)\ \varphi$	$\forall uv\,(u \in \llbracket\varphi\rrbracket \vee v \in \llbracket\psi\rrbracket \vee -Rwuv)$	$\varphi \oplus \psi$ [23, 36] $\varphi \oplus_1 \psi$ [2]
$\varphi\ (\sigma_6,+,s_5)\ \psi$ $= \psi\ (\sigma_3,+,s_6)\ \varphi$	$\exists uv\,(u \in \llbracket\varphi\rrbracket \wedge v \notin \llbracket\psi\rrbracket \wedge Ruwv)$	$\varphi \oslash \psi$ [23, 36]

Figure 7: The 2–ary gaggle connectives

Gaggle connective	Truth condition	Substructural connective
The symmetrical conjunction orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1,-,s_1)\ )$: residuations		
$\varphi\ (\sigma_1,-,s_1)\ \psi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge -Ruvw)$	$\varphi \circ \psi$ [7, Def. 5.2.3]
$\varphi\ (\sigma_2,+,s_2)\ \psi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Rwvu)$	
$\varphi\ (\sigma_3,+,s_2)\ \psi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Rvwu)$	
$\varphi\ (\sigma_4,-,s_1)\ \psi$ $= \psi\ (\sigma_1,-,s_1)\ \varphi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge -Rvuw)$	
$\varphi\ (\sigma_5,+,s_3)\ \psi$ $= \psi\ (\sigma_2,+,s_2)\ \varphi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee Rwuv)$	$\varphi \to \psi$ [7, Def. 5.2.3]
$\varphi\ (\sigma_6,+,s_3)\ \psi$ $= \psi\ (\sigma_3,+,s_2)\ \varphi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee Ruwv)$	
The symmetrical not–but orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1,-,s_6)\ )$: residuations		
$\varphi\ (\sigma_1,-,s_6)\ \psi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge -Ruvw)$	
$\varphi\ (\sigma_2,-,s_6)\ \psi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge -Rwvu)$	
$\varphi\ (\sigma_3,+,s_4)\ \psi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee Rvwu)$	
$\varphi\ (\sigma_4,-,s_5)\ \psi$ $= \psi\ (\sigma_1,-,s_6)\ \varphi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Rvuw)$	
$\varphi\ (\sigma_5,-,s_5)\ \psi$ $= \psi\ (\sigma_2,-,s_6)\ \varphi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Rwuv)$	
$\varphi\ (\sigma_6,+,s_4)\ \psi$ $= \psi\ (\sigma_3,+,s_4)\ \varphi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee Ruwv)$	
The symmetrical but–not orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1,-,s_5)\ )$: residuations		
$\varphi\ (\sigma_1,-,s_5)\ \psi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Ruvw)$	
$\varphi\ (\sigma_2,+,s_4)\ \psi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee Rwvu)$	
$\varphi\ (\sigma_3,-,s_6)\ \psi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge -Rvwu)$	
$\varphi\ (\sigma_4,-,s_6)\ \psi$ $= \psi\ (\sigma_1,-,s_5)\ \varphi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge -Rvuw)$	
$\varphi\ (\sigma_5,+,s_4)\ \psi$ $= \psi\ (\sigma_2,+,s_4)\ \varphi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee Rwuv)$	
$\varphi\ (\sigma_6,-,s_5)\ \psi$ $= \psi\ (\sigma_3,-,s_6)\ \varphi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Ruwv)$	

Figure 8: The 2–ary gaggle connectives

Gaggle connective	Truth condition	Substructural connective
The disjunction orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1,-,s_4)\ )$: dual residuations		
$\varphi\ (\sigma_1,-,s_4)\ \psi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee -Ruvw)$	$\varphi \oplus_3 \psi$ [2]
$\varphi\ (\sigma_2,+,s_5)\ \psi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge Rwvu)$	
$\varphi\ (\sigma_3,+,s_5)\ \psi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge Rvwu)$	$\varphi \prec_2 \psi$ [2]
$\varphi\ (\sigma_4,-,s_4)\ \psi$ $= \psi\ (\sigma_1,-,s_4)\ \varphi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee -Rvuw)$	
$\varphi\ (\sigma_5,+,s_6)\ \psi$ $= \psi\ (\sigma_2,+,s_5)\ \varphi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge Rwuv)$	$\varphi \succ_1 \psi$ [2]
$\varphi\ (\sigma_6,-,s_6)\ \psi$ $= \psi\ (\sigma_3,+,s_5)\ \varphi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge Ruwv)$	
The implication orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1,-,s_3)\ )$: dual residuations		
$\varphi\ (\sigma_1,-,s_3)\ \psi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee -Ruvw)$	$\varphi \supset_3 \psi$ [2]
$\varphi\ (\sigma_2,-,s_3)\ \psi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee -Rwvu)$	
$\varphi\ (\sigma_3,+,s_1)\ \psi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge Rvwu)$	$\varphi \otimes_2 \psi$ [2]
$\varphi\ (\sigma_4,-,s_2)\ \psi$ $= \psi\ (\sigma_1,+,s_3)\ \varphi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee -Rvuw)$	
$\varphi\ (\sigma_5,-,s_2)\ \psi$ $= \psi\ (\sigma_2,-,s_3)\ \varphi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee -Rwuv)$	$\varphi \subset_1 \psi$ [2]
$\varphi\ (\sigma_6,+,s_1)\ \psi$ $= \psi\ (\sigma_3,+,s_1)\ \varphi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge Ruwv)$	
The coimplication orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1,-,s_2)\ )$: dual residuations		
$\varphi\ (\sigma_1,-,s_2)\ \psi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee -Ruvw)$	$\varphi \subset_3 \psi$ [2]
$\varphi\ (\sigma_2,+,s_1)\ \psi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge Rwvu)$	
$\varphi\ (\sigma_3,-,s_3)\ \psi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee -Rvwu)$	$\varphi \supset_2 \psi$ [2]
$\varphi\ (\sigma_3,-,s_3)\ \psi$ $= \psi\ (\sigma_1,-,s_2)\ \varphi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \in [\![\psi]\!] \vee -Rvuw)$	
$\varphi\ (\sigma_5,+,s_1)\ \psi$ $= \psi\ (\sigma_2,+,s_1)\ \varphi$	$\exists uv\,(u \in [\![\varphi]\!] \wedge v \in [\![\psi]\!] \wedge Rwuv)$	$\varphi \otimes_1 \psi$ [2]
$\varphi\ (\sigma_6,-,s_2)\ \psi$ $= \psi\ (\sigma_3,-,s_3)\ \varphi$	$\forall uv\,(u \in [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee -Ruwv)$	

Figure 9: The 2–ary gaggle connectives

Gaggle connective	Truth condition	Substructural connective
The symmetrical disjunction orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1,+,s_4)\ )$: dual residuations		
$\varphi\ (\sigma_1,+,s_4)\ \psi$	$\forall uv\,(u \in \llbracket\varphi\rrbracket \lor v \in \llbracket\psi\rrbracket \lor Ruvw)$	$\varphi \oplus \psi$ [22]
$\varphi\ (\sigma_2,-,s_5)\ \psi$	$\exists uv\,(u \in \llbracket\varphi\rrbracket \land v \notin \llbracket\psi\rrbracket \land -Rwvu)$	$\varphi \mathrel{-\!\!<} \psi$ [22]
$\varphi\ (\sigma_3,-,s_5)\ \psi$	$\exists uv\,(u \in \llbracket\varphi\rrbracket \land v \notin \llbracket\psi\rrbracket \land -Rvwu)$	
$\varphi\ (\sigma_4,+,s_4)\ \psi$ $= \psi\ (\sigma_1,+,s_4)\ \varphi$	$\forall uv\,(u \in \llbracket\varphi\rrbracket \lor v \in \llbracket\psi\rrbracket \lor Rvuw)$	
$\varphi\ (\sigma_5,-,s_6)\ \psi$ $= \psi\ (\sigma_2,-,s_5)\ \varphi$	$\exists uv\,(u \notin \llbracket\varphi\rrbracket \land v \in \llbracket\psi\rrbracket \land -Rwuv)$	
$\varphi\ (\sigma_6,+,s_6)\ \psi$ $= \psi\ (\sigma_3,-,s_5)\ \varphi$	$\exists uv\,(u \notin \llbracket\varphi\rrbracket \land v \in \llbracket\psi\rrbracket \land -Ruwv)$	$\varphi \mathrel{>\!\!-} \psi$ [22]
The symmetrical implication orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1,+,s_3)\ )$: dual residuations		
$\varphi\ (\sigma_1,+,s_3)\ \psi$	$\forall uv\,(u \notin \llbracket\varphi\rrbracket \lor v \in \llbracket\psi\rrbracket \lor Ruvw)$	
$\varphi\ (\sigma_2,+,s_3)\ \psi$	$\forall uv\,(u \notin \llbracket\varphi\rrbracket \lor v \in \llbracket\psi\rrbracket \lor Rwvu)$	
$\varphi\ (\sigma_3,-,s_1)\ \psi$	$\exists uv\,(u \in \llbracket\varphi\rrbracket \land v \in \llbracket\psi\rrbracket \land -Rvwu)$	
$\varphi\ (\sigma_4,+,s_2)\ \psi$ $= \psi\ (\sigma_1,+,s_3)\ \varphi$	$\forall uv\,(u \in \llbracket\varphi\rrbracket \lor v \notin \llbracket\psi\rrbracket \lor Rvuw)$	
$\varphi\ (\sigma_5,+,s_2)\ \psi$ $= \psi\ (\sigma_2,+,s_3)\ \varphi$	$\forall uv\,(u \in \llbracket\varphi\rrbracket \lor v \notin \llbracket\psi\rrbracket \lor Rwuv)$	
$\varphi\ (\sigma_6,-,s_1)\ \psi$ $= \psi\ (\sigma_3,-,s_1)\ \varphi$	$\exists uv\,(u \in \llbracket\varphi\rrbracket \land v \in \llbracket\psi\rrbracket \land -Ruwv)$	
The symmetrical coimplication orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1,+,s_2)\ )$: dual residuations		
$\varphi\ (\sigma_1,+,s_2)\ \psi$	$\forall uv\,(u \in \llbracket\varphi\rrbracket \lor v \notin \llbracket\psi\rrbracket \lor Ruvw)$	
$\varphi\ (\sigma_2,-,s_1)\ \psi$	$\exists uv\,(u \in \llbracket\varphi\rrbracket \land v \in \llbracket\psi\rrbracket \land -Rwvu)$	
$\varphi\ (\sigma_3,+,s_3)\ \psi$	$\forall uv\,(u \notin \llbracket\varphi\rrbracket \lor v \in \llbracket\psi\rrbracket \lor Rvwu)$	
$\varphi\ (\sigma_4,+,s_3)\ \psi$ $= \psi\ (\sigma_1,+,s_2)\ \varphi$	$\forall uv\,(u \notin \llbracket\varphi\rrbracket \lor v \in \llbracket\psi\rrbracket \lor Rvuw)$	
$\varphi\ (\sigma_5,-,s_1)\ \psi$ $= \psi\ (\sigma_2,-,s_1)\ \varphi$	$\exists uv\,(u \in \llbracket\varphi\rrbracket \land v \in \llbracket\psi\rrbracket \land -Rwuv)$	
$\varphi\ (\sigma_6,+,s_2)\ \psi$ $= \psi\ (\sigma_3,+,s_3)\ \varphi$	$\forall uv\,(u \in \llbracket\varphi\rrbracket \lor v \notin \llbracket\psi\rrbracket \lor Ruwv)$	

Figure 10: The 2–ary gaggle connectives

Gaggle connective	Truth condition	Substructural connective
The stroke orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1,+,s_7)\ )$: Galois connections		
$\varphi\ (\sigma_1,+,s_7)\ \psi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge Ruvw)$	$\varphi \mid_3 \psi$ [1, 22]
$\varphi\ (\sigma_2,+,s_7)\ \psi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge Rwvu)$	
$\varphi\ (\sigma_3,+,s_7)\ \psi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge Rvwu)$	
$\varphi\ (\sigma_4,+,s_7)\ \psi$ $= \psi\ (\sigma_1,+,s_7)\ \varphi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge Rvuw)$	
$\varphi\ (\sigma_5,+,s_7)\ \psi$ $= \psi\ (\sigma_2,+,s_7)\ \varphi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge Rwuv)$	$\varphi \mid_1 \psi$ [1, 22]
$\varphi\ (\sigma_6,+,s_7)\ \psi$ $= \psi\ (\sigma_3,+,s_7)\ \varphi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge Ruwv)$	$\varphi \mid_2 \psi$ [1, 22]
The dagger orbit $\mathcal{O}_{\alpha_3}(\ (\sigma_1,-,s_8)\ )$: Galois connections		
$\varphi\ (\sigma_1,-,s_8)\ \psi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee -Ruvw)$	$\varphi\ \downarrow_3\ \psi$ [1, 22]
$\varphi\ (\sigma_2,-,s_8)\ \psi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee -Rwvu)$	
$\varphi\ (\sigma_3,-,s_8)\ \psi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee -Rvwu)$	
$\varphi\ (\sigma_4,-,s_8)\ \psi$ $= \psi\ (\sigma_1,-,s_8)\ \varphi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee -Rvuw)$	
$\varphi\ (\sigma_5,-,s_8)\ \psi$ $= \psi\ (\sigma_2,-,s_8)\ \varphi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee -Rwuv)$	$\varphi\ \downarrow_1\ \psi$ [1, 22]
$\varphi\ (\sigma_6,-,s_8)\ \psi$ $= \psi\ (\sigma_3,-,s_8)\ \varphi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee -Ruwv)$	$\varphi\ \downarrow_2\ \psi$ [1, 22]

Figure 11: The 2–ary gaggle connectives

Definition 34 (Logic)**.** A *logic* is a triple $\mathsf{L} = (\mathcal{L}, E, \models)$ where

- $\mathcal{L}$ is a *language* defined as a set of well-formed expressions built from a set of *connectives* C and a set of *atoms* $\mathbb{P}$;
- E is a *class of pointed models or frames*;
- $\models$ is a *satisfaction relation* which relates in a compositional manner elements of $\mathcal{L}$ to models of E by means of so-called *truth conditions*.

A $\mathcal{L}$–*consecution* is an expression of the form $\varphi \vdash \psi$, $\vdash \psi$ or $\varphi \vdash$, where $\varphi, \psi \in \mathcal{L}$. □

Our definition of a calculus and of an inference rule is taken from [32].

Gaggle connective	Truth condition	Substructural connective
The symmetrical stroke orbit $\mathcal{O}_{\alpha_3}$ ($(\sigma_1, -, s_7)$): dual Galois connections		
$\varphi\ (\sigma_1, -, s_7)\ \psi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Ruvw)$	
$\varphi\ (\sigma_2, -, s_7)\ \psi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Rwvu)$	
$\varphi\ (\sigma_3, -, s_7)\ \psi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Rvwu)$	
$\varphi\ (\sigma_4, -, s_7)\ \psi$ $= \psi\ (\sigma_1, -, s_7)\ \varphi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Rvuw)$	
$\varphi\ (\sigma_5, -, s_7)\ \psi$ $= \psi\ (\sigma_2, -, s_7)\ \varphi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Rwuv)$	
$\varphi\ (\sigma_6, -, s_7)\ \psi$ $= \psi\ (\sigma_3, -, s_7)\ \varphi$	$\exists uv\,(u \notin [\![\varphi]\!] \wedge v \notin [\![\psi]\!] \wedge -Ruwv)$	
The symmetrical dagger orbit $\mathcal{O}_{\alpha_3}$ ($(\sigma_1, +, s_8)$): dual Galois connections		
$\varphi\ (\sigma_1, +, s_8)\ \psi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Ruvw)$	
$\varphi\ (\sigma_2, +, s_8)\ \psi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Rwvu)$	
$\varphi\ (\sigma_3, +, s_8)\ \psi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Rvwu)$	
$\varphi\ (\sigma_4, +, s_8)\ \psi$ $= \psi\ (\sigma_1, +, s_8)\ \varphi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Rvuw)$	
$\varphi\ (\sigma_5, +, s_8)\ \psi$ $= \psi\ (\sigma_2, +, s_8)\ \varphi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Rwuv)$	
$\varphi\ (\sigma_6, +, s_8)\ \psi$ $= \psi\ (\sigma_3, +, s_8)\ \varphi$	$\forall uv\,(u \notin [\![\varphi]\!] \vee v \notin [\![\psi]\!] \vee Ruwv)$	

Figure 12: The 2–ary gaggle connectives

Definition 35 (Conservativity)**.** Let $\mathsf{L} = (\mathcal{L}, E, \models)$ and $\mathsf{L}' = (\mathcal{L}', E', \models')$ be two logics such that $\mathcal{L} \subseteq \mathcal{L}'$. We say that L' is a *conservative extension* of L when $\{\varphi \in \mathcal{L} \mid \models_{\mathsf{L}} \varphi\} = \mathcal{L} \cap \{\varphi' \in \mathcal{L}' \mid \models'_{\mathsf{L}'} \varphi'\}$. □

Definition 36 (Calculus and sequent calculus)**.** Let $\mathsf{L} = (\mathcal{L}, E, \models)$ be a logic. A *calculus* P for $\mathcal{L}$ is a set of elements of $\mathcal{L}$ called *axioms* and a set of *inference rules*. Most often, one can effectively decide whether a given element of $\mathcal{L}$ is an axiom. To be more precise, an *inference rule* R for $\mathcal{L}$ is a relation among elements of $\mathcal{L}$ such that there is a unique $l \in \mathbb{N}^*$ such that, for all $\varphi, \varphi_1, \ldots, \varphi_l \in \mathcal{L}$, one can effectively decide whether $(\varphi_1, \ldots, \varphi_l, \varphi) \in R$. The elements $\varphi_1, \ldots, \varphi_l$ are called the *premises* and φ is called the *conclusion* and we say that φ is a *direct consequence of* $\varphi_1, \ldots, \varphi_l$

by virtue of R. Let $\Gamma \subseteq \mathcal{L}$ and let $\varphi \in \mathcal{L}$. We say that φ is *provable* (from Γ) in P or a *theorem* of P, denoted $\vdash_{\mathsf{P}} \varphi$ (resp. $\Gamma \vdash_{\mathsf{P}} \varphi$), when there is a *proof* of φ (from Γ) in P, that is, a finite sequence of formulas ending in φ such that each of these formulas is:

1. either an instance of an axiom of P (or a formula of Γ);
2. or the direct consequence of preceding formulas by virtue of an inference rule R.

If $\mathcal{S}$ is a set of $\mathcal{L}$–consecutions, this set $\mathcal{S}$ can be viewed as a language. In that case, we call *sequent calculus for* $\mathcal{S}$ a calculus for $\mathcal{S}$.

Axioms and inference rules are often represented by means of *axiom schemas* and *inference rule schemas*, that is, expressions of the following form, depending on whether we deal with formulas of $\mathcal{L}$ or $\mathcal{L}$–consecutions:

Axiom schemas:

α $\qquad\qquad$ $A \models B$

Inference rule schemas:

$$\frac{\alpha_1 \quad \dots \quad \alpha_n}{\alpha} \qquad\qquad \frac{A_1 \models B_1 \quad \dots \quad A_n \models B_n}{A \models B}$$

where $\alpha_1, \dots, \alpha_n, \alpha$ are built up from *variables* often denoted $\varphi, \psi, \dots$ and the connectives of C and, likewise, $A_1, \dots, A_n, B_1, \dots, B_n, A, B$ are built up from *variables* often denoted $X, Y, \dots$ and the connectives of C. In this representation, inference rules and axioms schemas are closed by *uniform substitution*: each variable can be replaced uniformly by *any* well-formed expression of $\mathcal{L}$.

