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The Direct Scattering Map for the Intermediate Long Wave Equation

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THE DIRECT SCATTERING MAP FOR THE INTERMEDIATE LONG WAVE EQUATION

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

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ABSTRACT OF DISSERTATION

THE DIRECT SCATTERING MAP FOR THE INTERMEDIATE LONG WAVE EQUATION

In the early 1980's, Kodama, Ablowitz and Satsuma, together with Santini, Ablowitz and Fokas, developed the formal inverse scattering theory of the Intermediate Long Wave (ILW) equation and explored its connections with the Benjamin-Ono (BO) and KdV equations. The ILW equation

$$u_t + \frac{1}{\delta} u_x + 2uu_x + Tu_{xx} = 0,$$

models the behavior of long internal gravitational waves in stratified fluids of depth $0 < \delta < \infty$, where T is a singular operator which depends on the depth δ. In the limit $\delta \to 0$, the ILW reduces to the Korteweg de Vries (KdV) equation, and in the limit $\delta \to \infty$, the ILW (at least formally) reduces to the Benjamin-Ono (BO) equation.

While the KdV equation is very well understood, a rigorous analysis of inverse scattering for the ILW equation remains to be accomplished. There is currently no rigorous proof that the Inverse Scattering Transform outlined by Kodama *et al.* solves the ILW, even for small data. In this dissertation, we seek to help ameliorate this gap in knowledge by presenting a mathematically rigorous construction of the direct scattering map for the ILW's Inverse Scattering Transform.

KEYWORDS: Inverse Scattering Transform, Inverse Scattering, Intermediate Long Wave Equation, Dispersive Waves, Diffeo-Integral Equations, Partial Differential Equations

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November 18, 2020

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DEDICATION

To the University of Kentucky Graduate Student Congress, whose community of graduate students, professional students and postdocs helped sustain me through the myriad challenges of graduate school.

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TABLE OF CONTENTS

Acknowledgments iii

List of Figures vii

CHAPTER 1. INTRODUCTION 1
1.1 The Intermediate Long Wave Equation 1
1.2 Introduction to Inverse Scattering 2
1.3 Overview of the ILW Direct Scattering Map 5

CHAPTER 2. GREEN'S FUNCTIONS: LOWER BOUNDARY VALUES . . 12
2.1 Introduction 12
2.2 Motivation of the Green's Functions 16
2.3 Green's Function Contour Shift 22
2.4 Asymptotics of K^+ 26
2.5 Green's Function Singularity near $x = 0$ 28

CHAPTER 3. GREENS FUNCTIONS: MAPPING PROPERTIES 36
3.1 Introduction 36
3.2 Boundedness as a Convolution Operator 39
3.3 λ-Differentiability 45

CHAPTER 4. GREEN'S FUNCTIONS: ANALYTIC CONTINUATION . . . 51
4.1 Introduction 51
4.2 Analytic Extension to the Strip 54
4.3 The Continuous Limit 58
4.4 Exponential Cauchy Transform 61
4.5 Boundedness of the Exponentially Weighted Hilbert Transform 73

CHAPTER 5. JOST SOLUTIONS & THE DIRECT SCATTERING MAP . 78
5.1 Introduction 78
5.2 Existence and Continuity of Jost Solutions 81
5.3 Equivalence of Integral Equation Solutions and Jost Solutions 85
5.3.1 Jost Solutions Solve the Integral Equations 90
5.3.2 Integral Equation Solutions are Jost Solutions 97
5.4 The Direct Scattering Map 103

APPENDICES 112
APPENDIX 1. LAX REPRESENTATION 113
APPENDIX 2. JUMP RELATION 116
APPENDIX 3. HARMONIC ANALYSIS RESULTS 118
APPENDIX 4. NOTATION INDEX 123

Fundamental Notation 123
Chapter 1 Notation 126
Chapter 2 Notation 129
Chapter 3 Notation 132
Chapter 4 Notation 134
Chapter 5 Notation 137

Bibliography 139

Vita 141

LIST OF FIGURES

1.1 Diagram of the Inverse Scattering Method 3

2.1 Contours of integration Γ_L and Γ_R for G_L and G_R. 15
2.2 Ranges for the $W_{\pm 2}$, $W_{\pm 1}$, and W_0 branches of the W function. 19
2.3 W_0 Branch Cut 20
2.4 W_k ($k \neq 0$) Branch Cuts 20
2.5 Contours for shifting the contour of integration for G_L off of the real line. 24

5.1 Contour of integration for proof of Proposition 5.3.3 88

CHAPTER 1. INTRODUCTION

1.1 The Intermediate Long Wave Equation

The Intermediate Long Wave (ILW) equation

$$u_t + \frac{1}{\delta}u_x + 2uu_x + Tu_{xx} = 0, \tag{1.1}$$

models the behavior of long internal gravitational waves in stratified fluid of depth $0 < \delta < \infty$, where T denotes the singular integral operator given by

$$(Tf)(x) := \frac{1}{2\delta}\,\mathrm{p.\,v.}\int_{\mathbb{R}} \coth\Big(\frac{\pi}{2\delta}(y-x)\Big)\, f(y)\,\mathrm{d}y,$$

and p. v. denotes the Cauchy principal value, $\mathrm{p.\,v.}\int \mathrm{d}x = \lim_{\varepsilon \searrow 0}\int_{|x|>\varepsilon} dx$. In the limit $\delta \to 0$, the ILW reduces to the Korteweg de Vries (KdV) equation, and under the limit $\delta \to \infty$, Santini, Ablowitz and Fokas showed formally in their 1984 paper [12] that the ILW equation reduces to the Benjamin-Ono (BO) equation. As such, the ILW can be thought of as an intermediary between the two equations. While the KdV equation is very well understood, understanding the BO equation is still an area of active research.[1] Given the ILW's role as an intermediary between the two equations, a mathematically rigorous understanding of the ILW is of great interest to many mathematicians.

As discussed in the 2019 survey paper [14] by Jean-Claude Saut, the formal derivation of the ILW as a model for physical phenomenon was given by R. I. Joseph in [4]. Joseph based his derivation on the Whithan non-local equation derived by Gerald Whitham in his 1967 paper [17]. The modern form of the ILW was introduced by T. Kubota, D.R.S. Ko, and L.D. Dobbs in their 1978 paper [9]. Some atmospheric and oceanic applications of the ILW can be found in the 1978 paper by D.R. Christie *et*

[1] For more on the BO equation, please see Allen Wu's papers [18] and [19].

al. [1], the 1980 S.A. Maslow and L.G. Redekopp paper [10], the 1981 C. Gary Koop and Gerald Butler paper [8], and the 1984 N.N. Romanova paper [11].

1.2 Introduction to Inverse Scattering

First introduced in the late 1960's and early 1970's as a technique for studying the Korteweg-de Vries (KdV) equation, the inverse scattering method has since been adapted to find solutions to a number of other non-linear dispersive equations—equations whose solutions model dispersive wave phenomenon. Since non-linear dispersive equations arise naturally in many fields of science and engineering as a means to model the behavior of many important phenomenon, understanding non-linear dispersive equations is a very active area of research. The inverse scattering method has been very successful in solving certain types of non-linear dispersive equations.

The structure of the inverse scattering method can be thought of as a model for how problem solving often works. When confronted with a difficult problem about which very little is know, it is often helpful to find a way to relate that problem to another problem which you know how to solve. So, by solving the related problem and connecting its solution back to the original problem, you can find a solution for the original problem. This process is essentially how the inverse scattering method works.

For us in this dissertation, solving a non-linear dispersive equation means finding a mathematically rigorous procedure by which if someone tells you what a non-linear wave described by a particular non-linear dispersive equation looks like initially, you can determine how that wave looks in the future. We use the term "initial data" to refer to the initial state of a wave described by a given non-linear dispersive equation and the term "solution" to refer to a formula which provides a description for what the initial data looks like for each relevant time t. As mathematicians, our goal in solving a non-linear dispersive equation is to not only describe a map (*i.e.* procedure or function) which takes as input initial data for that non-linear dispersive equation and returns as output the corresponding description for how that initial data evolves in time, but to also understand this map deeply enough to be able to guarantee—at least under certain conditions—this map actually works. Such a map is often referred to as a solution map, as it provides the solution corresponding a given initial data (*i.e.* a solution map maps initial data to solutions).

In the inverse scattering method, we use certain properties briefly described below of given non-linear dispersive equation to construct a map $\mathscr{D}$, called the "direct scattering map," which maps arbitrary initial data of the equation to the initial data's so-called "scattering data" describing certain physical properties of the initial data. Using properties of the direct scattering map, we have tools to determine how the scattering data changes in time. As demonstrated below in Figure 1.1, by also constructing a second map $\mathcal{I}$, called the "inverse scattering map" which serves as the inverse for the direct scattering map —*i.e.* maps scattering data to its corresponding initial data—we can then find a solution to a non-linear dispersive equation by applying the inverse scattering map to the time dependent representation of the scattering data.

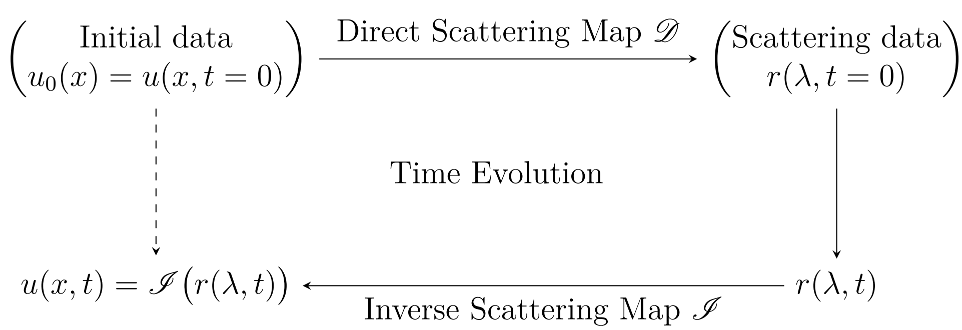


Figure 1.1: Diagram of the Inverse Scattering Method

The combination of the direct scattering map and its corresponding inverse scattering map is called an Inverse Scattering Transform, or IST for short. As such, using the Inverse Scattering Method to solve a given non-linear dispersive equation is often referred to as solving that non-linear dispersive equation by its Inverse Scattering Transform. Readers familiar with the method of solving linear partial differential equations by the Fourier Transform may recognize the Inverse Scattering Method as an analogous method for solving non-linear dispersive equations. However, an IST differs from the Fourier Transform in one very key way: while the Fourier Transform can be defined in a single set way that does not depend on the linear equation one wishes to solve, the definition for an IST depends *entirely* on the non-linear dispersive equation one hopes to solve using the Inverse Scattering Method. For this reason, the crux of solving a non-linear dispersive equation using the Inverse Scattering Method lies in the construction of that particular dispersive equation's corresponding IST, and proving that that IST is both well-defined and bi-Lipschitz continuous—*i.e.* both the direct scattering map and the inverse scattering map for a given IST are Lipschitz continuous.[2] Bi-Lipschitz continuity is desired as a property of an IST as it helps to ensure the solution map—*i.e.* map from initial data to the corresponding solution—is continuous in initial data, and hence, in the language of the theory of partial differential equations, well-posed. In other words, proving that an IST for a particular dispersive wave is Lipschitz continuous ensures that changing a given initial data slightly results in a similarly small change in the solution.

One can create an IST for a given dispersive equation if the equation has what is called a "*Lax representation*" of the equation. A Lax representation for a dispersive equation is a pair of time dependent operators L, B parametrized by a function $u(x,t)$ which satisfy the identity

$$\frac{d}{dt}L = BL - LB \tag{1.2}$$

if and only if u solves the given equation. In the literature, such L and B are

[2]Informally, Lipschitz continuity is essentially the requirement that the distance between two outputs of a given function or map is directly proportional to the distance between their corresponding inputs. Mathematically, we say that for any two spaces Y and Z, a map $f : Y \to Z$ is Lipschitz continuous if there exists some constant $C > 0$ so that $\|f(y_1) - f(y_2)\|_Z \leq C\|y_1 - y_2\|_Y$ for every $y_1, y_2 \in Y$.

sometimes refered to as a "*Lax pair*." Whenever L, B satisfy (1.2), one can show that the spectrum of L is time invariant. As such, (1.2) further implies

$$\begin{cases} \kappa(t) = \kappa(0) \\ \dfrac{\partial}{\partial t}\psi = B\psi \end{cases} \tag{1.3}$$

for all κ, Ψ satisfying $L\psi = \kappa\psi$, and the given equation is said to be an isospectral flow for the linear spectral problem $L\psi = \kappa\psi$. The linear spectral problem can be therefore solved at arbitrary time t by first solving it for an initial time t_0 (typically $t_0 = 0$) and using (1.3) to propagate in time. The model, then, for solving an equation by the Inverse Scattering Method is to construct the direct scattering map by solving linear spectral problem $L\Psi = \kappa\Psi$ for time t_0, and then use the operator B to propagate the scattering data in time. By constructing an inverse to the direct scattering map (*i.e.* the corresponding inverse scattering map), one can then recover a solution to the original dispersive equation from the time evolved scattering data.

1.3 Overview of the ILW Direct Scattering Map

While there are a series of papers from the late 1970's and early 1980's culminating in a paper by Y. Kodama, M.J. Ablowitz and J. Satsuma [6] and a subsequent paper by P.M. Santini, M.J. Ablowitz and A.S. Fokas [12] which formally describe the Inverse Scattering Transform (IST) for the ILW, little research has been done to place the Inverse Scattering Method for the ILW equation on a rigorous mathematical footing. At the time of writing this dissertation, the author found no results in the literature showing that the IST for the ILW equation is actually well-defined or bi-Lipschitz continuous—even for small data.

As described in Section 1.2, using the Inverse Scattering Method to solve the ILW entails constructing an invertible, bi-Lipschitz continuous map from initial data to the corresponding scattering data in such a way that linearizes the flow—*i.e.* the time dependence of the output of this map applied to initial data is determined by a linear differential equation. The "forward direction" of this map which takes initial data to scattering data is referred to as the "direct scattering map," and its inverse is referred to as the "inverse scattering map." This distinction is made as the process for constructing the direct scattering map for a given equation is often very different from the process for constructing the corresponding inverse scattering map. As previously mentioned, the combination of this direct scattering map with the corresponding inverse scattering map and the linear differential equation used to propagate the scattering data in time is called an Inverse Scattering Transform for the ILW.

What allows us to construct an IST for the ILW is fact that the ILW is an isospectral flow for the linear spectral problem

$$L_\delta(\Psi) := \frac{1}{i}\frac{\partial}{\partial x}\Psi^+ - \zeta\left(\Psi^+ - \Psi^-\right) = u\Psi^+, \tag{1.4}$$

which is a part of the Lax pair[3]

$$\frac{1}{i}\frac{\partial}{\partial x}\Psi^+ - \zeta\left(\Psi^+ - \Psi^-\right) = u\Psi^+ \tag{1.5a}$$

$$\frac{1}{i}\frac{\partial}{\partial t}\Psi^\pm + 2i\left(\zeta - \frac{1}{2\delta}\right) + \Psi_{xx} = \left[\pm iu_x - Tu_x + \eta\right]\Psi^\pm, \tag{1.5b}$$

where Ψ is a function analytic in the complex strip

$$\mathcal{S}_\delta := \{z \in \mathbb{C} \,:\, 0 < \operatorname{Im} z < 2\delta\}$$

with respective lower and upper boundary values

$$\Psi^+(x) := \lim_{y \searrow 0} \Psi(x+iy), \qquad \text{and} \qquad \Psi^-(x) := \lim_{y \nearrow 2\delta} \Psi(x+iy), \tag{1.6}$$

where we use the superscript notation $f^\pm$ throughout this dissertation to indicate lower and upper boundary values as shown above of functions f analytic on the

[3]Please see the appendix titled *Lax Representation* for a comparison of (1.5) with the ILW Lax pair typically given in the literature.

complex strip $\mathcal{S}_\delta$. The spectral parameter $\zeta \in (0,\infty)$ is itself parametrized by a second spectral parameter $\lambda \in \mathbb{R}$ as

$$\zeta(\lambda;\delta) := \frac{\lambda}{1-e^{-2\delta\lambda}},$$

where we use the notation $f(\,\cdot\,;t_1,\ldots,t_n) := f_{t_1,\ldots,t_n}$ to denote a family of functions "indexed" in a (possibly) uncountable sense by $t_1,\ldots,t_n$. The output of direct scattering map $\mathscr{D}$ for the ILW is determined by the lower boundary values M_1^+, M_e^+, N_1^+, N_e^+ of eigenfunctions M_1, M_e, N_1, N_e of L_δ which satisfy the asymptotic conditions given in (1.7). Such eigenfunctions are referred to in the literature as "Jost solutions," and, given their importance in the construction of the direct scattering map $\mathscr{D}$, we explicitly define Jost solutions as follows:

Definition 1.3.1 (Jost solutions)**.** The Jost solutions M_1, M_e, N_1, N_e are solutions to the linear spectral problem (1.4) whose lower boundary values M_1^+, M_e^+, N_1^+, N_e^+ as defined in (1.6) obey the following asymptotic conditions

$$\lim_{x\to-\infty}\langle x\rangle\left(M_1^+(x;\lambda,\delta)-1\right) = \lim_{x\to\infty}\langle x\rangle\left(N_1^+(x;\lambda,\delta)-1\right) = 0 \tag{1.7a}$$

$$\lim_{x\to-\infty}\langle x\rangle\left(M_e^+(x;\lambda,\delta)-e^{i\lambda x}\right) = \lim_{x\to\infty}\langle x\rangle\left(N_e^+(x;\lambda,\delta)-e^{i\lambda x}\right) = 0, \tag{1.7b}$$

where we use the notation $\langle x\rangle := \sqrt{1+|x|^2}$ to indicate a linear weight.

Additionally, we require the upper boundary values $M_{(\,\cdot\,)}^-$, $N_{(\,\cdot\,)}^-$ (where $(\,\cdot\,)$ represents either the subscript 1 or e) of $M_{(\,\cdot\,)}$, $N_{(\,\cdot\,)}$ to have a decomposition

$$M_1^- - 1 = M_1^{(1)} + M_1^{(2)} \qquad \text{and} \qquad N_1^- - 1 = N_1^{(1)} + N_1^{(2)}$$

$$M_e^- - e^{i\lambda x}\,e^{-2\delta\lambda} = M_e^{(1)} + M_e^{(2)} \qquad \text{and} \qquad N_e^- - e^{i\lambda x}\,e^{-2\delta\lambda} = N_e^{(1)} + N_e^{(2)}$$

satisfying

$$\langle x\rangle^{1+v}\left|M_{(\,\cdot\,)}^{(1)}(x)\right| \lesssim 1 \quad (\text{for } x \ll -1), \qquad \langle x\rangle^{1+v}\left|N_{(\,\cdot\,)}^{(1)}(x)\right| \lesssim 1 \quad (\text{for } x \gg 1),$$

and

$$\langle \cdot \rangle^\tau M^{(2)}_{(\cdot)}, \;\; \langle \cdot \rangle^\tau N^{(2)}_{(\cdot)} \in L^2(\mathbb{R})$$

for any $\upsilon \in \left(0, \frac{1}{2}\right)$ and $\tau \in [0,1)$, where we use the notation $a \lesssim b$ to indicate $a \leq C\, b$ for some constant $C > 0$.

For a given $u(x)$, the corresponding output $r = \mathscr{D}u$ of direct scattering map is given by $r(\lambda;\delta) = b(\lambda;\delta)/a(\lambda;\delta)$, where

$$\begin{aligned} b(\lambda) &:= \frac{i}{1 - 2\delta\zeta(-\lambda)} \int_{\mathbb{R}} e^{-i\lambda x} u(x)\, M_1^+(x;\lambda,\delta)\, \mathrm{d}x \\ a(\lambda) &:= 1 + \frac{i}{1 - 2\delta\zeta(\lambda)} \int_{\mathbb{R}} u(x)\, M_1^+(x;\lambda,\delta)\, \mathrm{d}x. \end{aligned}$$

In this dissertation, we prove the following result:

Theorem 1.3.2. *For sufficiently small $c_0 > 0$ the map*

$$\mathscr{D} : B_X(0, c_0) \to L^\infty(\mathbb{R})$$

$$u \mapsto r$$

is well-defined for all real λ, where X denotes the space $\langle \cdot \rangle^{-4} L^2(\mathbb{R})$, and $B_X(0,c_0)$ is the ball in the space X about zero with radius c_0.[4] *Further, $\mathscr{D}$ is Lipschitz continuous as a map from $B_X(0,c_0)$ to $L^\infty\big((-\infty,k] \cup [k,\infty)\big)$ for each fixed $k > 0$.*[5]

A crucial first step to showing that $\mathscr{D}$ is well-defined is showing that for each given $u \in B_X$ the corresponding Jost solutions both exist and are unique. To do so, one uses the (formal) Green's Functions G_L and G_R, given by

$$G_L(z;\lambda,\delta) = \lim_{\varepsilon \searrow 0} \frac{1}{2\pi} \int\limits_{\mathbb{R} - i\varepsilon} \frac{e^{iz\xi}}{\xi - \zeta(\lambda)\left(1 - e^{-2\delta\xi}\right)}\, \mathrm{d}\xi, \qquad \left(z \in \overline{\mathcal{S}}_\delta\right) \tag{1.8a}$$

[4]See Theorem 5.4.3 in Section 5.4 of Chapter 5.
[5]See Theorem 5.4.4 in Section 5.4 of Chapter 5.

and

$$G_R(z;\lambda,\delta)=\lim_{\varepsilon\searrow 0}\frac{1}{2\pi}\int\limits_{\mathbb{R}+i\varepsilon}\frac{e^{iz\xi}}{\xi-\zeta(\lambda)\left(1-e^{-2\delta\xi}\right)}\,\mathrm{d}\xi,\qquad \left(z\in\overline{\mathcal{S}}_\delta\right) \tag{1.8b}$$

to rewrite (1.4) with asymptotic conditions (1.7) as the integral equations

$$\begin{pmatrix} M_1^+(x;\lambda,\delta)\\ M_e^+(x;\lambda,\delta)\end{pmatrix}=\begin{pmatrix}1\\ e^{i\lambda x}\end{pmatrix}+\int_{\mathbb{R}}G_L^+(x-x';\lambda,\delta)u(x')\begin{pmatrix} M_1^+(x';\lambda,\delta)\\ M_e^+(x';\lambda,\delta)\end{pmatrix}\,\mathrm{d}x' \tag{1.9a}$$

$$\begin{pmatrix} N_1^+(x;\lambda,\delta)\\ N_e^+(x;\lambda,\delta)\end{pmatrix}=\begin{pmatrix}1\\ e^{i\lambda x}\end{pmatrix}+\int_{\mathbb{R}}G_R^+(x-x';\lambda,\delta)u(x')\begin{pmatrix} N_1^+(x';\lambda,\lambda)\\ N_e^+(x';\lambda,\lambda)\end{pmatrix}\,\mathrm{d}x', \tag{1.9b}$$

where we again use the "+" superscript to indicate the lower boundary values of functions analytic in the complex strip $\mathcal{S}_\delta$.

Formally, assuming that the solutions to equations (1.9) have analytic extensions to the strip $\mathcal{S}_\delta$ with upper boundary values M_1^-, M_e^-, N_1^-, and N_e^-, one can show through a simply heuristic computation that solutions to the integral equations (1.9) should satisfy the spectral problem (1.4) with asymptotic conditions (1.7). However, as discussed in Section 2.2,the integrand of G_L and G_R has exactly two poles along the real line as shown in —namely $\xi=0$ and $\xi=\lambda$—and countably many poles in the complex plane. Worse, the Fourier symbol $p(\xi)$ of G_L, G_R does not belong to any standard symbol class due to its radically different asymptotic behavior as $\xi\to-\infty$, and $\xi\to+\infty$, respectively. As such, it is hardly obvious that G_L and G_R are even remotely well defined as convolution operators. As such, before we can even begin to prove that the direct scattering map $\mathscr{D}$ is well-defined, we require a thorough understanding of the Green's functions G_L, G_R as convolution operators—indeed, such is the focus of Chapters 2 through 4 of this dissertation.

We begin our study of the Green's functions G_L, G_R in Chapter 2 by analyzing the properties of the lower boundary values G_L^+, G_R^+ as functions. Using a combination of a contour shift and several dyadic decompositions, we show that the boundary values

G_L^+ and G_R^+ can be represented as

$$(1.10)\qquad G_L^+(x;\lambda,\delta) = \begin{cases} K^+(x;\lambda,\delta), & x<0 \\ K^+(x;\lambda,\delta) + i\alpha(\lambda;\delta) + i\beta(\lambda;\delta)\, e^{i\lambda x}, & x>0 \end{cases}$$

where $G_L(z;\lambda,\delta) = \overline{G_R(-\operatorname{Re} z + i \operatorname{Im} z;\lambda,\delta)}$,

$$\alpha(\lambda;\delta) = \frac{1}{1-2\delta\lambda(\lambda)}, \qquad \beta(\lambda;\delta) = \frac{1}{1-2\delta\lambda(-\lambda)e^{-2\delta\lambda}},$$

and the function K^+ satisfies the properties

(i) $K^+(x) = \mathcal{O}\left(\frac{e^{-\pi|x|}}{x}\right)$ for $|x| \geq 1$, and

(ii) $|K^+(x)| \leq C + C\log\left(\frac{1}{|x|}\right)$ for $|x| < 1$.

Our study of the boundary values G_L^+, G_R^+ continues in Chapter 3 as we use 1.10 to understand the mapping properties of G_L^+, G_R^+ as convolution operators. More specifically, we study the operators $T_{L,\lambda,u}$, $T_{R,\lambda,u}$ given by

$$(1.11)\qquad T_{L,\lambda,u}f := G_L^+(\,\cdot\,;\lambda,\delta) * (uf) \qquad \text{and} \qquad T_{R,\lambda,u}f := G_R^+(\,\cdot\,;\lambda,\delta) * (uf)$$

and show that the are bounded operators acting the space $\langle\,\cdot\,\rangle\, L^\infty(\mathbb{R})$ whose operator norms depend only on the $\|u\|_X$ and not on λ. We further show in Chapter 3 that for every $f \in \langle\,\cdot\,\rangle\, L^\infty(\mathbb{R})$ and $u \in X$ the operators $T_{L,\lambda,u}$, $T_{R,\lambda,u}$ satisfy the asymptotic behavior

$$\lim_{x\to-\infty} T_{L,\lambda,u}f(x) = \lim_{x\to-\infty} T_{R,\lambda,u}f(x) = 0$$

when real $\lambda \neq 0$, and

$$\lim_{x\to-\infty} \langle x\rangle^{-1}\, T_{L,\lambda,u}f(x) = \lim_{x\to-\infty} \langle x\rangle^{-1}\, T_{R,\lambda,u}f(x) = 0$$

for every $\lambda \in \mathbb{R}$. That $T_{L,\lambda,u}$, $T_{R,\lambda,u}$ satisfy the above limits is a property we use later to prove solutions to the integral equations (1.9) satisfy the asymptotic conditions in (1.7).

The focus for our final chapter on the the Green's functions, Chapter 4, is analytically extending G_L^+, G_R^+ in the variable x to G_L and G_R defined on the complex strip $\mathcal{S}_\delta$ and showing that G_L, G_R have upper boundary values G_L^-, G_R^-. Analytically extending G_L^+, G_R^+ is important as it allows us to analytically extend the solutions M_1^+, M_e^+, N_1^+, N_e^+ to the integral equations 1.9, which is a prerequisite to showing that solutions to the integral equations 1.9 are Jost solutions. While, extending G_L^+, G_R^+ (as convolution operators) to the open strip $\mathcal{S}_\delta$ is straight forward, showing the existence of the upper boundaries G_L^-, G_R^- is considerably more involved, as it involves working with a singular operator that is reminiscent of the Hilbert transform. In fact, nearly the entirety of Chapter 4 is devoted to proving the existence (in an L^2 sense) of G_L^- and G_R^-.

Our analysis of the Green's functions in 2 through 4 allows us to finally prove in Section 5.3 of Chapter 5 the equivalence of Jost solutions and solutions to the integral equations (1.9). The big pay-off in proving this equivalence is that it allows us to consider the Jost solutions as solutions to Volterra type integral equations instead of an ordinary differential equation on a complex strip involving complex boundary values. Being able to do so is invaluable as the theory of Volterra type integral equations is far better understood (at least by this author) than the theory of such complex ordinary differential equations. Indeed, it is precisely by treating the Jost solutions as solutions to integral equations (1.9) that we are ultimately able in Chapter 5 to prove that $\mathscr{D}$ is well-defined and, at least for real λ values away from zero, $\mathscr{D}$ is also Lipschitz continuous.

CHAPTER 2. GREEN'S FUNCTIONS: LOWER BOUNDARY VALUES

2.1 Introduction

Proving the direct scattering map $\mathscr{D}$ for the Inverse Scattering Transform of the Intermediate Long Wave equation is both well-defined and Lipschitz continous hinges on our ability to reformulate the linear spectral problem (1.4) with prescribed asyptotic conditions (1.7) as the integral equations (1.9) and understand the behavior of the solutions to (1.9). Both require a deep understanding of the properties of the Green's functions G_L and G_R defined in equation (1.8) of Section 1.3. Indeed, this is the first of three chapters devoted solely to the study of G_L and G_R.

The focus of this chapter is to study the properties of the lower boundary values G_L^+ and G_R^+ as functions on $\mathbb{R}$, where the symbol $\mathbb{R}$ denotes the of all real numbers. In particular, we use a contour shift to derive the alternate formulas (2.4) and (2.4b) for G_L^+ and G_R^+ from Theorem 2.1.1 (Section 2.3), which we use to study the asymptotic properties of G_L^+, G_R^+ (Section 2.4) and the singularity both functions have at $x = 0$ (Section 2.4). We continue our study of the Green's functions in Chapters 3 and 4 where we use our analyses from this chapter to first study mapping properties of G_L^+, G_R^+ as convolution operators (Chapter 3), and then to prove that G_L^+, G_R^+ extend analytically in the variable x to the complex strip $\mathcal{S}_\delta$ (Chapter 4).

We summarize the primary results of this chapter below in Theorems 2.1.1 and 2.1.2.

Theorem 2.1.1 (Green's Function Representation)**.** *The Green's functions given* *above in* (1.8) *can be written as*

$$G_L^+(x;\lambda,\delta) = \begin{cases} K^+(x;\lambda,\delta) + i\big[\alpha(\lambda;\delta) + \beta(\lambda;\delta)e^{i\lambda x}\big]\chi_L(x) & \lambda \neq 0 \\ K^+(x;\lambda,\delta) + i\left[\frac{2}{3} + i\frac{x}{\delta}\right]\chi_L(x) & \lambda = 0 \end{cases} \tag{2.1a}$$

and

$$G_R^+(x;\lambda,\delta) = \begin{cases} K^+(x;\lambda,\delta) - i\big[\alpha(\lambda;\delta) + \beta(\lambda;\delta)e^{i\lambda x}\big]\chi_R(x) & \lambda \neq 0 \\ K^+(x;\lambda,\delta) - i\left[\frac{2}{3} + i\frac{x}{\delta}\right]\chi_R(x) & \lambda = 0 \end{cases} \tag{2.1b}$$

where $\chi_L := \chi_{(0,\infty)}$ *and* $\chi_R := \chi_{(-\infty,0)}$ *respectively denote the characteristic functions on the intervals* $(0,\infty)$ *and* $(-\infty,0)$,

$$\alpha(\lambda;\delta) := \frac{1}{1-2\delta\zeta} = \frac{1-e^{2\delta\lambda}}{2\delta\lambda e^{2\delta\lambda} + 1 - e^{2\delta\lambda}},$$

$$\beta(\lambda;\delta) := \frac{1}{1-2\delta\zeta^*} = \frac{1-e^{2\delta\lambda}}{1+2\delta\lambda - e^{2\delta\lambda}},$$

are respectively determined by the residues of the integrand of G_L^+ *and* G_R^+ *at* $\xi = 0$ *and* $\xi = \lambda$, ζ^* *is the non-linear reflection given by*

$$\zeta^* := \zeta(-\lambda) = \zeta\big(-\lambda(\zeta)\big)$$

and

$$K^+(x;\lambda,\delta) := \frac{e^{-\pi|x|}}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\frac{1}{\xi - \zeta(\lambda)\big(1-e^{-2\xi\delta}\big) + i\,\mathrm{sign}(x)\pi}\,\mathrm{d}\xi$$

results from shifting the contour of integration for the integral in G_L^+ *and* G_R^+.

Theorem 2.1.2. *Suppose* α, β, *and* K^+ *are as defined in Theorem 2.1.1. Then the functions* α *and* β *satisfy the following properties*

$$\lim_{\lambda\to 0}|\alpha(\lambda;\delta)| = \lim_{\lambda\to 0}|\beta(\lambda;\delta)| = \infty,$$

and

$$\lim_{\lambda\to 0}\big[\alpha(\lambda;\delta) + \beta(\lambda;\delta)e^{ix\lambda}\big] = \frac{2}{3} + i\frac{x}{\delta}. \tag{2.2}$$

Further, K^+ is uniformly bounded in λ and

$$K^+(x;\lambda,\delta) = \begin{cases} C\log_+\left(\frac{1}{|x|}\right) + \mathcal{O}(1), & |x| < 1 \\ \mathcal{O}\left(\frac{e^{-\pi|x|}}{|x|}\right), & |x| \geq 1 \end{cases} \tag{2.3}$$

for some constant $C \in \mathbb{C}$, where $\mathbb{C}$ denotes the set of all complex numbers and $\log_+$ *is the function defined by* $\log_+(x) := \max\big\{\log(x),\, 0\big\}$.

Remark 1. An important and immediate consequence of Theorem 2.1.2 and our work in Sections 2.4 and 2.5 is that K^+ can be written as

$$K^+(x;\lambda,\delta) = e^{-\pi|x|}k(x;\lambda,\delta)$$

where $k(\,\cdot\,;\lambda,\delta) \in L^2(\mathbb{R})$ and k is uniformly bounded in λ for all real λ. Unless necessary to avoid confusion, we commonly write $k(x;\lambda,\delta)$ as $k(x)$.

We begin this chapter by first motivating our choice of Green's functions in Section 2.2. Since we need to know the locations of the Green's functions' integrand singularities to be able to both justify the contours of integration for the Green's functions and to justify the representation theorem above (Theorem 2.1.1), locating these singularities is also a primary task for Section 2.2.

To simplify notation, throughout the remainder of this dissertation, we use notation

$$p(\xi;\lambda,\delta) := \xi - \zeta(\lambda)\big(1 - e^{-2\xi\delta}\big),$$

where $e^{i\xi x}/p(\xi;\lambda,\delta)$ is the integrand for G_L^+ and G_R^+. Since it is occasionally more useful to consider the Green's functions as parameterized by ζ rather than λ, we use the notation $p(\xi;\lambda,\delta)$ and $p(\xi;\zeta,\delta)$ interchangeably. Further, we will not always need to consider the affects of the parameters λ and δ in our subsequent analyses. In such cases, we often use the shorter notation $p(\xi)$ or $p(\xi;\lambda)$ *en lieu* of the more cumbersome $p(\xi;\lambda,\delta)$ or $p(\xi;\zeta,\delta)$.

As we see in Section 2.2, the only roots of $p(\xi;\lambda,\delta)$ in the complex strip

$$\mathcal{R}_\delta := \{z \in \mathbb{C} \ : \ -\pi/\delta \le \operatorname{Im} z \le \pi/\delta\}$$

are $\xi = 0$ and $\xi = \lambda$ (provided $\lambda \neq 0$). As such, we may use analyticity to write G_L^+ and G_R^+ as

$$G_L^+(x;\lambda,\delta) := \frac{1}{2\pi}\int_{\Gamma_L} e^{i\xi x}\frac{1}{p(\xi;\lambda,\delta)}\,\mathrm{d}\xi$$
$$G_R^+(x;\lambda,\delta) := \frac{1}{2\pi}\int_{\Gamma_R} e^{i\xi x}\frac{1}{p(\xi;\lambda,\delta)}\,\mathrm{d}\xi,$$

where the symbol Γ_L is used to denote a contour from $-\infty$ to ∞ along the real axis which is deformed in small circular arcs around $\xi = 0$ and $\xi = \lambda$ so that the contour bypasses these two real roots of p from below (Figure 2.1a), and Γ_R denotes the corresponding contour which bypasses the roots $\xi = 0$ and $\xi = \lambda$ from above (Figure 2.1b).

(a) Γ_L Contour (b) Γ_R Contour

Figure 2.1: Contours of integration Γ_L and Γ_R for G_L and G_R.

Remark 2. Throughout this dissertation, we commonly use the symbol "$\star$" as a place holder for both L and R. For example, if we write "$G_\star$ ($\star = L$, or R) are a bounded as a convolution operators," then what we mean is that "both G_L and G_R are bounded as convolution operators." As a further example of how we use this notational convention, please see the following two very important remarks.

Remark 3. When $\lambda = 0$, the function $\zeta(\lambda;\delta)$ is technically undefined. However, since $\lim_{\lambda\to 0}\zeta(\lambda;\delta) = \frac{1}{2\delta}$, $\lambda = 0$ is a removable singularity of ζ. As such, we define $\zeta(\lambda;\delta) := \frac{1}{2\delta}$. Further, the case $\lambda = 0$ also corresponds to the case when the two roots $\xi = 0$ and $\xi = \lambda$ of the function p—which are simple when $\lambda \neq 0$—coalesce to form a single double zero of p. Under the caveat that we define $\zeta(0;\delta) := \frac{1}{2\delta}$, a direct computation shows that the residue of $e^{ix\xi}/p(\xi;0,\delta)$ at $\xi = 0$ is $\frac{2}{3} + i\frac{x}{\delta}$—hence the piecewise (in λ) definition of G_L^+ and G_R^+ in (2.1). Thus, even though the residue sum

$$\mathcal{R}_\star(x;\lambda,\delta) := i\left[\alpha(\lambda;\delta) + \beta(\lambda;\delta)e^{i\lambda x}\right]\chi_\star \qquad (\star = L, \text{ or } R)$$

is not technically defined at $\lambda = 0$, it nonetheless makes sense for us to agree on the convention that

$$\mathcal{R}_\star(x;\lambda = 0;\delta) := \left[i\frac{2}{3} - \frac{x}{\delta}\right]\chi_\star. \qquad (\star = L, \text{ or } R)$$

Under this convention, (2.1) can be written slightly more succinctly as

$$G_L^+(x;\lambda,\delta) = K^+(x;\lambda,\delta) + \mathcal{R}_L(x;\lambda,\delta) \tag{2.4a}$$

$$G_R^+(x;\lambda,\delta) = K^+(x;\lambda,\delta) - \mathcal{R}_R(x;\lambda,\delta). \tag{2.4b}$$

Remark 4. Continuing to our discussion on the case of $\lambda = 0$ and the coalescing of the Green's function integrand poles $\xi = 0$, $\xi = \lambda$, under this scenario, we take the contours $\Gamma_\star$ ($\star = L$, or R) such that they have only one circular deformation away from the real line which allows them to bypass the single (double) pole at $\xi = 0$.

2.2 Motivation of the Green's Functions

To motivate our choice of Green's function for this linear spectral problem, suppose there exists a function G satisfying $L_\delta(G^+)(x) = \delta_0(x)$, where $\delta_0(x)$ denotes the Dirac delta function (not to be confused with the parameter δ). Formally, by taking the Fourier transform of both sides of $L_\delta(G^+)(x) = \delta_0(x)$, we have

$$1 = \xi\,\widehat{G}^+ - \zeta\left(\widehat{G}^+ - \widehat{G}^-\right) = \left[\xi - \zeta\left(1 - e^{-2\delta\xi}\right)\right]\widehat{G}^+ = p(\xi;\zeta,\delta)\,\widehat{G}^+,$$

where

$$
\begin{aligned}
p(\xi;\lambda,\delta) &= \xi-\zeta\left(1-e^{-2\delta\xi}\right)\\
&= \left(\frac{\xi}{1-e^{-2\delta\xi}}-\zeta\right)\left(1-e^{-2\delta\xi}\right)\\
&= \big(\zeta(\xi)-\zeta(\lambda)\big)\left(1-e^{-2\delta\xi}\right).
\end{aligned}
$$

However, this approach is somewhat problematic, given that the function p has roots $\xi=0$ and $\xi=\lambda$ along the real line. So, instead defining a single Green's function for the spectral problem (A.6a) based on taking the inverse Fourier transform of $1/p$, we instead define two different Green's functions G_L^+ and G_R^+ based on taking two different "Fourier inverse like" transforms of $1/p$ for which the respective contours of integration avoids the roots of p. Specifically,

$$
\begin{aligned}
G_L^+(x;\lambda,\delta) &:= \frac{1}{2\pi}\int_{\Gamma_L} e^{i\xi x}\frac{1}{p(\xi;\lambda,\delta)}\,d\xi\\
G_R^+(x;\lambda,\delta) &:= \frac{1}{2\pi}\int_{\Gamma_R} e^{i\xi x}\frac{1}{p(\xi;\lambda,\delta)}\,\mathrm{d}\xi
\end{aligned}
$$

where the contour Γ_L bypasses the roots $\xi=0$ and $\xi=\lambda$ from below, and the contour Γ_R bypasses the roots $\xi=0$ and $\xi=\lambda$ from above, as mentioned in the previous section.

Since $p(\xi;\lambda)$ can be rewritten as

$$
p(\xi;\lambda)=\left(\frac{\xi}{1-e^{-2\xi\delta}}-\zeta(\lambda)\right)\left(1-e^{-2\xi\delta}\right)=\big(\zeta(\xi)-\zeta(\lambda)\big)\left(1-e^{-2\xi\delta}\right),
$$

it is easy to check that both $\xi=0$ and $\xi=\lambda$ are roots of p. In the remainder of this section, we argue that these are the only roots of p in the strip

$$
\mathcal{R}_\delta := \left\{z\in\mathbb{C}\ :\ -\frac{\pi}{\delta}\le \operatorname{Im} z\le\frac{\pi}{\delta}\right\}
$$

as is defined in the Introduction of this chapter. That is, we show that the equation

$$
\xi-\zeta(\lambda)+\zeta(\lambda)e^{-2\xi\delta}=0 \tag{2.5}
$$

has exactly two solutions (in ξ) for $\xi \in \mathcal{R}_\delta$. With a little algebraic manipulation, equation (2.5) can be rewritten as

$$(2.6) \qquad \left(2\delta\xi - 2\delta\,\zeta(\lambda)\right) e^{2\delta\xi - 2\delta\,\zeta(\lambda)} = -2\delta\,\zeta(\lambda)\, e^{-2\delta\,\zeta(\lambda)}.$$

In order to "solve" (2.6), recall that the Lambert W function (which we henceforth refer to only as W) is defined to be the multivalued "inverse" of the function ze^z. So, to determine the number of solutions (2.6) has for $\xi \in \mathcal{R}_\delta$, we need to specify which branches of W are appropriate to apply to both sides of (2.6) given our restriction on ξ.

The following discussion of the branches of the complex W function is heavily inspired by Section 4 from the 1996 R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffery and D.E. Knuth paper [2]. As is the case with the standard complex exponential and logarithmic functions, to define the branches of W, the canonical approach is to make a branch cut along the negative real axis of the range of ze^z, and determine which curves in the range of W are mapped to the branch cut (*i.e.* the negative real axis).

To do so, set

$$\begin{aligned} z &:= we^w & \qquad w &:= W(z) \\ &:= x + iy & \qquad &:= t + is. \end{aligned}$$

Then, using Euler's formula to simply the equation

$$(x + iy) = (t + is)e^{t+is}$$

and taking real and imaginary parts, we obtain the system

$$\begin{cases} x = e^t\left(t\cos s - s\sin s\right) \\ y = e^t\left(t\sin s + s\cos s\right). \end{cases}$$

So, if $y = 0$, then either $s = 0$ or $t = -s\cot s$. Further, $x < 0$ if and only if $t\cos s - s\sin s < 0$. Now, since $t = -s\cot s$ has asymptotes at $s = k\pi$ ($k \in \mathbb{Z}\backslash\{0\}$), and the function $t\cos s - s\sin s$ has no roots, the inequality $t\cos s - s\sin s < 0$ holds precisely on the intervals

$$\left(\bigcup_{-k\in\mathbb{N}} \big((2k+1)\pi,\, 2k\pi\big)\right) \cup (-\pi,\, \pi) \cup \left(\bigcup_{k\in\mathbb{N}} \big(2k\pi,\, (2k+1)\big)\right)$$

As such, the only curves the function ze^z maps to the negative real axis are

$$\gamma_k(s) := -s\cot s + is, \qquad k \in \mathbb{Z},$$

whose respective domains are given by

$$\operatorname{domain}\gamma_k(s) := \begin{cases} \big((2k+1)\pi,\, 2k\pi\big), & -k \in \mathbb{N} \\ (-\pi,\, \pi), & k = 0 \\ \big(2k\pi,\, (2k+1)\pi\big), & k \in \mathbb{N} \end{cases}$$

and the curve whose graph is

$$(-\infty, -1) \cup \{\gamma_0(s) + is \;:\; -\pi < s \le 0\}.$$

As such, these curves form the boundary values between the ranges of the different branches of W. In particular, the ranges for the the principle branch W_0 and the W_{-1} and W_1 branches are shown below in Figure 2.2.

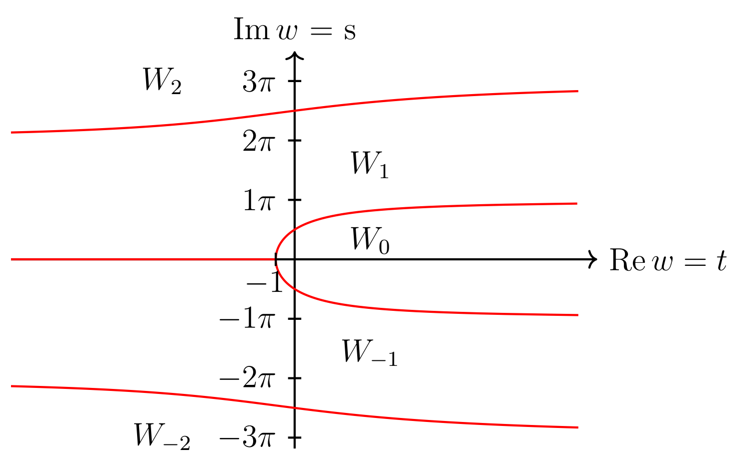


Figure 2.2: Ranges for the $W_{\pm 2}$, $W_{\pm 1}$, and W_0 branches of the W function.

Since the map $w \mapsto we^w$ takes $w = -1$ to $z = -e^{-1}$, on pages 17 and 18 in [2], Corless *et al.* proposetaking the branch cut which defines the principle branch of W along $\{z \in \mathbb{C} \ : \ -\infty < z \leq -e^{-1}\}$, and taking all other branch cuts along the entire negative $\operatorname{Re} z$-axis. They further take all branch cuts in such a way that the branch cuts are closed "on the top," as shown in Figures 2.3 and 2.4.

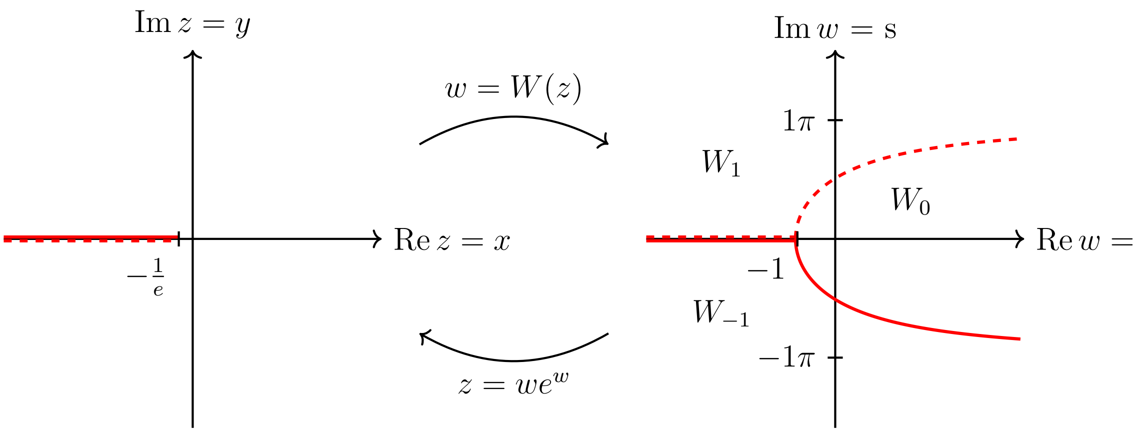


Figure 2.3: W_0 Branch Cut

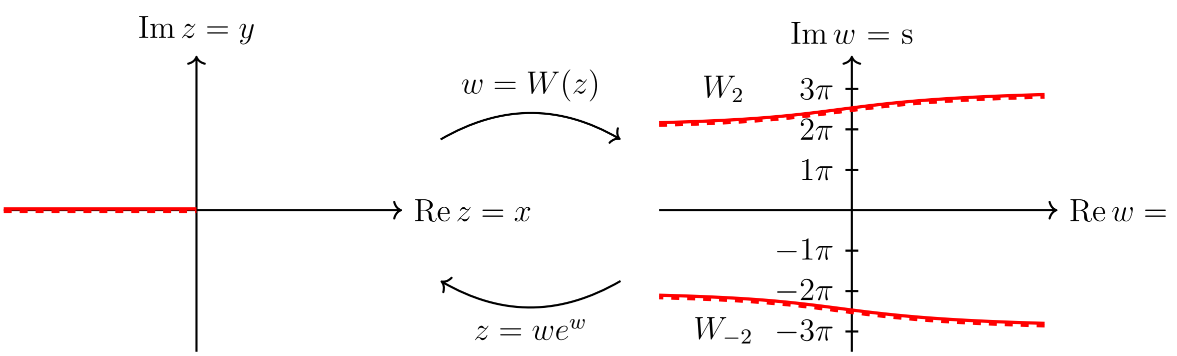


Figure 2.4: W_k $(k \neq 0)$ Branch Cuts

Corless *et al.* further argue in [2] that each branch $W_k : \mathbb{C} \to \operatorname{ran} W_k$ is bijective, which allows us to solve equation (2.6) and find

$$\xi = \frac{1}{2\delta} W_k\big(-2\delta\zeta(\lambda)\big) + \zeta(\lambda), \qquad k \in \zeta.$$

Consequently, the only roots of $p(\xi, \lambda)$ which could have a chance of living in $\mathcal{R}_\delta$ are those corresponding to the W_{-1}, W_0, and W_1 branches. Now, if $\xi = 0$, then

$$W_k\big(-2\delta\zeta(\lambda)\big) = -2\delta\zeta(\lambda).$$

Since ζ is a positive, strictly increasing function with $\zeta(0) = 1/(2\delta)$, if $\lambda < 0$, then $-1 < -2\delta\zeta(\lambda) < 0$, and if $\lambda \geq 0$, then $-2\delta\zeta(\lambda) \leq -1$. In other words, if $\lambda < 0$ the $\xi = 0$ root of p corresponds to the W_0 branch, but if $\lambda \geq 0$, then it corresponds to the W_{-1} branch.