An inference rule R' is *derivable from an inference rule R in* P when there is a finite sequence of rules $R_1, \dots, R_n$ of P, with at least one of them equal to R, such that $R' = R_1 \circ \dots \circ R_n$. □

Definition 37 (Truth, validity, logical consequence)**.** Let $\mathsf{L} = (\mathcal{L}, E, \models)$ be a logic. Let $M \in E$, $\varphi \in \mathcal{L}$, R be an inference rule for $\mathcal{L}$ and S, S' be either inference rules for $\mathcal{L}$ or formulas of $\mathcal{L}$. If Γ is a set of formulas or inference rules, we write $M \models \Gamma$ when for all $\varphi \in \Gamma$, we have $M \models \varphi$. Then, we say that

- φ is *true* (*satisfied*) at M or M is a *model* of φ when $M \models \varphi$;
- φ is *valid*, denoted $\models_{\mathsf{L}} \varphi$, when for all models $M \in E$, we have $M \models \varphi$;
- R is *true* (*satisfied*) at M or M is a *model* of R, denoted $M \models R$, when for all $(\varphi_1, \dots, \varphi_l, \varphi) \in R$, if $M \models \varphi_i$ for all $i \in \{1, \dots, l\}$, then $M \models \varphi$.

An inference rule R is *equivalent* to another inference rule R' iff for all $M \in E$, $M \models R$ iff $M \models R'$. $\square$

Definition 38 (Soundness and completeness)**.** Let $\mathsf{L} = (\mathcal{L}, E, \models)$ be a logic. Let P be a calculus for $\mathcal{L}$. Then,

- P is *sound* for the logic L when for all $\varphi \in \mathcal{L}$, if $\vdash_{\mathsf{P}} \varphi$, then $\models_{\mathsf{L}} \varphi$.
- P is *(strongly) complete* for the logic L when for all $\varphi \in \mathcal{L}$ (and all $\Gamma \subseteq \mathcal{L}$), if $\models_{\mathsf{L}} \varphi$, then $\vdash_{\mathsf{P}} \varphi$ (resp. if $\Gamma \models_{\mathsf{L}} \varphi$, then $\Gamma \vdash_{\mathsf{P}} \varphi$). $\square$

7.2 Structures and consecutions

In order to provide a sound and complete calculus for a gaggle logic based on a set of connectives $\mathsf{C} \subseteq \mathbb{C}$, we will need to resort to the connectives of C which are in the orbits of the free action $\alpha_n * \beta_n$ (for appropriate ns). We introduce these extra connectives in the language as *structural* connectives: they will appear in the proof derivations but not in the formulas proved by the calculus.

Definition 39 (Structural connectives)**.** *(Gaggle) structural connectives*, denoted $[\mathbb{C}]$, are a copy of the connectives: for all $\mathsf{C} \subseteq \mathbb{C}$,

$$[\mathsf{C}] \triangleq \{[\circledast] \mid \circledast \in \mathsf{C}\}.$$

Structural connectives are denoted $[p], [p_1], [p_2], \ldots$ and $[\circledast], [\circledast_1], [\circledast_2], \ldots$ For all $\circledast = (\sigma, \pm, s) \in \mathbb{C}$, the *arity*, *signature*, *tonicity signature*, *quantification signature* of $[\circledast]$ are the same as $\circledast$.

We also introduce the *(Boolean) structural connective* $\,,\,$. $\square$

Definition 40 (Structural gaggle language and consecutions)**.** The *structural gaggle language* $[\mathcal{L}]$ is the smallest set that contains the gaggle language $\mathcal{L}$, the structures $*\varphi$ for all $\varphi \in \mathcal{L}$ as well as $[\mathbb{P}]$ and that is closed under the structural connectives of $[\mathbb{C}] \cup \{\,,\,\}$.

A $\mathcal{L}$*–consecution* (resp. $[\mathcal{L}]$*–consecution*) is an expression of the form $\varphi \vdash \psi$ (resp. $X \vdash Y$), where $\varphi, \psi \in \mathcal{L}$ (resp. $X, Y \in [\mathcal{L}]$). The set of all (Boolean) $\mathcal{L}$–consecutions (resp. $[\mathcal{L}]$–consecutions) is denoted $\mathcal{S}$ (resp. $[\mathcal{S}]$) and the set of all $\mathcal{L}^0$–consecutions is denoted $\mathcal{S}^0$. If $\mathsf{C} \subseteq \mathbb{C}$ then an element of $[\mathcal{L}]_{\mathsf{C}}$ (resp. $\mathcal{S}^0_{\mathsf{C}}$, $\mathcal{S}_{\mathsf{C}}$, $[\mathcal{S}]_{\mathsf{C}}$) is an element of $[\mathcal{L}]$ (resp. $\mathcal{S}^0$, $\mathcal{S}$, $[\mathcal{S}]$) which contains only connectives of $[\mathsf{C}]$.

Elements of $\mathcal{L}$ (resp. $[\mathcal{L}]$ and $[\mathcal{S}]$) are called *formulas* (resp. *structures* and *consecutions*); they are denoted $\varphi, \psi, \alpha, \ldots$ (resp. $X, Y, A, B, \ldots$ and $X \vdash Y, A \vdash B, \ldots$). $\square$

Definition 41 (Boolean negation)**.** Let $X \in [\mathcal{L}]$ be a structure. The *Boolean negation of* X, denoted $*X$, is defined inductively as follows:

$$*X \triangleq \begin{cases} [-\circledast]\,(X_1,\ldots,X_n) & \text{if } X = [\circledast]\,(X_1,\ldots,X_n) \\ (*X_1 \;,\; *X_2) & \text{if } X = (X_1 \;,\; X_2) \\ \varphi & \text{if } X = *\varphi \\ *\varphi & \text{if } X = \varphi \in \mathcal{L} \end{cases}$$

where $-\circledast$ was defined in Definition 20. □

Note that from that definition, for all structures $X \in [\mathcal{L}]$, it follows that $**X = X$.

Definition 42 (Formula associated to a structure)**.** We define inductively the function τ_0 and τ_1 from structures of $[\mathcal{L}]$ to formulas of $\mathcal{L}$ as follows: for all $i \in \{0,1\}$, all $\circledast = (\sigma, \pm, (Æ, (\pm_1,\ldots,\pm_n)))$,

$$\begin{array}{rcl} \tau_i(\varphi) & \triangleq & \varphi \\ \tau_i(*\varphi) & \triangleq & \neg\varphi \\ \tau_0\,(X \;,\; Y) & \triangleq & (\tau_0(X) \wedge \tau_0(Y)) \\ \tau_1\,(X \;,\; Y) & \triangleq & (\tau_1(X) \vee \tau_1(Y)) \\ \tau_i\,([\circledast]\,(X_1,\ldots,X_n)) & \triangleq & \circledast(\tau_{i_1}(X_1),\ldots,\tau_{i_n}(X_n)) \end{array}$$

where for all $j \in [\![1;n]\!]$, $\tau_{i_j}(X_j) \triangleq \begin{cases} \tau_i(X_j) & \text{if } \pm_j = + \\ \tau_{1-i}(X_j) & \text{if } \pm_j = - \end{cases}$.

Then, we define the function τ from $[\mathcal{L}]$–consecutions of $[\mathcal{S}]$ to $\mathcal{L}$–consecutions of $\mathcal{S}$ as follows:

$$\tau(X \models Y) \quad \triangleq \quad \tau_0(X) \models \tau_1(Y)$$ □

Instead of a single structural connective $\;,\;$, we could introduce two Boolean structural connectives $[\wedge]$, $[\vee]$ as a copy of the Boolean connectives $\wedge, \vee$, like for the other gaggle connectives $\circledast$. This would not be usual but in line with our approach. This would greatly simplify the definition of the function τ since the interpretation of the structural connectives would then not be context-dependent as here. In particular one would not need two functions τ_0 and τ_1. We proceed as follows on the one hand in order to stay in line with current practice and on the other hand because it simplifies the subsequent calculus $\mathsf{GGL_C}$ of Figure 13: we use one structural connective $(\;,\;)$ instead of two ($[\wedge]$ and $[\vee]$). This said, it would be easily possible to adapt and rewrite the calculus $\mathsf{GGL_C}$ with these two structural connectives $[\wedge]$ and $[\vee]$: the structural connective $\;,\;$ would need to be replaced by $[\wedge]$ in the premise of (dr_2) and in $(\mathsf{B}\vdash), (\mathsf{CI}\vdash), (\mathsf{K}\vdash), (\wedge\vdash)$ and by $[\vee]$ in the conclusion of (dr_2) and in $(\vdash\mathsf{B}), (\vdash\mathsf{CI}), (\vdash\mathsf{K}), (\vdash\vee)$ (see below).

Definition 43 (Interpretation of gaggle structures and consecutions)**.** Let $\mathsf{C} \subseteq \mathbb{C}$ and let $M = (W, \mathcal{R})$ be a C–model. We extend the interpretation function $[\![\cdot]\!]^M$ of $\mathcal{L}_\mathsf{C}$ in M to $\mathcal{L}_\mathsf{C}$–consecutions of $\mathcal{S}_\mathsf{C}$ as follows: for all $\varphi, \psi \in \mathcal{L}_\mathsf{C}$ and all $w \in W$, we have that $w \in [\![\varphi \models \psi]\!]^M$ iff if $w \in [\![\varphi]\!]^M$ then $w \in [\![\psi]\!]^M$, we have that $w \in [\![\ \models \psi]\!]^M$ iff $w \in [\![\psi]\!]^M$ and we have that $w \in [\![\varphi \models\ ]\!]^M$ iff $w \notin [\![\varphi]\!]^M$. We then extend in a natural way the interpretation function $[\![\cdot]\!]^M$ of $\mathcal{L}_\mathsf{C}$ in M to $[\mathcal{L}]_\mathsf{C}$–consecutions of $[\mathcal{S}]_\mathsf{C}$ as follows: for all $X \in \mathcal{L}_\mathsf{C}$, all $X \models Y \in [\mathcal{S}]_\mathsf{C}$ and all $w \in W$, we have that $w \in [\![X \models Y]\!]^M$ if, and only if, $w \in [\![\tau(X \models Y)]\!]^M$. If $\mathcal{E}_\mathsf{C}$ is a class of C–models, then the satisfaction relation $\Vdash\ \subseteq \mathcal{E}_\mathsf{C} \times [\mathcal{S}]_\mathsf{C}$ is defined like for formulas of $\mathcal{L}$. □

7.3 Our display calculus

We introduce a calculus for *Boolean* basic gaggle logics. Our calculus is defined relatively to an orbit/family of connectives. This means that if we have a basic gaggle logic defined on the basis of some connectives C and if we want to obtain a sound and complete calculus for that logic, we need to consider in the proof system the following associated set of connectives:

$$\mathcal{O}(\mathsf{C}) \triangleq \bigcup_{\otimes \in \mathsf{C}} \left\{ \mathcal{O}_{\alpha_n * \beta_n}(\otimes) \mid a(\otimes) = n \right\} \tag{4}$$

This set of connectives $\mathcal{O}(\mathsf{C})$ is stable under the free action $\alpha_n * \beta_n$: for all $\otimes \in \mathcal{O}(\mathsf{C})$, we have that $\mathcal{O}_{\alpha_n * \beta_n}(\otimes) \subseteq \mathcal{O}(\mathsf{C})$. This is because in the completeness proof, we need to apply the abstract law of residuation for any arguments j (associated to the residuation operator of Definition 18) and consider the Boolean negation for each connective. This entails that we must consider the orbits of the connectives of C under the free action $\alpha_n * \beta_n$.

Definition 44. Let $\mathsf{C} \subseteq \mathbb{C}$. We denote by GGL_C the calculus of Figure 13 where the introduction rules $(\vdash \otimes)$ and $(\otimes \vdash)$ are defined for the connectives $\otimes$ of C and where the rule (dr_1) is defined for the elements τ of an arbitrary set of generators of $\mathfrak{S}_{n+1}$ (for each n ranging over the arities of the connectives of C). □

Theorem 45 (Soundness and strong completeness)**.** *Let $\mathsf{C} \subseteq \mathbb{C}$ be such that $\mathcal{O}(\mathsf{C}) = \mathsf{C}$. The calculus GGL_C is sound and strongly complete for the Boolean basic gaggle logic $(\mathcal{S}_\mathsf{C}, \mathcal{M}_\mathsf{C}, \Vdash)$.*

Proof: See the Appendix, Section B. □

Some comments about the rules of the calculus GGL_C are needed.

Structural rules:

$$\frac{((X\ ,\ Y)\ ,\ Z) \vdash U}{(X\ ,\ (Y\ ,\ Z)) \vdash U}\ (\mathsf{B}\vdash) \qquad \frac{(X\ ,\ Y) \vdash U}{(Y\ ,\ X) \vdash U}\ (\mathsf{CI}\vdash)$$

$$\frac{X \vdash U}{(X\ ,\ Y) \vdash U}\ (\mathsf{K}\vdash) \qquad \frac{(X\ ,\ X) \vdash U}{X \vdash U}\ (\mathsf{WI}\vdash)$$

$$\frac{U \vdash \varphi \quad \varphi \vdash V}{U \vdash V}\ cut$$

Display rules:

$$\frac{S\left([\circledast], X_1, \ldots, X_n, X_{n+1}\right)}{S\left([\tau\circledast], X_{\tau(1)}, \ldots, X_{\tau(n)}, X_{\tau(n+1)}\right)}\ (\mathsf{dr}_1) \qquad \frac{(X\ ,\ Y) \vdash Z}{X \vdash (Z\ ,\ *Y)}\ (\mathsf{dr}_2)$$

Introduction rules:

$$\frac{U \vdash\ *\varphi}{U \vdash \neg\varphi}\ (\vdash\neg) \qquad \frac{*\varphi \vdash U}{\neg\varphi \vdash U}\ (\neg\vdash)$$

$$\frac{X \vdash \varphi \quad Y \vdash \psi}{(X\ ,\ Y) \vdash (\varphi \wedge \psi)}\ (\vdash\wedge) \qquad \frac{(\varphi\ ,\ \psi) \vdash U}{(\varphi \wedge \psi) \vdash U}\ (\wedge\vdash)$$

$$\frac{U \vdash (\varphi\ ,\ \psi)}{U \vdash (\varphi \vee \psi)}\ (\vdash\vee) \qquad \frac{\varphi \vdash X \quad \psi \vdash Y}{(\varphi \vee \psi) \vdash (X\ ,\ Y)}\ (\vee\vdash)$$

$$\frac{U_1 \vdash V_1 \quad \ldots \quad U_n \vdash V_n}{S\left([\circledast], X_1, \ldots, X_n, \circledast(\varphi_1, \ldots, \varphi_n)\right)}\ (\vdash\circledast) \qquad \frac{S\left([\circledast], \varphi_1, \ldots, \varphi_n, U\right)}{S\left(\circledast, \varphi_1, \ldots, \varphi_n, U\right)}\ (\circledast\vdash)$$

In rules $(\vdash\circledast)$ and $(\circledast\vdash)$, for all $\circledast = (\sigma, \pm, (Æ, (\pm_1, \ldots, \pm_n))) \in \mathsf{C}$:

- for all $j \in [\![1;n]\!]$, we set $U_j \vdash V_j \triangleq \begin{cases} X_j \vdash \varphi_j & \text{if } \pm_j \pm (Æ) = - \\ \varphi_j \vdash X_j & \text{if } \pm_j \pm (Æ) = + \end{cases}$
  such that, in rule $(\vdash\circledast)$, for all j X_j is not empty and if φ_j is empty for some j then $\circledast(\varphi_1, \ldots, \varphi_n)$ is also empty.
- for all $\star \in \{\circledast, [\circledast]\}$, $S(\circledast, X_1, \ldots, X_n, X) \triangleq \begin{cases} \star(X_1, \ldots, X_n) \vdash X & \text{if } Æ = \exists \\ X \vdash\ \star(X_1, \ldots, X_n) & \text{if } Æ = \forall \end{cases}$.

If X is empty then $*X$ is empty and $(X\ ,\ Y)$ and $(Y\ ,\ X)$ are equal to Y.

Figure 13: Calculus $\mathsf{GGL}_{\mathsf{C}}$

• The axioms and inference rules for atoms p are special instances of the rules $(\vdash \circledast)$ and $(\circledast \vdash)$ of Figure 13. With $\circledast = p$, we have that $n = 0$ and, replacing $\circledast$ with p in $(\vdash \circledast)$, we obtain the inference rules below. Note that $(\vdash p)$ is in fact an axiom.

$$\frac{}{S([p],p)}\ (\vdash p) \qquad\qquad \frac{S\,([p]\,,X)}{S\,(p,X)}\ (p\vdash)$$

where, if $\circledast$ is p or $[p]$, then $S(\circledast, X) \triangleq \begin{cases} \circledast \models X & \text{if } Æ = \exists \\ X \models \circledast & \text{if } Æ = \forall \end{cases}$.

Hence, for all $p = (1, \pm, Æ)$, if $Æ = \exists$ then $(\vdash p)$ and $(p \vdash)$ rewrite as follows:

$$\frac{}{[p] \models p}\ (\vdash p) \qquad\qquad \frac{[p] \models X}{p \models X}\ (p\vdash) \tag{5}$$

and if $Æ = \forall$ then $(\vdash p)$ and $(p \vdash)$ rewrite as follows:

$$\frac{}{p \models [p]}\ (\vdash p) \qquad\qquad \frac{X \models [p]}{X \models p}\ (p\vdash) \tag{6}$$

Note that in both cases, the standard axiom $p \models p$ is derivable by applying $(p \vdash)$ once again to $[p] \models p$ or $p \models [p]$. If $[p]$ is replaced by $\mathbf{I}$ and p by $\top$ in the first pair and if $[p]$ is replaced by $\mathbf{I}$ and p by $\bot$ in the second pair then we obtain respectively the operational rules $(\models \top)$, $(\top \models)$, $(\bot \models)$ and $(\models \bot)$ of Kracht [27] and Belnap [6]. This is meaningful since truth constants can be seen as special atoms, those that are always true or always false. Then, one needs, like in the calculus **DLM** of Kracht [27], to impose some conditions on these atoms by means of particular structural inference rules so that these special atoms $\top$ and $\bot$ do behave as truth constants, as intended. Note that the reading of $\mathbf{I}$, either as $\top$ or as $\bot$, is clearly separated here by means of two structural constants, whereas in the literature it is disambiguated depending on the context, whether it is in antecedent part or consequent part of a consecution. Alternatively, one can easily prove (by extending the proof of Section **??**) that adding the following axioms to our calculus $\mathsf{GGL_C}$ is enough to capture the standard truth constants $\top$ and $\bot$:

$$\frac{}{\bot \models}\ (\bot\vdash) \qquad\qquad \frac{}{\models \top}\ (\vdash \top)$$

• The Boolean operator $*$ transforms the structures on which it is applied. It does not function as an operator applied externally on structures, it modifies them internally. Hence, for example, for any structure $[\circledast]\,(X_1, \ldots, X_n)$, $*\,[\circledast]\,(X_1, \ldots, X_n)$ is equal to $[-\circledast]\,(X_1, \ldots, X_n)$. In that sense, it is formally different from the usual structural connective $*$ used in display logics, even if its semantic meaning is the same (it behaves as a Boolean negation). Moreover, because by Definition 41 $**X = X$, the following rule is a reformulation of the display rule (dr_2) (premise and conclusion are turned upside down):

$$\frac{X \vdash (Y \;,\; Z)}{\overline{(X \;,\; *Z) \vdash Y}}$$

• Because of our convention that if X is empty then $(X \;,\; Y)$ and $(Y \;,\; X)$ are equal to Y, the following rules are specific instances of the display rule (dr_2):

$$\frac{(X \;,\; Y) \vdash}{\overline{X \vdash\; *Y}} \qquad\qquad \frac{\vdash (Z \;,\; Y)}{\overline{*Y \vdash Z}}$$

Likewise, if $\circledast = (\sigma, \pm, (Æ, (\pm_1, \ldots, \pm_n)))$ is such that, for example, $Æ = \exists$ and $\pm_j = +$, then the following rule is an instance of the rule $(\vdash \circledast)$, because of our conventions about empty structures in the rule $(\vdash \circledast)$:

$$\frac{U_1 \vdash V_1 \quad \ldots \quad X_j \vdash \quad \ldots \quad U_n \vdash V_n}{[\circledast]\,(X_1, \ldots, X_j, \ldots, X_n) \vdash} \tag{7}$$

• The introduction rule $(\vdash \circledast)$ of our calculus is a direct translation in gaggle logics of the tonicity relations of Theorem 10. Likewise, the structural rule (dr_1) is a translation and a generalization of the abstract law of residuation of Theorem 10 (see Proposition 27).

• As shown in Example 26, $\wedge$ and $\vee$ can be formalized by the gaggle connectives $(\sigma_1, +, s_1)$ and $(\sigma_1, -, s_4)$ if these are interpreted on identity ternary relations (which can be obtained by imposing the validity of the classic structural rules involving these connectives). Hence, unsurprisingly, rules $(\vdash \vee)$ and $(\wedge \vdash)$ are instances of the (gaggle) rule $(\circledast \vdash)$ and rules $(\vdash \wedge)$ and $(\vee \vdash)$ are also instances of the (gaggle) rule $(\vdash \circledast)$.

This said, one could equivalently replace $(\vdash \wedge)$ and $(\vee \vdash)$ by their extensional/additive version $(\vdash \wedge)'$ and $(\vee \vdash)'$ of Proposition 46 and still obtain the completeness of the resulting calculus. In fact, completeness still holds if one also removes the contraction rule $(\mathsf{WI} \vdash)$ because a contraction is hidden in the extensional/additive version of the conjunction and disjunction rule. Yet, one needs the contraction rule $(\mathsf{WI} \vdash)$ explicitly to prove cut elimination, in particular for condition $(C8)$ with the conjunction case (see Theorem 49). So, we prefer to take in our calculus the intensional/multiplicative version $(\vdash \wedge)$ and $(\vee \vdash)$ of the conjunction and disjunction rules because they are instances of the general rules $(\circledast\vdash)$ and $(\vdash\circledast)$ for gaggle connectives.

• Our calculus has the subformula property, but not the substructure property: every formula appearing in a cut–free proof of a consecution is a subformula of a formula of the final consecution.

• In the calculus $\mathsf{GGL_C}$, we do not need to consider *all* permutations τ of the symmetric group $\mathfrak{S}_{n+1}$. In fact, it suffices to consider only a set of generators of $\mathfrak{S}_{n+1}$ because rules for any permutations are derivable from these rules for generators as the following proposition shows. One could naturally consider transpositions because they generate the symmetric group and correspond to residuation operations. One could consider as well other generators of the symmetric group $\mathfrak{S}_{n+1}$, such as the pair $\{(n\ n+1),(1\ 2\ \ldots\ n+1)\}$ or the set of generators $\{(1\ 2),(2\ 3),\ldots,(i\ i+1),\ldots,(n\ n+1)\}$ or $(1\ 2)$ together with the 3–cycles (see Section 3). Hence, one can reduce the number of inference rules (dr_1) from $(n+1)!$ to 2: it suffices to define the calculus $\mathsf{GGL_C}$ only with the rules (dr_1) where $\tau=(n\ n+1)$ and $\tau=(1\ 2\ \ldots\ n+1)$ for example. Indeed, the rules (dr_1) with $\tau\in\mathfrak{S}_{n+1}$ different from $(n\,n+1)$ and $(1\,2\,\ldots\,n+1)$ are all derivable from these two rules since these two cycles generate $\mathfrak{S}_{n+1}$.

Proposition 46. *Let $C\subseteq\mathbb{C}$ and let $\circledast\in C$ be a n-ary connective. The following rules are all derivable in $\mathsf{GGL_C}$.*

$$\frac{X\vdash Y}{\overline{*Y\vdash *X}}\ (dr_2')\qquad\qquad \frac{S([\circledast],X_1,\ldots,X_j,\ldots,X_n,X)}{S([s_j\circledast],X_1,\ldots,*X_j,\ldots,X_n,X)}\ (sw^j)$$

$$\frac{*X\vdash Y}{\overline{*Y\vdash X}}\ (dr_2'')\qquad\qquad \frac{X\vdash *Y}{\overline{Y\vdash *X}}\ (dr_2''')$$

$$\frac{U\vdash((X\ ,\ Y)\ ,\ Z)}{\overline{U\vdash(X\ ,\ (Y\ ,\ Z))}}\ (\vdash B)\qquad\qquad \frac{U\vdash(X\ ,\ Y)}{U\vdash(Y\ ,\ X)}\ (\vdash CI)$$

$$\frac{U\vdash X}{U\vdash(X\ ,\ Y)}\ (\vdash K)\qquad\qquad \frac{U\vdash(X\ ,\ X)}{U\vdash X}\ (\vdash WI)$$

$$\frac{U\vdash\varphi\quad U\vdash\psi}{U\vdash(\varphi\wedge\psi)}\ (\vdash\wedge)'\qquad\qquad \frac{\varphi\vdash U\quad \psi\vdash U}{(\varphi\vee\psi)\vdash U}\ (\vee\vdash)'$$

The rule (dr_2') *is called the* Boolean negation rule *and the rule* (sw^j)*, for $j\in[\![1;n]\!]$, is called the* switch rule w.r.t. the j^{th} argument. *The rule* (dr_1) *is also derivable in* GGL_C, for all $\tau\in\mathfrak{S}_{n+1}$.

Proof: See the Appendix, Section A. □

8 Cut elimination and displayability

In this section, we prove that the cut rule can be eliminated from any proof of $\mathsf{GGL}_{\mathsf{C}}$. This result relies on the fact that our gaggle calculi are in fact display calculi and enjoy the display property: every substructure of a consecution provable in $\mathsf{GGL}_{\mathsf{C}}$ can be displayed as the sole antecedent or consequent of a provably equivalent consecution. In display calculi [6], the antecedent or consequent position depends on the kind of position in which the given substructure appears in the consecution: either in "antecedent part" or in "consequent part". In standard display logics, these two related notions are defined on the basis of the parity of the number of structural connectives $*$ that occur in front of the given substructure (odd or even). Since our framework is more abstract, we reformulate these two notions in a more abstract form based on the tonicity of the connectives that occur in front of the substructure. This leads us to define the following notions of 'protoantecedant part' and 'protoconsequent part'. A similar notion was defined by Goré [21] without Boolean structural connectives.

Definition 47 (Protoantecedent and protoconsequent part)**.** Let $X, Y, Z \in [\mathcal{L}]$ be structures. If Z is a substructure of X, then $tn(X, Z)$ is defined inductively as follows:

- if $X = Z$ then $tn(X, Z) \triangleq +$;
- if $X = *Y$ and Z appears in Y then $tn(X, Z) \triangleq -tn(Y, Z)$;
- if $X = (X_1\ ,\ X_2)$ and Z appears in X_j then $tn(X, Z) \triangleq tn(X_j, Z)$;
- if $X = [\circledast](X_1, \dots, X_n)$ and Z appears in X_j then $tn(X, Z) \triangleq tn(\circledast, j) tn(X_j, Z)$.

If $X \vdash Y$ is a $[\mathcal{L}]$–consecution, then X is called the *antecedent* and Y is called the *consequent* of $X \vdash Y$. If Z is a substructure of X or Y, Z is called a *protoantecedent part* (resp. *protoconsequent part*) of $X \vdash Y$ when $tn(X, Z) = +$ or $tn(Y, Z) = -$ (resp. $tn(X, Z) = -$ or $tn(Y, Z) = +$). □

Proposition 48 (Display property)**.** *Let $\mathsf{C} \subseteq \mathbb{C}$. For all $[\mathcal{L}]$–consecutions $X \vdash Y$ provable in $\mathsf{GGL}_{\mathsf{C}}$ and for all substructure Z of $X \vdash Y$,*

- *if Z is protoantecedent part of $X \vdash Y$ then there exists a structure $W \in [\mathcal{L}]$ such that $Z \vdash W$ is provably equivalent to $X \vdash Y$ in $\mathsf{GGL}_{\mathsf{C}}$;*
- *if Z is protoconsequent part of $X \vdash Y$ then there exists a structure $W \in [\mathcal{L}]$ such that $W \vdash Z$ is provably equivalent to $X \vdash Y$ in $\mathsf{GGL}_{\mathsf{C}}$.*

Hence, $\mathsf{GGL}_{\mathsf{C}}$ is a display calculus.

Proof: It follows from an inductive application of the display rules (dr_1) and (dr_2) on each substructure of X (or Y) containing Z, from the outermost one to the innermost one (Z itself). We use (dr_1) if we have to 'unfold' a structural gaggle connective $[\circledast]$ and (dr_2) (or one of its derived rules) if we have to 'unfold' the structural Boolean connective $\,,\,$. □

Theorem 49 (Cut–elimination)**.** *Let $\mathsf{C} \subseteq \mathbb{C}$. The calculus $\mathsf{GGL}_{\mathsf{C}}$ is cut–eliminable: it is possible to eliminate all occurrences of the cut rule from a given proof in order to obtain a cut-free proof of the same consecution.*

Proof: See the Appendix, Section C. □

As usual in proof theory and ever since Gentzen [18], the fact that the cut rule can be eliminated from any proof is of practical and theoretical importance and we easily obtain a number of significant results about our logics. This also holds in our setting.

Theorem 50 (Conservativity)**.** *If $\mathsf{C} \subseteq \mathsf{C}' \subseteq \mathbb{C}$ then the logic $\left(\mathcal{S}_{\mathsf{C}'}, \mathcal{M}_{\mathsf{C}'}, \Vdash\right)$ is a conservative extension of the logic $\left(\mathcal{S}_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash\right)$.*

Proof: It is standard because our calculi have the subformula property. See for example [39] for details. □

Theorem 51 (Soundness and strong completeness)**.** *Let $\mathsf{C} \subseteq \mathbb{C}$. The calculus $\mathsf{GGL}_{\mathsf{C}}$ is sound and strongly complete for the Boolean basic gaggle logic $(\mathcal{S}_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$.*

Proof: Since any proof of a consecution $\varphi \vdash \psi \in \mathcal{S}_{\mathsf{C}}$ can be cut–free and our calculus has the subformula property, it contains only the introduction rules $(\vdash \circledast)$ for the connectives of C. (The introduction rules for the other connectives of $\mathcal{O}(\mathsf{C}) - \mathsf{C}$ were needed in the initial completeness proof before the cut elimination theorem for Lemma 68.) □

The difference between the above theorem and Theorem 45 is that the set of connectives C considered is not assumed to be such that $\mathsf{C} = \mathcal{O}(\mathsf{C})$ (we recall that $\mathcal{O}(\mathsf{C})$ is defined by Expression (4)). Thanks to cut–elimination, the completeness result also holds if we do not have equality. This said, all connectives of $\mathcal{O}(\mathsf{C})$ do appear in the calculus, but only as structural connectives.

9 Calculi for gaggle logics

Until now, our calculi are sound and complete for logics including the Boolean connectives. However, we would like to obtain calculi for plain gaggle logics, without Boolean connectives. Indeed, we consider the latter to be more primitive than Boolean gaggle logics because even the Boolean connectives can be seen as particular gaggle connectives, interpreted over special relations (identity relations, see Example 26). These special relations are obtained at the proof-theroretical level by imposing the validity of Gentzen's structural rules. So, in this section, we are going to define sound and complete calculi for (plain) gaggle logics, without Boolean connectives.

Definition 52. Let $\mathsf{C} \subseteq \mathbb{C}$. We denote by $\mathsf{GGL}^0_\mathsf{C}$ the calculus of Figure 14 where the introduction rules $(\vdash \circledast)$ and $(\circledast \vdash)$ are defined for the connectives $\circledast$ of C and where the rule (dr_1) is defined for the elements τ of an arbitrary set of generators of $\mathfrak{S}_{n+1}$ (for each n ranging over the arities of the connectives of C). $\square$

Note that $\left(\mathsf{dr}'_2\right)$ (introduced in Proposition 46) is in $\mathsf{GGL}^0_\mathsf{C}$ instantiated with gaggle connectives. More precisely, in $\mathsf{GGL}^0_\mathsf{C}$, an application of $\left(\mathsf{dr}'_2\right)$ is of the following form:

$$\frac{[\circledast]\,(X_1,\ldots,X_m) \vdash [\circledast']\,(X'_1,\ldots,X'_n)}{[-\circledast']\,(X'_1,\ldots,X'_n) \vdash [-\circledast]\,(X_1,\ldots,X_m)} \qquad \frac{\circledast(\varphi_1,\ldots,\varphi_m) \vdash \circledast'\,(\varphi'_1,\ldots,\varphi'_n)}{*\circledast'\,(\varphi'_1,\ldots,\varphi'_n) \vdash *\circledast(\varphi_1,\ldots,\varphi_m)}$$

An equivalent axiomatization of $\mathsf{GGL}^0_\mathsf{C}$ is obtained if we replace rule $\left(\mathsf{dr}'_2\right)$ by the switch rule $\left(\mathsf{sw}^j\right)$ of Proposition 46, for each $j \in [\![1;n]\!]$:

$$\frac{S([\circledast]\,,X_1,\ldots,X_j,\ldots,X_n,X)}{S([s_j\circledast]\,,X_1,\ldots,*X_j,\ldots,X_n,X)}\;\left(\mathsf{sw}^j\right).$$

This is due to the fact that the switch rule is derivable in $\mathsf{GGL}^0_\mathsf{C}$ and, vice versa, $\left(\mathsf{dr}'_2\right)$ is derivable from the switch rule and (dr_1) thanks to Proposition 30.

The main difference between GGL_C and $\mathsf{GGL}^0_\mathsf{C}$ lies in the fact that the introduction rules for the Boolean connectives have been removed as well as the structural rules.

Display rules:

$$\frac{S\left([\circledast], X_1, \ldots, X_n, X_{n+1}\right)}{S\left(\left[\tau\circledast\right], X_{\tau(1)}, \ldots, X_{\tau(n)}, X_{\tau(n+1)}\right)}\ (\mathsf{dr}_1) \qquad \frac{X \vdash Y}{*Y \vdash *X}\ \left(\mathsf{dr}_2'\right)$$

Introduction rules:

$$\frac{U_1 \vdash V_1 \quad \ldots \quad U_n \vdash V_n}{S\left([\circledast], X_1, \ldots, X_n, \circledast\left(\varphi_1, \ldots, \varphi_n\right)\right)}\ (\vdash \circledast) \qquad \frac{S\left([\circledast], \varphi_1, \ldots, \varphi_n, U\right)}{S\left(\circledast, \varphi_1, \ldots, \varphi_n, U\right)}\ (\circledast \vdash)$$

In rules $(\vdash \circledast)$ and $(\circledast \vdash)$, for all $\circledast = (\sigma, \pm, (Æ, (\pm_1, \ldots, \pm_n))) \in \mathsf{C}$:

- for all $j \in [\![1; n]\!]$, we set $U_j \vdash V_j \triangleq \begin{cases} X_j \vdash \varphi_j & \text{if } \pm_j \pm (Æ) = - \\ \varphi_j \vdash X_j & \text{if } \pm_j \pm (Æ) = + \end{cases}$

  such that, in rule $(\vdash \circledast)$, for all j X_j is not empty and with the convention that if φ_j is empty for some j then $\circledast(\varphi_1, \ldots, \varphi_n)$ is also empty.
- for all $\star \in \{\circledast, [\circledast]\}$, $S(\star, X_1, \ldots, X_n, X) \triangleq \begin{cases} \star(X_1, \ldots, X_n) \vdash X & \text{if } Æ = \exists \\ X \vdash \star(X_1, \ldots, X_n) & \text{if } Æ = \forall. \end{cases}$

Figure 14: Calculus $\mathsf{GGL}^0_{\mathsf{C}}$

Theorem 53 (Soundness and strong completeness)**.** *Let $\mathsf{C} \subseteq \mathbb{C}$. The calculus $\mathsf{GGL}^0_{\mathsf{C}}$ is sound and strongly complete for the basic gaggle logic $(\mathcal{S}^0_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$.*

Proof: See the Appendix, Section C. □

Goré [21] introduces a calculus $\delta\mathbf{OP}$ which is basically our calculus $\mathsf{GGL}^0_{\mathsf{C}}$ without the rule $\left(\mathsf{dr}_2'\right)$. Restall [41] establishes connections between gaggle theory and display logics and sketches a similar calculus (without proving condition $(C8)$). This difference between our and their calculi is due to the fact that they do not deal with Boolean negation and do not consider it in their approach and framework. As one can notice, this complicates the proofs tremendously even if the addition in the calculi is minimal. This said, Goré [21] recognizes the dual character, in a proof–theoretical sense, of pairs of traces which are obtained from each other by multiplying every argument of the trace by $-$. This leads him to introduce the function/connective f^{Δ} of trace $-t$ associated to a function f of trace t. However, he does not make the connection between this function/connective f^{Δ} and the Boolean negation of f as we do (see Definition 20 and Proposition 29). Therefore, he proves the soundness and completeness of his calculus but with respect to two distinct yet dual semantics based on Dunn's tonoids. As such, he does not connect his algebraic semantics with the Kripke–style relational semantics (elicited by Dunn) explicitly as we do. A similar observation regarding the role of Boolean negation in his and our work was already made in [2].

Theorem 54 (Decidability)**.** *Let $\mathsf{C} \subseteq \mathbb{C}$ and let $\varphi, \psi \in \mathcal{L}^0_{\mathsf{C}}$. The problem of determining whether φ or $\varphi \models \psi$ are valid in the logics $(\mathcal{L}^0_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$ and $(\mathcal{S}^0_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$ (respectively) is decidable.*

Proof: It suffices to observe that the set of consecutions that can lead to a *cut-free* proof of $\varphi \vdash \psi$ in $\mathsf{GGL}^0_{\mathsf{C}}$ is finite. The problem of finding a proof of $\varphi \vdash \psi$ thus boils down to a graph reachability problem in a finite graph whose edges are labeled by the rules. This problem is decidable. We then obtain the result by the completeness of $\mathsf{GGL}^0_{\mathsf{C}}$ for $(\mathcal{L}^0_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$ and $(\mathcal{S}^0_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$ of Theorem 53. □

10 Logics defining groups and groups defining logics

In this section, we are going to show how notions of groups arise naturally from our gaggle logics and how gaggle logics can be canonically defined from groups thanks to our connections with group theory.

10.1 Groups defined from logics

One problem solved in this article is the following: given an arbitrary basic gaggle logic (Boolean or not) defined by a set C of (gaggle) connectives, how do we compute and define uniformly a sound and complete calculus for that logic ? Theorems 51 and 53 of the previous sections have solved it. However, we needed in our calculi to introduce *all* connectives of $\mathcal{O}(\mathsf{C})$ (defined by Expression (4)) either as logical connectives in Theorem 45 or as structural connectives in Theorems 51 and 53. In this section, we are going to show that we can in fact limit further the connectives considered and not take the full orbits $\mathcal{O}(\mathsf{C})$ of C under the action $\alpha_n * \beta_n$. For that, we need to explore a bit more the proof–theoretical aspects of our gaggle logics in light of our connections with group theory.

We have introduced actions on the set of gaggle connectives. Even if we know how a permutation, the Boolean negation and their combinations act on connectives,

$$\frac{S\left([\circledast], X_1, \ldots, X_n, X_{n+1}\right)}{S\left([\tau\circledast], \pm^1 X_{\overline{\tau}(1)}, \ldots, \pm^n X_{\overline{\tau}(n)}, \pm^{n+1} X_{\overline{\tau}(n+1)}\right)}\ (\mathsf{dr}_3)$$

where $\tau \in \mathfrak{S}_{n+1} * P_{(+,-)}$ and

if $\tau = \tau_0 - \tau_1 \ldots - \tau_m$ with $m \geq 1$ then $\overline{\tau} \triangleq \tau_0\tau_1 \ldots \tau_m$ and for all $j \in [\![1; n+1]\!]$,
$\pm^j \triangleq \pm^j_1 \pm^j_2 \ldots \pm^j_m$ with, for all $i \in [\![1; m]\!]$, $\pm^j_i \triangleq \begin{cases} * & \text{if } j = \tau_i\tau_{i+1}\ldots\tau_m(n+1) \\ \text{empty} & \text{otherwise} \end{cases}$;
if $\tau = \tau_0 - \tau_1 \ldots - \tau_{m-1}-$ with $m \geq 1$ then replace τ with $\tau_0 - \tau_1 \ldots - \tau_{m-1} - 1$;
if $\tau = -\tau_1 \ldots - \tau_{m-1} - \tau_m$ with $m \geq 1$ then replace τ with $1 - \tau_1 \ldots - \tau_{m-1} - \tau_m$;
if $\tau \in \mathfrak{S}_{n+1}$ then $\overline{\tau} \triangleq \tau$ and $\pm^1, \ldots, \pm^{n+1}$ are empty;
if $\tau = -$ then $\overline{\tau} = 1$ and $\pm^1, \ldots, \pm^n$ are empty and $\pm^{n+1} = -$.