On the other hand, if $\xi = \lambda$, then

$$W_k\big(-2\delta\zeta(\lambda)\big) = 2\delta\lambda - 2\delta\zeta(\lambda).$$

Let $g(\lambda) := 2\delta\lambda - 2\delta\zeta(\lambda)$. Note that

$$\begin{cases} 0 < \zeta'(\lambda) < \frac{1}{2}, & \text{for } \lambda < 0 \\ \zeta'(0) = \frac{1}{2}, & \\ \frac{1}{2} < \zeta'(\lambda) < 1, & \text{for } \lambda > 0 \end{cases}$$

which implies g is a strictly increasing function. Moreover, since $\lim_{\lambda\to\infty} g(\lambda) = 0$ and $g(0) = -1$, we see that

$$\begin{cases} 2\delta\lambda - 2\delta\zeta(\lambda) < -1, & \lambda < 0 \\ 2\delta\lambda - 2\delta\zeta(\lambda) = -1, & \lambda = 0 \\ -1 < 2\delta\lambda - 2\delta\zeta(\lambda) < 0, & \lambda > 0 \end{cases}$$

Therefore, if $\lambda < 0$ then the $\xi = \lambda$ zero of p corresponds to the W_{-1} branch, but if $\lambda \geq 0$, it corresponds to the W_0. Moreover, given that $-2\delta\zeta(\lambda)$ lies on the negative real axis and hence on the branch cut used for W_k, $k \neq 0$, each value of $2\delta\xi - 2\delta\zeta(\lambda) = W_k\big(-2\delta\zeta(\lambda)\big)$ $(k \neq 0)$ lies on the boundary between the respective ranges of the W branches. In particular, given that the strip $\big\{z \;:\; -2\pi \leq \operatorname{Im} z \leq 2\pi\big\}$ does not contain any part of the boundary between $\operatorname{ran} W_{-2}$ and $\operatorname{ran} W_{-1}$ or any part of the boundary between $\operatorname{ran} W_1$ and $\operatorname{ran} W_2$, this strip contains exactly two values of $W\big(-2\delta\zeta(\lambda)\big)$. Since

$$\mathcal{R}_\delta = \frac{1}{2\delta}\big\{z \;:\; -2\pi \leq \operatorname{Im} z \leq 2\pi\big\} + \zeta(\lambda),$$

the strip $\mathcal{R}\delta$ contains exactly two roots of p—precisely as claimed.

2.3 Green's Function Contour Shift

Now that we know the integrand for G_L^+, G_R^+ has exactly two poles in a strip about the real line, we are now able to use contour shifts to derive representation formulas for G_L^+ and G_R^+ which we use to study the mapping properties of these operators.

To simplify our analyses, we first note that G_L^+ and G_R^+ have the useful scaling (in δ) and conjugate properties shown in Proposition 2.3.1.

Proposition 2.3.1. *The Green's Functions G_L^+ and G_R^+ satisfy the following identities:*

(i) $G_\star^+(x;\lambda,\delta) = G_\star^+(x/\delta;\delta\lambda,1)$

(ii) $G_R^+(x;\lambda,\delta) = \overline{G_L^+(-x;\lambda,\delta)}$,

where, as mentioned in Remark 2, we use the $\star$ in the notation in $G_\star^+$ is a stand-in for either L, or R. That is, both G_L^+ and G_R^+ satisfy identity (i).

Proof. To prove (i), observe that

$$\zeta(\xi;\delta) = \frac{\xi e^{\delta\xi}}{e^{\delta\xi}-e^{-\delta\xi}} = \delta^{-1}\frac{\delta\xi e^{\delta\xi}}{e^{\delta\xi}-e^{-\delta\xi}} = \delta^{-1}\zeta(\delta\xi;1).$$

As such

$$p(\xi;\lambda,\delta) = \big(\zeta(\xi;\lambda)-\zeta(\lambda)\big)\left(1-e^{-2\delta\xi}\right) = \delta^{-1}p(\delta\xi;\delta\lambda,1). \tag{2.7}$$

Recalling that $G_\star^+$ can be written in the form

$$G_L^+(x;\lambda,\delta) = \frac{1}{2\pi}\int_{\mathbb{R}-i0}\frac{e^{ix\xi}}{p(\xi,\lambda;\delta)}\,\mathrm{d}\xi, \qquad G_R^+(x;\lambda,\delta) = \frac{1}{2\pi}\int_{\mathbb{R}+i0}\frac{e^{ix\xi}}{p(\xi,\lambda;\delta)}\,\mathrm{d}\xi,$$

where we use the convention that $(\,\cdot\pm i0)$ implies a limit involving $(\,\cdot\pm i\varepsilon)$ as $\varepsilon\searrow 0$, equation 2.7 implies that the Green's Functions $G_\star^+$ satisfy scaling identity (i), as

$$\frac{1}{2\pi}\int_{\mathbb{R}\mp i0}\frac{e^{i\left(\frac{x}{\delta}\right)(\delta\xi)}}{p(\xi,\lambda;\delta)}\,\mathrm{d}\xi = \frac{1}{2\pi}\,\delta\int_{\mathbb{R}\mp i0}\frac{e^{i\left(\frac{x}{\delta}\right)(\delta\xi)}}{p(\delta\xi,\delta\lambda;1)}\,\mathrm{d}\xi = \frac{1}{2\pi}\int_{\mathbb{R}\mp i0}\frac{e^{i\left(\frac{x}{\delta}\right)\xi}}{p(\xi,\delta\lambda;1)}\,\mathrm{d}\xi.$$

Further, the computation

$$G_R^+(x;\lambda,\delta) = \frac{1}{2\pi}\int\limits_{\mathbb{R}+i0} \frac{\overline{e^{i(-x)\xi}}}{p(\xi,\lambda)}\,\mathrm{d}\xi = \frac{1}{2\pi}\overline{\int\limits_{\mathbb{R}-i0} \frac{e^{i(-x)\xi}}{p(\xi,\lambda)}\,\mathrm{d}\xi} = \overline{G_L^+(-x;\lambda,\delta)},$$

verifies identity (ii) and completes this proof. □

Remark 5. In Chapter 4, we show that $G_\star^+$ ($\star = L$, or R) has an analytical extension $G_\star$ to the strip

$$\mathcal{S}_\delta = \{z \in \mathbb{C}\ :\ 0 < \operatorname{Im} z < 2\delta\}$$

with upper boundary value $G_\star^-$. Since the arguments in the above proof still hold when x is replaced with $x + iy$, ($x \in \mathbb{R}$ and $0 < y < 2\delta$), we take without proof that the functions $G_\star$ satisfy the corresponding identities

(i) $G_\star(z;\lambda,\delta) = G_\star(z/\delta;\delta\lambda,1)$

(ii) $G_R(x+iy;\lambda,\delta) = \overline{G_L(-x+iy;\lambda,\delta)}$,

for $z \in \mathcal{S}_\delta$, $x \in \mathbb{R}$, and $0 < y < 2\delta$.

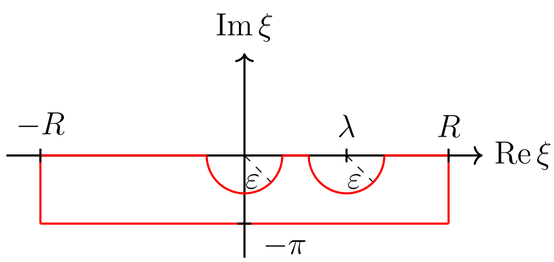
Remark 6. Proposition 2.3.1 in conjunction with Remark 5 allows us to focus our analysis on $G_L(z;\lambda,1)$ on its boundary values $G_L^\mp(x;\lambda,1)$ and deduce the corresponding results for $G_L(z;\lambda,\delta)$ and $G_R(z;\lambda,\delta)$. As such, unless otherwise explicitly stated, we take $\delta = 1$ throughout the remainder of this dissertation and commonly suppress the δ dependence of the functions we analyze. For example, we will often write $G_L^+(x)$ or $G_L^+(x;\lambda)$ instead of $G_L^+(x;\lambda,\delta)$.

The discussion that follows is based on integrating $e^{ix\xi}/p(\xi)$ around the contour shown in Figure 2.5 under the assumption that $\lambda \neq 0$. Recalling from Section 2.2 that the only roots of p contained in the strip

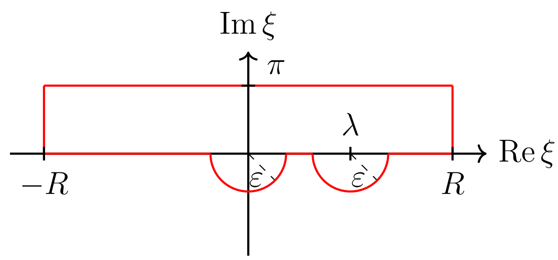
$$\mathcal{S}_1 = \left\{\xi \in \mathbb{C}\ :\ -\pi \leq \operatorname{Im}\xi \leq \pi\right\}$$

are $\xi = 0$ and $\xi = \lambda$, integrating $e^{ix\xi}/p(\xi)$ around such a contour is allowed. In the case where $\lambda = 0$, then the contours shown in Figure 2.5 contain a single circular arc arround the pole at $\xi = 0$.

The function $e^{ix\xi}$ has an analytic continuation to the upper complex ξ plane for $x > 0$ and an analytic continuation to the lower complex ξ plane when $x < 0$. So, for fixed $\varepsilon > 0$ with $\varepsilon < \min\{\lambda/2, \pi\}$, define the counter-clockwise oriented contours $\Gamma(R, \varepsilon, x, \lambda)$ as shown below in Figure 2.5 and note that the integrand of $G_L^+(x, \lambda, 1)$ is analytic along $\Gamma(R, \varepsilon, x, \lambda)$.



(a) The contour $\Gamma(R, \varepsilon, x, \lambda)$ for $x < 0$



(b) The contour $\Gamma(R, \varepsilon, x, \lambda)$ for $x > 0$

Figure 2.5: Contours for shifting the contour of integration for G_L off of the real line.

By the reverse triangle inequality,

$$|p(\xi)| \geq \left| |\xi| - \zeta(\lambda) \left|1 - e^{-2\xi}\right| \right|.$$

Since

$$\left|1 - e^{-2\xi}\right|^2 = \left(1 - e^{-2\xi}\right)\left(1 - e^{-2\bar{\xi}}\right) = 1 - 2\operatorname{Re} e^{-2\xi} + e^{-4\operatorname{Re}\xi},$$

and

$$\operatorname{Re} e^{-2\xi} = e^{\mp 2R} \cos(-2y)$$

for $\xi = \pm R + iy$ ($y \in \mathbb{R}$), we find

$$\lim_{R\to\infty} \left|p(\pm R + iy)\right| \geq \lim_{R\to\infty} \left| \sqrt{R^2 + y^2} - \zeta(\lambda)\left(1 - e^{\mp 2R}\cos(-2y) + e^{\mp 4R}\right)^{\frac{1}{2}} \right| = \infty.$$

Thus,

$$\lim_{R\to\infty} \left| \frac{e^{ix(\pm R + iy)}}{p(\pm R + iy)} \right| = 0, \tag{2.8}$$

which implies that the contributions of the sides of $\Gamma(R,\varepsilon,x,\lambda)$ to the overall value of the integral $\int_\Gamma e^{ix\xi}/p(\xi)\,\mathrm{d}\xi$ goes to zero as we take $R\to\infty$. Cauchy's theorem in conjunction with (2.8) consequently tell us that

$$G_L^+(x;\lambda) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}-i\pi} \frac{e^{ix\xi}}{p(\xi)}\,\mathrm{d}\xi, & x<0 \\[2ex] i\,\mathrm{Res}_{\xi=0}\left(\dfrac{e^{ix\xi}}{p(\xi)}\right) + i\,\mathrm{Res}_{\xi=\lambda}\left(\dfrac{e^{ix\xi}}{p(\xi)}\right)\dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}+i\pi}\frac{e^{ix\xi}}{p(\xi)}\,\mathrm{d}\xi, & x>0 \end{cases} \tag{2.9}$$

where

$$\mathrm{Res}_{\xi=0}\left(\frac{e^{ix\xi}}{p(\xi)}\right) = \frac{1-e^{2\lambda}}{2\lambda e^{2\lambda}+1-e^{2\lambda}}, \qquad \text{and} \qquad \mathrm{Res}_{\xi=\lambda}\left(\frac{e^{ix\xi}}{p(\xi)}\right) = \frac{1-e^{2\lambda}}{1+2\lambda-e^{2\lambda}}\,e^{ix\lambda}.$$

We claim that

$$K^+(x;\lambda) := \frac{e^{-\pi|x|}}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\frac{1}{p(\xi;\lambda,1)+i\pi\,\mathrm{sign}(x)}\,\mathrm{d}\xi = \frac{1}{2\pi}\int_{\mathbb{R}+i\pi\,\mathrm{sign}(x)}\frac{e^{ix\xi}}{p(\xi;\lambda,1)}\,\mathrm{d}\xi.$$

Indeed, the claim follows from (2.10) and

$$e^{ix(\xi+i\pi\,\mathrm{sign}(x))} = e^{-\pi\,x\,\mathrm{sign}(x)}\,e^{ix\xi} = e^{-\pi|x|}\,e^{ix\xi}.$$

By defining for $\lambda\neq 0$,

$$\alpha(\lambda) := \mathrm{Res}_{\xi=0}\left(\frac{e^{ix\xi}}{p(\xi)}\right) \qquad \text{and} \qquad \beta(\lambda) := e^{-ix\lambda}\,\mathrm{Res}_{\xi=\lambda}\left(\frac{e^{ix\xi}}{p(\xi)}\right),$$

noting that

$$\mathrm{Res}_{\xi=0}\left(\frac{e^{ix\xi}}{p(\xi)}\right) = \frac{2}{3}+ix$$

when $\lambda=0$, and repeating the arguments given this section for G_R^+, equations 2.4 follow. Simple but tedious computations verify that α and β satisfy the limits from Theorem 2.1.2.

2.4 Asymptotics of K^+

A cursory inspection of the integrand of $G^+_\star$ ($\star = L$, or R) indicates that $G^+_\star$ behavior differs wildly for $x \in \mathbb{R}$ near zero, and $|x| > c$, where $c > 0$ is any fixed constant. While $1/p(\xi)$ decays roughly exponentially as $\xi \to -\infty$, the fact that $1/p(\xi)$ decays as $1/\xi$ for $\xi \gg 1$ means the oscillatory term $e^{ix\xi}$ in the integrand of $G^+_\star$ is imperative for the contour integral in $G^+_\star$ of $e^{ix\xi}/p(\xi)$ to even have a chance for convergence. In this section we study the "nice" case of $|x| > 1$ and show that not only does the integral in $G^+_\star$ converge in this scenario, it is rapidly decaying (as long as x stays away from zero). In the Section 2.5, we study the behavior of $G^+_\star$ for x near zero and show $G^+_\star$ has at worst a log type singularity at $x = 0$. Taken together, the results from this section combined with the results from Section 2.5 constitute a proof of Theorem 2.1.2 from the introduction of this chapter.

Key to the analysis in both this section and Section 2.5 are the representation formulas (2.4) proven in Section 2.3. In particular, (2.4) allows us to reduce our analyses to a thorough study of K^+

To understand the properties of K^+, we study the convergence of the integral

$$\int_{\Sigma_{\mathrm{sign}(x)}} \frac{e^{ix\xi}}{p(\xi)}\,\mathrm{d}\xi,$$

where $\Sigma_{\mathrm{sign}(x)} := \mathbb{R} + i\,\mathrm{sign}(x)\pi$. Let $\Sigma\big(R,\ \mathrm{sign}(x)\big)$ denote the contour $(-R, R) + i\,\mathrm{sign}(x)\pi$. Recall that $p(\xi)$ can be written as

$$p(\xi) = \xi - \zeta(\lambda)\left(1 - e^{-2\xi}\right),$$

which implies $p'(\xi) = 1 - 2\zeta(\lambda)\, e^{-2\xi}$. In which case

$$\int_{\Sigma\left(R,\ \mathrm{sign}(x)\right)} \frac{e^{ix\xi}}{p(\xi)}\,\mathrm{d}\xi = \frac{e^{ix\xi}}{p(\xi)}\Bigg|_{-R+i\,\mathrm{sign}(x)\pi}^{R+i\,\mathrm{sign}(x)\pi} - \frac{1}{ix}\int_{\Sigma\left(R,\ \mathrm{sign}(x)\right)} e^{ix\xi}\,\frac{p'(\xi)}{\big(p(\xi)\big)^2}\,\mathrm{d}\xi.$$

Now

$$p'(t \pm i\pi) = 1 - 2\zeta(\lambda)e^{-2t}e^{\mp 2\pi i} = 1 - 2\zeta(\lambda)e^{-2t} = p'(t), \qquad t \in \mathbb{R}.$$

Further

$$\zeta(t \pm i\pi) - \zeta(\lambda) = \frac{t}{1-e^{-2t}} - \zeta(\lambda) \pm \frac{i\pi}{1-e^{-2t}} = \zeta(t) - \zeta(\lambda) \pm \frac{i\pi}{1-e^{-2t}}, \qquad t \in \mathbb{R},$$

which implies

$$\begin{aligned} p(t \pm i\pi) &= \big(\zeta(t \pm i\pi) - \zeta(\lambda)\big)\left(1 - e^{-2t}\right) \\ &= \big(\zeta(t) - \zeta(\lambda)\big)\left(1 - e^{-2t}\right) \pm i\pi = p(t) \pm i\pi. \end{aligned} \tag{2.10}$$

Combining the two calculations above, we see that

$$\frac{1}{ix} \int\limits_{\Sigma\left(R,\ \mathrm{sign}(x)\right)} e^{ix\xi} \frac{p'(\xi)}{\big(p(\xi)\big)^2} \,\mathrm{d}\xi = \frac{e^{-|x|\pi}}{ix} \int_{-R}^{R} e^{ixt} \frac{p'(t)}{\big(p(t) + i\,\mathrm{sign}(x)\pi\big)^2} \,\mathrm{d}t.$$

Thus,

$$\lim_{R \to \infty} \frac{1}{\big|p\big(\pm R + i\,\mathrm{sign}(x)\pi\big)\big|} = 0$$

and, formally,

$$\left| \int\limits_{\Sigma_{\mathrm{sign}(x)}} \frac{e^{ix\xi}}{p(\xi)} \,\mathrm{d}\xi \right| \le \frac{e^{-|x|\pi}}{|x|} \int_{\mathbb{R}} \frac{|p'(t)|}{p(t)^2 + \pi^2} \,\mathrm{d}t, \tag{2.11}$$

where we proceed to establish the convergence of the integral on the right-hand side of (2.11). Note that $p'(t) = 0$ only when $t = \frac{1}{4}\ln\big(\zeta(\lambda)\big)$. Further, given $p''(t) = 4\zeta(\lambda)\, e^{-2t} > 0$ for all real t,

$$|p'(t)| = \begin{cases} -p'(t), & t < t_0 \\ p'(t), & t \ge t_0, \end{cases}$$

where $t_0 := \frac{1}{4}\ln\big(\zeta(\lambda)\big)$. Given this fact, we now proceed to evaluate the integral on the right-hand side of (2.11) through u-substitution by setting $u = p(t)$. Observe that

$$\int \frac{p'(t)}{p(t)^2 + \pi^2} \,\mathrm{d}t = \int \frac{1}{u^2 + \pi^2} = \frac{1}{\pi} \arctan\left(\frac{u}{\pi}\right) + C = \frac{1}{\pi} \arctan\left(\frac{p(t)}{\pi}\right) + C. \tag{2.12}$$

So, for $R > |t_0|$,

$$\int_{-R}^{t_0} \frac{|p'(t)|}{p(t)^2+\pi^2}\,dt = -\int_{-R}^{t_0} \frac{p'(t)}{p(t)^2+\pi^2} = -\frac{1}{\pi}\left[\arctan\left(\frac{p(t_0)}{\pi}\right) - \arctan\left(\frac{p(-R)}{\pi}\right)\right],$$

and

$$\int_{t_0}^{R} \frac{|p'(t)|}{p(t)^2+\pi^2}\,dt = \frac{1}{\pi}\left[\arctan\left(\frac{p(R)}{\pi}\right) - \arctan\left(\frac{p(t_0)}{\pi}\right)\right].$$

Now

$$\lim_{R\to\infty} p(\pm R) = \lim_{R\to\infty}\left[\pm R - \zeta(\lambda)\left(1-e^{\mp 2R}\right)\right] = \infty,$$

which implies that

$$\begin{aligned}
\lim_{R\to\infty}\int_{-R}^{R} \frac{|p'(t)|}{p(t)^2+\pi^2}\,dt &= \lim_{R\to\infty}\left(\int_{-R}^{t_0} \frac{|p'(t)|}{p(t)^2+\pi^2}\,dt + \int_{t_0}^{R} \frac{|p'(t)|}{p(t)^2+\pi^2}\,dt\right)\\
&= \frac{1}{\pi}\left[\lim_{R\to\infty}\arctan\left(\frac{p(R)}{\pi}\right) + \lim_{R\to\infty}\arctan\left(\frac{p(-R)}{\pi}\right)\right.\\
&\qquad \left. - 2\arctan\left(\frac{p(t_0)}{\pi}\right)\right]\\
&= 1 - \frac{2}{\pi}\arctan\left(\frac{p(t_0)}{\pi}\right)
\end{aligned}$$

Since

$$p(t_0) = \frac{1}{4}\ln\zeta(\lambda) - \zeta(\lambda) + \left(\zeta(\lambda)\right)^{\frac{1}{2}},$$

$p(t_0) \to -\infty$ as $\lambda \to \pm\infty$, and

$$\lim_{\lambda\to\pm\infty}\int_{\mathbb{R}} \frac{|p'(t)|}{p(t)^2+\pi^2}\,dt = 2.$$

Putting everything together, we see that

$$K^+(x;\lambda) = \mathcal{O}\left(\frac{e^{-|x|\pi}}{|x|}\right) \tag{2.13}$$

for $|x| \geq 1$.

2.5 Green's Function Singularity near $x = 0$

As we continue our analysis of K^+, recall that our ultimate goal for this section is to finish the proof of Theorem 2.1.2. To determine the properties of $G_\star^+$ ($\star = L$, or R)

near x, it suffices to analyze the integrability of K^+ for small $|x|$. The discussion that follows within this section is based on the unpublished notes of Prof. Allen Wu.

We take without loss of generality $0 < x < 1$ and consider the integral

$$K_\zeta(x) := \int_{\mathbb{R}} \frac{e^{ix\xi}}{\xi - \zeta\left(1 - e^{-2\xi}\right) + i\pi}\, \mathrm{d}\xi$$

for the three cases $0 < \zeta < \zeta_0$, $\zeta_0 \le \zeta \le \zeta_1$ and $\zeta > \zeta_1$, where $\zeta_0 > 1$ is chosen sufficiently small, and $\zeta_1 \gg 1$ is chosen sufficiently large. In each of these three cases, we perform a dyadic decomposition on $K_\zeta(x)$ so that

$$K_\zeta(x) = \sum_q K_q(x), \tag{2.14}$$

where

$$K_q(x) := \int_{\mathbb{R}} \frac{e^{ix\xi}\chi\left(2^{-q}x\xi\right)}{\xi - \zeta\left(1 - e^{-2\xi}\right) + i\pi}\, \mathrm{d}\xi.$$

and χ is an even, smooth function with compact support near $|\xi| = 1$ so that $\sum_q \chi(2^{-q}\xi) = 1$ for $\xi \neq 0$.

The main result of our analyses is summarized in Theorem 2.5.1. Though, the three major cases in the proof of Theorem 2.5.1 are shown in the proofs of Lemmas 2.5.2 through 2.5.3.

Theorem 2.5.1. *Suppose $|x| < 1$. Then*

$$|K_\zeta(x)| \le C + C\big|\log|x|\big| \tag{2.15}$$

for all $\zeta > 0$.

Proof. Since analogous results to Lemmas 2.5.2 through 2.5.4 also hold for $-1 < x < 0$, Theorem 2.5.1 is an immediate consequence of these three lemmas. □

Lemma 2.5.2. *For $\zeta_0 := \zeta(\lambda_0)$ and λ_0 chosen sufficiently small that*

$$1 - \frac{\lambda}{\lambda+1}\frac{e^{-2}e^{-2\lambda} - 1}{e^{-2\lambda} - 1} > \frac{1}{2}$$

and

$$1-\frac{\lambda}{\lambda-1}\frac{e^{2}e^{-2\lambda}-1}{e^{-2\lambda}-1}<-1$$

whenever $\lambda<\lambda_0$*, the bound* (2.15) *holds for all* $0<\zeta<\zeta_0$ *and* $0<x<1$*.*

Proof. Notice that $\chi\left(2^{-q}x\xi\right)$ is non-zero only for $|2^{-q}x\xi|\approx 1$. Thus, for $2^{-q}x\xi\in \operatorname{supp}\chi$, $|\xi|\approx 2^{q}/x$. For a given $\zeta=\zeta(\lambda)$ and x, there are at most five values of q for which

$$\frac{1}{2}<\frac{|\xi|}{|\lambda|}<2 \qquad \text{and} \qquad |\xi|\approx\frac{2^{q}}{x}$$

We first estimate K_q for these values of q as follows.

Observe that $\dfrac{1-e^{-2\xi}}{\xi}$ is a positive, decreasing function. If $\xi>\lambda+1$, then

$$\frac{\xi-\zeta\left(1-e^{-2\xi}\right)}{\xi}=1-\frac{\lambda}{\xi}\frac{1-e^{-2\xi}}{1-e^{-2\lambda}}>1-\frac{\lambda}{\lambda+1}\frac{e^{-2}e^{-2\lambda}-1}{e^{-2\lambda}-1}>\frac{1}{2}.$$

It follows that

$$\left|\xi-\zeta\left(1-e^{-2\xi}\right)\right|>\frac{1}{2}|\xi| \tag{2.16}$$

when $\xi>\lambda+1$.

On the other hand, if $\xi<\lambda-1$,

$$\frac{\xi-\zeta\left(1-e^{-2\xi}\right)}{\xi}=1-\frac{\lambda}{\xi}\frac{1-e^{-2\xi}}{1-e^{-2\lambda}}<1-\frac{\lambda}{\lambda-1}\frac{e^{2}e^{-2\lambda}-1}{e^{-2\lambda}-1}<-1. \tag{2.17}$$

Thus, it follows from (2.16) and (2.17) that

$$\left|\xi-\zeta\left(1-e^{-2\xi}\right)\right|>\frac{1}{2}|\xi|$$

when $|\xi-\lambda|>1$. Thus

$$|K_q(x)|\leq C+C\int_{|\xi-\lambda|>1}\frac{\chi(2^{-q}x\xi)}{|\xi|+\pi}\,\mathrm{d}\xi\leq C+C\int_{\mathbb{R}}\frac{\chi(2^{-q}x\xi)}{|\xi|+\pi}\,\mathrm{d}\xi\leq C. \tag{2.18}$$

For the remainder of this proof, we assume $|\xi|<\frac{1}{2}|\lambda|$, or $|\xi|>2|\lambda|$. In particular, we have $|\xi-\lambda|>1$ and $\left|\xi-\zeta\left(1-e^{-2\xi}\right)\right|>\frac{1}{2}|\xi|$.

Next, we focus all all those q's satisfying $2^q \lesssim x$. For such q's,

$$|K_q(x)| \leq \int_{\mathbb{R}} \frac{\chi\left(2^{-q}x\xi\right)}{\pi} \,\mathrm{d}\xi \leq C2^q x. \tag{2.19}$$

The sum of these terms are bounded by

$$\sum_{2^q \leq x} C2^q/x \leq C.$$

We now focus on all those q's satisfying $x \lesssim 2^q \lesssim 1$. For such q's,

$$|K_q(x)| \leq C \int_{\mathbb{R}} \frac{\chi\left(2^{-q}x\xi\right)}{|\xi|} \,\mathrm{d}\xi \leq C. \tag{2.20}$$

The sum of these terms gives

$$C \sum_{x \lesssim 2^q \lesssim 1} 1 \leq C|\log x|. \tag{2.21}$$

Lastly, we consider the K_q terms for which $q \gtrsim 1$. For these values of q, we integrate by parts and ignore a factor of i to get

$$\begin{aligned}
K_q(x) &= K_{1q}(x) + K_{2q}(x) \\
&= 2^{-q} \int_{\mathbb{R}} \frac{e^{ix\xi}}{\xi - \zeta(1 - e^{-2\xi}) + i\pi} \chi'(2^{-q}x\xi) \,\mathrm{d}\xi \\
&\quad + \int_{\mathbb{R}} \frac{1}{x} e^{ix\xi} \chi\left(2^{-q}x\xi\right) \left[\frac{1}{\xi - \zeta(1 - e^{-2\xi}) + i\pi}\right]' \mathrm{d}\xi.
\end{aligned}$$

Observe that

$$|K_{1q}(x)| \leq C2^{-q} \int_{\mathbb{R}} \frac{|\chi'(2^{-q}x\xi)|}{|\xi|} \,\mathrm{d}\xi \leq 22^{-q}. \tag{2.22}$$

We can compute K_{2p} and write it as

$$2^{-q} \int_{\mathbb{R}} \chi\left(2^{-q}x\xi\right) \frac{1}{\left[\xi - \zeta\left(1 - e^{-2\xi}\right) + i\pi\right]^2} \frac{1}{x} \,\mathrm{d}\xi \tag{2.23}$$

Using the condition $2^{-q}x|\xi| \approx 1$ whenever $2^{-q}x\xi \in \operatorname{supp}\chi$, we bound (2.23) by

$$2^{-q} \int_{\mathbb{R}} \chi\left(2^{-q}x\xi\right) \left|\frac{\xi(1 - 2\zeta e^{-2\xi})}{\left[\xi - \zeta\left(1 - e^{-2\xi}\right) + i\pi\right]^2}\right| \mathrm{d}\xi \tag{2.24}$$

Let us first assume $|\xi| < \dfrac{1}{2}|\lambda|$. In this case, we obviously have $|\xi - \lambda| > 1$ and $\left|\xi - \zeta\left(1 - e^{-2\xi}\right)\right| \geq \dfrac{1}{2}$. Furthermore, we claim that $\left|\zeta e^{-2\xi}\right| \leq C$. In fact, for $s = \xi - \lambda/2 > 0$, we have

$$\left|\zeta e^{-2\xi}\right| \leq C + \left|\zeta\left(1 - e^{-2\xi}\right)\right| \quad \leq C\left|\frac{\lambda}{1 - e^{-2\lambda}}\left(1 - e^{-2(\lambda/2+s)}\right)\right| \leq C + C\left|\lambda e^{\lambda} e^{-2s}\right| \leq C.$$

Since $\lambda < \lambda_0$ is sufficiently negative, (2.24) is bounded by

$$C2^{-q}\int_{\mathbb{R}} \chi\left(2^{-q}x\xi\right)\left|\frac{\xi}{\xi^2}\right| d\xi \leq 2^{-q}. \tag{2.25}$$

Finally, we assume $|\xi| > 2|\lambda|$. The part of ξ that is positive is obviously bounded by $C2^{-q}$. We focus on the part of ξ such that $\xi < 2\lambda$. Since for this part $\xi - \lambda < -1$, with the similar estimates above, we claim that

$$-\zeta(1 - e^{-2\xi}) > -2\xi. \tag{2.26}$$

We further claim that

$$\left|\frac{\xi}{\zeta(1 - e^{-2\xi})}\right| \leq \frac{C}{|\xi|}.$$

In fact, for $s = 2\lambda - \xi > 0$, we have

$$\begin{aligned}
\left|\frac{\xi}{\zeta(1 - e^{-2\xi})}\right| &\leq \left|\frac{1 - e^{-2\lambda}}{\lambda}\frac{(2\lambda - s)^2}{1 - e^{-2(2\lambda - s)}}\right| \\
&\leq C\left|e^{2\lambda}e^{-2s}\left(4\lambda - 4s + \frac{s^2}{\lambda}\right)\right| \\
&\leq C.
\end{aligned}$$

Again, remembering that $\lambda < \lambda_0$ is sufficiently negative, with the estimates above we see that (2.24) is bounded by

$$\begin{aligned}
C2^{-q}\int_{\mathbb{R}} \chi\left(2^{-q}x\xi\right)\frac{\left|\xi\left(1 - 2\zeta e^{-2\xi}\right)\right|}{\left|\zeta\left(1 - e^{-2\xi}\right)\right|^2}\,\mathrm{d}\xi &\leq C2^{-q}\int_{\mathbb{R}} \chi\left(2^{-q}x\xi\right)\frac{1}{|\xi|}\,\mathrm{d}\xi \\
&\leq C2^{-q}.
\end{aligned}$$

In summary, we have $|K_q(x)| \leq C2^{-q}$ for $2^q \gtrsim 1$ whose the sum gives $\sum_{2^q \gtrsim 1} C2^{-q}$. Combining all the above estimates, we get the desired result. □

Lemma 2.5.3. *Let $\xi_0 > 1$ be sufficiently large that*

$$\frac{1}{2} < \frac{1 - e^{-2\xi} - 2\xi e^{-2\xi}}{(1 - e^{-2\xi})^2} < 2$$

for $\xi > \xi_0$ and let $_1 > \xi_0$. Then, the estimate (2.15) *holds for all for all* $0 < x < 1$ *and $\zeta > \zeta_1 := \zeta(\lambda_1)$.*

Proof. By the mean value theorem,

$$\frac{\xi}{1 - e^{-2\xi}} - \frac{\lambda}{1 - e^{-2\lambda}} = \left. \frac{1 - e^{-2\xi} - 2\xi e^{-2\xi}}{\left(1 - e^{-2\xi}\right)^2} \right|_{\xi = \theta}. \tag{2.27}$$

Here θ is between ξ and λ. Let $\chi_{\xi_0+}(\xi)$ be a smooth cutoff function that is 1 for $\xi > \xi_0 + 1$ and 0 for $\xi \leq \xi_0$, with gradient bounded by 2. We write

$$\begin{aligned} K_\zeta(x) &= K_{1\zeta}(x) + K_{2\zeta}(x) \\ &:= \int_{\mathbb{R}} \frac{e^{ix\xi}\left(1 - \xi_{\xi_0+}(\xi)\right)}{\xi - \zeta\left(1 - e^{-2\xi}\right) + i\pi} \, \mathrm{d}\xi + \int_{\mathbb{R}} \frac{e^{ix\xi}\left(\xi_{\xi_0+}(\xi)\right)}{\xi - \zeta\left(1 - e^{-2\xi}\right) + i\pi} \, \mathrm{d}\xi \end{aligned}$$

Take some $\lambda_1 > \xi_0$ and estimate $K_{1\zeta}$ by

$$\begin{aligned} |K_{1\zeta}(\xi)| &\leq C + \int_{-\infty}^{-1} \frac{1}{\left(e^{-2\xi} - 1\right)\left(\zeta - \frac{\xi}{1 - e^{-2\xi}}\right)} \, \mathrm{d}\xi \\ &\leq C + \frac{1}{\zeta(\lambda_1) - \zeta(-1)} \int_{-\infty}^{-1} \frac{1}{e^{-2\xi} - 1} \, \mathrm{d}\xi. \end{aligned}$$

To estimate $K_{2\zeta}$, we take a dyadic decomposition

$$K_{1\zeta}(\xi) = \sum_q K_q(x) = \sum_q \int_{\mathbb{R}} \frac{e^{ix\xi}\chi_{\xi_0+}(\xi)}{\xi - \zeta\left(1 - e^{-2\xi}\right) + i\pi} \chi\left(e^{-q}x(\xi - \lambda)\right) \mathrm{d}\xi.$$

for $2^q \gtrsim x$, $|K_q(x)| \leq C 2^q / x$. These terms sum to C. For $x \lesssim 2^q \lesssim 1$, we notice that since $\xi, \lambda \geq \xi_0$,

$$\left|\xi - \zeta\left(1 - e^{-2\xi}\right)\right| = \left|\left(1 - e^{-2\xi}\right)\left(\frac{\xi}{1 - e^{-2\xi}} - \frac{\lambda}{1 - e^{-2\lambda}}\right)\right| \approx |\xi - \lambda|.$$

Thus

$$|K_q(x)| \leq C \int_{\mathbb{R}} \frac{1}{|\xi - \lambda|} \chi\left(2^{-q}x(\xi - \lambda)\right) \mathrm{d}\xi \leq C,$$

which implies

$$\sum_{x \lesssim 2^q \lesssim 1} |K_q(x)| \leq C|\log x|.$$

For $2^q \gtrsim 1$, we integrate K_p by parts to obtain

$$\int_{\mathbb{R}} e^{ix\xi} \frac{\chi\big(2^{-q}x(\xi-\lambda)\big)}{\xi - \zeta\big(1-e^{-2\xi}\big) + i\pi} \frac{\chi_{\xi_0+}(\xi)'}{x} \,\mathrm{d}\xi\, d\xi \tag{2.28}$$

$$+ 2^{-q} \int_{\mathbb{R}} \frac{e^{ix\xi}\chi_{\xi_0+}}{\xi - \zeta\big(1-e^{-2\xi}\big) + i\pi} \frac{\chi'_{\xi_0+}(\xi)}{x} \,\mathrm{d}\xi \chi'\big(2^{-q}x(\xi-\lambda)\big)\,\mathrm{d}\xi \tag{2.29}$$

$$- \int_{\mathbb{R}} e^{ix\xi}\chi_{\xi_0+}(\xi) \frac{1-2\zeta e^{-2\xi}}{\big(\xi - \zeta\big(1-e^{-2\xi}\big) + i\pi\big)^2} \frac{\chi\big(2^{-q}x(\xi-\lambda)\big)}{x}\,\mathrm{d}\xi \tag{2.30}$$

Noticing the condition $\frac{1}{x} \approx 2^{-q}|\xi - \lambda|$, (2.28) is bounded by

$$C2^{-q} \int_{\mathbb{R}} \frac{|\xi-\lambda|}{|\xi-\lambda|} |\chi'_{\xi_0+}(\xi)|\, d\xi \leq C2^{-q}.$$

(2.29) is bounded by

$$C2^{-q} \int_{\mathbb{R}} \frac{|\chi'(2^{-q}x(\xi-\lambda))|}{|\xi-\lambda|}\,\mathrm{d}\xi \leq C2^{-q}.$$

(2.30) is bounded by

$$\begin{aligned}
&C2^{-q} \int_{\mathbb{R}} \chi_{\xi_0+}(\xi) \frac{|1-2\zeta e^{-2\xi}|}{|\xi-\lambda|} \chi(2^{-q}x(\xi-\lambda))\,\mathrm{d}\xi \\
&\quad\leq C2^{-q} \int_{\mathbb{R}} \frac{\chi(2^{-q}x(\xi-\lambda))}{|\xi-\lambda|}\,\mathrm{d}\xi + C2^{-q} \int_0^\infty \frac{\lambda e^{-2\lambda} e^{-2(\xi-\lambda)}}{|\xi-\lambda|} \chi(2^{-q}x(\xi-\lambda))\,\mathrm{d}\xi \\
&\quad\leq C2^{-q} + C2^{-q}\lambda e^{-2\lambda} \int_{-2^{-q}x\lambda}^\infty \frac{e^{-\xi 2^{q+1}/x}}{|\xi|} \chi(\xi)\,\mathrm{d}\xi
\end{aligned}$$

$$\leq C2^{-q} + C2^{-q}\lambda e^{-2\lambda} \int_{-2^{-q}x\lambda}^0 \frac{e^{-\xi 2^{q+1}/x}}{|\xi|} \chi(\xi)\,\mathrm{d}\xi \tag{2.31}$$

Let's say χ is supported between $\frac{1}{2}$ and 2, then the integral in (2.31) is nonzero only when $2^{-q}x\lambda \geq \frac{1}{2}$. If $\frac{1}{2} \leq 2^{-q}x\lambda \leq 2$, the integral in (2.31) is bounded by

$$C \int_{-2^{-q}x\lambda}^0 e^{-\xi 2^{q+1}/x}\, d\xi \leq C2^{-(q+1)} x e^{2\lambda}.$$

Thus (2.31) is bounded by

$$C2^{-q} + C2^{-q}\lambda 2^{-(q+1)}x \le C2^{-q}.$$

If $2^{-q}x\lambda \ge 2$, the integral in (2.31) is bounded by

$$C\int_{-2}^{0} e^{-\xi 2^{q+1}/x}\, d\xi \le C2^{-(q+1)}xe^{2\,\cdot\, 2^{q+1}/x}.$$

Thus (2.31) is bounded by

$$C2^{-q} + C2^{-q}\frac{\lambda e^{-2\lambda}}{(2^{q+1}/x)e^{-2\,\cdot\, 2^{q+1}/x}} \le C2^{-q}.$$

The last step above follows from the condition $2^{q+1}/x \le \lambda$, and the fact that $\lambda e^{-2\lambda}$ is decreasing for $\lambda > \frac{1}{2}$.

Therefore, the sum $\sum_{2^q \gtrsim 1} |K_q(x)| \le C$. As a consequence,

$$|K_\zeta(x)| \le C + C|\log x|$$

for all $\zeta > \zeta_1$, and $0 < x < 1$. □

Lemma 2.5.4. *Inequality* (2.15) *holds for every ζ between ζ_0 and ζ_1, where ζ_0 and ζ_1 are as respectively defined in the statements of Lemmas 2.5.2 and 2.5.3.*

Proof. Now that we have the uniform estimates on G_ζ for $\zeta < \zeta_0 = \zeta(\lambda_0)$ and $\zeta > \zeta_1 = \zeta(\lambda_1)$, we can fill the gap $\zeta_0 < \zeta < \zeta_1$. We take the dyadic sum

$$\sum_q K_q(x) = \sum_q \int_{\mathbb{R}} \frac{e^{ix\xi}\chi\left(2^{-q}x\xi\right)\left(1 - \chi_{\lambda_0\lambda_1}(\xi)\right)}{\xi - \zeta\left(1 - e^{-2\xi}\right) + i\pi}\, \mathrm{d}\xi. \tag{2.32}$$

Notice we have cut off a piece from $\lambda_0 - 1$ to $2\zeta_1$ by the cutoff function $\chi_{\lambda_0\lambda_1}(\xi)$. Of course, the piece that's cut off is uniformly bounded. Observe that for $\xi < \lambda_0 - 1$ and $\zeta > \zeta_0$,

$$\left|\xi - \zeta\left(1 - e^{-2\xi}\right)\right| = \left|\left(1 - e^{-2\xi}\right)\left(\zeta(\xi) - \zeta\right)\right| \ge Ce^{-2\xi},$$

while for $\xi > 2\zeta_1$ and $\zeta < \zeta_1$,

$$\left|\xi - \zeta\left(1 - e^{-2\xi}\right)\right| \approx |\xi|, \quad 2\zeta e^{-2\xi} < \frac{1}{2}.$$

A similar argument as above gives the uniform estimates. □

CHAPTER 3. GREENS FUNCTIONS: MAPPING PROPERTIES

3.1 Introduction

One of the consequences of (2.2) along with (1.8) is that $G_\star^+(x;\lambda)$ ($\star = L$, or R) grows linearly as a function of $x \in \mathbb{R}$ when $\lambda = 0$. This fact aligns with intuition, since the two simple poles $\xi = 0$ and $\xi = \lambda$ of $1/p$ coalesce into a single double pole as $\lambda \to 0$. Since r is a function of λ, in order to prove the direct scattering map $\mathscr{D} : u \mapsto r$ is well-defined and Lipschitz continuous, we need estimates on the solutions to the integral equations (1.9)—and hence estimates on the Green's functions—which are uniform in λ. This will allow us to obtain similarly uniform estimates on the scattering data derived from the Jost solutions. Given the linear growth of $G_\star^+$ as $\lambda \to 0$, in order to obtain λ-independent estimates while studying the mapping properties of $G_\star^+$, we commonly work over polynomially weighted L^p ($1 \leq p \leq \infty$) weighted spaces defined in definitions 3.1.1 and 3.1.2. In order to avoid introducing poles when weighting by the reciprical of a polynomial, we introduce the notation $\langle x \rangle := \sqrt{1+x^2}$ to represent a linear weight.

Definition 3.1.1 ($L^{p,s}$)**.** For $1 < p < \infty$ and $s \in (-\infty, \infty)$, we define $L^{p,s}(\mathbb{R})$ to be the space of all measurable functions f with the property that $\langle \cdot \rangle^s f \in L^p(\mathbb{R})$, and associate with $L^{p,s}(\mathbb{R})$ the norm $\| \cdot \|_{L^{p,s}}$ given by

$$\|f\|_{L^{p,s}} := \left(\int_{\mathbb{R}} \langle x \rangle^{sp} |f(x)|^p \right)^{1/p}.$$

for each $f \in L^{p,s}(\mathbb{R})$.

Remark 7. In accordance with the notation used in [3], we use $L^{p,\infty}$ to denote the space *weak*-L^p. The space $L^{p,\infty}$ should be thought of completely seperately from $L^{p,s}$ and should not be considered as a limit (in s) of $L^{p,s}$ spaces. For a definition of $L^{p,\infty}$, please see the appendix titled "Harmonic Analysis Results."

Definition 3.1.2 ($\langle\,\cdot\,\rangle^s L^p$)**.** Let $1 < p \le \infty$. Then we use the notation $\langle\,\cdot\,\rangle^s L^p(\mathbb{R})$ to indicate the collection of measurable functions f with $\langle\,\cdot\,\rangle^{-s} f \in L^p()$. Specifically,

$$\langle\,\cdot\,\rangle^s L^p(\mathbb{R}) := \left\{ \langle\,\cdot\,\rangle^s f \;:\; f \in L^p(\mathbb{R}) \right\}.$$

The norm $\|\cdot\|_{\langle\,\cdot\,\rangle L^p}$ on the space $\langle\,\cdot\,\rangle L^p(\mathbb{R})$ is defined by $\|f\|_{\langle\,\cdot\,\rangle L^p} := \|\langle\,\cdot\,\rangle^{-p} f\|_{L^p}$.

Remark 8. Since we commonly work in the space $\langle\,\cdot\,\rangle L^\infty(\mathbb{R})$, it is worth highlighting the fact that $\langle\,\cdot\,\rangle L^\infty(\mathbb{R})$ is the collection of all measurable functions f for which $\langle\,\cdot\,\rangle f$ is essentially bounded. That is,

$$\|f\|_{\langle\,\cdot\,\rangle L^\infty} = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |\langle x\rangle^{-1} f(x)|$$

is finite for all $f \in \langle\,\cdot\,\rangle L^\infty(\mathbb{R})$.

Remark 9. In cases involving functions of multiple variables or parameters, we sometimes use a subscript in conjunction with function space notation to avoid confusion. For example, $f \in L^p_\xi(\mathbb{R})$ indicates that the function f is L^p integrable with respect to the variable ξ.

In addition to considering the mapping properties of $G^+_\star$, we also consider in this chapter the related operators $T_{\star,\lambda,u}$ (commonly denoted as $T_\star$ or $T_{\star,\lambda}$, for short) given by

$$T_{\star,\lambda,u} f(x) := \left[G^+_\star(\,\cdot\,;\lambda)\right] * (u\,f)(x), \qquad (\star = L, \text{ or } R) \tag{3.1}$$

as the integral equations (1.9) can be reformulated as

$$\begin{pmatrix} 1 \\ e^{i\lambda x} \end{pmatrix} = (I - T_{L,\lambda,u}) \begin{pmatrix} M^+_1(x;\lambda,\delta) \\ M^+_e(x;\lambda,\delta) \end{pmatrix} \tag{3.2a}$$

$$\begin{pmatrix} 1 \\ e^{i\lambda x} \end{pmatrix} = (I - T_{R,\lambda,u}) \begin{pmatrix} N^+_1(x;\lambda,\delta) \\ N^+_e(x;\lambda,\delta) \end{pmatrix} \tag{3.2b}$$

In considering a space of potentials u, we need to compensate for the logarithmic singularity of $G_\star^+$ at $x = 0$ as well as the linear growth (in x) of $G_\star^+$. In particular, the linear growth of the sum of the residue terms of $G_\star^+$ for $\lambda = 0$ forces us to allow solutions to (3.2) which grow at most linearly (*i.e.* are $\langle \cdot \rangle L^\infty$.) As we see in Proposition 3.2.2, requiring $\langle \cdot \rangle^2 u$ is L^1 *and* also requiring that the convolution of $\langle \cdot \rangle u$ with $\log_+ \left(\frac{1}{|x|}\right)$ be essentially bounded ensures that $T_{\star,\lambda}$ is a bounded operator on $\langle \cdot \rangle L^\infty(\mathbb{R})$ for all real λ. However, in order to ensure that solutions to (3.2) satisfy the asymptotic conditions imposed on the Jost solutions—necessary to prove the equivalence of the Jost solutions and solutions to (3.2)—we need u to satisfy the even stronger decay condition that $\langle \cdot \rangle^3 u$ is L^1. Please see Section 5.3 for why such strong decay is needed.

In proving the equivalence of Jost solutions and solutions to the integral equations 3.2 in Section 5.3, we also need the upper boundary values of solutions to the integral equations 1.9 to exist in an L^2 sense. For reasons that become apparent in Sections 4.4 and 4.5, doing so also requires u to be L^2 integrable. Keeping our entire list of desired properties for u in mind, we select the space X of potentials u as follows:

Definition 3.1.3. Denote by X the space of all measurable functions u for which

$$\|u\|_X := \left\|\langle \cdot \rangle^4 u\right\|_{L^2}$$

is finite. That is, $X = \langle \cdot \rangle^{-4} L^2(\mathbb{R}) = L^{2,4}(\mathbb{R})$.

Remark 10. To see that $u \in X$ actually has all of the required properties, we first note that by the Cauchy-Schwarz inequality,

$$\left\|\langle \cdot \rangle^3 u\right\|_{L^1} = \left\langle \langle \cdot \rangle^{-1}, \langle \cdot \rangle^4 u \right\rangle_{L^2} = \left\|\langle \cdot \rangle^{-1}\right\|_{L^2} \left\|\langle \cdot \rangle^4 u\right\|_{L^2} < \infty,$$

where we use $\langle \cdot, \cdot \rangle_{L^2}$ to denote the L^2 inner product. Hence, $u \in X$ implies $u \in L^{1,2}(\mathbb{R})$. A similar use of the Cauchy-Schwarz inequality also shows that the convolution of $\log_+ \left(\frac{1}{|x|}\right)$ with $\langle \cdot \rangle u$ is bounded by $4\|u\|_X$ for all $x \in \mathbb{R}$.

We explore in Section 3.2 the boundedness and asymptotic properties of $G_\star^+$ and $T_\star$ needed to that the direct scattering map $\mathscr{D}$ is both well-defined and Lipschitz continuous. In Section 3.3 we continue our exploration of the mapping properties of both $G_\star$ and $T_\star$ by considering their continuity and differentiability in λ, which we use while verifying the Lipschitz continuity of $\mathscr{D}$.

3.2 Boundedness as a Convolution Operator

As a warm-up, we begin our study of the Green's functions mapping properties by studying in Proposition 3.2.1 the mapping properties of G_L^+, G_R^+ as convolution operators under the constraint $\lambda \neq 0$.

Proposition 3.2.1. *For each fixed $\lambda \in \mathbb{R}\backslash\{0\}$, $G_\star^+$ ($\star = L$, or R) are bounded as convolution operators from $L^1(\mathbb{R}) \cap L^p$ ($1 < p \leq 2$) with*

$$\|G_\star^+ * f\|_{L^\infty} \lesssim_\lambda \|f\|_{L^1 \cap L^p},$$

where the implied constant depends on λ when $|\lambda| < 1$. Further,

$$\lim_{x \to -\infty} \big(G_L^+ * f\big)(x) = \lim_{x \to +\infty} \big(G_R^+ * f\big)(x) = 0 \tag{3.3}$$

whenever the spectral parameter λ is both real and non-zero.