Figure 15: Rule (dr_3)

we still do not know how their combination and iteration operate at the proof–theoretical level. Indeed, we have a rule (dr_1) for permutations $\tau_1, \ldots, \tau_n$ and a rule (dr'_2) for Boolean negation $-$, yet we do not have a rule combining both, for elements $\tau_0 - \tau_1 \ldots - \tau_m$ of the free group $\mathfrak{S}_{n+1} * P_{(+,-)}$. Such a rule is defined in Figure 15. One can easily prove that rule (dr_3) is valid and derives from (dr_1) and (dr'_2) in $\mathsf{GGL}^0_{\mathsf{C}}$. Conversely, with $\tau \in \mathfrak{S}_{n+1}$, we recover rule (dr_1) and with $\tau = -$ we recover rule (dr'_2). (The term "empty" could be replaced by $**$.)

Now, let us be given a set of connectives $\mathsf{C} \subseteq \mathbb{C}$ and assume without loss of generality that all connectives of C belong to the same orbit $\mathcal{O}(\mathsf{C}) = \mathcal{O}_{\alpha_n * \beta_n}(\circledast)$ (for some $\circledast \in \mathsf{C}$). What we would want in (dr_1) is to be able to 'go' from one connective $\circledast$ of C to an arbitrary other connective $\circledast'$ of C. By transitivity of the action $\alpha_n * \beta_n$, this is possible in $\mathcal{O}(\mathsf{C})$: given any two connectives $\circledast, \circledast' \in \mathsf{C}$, there is an element of the group $g \in \mathfrak{S}_{n+1} * P_{(+,-)}$ such that $\circledast' = \alpha_n * \beta_n(g, \circledast)$. This leads us to define a special subset G of $\mathfrak{S}_{n+1} * P_{(+,-)}$ such that for all $\circledast, \circledast' \in \mathsf{C}$ there is $g \in G$ such that $\circledast' = \alpha_n * \beta_n(g, \circledast)$. We want this set G to be a group. Indeed, informally, its composition operation should be associative, because of the definition of an action group, and every element g of G should have an inverse: if $\circledast' = \alpha_n * \beta_n(g, \circledast)$ then there should be a g^{-1} such that $\circledast = \alpha_n * \beta_n(g^{-1}, \circledast')$. This leads us to the following definition:

Definition 55 (Group associated to a set of connectives)**.** Let $\mathsf{C} \subseteq \mathbb{C}$. A *group associated to* C is a group G such that for all $n \in \mathbb{N}^*$, all $\circledast, \circledast' \in \mathsf{C} \cap \mathbb{C}_n$, there is $g \in G$ such that $\circledast' = \alpha_n * \beta_n(g, \circledast)$. □

Implicitly, note that $G \subseteq \bigcup_{n \in \mathbb{N}} \left\{ \mathfrak{S}_{n+1} * P_{(+,-)} \mid a(\circledast) = n, \circledast \in \mathsf{C} \right\}$. A group associated to a set of connectives always exists because the free group $\left\langle \bigcup_{n \in \mathbb{N}^*} \left\{ g \in \mathfrak{S}_{n+1} * P_{(+,-)} \mid \circledast' = g \circledast' \text{ for some } \circledast, \circledast' \in \mathsf{C} \cap \mathbb{C}_n \right\} \right\rangle$ satisfies the required condition. It is not in general unique because the action $\alpha_n * \beta_n$ is not faithful: we proved in Proposition 32 (item 1) that $-r_j - r_j - r_j - r_j \circledast = \circledast$.

Definition 56. Let $\mathsf{C} \subseteq \mathbb{C}$ and let G be a group associated to C. We denote by $\mathsf{GL}^0_{\mathsf{C},G}$ (resp. $\mathsf{GL}_{\mathsf{C},G}$) the calculus of Figure 14 (resp. Figure 13) where the introduction rules $(\vdash \circledast)$ and $(\circledast \vdash)$ are defined for the connectives $\circledast$ of C and where rules (dr_1) and $\left(\mathsf{dr}'_2\right)$ (resp. only (dr_1)) are replaced by rule (dr_3) which is defined for elements τ belonging to a set of generators of the group G. □

Theorem 57 (Soundness and strong completeness)**.** *Let $\mathsf{C} \subseteq \mathbb{C}$ and let G be a group associated to C. The calculus $\mathsf{GL}^0_{\mathsf{C},G}$ ($\mathsf{GL}_{\mathsf{C},G}$) is sound and strongly complete for the (Boolean) basic gaggle logic $(\mathcal{S}^0_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$ (resp. $(\mathcal{S}_{\mathsf{C}}, \mathcal{M}_{\mathsf{C}}, \Vdash)$).*

Proof: See the Appendix, Section C. □

Example 58. The symmetric group $\mathfrak{S}_3$ is a group associated to the connectives of the Lambek calculus [30] and update logic [2]. However, there is a simpler and smaller group associated. Indeed, the alternating group $\mathfrak{A}_3$, generated by the 3–cycle (123) (or (132) = (123) ∘ (123), see Section 3) is another group associated to the connectives of the Lambek calculus and update logic. This confirms an observation already made in [2] about the central role played by ternary cycles in update logic and substructural logics in general. The free group $\mathfrak{A}_3 * P_{(+,-)}$ is a group associated to the connectives of *dual* update logic [2], because the dual connectives of dual update logic are definable from the connectives of update logic thanks to Boolean negation (see [2, Proposition 16]).

10.2 Logics defined from groups

According to Cayley's theorem, every finite group of cardinal $n+1$ is isomorphic to a subgroup of the symmetric group $\mathfrak{S}_{n+1}$. Now, the restriction of the action α_n to any subgroup G of $\mathfrak{S}_{n+1}$ is also an action of G on $\mathbb{C}_n$. Therefore, every finite group G of cardinal $n+1$ induces a canonical group action α of G on $\mathbb{C}_n$ defined for all

$g \in G$ and $\circledast \in \mathbb{C}_n$ by $\alpha(g, \circledast) = \alpha_n(\varphi(g), \circledast)$, where φ is an isomorphism between G and the subgroup of $\mathfrak{S}_{n+1}$. Every finite group therefore defines a set of connectives obtained by considering the orbit of an arbitrary connective $\circledast \in \mathbb{C}$ by this canonical group action α. In other words, every finite group defines a class of logics. These logics provide a certain perspective on the whole set of gaggle connectives.

11 Conclusion

In this article we have introduced a uniform method to automatically compute sound and strongly complete calculi for a wide class of non–classical logics, basic gaggle logics. These calculi are display calculi and enjoy the cut elimination. This allowed us to prove in particular that basic gaggle logics are decidable. We further restrained the structural connectives needed in our calculi by introducing the notion of group associated to a set of connectives. We also established connections between gaggle theory and group theory. We showed that Dunn's abstract law of residuation corresponds to an action of transpositions of the symmetric group on the set of gaggle connectives and that Dunn's families of connectives are orbits of the same action of the symmetric group. Other operations on connectives, such as dual and Boolean negation, were also reformulated in terms of actions of groups and their combination was defined by means of free groups and free products.

Based on our connection with group theory, we argued that there are more 'basic' operations on connectives than Dunn's abstract law of residuation, based on cycles of the symmetric group rather than transpositions (which are cycles anyway), because every permutation factorizes uniquely into disjoint cycles. Residuation is still central because it corresponds to the action of transpositions of the symmetric group and transpositions generate it as well. Yet, there are many other generators and ways to present and represent the symmetric groups and its subgroups. What really matters from a proof-theoretical perspective is the set of generators of the groups that we consider and how groups can be presented. That is why the results in group theory regarding the presentation and classification of finite groups have now become quite relevant for the study of various (gaggle) logics.

Our connections with the theory of groups enable to study the structure of gaggle connectives in a very modular and systematic way, using bridges from algebra such as Cayley's theorem. Thanks to this bridge, each finite group can be seen as a set of operations acting on the set of connectives. Hence, each group generates and defines gaggle logics. Thus, the structure of the gaggle connectives can be studied under a variety of different viewpoints by means of different logics that correspond to the wide range of finite groups that can act on the connectives. This is similar to what happens in mathematics where the structure of (vectorial, Euclidean, *etc.*) spaces can be studied by different geometries corresponding to different groups of transformation acting on it: Euclidean geometry with the isometric group, hyperbolic geometry with the Lorentz group, affine geometry with the affine group, *etc.*

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A Proofs of propositions 30, 32 and 46

Proposition 30. *If $\circledast \in \mathbb{C}_n$ is a n–ary connective, then for all $j \in [\![1;n]\!]$,*

- $s_j \circledast = r_j - r_j \circledast$
- $d_j \circledast = r_j - r_j - \circledast$
- $d \circledast = s_1 \ldots s_n - \circledast$.

Proof: Let $\circledast = (\sigma, \pm, (Æ, (\pm_1, \ldots, \pm_n))) \in \mathbb{C}_n$. Then,

$$\begin{aligned} r_j \circledast =& (\sigma, - \pm_j \pm, (- \pm_j Æ, (- \pm_j \pm_1, \ldots, \pm_j, \ldots, - \pm_j \pm_n))) \\ -r_j \circledast =& (\sigma, \pm_j \pm, (\pm_j Æ, (\pm_j \pm_1, \ldots, -\pm_j, \ldots, \pm_j \pm_n))) \\ r_j - r_j \circledast =& (\sigma, \pm, (Æ, (\pm_1, \ldots, -\pm_j, \ldots, \pm_n))) \end{aligned}$$

Moreover,

$$\begin{aligned} -\circledast =& (\sigma, -\pm, (-Æ, (-\pm_1, \ldots, -\pm_n))) \\ r_j - \circledast =& ((j\ n+1) \circ \sigma, - \pm_j \pm, (- \pm_j Æ, (- \pm_j \pm_1, \ldots, -\pm_j, \ldots, - \pm_j \pm_n))) \\ -r_j - \circledast =& ((j\ n+1) \circ \sigma, (\pm_j \pm, (\pm_j Æ, (\pm_j \pm_1, \ldots, \pm_j, \ldots, \pm_j \pm_n)))) \\ r_j - r_j - \circledast =& (\sigma, -\pm, (-Æ, (-\pm_1, \ldots, \pm_j, \ldots, -\pm_n))) \end{aligned}$$

$\square$

Proposition 32. *Let $n \in \mathbb{N}^*$, $j \in [\![1;n]\!]$ and let us define $G_j = \langle r_j \rangle * P_{(+,-)}$. Since G_j is a subgroup of $\mathfrak{S}_{n+1} * P_{(+,-)}$, let us denote by α_{G_j} the action of G_j on $\mathbb{C}_n$ induced by the free action $\alpha_n * \beta_n$. Then, for all connectives $\circledast$ of arity n,*

1. *$\mathcal{O}_{\alpha_{G_j}}(\circledast)$ is isomorphic to a cyclic group of order 8.*
2. *$\{\mathcal{O}_{\alpha_n * \beta_n}(\circledast), \mathcal{O}_{\alpha_n * \beta_n}(\sim \circledast)\}$ forms a partition of the set $\mathbb{C}_n$ of connectives of arity n. Moreover, the mapping $\tilde{\cdot} : \mathcal{O}_{\alpha_n * \beta_n}(\circledast) \to \mathcal{O}_{\alpha_n * \beta_n}(\sim \circledast)$, $x \mapsto \sim x$ is involutive.*
3. *For all $n \in \mathbb{N}^*$, the free action $\alpha_n * \beta_n * \gamma_n$ on the set of connectives $\mathbb{C}_n$ is transitive.*

Proof: For the first item, it suffices to prove that for all connectives $\circledast$ of arity n and all $j \in [\![1;n]\!]$, $-r_j - r_j - r_j - r_j\circledast = \circledast$. Let $\circledast = (\sigma, \pm, (Æ, (\pm_1, \dots, \pm_n)))$ and let r_j be the transposition $(j\ n+1)$. (See also Figure 2 for an example.)

$$
\begin{aligned}
\circledast &= (\sigma, \pm, (Æ, (\pm_1, \dots, \pm_j, \dots, \pm_n)))\\
r_j\circledast &= (r_j \circ \sigma, - \pm_j \pm, (- \pm_j Æ, (- \pm_j \pm_1, \dots, \pm_j, \dots, - \pm_j \pm_n)))\\
-r_j\circledast &= (r_j \circ \sigma, \pm_j\pm, (\pm_j Æ, (\pm_j\pm_1, \dots, -\pm_j, \dots, \pm_j\pm_n)))\\
r_j - r_j\circledast &= (\sigma, \pm, (Æ, (\pm_1, \dots, -\pm_j, \dots, \pm_n)))\\
-r_j - r_j\circledast &= (\sigma, -\pm, (-Æ, (-\pm_1, \dots, \pm_j, \dots, -\pm_n)))\\
r_j - r_j - r_j\circledast &= (r_j \circ \sigma, \pm_j\pm, (\pm_j Æ, (\pm_j\pm_1, \dots, \pm_j, \dots, \pm_j\pm_n)))\\
-r_j - r_j - r_j\circledast &= (r_j \circ \sigma, - \pm_j \pm, (- \pm_j Æ, (- \pm_j \pm_1, \dots, -\pm_j, \dots, - \pm_j \pm_n)))\\
r_j - r_j - r_j - r_j\circledast &= (\sigma, -\pm, (-Æ, (-\pm_1, \dots, -\pm_j, \dots, -\pm_n)))\\
-r_j - r_j - r_j - r_j\circledast &= (\sigma, \pm, (Æ, (\pm_1, \dots, \pm_j, \dots, \pm_n))) = \circledast.
\end{aligned}
$$

For the second item, one should first observe that $\mathcal{O}_{\alpha_n * \beta_n}(\circledast) \cap \mathcal{O}_{\alpha_n * \beta_n}(\sim \circledast) = \varnothing$ $(*)$. Indeed, for all $\circledast' = (\sigma', \pm', (Æ', (\pm'_1, \dots, \pm'_n))) \in \mathcal{O}_{\alpha_n * \beta_n}(\circledast)$, we have that $\pm' \pm (Æ') = \pm \pm (Æ)$ but at the same time, for all $\circledast' = (\sigma', \pm', (Æ', (\pm'_1, \dots, \pm'_n))) \in \mathcal{O}_{\alpha_n * \beta_n}(\sim \circledast)$, we also have that $\pm' \pm (Æ') = - \pm \pm (Æ)$. Now, we prove that for all $\circledast' = (\sigma', \pm', (Æ', (\pm'_1, \dots, \pm'_n)))$, if $\pm' \pm (Æ') = \pm \pm (Æ)$ then $\circledast' \in \mathcal{O}_{\alpha_n * \beta_n}(\circledast)$, and $\circledast' \in \mathcal{O}_{\alpha_n * \beta_n}(\sim \circledast)$ otherwise. First, assume that $\pm' \pm (Æ') = \pm \pm (Æ)$. Then, we define $\circledast'' = \sigma\sigma'^{-}\circledast'$. So, $\circledast'' = (\sigma, \pm'', (Æ'', (\pm''_1, \dots, \pm''_n)))$ and we still have that $\pm'' \pm (Æ'') = \pm' \pm (Æ') = \pm \pm (Æ)$. If $\pm'' = \pm$, then it only suffices to switch the tonicity of the arguments $j_1, \dots, j_k$ of $\circledast''$ such that $\pm''_j \neq \pm_j$. This can be done by applying the switch operation for the arguments $j_1, \dots, j_k$ to $\circledast''$. We then obtain that $s_{j_1}s_{j_2}\dots s_{j_k}\circledast'' = \circledast$. Thus, $s_{j_1}s_{j_2}\dots s_{j_k}\sigma\sigma'^{-}\circledast' = \circledast$. Second, assume that $\pm' \pm (Æ') = - \pm \pm (Æ)$. Then, we define $\circledast'' =\sim \circledast = (\sigma', \pm', (Æ'', (\pm'_1, \dots, \pm'_n)))$ and we have that $\pm' \pm (Æ'') = \pm'(- \pm (Æ')) = \pm \pm (Æ)$. So, we proceed like in the first case. We then obtain that there are $i_1, \dots, i_l \in [\![1;n]\!]$ such that $s_{i_1}s_{i_2}\dots s_{i_l}\sigma\sigma'^{-} \sim \circledast' = \circledast$. So, we have proved that for all $\circledast' = (\sigma', \pm', (Æ', (\pm'_1, \dots, \pm'_n)))$, it holds that $\circledast' \in \mathcal{O}_{\alpha_n * \beta_n}(\circledast)$ iff $\pm' \pm (Æ') = \pm \pm (Æ)$. This entails that $|\mathcal{O}_{\alpha_n * \beta_n}(\circledast)| = |\mathcal{O}_{\alpha_n * \beta_n}(\sim \circledast)| = (n+1)! \cdot 2^{n+1} = \frac{|\mathbb{C}_n|}{2}$. Therefore, $\mathcal{O}_{\alpha_n * \beta_n}(\circledast) \cup \mathcal{O}_{\alpha_n * \beta_n}(\sim \circledast) = \mathbb{C}_n$ and together with (*), we have that $\{\mathcal{O}_{\alpha_n * \beta_n}(\circledast), \mathcal{O}_{\alpha_n * \beta_n}(\sim \circledast)\}$ forms a partition of $\mathbb{C}_n$.

The third item follows easily from the second item. □

Proposition 46. *Let $C \subseteq \mathbb{C}$ and let $\circledast \in C$ be a n-ary connective. The following rules* *are all derivable in* $\mathsf{GGL_C}$.

$$\dfrac{X \vdash Y}{*Y \vdash *X}\ (\mathit{dr}_2') \qquad \dfrac{S([\circledast], X_1, \ldots, X_j, \ldots, X_n, X)}{S([s_j\circledast], X_1, \ldots, *X_j, \ldots, X_n, X)}\ (\mathit{sw}^j)$$

$$\dfrac{*X \vdash Y}{*Y \vdash X}\ (\mathit{dr}_2'') \qquad \dfrac{X \vdash *Y}{Y \vdash *X}\ (\mathit{dr}_2''')$$

$$\dfrac{U \vdash ((X\ ,\ Y)\ ,\ Z)}{U \vdash (X\ ,\ (Y\ ,\ Z))}\ (\vdash \mathit{B}) \qquad \dfrac{U \vdash (X\ ,\ Y)}{U \vdash (Y\ ,\ X)}\ (\vdash \mathit{CI})$$

$$\dfrac{U \vdash X}{U \vdash (X\ ,\ Y)}\ (\vdash \mathit{K}) \qquad \dfrac{U \vdash (X\ ,\ X)}{U \vdash X}\ (\vdash \mathit{WI})$$

$$\dfrac{U \vdash \varphi \quad U \vdash \psi}{U \vdash (\varphi \wedge \psi)}\ (\vdash \wedge)' \qquad \dfrac{\varphi \vdash U \quad \psi \vdash U}{(\varphi \vee \psi) \vdash U}\ (\vee \vdash)'$$

The rule (dr_2') *is called the* Boolean negation rule *and the rule* (sw^j), *for* $j \in [\![1;n]\!]$, *is called the* switch rule w.r.t. the j^{th} argument. *The rule* (dr_1) *is also derivable in* $\mathsf{GGL_C}$, for all $\tau \in \mathfrak{S}_{n+1}$.