Proof. From our previous work in Section 2.3 proving Theorem 2.1.1, we have

$$G_L^+(x;\lambda,\delta) = K^+(x;\lambda) + \big[i\alpha(\lambda) + i\beta(\lambda)\, e^{ix\lambda}\big]\chi_+(x) \tag{3.4}$$

and

$$G_R^+(x;\lambda,\delta) = K^+(x;\lambda) - \big[i\alpha(\lambda) + i\beta(\lambda)\, e^{ix\lambda}\big]\chi_-(x) \tag{3.5}$$

where

$$K^+(x;\lambda) := \frac{e^{-\pi|x|}}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \frac{1}{p(\xi;\lambda) + i\pi \operatorname{sign}(x)}\, \mathrm{d}\xi.$$

and $\chi_- := \chi_{(-\infty,0)}$, $\chi_+ := \chi_{(0,\infty)}$ denote the respective characteristic functions on the open intervals $(-\infty,0)$, $(0,\infty)$. Although the integral for K^+ is conditionally convergent, it avoids zeros of the symbol p, and may be understood through the L^q theory of the Fourier transform, since the integrand belongs to L^q for any $q>1$. Moreover, it follows from the Hausdorff-Young inequality and dominated convergence that $\lim_{h\to 0}\|K(\,\cdot\,+h)^+ - K^+\|_{L^{q'}} = 0$ for any $q\in(1,2]$. As a consequence, the convolutions

$$
\begin{aligned}
(3.6)\quad G_L^+(\,\cdot\,;\lambda)*f(x) &= \int_{\mathbb{R}} G_L^+(x-x',\lambda)f(x')\,\mathrm{d}x' \\
&= \int_{-\infty}^{x} K^+(x-x',\lambda)f(x')\,\mathrm{d}x' + \int_x^{\infty} K^+(x-x',\lambda)f(x')\,\mathrm{d}x' \\
&\quad + i\alpha(\lambda)\int_{-\infty}^{x} f(x')\,\mathrm{d}x' + i\beta(\lambda)e^{i\lambda x}\int_{-\infty}^{x} e^{-i\lambda x'}f(x')\,\mathrm{d}x',
\end{aligned}
$$

and

$$
\begin{aligned}
(3.7)\quad G_R^+(\,\cdot\,;\lambda)*f(x) &= \int_{-\infty}^{x} K^+(x-x',\lambda)f(x')\,\mathrm{d}x' + \int_x^{\infty} K^+(x-x',\lambda)f(x')\,\mathrm{d}x' \\
&\quad - i\alpha(\lambda)\int_x^{\infty} f(x')\,\mathrm{d}x' - i\beta(\lambda)e^{i\lambda x}\int_x^{\infty} e^{-i\lambda x'}f(x')\,\mathrm{d}x',
\end{aligned}
$$

define bounded continuous functions for any $f\in L^1(\mathbb{R})\cap L^p(\mathbb{R})$ for any $p\in(1,2]$ with

$$
(3.8)\qquad \left\|G_\star^+ * f\right\|_{L^\infty(\mathbb{R})} \lesssim_\lambda \|f\|_{L^1\cap L^p}.
$$

For $f\in C_0^\infty(\mathbb{R})$, it is easy to see that

$$
\lim_{x\to-\infty} \alpha(\lambda)\int_{-\infty}^{x} f(x')\,\mathrm{d}x' + \beta(\lambda)e^{i\lambda x}\int_{-\infty}^{x} e^{-i\lambda x'}f(x')\,\mathrm{d}x' = 0
$$

and

$$
\lim_{x\to\infty} \alpha(\lambda)\int_x^{\infty} f(x')\,\mathrm{d}x' + \beta(\lambda)e^{i\lambda x}\int_x^{\infty} e^{-i\lambda x'}f(x')\,\mathrm{d}x' = 0.
$$

Further, the Dominated Convergence Theorem implies

$$
\lim_{x\to\pm\infty} K^+ * f(x) = \lim_{x\to\pm\infty}\int_{\mathbb{R}} K^+(x-x')f(x')\,\mathrm{d}x = 0,
$$

as $|K^+(x')f(x-x')| \le \|f\|_{L^\infty}|K^+(x)| \in L^1_x(\mathbb{R})$, and $K^+(x) = \mathcal{O}\left(e^{-|x|}\right)$ for $|x| \ge 1$.

It now follows from a density argument and (3.8) that if $p \in (1,2]$, then

$$\lim_{x\to-\infty}\left(G_L^+ * f\right)(x) = \lim_{x\to+\infty}\left(G_R^+ * f\right)(x) = 0 \tag{3.9}$$

for any $f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$. □

We now turn our focus towards the operators $T_{\star,\lambda,u}$ ($\star = L$, or R). In Proposition 3.2.2 we prove that as an operator on the space $\langle\,\cdot\,\rangle\, L^\infty(\mathbb{R})$, the operators $T_{\star,\lambda,u}$ are uniformly bounded in λ, whose operator norms depend only on the norm of their corresponding potential u. As a reminder, the function space for potentials u is $X := L^{2,4}(\mathbb{R}) = \langle\,\cdot\,\rangle^{-4} L^2(\mathbb{R})$. Beginning in Proposition 3.2.2, we also introduce the notation $Y \circlearrowleft$ to denote a map from Y into Y, and the notation $\|\,\cdot\,\|_{Y\to Z}$ to denote the implied operator norm for an operator which maps from Y to Z. Thus, the notation $\|\,\cdot\,\|_{\langle\,\cdot\,\rangle L^\infty\circlearrowleft}$ used in Equation (3.10) of Proposition 3.2.2 denotes the operator norm for an operator which maps from the space $\langle\,\cdot\,\rangle\, L^\infty(\mathbb{R})$ into itself.

Proposition 3.2.2. *Consider the operators $T_{\star,\lambda,u}$ ($\star = L$, or R) given by*

$$\big(T_{\star,\lambda,u}f\big)(x) := \int_{\mathbb{R}} G_\star^+(x-x';\lambda)\, u(x') f(x')\, \mathrm{d}x'.$$

For every $u \in X$, operators $T_{\star,\lambda,u} : \langle\,\cdot\,\rangle\, L^\infty(\mathbb{R}) \to \langle\,\cdot\,\rangle\, L^\infty(\mathbb{R})$ are bounded uniformly in $\lambda \in \mathbb{R}$ with

$$\|T_{\star,\lambda,u}\|_{\langle\,\cdot\,\rangle L^\infty\circlearrowleft} \lesssim \|u\|_X. \tag{3.10}$$

Proof. To simplify notation, we write $T_\star$ instead of $T_{\star,\lambda,u}$ throughout this proof. We begin by noting that

$$\begin{aligned}|T_\star f(x)| &= \left|\int_{\mathbb{R}} G_\star^+(x-x';\lambda)\big(u(x')\,\langle x'\rangle\big)\big(\,\langle x'\rangle^{-1} f(x')\big)\,\mathrm{d}x'\right| \\ &\le \|f\|_{\langle\,\cdot\,\rangle L^\infty} \int_{\mathbb{R}} \left|G_\star^+(x-x';\lambda)\right| \langle x'\rangle\, |u(x')|\,\mathrm{d}x'.\end{aligned}$$

Hence

$$\|T_\star\|_{\langle\,\cdot\,\rangle L^\infty \circlearrowleft} \le \sup_{x\in\mathbb{R}} \langle x\rangle^{-1} \int_{\mathbb{R}} \left|G_\star^+(x-x';\lambda)\right| \langle x'\rangle \, |u(x')| \, \mathrm{d}x'. \tag{3.11}$$

Recall from Theorem 2.1.1 that

$$G_L^+(x;\lambda) = K^+(x;\lambda) + \left[i\alpha(\lambda) + i\beta(\lambda)e^{ix\lambda}\right]\chi_L(x) \tag{3.12a}$$

$$G_R^+(x;\lambda) = K^+(x;\lambda) - \left[i\alpha(\lambda) + i\beta(\lambda)e^{ix\lambda}\right]\chi_R(x) \tag{3.12b}$$

where we define $\chi_L := \chi_{\mathbb{R}^+}$, $\chi_R := \chi_{\mathbb{R}^-}$, and $\chi_{\mathbb{R}^+}$, $\chi_{\mathbb{R}^-}$ respectively denote the characteristic functions on the intervals $(0,\infty)$ and $(-\infty,0)$. According to Theorem 2.1.2

$$\lim_{\lambda\to 0} \alpha(\lambda) + \beta(\lambda)\, e^{ix\lambda} = \frac{2}{3} + ix,$$

which implies that for sufficiently small $\varepsilon > 0$,

$$\left|\alpha(\lambda) + \beta(\lambda)\, e^{ix\lambda}\right| \lesssim_\varepsilon \begin{cases} 1, & |\lambda| \ge \varepsilon \\ 1 + |x|, & |\lambda| < \varepsilon \end{cases} \tag{3.13}$$

Further, since Theorem 2.1.2 also states that

$$|K^+(x-x';\lambda)| \lesssim 1 + \log_+\left(\frac{1}{|x-x'|}\right)$$

we see from (3.12) that

$$|G_\star^+(x-x';\lambda)| \lesssim 1 + |x-x'| + \log_+(1/|x-x'|) \tag{3.14}$$

where the implied constant is λ independent.

Thus, the operator norm of $T_\star$ is bounded by

$$\sup_{x\in\mathbb{R}} \langle x\rangle^{-1} \int_{\mathbb{R}} \left[1 + |x-x'| + \log_+\left(\frac{1}{|x-x'|}\right)\right] \langle x'\rangle \, |u(x')| \, \mathrm{d}x'. \tag{3.15}$$

To estimate (3.15), first note that since $\langle x\rangle \ge 1$ for $x \in \mathbb{R}$, we have $\langle x'\rangle \langle x\rangle^{-1} \le \langle x'\rangle$. Further,

$$\langle x\rangle^{-1} |x-x'| = \frac{|x-x'|}{\langle x\rangle \langle x'\rangle} \langle x\rangle^2 \lesssim \langle x'\rangle^2 .$$

and, lastly, $\langle x\rangle^{-1}\langle x'\rangle \lesssim 1$ whenever $|x-x'|\le 1$. As such,

$$\begin{aligned}\|T_\star\|_{\langle\,\cdot\,\rangle L^\infty\circlearrowleft} &\lesssim \int_{\mathbb{R}} \langle x'\rangle\,|u(x')|\,\mathrm{d}x' \\ &\qquad + \operatorname*{ess\,sup}_{x\in\mathbb{R}} \int_{|x-x'|\le 1} \log\left(\frac{1}{|x-x'|}\right)|u(x')|\,\mathrm{d}x' \\ &\lesssim \|u\|_X,\end{aligned}$$

by Remark 10. □

If we don't need to worry about non-zero λ, then we can improve Proposition 3.2.2 slightly by proving that $T_{\star,\lambda,u}$ is actually a bounded operator on (unweighted) essentially bounded, measurable functions. We do this next in Proposition 3.2.3.

Proposition 3.2.3. *For real $\lambda \neq 0$ and $u\in X$, the operators $T_{\star,\lambda,u}$ ($\star = L$, or R) map from $L^\infty(\mathbb{R})$ to $L^\infty(\mathbb{R})$ with*

$$\|T_\star\|_{L^\infty\to L^\infty} \lesssim_\lambda \|u\|_X, \tag{3.16}$$

where the implied constant depends on λ.

Proof. By repeating our work in the proof of Proposition 3.2.2, we obtain the following estimate

$$\begin{aligned}\|T_\star f\|_{L^\infty} &\lesssim_\lambda \|\langle x\rangle\,u\|_{L^1}\|f\|_{\langle x\rangle L^\infty} \\ &\qquad + \left[\operatorname*{ess\,sup}_{x\in\mathbb{R}}\left(\int_{|x-x'|\le 1}\log\left(\frac{1}{|x-x'|}\right)\langle x'\rangle\,|u(x')|\,\mathrm{d}y\right)\right],\end{aligned} \tag{3.17}$$

which holds for $\lambda\in(-\infty,0)\cup(0,\infty)$. Since $|\langle x\rangle\; u(x)| \le \left|\langle x\rangle^2\; u(x)\right|$, we see that $\|\langle x\rangle\,u\|_{L^1}\le\|u\|_X$. Hence, Proposition 3.2.3 follows from (3.17). □

Proposition 3.2.4. *The operators $T_{\star,\lambda,u}$ satisfy the asymptotic conditions*

$$\lim_{x\to-\infty}\langle x\rangle\,T_{L,\lambda,u}f(x) = \lim_{x\to+\infty}\langle x\rangle\,T_{R,\lambda,u}f(x) = 0 \tag{3.18}$$

for every real λ, $u\in X$ and $f\in\langle\,\cdot\,\rangle\,L^\infty(\mathbb{R})$. Alternatively stated, the limits

$$\lim_{x\to-\infty} T_{L,\lambda,u}f(x), \qquad \textit{and} \qquad \lim_{x\to+\infty} T_{R,\lambda,u}f(x)$$

converge to zero faster than $1/x$.

Proof. We prove T_L satisfies asymptotic conditions (3.18) noting that the corresponding proof for T_R is similar. Since G_L^+ experiences linear growth for $|x| \gg 1$ only when $\lambda = 0$ and is otherwise bounded for large $|x|$, we consider the $\lambda = 0$ case first.

Recall from Remark 1 in Section 2.1 that the function $K^+(x;\lambda)$ can be written in the form

$$K^+(x;\lambda) = e^{-\pi|x|}k(x;\lambda) \tag{3.19}$$

where $k(\,\cdot\,;\lambda) \in L^2(\mathbb{R})$ for all real λ, and k is uniformly bounded in λ. Consequently,

$$\begin{aligned} T_{L,0,u}f(x) = &\int_{-\infty}^{x} \left(i\frac{2}{3} - (x - x')\right) u(x')f(x')\,\mathrm{d}x' \\ &+ \int_{\mathbb{R}} e^{-\pi|x-x'|}u(x')f(x')\,\mathrm{d}x', \end{aligned}$$

which implies

$$|T_{L,0,u}f(x)| \lesssim I_1 + I_2,$$

where

$$\begin{aligned} I_1(x) &:= \int_{\mathbb{R}} \langle x - x'\rangle\,|u(x')|\,\langle x'\rangle\,\mathrm{d}x' \\ I_2(x) &:= \int_{\mathbb{R}} \langle x - x'\rangle^{-2}\,|k(x - x')||u(x')|\,\langle x'\rangle\,\mathrm{d}x', \end{aligned}$$

where we use the fact that $f \in \langle\,\cdot\,\rangle\, L^\infty(\mathbb{R})$ to estimate $f \lesssim \langle\,\cdot\,\rangle$ and note that that $e^{-\pi|x|} \lesssim \langle x\rangle^{-N}$ for all whole numbers N (in I_2, we select $N = 2$).

To bound I_1, we assume $x < 0$ and note that this implies $\langle x'\rangle \geq \langle x\rangle$ and $\langle x'\rangle^{-1} \leq \langle x\rangle^{-1}$ since the function $\langle\,\cdot\,\rangle$ is strictly decreasing on the interval $(\infty, 0)$. Further, it is straightforward to show that $\langle x - x'\rangle \leq \langle x\rangle + \langle x'\rangle \leq 2\,\langle x'\rangle \lesssim \langle x'\rangle$. Hence

$$|I_1(x)| \lesssim \int_{-\infty}^{x} \left(\langle x'\rangle^2\,\langle x'\rangle^{-3}\right)\left(\langle x'\rangle^3\,|u(x')|\right)\,\mathrm{d}x' \lesssim \langle x\rangle^{-1}\,\left\|\langle\,\cdot\,\rangle^3\,u\right\|_{L^1}.$$

Since the Dominated Convergence Theorem implies that the integral of $\langle \cdot \rangle^3 |u| \chi_{(-\infty,x)}$ goes to zero as $x \to -\infty$, we see that

$$\lim_{x\to-\infty} \langle x \rangle I_1(x) = 0.$$

Similarly,

$$\begin{aligned} |I_2(x)| &\lesssim \langle x \rangle^{-2} \int_{-\infty}^{x} \frac{\langle x \rangle^2}{\langle x - x' \rangle^2 \langle x' \rangle^2} |k(x-x')| \langle x' \rangle^2 |u(x')| \, \mathrm{d}x' \\ &\lesssim \langle x \rangle^{-2} \|k\|_{L^2} \|u\|_{L^{2,2}}, \end{aligned} \tag{3.20}$$

as

$$\frac{\langle x \rangle}{\langle x - x' \rangle \langle x' \rangle} \le \frac{1}{\langle x - x' \rangle} \le 1$$

when $x < 0$ and $x' < x$. Estimate (3.20) therefore implies

$$\lim_{x\to-\infty} \langle x \rangle I_2(x) = \lim_{x\to-\infty} \langle x \rangle I_1(x) = 0,$$

which in turn implies $T_{L,0,u}f(x)$ satifies the asymptotic condition (3.18)

Since the sum of the residue terms in G_L^+ is bounded (in x) for all fixed $\lambda \neq 0$, a slight modification of the above argument shows that $T_{L,\lambda,u}f(x)$ also satisfies (3.18) when $\lambda \neq 0$. □

3.3 λ-Differentiability

We begin this section by proving a useful variant of Young's inequality (Technical Lemma 3.3.1) that we use in a number of proofs in this dissertation. We then prove that that the operators $T_{\star,\lambda,u}$ are continuous in the spectral parameter λ (Proposition 3.3.2). The remainder of this section the focuses on proving the λ-differentiability of $G_\star^+$. We direclty use the results from all four propositions in this section in proving that the Jost solution boundary M_1^+ is differentiable in λ, and hence has a linearization in λ. The λ linearization of M_1^+ is ultimately used in the proof that the direct scattering map $\mathscr{D}$ is Lipschitz continuous.

Since the proof of the λ-continuity of $T_{\star,\lambda,u}$ calls Technical Lemma 3.3.1, we start with the proof of that lemma.

Technical Lemma 3.3.1. *For $f \in \langle \cdot \rangle^s L^1(\mathbb{R})$ and $\langle \cdot \rangle^s g \in L^\infty(\mathbb{R})$ the inequality*

$$\|f * g\|_{\langle \cdot \rangle^s L^\infty} \leq \|f\|_{\langle \cdot \rangle^s L^1} \| \langle \cdot \rangle^s g\|_{L^\infty} \tag{3.21}$$

holds for $s \geq 0$. Alternatively, if $\langle \cdot \rangle^s f \in L^1(\mathbb{R})$ and $g \in \langle \cdot \rangle^s L^\infty(\mathbb{R})$ the estimate

$$\|f * g\|_{\langle \cdot \rangle^s L^\infty} \leq \|\langle \cdot \rangle^s f\|_{L^1} \|g\|_{\langle \cdot \rangle^s L^\infty} \tag{3.22}$$

holds instead for $s \geq 0$.

Proof. It is straightforward to show that

$$\frac{\langle x' \rangle}{\langle x - x' \rangle \langle x \rangle} \leq 1,$$

for all $x, x' \in \mathbb{R}$, as

$$\left(\frac{\langle x' \rangle}{\langle x - x' \rangle \langle x \rangle} \right)^2 = \frac{1 + (x')^2}{1 + (x - x')^2 + x^2 + x^2 (x - x')^2} \leq \frac{1 + (x')^2}{1 + (x')^2}.$$

As such, we find for $s > 0$

$$\begin{aligned} \|f * g\|_{\langle \cdot \rangle^s L^\infty} &= \sup_{x \in \mathbb{R}} \langle x \rangle^{-s} \int_{\mathbb{R}} \left| f(x')\, g(x - x') \right| \mathrm{d}x' \\ &= \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} \left| \left[\langle x' \rangle^{-s} f(x') \right] \left[\langle x - x' \rangle g(x - x') \right] \right| \left(\frac{\langle x' \rangle}{\langle x - x' \rangle \langle x \rangle} \right)^s \mathrm{d}x' \\ &\leq \left\| \left[\langle \cdot \rangle^{-s} f \right] * \left[\langle \cdot \rangle^s g \right] \right\|_{L^\infty} \\ &\leq \|f\|_{\langle \cdot \rangle^s L^1} \| \langle \cdot \rangle^s g\|_{L^\infty} \end{aligned}$$

by Minkowski's integral inequality [3, Theorem 1.2.10]. If $s = 0$, then (3.21) automatically holds by [3, Theorem 1.2.10]. An analogous argument also verifies (3.22). □

Proposition 3.3.2. *For $u \in X \cap \langle \cdot \rangle^{-2} L^\infty(\mathbb{R})$, the operator $T_{\star,\lambda,u} : \langle \cdot \rangle L^\infty(\mathbb{R}) \to \langle \cdot \rangle L^\infty(\mathbb{R})$ given by*

$$T_{\star,\lambda,u} : f \mapsto \left[G_\star^+(\cdot; \lambda) \right] * (u\, f)$$

is continuous in the parameter $\lambda \in \mathbb{R}$ in the sense the limit

$$\lim_{h\to 0} \|T_{\star,\lambda+h,u} - T_{\star,\lambda,u}\|_{\langle\,\cdot\,\rangle L^\infty \circlearrowleft} = 0$$

holds pointwise for each fixed $\lambda \in \mathbb{R}$.

Proof. To simplify notation, let $T_{\star,\lambda,u}$ be denoted by T_λ. Then, we see from Technical Lemma 3.3.1 that

$$\begin{aligned} \|(T_{\lambda+h} - T_\lambda) f\|_{\langle\,\cdot\,\rangle L^\infty} &= \left\| \left[G_\star^+(\,\cdot\,,\lambda+h) - G_\star^+(\,\cdot\,,\lambda)\right] * uf \right\|_{\langle\,\cdot\,\rangle L^\infty} \\ &\le \left\|G_\star^+(\,\cdot\,,\lambda+h) - G_\star^+(\,\cdot\,,\lambda)\right\|_{\langle\,\cdot\,\rangle L^\infty} \|\langle\,\cdot\,\rangle\, uf\|_{L^\infty} \end{aligned} \tag{3.23}$$

for all $f \in \langle\,\cdot\,\rangle\, L^\infty(\mathbb{R})$ as $u \in X \cap \langle\,\cdot\,\rangle^{-2} L^\infty(\mathbb{R})$ implies $\langle\,\cdot\,\rangle\, u\, f \in \langle\,\cdot\,\rangle\, L^\infty(\mathbb{R})$. Noting that the argument in the proof of Proposition 3.3.3 (which does not depend on this Proposition) implies

$$\lim_{h\to\infty} \left\|G_\star^+(\,\cdot\,,\lambda+h) - G_\star^+(\,\cdot\,,\lambda)\right\|_{\langle\,\cdot\,\rangle L^\infty} = 0, \tag{3.24}$$

Proposition 3.3.2 follows from (3.23) and (3.24). □

Remark 11. The notation in proofs of Propositions 3.3.3 through 3.3.5 can get unnecessarily complicated. To avoid this, in these proof we use the notation $G(x, \lambda)$ and $G(\lambda)$ as stand-ins for $G_\star^+(x; \lambda)$.

Proposition 3.3.3. *For each fixed $x \neq 0$ and $\lambda \in \mathbb{R}$, the Green's function boundary $G_\star^+$ ($\star = L$, or R) is differentiable in the spectral parameter λ, and*

$$\frac{\partial}{\partial\lambda} G_\star^+(x;\lambda) = \frac{1}{2\pi} \int_{\Gamma_\star} e^{ix\xi} \left(\frac{\partial}{\partial\lambda} \frac{1}{p(\xi;\lambda)}\right) \mathrm{d}\xi.$$

Remark 12. Since

$$\frac{\partial}{\partial\lambda}\frac{1}{p(\xi;\lambda)} = \frac{\zeta'(\lambda)}{\Big(\xi - \zeta(\lambda)\big(1 - e^{-2\xi}\big)\Big)^2},$$

the function $\frac{\partial}{\partial\lambda}\frac{1}{p(\xi;\lambda)}$ decays exponentially to zero as $\xi \to -\infty$, and, for $\xi > 0$, decays like $1/\xi^2$. As such, the integral

$$\int_{\Gamma_\star} \left(\frac{\partial}{\partial\lambda}\frac{1}{p(\xi;\lambda)}\right) \mathrm{d}\xi.$$

converges absolutely.

Proof of Proposition 3.3.3. In accordance with Remark 11, we write $G^+_\star(x;\lambda)$ as either $G(x,\lambda)$ or as $G(\lambda)$. We further define G_h to be the difference quotient

$$G_h := \frac{G(\lambda+h)-G(\lambda)}{h}.$$

and seek to prove that $\lim_{h\to 0} G_h$ converges. For $|h|>0$ sufficiently small, we may assume by analyticity that $G(\lambda)$ and $G(\lambda+h)$ share the same contour of integration Γ. If $\lambda=0$, the contour Γ runs along with a single small semi-circular detour below the real axis to avoid passing through $\xi=0$ and $\xi=h$. In which case

$$G_h(\lambda) = \frac{1}{2\pi}\int_\Gamma e^{ix\xi}\,\frac{1}{h}\left[\frac{1}{p(\xi;\lambda+h)} - \frac{1}{p(\xi;\lambda)}\right] \mathrm{d}\xi = \int_\Gamma e^{ix\xi}\left(\frac{1}{p_\lambda(\xi)}\right)_h \mathrm{d}\xi,$$

where we define

$$\left(\frac{1}{p_\lambda(\xi)}\right)_h := \frac{1}{h}\left[\frac{1}{p(\xi;\lambda+h)} - \frac{1}{p(\xi;\lambda)}\right]$$

as the difference quotient of $1/p$ with respect to λ. Since

$$\left(\frac{1}{p_\lambda(\xi)}\right)_h = \frac{1}{h}\frac{\big[\zeta(\lambda+h)-\zeta(h)\big]\big(1-e^{-2\xi}\big)}{p(\xi;\lambda+h)\,p(\xi;\lambda)}$$

decays exponentially as $\xi\to-\infty$ and decays as $1/\xi^2$ for positive ξ,

$$\left(\frac{1}{p_\lambda(\xi)}\right)_h \in L^1(\Gamma)$$

for each fixed $h\neq 0$. Further, using the continuity of ζ and the reverse triangle inequality, we also have

$$\left|\left(\frac{1}{p_\lambda(\xi)}\right)_h\right| \lesssim_\lambda \frac{\big|1-e^{-2\xi}\big|}{|\xi|\big|\xi-\zeta(\lambda)\big(1-e^{-2\xi}\big)\big|} =: \iota(\xi).$$

Since ι is continuous in ξ on Γ and $\iota(\xi)=\mathcal{O}(1/\xi^2)$ for large $|\xi|$, one application of the Dominated Convergence Theorem completes this proof. □

Proposition 3.3.4. *The partial derivative $\frac{\partial}{\partial\lambda}G^+_\star$ ($\star = L$, or R) of the Green's function boundary value $G^+_\star$ lies in $\langle x\rangle^s L^1_x(\mathbb{R})$ for $s>3$ and all $\lambda\in\mathbb{R}$. If $\lambda\in\mathbb{R}$ and $\lambda\neq 0$, then $\frac{\partial}{\partial\lambda}G^+_\star \in \langle x\rangle^s L^1_x(\mathbb{R})$ for $s>2$.*

Proof. Define

$$g(\xi;\lambda) := e^{ix\xi}\,\frac{\partial}{\partial\lambda}\frac{1}{p},$$

Through direct computation, one can show

$$\operatorname{Res}_{\xi=0} g = \frac{2e^{2\lambda}\big(e^{2\lambda}-2\lambda-1\big)}{\big(1-e^{2\lambda}+2\lambda e^{2\lambda}\big)^2}, \tag{3.25}$$

$$\operatorname{Res}_{\xi=\lambda} g = \frac{2e^{2\lambda}-2-4\lambda e^{2\lambda}+i\big[x-2xe^{2\lambda}+xe^{4\lambda}+2x\lambda-2x\lambda e^{2\lambda}\big]}{\big(e^{2\lambda}-2\lambda-1\big)^2}\,e^{ix\lambda}, \tag{3.26}$$

and

$$\lim_{\lambda\to 0}\big[\operatorname{Res}_{\xi=0} g+\operatorname{Res}_{\xi=\lambda} g\big] = -\frac{1}{2}\,x^2+i\,\frac{1}{3}\,x. \tag{3.27}$$

In particular, equations (3.25), (3.26), and (3.27) imply

$$\begin{gathered}\big|\operatorname{Res}_{\xi=0} g\big| \lesssim_\lambda 1, \qquad \big|\operatorname{Res}_{\xi=\lambda} g\big| \lesssim_\lambda |x|,\\ \lim_{\lambda\to 0}\big[\operatorname{Res}_{\xi=0} g+\operatorname{Res}_{\xi=\lambda} g\big] = \mathcal{O}(x^2)\end{gathered} \tag{3.28}$$

Further, on our work in the proof of Lemma 3.3.3 also implies that

$$\frac{\partial}{\partial\lambda}\frac{1}{p} \in L^1_\xi(\mathbb{R}\pm i\pi). \tag{3.29}$$

As such, after applying the contour shift demonstrated in Figure 2.5 to the contour of integration for $\frac{\partial}{\partial\lambda}G_L^+$, Lemma 3.3.4 is an immediate consequence of (3.28) and (3.29). □

Proposition 3.3.5. *The difference quotient*

$$G_h(x;\lambda) := \frac{G_L^+(x;\lambda+h)-G_L^+(x;\lambda)}{h}$$

converges to $\frac{\partial}{\partial\lambda}G_L^+$ *in* $\langle x\rangle^s L^1_x(\mathbb{R})$ *for* $s>3$ *and all real* λ*. If* λ *is real and non-zero, then this convergence happens in* $\langle x\rangle^s L^1_x(\mathbb{R})$ *for* $s>2$.

Proof. Direct computation yields the following results

$$\begin{aligned}
\operatorname{Res}_{\xi=0}\left[e^{ix\xi}\left(\frac{1}{p_\lambda(\xi)}\right)_h\right] &= \frac{2e^{2\lambda}\big(h\,e^{2(\lambda+h)}+\lambda-e^{2h}(\lambda+h)\big)}{h\big(1+e^{2\lambda}(2\lambda-1)\big)\big(1+e^{2(\lambda+h)(2\lambda+2h-1)}\big)}\\
\operatorname{Res}_{\xi=\lambda}\left[e^{ix\xi}\left(\frac{1}{p_\lambda(\xi)}\right)_h\right] &= -\frac{1}{h}\frac{e^{2\lambda}-1}{e^{2\lambda}-2\lambda-1}\,e^{ix\lambda}\\
\operatorname{Res}_{\xi=\lambda+h}\left[e^{ix\xi}\left(\frac{1}{p_\lambda(\xi)}\right)_h\right] &= \frac{1}{h}\frac{e^{2(\lambda+h)}-1}{e^{2(\lambda+h)}-2(\lambda+h)-1}\,e^{ix(\lambda+h)}
\end{aligned}$$

As such, through further direct computation, we find the limits

$$\lim_{h\to 0}\operatorname{Res}_{\xi=0}\left\{\left(e^{ix\xi}\frac{\partial}{\partial\lambda}\frac{1}{p_\lambda(\xi)}\right)-\operatorname{Res}_{\xi=0}\left[e^{ix\xi}\left(\frac{1}{p_\lambda(\xi)}\right)_h\right]\right\}=0,$$

and

$$\lim_{h\to 0}\operatorname{Res}_{\xi=\lambda}\left\{\left(e^{ix\xi}\frac{\partial}{\partial\lambda}\frac{1}{p_\lambda(\xi)}\right)-\sum_{k\in\{\lambda,\lambda+h\}}\operatorname{Res}_{\xi=k}\left[e^{ix\xi}\left(\frac{1}{p_\lambda(\xi)}\right)_h\right]\right\}=0$$

hold pointwise for each $x\in\mathbb{R}$. Thus, since $\left(\frac{1}{p_\lambda(\xi)}\right)_h \in L^1_\xi(\mathbb{R}\pm i\pi)$ implies by Fourier theory that $\int_{\mathbb{R}\pm i\pi} e^{ix\xi}\left(\frac{1}{p_\lambda(\xi)}\right)_h \mathrm{d}\xi$ is continuous and bounded (in $x\in\mathbb{R}$), we may complete this proof by doing a contour shift and then applying Dominated Convergence to $\int_{\mathbb{R}} \langle x\rangle^{-s}\, G_h(x)\,\mathrm{d}x$. □

4.1 Introduction

In order to work with the integral equations (1.9) instead of the the linear spectral problem (with prescribed asymptotics) (1.4), we need to know that the two are equivalent in the sense that solutions to one solve the other and *vise versa*. As we explore further in Chapter 5, doing so requires us to first understand the analytic properties of the Green's functions $G^+_\star$ ($\star = L$, or R), taken as convolution operators, where we use $G^+_\star$ in accordance with Remark 2 as a shorthand to refer to both G^+_L and G^+_R. Specifically, we need to show the existence of functions $G_\star(z = x + iy; \lambda)$ analytic in the strip $\{z \in \mathbb{C} \ : \ 0 < \operatorname{Im} z < 2\}$ with respective lower and upper boundary values $G^+_\star$ and $G^-_\star$. As we see in Section 4.2, showing that $G^+_\star$ taken as a convolution operator extends analytically (in x) to the strip $\{z \in \mathbb{C} \ : \ 0 < \operatorname{Im} z < 2\}$ is straightforward. However, proving the existence of an upper boundary value $G^-_\star$ defined along the line $\operatorname{Im} z = 2$ is much more delicate and is the primary focus of this chapter. Indeed, the principle result of this chapter is summarized in the theorem below:

Theorem 4.1.1. *The Green's Functions $G^+_\star$ ($\star = L$, or R) extend to functions $G_\star$ analytic on the strip $\mathcal{S}_1 = \{z \in \mathbb{C} \ : \ 0 < \operatorname{Im} z < 2\}$. For $f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ ($1 < p \leq 2$), the limit*

$$\lim_{y \nearrow 2} G^+_\star(\,\cdot\, + iy) * f = G^-_\star * f \tag{4.1}$$

converges both pointwise almost everywhere (a.e.) and in $L^{p,1}$ for all real λ. If $\lambda \neq 0$, then the limit (4.1) *converges pointwise a.e. and in L^p. Moreover, $G^-_\star$ is a bounded* *convolution type operator on L^p given by*

$$G_L^- * f(x) = \mathfrak{C}(\,\cdot\,,2) * f(x) + Ef(x) - \frac{1}{2}f(x) \tag{4.2a}$$
$$+\, i\,\alpha(\lambda)\int_{-\infty}^{x} f(x')\,\mathrm{d}x' + i\,\beta(\lambda)\,e^{i\lambda x}\,e^{-2\lambda}\int_{-\infty}^{x} e^{i\lambda x'} f(x')\,\mathrm{d}x'$$
$$= \underbrace{\Big[\mathfrak{C}(\,\cdot\,,2) + \mathcal{R}_L(\,\cdot\,+i2;\lambda)\Big]}_{\textit{continuous operator}} * f(x) + \underbrace{\Big[E - \frac{1}{2}\Big]}_{\textit{singular operator}} f(x)$$

$$G_R^- * f(x) = \mathfrak{C}(\,\cdot\,,2) * f(x) + Ef(x) - \frac{1}{2}f(x) \tag{4.2b}$$
$$-\, i\,\alpha(\lambda)\int_{-\infty}^{x} f(x')\,\mathrm{d}x' - i\,\beta(\lambda)\,e^{i\lambda x}\,e^{-2\lambda}\int_{-\infty}^{x} e^{i\lambda x'} f(x')\,\mathrm{d}x'$$
$$= \underbrace{\Big[\mathfrak{C}(\,\cdot\,,2) - \mathcal{R}_R(\,\cdot\,+i2;\lambda)\Big]}_{\textit{continuous operator}} * f(x) + \underbrace{\Big[E - \frac{1}{2}\Big]}_{\textit{singular operator}} f(x)$$

where

$$Ef(x) := \frac{1}{2\pi i}\,\mathrm{p.\,v.}\int_{\mathbb{R}} \frac{e^{-\pi|x-x'|}}{x-x'} f(x')\,\mathrm{d}x',$$

the convolution operator $\mathfrak{C}$ is defined as

$$\mathfrak{C}(x,y) := \frac{1}{2\pi} e^{-\pi|x|}\, e^{-\operatorname{sign}(x)\, i\pi y}\int_{\mathbb{R}} e^{ix\xi}\rho\big(\xi, y, \operatorname{sign}(x)\big)\,\mathrm{d}\xi, \tag{4.3}$$

for

$$\rho\big(\xi, y, \operatorname{sign}(x);\lambda\big) := \begin{cases} \dfrac{e^{-y\xi}}{p(\xi;\lambda) + \operatorname{sign}(x) i\pi}, & \xi > 0 \\[2ex] \dfrac{1}{\zeta(\lambda)}\,\dfrac{\big(\zeta(\lambda) - \xi - \operatorname{sign}(x)\, i\pi\big)e^{(2-y)\xi}}{p(\xi;\lambda) + \operatorname{sign}(x)\, i\pi}, & \xi < 0. \end{cases} \tag{4.4}$$

and

$$\mathcal{R}_\star(x+iy;\lambda) := i\,\big[\alpha(\lambda) + \beta(\lambda)e^{i\lambda x}e^{-\lambda y}\big]\,\chi_\star \tag{4.5}$$

is the convolution operator defined in Remark 3 from Chapter 2.

Remark 13. As we see in Section 4.2, the effect of choosing the function space for f as $L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ for $p \in (1,2]$ is to ensure that $\|G_\star(\,\cdot\,+iy) * f\|_{L^\infty} \lesssim \|f\|_{L^1\cap L^p}$. That is, for fixed $y \in (0,2)$, the convolution operator $G_\star(\,\cdot\, + iy)$ is a bounded operator from $L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ into $L^\infty(\mathbb{R})$.

Given its resemblance to the Hilbert transform, we often refer to E in this document as the exponentially weighted Hilbert transform. Equation (4.2a) results from decomposing the operator K from equations (4.7) and (4.8) into a convolution operator R which is well behaived under the limit $y \nearrow 2$ plus a singular Cauchy transform like operator. The later is resposible for the $Ef(x) - \frac{1}{2}f(x)$ term in equation (4.2a).

In Chapter 5 as we show the equivalence of the linear spectral problem and the integral equations (1.9), we repeatedly make use of the fact that G_L^- can be decomposed as indicated in (4.2)—as a continuous operator plus a singular operator. More specifically, if M is a function analytic on the complex $\mathcal{S}_1$ with a lower boundary value $M^+ \in \langle \cdot \rangle\, L^\infty(\mathbb{R})$ so that

$$M(x+iy) = M_0(x) + G_L(\,\cdot\, + iy) * \big(u\, M^+\big)(x),$$

for some sufficiently reasonable forcing function M_0 and some $u \in X$, then, as a consequence of Theorem 4.1.1, we may decompose M as $M(z) = M_c(z) + M_s(z)$, where $M_c(z)$ has the continuous upper boundary value $M_c^-(x)$ given by

$$M_c^-(x) = M_0^-(x) + \left[\mathfrak{C}(\,\cdot\,, 2) + \mathcal{R}_R(\,\cdot\, + i2; \lambda)\right] * \left[u\, M_c^+\right](x)$$

and $M_s(z)$ has a "singular" upper boundary value $M_s^-(x)$ only in L^2 sense which is given by

$$M_s^-(x) = \left[E - \frac{1}{2}\right]\left[u\, M_c^+\right](x).$$

Of course, a function N analytic on $\mathcal{S}_1$ with an analogous property involving G_R (*i.e.* $N = N_0 + G_R * uN$) will also have the same sort of decomposition based on (4.2b). This property is the motivation for property (iv) in Definition 5.3.1.

As a final (informal) remark, throughout this chapter—and particularly in Sections 4.4 and 4.5—we repeatedly make use of results found in Loukas Grafakos' book

Classical Fourier Analysis ([3]). In the appendix titled Harmonic Analysis Results, we provide statements of these results without proof.

4.2 Analytic Extension to the Strip

A natural candidate for the analytic extension of $G_\star$ to the open strip $\mathcal{S}_1 = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < 2\}$ is

$$G_\star(x+iy) := \frac{1}{2\pi}\int_{\Gamma_\star} e^{ix\xi}\frac{e^{-y\xi}}{\xi - \zeta\left(1-e^{-2\xi}\right)}\,\mathrm{d}\xi, \qquad (\star = L, \text{ or } R) \tag{4.6}$$

which is a convergent integral for $0 < y < 2$ owing to the new factor $e^{-y\xi}$. A straightforward argument with the dominated convergence theorem shows that $G_\star(z)$ is continuous on the open strip $\mathcal{S}_1$. It then follows from Morera's theorem and (4.6) that $G_\star(z)$ is analytic in z.

Next we consider convolution of $G_\star(\,\cdot\, + iy)$ with L^1 functions. Since the integral (4.6) is absolutely convergent for y in compact subintervals of $(0,2)$, it follows that for any $f \in L^1(\mathbb{R})$, $G_\star(\,\cdot\, + iy) * f$ obeys the uniform in λ bound

$$\|G_\star(\,\cdot\, + iy;\lambda) * f\|_{\langle\,\cdot\,\rangle L^\infty} \lesssim_y \|f\|_{L^1}$$

where the implied constant has the same uniformity. Another application of Morera's theorem shows that, for any $f \in C_0^\infty(\mathbb{R})$, the convolution $G_\star(\,\cdot\, + iy) * f$ defines an analytic function of z in $\mathcal{S}_1$. Finally let $f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ for some $p \in (1,2]$, and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence from $C_0^\infty(\mathbb{R})$ converging to f in $L^1 \cap L^p$. Then $\left(G_\star(\,\cdot\, + iy) * f_n\right)(x)$ converges uniformly to $\left(G_\star(\,\cdot\, + iy) * f\right)(x)$ on compact subsets of $\mathcal{S}_1$, so $G_\star(\,\cdot\, + iy) * f$ is also analytic in $\mathcal{S}_1$.

It remains to show that $G_\star$ as a convolution operator has an upper boundary value $G_\star^-$ and to obtain an effective formula for $G_\star^-$. Using the "boxcar" contour in

Figure 2.5 again we can compute

$$\text{(4.7a)} \qquad G_L(x+iy;\lambda) = K(x+iy;\lambda) + i\big[\alpha(\lambda) + \beta(\lambda)e^{i\lambda x}e^{-\lambda y}\big]\chi_L(x)$$
$$= K(x+iy;\lambda) + \mathcal{R}_L(x+iy;\lambda)$$

$$\text{(4.7b)} \qquad G_R(x+iy;\lambda) = K(x+iy;\lambda) - i\big[\alpha(\lambda) + \beta(\lambda)e^{i\lambda x}e^{-\lambda y}\big]\chi_R(x)$$
$$= K(x+iy;\lambda) - \mathcal{R}_R(x+iy;\lambda)$$

Here

$$\text{(4.8)} \qquad K(x+iy;\lambda) := \frac{e^{-\pi|x|}\exp\big(-i\pi y\,\mathrm{sign}(x)\big)}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\frac{e^{-y\xi}}{p(\xi;\lambda)+i\pi\,\mathrm{sign}(x)}\,\mathrm{d}\xi$$

is defined by a convergent integral for $y \in (0,2)$, so that $K(x+iy)$ is actually a bounded continuous function. We can now study boundary values for the convolution of $G_L(\,\cdot\,+iy)$ with a function on the line as $y \nearrow 2$. Note that

$$K(x+iy;\lambda) = \frac{1}{2\pi}\int\limits_{\Sigma_{\mathrm{sign}(x)}} \frac{e^{i(x+iy)\xi}}{p(\xi)}\,\mathrm{d}\xi,$$

where $\Sigma_{\mathrm{sign}(x)}$ is the contour defined at the beginning of Subsection 2.4, as

$$i(x+iy)\big(\xi+i\pi\,\mathrm{sign}(x)\big) = \big(ix\xi - y\xi\big) + \big(-\pi x\,\mathrm{sign}(x) - i\pi y\,\mathrm{sign}(x)\big).$$

In our analysis of this limit, we first claim that for any $f \in L^1 \cap L^p$ $(1 < p \leq 2)$, we can rewrite the convolutions $G_\star * f$ as

$$\begin{aligned}
\text{(4.9a)} \qquad \big(G_L(\,\cdot\,+iy)*f\big)(x) &= i\alpha(\lambda)\int_{-\infty}^{x} f(x')\,\mathrm{d}x' \\
&\quad + i\beta(\lambda)\,e^{i\lambda x}e^{-\lambda y}\int_{-\infty}^{x} e^{-i\lambda x'} f(x')\,\mathrm{d}x' \\
&\quad + \int_{\mathbb{R}} \mathfrak{C}(x-x',y) f(x')\,\mathrm{d}x' \\
&\quad + \frac{e^{-i\pi y}}{2\pi i}\int_{-\infty}^{x} e^{-\pi|x-x'|}\frac{1}{(x-x')-i(2-y)} f(x')\,\mathrm{d}x' \\
&\quad + \frac{e^{i\pi y}}{2\pi i}\int_{x}^{\infty} e^{-\pi|x-x'|}\frac{1}{(x-x')-i(2-y)} f(x')\,\mathrm{d}x'
\end{aligned}$$

$$\begin{aligned}
\text{(4.9b)} \qquad \big(G_R(\,\cdot\,+iy)*f\big)(x) &= i\alpha(\lambda)\int_{-\infty}^{x} f(x')\,\mathrm{d}x' \\
&\quad + i\beta(\lambda)\,e^{i\lambda x}e^{-\lambda y}\int_{-\infty}^{x} e^{-i\lambda x'} f(x')\,\mathrm{d}x' \\
&\quad + \int_{\mathbb{R}} \mathfrak{C}(x-x',y) f(x')\,\mathrm{d}x' \\
&\quad + \frac{e^{-i\pi y}}{2\pi i}\int_{-\infty}^{x} e^{-\pi|x-x'|}\frac{1}{(x-x')-i(2-y)} f(x')\,\mathrm{d}x' \\
&\quad + \frac{e^{i\pi y}}{2\pi i}\int_{x}^{\infty} e^{-\pi|x-x'|}\frac{1}{(x-x')-i(2-y)} f(x')\,\mathrm{d}x'
\end{aligned}$$

where $\mathfrak{C}$ is as defined in equations (4.3) and (4.4) above. Indeed, one can see directly from (4.7)

$$\begin{aligned}
\big(G_L(\,\cdot\,+iy)*f\big)(x) &= \big(K(\,\cdot\,+iy)*f\big)(x) + \mathcal{R}_L(\,\cdot\,+iy)*f \\
\big(G_R(\,\cdot\,+iy)*f\big)(x) &= \big(K(\,\cdot\,+iy)*f\big)(x) + \mathcal{R}_R(\,\cdot\,+iy)*f.
\end{aligned}$$

Further, since

$$\begin{aligned}
\big(K(\,\cdot\,+iy)*f\big)(x) = \frac{e^{-i\pi y}}{2\pi i}&\int_{-\infty}^{x}\left(e^{-\pi|x-x'|}\int_{\mathbb{R}} e^{ix\xi}\frac{e^{-y\xi}}{p(\xi;\lambda)+i\pi}\,\mathrm{d}\xi\right) f(x')\,\mathrm{d}x' \\
+ \frac{e^{i\pi y}}{2\pi i}&\int_{x}^{\infty}\left(e^{-\pi|x-x'|}\int_{\mathbb{R}} e^{ix\xi}\frac{e^{-y\xi}}{p(\xi;\lambda)-i\pi}\,\mathrm{d}\xi\right) f(x')\,\mathrm{d}x'
\end{aligned}$$

the identity

$$\frac{e^{-y\xi}}{p(\xi,\lambda)\pm i\pi} = \frac{1}{\zeta(\lambda)}e^{(2-y)\xi}\chi_{\mathbb{R}^-}(\xi) + \rho\big(\xi, y, \pm 1; \lambda\big), \tag{4.10}$$

where $\chi_{\mathbb{R}^-}$ denotes the characteristic function of $(-\infty, 0)$, along with the Fourier identity

$$\int_{\mathbb{R}} e^{ix\xi}\frac{1}{\zeta(\lambda)}e^{(2-y)\xi}\chi_{\mathbb{R}^-}(\xi)\,\mathrm{d}\xi = \frac{1}{ix+(2-y)}$$

imply equations (4.9) hold.

To verify identity (4.10) note that

$$\begin{aligned}\zeta &= \big(-\zeta+\zeta\, e^{-2\xi}+\zeta\big)e^{2\xi}\\ &= \big[\xi-\zeta\big(1-e^{-2\xi}\big)\pm i\pi-\xi\mp i\pi\big]e^{2\xi},\end{aligned}$$

which implies

$$\begin{aligned}&\frac{1}{\zeta}e^{(2-y)\xi}+\frac{1}{\zeta}\frac{\big(\zeta-\xi\mp i\pi\big)e^{(2-y)\xi}}{\xi-\zeta(1-e^{2\xi})\pm i\pi}\\ &\qquad=\frac{1}{\zeta}\Big\{\big[\xi-\zeta\big(1-e^{-2\xi}\big)\pm i\pi-\xi\mp i\pi\big]e^{2\xi}\Big\}\frac{e^{-y\xi}}{p(\xi;\lambda)\pm i\pi}\\ &\qquad=\frac{e^{-y\xi}}{p(\xi;\lambda)\pm i\pi},\end{aligned}$$

as claimed.

By introducing the notation

$$\begin{aligned}\left(\mathcal{E}_\varepsilon f\right)(x) := \frac{e^{-i\pi(2-\varepsilon)}}{2\pi i}&\int_{-\infty}^{x}\frac{e^{-\pi|x-x'|}}{(x-x')-i\varepsilon}f(x')\,\mathrm{d}x'\\ +\frac{e^{i\pi(2-\varepsilon)}}{2\pi i}&\int_{x}^{\infty}\frac{e^{-\pi|x-x'|}}{(x-x')-i\varepsilon}f(x')\,\mathrm{d}x',\end{aligned}$$

equation (4.9) can be rewritten more simply as

$$\big(G_L(\,\cdot\,+iy)*f\big)(x) = \big[\mathfrak{C}(\,\cdot\,,y)+\mathcal{R}_L(\,\cdot\,+iy)\big]*f(x)+\mathcal{E}_{(2-y)}f(x) \tag{4.11a}$$

$$\big(G_R(\,\cdot\,+iy)*f\big)(x) = \big[\mathfrak{C}(\,\cdot\,,y)-\mathcal{R}_R(\,\cdot\,+iy)\big]*f(x)+\mathcal{E}_{(2-y)}f(x) \tag{4.11b}$$

Since the integrands of the residue terms

$$\mathcal{R}_L(\,\cdot\,+iy)*f(x) = i\alpha\int_{-\infty}^{x} f(x')\,\mathrm{d}x' + i\beta(\lambda)\,e^{\lambda(ix-y)}\int_{-\infty}^{x} e^{-i\lambda x'}f(x')\,\mathrm{d}x'$$
$$\mathcal{R}_R(\,\cdot\,+iy)*f(x) = i\alpha\int_{-\infty}^{x} f(x')\,\mathrm{d}x' + i\beta(\lambda)\,e^{\lambda(ix-y)}\int_{x}^{\infty} e^{-i\lambda x'}f(x')\,\mathrm{d}x'$$

in (4.11) do not involve y, they are certainly well behaved under the limit $y \nearrow 2$. Further, the purpose of decomposing the convolution operator G_L as shown in (4.11) is to isolate the singularity that results under the limit $y \nearrow 2$. Indeed, a cursory inspection of the convolution operator $\mathfrak{C}$ leads one to believe that it is well behaved under this limit, which is something we discuss further in Section 4.3.

The $\mathcal{E}_{(2-y)}f(x)$ terms in (4.11) captures the singular portion of K under the $y \nearrow 2$ limit. Understanding the behavior of these terms under this limit involves much more delicate analysis and is the subject of Sections 4.4 and 4.5.

Remark 14. That $\mathcal{R}_\star$ grows linearly for $\lambda = 0$ is the single reason why we can only guarantee the limit (4.1) converges as an L^p limit for functions with sufficient decay (*i.e.* $L^{p,1}$) and not for all L^p functions when $\lambda = 0$.

4.3 The Continuous Limit

With a little work, it is straightforward to observe that ρ decays exponentially as $|\xi| \to \infty$. So, since ρ is also bounded, the exponential decay of ρ means that ρ is certainly in $L^1(\mathbb{R})$ and thus $\check{\rho} \in L^\infty(\mathbb{R})$. While the singularity of ρ at $\xi = 0$ means that it is very unlikely ρ is L^1, the exponential factor $e^{-\pi|\,\cdot\,|}$ in $\mathfrak{C}(\,\cdot\,,y)$, where $2\pi\,\mathfrak{C}(x,y) = e^{\pi|x|}e^{-i\,\mathrm{sign}(x)\,\pi\,y}\,\rho(x)$ does make $\mathfrak{C}$ an L^1 function for all $y \in [0,2]$. As such, proving the L^p convergence of $\mathfrak{C}(\,\cdot\,,y)*f$ to $\mathfrak{C}(\,\cdot\,,2)*f$ as $y \nearrow 2$ boils down to using just two tools: [3, Theorem 1.2.10][1] and the Dominated Convergence Theorem. Proving the pointwise (*a.e.*) convergence of this limit comprises using uniform (in y)

[1]See the appendix titled "Harmonic Analysis Results."

estimates on $\mathfrak{C}$ and a density argument. We do all of this in the proof of the following result.

Theorem 4.3.1. *For $f \in L^p(\mathbb{R})$ with $1 < p < \infty$, the limit*

$$\lim_{y \nearrow 2} \big(\mathfrak{C}(\,\cdot\,, y) * f\big)(x) = \big(\mathfrak{C}(\,\cdot\,, 2) * f\big)(x) \tag{4.12}$$

holds both as a pointwise a.e. limit and as an L^p limit.