Proof:

(dr_2'):

$$\dfrac{\dfrac{\dfrac{X \vdash Y}{(X\ ,\ *Y) \vdash}\ (\mathsf{dr}_2)}{(*Y\ ,\ X) \vdash}\ (\mathsf{CI} \vdash)}{*Y \vdash *X}\ (\mathsf{dr}_2)$$

(dr_2''):

$$\dfrac{\dfrac{\dfrac{*X \vdash Y}{(*X\ ,\ *Y) \vdash}\ (\mathsf{dr}_2)}{(*Y\ ,\ *X) \vdash}\ (\mathsf{CI} \vdash)}{*Y \vdash X}\ (\mathsf{dr}_2)$$

(dr_2'''):

$$\dfrac{\dfrac{\dfrac{X \vdash *Y}{(X\ ,\ Y) \vdash}\ (\mathsf{dr}_2)}{(Y\ ,\ X) \vdash}\ (\mathsf{CI} \vdash)}{Y \vdash *X}\ (\mathsf{dr}_2)$$

(sw^j):

$$\dfrac{\dfrac{\dfrac{\dfrac{S(\circledast, X_1, \ldots, X_j, \ldots, X_n, X)}{S(r_j\circledast, X_1, \ldots, X, \ldots, X_n, X_j)}\ (\mathsf{dr}_1)}{S(-r_j\circledast, X_1, \ldots, X, \ldots, X_n, *X_j)}\ (\mathsf{dr}_2')}{S(r_j - r_j\circledast, X_1, \ldots, *X_j, \ldots, X_n, X)}\ (\mathsf{dr}_1)}{S(s_j\circledast, X_1, \ldots, *X_j, \ldots, X_n, X)}\ \text{Rewrite}$$

$(\vdash \mathsf{K})$:

$$\dfrac{\dfrac{\dfrac{\dfrac{U \vdash X}{*X \vdash *U}\ (\mathsf{dr}_2')}{(*X\ ,\ *Y) \vdash *U}\ (\mathsf{K} \vdash)}{U \vdash *(*X\ ,\ *Y)}\ (\mathsf{dr}_2''')}{U \vdash (X\ ,\ Y)}\ \text{Rewrite}$$

$(\vdash \mathsf{CI}):$

$$\begin{array}{cl}
U \vdash (X \,,\, Y) & \\ \hline
(U \,,\, *(X \,,\, Y)) \vdash & (\mathsf{dr}_2) \\ \hline
(U \,,\, (*X \,,\, *Y)) \vdash & \text{Rewrite} \\ \hline
((*X \,,\, *Y) \,,\, U) \vdash & (\mathsf{CI} \vdash) \\ \hline
(*X \,,\, *Y) \vdash *U & (\mathsf{dr}_2) \\ \hline
(*Y \,,\, *X) \vdash *U & (\mathsf{CI} \vdash) \\ \hline
((*Y \,,\, *X) \,,\, U) \vdash & (\mathsf{dr}_2) \\ \hline
(U \,,\, (*Y \,,\, *X)) \vdash & (\mathsf{CI} \vdash) \\ \hline
U \vdash *(*Y \,,\, *X) & (\mathsf{dr}_2) \\ \hline
U \vdash (Y \,,\, X) & \text{Rewrite}
\end{array}$$

$(\vdash \mathsf{B}):$

$$\begin{array}{cl}
U \vdash ((X \,,\, Y) \,,\, Z) & \\ \hline
*((X \,,\, Y) \,,\, Z) \vdash *U & (\mathsf{dr}_2') \\ \hline
((*X \,,\, *Y) \,,\, *Z) \vdash *U & \text{Rewrite} \\ \hline
(*X \,,\, (*Y \,,\, *Z)) \vdash *U & (\mathsf{B} \vdash) \\ \hline
U \vdash *(*X \,,\, (*Y \,,\, *Z)) & (\mathsf{dr}_2') \\ \hline
U \vdash (X \,,\, (Y \,,\, Z)) & \text{Rewrite}
\end{array}$$

$(\vdash \mathsf{WI}):$

$$\begin{array}{cl}
U \vdash (X \,,\, X) & \\ \hline
(*X \,,\, *X) \vdash *U & (\mathsf{dr}_2') \\ \hline
*X \vdash *U & (\mathsf{WI} \vdash) \\ \hline
U \vdash X & (\mathsf{dr}_2')
\end{array}$$

$(\vdash \wedge)':$

$$\begin{array}{cl}
U \vdash \varphi \quad U \vdash \psi & \\ \hline
(U \,,\, U) \vdash (\varphi \wedge \psi) & (\vdash \wedge) \\ \hline
U \vdash (\varphi \wedge \psi) & (\mathsf{WI} \vdash)
\end{array}$$

$(\vee \vdash)':$

$$\begin{array}{cl}
\varphi \vdash U \quad \psi \vdash U & \\ \hline
(\varphi \vee \psi) \vdash (U \,,\, U) & (\vee \vdash) \\ \hline
(\varphi \vee \psi) \vdash U & (\vdash \mathsf{WI})
\end{array}$$

The last rewriting part in the proof of (sw^j) is due to Proposition 30. □

B Proof of theorem 45

Theorem 45 (Soundness and strong completeness)**.** *Let $C \subseteq \mathbb{C}$ be such that $\mathcal{O}(C) = C$. The calculus GGL_C is sound and strongly complete for the Boolean basic gaggle logic $(\mathcal{S}_C, \mathcal{M}_C, \Vdash)$.*

In this section, $\mathsf{C} \subseteq \mathbb{C}$ is such that $\mathcal{O}(\mathsf{C}) = \mathsf{C}$. We provide the soundness and completeness proofs of Theorem 45. We adapt the proof methods introduced in [2], based on a Henkin construction, to our more abstract and general setting. We start by the soundness proof.

Lemma 59. *The calculus GGL_C is sound for the Boolean basic gaggle logic $(\mathcal{S}_C, \mathcal{M}_C, \Vdash)$.*

Proof: We only need to prove the soundness for the rules (dr_1) and $(\vdash \circledast)$, the soundness of the other rules being standard. The soundness of the inference rule $(\vdash \circledast)$ follows directly from item 1 of Theorem 10, the soundness of rule (dr_1) follows from an iterative application of item 2 of Theorem 10 (or Proposition 27) by the decomposition of permutations into cycles or transpositions. □

The completeness proof uses a canonical model built up from maximal GGL_C–consistent sets. First, we define the notions of GGL_C–consistent set and maximal GGL_C–consistent set. In the sequel, by abuse of notation and to ease the presentation, when we write $\varphi \vdash \psi$ we mean that $\varphi \vdash \psi$ is provable in the calculus GGL_C.

Definition 60 ((Maximal) GGL_C–consistent set)**.**

- A *GGL_C–consistent set* is a subset Γ of $\mathcal{L}_\mathsf{C}$ such that there are no $\varphi_1, \dots, \varphi_n \in \Gamma$ such that $\varphi_1 \ , \ \dots \ , \ \varphi_n \vdash$. If $\varphi \in \mathcal{L}_\mathsf{C}$, we also say that φ is *GGL_C–consistent* when the set $\{\varphi\}$ is GGL_C–consistent.
- A *maximal GGL_C–consistent set* is a GGL_C–consistent set Γ of $\mathcal{L}_\mathsf{C}$ such that there is no $\varphi \in \mathcal{L}_\mathsf{C}$ satisfying both $\varphi \notin \Gamma$ and $\Gamma \cup \{\varphi\}$ is GGL_C–consistent. □

Lemma 61 (Cut lemma)**.** *Let Γ be a maximal GGL_C–consistent set. For all $\varphi_1, \dots, \varphi_n \in \Gamma$ and all $\varphi \in \mathcal{L}$, if $\varphi_1 \ , \ \dots \ , \ \varphi_n \vdash \varphi$ then $\varphi \in \Gamma$.*

Proof: First, we show that $\Gamma \cup \{\varphi\}$ is GGL_C–consistent. Assume towards a contradiction that it is not the case. Then, there are $\psi_1, \dots, \psi_m \in \Gamma$ such that $\psi_1 \ , \ \dots \ , \ \psi_m \ , \ \varphi \vdash$. Then, by the rules (dr_2) and $(\mathsf{Cl} \vdash)$, we have that $\varphi \vdash \ast(\psi_1 \ , \ \dots \ , \ \psi_m)$. Now, by assumption, $\varphi_1 \ , \ \dots \ , \ \varphi_n \vdash \varphi$. Therefore, by the cut rule, we have that $\varphi_1 \ , \ \dots \ , \ \varphi_n \vdash \ast(\psi_1 \ , \ \dots \ , \ \psi_m)$. Then, by the rules (dr_2) and $(\mathsf{B} \vdash)$, we have that $\varphi_1 \ , \ \dots \ , \ \varphi_n \ , \ \psi_1 \ , \ \dots \ , \ \psi_m \vdash$. However, $\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_m \in \Gamma$. This entails that Γ is not GGL_C–consistent, which is impossible. Thus, $\Gamma \cup \{\varphi\}$ is GGL_C–consistent. Now, since Γ is a *maximal* GGL_C–consistent set, this implies that $\varphi \in \Gamma$. □

Lemma 62 (Lindenbaum lemma)**.** *Any GGL_C–consistent set can be extended into a maximal GGL_C–consistent set.*

Proof: Let $\varphi_1, \varphi_2, \dots, \varphi_n, \dots$ be an enumeration of $\mathcal{L}_\mathsf{C}$ (it exists because C is countable). We define the sets Γ_n inductively as follows:

$$\Gamma_0 \triangleq \Gamma$$

$$\Gamma_{n+1} \triangleq \begin{cases} \Gamma_n \cup \{\varphi_n\} & \text{if } \Gamma_n \cup \{\varphi_n\} \text{ is } \mathsf{GGL}_\mathsf{C}\text{–consistent} \\ \Gamma_n & \text{otherwise.} \end{cases}$$

Then, we define the subset Γ^+ of $\mathcal{L}$ as follows: $\Gamma^+ = \bigcup_{n \in \mathbb{N}} \Gamma_n$.

We show that Γ^+ is a maximal $\mathsf{GGL_C}$–consistent set. Clearly, for all $n \in \mathbb{N}$, Γ_n is $\mathsf{GGL_C}$–consistent by definition of Γ_n. So, if Γ^+ was not $\mathsf{GGL_C}$–consistent, there would be a $n_0 \in \mathbb{N}$ such that Γ_{n_0} is not $\mathsf{GGL_C}$–consistent, which is impossible. Now, assume towards a contradiction that Γ^+ is not a *maximal* $\mathsf{GGL_C}$–consistent set. Then, there is $\varphi \in \mathcal{L}_\mathsf{C}$ such that $\varphi \notin \Gamma^+$ and $\Gamma \cup \{\varphi\}$ is $\mathsf{GGL_C}$–consistent. But there is $n_0 \in \mathbb{N}$ such that $\varphi = \varphi_{n_0}$. Because $\varphi \notin \Gamma^+$, we also have that $\varphi_{n_0} \notin \Gamma_{n_0+1}$. So, $\Gamma_{n_0} \cup \{\varphi_{n_0}\}$ is not $\mathsf{GGL_C}$–consistent by definition of Γ^+. Therefore, $\Gamma^+ \cup \{\varphi\}$ is not $\mathsf{GGL_C}$–consistent either, which is impossible. □

Lemma 63. *The following consecutions are provable in GGL: for all $\varphi, \varphi' \in \mathcal{L}$, all $\circledast = (\sigma, \pm, (\exists, (\pm_1, \ldots, \pm_j, \ldots, \pm_n)))$,*

$$\varphi \vdash \varphi \tag{8}$$
$$((\varphi \vee \varphi') \wedge (\varphi \vee \neg\varphi')) \vdash \varphi \tag{9}$$
$$\varphi \vdash ((\varphi \wedge \neg\varphi') \vee (\varphi \wedge \varphi')) \tag{10}$$

if $\pm_j = +$ *then*
$$\circledast(\varphi_1, \ldots, \varphi_j \vee \varphi'_j, \ldots, \varphi_n) \vdash (\circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \vee \circledast(\varphi_1, \ldots, \varphi'_j, \ldots, \varphi_n)) \tag{11}$$
if $\pm_j = -$ *then*
$$\circledast(\varphi_1, \ldots, \varphi_j \wedge \varphi'_j, \ldots, \varphi_n) \vdash (\circledast(\varphi_1, \ldots, \varphi_j, \ldots, \varphi_n) \vee \circledast(\varphi_1, \ldots, \varphi'_j, \ldots, \varphi_n)) \tag{12}$$

$$(\varphi\ ,\ \neg\psi) \vdash \quad \textit{iff}\ \varphi \vdash \psi \tag{13}$$

Proof: The proof of Expression (8) is by induction on φ. The proof of Expression

(9) is:

$$
\begin{prooftree}
\AxiomC{$\varphi \vdash \varphi$}
\RightLabel{$\mathsf{K}\,,$}
\UnaryInfC{$\varphi \vdash (\varphi \,,\, *(\varphi \vee \neg\varphi'))$}
\AxiomC{$\varphi \vdash \varphi$}
\RightLabel{$\mathsf{K}\,,$}
\UnaryInfC{$\varphi \vdash (\varphi \,,\, *\varphi')$}
\AxiomC{$\varphi' \vdash \varphi'$}
\RightLabel{$(\mathsf{dr}_2)\,,(\mathsf{dr}_2)$}
\UnaryInfC{$*\varphi' \vdash\, *\varphi'$}
\RightLabel{$(\neg \vdash)$}
\UnaryInfC{$\neg\varphi' \vdash\, *\varphi'$}
\RightLabel{$(\vdash \mathsf{Cl})\,,\mathsf{K}\,,$}
\UnaryInfC{$\neg\varphi' \vdash (\varphi \,,\, *\varphi')$}
\RightLabel{$(\vee \vdash)'$}
\BinaryInfC{$(\varphi \vee \neg\varphi') \vdash (\varphi \,,\, *\varphi')$}
\RightLabel{(dr_2)}
\UnaryInfC{$((\varphi \vee \neg\varphi') \,,\, \varphi') \vdash \varphi$}
\RightLabel{$(\mathsf{Cl} \vdash)$}
\UnaryInfC{$(\varphi' \,,\, (\varphi \vee \neg\varphi')) \vdash \varphi$}
\RightLabel{(dr_2)}
\UnaryInfC{$\varphi' \vdash (\varphi \,,\, *(\varphi \vee \neg\varphi'))$}
\RightLabel{$(\vee \vdash)'$}
\BinaryInfC{$(\varphi \vee \varphi') \vdash (\varphi \,,\, *(\varphi \vee \neg\varphi'))$}
\RightLabel{(dr_2)}
\UnaryInfC{$((\varphi \vee \varphi') \,,\, (\varphi \vee \neg\varphi')) \vdash \varphi$}
\RightLabel{$(\wedge \vdash)$}
\UnaryInfC{$((\varphi \vee \varphi') \wedge (\varphi \vee \neg\varphi')) \vdash \varphi$}
\end{prooftree}
$$

and the proof of Expression (10) is:

$$
\begin{prooftree}
\AxiomC{$\varphi \vdash \varphi$}
\RightLabel{$(\mathsf{K} \vdash)$}
\UnaryInfC{$(\varphi \,,\, *(\varphi \wedge \neg\varphi')) \vdash \varphi$}
\AxiomC{$(\varphi \vdash \varphi)$}
\RightLabel{$(\mathsf{K} \vdash)$}
\UnaryInfC{$(\varphi \,,\, *\varphi') \vdash \varphi$}
\AxiomC{$\varphi' \vdash \varphi'$}
\RightLabel{$(\mathsf{dr}_2)\,,(\mathsf{dr}_2)$}
\UnaryInfC{$*\varphi' \vdash\, *\varphi'$}
\RightLabel{$(\vdash \neg)$}
\UnaryInfC{$*\varphi' \vdash \neg\varphi'$}
\RightLabel{$(\mathsf{Cl} \vdash)\,,(\mathsf{K} \vdash)$}
\UnaryInfC{$(\varphi \,,\, *\varphi') \vdash \neg\varphi'$}
\RightLabel{$(\vdash \wedge)'$}
\BinaryInfC{$(\varphi \,,\, *\varphi') \vdash (\varphi \wedge \neg\varphi')$}
\RightLabel{(dr_2)}
\UnaryInfC{$(\varphi \,,\, *(\varphi \wedge \neg\varphi')) \vdash \varphi'$}
\RightLabel{$(\mathsf{K} \vdash)$}
\BinaryInfC{$(\varphi \,,\, *(\varphi \wedge \neg\varphi')) \vdash (\varphi \wedge \varphi')$}
\RightLabel{(dr_2)}
\UnaryInfC{$\varphi \vdash ((\varphi \wedge \neg\varphi') \,,\, (\varphi \wedge \varphi'))$}
\RightLabel{$(\vdash \vee)$}
\UnaryInfC{$\varphi \vdash ((\varphi \wedge \neg\varphi') \vee (\varphi \wedge \varphi'))$}
\end{prooftree}
$$

Proof of Expression (11). Assume that $\pm_j = +$. Then,

$$
\begin{prooftree}
\AxiomC{$[\circledast]\,(\varphi_1,\ldots,\varphi_n) \vdash\, \circledast\,(\varphi_1,\ldots,\varphi_n)$}
\RightLabel{$(\vdash \mathsf{K})$}
\UnaryInfC{$[\circledast]\,(\varphi_1,\ldots,\varphi_n) \vdash (\circledast(\varphi_1,\ldots,\varphi_j,\ldots,\varphi_n) \,,\, \circledast(\varphi_1,\ldots,\varphi'_j,\ldots,\varphi_n))$}
\RightLabel{$(\vdash \vee)$}
\UnaryInfC{$[\circledast]\,(\varphi_1,\ldots,\varphi_n) \vdash (\circledast(\varphi_1,\ldots,\varphi_j,\ldots,\varphi_n) \vee \circledast(\varphi_1,\ldots,\varphi'_j,\ldots,\varphi_n))$}
\RightLabel{(dr_1)}
\UnaryInfC{$\varphi_j \vdash\, [\tau_j \circledast]\,(\varphi_1,\ldots,(\circledast(\varphi_1,\ldots,\varphi_j,\ldots,\varphi_n) \vee \circledast(\varphi_1,\ldots,\varphi'_j,\ldots,\varphi_n)),\ldots,\varphi_n)$}
\end{prooftree}
$$

Likewise, we prove that:

$$\varphi'_j \vdash [\tau_j\circledast]\,(\varphi_1,\ldots,\circledast(\varphi_1,\ldots,\varphi_j,\ldots,\varphi_n)\vee\circledast(\varphi_1,\ldots,\varphi'_j,\ldots,\varphi_n),\ldots,\varphi_n).$$

So, by $(\vee\vdash)'$, we obtain that:

$$\varphi_j\vee\varphi'_j \vdash [\tau_j\circledast]\,(\varphi_1,\ldots,\circledast(\varphi_1,\ldots,\varphi_j,\ldots,\varphi_n)\vee\circledast(\varphi_1,\ldots,\varphi'_j,\ldots,\varphi_n),\ldots,\varphi_n).$$

Thus, by (dr_1) and $(\circledast\vdash)$, we obtain that:

$$\circledast(\varphi_1,\ldots,\varphi_j\vee\varphi'_j,\ldots,\varphi_n) \vdash \circledast\,(\varphi_1,\ldots,\varphi_j,\ldots,\varphi_n)\vee\circledast(\varphi_1,\ldots,\varphi'_j,\ldots,\varphi_n).$$

Proof of Expression (12). Assume that $\pm_j = -$. Then,

$$\dfrac{\dfrac{\dfrac{[\circledast]\,(\varphi_1,\ldots,\varphi_n)\vdash\circledast\,(\varphi_1,\ldots,\varphi_n)}{[\circledast]\,(\varphi_1,\ldots,\varphi_n)\vdash(\circledast(\varphi_1,\ldots,\varphi_j,\ldots,\varphi_n)\ ,\ \circledast(\varphi_1,\ldots,\varphi'_j,\ldots,\varphi_n))}\,(\vdash\mathsf{K})}{[\circledast]\,(\varphi_1,\ldots,\varphi_n)\vdash(\circledast(\varphi_1,\ldots,\varphi_j,\ldots,\varphi_n)\vee\circledast(\varphi_1,\ldots,\varphi'_j,\ldots,\varphi_n))}\,(\vdash\vee)}{[\tau_j\circledast]\,(\varphi_1,\ldots,(\circledast(\varphi_1,\ldots,\varphi_j,\ldots,\varphi_n)\vee\circledast(\varphi_1,\ldots,\varphi'_j,\ldots,\varphi_n)),\ldots,\varphi_n)\vdash\varphi_j}\,(\mathsf{dr}_1)$$

Likewise, we prove that:

$$[\tau_j\circledast]\,(\varphi_1,\ldots,(\circledast(\varphi_1,\ldots,\varphi_j,\ldots,\varphi_n)\vee\circledast(\varphi_1,\ldots,\varphi'_j,\ldots,\varphi_n)),\ldots,\varphi_n)\vdash\varphi'_j.$$

So, by $(\vdash\wedge)'$, we obtain that:

$$[\tau_j\circledast]\,(\varphi_1,\ldots,(\circledast(\varphi_1,\ldots,\varphi_j,\ldots,\varphi_n)\vee\circledast(\varphi_1,\ldots,\varphi'_j,\ldots,\varphi_n)),\ldots,\varphi_n)\vdash\varphi_j\wedge\varphi'_j.$$

Thus, by (dr_1) and $(\circledast\vdash)$, we obtain that:

$$\circledast(\varphi_1,\ldots,\varphi_j\wedge\varphi'_j,\ldots,\varphi_n)\vdash(\circledast(\varphi_1,\ldots,\varphi_j,\ldots,\varphi_n)\vee\circledast(\varphi_1,\ldots,\varphi'_j,\ldots,\varphi_n)).$$

Proof of Expression (13):

$$
\begin{prooftree}
\AxiomC{$\varphi \vdash \psi$}
\RightLabel{(dr_2')}
\UnaryInfC{$*\psi \vdash *\varphi$}
\RightLabel{$(\neg\vdash)$}
\UnaryInfC{$\neg\psi \vdash *\varphi$}
\RightLabel{(dr_2)}
\UnaryInfC{$(\neg\psi \,,\, \varphi) \vdash$}
\RightLabel{$(\mathsf{Cl}\vdash)$}
\UnaryInfC{$(\varphi \,,\, \neg\psi) \vdash$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$\varphi \,,\, \neg\psi \vdash$}
\RightLabel{(dr_2)}
\UnaryInfC{$\varphi \vdash *\neg\psi$}
\RightLabel{$(\neg\vdash)$}
\UnaryInfC{$\varphi \vdash \neg\neg\psi$}
\AxiomC{$\psi \vdash \psi$}
\RightLabel{(dr_2')}
\UnaryInfC{$*\psi \vdash *\psi$}
\RightLabel{$(\vdash\neg)$}
\UnaryInfC{$*\psi \vdash \neg\psi$}
\RightLabel{(dr_2'')}
\UnaryInfC{$*\neg\psi \vdash \psi$}
\RightLabel{$(\neg\vdash)$}
\UnaryInfC{$\neg\neg\psi \vdash \psi$}
\RightLabel{cut}
\BinaryInfC{$\varphi \vdash \psi$}
\end{prooftree}
$$

□

Lemma 64. *Let* $\circledast(\varphi_1, \ldots, \varphi_n) \in \mathcal{L}$ *with* $\circledast = (\sigma, \pm, (\exists, (\pm_1, \ldots, \pm_n)))$. *If* $\circledast(\varphi_1, \ldots, \varphi_n)$ *is* $\mathsf{GGL}_{\mathsf{C}}$*–consistent then* $\pm_1\varphi_1, \ldots, \pm_n\varphi_n$ *are* $\mathsf{GGL}_{\mathsf{C}}$*–consistent, where* $\pm_j\varphi_j \triangleq \begin{cases} \varphi_j & \textit{if } \pm_j = + \\ \neg\varphi_j & \textit{if } \pm_j = - \end{cases}$.