Proof. Recall that

$$\begin{aligned}\mathfrak{C}(x,y) &:= \frac{1}{2\pi} e^{-\pi|x|}\, e^{-\operatorname{sign}(x)\, i\pi y} \int_{\mathbb{R}} e^{ix\xi} \rho\big(\xi, y, \operatorname{sign}(x)\big)\, \mathrm{d}\xi \\ &= \frac{1}{2\pi} e^{-\pi|x|}\, e^{-\operatorname{sign}(x)\, i\pi y} \rho\big(x, y, \operatorname{sign}(x)\big)\end{aligned}$$

and

$$\rho\big(\xi, y, \operatorname{sign}(x); \lambda\big) := \begin{cases} \dfrac{e^{-y\xi}}{p(\xi;\lambda) + \operatorname{sign}(x) i\pi}, & \xi > 0 \\[2ex] \dfrac{1}{\lambda} \dfrac{\big(\lambda - \xi - \operatorname{sign}(x)\, i\pi\big) e^{(2-y)\xi}}{p(\xi;\lambda) + \operatorname{sign}(x)\, i\pi}, & \xi < 0. \end{cases}$$

As such, for each $x \in \mathbb{R}$,

$$\begin{aligned} &2\pi|\mathfrak{C}(x,y) - \mathfrak{C}(x,2)| \\ &\quad\le e^{-\pi|x|} \Big| e^{-\operatorname{sign}(x)\, i\pi y} - e^{-\operatorname{sign}(x)\, 2i\pi} \Big| \int_{\mathbb{R}} \big|\rho\big(\xi, y, \operatorname{sign}(x)\big)\big|\, \mathrm{d}\xi \\ &\qquad + e^{\pi|x|} e^{-\operatorname{sign}(x)\, 2i\pi} \int_{\mathbb{R}} \Big|\rho\big(\xi, y, \operatorname{sign}(x)\big) - \rho\big(\xi, 2, \operatorname{sign}(x)\big)\Big|\, \mathrm{d}\xi \end{aligned} \tag{4.13}$$

Now

$$\big|\rho\big(\xi, y, \operatorname{sign}(x); \lambda\big)\big| = \begin{cases} \big(p(\xi)^2 + \pi^2\big)^{-\frac{1}{2}}\, e^{-y\xi}, & \xi > 0 \\ \frac{1}{\lambda} \left(\frac{(\lambda-\xi)^2 + \pi^2}{p(\xi)^2 + \pi^2}\right)^{\frac{1}{2}} e^{(2-y)\xi}, & \xi < 0 \end{cases}$$

Hence $\frac{\partial}{\partial y}|r| < 0$ for $\xi > 0$ and $\frac{\partial}{\partial y}|r| > 0$ for $\xi < 0$, which implies

$$\begin{aligned} &\big|\rho\big(\xi, y, \operatorname{sign}(x); \lambda\big)\big| \\ &\qquad \le \big|\rho\big(\xi, y = 1, \operatorname{sign}(x); \lambda\big)\big| \chi_{\mathbb{R}^+} + \big|\rho\big(\xi, y = 2, \operatorname{sign}(x); \lambda\big)\big| \chi_{\mathbb{R}^-} \in L^1(\mathbb{R}), \end{aligned}$$

as $\left|\rho\big(\xi, y = 1, \operatorname{sign}(x); \lambda\big)\right|\chi_{\mathbb{R}^+}$ decays according to $O\big(e^{-\xi}\big)$ as $\xi \to +\infty$ and $\left|\rho\big(\xi, y = 2, \operatorname{sign}(x); \lambda\big)\right|\chi_{\mathbb{R}^-}$ decays according to $O\big(e^{-2\xi}\big)$ as $\xi \to -\infty$. Thus,

$$\int_{\mathbb{R}} \left|\rho\big(\xi, y, \operatorname{sign}(x)\big)\right| \mathrm{d}\xi \leq C$$

where $C > 0$ is some constant independent of y, and DCT $\implies$

$$\int_{\mathbb{R}} \Big|\rho\big(\xi, y, \operatorname{sign}(x)\big) - \rho\big(\xi, 2, \operatorname{sign}(x)\big)\Big| \mathrm{d}\xi \to 0$$

as $y \nearrow 2$, which, by (4.13), means $|\mathfrak{C}(x, y) - \mathfrak{C}(x, 2)| \to 0$ as $y \nearrow 2$ for each fixed $x \in \mathbb{R}$. Moreover, since $e^{-\pi|x|}\rho(x)$ is dominated by an L^1 function that is independent of y, the Dominated Convergence Theorem further implies

$$\|\mathfrak{C}(x, y) - \mathfrak{C}(x, 2)\|_{L^1} \to 0$$

as $y \nearrow 2$. Thus, by [3, Theorem 1.2.10]

$$\big\|\big(\mathfrak{C}(\,\cdot\,, y) - \mathfrak{C}(\,\cdot\,, 2)\big) * f\big\|_{L^p} \leq \|\mathfrak{C}(\,\cdot\,, y) - \mathfrak{C}(\,\cdot\,, 2)\|_{L^1}\|f\|_{L^p} \tag{4.14}$$

which implies the limit (4.12) holds as an L^p limit.

For pointwise limit, first take $h \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then

$$\big|\big(\mathfrak{C}(\,\cdot\,, y) - \mathfrak{C}(\,\cdot\,, 2)\big) * h\big| \leq \|h\|_{L^\infty}\|\mathfrak{C}(\,\cdot\,, y) - \mathfrak{C}(\,\cdot\,, 2)\|_{L^1} \to 0,$$

by our previous work.

We use a density argument to finish this proof. For $f \in L^p(\mathbb{R})$ with $p \geq 1$, we may approximate f by a bounded function $g \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ so that

$$\|f - g\|_{L^p} < r^p,$$

where $0 < r \ll 1$ is some arbitrarily small number. By Chebyshev's inequality and [3, Theorem 1.2.10],

$$\begin{aligned} & m\Big(\big\{x \in \mathbb{R} \,:\, \big(\mathfrak{C}(\,\cdot\,, y) - \mathfrak{C}(\,\cdot\,, 2)\big) * (f - g)(x) > r\big\}\Big) \\ &\leq \frac{1}{r^p}\big\|\big(\mathfrak{C}(\,\cdot\,, y) - \mathfrak{C}(\,\cdot\,, 2)\big) * (f - g)\big\|_{L^p} \\ &\leq \|\mathfrak{C}(\,\cdot\,, y) - \mathfrak{C}(\,\cdot\,, 2)\|_{L^1}\,, \end{aligned} \tag{4.15}$$

where $m(\,\cdot\,)$ denotes the Lebesgue measure on $\mathbb{R}$. Given

$$\begin{aligned}\big|\big(\mathfrak{C}(\,\cdot\,,y) - \mathfrak{C}(\,\cdot\,,2)\big) * f(x)\big| \\ \leq \big|\big(\mathfrak{C}(\,\cdot\,,y) - \mathfrak{C}(\,\cdot\,,2)\big) * (f-g)(x)\big| + \big|\big(\mathfrak{C}(\,\cdot\,,y) - \mathfrak{C}(\,\cdot\,,2)\big) * g(x)\big|,\end{aligned}$$

pointwise *a.e.* convergence follows, as (4.15) implies $\big(\mathfrak{C}(\,\cdot\,,y) - \mathfrak{C}(\,\cdot\,,2)\big) * (f-g)(x)$ converges pointwise *a.e.* to zero. □

4.4 Exponential Cauchy Transform

As mentioned at the end of Section 4.2, the crux of proving Theorem 4.1.1 involves analyzing the behavior of the $\mathcal{E}_{(2-y)}f(x)$ terms in (4.11), which we write here as

$$\begin{aligned}\left(\mathcal{E}_\varepsilon f\right)(x) := \frac{e^{-i\pi(2-\varepsilon)}}{2\pi i}\int_{-\infty}^{x} \frac{e^{-\pi|x-x'|}}{(x-x')-i\varepsilon} f(x')\,\mathrm{d}x' \\ + \frac{e^{i\pi(2-\varepsilon)}}{2\pi i}\int_{x}^{\infty} \frac{e^{-\pi|x-x'|}}{(x-x')-i\varepsilon} f(x')\,\mathrm{d}x',\end{aligned}$$

for $\varepsilon = 2-y$. The exponential coefficient in front of each integral in the above definition of $\mathcal{E}_\varepsilon f(x)$ add undesirable complexity to analyzing $\mathcal{E}_\varepsilon f(x)$. As such, we introduce the exponentially weighted Cauchy Transform E_ε defined by

$$\begin{aligned}\left(E_\varepsilon f\right)(x) &:= \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{e^{-\pi|x-x'|}}{(x-x')-i\varepsilon} f(x')\,dx' \\ &= \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-\pi|x-x'|}\frac{(x-x'+i\varepsilon)}{(x-x')^2-\varepsilon^2} f(x')\,dx'\end{aligned}$$

In Lemma 4.4.2, we establish that $\mathcal{E}_\varepsilon$ and E_ε share the same pointwise *i.e.* and L^p limits (as $\varepsilon \searrow 0$). In the remainder of this section we therefore focus our attention on the exponentially weighted Cauchy transform E_ε. In particular, we prove the following theorem:

Theorem 4.4.1. *For $f \in L^p(\mathbb{R})$ with $1 < p < \infty$, the limit*

$$\lim_{\varepsilon\searrow 0} E_\varepsilon f(x) = Ef(x) - \frac{1}{2}f(x)$$

holds both as a pointwise a.e. limit and as an L^p limit.

In the process of proving Theorem 4.4.1, we assume that the exponentially weighted Hilbert transform E is a bounded linear operator on $L^p(\mathbb{R})$—a fact that we later prove in Section 4.5. The boundedness of E in conjunction with Theorem 4.4, Theorem 4.3.1 and equation (4.2a) verify the existence and boundedness of the L^p operators $G_\star^-$.

Lemma 4.4.2. *For $f \in L^p(\mathbb{R})$, the limit*

$$(E_\varepsilon - \mathcal{E}_\varepsilon) f(x) \to 0 \tag{4.16}$$

holds pointwise a.e. and in L^p for $p \geq 1$.

Proof. We begin by proving the L^p convergence. With that goal in mind, we rewrite $(E_\varepsilon - \mathcal{E}_\varepsilon)(f)(x)$ as

$$\begin{aligned} 2\pi i \big(E_\varepsilon - \mathcal{E}_\varepsilon\big)\big(f(x)\big) &= \big(1 - e^{-i\pi(2-\varepsilon)}\big) \int_{-\infty}^{x} \frac{e^{-\pi|x-x'|}}{(x-x') - i\varepsilon} f(x')\,\mathrm{d}x' \\ &\quad + \big(1 - e^{i\pi(2-\varepsilon)}\big) \int_{x}^{\infty} \frac{e^{-\pi|x-x'|}}{(x-x') - i\varepsilon} f(x')\,\mathrm{d}x' \\ &= \left(\big(1 - e^{-i\pi(2-\varepsilon)}\big) \frac{e^{-\pi|\,\cdot\,|}}{(\,\cdot\,) - i\varepsilon} \chi_{\mathbb{R}^+}\right) * f \\ &\quad + \left(\big(1 - e^{i\pi(2-\varepsilon)}\big) \frac{e^{-\pi|\,\cdot\,|}}{(\,\cdot\,) - i\varepsilon} \chi_{\mathbb{R}^-}\right) * f, \end{aligned} \tag{4.17}$$

where $\chi_{R^\pm}$ respectively denote the characteristic functions for the negative and positive real half-lines, and use Theorem 1.2.10 from [3], which states that the L^p operator norm of a convolution operator is less than or equal to the L^1 norm of its kernel. Indeed,

$$\begin{aligned} \left\| \frac{e^{-\pi|\,\cdot\,|}}{(\,\cdot\,) - i\varepsilon} \chi_{\mathbb{R}^-} \right\|_{L^1(\mathbb{R})} &\leq \int_{-\infty}^{-1} e^{-\pi|x|}\,\mathrm{d}x + \int_{-1}^{0} \frac{1}{\sqrt{x^2 + \varepsilon^2}}\,\mathrm{d}x \\ &\leq \frac{1 - e^{-\pi}}{\pi} + \log\left[x + \sqrt{x^2 + \varepsilon^2}\right]_{-1}^{0} \\ &= \frac{1 - e^{-\pi}}{\pi} + \log(\varepsilon) - \log\left(\sqrt{1 + \varepsilon^2} - 1\right), \end{aligned}$$

from which it follows that

$$\left\| \left(1 - e^{i\pi(2-\varepsilon)}\right) \frac{e^{-\pi|\,\cdot\,|}}{(\,\cdot\,) - i\varepsilon}\chi_{\mathbb{R}^-} \right\|_{L^1(\mathbb{R})} \to 0, \qquad \text{as } \varepsilon \to 0. \tag{4.18}$$

A similar argument shows that

$$\left\| \left(1 - e^{-i\pi(2-\varepsilon)}\right) \frac{e^{-\pi|\,\cdot\,|}}{(\,\cdot\,) - i\varepsilon}\chi_{\mathbb{R}^+} \right\|_{L^1(\mathbb{R})} \to 0, \qquad \text{as } \varepsilon \to 0. \tag{4.19}$$

Thus, it follows from [3, Theorem 1.2.10] and Equations (4.18) and (4.19), (4.17) that

$$\left\| \left(E_\varepsilon - \mathcal{E}_\varepsilon\right) f \right\|_{L^p(\mathbb{R})} \le \frac{1}{2\pi} \left\| E_\varepsilon - \mathcal{E}_\varepsilon \right\|_{L^1(\mathbb{R})} \|f\|_{L^p(\mathbb{R})} \to 0$$

as $\varepsilon \to 0$, where $E_\varepsilon - \mathcal{E}_\varepsilon$ on the right-hand side of the above inequality is used to denote the kernel of the convolution operator $E_\varepsilon - \mathcal{E}_\varepsilon$.

To prove pointwise *a.e.* convergence, we prove the result for bounded functions and use Chebyshev's inequality to extend by density. Let $h \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$, for $p \ge 1$ Using Euler's formula, it is easy to see that

$$\begin{aligned} 2\pi i\left(E_\varepsilon - \mathcal{E}_\varepsilon\right)h(x) = \left[1 - \cos\left(\pi(2-\varepsilon)\right)\right] &\int_{\mathbb{R}} \frac{e^{-\pi|x-x'|}}{(x-x') - i\varepsilon} h(x')\, dx' \\ + i \sin\left(\pi(2-\varepsilon)\right) &\int_{-\infty}^{x} \frac{e^{-\pi|x-x'|}}{(x-x') - i\varepsilon} h(x')\, \mathrm{d}x' \\ - i \sin\left(\pi(2-\varepsilon)\right) &\int_{x}^{\infty} \frac{e^{-\pi|x-x'|}}{(x-x') - i\varepsilon} h(x')\, \mathrm{d}x'. \end{aligned}$$

Now,

$$\begin{aligned} &\left| \sin\left(\pi(2-\varepsilon)\right) \int_{-\infty}^{x} \frac{e^{-\pi|x-x'|}}{(x-x') - i\varepsilon} h(x')\, \mathrm{d}x' \right| \\ &\quad = \left| \sin\left(\pi(2-\varepsilon)\right) \left(\int_{\infty}^{x-1} + \int_{x-1}^{x} \right) \frac{e^{-\pi|x-x'|}}{(x-x') - i\varepsilon} h(x')\, \mathrm{d}x' \right| \\ &\quad \le \left| \sin\left(\pi(2-\varepsilon)\right) \right| \left(C + C \int_{x-1}^{x} \frac{dx'}{\sqrt{(x-x')^2 + \varepsilon^2}} \right) \\ &\quad \le \left| \sin\left(\pi(2-\varepsilon)\right) \right| \left(C + C \log|\varepsilon| + C \log\left| -1 + \sqrt{1^2 + \varepsilon^2} \right| \right) \end{aligned}$$

and

$$\begin{aligned} &\left| \sin\left(\pi(2-\varepsilon)\right) \int_{x}^{\infty} \frac{e^{-\pi|x-x'|}}{(x-x') - i\varepsilon} h(x')\, \mathrm{d}x' \right| \\ &\quad \le \left| \sin\left(\pi(2-\varepsilon)\right) \right| \left(C + C \log\left| 1 + \sqrt{1^2 + \varepsilon^2} \right| + C \log|\varepsilon| \right), \end{aligned}$$

where the constants above depend only on the L^∞ norm of h. It therefore follows from an application of L'Hôpital's rule that

$$\lim_{\varepsilon\to 0}\left|\left[1-\cos\left(\pi(2-\varepsilon)\right)\right]\int_{\mathbb{R}}\frac{e^{-\pi|x-x'|}}{(x-x')-i\varepsilon}h(x')\,dx'\right|=0$$
$$\lim_{\varepsilon\to 0}\left|\sin\left(\pi(2-\varepsilon)\right)\int_{-\infty}^{x}\frac{e^{-\pi|x-x'|}}{(x-x')-i\varepsilon}h(x')\,\mathrm{d}x'\right|=0$$
$$\lim_{\varepsilon\to 0}\left|\sin\left(\pi(2-\varepsilon)\right)\int_{x}^{\infty}\frac{e^{-\pi|x-x'|}}{(x-x')-i\varepsilon}h(x')\,\mathrm{d}x'\right|=0,$$

which implies

$$\lim_{\varepsilon\to 0}\left(E_\varepsilon-\mathcal{E}_\varepsilon\right)h(x)=0,$$

for every $h\in L^p(\mathbb{R})\cap L^\infty(\mathbb{R})$ $(p\geq 1)$ and $x\in\mathbb{R}$. The result therefore follows from a density argument analogous to the one given at the end of the proof for Theorem 4.3.1. □

The first piece to proving Theorem 4.4 is establishing that it holds for sufficiently "nice" functions. We do this next in Lemma 4.4.3.

Lemma 4.4.3. *Let $f\in\mathscr{S}(\mathbb{R})$ and use E to denote the operator given by*

$$Ef(x)=\frac{1}{2\pi i}\,\mathrm{p.\,v.}\int_{\mathbb{R}}\frac{f(x')e^{-\pi|x-x'|}}{x-x'}\,dx',$$

where we use $\mathscr{S}(\mathbb{R})$ to denote the space of all Schwartz class functions on $\mathbb{R}$. Then, for each $x\in\mathbb{R}$ the following pointwise limit holds

$$\lim_{\varepsilon\to 0}E_\varepsilon f(x)=Ef(x)-\frac{1}{2}f(x).$$

Proof. We follow Terence Tao's proof of the Plemelj formulae [16]. As in Tao's proof, we use translation invariance to take $x=0$ and reduce the proof to showing

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{f(x')e^{-\pi|x'|}}{-x'-i\varepsilon}\,dx'+\frac{1}{2}f(0)-\frac{1}{2\pi i}\int_{|x'|>\varepsilon}\frac{f(x')e^{-\pi|x'|}}{-x'}\,dx'=0$$

By multiplying by $2\pi i$ and introducing the change of variables $x'=\varepsilon w$, we further reduce the proof to showing

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}}f(\varepsilon w)e^{-\pi\varepsilon|w|}\left(\frac{1}{-w-i}-\frac{\chi_{|w|>1}}{-w}\right)dw-\pi i f(0)=0,\tag{4.20}$$

where $\chi_{|w|>1}$ denotes the characteristic function on the set $|w| > 1$. As Tao notes, direct computation yields

$$\int_{\mathbb{R}} \left(\frac{1}{-w-i} - \frac{\chi_{|w|>1}}{-w} \right) dw = \pi i.$$

Thus, (4.20) can be rewritten as

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} \left(f(\varepsilon w) e^{-\pi\varepsilon|w|} + f(0) \right) \left(\frac{1}{-w-i} - \frac{\chi_{|w|>1}}{-w} \right) dw = 0. \tag{4.21}$$

Since (4.21) holds by Dominated Convergence, the result follows. □

Remark 15. E is a bounded operator on $L^p(\mathbb{R})$ for $1 < p < \infty$—which is something we show in subsection 4.5 after discussing the pointwise and L^p convergence of E_ε.

Taking our lead from Lemma 4.4.3, we decompose E_ε as

$$(E_\varepsilon f)(x) = (\mathcal{E}_\varepsilon * f)(x) - \frac{1}{2}(\mathcal{P}_\varepsilon * f)(x), \tag{4.22}$$

where

$$\mathcal{E}_\varepsilon(y) := \frac{1}{2\pi i} \frac{y}{y^2+\varepsilon^2} e^{-\pi|y|}, \qquad \text{and} \qquad \mathcal{P}_\varepsilon(y) := \frac{1}{\pi} \frac{\varepsilon}{y^2+\varepsilon^2} e^{-\pi|y|}.$$

We define the truncated exponentially weighted Hilbert transform $E^{(\varepsilon)}$ by

$$\left(E^{(\varepsilon)} f\right)(x) := \frac{1}{2\pi i} \int_{|y|\geq\varepsilon} \frac{e^{-\pi|y|}}{y} f(x-y)\,\mathrm{d}y,$$

and note that by definition

$$(Ef)(x) := \lim_{\varepsilon \searrow 0} \left(E^{(\varepsilon)} f\right)(x)$$

for $f \in \mathscr{S}(\mathbb{R})$.

Remark 16. Before continuing, it is worth taking a brief respite to consider our strategy for the analysis which follows. In (4.22), we break E_ε into its (effective) real and imaginary parts $\mathcal{E}_\varepsilon$ and $\mathcal{P}_\varepsilon$ (which should both be thought of as convolution operators), respectively, and show they respectively converge both *a.e.* and in L^p to the exponentially waited Hilbert transform and the identity operator. Doing the former involves first showing that the convolution operator $\mathcal{E}_\varepsilon$ and the truncated exponentially weighted Hilbert transform $E^{(\varepsilon)}$ (when applied to an L^p function) share the same pointwise *a.e.* and L^p limits (see Theorem 4.4.5. We then show that pointwise *a.e.* and L^p limits $E^{(\varepsilon)}f \to Ef$ are actually well defined for general L^p functions f.

Since $\mathcal{P}_\varepsilon$ is essentially the Poisson kernel with an exponential weight, it is the easier of the two operators to understand. As such, we first turn our attention to understanding its pointwise *a.e.* and L^p limits.

Theorem 4.4.4. *The limit $\mathcal{P}_\varepsilon * f \to f$ $(\varepsilon \searrow 0)$ converges pointwise a.e. and in L^p for $f \in L^p(\mathbb{R})$, $p \geq 1$.*

Proof. Let $P_\varepsilon(y) = \frac{1}{\pi}\frac{\varepsilon}{y^2+\varepsilon^2}$ denote the Poisson kernel. Since the family P_ε is an approximate identity, to prove Theorem 4.4.4 it suffices by [3, Theorem 1.2.19] to consider the pointwise *a.e.* and L^p limits of $\left(P_\varepsilon - \mathcal{P}_\varepsilon\right) * f$.]

Since,

$$P_\varepsilon - \mathcal{P}_\varepsilon = \frac{1}{\pi}\frac{\varepsilon}{y^2+\varepsilon^2}\left(1 - e^{-\pi|y|}\right)$$

is radially symmetric and non-negative, to compute the L^1 norm of $P_\varepsilon - \mathcal{P}_\varepsilon$, it suffices to integrate $P_\varepsilon - \mathcal{P}_\varepsilon$ on the half-line $(0, \infty)$. Using integration by parts, we find

$$\int_0^\infty (P_\varepsilon - \mathcal{P}_\varepsilon)\,\mathrm{d}y = \frac{1}{\pi}\left(1 - e^{-\pi}\right)\arctan\left(\frac{y}{\varepsilon}\right)\Big|_0^\infty - \int_0^\infty \arctan\left(\frac{y}{\varepsilon}\right) e^{-\pi y}\, dy.$$

Hence, by Dominated Convergence,

$$\lim_{\varepsilon\to 0}\int_0^\infty (P_\varepsilon - \mathcal{P}_\varepsilon)\,\mathrm{d}y = 0,$$

which implies $\|P_\varepsilon - \mathcal{P}_\varepsilon\|_{L^1} \to 0$ as $\varepsilon \searrow 0$. As such, an argument analogous to the one in the proof of Lemma 4.4.2 allows us to complete this proof. □

Following the method employed in the proofs of [3, Theorem 4.1.12, Theorem 5.1.5] to study the Hilbert transform, we show that $E^{(\varepsilon)}$ converges *a.e.* and in L^p to

E and $\mathcal{E}_\varepsilon * f - E^{(\varepsilon)}(f) \to 0$ *a.e.* and in L^p. In doing so, we use [3, Theorem 1.2.21] and [3, Corollary 2.1.19], whose statements can be found in the Harmonic Analysis Results appendix.

Theorem 4.4.5. *Let* $1 < p < \infty$. *For any* $f \in L^p(\mathbb{R})$, *we have*

$$\mathcal{E}_\varepsilon * f - E^{(\varepsilon)}(f) \to 0$$

in L^p *and almost everywhere as* $\varepsilon \to 0$.

Proof. Let

$$Q_\varepsilon(y) = e^{-\pi|y|}\,\varepsilon^{-1}\psi(y/\varepsilon),$$

where

$$\psi(t) = \begin{cases} \dfrac{t}{t^2+1} - \dfrac{1}{t}, & |t| \geq 1 \\ \\ \dfrac{t}{t^2+1}, & |t| < 11. \end{cases}$$

As noted in [3], the function ψ has integral zero with a radially decreasing majorant Ψ given by

$$\Psi(t) = \begin{cases} \dfrac{t}{t^2+1}, & |t| \geq 1 \\ \\ 1, & |t| < 1. \end{cases}$$

Moreover, since the calculation

$$\varepsilon^{-1}\psi\left(\frac{y}{\varepsilon}\right) = \varepsilon^{-1}\left(\frac{\frac{y}{\varepsilon}}{\left(\frac{y}{\varepsilon}\right)^2 + 1^2} - \varepsilon\frac{\chi_{|y|\geq\varepsilon}}{y}\right) = \frac{y}{y^2+\varepsilon^2} - \frac{\chi_{|y|\geq\varepsilon}}{y}$$

implies

$$\mathcal{E}_\varepsilon * f - E^{(\varepsilon)}(f) = \frac{1}{2\pi i}\,Q_\varepsilon * f,$$

the result follows from Theorem [3, Theorem 1.2.21] and Corollary [3, Corollary 2.1.19] (with $c = 0$). □

To prove the L^p and pointwise *a.e.* convergence of the convolution operator $\mathcal{E}_\varepsilon$ to the exponentially weighted Hilbert transform E, it remains to show that $E^{(\varepsilon)}$ converges to E. We do so in Theorem 4.4.9. However, we must first establish the Cotlar type inequality in Lemma 4.4.6 for the maximal operator $E^* := \left(E^{(\varepsilon)}\right)^*$ associated with the with exponential Cauchy transform $\left\{E^{(\varepsilon)}\right\}$, as we use this result in the proof of Theorem 4.4.9.