Proof: We prove it by contraposition. If $\pm_j\varphi_j$ is $\mathsf{GGL}_{\mathsf{C}}$–inconsistent then $\pm_j\varphi_j \vdash$. If $\pm_j = +$ then $\varphi_j \vdash$. If $\pm_j = -$ then $\neg\varphi_j \vdash$ and therefore $\vdash \varphi_j$ by the cut rule because $\neg\neg\varphi_j \vdash \varphi_j$ is provable. So, in both cases, applying Rule $(\vdash \circledast)$, we obtain that $\circledast(\varphi_1, \ldots, \varphi_n) \vdash$ and thus $\circledast(\varphi_1, \ldots, \varphi_n)$ is $\mathsf{GGL}_{\mathsf{C}}$–inconsistent. □

Definition 65 (Canonical model)**.** The *canonical model* is the tuple $(W^c, \mathcal{R}^c)$ where W^c is the set of all maximal $\mathsf{GGL}_{\mathsf{C}}$–consistent sets of $\mathcal{L}_{\mathsf{C}}$ and $\mathcal{R}^c$ is a set of relations $R_\circledast$ over W^c, associated to the connectives $\circledast \in \mathsf{C}$ and defined by:

- if $\circledast = p$ then $\Gamma \in R_p^{\pm}$ iff $p \in \Gamma$ (where $p = (1, \pm, Æ)$);
- if $\circledast = (\sigma, \pm, (\exists, (\pm_1, \ldots, \pm_n)))$ then $(\Gamma_1, \ldots, \Gamma_{n+1}) \in R_\circledast^{\pm\sigma}$ iff for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}_{\mathsf{C}}$, if $\varphi_1 \pitchfork \Gamma_1$ and ... and $\varphi_n \pitchfork \Gamma_n$ then $\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma_{n+1}$;
- if $\circledast = (\sigma, \pm, (\forall, (\pm_1, \ldots, \pm_n)))$ then $(\Gamma_1, \ldots, \Gamma_{n+1}) \notin R_\circledast^{\pm\sigma}$ iff for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}_{\mathsf{C}}$, if $\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma_{n+1}$ then $\varphi_1 \pitchfork \Gamma_1$ or ... or $\varphi_n \pitchfork \Gamma_n$;

where for all $j \in [\![1; n]\!]$, $\varphi_j \pitchfork \Gamma_j \triangleq \begin{cases} \varphi_j \in \Gamma_j & \text{if } \pm_j = + \\ \varphi_j \notin \Gamma_j & \text{if } \pm_j = - \end{cases}$. □

Lemma 66 (Truth lemma)**.** *For all* $\varphi \in \mathcal{L}$*, for all maximal* $\mathsf{GGL}_{\mathsf{C}}$*–consistent sets* Γ*, we have that* $M^c, \Gamma \Vdash \varphi$ *iff* $\varphi \in \Gamma$*.*

Proof: By induction on φ. The base case $\varphi = p \in \mathbb{P}$ holds trivially by definition of M^c.

- Case $\neg\varphi$.

Assume that $\neg\varphi \in \Gamma$ and assume towards a contradiction that it is not the case that $M^c, \Gamma \Vdash \neg\varphi$. Then, $M^c, \Gamma \Vdash \varphi$. So, by Induction Hypothesis, $\varphi \in \Gamma$. Now, $\varphi \ , \ \neg\varphi \vdash$ and $\neg\varphi \in \Gamma$ by assumption. Thus, Γ is not $\mathsf{GGL_C}$–consistent, which is impossible. Therefore, $M^c, \Gamma \Vdash \neg\varphi$.

Conversely, assume that $M^c, \Gamma \Vdash \neg\varphi$. Then, it is not the case that $M^c, \Gamma \Vdash \varphi$, so, by Induction Hypothesis, $\varphi \notin \Gamma$. Since Γ is a maximal $\mathsf{GGL_C}$–consistent set, this implies that $\Gamma \cup \{\varphi\}$ is not $\mathsf{GGL_C}$–consistent. So, there are $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\varphi_1 \ , \ \ldots \ , \ \varphi_n \ , \ \varphi \vdash$. Thus, $\varphi_1 \ , \ \ldots \ , \ \varphi_n \vdash * \varphi$ and also by $(\vdash \neg)$, $\varphi_1 \ , \ \ldots \ , \ \varphi_n \vdash \neg\varphi$. Therefore, $\neg\varphi \in \Gamma$ by the cut lemma.

- Case $(\varphi \vee \psi)$.

We prove the following fact. It will prove the induction step because $M^c, \Gamma \Vdash \varphi \vee \psi$ iff $M^c, \Gamma \Vdash \varphi$ or $M^c, \Gamma \Vdash \psi$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$ by induction hypothesis.

Fact 67. *For all maximal $\mathsf{GGL_C}$–consistent sets Γ, $(\varphi \vee \psi) \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$.*

Without loss of generality, assume that $\varphi \in \Gamma$. Then, $\varphi \vdash \varphi$ implies $\varphi \vdash \varphi \vee \psi$ by $\mathsf{K}_,$ and $(\vdash \vee)$. So, by the cut lemma, $(\varphi \vee \psi) \in \Gamma$ since $\varphi \in \Gamma$. Conversely, we prove that $(\varphi \vee \psi) \in \Gamma$ implies that $\varphi \in \Gamma$ or $\psi \in \Gamma$. Assume that $(\varphi \vee \psi) \in \Gamma$ and assume towards a contradiction that $\varphi \notin \Gamma$ and $\psi \notin \Gamma$. Then, because Γ is a maximal $\mathsf{GGL_C}$–consistent set, there are $\varphi_1, \ldots, \varphi_m \in \Gamma$ and $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\varphi_1 \ , \ \ldots \ , \ \varphi_m \ , \ \varphi \vdash$ and $\psi_1 \ , \ \ldots \ , \ \psi_n \ , \ \psi \vdash$. Thus, by $(\mathsf{K} \vdash)$, $(\mathsf{B} \vdash)$ and $(\mathsf{CI} \vdash)$, we have that $\varphi_1 \ , \ \ldots \ , \ \varphi_m \ , \ \psi_1 \ , \ \ldots \ , \ \psi_n \ , \ \varphi \vdash$ and $\varphi_1 \ , \ \ldots \ , \ \varphi_m \ , \ \psi_1 \ , \ \ldots \ , \ \psi_n \ , \ \psi \vdash$. Then, by rule (dr_2), we have that $\varphi \vdash * (\varphi_1 \ , \ \ldots \ , \ \varphi_m \ , \ \psi_1 \ , \ \ldots \ , \ \psi_n)$ and $\psi \vdash * (\varphi_1 \ , \ \ldots \ , \ \varphi_m \ , \ \psi_1 \ , \ \ldots \ , \ \psi_n)$. So, by rule $(\vee \vdash)'$, $(\varphi \vee \psi) \vdash * (\varphi_1 \ , \ \ldots \ , \ \varphi_m \ , \ \psi_1 \ , \ \ldots \ , \ \psi_n)$ and by rule (dr_2) and $(\mathsf{B} \vdash)$, $(\varphi \vee \psi) \ , \ \varphi_1 \ , \ \ldots \ , \ \varphi_m \ , \ \psi_1 \ , \ \ldots \ , \ \psi_n \vdash$. However, $(\varphi \vee \psi), \varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_n \in \Gamma$. Therefore, Γ is not $\mathsf{GGL_C}$–consistent, which is impossible. Thus, $\varphi \in \Gamma$ or $\psi \in \Gamma$.

- Case $(\varphi \wedge \psi)$.

We prove that $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$. This will prove this induction step because $M^c, \Gamma \Vdash \varphi \wedge \psi$ iff $M^c, \Gamma \Vdash \varphi$ and $M^c, \Gamma \Vdash \psi$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$ by induction hypothesis. Assume that $\varphi \in \Gamma$ and $\psi \in \Gamma$. Then, since $\varphi \ , \ \psi \vdash \varphi \wedge \psi$ is provable, we have by the cut lemma that $\varphi \wedge \psi \in \Gamma$. Conversely, assume that $\varphi \wedge \psi \in \Gamma$ and assume towards a contradiction that $\varphi \notin \Gamma$. Since Γ is a maximal $\mathsf{GGL_C}$–consistent set, there is $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\varphi_1 \ , \ \ldots \ , \ \varphi_n \ , \ \varphi \models$. Now, by rule $(\mathsf{K} \vdash)$, we have that $\varphi_1 \ , \ \ldots \ , \ \varphi_n \ , \ \varphi \ , \ \psi \models$. Therefore, by rule $\mathsf{B}_\vee$, $\varphi_1 \ , \ \ldots \ , \ \varphi_n \ , \ (\varphi \ , \ \psi) \models$. Then, by rules $(\mathsf{Cl} \vdash)$ (dr_2), we have that $(\varphi \ , \ \psi) \models \ * (\varphi_1 \ , \ \ldots \ , \ \varphi_n)$. So, by rule $(\wedge \vdash)$, we have that $(\varphi \wedge \psi) \models \ * (\varphi_1 \ , \ \ldots \ , \ \varphi_n)$. Then, again by rules $(\mathsf{Cl} \vdash)$ and (dr_2), we obtain $\varphi_1 \ , \ \ldots \ , \ \varphi_n \ , \ (\varphi \ , \ \psi) \models$. Since $(\varphi \wedge \psi) \in \Gamma$ and $\varphi_1, \ldots, \varphi_n \in \Gamma$, this entails that Γ is not $\mathsf{GGL_C}$–consistent, which is impossible. Therefore, $\varphi \in \Gamma$. Likewise, we prove that $\psi \in \Gamma$.

- Case $\circledast(\varphi_1, \ldots, \varphi_n)$ with $\circledast = (\sigma, \pm, (Æ, (\pm_1, \ldots, \pm_n)))$.

First, we deal with the subcase $Æ = \exists$.

Assume that $\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma$. We have to show that $M^c, \Gamma \Vdash \circledast (\varphi_1, \ldots, \varphi_n)$, *i.e.*, there are $\Gamma_1, \ldots, \Gamma_n \in M^c$ such that $R^{\pm\sigma}_{\circledast} \Gamma_1 \ldots \Gamma_n \Gamma$ and $\Gamma_1 \pitchfork [\![\varphi_1]\!]$ and $\ldots$ and $\Gamma_n \pitchfork [\![\varphi_n]\!]$. We build these maximal $\mathsf{GGL_C}$–consistent sets $\Gamma_1, \ldots, \Gamma_n$ thanks to (pseudo) Algorithm 1 (because it does not terminate). This algorithm is such that if $\circledast (\bowtie_1 \pm_1 \Gamma_1, \ldots, \bowtie_n \pm_n \Gamma_n) \in \Gamma$ then for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}$, there are $(\pm'_1, \ldots, \pm'_n) \in \{+, -\}^n$ such that $\circledast ((\bowtie_1 \pm_1 \Gamma^m_1) \times_1 (\pm'_1 \varphi^m_1), \ldots, (\bowtie_n \pm_n \Gamma^m_n) \times_n (\pm'_n \varphi^m_n)) \in \Gamma$. This is due to Expressions (9), (10) and Expressions (11), (12) of Lemma 63. What happens is that each $\bowtie_j \pm_j \Gamma_j$ is decomposed into disjunctions $((\bowtie_j \pm_j \Gamma_j) \wedge \varphi_n) \vee ((\bowtie_j \pm_j \Gamma_n) \wedge \neg\varphi_n)$ and conjunctions $((\bowtie_j \pm_j \Gamma_j) \vee \varphi_n) \wedge ((\bowtie_j \pm_j \Gamma_j) \vee \neg\varphi_n)$ depending on whether $\pm_j = +$ or $\pm_j = -$. Then, each decomposition of $\bowtie_j \pm_j \Gamma_n$ is replaced in Expression $\circledast (\bowtie_1 \pm_1 \Gamma_1, \ldots, \bowtie_n \pm_n \Gamma_n)$. This is possible thanks to rule $(\vdash \circledast)$ and this yields a new expression $(*)$. This new expression $(*)$ belongs to Γ because Γ is a maximal $\mathsf{GGL_C}$–consistent set, by the cut lemma. Then, we decompose again $(*)$ iteratively by applying Expressions (11) or (12). For each decomposition, at least one disjunct belongs to Γ because $\varphi \vee \psi \in \Gamma$ implies that either $\varphi \in \Gamma$ or $\psi \in \Gamma$ by Fact 67. Finally, after having decomposed each argument of $\circledast$, we obtain that there is $(\pm'_1, \ldots, \pm'_n) \in \{+, -\}^n$ such that $\circledast ((\bowtie_1 \pm_1 \Gamma^m_1) \times_1 (\pm'_1 \varphi^m_1), \ldots, (\bowtie_n \pm_n \Gamma^m_n) \times_n (\pm'_n \varphi^m_n)) \in \Gamma$.

Now, let $m \geq 0$ be fixed and assume that Γ^m_j is $\mathsf{GGL_C}$–consistent. Then, $\circledast ((\bowtie_1 \pm_1 \Gamma^m_1) \times_1 (\pm'_1 \varphi^m_1), \ldots, (\bowtie_n \pm_n \Gamma^m_n) \times_n (\pm'_n \varphi^m_n))$ is $\mathsf{GGL_C}$–consistent because it belongs to the $\mathsf{GGL_C}$–consistent set Γ^m_j. Thus, by Lemma 64, for all $j \in [\![1; n]\!]$, if $\pm_j = +$ then $\bigwedge \Gamma^m_j \wedge \pm'_j \varphi^m_j$ is $\mathsf{GGL_C}$–consistent and if $\pm_j = -$ then $\bigwedge \Gamma^m_j \wedge (-\pm'_j) \varphi^m_j$ is $\mathsf{GGL_C}$–consistent. That is, in both cases, Γ^{m+1}_j is $\mathsf{GGL_C}$–consistent. We have proved by induction that for all $m \geq 0$, Γ^m_j is $\mathsf{GGL_C}$–consistent. Thus, $\Gamma_1, \ldots, \Gamma_n$ are $\mathsf{GGL_C}$–

Algorithm 1

Require: $(\varphi_1, \ldots, \varphi_n) \in \mathcal{L}^n_{\mathsf{C}}$ and a maximal $\mathsf{GGL_C}$–consistent set Γ such that $\circledast(\varphi_1, \ldots, \varphi_n) \in \Gamma$ with $\circledast = (\sigma, \pm, (\exists, (\pm_1, \ldots, \pm_n)))$.

Ensure: A n–tuple of maximal $\mathsf{GGL_C}$–consistent sets $(\Gamma_1, \ldots, \Gamma_n)$ such that $R^{\pm\sigma}_{\circledast}\Gamma_1 \ldots \Gamma_n\Gamma$ and $\pm_1\varphi_1 \in \Gamma_1, \ldots, \pm_n\varphi_n \in \Gamma_n$.

Let $(\varphi^0_1, \ldots, \varphi^0_n), \ldots, (\varphi^m_1, \ldots, \varphi^m_n), \ldots$ be an enumeration of $\mathcal{L}^n_{\mathsf{C}}$;

$\Gamma^0_1 := \{\pm_1\varphi_1\}; \ldots; \Gamma^0_n := \{\pm_n\varphi_n\}$;

for all $m \geq 0$ **do**
  for all $(\pm'_1, \ldots, \pm'_n) \in \{+, -\}^n$ **do**
    if $\circledast\left(\left(\ltimes_1 \pm_1 \Gamma^m_1\right) \times_1 \left(\pm'_1\varphi^m_1\right), \ldots, \left(\ltimes_n \pm_n \Gamma^m_n\right) \times_n \left(\pm'_n\varphi^m_n\right)\right) \in \Gamma$ **then**
      $\Gamma^{m+1}_1 := \Gamma^m_1 \cup \{(\pm_1\pm'_1)\varphi^m_1\}$;
      $\vdots$
      $\Gamma^{m+1}_n := \Gamma^m_n \cup \{(\pm_n\pm'_n)\varphi^m_n\}$;
    end if
  end for
end for

$\Gamma_1 := \bigcup_{m\geq 0} \Gamma^m_1; \ldots; \Gamma_n := \bigcup_{m\geq 0} \Gamma^m_n$;

where for all $\varphi \in \mathcal{L}$, $\pm\varphi \triangleq \begin{cases} \varphi & \text{if } \pm = + \\ \neg\varphi & \text{if } \pm = - \end{cases}$; for all $j \in [\![1; n]\!]$, $\times_j \triangleq \begin{cases} \wedge & \text{if } \pm_j = + \\ \vee & \text{if } \pm_j = - \end{cases}$ and $\ltimes_j \pm_j \Gamma^m_j \triangleq \begin{cases} \bigwedge \left\{\varphi \mid \varphi \in \Gamma^m_j\right\} & \text{if } \pm_j = + \\ \bigvee \left\{\neg\varphi \mid \varphi \in \Gamma^m_j\right\} & \text{if } \pm_j = - \end{cases}$.

consistent. Moreover, for all $j \in [\![1; n]\!]$, Γ_j are *maximally* $\mathsf{GGL_C}$–consistent because by construction for all $\varphi \in \mathcal{L}$ either $\varphi \in \Gamma_j$ or $\neg\varphi \in \Gamma_j$.

Finally, we prove that $R^{\pm\sigma}_{\circledast}\Gamma_1 \ldots \Gamma_n\Gamma$, that is, we prove that for all $\psi_1, \ldots, \psi_n \in \mathcal{L}$ if $\psi_1 \pitchfork \Gamma_1$ and $\ldots$ and $\psi_n \pitchfork \Gamma_n$ then $\circledast(\psi_1, \ldots, \psi_n) \in \Gamma$, that is, since $\Gamma_1, \ldots, \Gamma_n$ are maximally $\mathsf{GGL_C}$–consistent sets, if $\pm_1\psi_1 \in \Gamma_1$ and $\ldots$ and $\pm_n\psi_n \in \Gamma_n$ then $\circledast(\psi_1, \ldots, \psi_n) \in \Gamma$. Assume that $\pm_1\psi_1 \in \Gamma_1$ and $\ldots$ and $\pm_n\psi_n \in \Gamma_n$, we are going to prove that $\circledast(\psi_1, \ldots, \psi_n) \in \Gamma$. Now $(\psi_1, \ldots, \psi_n) \in \mathcal{L}^n$, so there is $m_0 \geq 0$ such that $(\varphi_1^{m_0},\ldots,\varphi_n^{m_0}) = (\psi_1,\ldots,\psi_n)$. Since $\Gamma_1^{m_0+1} \subseteq \Gamma_1$ and ... and $\Gamma_n^{m_0+1} \subseteq \Gamma_n$, we have that the tuple $(\pm_1',\ldots,\pm_n')$ satisfying the condition of line 8 of Algorithm 1 is $(+,\ldots,+)$, because of the way $\Gamma_1^{m_0+1}$,..., $\Gamma_n^{m_0+1}$ are defined. So, the condition of line 8, which is fulfilled, is $\circledast\left(\left(\bowtie_1 \pm_1 \Gamma_1^{m_0}\right)\times_1 \varphi_1^{m_0},\ldots,\left(\bowtie_n \pm_n \Gamma_n^{m_0}\right)\times_n \varphi_n^{m_0}\right) \in \Gamma$. Then, for all $j \in [\![1;n]\!]$, if $\pm_j = +$ then $\left(\bowtie_j \pm_j \Gamma_j^{m_0}\right)\times_j \varphi_j^{m_0} \models \varphi_j^{m_0}$ and if $\pm_j = -$ then $\varphi_j^{m_0} \models \left(\bowtie_j \pm_j \Gamma_j^{m_0}\right)\times_j \varphi_j^{m_0}$. Therefore, applying rule $(\vdash \circledast)$, we obtain that $\circledast\left(\left(\bowtie_1 \pm_1 \Gamma_1^{m_0}\right)\times_1 \varphi_1^{m_0},\ldots,\left(\bowtie_n \pm_n \Gamma_n^{m_0}\right)\times_n \varphi_n^{m_0}\right) \models \circledast\left(\varphi_1^{m_0},\ldots,\varphi_n^{m_0}\right)$ is provable. Since we have proved that $\circledast\left(\left(\bowtie_1 \pm_1 \Gamma_1^{m_0}\right)\times_1 \varphi_1^{m},\ldots,\left(\bowtie_n \pm_n \Gamma_n^{m_0}\right)\times_n \varphi_n^{m_0}\right) \in \Gamma$, we obtain by the cut lemma that $\circledast\left(\varphi_1^{m_0},\ldots,\varphi_n^{m_0}\right) \in \Gamma$ as well, that is $\circledast\left(\psi_1,\ldots,\psi_n\right) \in \Gamma$.

Conversely, assume that $M^c,\Gamma \Vdash \circledast(\varphi_1,\ldots,\varphi_n)$, we are going to show that $\circledast(\varphi_1,\ldots,\varphi_n) \in \Gamma$. By definition, we have that there are $\Gamma_1,\ldots,\Gamma_n \in M^c$ such that $R_\circledast^{\pm\sigma}\Gamma_1\ldots\Gamma_n\Gamma$ and $\Gamma_1 \pitchfork [\![\varphi_1]\!]$ and ... and $\Gamma_n \pitchfork [\![\varphi_n]\!]$. By Induction Hypothesis, we have that $\varphi_1 \pitchfork \Gamma_1$ and ... and $\varphi_n \pitchfork \Gamma_n$. Then, by definition of $R_\circledast^{\pm\sigma}$ in Definition 65, we have that $\circledast(\varphi_1,\ldots,\varphi_n) \in \Gamma$.

Second, we deal with the subcase Æ = ∀.

Assume that $\circledast(\varphi_1,\ldots,\varphi_n) \in \Gamma$. We have to show that $M^c,\Gamma \Vdash \circledast(\varphi_1,\ldots,\varphi_n)$, *i.e.* for all $\Gamma_1,\ldots,\Gamma_n \in M^c$, $(\Gamma_1,\ldots,\Gamma_n,\Gamma) \in R_\circledast^{\pm\sigma}$ or $\Gamma_1 \pitchfork [\![\varphi_1]\!]$ or ... or $\Gamma_n \pitchfork [\![\varphi_n]\!]$. Assume that $(\Gamma_1,\ldots,\Gamma_n,\Gamma) \notin R_\circledast^{\pm\sigma}$. Then, since $\circledast(\varphi_1,\ldots,\varphi_n) \in \Gamma$, we have by Definition 65 that $\varphi_1 \pitchfork \Gamma_1$ or ... or $\varphi_n \pitchfork \Gamma_n$. So, by Induction Hypothesis, we have that $\Gamma_1 \pitchfork [\![\varphi_1]\!]$ or ... or $\Gamma_n \pitchfork [\![\varphi_n]\!]$.

Conversely, we reason by contraposition and we assume that $\circledast(\varphi_1,\ldots,\varphi_n) \notin \Gamma$. We are going to show that $M^c,\Gamma \Vdash -\circledast(\varphi_1,\ldots,\varphi_n)$ (we recall that $-\circledast$ is a connective of $\mathbb{C}$), which will prove that it is not the case that $M^c,\Gamma \Vdash \circledast(\varphi_1,\ldots,\varphi_n)$ by Proposition 29. First, we prove that $\neg\circledast(\varphi_1,\ldots,\varphi_n) \models -\circledast(\varphi_1,\ldots,\varphi_n)$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi_1 \models \varphi_1 \quad \ldots \quad \varphi_n \models \varphi_n}{[-\circledast](\varphi_1,\ldots,\varphi_n) \models -\circledast(\varphi_1,\ldots,\varphi_n)}(\vdash\circledast)}{*[\circledast](\varphi_1,\ldots,\varphi_n) \models -\circledast(\varphi_1,\ldots,\varphi_n)}\text{Rewrite}}{*-\circledast(\varphi_1,\ldots,\varphi_n) \models [\circledast](\varphi_1,\ldots,\varphi_n)}(\mathsf{dr}_2'')}{*-\circledast(\varphi_1,\ldots,\varphi_n) \models \circledast(\varphi_1,\ldots,\varphi_n)}(\circledast\vdash)}{*\circledast(\varphi_1,\ldots,\varphi_n) \models -\circledast(\varphi_1,\ldots,\varphi_n)}(\mathsf{dr}_2'')}{\neg\circledast(\varphi_1,\ldots,\varphi_n) \models -\circledast(\varphi_1,\ldots,\varphi_n)}(\neg\vdash)$$

Then, by Fact 67 and because $\models (\varphi \vee \neg\varphi)$ is provable, we have that

$\neg\circledast(\varphi_1,\ldots,\varphi_n)\in\Gamma$ or $\circledast(\varphi_1,\ldots,\varphi_n)\in\Gamma$. So, by assumption, $\neg\circledast(\varphi_1,\ldots,\varphi_n)\in\Gamma$. Therefore, by the cut lemma, since $\neg\circledast(\varphi_1,\ldots,\varphi_n)\vdash -\circledast(\varphi_1,\ldots,\varphi_n)$ we have that $-\circledast(\varphi_1,\ldots,\varphi_n)\in\Gamma$. Hence, this case boils down to the case Æ $=\exists$ because $-\circledast=(\sigma,-\pm,(\exists,(-\pm_1,\ldots,-\pm_n)))$. This case has been proved in the previous item and we thus have that $M^c,\Gamma\Vdash -\circledast(\varphi_1,\ldots,\varphi_n)$. □

We finally prove that the canonical model is indeed a C–model. For that, we need to prove the following lemma:

Lemma 68. *Let $\circledast\in C$ be a connective of arity $n\in\mathbb{N}$. Then, for all $\circledast'\in\mathcal{O}_{\alpha_n*\beta_n}(\circledast)$, we have that $R_\circledast=R_{\circledast'}$.*

Proof: We prove this lemma using the following two facts: for all $\circledast\in\mathbb{C}$, all transpositions $\tau_j=(j\ n+1)$,

$$\text{if } \circledast=(\sigma,\pm,(\exists,(\pm_1,\ldots,\pm_n)))\text{ then } \circledast(\varphi_1,\ldots,\tau_j\circledast(\varphi_1,\ldots,\varphi_n),\ldots,\varphi_n)\vdash\varphi_j \quad (14)$$

$$\text{if } \circledast=(\sigma,\pm,(\forall,(\pm_1,\ldots,\pm_n)))\text{then } \varphi_j\vdash\circledast(\varphi_1,\ldots,\tau_j\circledast(\varphi_1,\ldots,\varphi_n),\ldots,\varphi_n) \quad (15)$$

Expressions (14) and (15) are proved by a direct application of (dr_1) with τ_j and then $(\circledast\vdash)$ to the provable consecution $[\tau_j\circledast](\varphi_1,\ldots,\varphi_n)\vdash\tau_j\circledast(\varphi_1,\ldots,\varphi_n)$ if Æ$(\tau_j\circledast)=\exists$ and $\tau_j\circledast(\varphi_1,\ldots,\varphi_n)\vdash[\tau_j\circledast](\varphi_1,\ldots,\varphi_n)$ if Æ$(\tau_j\circledast)=\forall$.

First, we prove that for all $\circledast'\in\mathcal{O}_{\alpha_n}(\circledast)$, we have that $R_\circledast=R_{\circledast'}$. For that, it suffices to prove that for all transpositions $\tau_j=(j\ n+1)$, we have that $R_{\tau_j\circledast}=R_\circledast$ because the transpositions generate the symmetric group. Proving $R_\circledast\subseteq R_{\tau_j\circledast}$ or $R_{\tau_j\circledast}\subseteq R_\circledast$ for all $\tau_j=(j\ n+1)$ is enough, because by double inclusion we then have that $R_\circledast\subseteq R_{\tau_j\circledast}\subseteq R_{\tau_j\tau_j\circledast}=R_\circledast$ and thus $R_\circledast=R_{\tau_j\circledast}$.

- Case $\circledast=(\sigma,\pm,(\exists,(\pm_1,\ldots,\pm_{j-1},+,\pm_{j+1},\ldots,\pm_n)))$. Then, $\tau_j\circledast=(\tau_j\sigma,-\pm,(\forall,(-\pm_1,\ldots,-\pm_{j-1},+,-\pm_{j+1},\ldots,-\pm_n)))$.

Assume that $(\Gamma_1,\ldots,\Gamma_{n+1})\in R^{\pm\sigma}_\circledast$. We are going to show that $(\Gamma_1,\ldots,\Gamma_{n+1})\in R^{\pm\sigma}_{\tau_j\circledast}$, *i.e.* $(\Gamma_1,\ldots,\Gamma_{n+1})\notin R^{-\pm\sigma}_{\tau_j\circledast}$, *i.e.* $(\Gamma_1,\ldots,\Gamma_{j-1},\Gamma_{n+1},\Gamma_{j+1},\ldots,\Gamma_n,\Gamma_j)\notin R^{-\pm\tau_j\sigma}_{\tau_j\circledast}$. Let $\varphi_1,\ldots,\varphi_n\in\mathcal{L}_\mathsf{C}$ and assume that $\tau_j\circledast(\varphi_1,\ldots,\varphi_n)\in\Gamma_j$ and $\varphi_1\pitchfork\Gamma_1$ and … and $\varphi_n\pitchfork\Gamma_n$ where $\varphi_i\pitchfork\Gamma_i\triangleq\begin{cases}\varphi_i\in\Gamma_i & \text{if } \pm_i=+\\ \varphi_i\notin\Gamma_i & \text{if } \pm_i=-\end{cases}$. We want to prove that $\varphi_j\in\Gamma_{n+1}$.

Since $(\Gamma_1,\ldots,\Gamma_n,\Gamma_{n+1})\in R^{\pm\sigma}_\circledast$ and $\varphi_1\pitchfork\Gamma_1$ and … and $\tau_j\circledast(\varphi_1,\ldots\varphi_n)\in\Gamma_j$ and … and $\varphi_n\pitchfork\Gamma_n$, we have that $M^c,\Gamma_{n+1}\Vdash\circledast(\varphi_1,\ldots,\tau_j(\varphi_1,\ldots,\varphi_n),\ldots,\varphi_n)$. So, by the truth lemma, $\circledast(\varphi_1,\ldots,\tau_j\circledast(\varphi_1,\ldots,\varphi_n),\ldots,\varphi_n)\in\Gamma_{n+1}$. Now, by Expression (14), $\circledast(\varphi_1,\ldots,\tau_j\circledast(\varphi_1,\ldots,\varphi_n),\ldots,\varphi_n)\vdash\varphi_j$. Therefore, $\varphi_j\in\Gamma_{n+1}$ by the cut lemma.

• Case $\circledast = (\sigma, \pm, (\exists, (-, \dots, -)))$. Then, $\tau_j \circledast = (\tau_j \sigma, \pm, (\exists, (-, \dots, -)))$.

Assume that $(\Gamma_1, \dots, \Gamma_{n+1}) \in R^{\pm\sigma}_{\circledast}$, *i.e.* for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_\mathsf{C}$, if $\varphi_1 \notin \Gamma_1$ and ... and $\varphi_n \notin \Gamma_n$ then $\circledast(\varphi_1, \dots, \varphi_n) \in \Gamma_{n+1}$ (1). We are going to show that $(\Gamma_1, \dots, \Gamma_{n+1}) \in R^{\pm\sigma}_{\tau_j\circledast}$, *i.e.* $(\Gamma_1, \dots, \Gamma_{n+1}, \dots \Gamma_n, \Gamma_j) \in R^{\pm\tau_j\sigma}_{\tau_j\circledast}$, *i.e.* for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_\mathsf{C}$, if $\varphi_1 \notin \Gamma_1$ and ... and $\varphi_j \notin \Gamma_{n+1}$ and ... and $\varphi_n \notin \Gamma$, then $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_j$. Assume that $\varphi_1 \notin \Gamma_1$ and ... and $\varphi_j \notin \Gamma_{n+1}$ and ... and $\varphi_n \notin \Gamma_n$. We want to prove that $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_j$.

Since $\varphi_j \notin \Gamma_{n+1}$, we have that $\circledast(\varphi_1, \dots, \tau_j \circledast (\varphi_1, \dots, \varphi_n), \dots, \varphi_n) \notin \Gamma_{n+1}$ because of the cut lemma since $\circledast(\varphi_1, \dots, \tau_j \circledast (\varphi_1, \dots, \varphi_n), \dots, \varphi_n) \vdash \varphi_j$ by Expression (14). Then, either $\varphi_1 \in \Gamma_1$ or $\varphi_2 \in \Gamma_2$ or ... or $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_j$ or $\varphi_{j+1} \in \Gamma_{j+1}$ or ... or $\varphi_n \in \Gamma_n$, because of (1). However, $\varphi_1 \notin \Gamma_1$,..., $\varphi_{j-1} \notin \Gamma_{j-1}$, $\varphi_{j+1} \notin \Gamma_{j+1}$, ..., $\varphi_n \notin \Gamma_n$. Therefore, $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_j$.

• Case $\circledast = (\sigma, \pm, (\forall, (\pm_1, \dots \pm_{j-1}, +, \pm_{j+1}, \dots, \pm_n)))$. Then, $\tau_j \circledast = (\tau_j \sigma, -\pm, (\exists, (-\pm_1, \dots, -\pm_{j-1}, +, -\pm_{j+1}, \dots, -\pm_n)))$.

Assume that $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \notin R^{\pm\sigma}_{\circledast}$. We are going to show that $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \notin R^{\pm\sigma}_{\tau_j\circledast}$, *i.e.* $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \in R^{-\pm\sigma}_{\tau_j\circledast}$ *i.e.* $(\Gamma_1, \dots, \Gamma_{n+1}, \Gamma_{j+1}, \dots, \Gamma_n, \Gamma_j) \in R^{-\pm\tau_j\sigma}_{\tau_j\circledast}$ *i.e.* for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_\mathsf{C}$, if $\varphi_1 \pitchfork \Gamma_1$ and ... and $\varphi_j \in \Gamma_{n+1}$ and $\varphi_{j+1} \pitchfork \Gamma_{j+1}$ and ... and $\varphi_n \pitchfork \Gamma_n$ then $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_j$ where $\varphi_i \pitchfork \Gamma_i \triangleq \begin{cases} \varphi_i \in \Gamma_i & \text{if } -\pm_i = + \\ \varphi_i \notin \Gamma_i & \text{if } -\pm_i = - \end{cases}$. Assume that $\varphi_1 \pitchfork \Gamma_1$ and ... and $\varphi_j \in \Gamma_{n+1}$ and $\varphi_{j+1} \pitchfork \Gamma_{j+1}$ and ... and $\varphi_n \pitchfork \Gamma_n$. We want to show that $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_j$.

Since $\varphi_j \in \Gamma_{n+1}$ and $\varphi_j \vdash \circledast (\varphi_1, \dots, \tau_j \circledast (\varphi_1, \dots, \varphi_n), \dots, \varphi_n)$ by Expression (15), we have by the cut lemma that $\circledast(\varphi_1, \dots, \tau_j \circledast (\varphi_1, \dots, \varphi_n), \dots \varphi_n) \in \Gamma_{n+1}$. So, $M^c, \Gamma_{n+1} \Vdash \circledast (\varphi_1, \dots, \tau_j \circledast (\varphi_1, \dots, \varphi_n), \dots, \varphi_n)$ by the truth lemma. That is, for all $\Gamma'_1, \dots, \Gamma'_n \in M^c$, either $(\Gamma'_1, \dots, \Gamma'_n, \Gamma_{n+1}) \in R^{\pm\sigma}_{\circledast}$ or not $\varphi_1 \pitchfork \Gamma_1$ or ... or $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_j$ or ... or not $\varphi_n \pitchfork \Gamma_n$ ($\varphi_i \pitchfork \Gamma_i$ is defined above). Take $(\Gamma'_1, \dots, \Gamma'_n) = (\Gamma_1, \dots, \Gamma_n)$. Then, by assumption, $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \notin R^{\pm\sigma}_{\circledast}$ and $\varphi_1 \pitchfork \Gamma_1$ and ... and $\varphi_{j-1} \pitchfork \Gamma_{j-1}$ and $\varphi_{j+1} \pitchfork \Gamma_{j+1}$ and ... and $\varphi_n \pitchfork \Gamma_n$. Therefore, $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_j$.

• Case $\circledast = (\sigma, \pm, (\forall, (-, \dots, -)))$. Then, $\tau_j \circledast = (\tau_j \sigma, \pm, (\forall, (-, \dots, -)))$.

Assume that $(\Gamma_1, \dots, \Gamma_{n+1}) \notin R^{\pm\sigma}_{\circledast}$, *i.e.* for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_\mathsf{C}$, if $\circledast(\varphi_1, \dots, \varphi_n) \in \Gamma_{n+1}$ and $\varphi_1 \in \Gamma_1$ and ...and $\varphi_n \in \Gamma_n$ then $\varphi_j \notin \Gamma_j$ (2). We are going to show that $(\Gamma_1, \dots, \Gamma_{n+1}) \notin R^{\pm\sigma}_{\tau_j\circledast}$, *i.e.* $(\Gamma_1, \dots, \Gamma_{n+1}, \dots, \Gamma_n, \Gamma_j) \notin R^{\pm\tau_j\sigma}_{\tau_j\circledast}$ *i.e.* for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_\mathsf{C}$ if $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_j$ and $\varphi_1 \in \Gamma_1$ and ... and $\varphi_n \in \Gamma_n$ then $\varphi_j \notin \Gamma_{n+1}$. Assume that $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_j$ (3) and $\varphi_1 \in \Gamma_1$ and ... and $\varphi_n \in \Gamma_n$. We want to prove that $\varphi_j \notin \Gamma_{n+1}$.

Assume towards a contradiction that $\varphi_j \in \Gamma_{n+1}$. Then, by Expression (15) and the cut lemma, $\circledast(\varphi_1, \dots, \tau_j \circledast (\varphi_1, \dots, \varphi_n), \dots, \varphi_n) \in \Gamma_{n+1}$. Now, $\varphi_1 \in \Gamma_1$ and $\dots$ and $\varphi_{j-1} \in \Gamma_{j-1}$ and $\varphi_{j+1} \in \Gamma_{j+1}$ and $\dots$ and $\varphi_n \in \Gamma_n$. So, by (2), because $(\Gamma_1, \dots, \Gamma_{n+1}) \notin R^{\pm\sigma}_{\circledast}$, we have that $\tau_j \circledast (\varphi_1, \dots, \varphi_n) \notin \Gamma_j$. This contradicts (3).

Second, we prove that $R_{\circledast} = R_{-\circledast}$. Again, it suffices to prove that $R_{\circledast} \subseteq R_{-\circledast}$.

- Case $\circledast = (\sigma, \pm, (\exists, (\pm_1, \dots, \pm_n)))$. Then, $-\circledast = (\sigma, -\pm, (\forall, (-\pm_1, \dots, -\pm_n)))$.

$(\Gamma_1, \dots, \Gamma_{n+1}) \in R^{\pm\sigma}_{\circledast}$ iff for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_{\mathsf{C}}$, if $\varphi_1 \pitchfork \Gamma_1$ and $\dots$ and $\varphi_n \pitchfork \Gamma_n$ then $\circledast(\varphi_1, \dots, \varphi_n) \in \Gamma_{n+1}$ where $\varphi_j \pitchfork \Gamma_j = \begin{cases} \varphi_j \in \Gamma_j & \text{if } \pm_j = + \\ \varphi_j \notin \Gamma_j & \text{if } \pm_j = - \end{cases}$. We are going to show that $(\Gamma_1, \dots, \Gamma_{n+1}) \in R^{\pm\sigma}_{-\circledast}$, *i.e.* $(\Gamma_1, \dots, \Gamma_{n+1}) \notin R^{-\pm\sigma}_{-\circledast}$ *i.e.* for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_{\mathsf{C}}$, if $- \circledast (\varphi_1, \dots, \varphi_n) \in \Gamma_{n+1}$ then $\varphi_1 \pitchfork' \Gamma_1$ or $\dots$ or $\varphi_n \pitchfork' \Gamma_n$ (1) where $\varphi_j \pitchfork' \Gamma_j = \begin{cases} \varphi_j \in \Gamma_j & \text{if } -\pm_j = + \\ \varphi_j \notin \Gamma_j & \text{if } -\pm_j = - \end{cases}$. So, for all j, $\varphi_j \pitchfork' \Gamma_j$ is (not $\varphi_j \pitchfork \Gamma_j$). Therefore, (1) holds iff if $\circledast(\varphi_1, \dots, \varphi_n) \notin \Gamma_{n+1}$ and $\varphi_1 \pitchfork \Gamma_1$ and $\dots$ and $\varphi_n \pitchfork \Gamma_n$ then not $\varphi_j \pitchfork \Gamma_j$

iff if $\varphi_1 \pitchfork \Gamma_1$ and $\dots$ and $\varphi_n \pitchfork \Gamma_n$ then $\circledast(\varphi_1, \dots, \varphi_n) \in \Gamma_{n+1}$

iff $(\Gamma_1, \dots, \Gamma_{n+1}) \in R^{\pm\sigma}_{\circledast}$ which holds by assumption.

- Case $\circledast = (\sigma, \pm, (\forall, (\pm_1, \dots, \pm_n)))$. It is proved like the previous case. □

Proof: (Completeness proof) We prove that for all sets $\Gamma \subseteq \mathcal{S}_{\mathsf{C}}$ and all $S = \varphi \vdash \psi \in \mathcal{S}_{\mathsf{C}}$, if $\Gamma \Vdash S$ holds then S is provable from Γ in $\mathsf{GGL}_{\mathsf{C}}$. We reason by contraposition. Assume that S is not provable from Γ in $\mathsf{GGL}_{\mathsf{C}}$. That is, there is no proof of $\varphi \vdash \psi$ in $\mathsf{GGL}_{\mathsf{C}}$ from Γ. Thus, it is not the case that $(\varphi \ , \ \neg\psi) \vdash$ is provable in $\mathsf{GGL}_{\mathsf{C}} \cup \Gamma$ by Expression (13). Hence, $\{\varphi, \neg\psi\}$ is $\mathsf{GGL}_{\mathsf{C}} \cup \Gamma$–consistent (we can naturally adapt the definition of $\mathsf{GGL}_{\mathsf{C}}$–consistency to define the notion of $\mathsf{GGL}_{\mathsf{C}} \cup \Gamma$–consistency). So, by Lemma 62 (where $\mathsf{GGL}_{\mathsf{C}}$–consistency is replaced by $\mathsf{GGL}_{\mathsf{C}} \cup \Gamma$–consistency), it can be extended into a maximal $\mathsf{GGL}_{\mathsf{C}} \cup \Gamma$–consistent set Γ' such that $\{\varphi, \neg\psi\} \subseteq \Gamma'$. Now, Γ' is also $\mathsf{GGL}_{\mathsf{C}}$–consistent, so it is a state of the canonical model M^c. Then, by the truth Lemma 66, we have that $(M^c, \Gamma') \Vdash \varphi$ and $(M^c, \Gamma') \Vdash \neg\psi$, so it is not the case that $(M^c, \Gamma') \Vdash S$. Moreover, by the cut Lemma 61 and because Γ' is also $\mathsf{GGL}_{\mathsf{C}} \cup \Gamma$–consistent, we also have that $(M^c, \Gamma') \Vdash \Gamma$. Hence, we have found a pointed model (M^c, Γ'), which is indeed a C–model according to Lemma 68, such that $(M^c, \Gamma') \Vdash \Gamma$ but not $(M^c, \Gamma') \Vdash S$. That is, it is not the case that $\Gamma \Vdash S$. □

C Proofs of theorems 49, 53 and 57

Theorem 49 (Cut–elimination). *Let $C \subseteq \mathbb{C}$. The calculus GGL_C is cut–eliminable: it is possible to eliminate all occurrences of the cut rule from a given proof in order to obtain a cut-free proof of the same consecution.*

Proof: Since $\mathsf{GGL_C}$ is a display calculus in the general sense of Ciabattoni & Ramanayake [9], we only need to prove that it satisfies the conditions $(C2)$–$(C8)$ spelled out in [9] as proved by Belnap [6]. Note that condition $(C1)$ is not needed in Belnap's proof [6]. The conditions $(C2)$–$(C7)$ are easily checked on each rule of $\mathsf{GGL_C}$. It remains to prove condition $(C8)$. It has already been proved in the literature for the Boolean connectives so we only prove it for the gaggle connectives. Instead of proving it in the general case, we prove it for $n = 2$ with $\circledast = (\sigma, \pm, (\exists, (+, -)))$. This should provide the reader with the main ideas underlying the proof in the general case. Basically, we display each subformula of the cut formula using the display rule (dr_1) and we apply the cut rule on each subformula.

$$\dfrac{\dfrac{X_1 \vdash \varphi_1 \quad \varphi_2 \vdash X_2}{[\circledast]\,(X_1, X_2) \vdash \circledast\,(\varphi_1, \varphi_2)}\,(\vdash \circledast) \quad \dfrac{[\circledast]\,(\varphi_1, \varphi_2) \vdash U}{\circledast(\varphi_1, \varphi_2) \vdash U}\,(\circledast \vdash)}{[\circledast]\,(X_1, X_2) \vdash U}\; cut\,(\circledast(\varphi_1, \varphi_2))$$

is transformed into

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{X_1 \vdash \varphi_1 \quad \dfrac{[\circledast]\,(\varphi_1, \varphi_2) \vdash U}{\varphi_1 \vdash [r_1\circledast]\,(U, \varphi_2)}\,(\mathsf{dr}_1)}{X_1 \vdash [r_1\circledast]\,(U, \varphi_2)}\,cut\,(\varphi_1)}{[\circledast]\,(X_1, \varphi_2) \vdash U}\,(\mathsf{dr}_1)}{[r_2\circledast]\,(X_1, U) \vdash \varphi_2}\,(\mathsf{dr}_1) \quad \varphi_2 \vdash X_2}{[r_2\circledast]\,(X_1, U) \vdash X_2}\,cut\,(\varphi_2)}{[\circledast]\,(X_1, X_2) \vdash U}\,(\mathsf{dr}_1)$$

We proceed similarly for the rules concerning the Boolean connectives $\neg, \wedge, \vee$ using the Boolean display rule (dr_2). $\square$

Theorem 53 (Soundness and strong completeness). *Let $C \subseteq \mathbb{C}$. The calculus GGL^0_C is sound and strongly complete for the basic gaggle logic $(\mathcal{S}^0_C, \mathcal{M}_C, \Vdash)$.*

Proof: We are going to perform a backward proof search and analyze the structure of a cut-free proof in $\mathsf{GGL_C}$ which ends up in a consecution of the following form, where $\varphi_1, \ldots, \varphi_k, \varphi'_1, \ldots, \varphi'_l \in \mathcal{L}^0_C$ do not contain Boolean connectives:

$$\circledast(\varphi_1, \ldots, \varphi_k) \vdash \circledast'\,(\varphi'_1, \ldots, \varphi'_l).$$

Our aim is, via that analysis, to transfom the proof in $\mathsf{GGL_C}$ of the above consecution into a proof in $\mathsf{GGL}^0_\mathsf{C}$ of the same consecution. This will prove the theorem.

Before proceeding further, note that the following rules are particular instances of $(\mathsf{K}\vdash)$ and $(\vdash\mathsf{K})$ (with X empty):

$$\frac{\vdash U}{Y \vdash U}\ (\mathsf{K}\vdash)' \qquad\qquad \frac{U \vdash}{U \vdash Y}\ (\vdash\mathsf{K})'$$

Since the proof is cut-free and the final consecution does not contain Boolean connectives, the Boolean rules $(\wedge\vdash)$, $(\vdash\wedge)$, $(\vee\vdash)$, $(\vdash\vee)$, $(\vdash\neg)$ and $(\neg\vdash)$ have not been applied in the proof. Indeed, a property of our *cut-free* calculus $\mathsf{GGL_C}$ is that once a (Boolean) connective is introduced in a proof it stays present in the proof. Because the conclusion of our proof does not contain Boolean connective, this entails that the Boolean rules have not been used.

Stage A: rules $(\circledast\vdash)$ and (dr_1). We start with a proof in $\mathsf{GGL_C}$ whose conclusion is of the form $\circledast(\varphi_1,\ldots,\varphi_k)\vdash\circledast'(\varphi'_1,\ldots,\varphi'_l)$ and we analyse its proof backwards and determine which rule(s) can be used as we proceed bottom–up. At the begining, it is not possible to apply rule $(\vdash\circledast)$ because the antecedent and the consequent of the consecution are both formulas. On the other hand, it is possible to apply rule (dr_2) or $(\mathsf{WI}\vdash)$ right at the beginning and in that case we go directly to stage B. Otherwise, it is also possible to apply the rules $(\circledast\vdash)$ and (dr_1) (possibly iteratively). We then obtain an expression of the form $S([\circledast_1], X_1,\ldots,X_m,\circledast_2(\psi_1,\ldots,\psi_n))$ or $S([\circledast_1], X_1,\ldots,X_m,[\circledast_2](Y_1,\ldots,Y_n))$ where $X_1,\ldots,X_m,Y_1,\ldots,Y_n$ belong to the language $\mathcal{L}^X$ built up from formulas φ, structural atoms and structural connectives $[\circledast]$. Hence, at the end of that stage, we have a consecution of the form $[\circledast_1](X_1,\ldots,X_n)\vdash\circledast_2(\psi_1,\ldots,\psi_n)$ (1) or $\circledast_1(\varphi_1,\ldots,\varphi_m)\vdash[\circledast_2](Y_1,\ldots,Y_n)$ (2) or $[\circledast_1](X_1,\ldots,X_m)\vdash[\circledast_2](Y_1,\ldots,Y_n)$ (3). Without loss of generality, let us deal with case (1) in what follows.

We can then go to stage B or to stage C.

Stage B: rules (dr_2) or $(\mathsf{WI}\vdash)$ and then structural rules. If rule (dr_2) is applied, we obtain

$$\frac{([\circledast_1](X_1,\ldots,X_m)\ ,\ *\circledast_2(\psi_1,\ldots,\psi_n))\vdash}{[\circledast_1](X_1,\ldots,X_m)\vdash\ \circledast_2(\psi_1,\ldots,\psi_n)}\ (\mathsf{dr}_2)$$

or

$$\frac{\vdash (\circledast_2(\psi_1,\dots,\psi_n) \;,\; *[\circledast_1](X_1,\dots,X_m))}{[\circledast_1](X_1,\dots,X_m) \vdash \circledast_2(\psi_1,\dots,\psi_n)}\;(\mathsf{dr}_2).$$

If rule $(\mathsf{Cl}\vdash)$ is applied, we obtain

$$\frac{([\circledast_1](X_1,\dots,X_m) \;,\; [\circledast_1](X_1,\dots,X_m)) \vdash \circledast_2(\psi_1,\dots,\psi_n)}{[\circledast_1](X_1,\dots,X_m) \vdash \circledast_2(\psi_1,\dots,\psi_n)}\;(\mathsf{Wl}\vdash).$$

In both cases, we obtain a premise including the structural connective $\,,\,$. This means that we cannot apply rules (dr_1), $(\vdash\circledast)$ or $(\circledast\vdash)$ for the moment. We must use the other rules, the structural rules and (dr_2), in order to apply one of these rules. Indeed, for the proof to terminate, we have to apply these rules in order to reduce the complexity of the consecution. Since the structural rules and (dr_2) do not change the constituants of a consecution, the consecutions that we can obtain as a result of applying these rules in order to be able to apply rules (dr_1), $(\vdash\circledast)$ or $(\circledast\vdash)$ again are the following:

1. $\vdash \circledast_2(\psi_1,\dots,\psi_n)$
2. $[\circledast_1](X_1,\dots,X_m) \vdash$
3. $*\circledast_2(\psi_1,\dots,\psi_n) \vdash [-\circledast_1](X_1,\dots,X_n)$
4. $[\circledast_1](X_1,\dots,X_m) \vdash \circledast_2(\psi_1,\dots,\psi_n)$.

For each case, we replace the existing derivation by the following derivation:

1.

$$\frac{\vdash \circledast_2(\psi_1,\dots,\psi_n)}{[\circledast_1](X_1,\dots,X_m) \vdash \circledast_2(\psi_1,\dots,\psi_n)}\;(\mathsf{K}\vdash)'$$

2.

$$\frac{[\circledast_1](X_1,\dots,X_m) \vdash}{[\circledast_1](X_1,\dots,X_m) \vdash \circledast_2(\psi_1,\dots,\psi_n)}\;(\vdash\mathsf{K})'$$

3.

$$\frac{*\circledast_2(\psi_1,\dots,\psi_n) \vdash [-\circledast_1](X_1,\dots,X_m)}{[\circledast_1](X_1,\dots,X_m) \vdash \circledast_2(\psi_1,\dots,\psi_n)}\;(\mathsf{dr}_2')$$

4. We simply remove the existing derivation.

So, for all cases the Boolean display rule (dr_2) and the structural rules have been eliminated. In all cases, the proof (considered so far) can be transfomed into a proof where (dr_2) has been eliminated and replaced by $\left(\mathsf{dr}_2'\right)$, $(\mathsf{K}\vdash)'$ and $(\vdash\mathsf{K})'$.

In all cases, the last premise ends up to be a consecution of the form $S([\circledast_1], X_1, \ldots, X_m, \circledast_2(\psi_1, \ldots, \psi_n))$ or $\circledast_1(\varphi_1, \ldots, \varphi_m) \vdash \circledast_2(\psi_1, \ldots, \psi_n)$ (possibly with $\circledast_2(\psi_1, \ldots, \psi_n)$ empty). Then, we go to stage C.

Stage C: rules (dr_1) or $(\vdash\circledast)$. If rule (dr_1) is applied then we go back to stage A.

If rule $(\vdash\circledast)$ is applied,

$$\frac{U_1 \vdash V_1 \quad \ldots \quad U_n \vdash V_n}{S([\circledast_1], X_1, \ldots, X_n, \circledast_2(\psi_1, \ldots, \psi_n))}\ (\vdash\circledast)$$

then for all $j \in [\![1;n]\!]$, $U_j \vdash V_j$ are of the form $X_j \vdash \psi_j$ or $\varphi_j \vdash X_j$ where $X_j \in \mathcal{L}^X$. So, we apply inductively stages A, B and C to each $U_j \vdash V_j$.

Hence, applying these stages recursively, we are able to eliminate all structural rules and the Boolean display rule (dr_2) from the proof and replace them with the rules $\left(\mathsf{dr}_2'\right)$, $(\mathsf{K}\vdash)'$ and $(\vdash\mathsf{K})'$.

Stage D. At this stage we have transformed our initial proof in $\mathsf{GGL_C}$ into a proof in the calculus consisting in the rules $(\vdash\circledast)$, $(\circledast\vdash)$, (dr_1), $\left(\mathsf{dr}_2'\right)$, $(\mathsf{K}\vdash)'$ and $(\vdash\mathsf{K})'$. A requirement of rule $(\mathsf{K}\vdash)'$ ($(\vdash\mathsf{K})'$) is that the antecedent (resp. consequent) of its premisse is empty. If we examine the other rules, we notice that an empty antecedent can only appear in rule $(\vdash\circledast)$ if one of its premise already contains an empty antecedent (see Expression (7)). As a matter of fact, because of our axioms (see Expressions (5) and (6)) and the other rules, this can never happen. Hence, rules $(\mathsf{K}\vdash)'$ and $(\vdash\mathsf{K})'$ are in fact never used in a proof. Therefore, the proof that we eventually obtain is actually a proof in $\mathsf{GGL}^0_\mathsf{C}$. □

Theorem 57 (Soundness and strong completeness)**.** *Let $\mathsf{C} \subseteq \mathbb{C}$ and let G be a group associated to C. The calculus $\mathsf{GL}^0_{\mathsf{C},G}$ ($\mathsf{GL}_{\mathsf{C},G}$) is sound and strongly complete for the (Boolean) basic gaggle logic $(\mathcal{S}^0_\mathsf{C}, \mathcal{M}_\mathsf{C}, \Vdash)$ (resp. $(\mathcal{S}_\mathsf{C}, \mathcal{M}_\mathsf{C}, \Vdash)$).*

Proof: We assume that we have a proof of a consecution $\circledast_1(\varphi_1, \ldots, \varphi_m) \vdash \circledast_2(\psi_1, \ldots, \psi_n) \in \mathcal{S}^0_\mathsf{C}$ in $\mathsf{GGL}^0_\mathsf{C}$ and we show that we can transform this proof into a proof of the same consecution in $\mathsf{GL}^0_{\mathsf{C},G}$. For that, we analyse the proof and perform a backward proof search. The first rule that we can apply (backwards) is $(\circledast \vdash)$ and we arrive at a consecution of the form $S([\circledast], \varphi_1, \dots, \varphi_n, U)$. Then, we can directly apply $(\circledast \vdash)$ or a sequence of display rules in order to apply $(\circledast \vdash)$. In both cases, we arrive at a consecution of the form $S([\circledast'], X_1, \dots, X_n, \circledast(\varphi_1, \dots, \varphi_n))$ with $\circledast' \in \mathsf{C}$ (in order to apply $(\vdash \circledast)$). Since both $\circledast \in \mathsf{C}$ and $\circledast' \in \mathsf{C}$, the sequence of display rules is equivalent to a single application of rule (dr_3) and it suffices to replace this sequence by a single application of rule (dr_3) to obtain a proof in $\mathsf{GL}^0_{\mathsf{C},G}$. Then, we repeat this process inductively to the premises of the instance of the rule $(\vdash \circledast)$ applied. Hence, we obtain the result for $\mathsf{GL}^0_{\mathsf{C},G}$.

As for $\mathsf{GL}_{\mathsf{C},G}$, it suffices to observe that (dr_3) is derivable from (dr_1) and (dr_2) and that, vice versa, (dr_1) is derivable from (dr_3). $\square$

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