Lemma 4.4.6 (Cotlar Type Inequality)**.** *Let E^* given by*

$$E^*f(x) := \left(E^{(\varepsilon)}\right)^* f(x) := \sup_{\varepsilon>0} \left\{\left|E^{(\varepsilon)}f(x)\right|\right\}.$$

denote the maximal operator associated with the operator family $\left\{E^{(\varepsilon)}\right\}$. If $f \in \mathscr{S}(\mathbb{R})$, then

$$E^*f(x) \le MEf(x) + CMf(x), \tag{4.23}$$

where C is independent of f and M denotes the Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{r>0} \left\{\frac{1}{B(0,r)} \int_{B(0,r)} |f(x-x')| \, \mathrm{d}x'\right\}.$$

In our proof of Lemma 4.4.6 we use the standard analysis result presented in Proposition 4.4.7.

Proposition 4.4.7. *Suppose $f \in L^1_{loc}(\mathbb{R}^n)$, where L^1_{loc} denotes the space of all locally L^1 integrable functions on $\mathbb{R}$. If $\phi : \mathbb{R}^n \to \mathbb{R}$ is nonnegative, radial, radially decreasing, and integrable, then*

$$\sup_{\varepsilon>0} \{|\phi_\varepsilon * f(x)|\} \le \|\phi\|_1 Mf(x), \tag{4.24}$$

where $\phi_\varepsilon(x) := \frac{1}{\varepsilon}\phi\left(\frac{x}{\varepsilon}\right)$.

Proof of Proposition 4.4.7. We first show (4.24) holds for simple functions, then extend (4.24) to arbitrary functions ϕ satisfying the the hypotheses of Proposition 4.4.7 by density using the Monotone Convergence Theorem. □

We are now ready to prove Lemma 4.4.6:

Proof of Lemma 4.4.6. It suffices to show

$$|E^{(\varepsilon)} f(x)| \leq MEf(x) + CMf(x)$$

for every $\varepsilon > 0$. Choose $\phi \in \mathscr{S}(\mathbb{R})$ satisfying the hypotheses of Proposition 4.4.7 so that $\|\phi\|_1 = 1$ and $\operatorname{supp}\phi = \left(-\frac{1}{2}, \frac{1}{2}\right)$. We decompose $\chi_{\{|\,\cdot\,|>\varepsilon\}} \frac{e^{-\pi|\,\cdot\,|}}{(\,\cdot\,)}$ (the integrand of the truncated exponentially weighted Hilbert transform) as

$$\chi_{\{|\,\cdot\,|>\varepsilon\}} \frac{e^{-\pi|\,\cdot\,|}}{(\,\cdot\,)} = E\phi_\varepsilon + \left(\frac{e^{-\pi|\,\cdot\,|}}{(\,\cdot\,)} \chi_{\{|\,\cdot\,|>\varepsilon\}} - E\phi_\varepsilon\right). \tag{4.25}$$

Thus, by taking the convolution of both sides of (4.25) with f, we find

$$|E^{(\varepsilon)} f(x)| \leq \frac{1}{2\pi} |(E\phi_\varepsilon) * f(x)| + \frac{1}{2\pi} \left|\frac{e^{-\pi|\,\cdot\,|}}{(\,\cdot\,)} \chi_{\{|\,\cdot\,|>\varepsilon\}} - E\phi_\varepsilon\right| * |f|(x). \tag{4.26}$$

Note that by using two applications of Fubini's Theorem in conjunction with two applications of Dominated Convergence, one can show $(E\phi_\varepsilon) * f(x) = \phi_\varepsilon * (Ef)(x)$. As such, we therefore see from Proposition 4.4.7 that first term on the right-hand side of (4.26) satisfies

$$\frac{1}{2\pi} |(E\phi_\varepsilon) * f(x)| \leq MEf(x).$$

As we consider the second term in the right-hand side of (4.26), we initially take $\varepsilon = 1$ and examine the case $|\,\cdot\,| < 1$ and $|\,\cdot\,| \geq 1$ separately. Assume $|w| \geq 1$ and observe that

$$\begin{aligned}\left|\frac{e^{-\pi|w|}}{w} - E\phi(w)\right| &= \left|\frac{e^{-\pi|w|}}{w} - \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(x') \frac{e^{-\pi|w-x'|}}{w-x'} \, \mathrm{d}x'\right| \\ &\leq \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(x') \left|\frac{e^{-\pi|w|}}{w} - \frac{e^{-\pi|w-x'|}}{w-x'}\right| \mathrm{d}x'.\end{aligned}$$

Since $x' \in \operatorname{supp}\phi$ only when $|x'| \leq \frac{1}{2}$, we have

$$\left|e^{-\pi|w|}(w-x') - e^{-\pi|w-x'|}(w)\right| \leq 2|w| g(w) + e^{-\pi|w|}|x'|,$$

where

$$g(w) = \begin{cases} e^{-\pi\left|w+\frac{1}{2}\right|}, & w \geq 0 \\ e^{-\pi\left|w-\frac{1}{2}\right|}, & w < 0. \end{cases}$$

Hence

$$\left|\frac{e^{-\pi|w|}}{w} - \frac{e^{-\pi|w-x'|}}{w-x'}\right| \leq \frac{2|w|\, g(w) + e^{-\pi|w|}|x'|}{|w|\,|w-x'|}.$$

Further,

$$\frac{|w|}{|w-x'|} \leq \frac{|w|}{\left|w - \frac{1}{2}\mathrm{sign}(w)\right|} \leq 2,$$

which implies

$$\frac{1}{|w||w-x'|} \leq \frac{2}{w^2},$$

and

$$\left|\frac{e^{-\pi|w|}}{w} - \frac{e^{-\pi|w-x'|}}{w-x'}\right| \leq \left(\frac{4|w|g(w)}{w^2} + \frac{2e^{-\pi|w|}|x'|}{w^2}\right).$$

As such,

$$\left|\frac{e^{-\pi|w|}}{w-i\varepsilon} - E\phi_\varepsilon(w)\right| \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(x') \left(\frac{4|w|g(w)}{w^2} + \frac{2e^{-\pi|w|}|x'|}{w^2}\right) \mathrm{d}x' \leq \frac{4|w|g(u) + e^{-\pi|w|}}{w^2} \leq \frac{C}{w^2},$$

for $|w| \geq 1$.

On the other hand, for $|w| < 1$,

$$\begin{aligned} \left|\frac{e^{-\pi|w|}}{w}\chi_{|w|>1} - E\phi(w)\right| &= \left| - E\phi(w)\right| \\ &= \left|\mathrm{p.\,v.}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\phi(w-x')}{x'} e^{-\pi|x'|}\,\mathrm{d}x'\right| \\ &= \left|\mathrm{p.\,v.}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\phi(w-x') - \phi(w)}{x'} e^{-\pi|x'|}\,\mathrm{d}x'\right| \\ &\leq C\|\phi'\|_{L^\infty} \\ &= C, \end{aligned}$$

as

$$\mathrm{p.\,v.}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\phi(w)}{x'} e^{-\pi|x'|}\,\mathrm{d}x = 0.$$

Combining the two cases, it follows from Proposition 4.4.7

$$\left|\frac{e^{-\pi|\,\cdot\,|}}{(\,\cdot\,)} - E\phi\right| * |f|(x) \leq CMf(x). \tag{4.27}$$

Finally, to verify that (4.27) holds for arbitrary ε $(0 < \varepsilon < 1)$, we use a dilation argument. Define h and g by

$$h(w,\varepsilon) := \left|\frac{e^{-\pi|w|}}{w}\,\chi_{|w|>\varepsilon} - E\phi_\varepsilon(w)\right|, \qquad \text{and} \qquad g(w) := \left|\frac{e^{-\pi|w|}}{w}\right|\chi_{|w|>1} + |E\phi(w)|\,.$$

Then

$$g_\varepsilon(w) = \frac{1}{\varepsilon}g\left(\frac{w}{\varepsilon}\right) = \left|\frac{e^{-\pi|w/\varepsilon|}}{w}\right|\chi_{|w|>\varepsilon} + |E\phi_\varepsilon(w)|\,.$$

For $f \in \mathscr{S}(\mathbb{R})$ and $f^\varepsilon(x) := f(\varepsilon x)$,

$$\begin{aligned} g * f^\varepsilon\left(\varepsilon^{-1}x\right) &= \int_{\mathbb{R}} f(\varepsilon y)\, g\left(\varepsilon^{-1}x - y\right)\, \mathrm{d}y \\ &= \int_{\mathbb{R}} f(y)\varepsilon^{-1} g\left(\varepsilon^{-1}x - \varepsilon^{-1}y\right)\, \mathrm{d}y \\ &= g_\varepsilon * f(x), \end{aligned}$$

and

$$\begin{aligned} Mf^\varepsilon(x) &= \sup \frac{1}{|B(0,r)|}\int_{B(0,r)} |f(\varepsilon x - \varepsilon y)|\, \mathrm{d}y \\ &= \sup \frac{1}{|B(0,\varepsilon r)|}\int_{B(0,\varepsilon r)} |f(\varepsilon x - \varepsilon y)|\, \mathrm{d}y \\ &= Mf(\varepsilon x). \end{aligned}$$

Hence, by (4.27)

$$h(\,\cdot\,,\varepsilon) * |f|(x) \leq g_\varepsilon * |f|(x) = g * |f^\varepsilon|(\varepsilon^{-1}x) \leq CMf^\varepsilon(\varepsilon^{-1}x) = CMf(x),$$

from which the result follows. □

One final tool we need to prove Theorem 4.4.9 is Theorem 2.1.14 from [3], which we present without proof below in Theorem 4.4.8

Theorem 4.4.8 (Theorem 2.1.14 from [3]). *Let (X, μ) and (Y, ν) be measure spaces and let $0 < p, q < \infty$. Suppose for every $\varepsilon > 0$, T_ε is a linear operator defined on $L^P(X, \nu)$ with values in the set of measurable functions on Y, and D is a dense subspace of $L^p(X)$. Define a sublinear operator*

$$T^*(f)(x) := \sup_{\varepsilon > 0} |T_\varepsilon(f)(x)| .$$

Suppose that for some $B > 0$ and all $f \in L^p(X)$ we have

$$\|T^*(f)\|_{L^{q,\infty}} \leq B\|f\|_{L^p}$$

and that for all $f \in D$

$$\lim_{\varepsilon \to 0} T_\varepsilon(f) = T(f) \tag{4.28}$$

exists and is finite ν-a.e. (and defines a linear operator on D). Then for all functions f in $L^p(X)$ the limit (4.28) *exists and is finite ν-a.e., and defines a linear operator T on $L^p(X)$ (uniquely extending T defined on D) that satisfies*

$$\|T(f)\|_{L^{q,\infty}} \leq B\|f\|_{L^p}.$$

Theorem 4.4.9. *For all p $(1 < p < \infty)$ there exists a constant C_p depending only on p such that*

$$\|E^* f\|_{L^p} \leq C_p\|f\|_{L^p}, \qquad \forall f \in L^p(\mathbb{R}). \tag{4.29}$$

Moreover, for all $f \in L^p(\mathbb{R})$, $E^{(\varepsilon)} f$ converges to Ef pointwise a.e. and in L^p.

Proof. Inequality (4.29) is an immediate consequence of Lemma 4.4.6, Theorem 4.5.4 from Subsection 4.5, of the fact that convergence holds for Schwartz class functions, and of Theorem 4.4.8. The L^p convergences follows from the almost every pointwise convergence and the dominated convergence theorem combined with the Cotlar inequality (4.23). □

4.5 Boundedness of the Exponentially Weighted Hilbert Transform

Proving that the exponentially weighted Hilbert transform E is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$ is a multistep process in which we first compute the Fourier multiplier of E (Lemma 4.5.1), use the Fourier multiplier of E to show that E is strong type $(2,2)$ (Theorem 4.5.2), prove E is weak type $(1,1)$ (Theorem 4.5.3), and then use these results to ultimately prove E is strong type (p,p) for $p > 1$ (Theorem 4.5.4). Note that an operator is said to be weak type (p,q) if it is a bounded operator from L^p to *weak*-L^q (*i.e.* $L^{q,\infty}$) and strong type (p,q) if it is a bounded operator from L^p to L^q. For a defintion of the space $L^{p,\infty}$, please see the "Harmonic Analysis Results" appendix.

Lemma 4.5.1. *The Exponentially weighted Hilbert transform E has Fourier multiplier*

$$m_E(\xi) = \frac{1}{\pi}\arctan(\xi/\pi).$$

Proof. In order to compute the Fourier multiplier of E, it suffices to compute the Fourier transform of the function $x^{-1}e^{-\pi|x|}$, which is $2\pi i$ times the convolution kernel of E. Letting θ denote the Heaviside function, we write $x^{-1}e^{-\pi|x|}$ as

$$\frac{e^{\pi|x|}}{x} = \frac{e^{\pi x}}{x}\theta(-x) + \frac{e^{-\pi x}}{x}\theta(x), \tag{4.30}$$

and compute the Fourier transform of two terms on the right-hand side of (4.30) separately. By further splitting $\frac{e^{\pi x}}{x}\theta(-x)$ into even and odd parts, we obtain

$$\begin{aligned}\mathcal{F}\left(\frac{e^{\pi(\,\cdot\,)}}{(\,\cdot\,)}\theta(-\,\cdot\,)\right) &= \mathcal{F}^{(c)}\left(\frac{e^{\pi(\,\cdot\,)}\theta(-\,\cdot\,) - e^{-\pi(\,\cdot\,)}\theta(\,\cdot\,)}{2(\,\cdot\,)}\right)(\xi) \\ &\quad + i\,\mathcal{F}^{(s)}\left(\frac{e^{\pi(\,\cdot\,)}\theta(-\,\cdot\,) + e^{-\pi(\,\cdot\,)}\theta(\,\cdot\,)}{2(\,\cdot\,)}\right)(\xi),\end{aligned} \tag{4.31}$$

where $\mathcal{F}^{(c)}$ and $\mathcal{F}^{(s)}$ respectively denote the Fourier cosine and Fourier sine transformations. Direct computation yields

$$\begin{aligned}\mathcal{F}^{(c)}\left(\frac{e^{\pi(\,\cdot\,)}\theta(-\,\cdot\,)-e^{-\pi(\,\cdot\,)}\theta(\,\cdot\,)}{2(\,\cdot\,)}\right)(\xi) &= \frac{1}{\pi}\int_0^\infty \frac{e^{\pi\,x}\theta(-x)-e^{-\pi\,x}\theta(x)}{2x}\cos(\xi x)\,\mathrm{d}x \\ &= \frac{1}{2}\big(\log(\xi^2+\pi^2)+2\gamma\big)\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}^{(s)}\left(\frac{e^{\pi(\,\cdot\,)}\theta(-\,\cdot\,)+e^{-\pi(\,\cdot\,)}\theta(\,\cdot\,)}{2(\,\cdot\,)}\right)(\xi) &= \frac{1}{2\pi}\int_0^\infty \frac{e^{\pi\,x}\theta(-x)+e^{-\pi\,x}\theta(x)}{2x}\sin(\xi x)\,\mathrm{d}x \\ &= \frac{1}{2}\big(\arctan(\xi/\pi)\big),\end{aligned}$$

where γ denotes the Euler-Mascheroni constant, which implies

$$\mathcal{F}\left(\frac{e^{\pi(\,\cdot\,)}}{(\,\cdot\,)}\theta(-\,\cdot\,)\right)(\xi) = \gamma + \frac{1}{2}\log(\xi^2+\pi^2) + i\,\arctan(\xi/\pi). \tag{4.32}$$

A similar computation also shows

$$\mathcal{F}\left(\frac{e^{-\pi(\,\cdot\,)}}{(\,\cdot\,)}\theta(\,\cdot\,)\right)(\xi) = -\gamma - \frac{1}{2}\log(\xi^2+\pi^2) + i\,\arctan(\xi/\pi), \tag{4.33}$$

from which the result follows. □

Theorem 4.5.2. *E is strong type $(2,2)$.*

Proof. This result is an immediate consequence of Lemma 4.5.1, Plancherel's Theorem, and the density of $\mathscr{S}(\mathbb{R})$ in $L^2(\mathbb{R})$. □

Theorem 4.5.3. *E is weak type $(1,1)$. That is, E is a bounded operator from $L^1(\mathbb{R})$ into $L^{1,\infty}(\mathbb{R})$, where $L^{1,\infty}$ denotes weak L^1.*

Proof. Fix $\Lambda > 0$, let f be Schwartz class, and assume without loss of generality that $f \in \mathscr{S}(\mathbb{R})$ is real-valued and nonnegative (otherwise, we can decompose f in the appropriate pieces). Let $\{I_j\}$ be the sequence sequence of dyadic intervals in the

Calderón-Zygmund decomposition of f at height Λ. For $\Omega := \bigcup_j I_j$, define

$$(4.34)\qquad g(x) := \begin{cases} f(x), & x \notin \Omega \\ \dfrac{1}{|I_j|}\displaystyle\int_{I_j} f, & x \in I_j \end{cases}, \qquad \text{and} \qquad b(x) := \sum_j b_j(x),$$

where

$$b_j(x) = \left(f(x) - \frac{1}{|I_j|}\int_{I_j} f \right) \chi_{I_j}(x).$$

Note that $f = g + b$ and

$$(4.35)\qquad g(x) \leq 2\Lambda \quad \forall x \in \mathbb{R}.$$

To show that Eg and Eb are well defined, it suffices to bound $d_{Eg}(\Lambda)$ and $d_{Eb}(\Lambda)$ in terms of only Λ and $\|f\|_1$, where the notation

$$d_h(\Lambda) := \left|\left\{x \in \mathbb{R} \,:\, |f(x)| > \Lambda\right\}\right|$$

is used to denote the *distributional function* of a function h. Using Chebyshev's inequality, the L^2 boundedness of E found in Theorem 4.5.3 and equations (4.35) and (4.34), we find

$$(4.36)\qquad \begin{aligned} d_{Eg}(\Lambda) &\leq \frac{1}{\Lambda^2}\int_{\mathbb{R}} Eg(x)^2 \,\mathrm{d}x \\ &\leq \frac{C}{\Lambda^2}\int_{\mathbb{R}} g(x)^2 \,\mathrm{d}x \\ &\leq \frac{C}{\Lambda^2}\int_{\mathbb{R}} g(x) \,\mathrm{d}x \\ &= \frac{C}{\Lambda^2}\left(\int_{\Omega} g + \int_{\mathbb{R}\setminus\Omega} g\right) \\ &\leq \frac{C}{\Lambda}\|f\|_1. \end{aligned}$$

On the other hand, for Eb, let $\Omega^* = \bigcup_j 2I_j$. Using the Calderón-Zygmund Covering Lemma and Chebyshev's inequality we find

$$(4.37)\qquad d_{Eb}(\Lambda) \leq |\Omega^*| + |\{x \notin \Omega^* \,:\, |Eb(x)| > \Lambda\}| \leq \frac{2}{\Lambda}\|f\|_1 + \frac{1}{\Lambda}\int_{\mathbb{R}\setminus\Omega^*} |Eb(x)| \,\mathrm{d}x.$$

So, to show that Eb is well defined, we need to bound the integral on the right hand side of the above inequality by $\|f\|_1$. To that end, note that if $x \notin \Omega^*$, then for each j, $x \notin 2I_j$ and

$$Eb_j(x) = \frac{1}{2\pi} p.v. \int_{\mathbb{R}} \frac{e^{-\pi|x-x'|}}{x-x'} b_j(x')\,\mathrm{d}x' = \frac{1}{2\pi} \int_{I_j} \frac{e^{-\pi|x-x'|}}{x-x'} b_j(x')\,\mathrm{d}x < \infty,$$

as $\operatorname{supp} b_j \subseteq I_j$. Since E is a tempered distribution (Lemma 4.5.1) and $f \in \mathscr{S}(\mathbb{R})$ means that $Ef \in L^2(\mathbb{R})$ and, hence $\sum_j Eb_j$ converges to Eb in the L^2 norm, it follows that

$$|Eb(x)| \le \sum_j |Eb_j(x)| \quad a.e.$$

As such, proving Eb is well defined reduces to showing

$$\int_{\mathbb{R}\setminus\Omega^*} \sum_j |Eb_j(x)|\,\mathrm{d}x \le C\|f\|_1. \tag{4.38}$$

If we let c_j denote the center of I_j, then, for $x \notin \Omega^*$, since b_j has zero average

$$\begin{aligned}
\left| \int_{I_j} \frac{e^{-\pi|x-x'|}}{x-x'} b_j(x')\,\mathrm{d}x' \right| &= \left| \int_{I_j} e^{-\pi|x-x'|} \left(\frac{b_j(x')}{x-x'} - \frac{b_j(x')}{x-c_j} \right) \mathrm{d}x' \right| \\
&\le \int_{I_j} e^{-\pi|x-x'|} \left| \frac{b_j(x')(x'-c_j)}{(x-x')(x-c_j)} \right| \mathrm{d}x' \\
&\le \int_{I_j} |b_j(x')| \frac{|I_j|}{(x-c_j)^2}\,\mathrm{d}x'
\end{aligned}$$

as $|x-x'| \ge |x-c_j|/2$ and $|x'-c_j| \le |I_j|/2$. Moreover,

$$\int_{\mathbb{R}\setminus\Omega^*} \frac{|I_j|}{(x-c_j)^2}\,\mathrm{d}x' \le \int_{\mathbb{R}\setminus I_j} \frac{|I_j|}{(x-c_j)^2}\,\mathrm{d}x' \le 4, \tag{4.39}$$

and so, by Fubini's Theorem,

$$\begin{aligned}
\sum_j \int_{\mathbb{R}\setminus\Omega^*} |Eb_j(x)|\,\mathrm{d}x &\le \frac{1}{2\pi} \sum_j \int_{\mathbb{R}\setminus\Omega^*} \left| \int_{I_j} e^{-\pi|x-x'|} \frac{b_j(x')}{x-x'}\,\mathrm{d}x' \right| \mathrm{d}x \\
&\le \frac{1}{2\pi} \sum_j \int_{\mathbb{R}\setminus\Omega^*} \int_{I_j} |b_j(x')| \frac{|I_j|}{(x-c_j)^2}\,\mathrm{d}x'\,\mathrm{d}x \\
&\le \frac{2}{\pi} \sum_j \int_{I_j} |b_j(x')|\,\mathrm{d}x' \\
&\le \frac{4}{\pi}\|f\|_1.
\end{aligned} \tag{4.40}$$

Putting everything together, we find

$$d_{Ef}(\Lambda) \leq d_{Eg}\left(\frac{\Lambda}{2}\right) + d_{Eb}\left(\frac{\Lambda}{2}\right) \leq \frac{C}{\Lambda}\|f\|_1, \tag{4.41}$$

where $C > 0$ is independent of Λ and f. Since f is Schwartz class, we can therefore extend the inequality (4.41) to L^1 *via* density to conclude that E is weak (1,1). □

Theorem 4.5.4. *The operator E is strong type (p, p) for $p > 1$.*

Proof. Theorems 4.5.2 and 4.5.3 in conjunction with the Marcinkiewicz Interpolation Theorem [3, Theorem 1.3.2] immediately imply that E is strong type (p, p) for $p \in (1, 2]$. As such, it remains only to show that E is strong type (p, p) for $p > 2$.

Let E' denote the adjoint of E. By density, we need only consider $f \in \mathscr{S}(\mathbb{R})$. Fix $p > 2$ and let q denote its Hölder Conjugate. Note that the map

$$g \mapsto \int_{\mathbb{R}} Ef\bar{g} =: \langle Ef, g\rangle$$

is a linear functional on L^q with norm $\|Ef\|_p$. As such, we see by Hölder's inequality that

$$\begin{aligned}
\|Ef\|_p &= \sup_{\|g\|_q=1} |\langle Ef, g\rangle| \\
&= \sup_{\|g\|_q=1} |\langle f, E'g\rangle| \\
&\leq \|f\|_p \|E\|_q \\
&\leq C\|f\|_p,
\end{aligned}$$

as $p > 2$ implies $1 < q < 2$, which means E is strong type (q, q), and Theorem 5 from Chapter VII.3 of [20] implies $\|E\|_q = \|E'\|_q$. □

CHAPTER 5. JOST SOLUTIONS & THE DIRECT SCATTERING MAP

5.1 Introduction

Following our deep dive into the properties of the Green's functions, let us take a brief respite to see where we are in the process of understanding the direct map for the Intermediate Long Wave equation Inverse Scattering Transform. In Section 1.3 we introduced the notion of a Jost Solution, which we now repeat for reference:

Definition 5.1.1 (Jost solutions)**.** Recall the linear spectral problem

$$L_\delta(\Psi) := \frac{1}{i}\frac{\partial}{\partial x}\Psi^+ - \zeta\left(\Psi^+ - \Psi^-\right) = u\Psi^+, \tag{5.1}$$

The Jost solutions M_1, M_e, N_1, N_e are solutions to the linear spectral problem (5.1) whose lower boundary values M_1^+, M_e^+, N_1^+, N_e^+ as defined in (1.6) obey the following asymptotic conditions

$$\lim_{x\to-\infty} \langle x\rangle \left(M_1^+(x;\lambda,\delta) - 1\right) = \lim_{x\to\infty} \langle x\rangle \left(N_1^+(x;\lambda,\delta) - 1\right) = 0 \tag{5.2a}$$

$$\lim_{x\to-\infty} \langle x\rangle \left(M_e^+(x;\lambda,\delta) - e^{i\lambda x}\right) = \lim_{x\to\infty} \langle x\rangle \left(N_e^+(x;\lambda,\delta) - e^{i\lambda x}\right) = 0 \tag{5.2b}$$

.

Additionally, we require the upper boundary values $M_{(\,\cdot\,)}^-$, $N_{(\,\cdot\,)}^-$ (where $(\,\cdot\,)$ represents either the subscript 1 or e) of $M_{(\,\cdot\,)}$, $N_{(\,\cdot\,)}$ to have a decomposition

$$M_1^- - 1 = M_1^{(1)} + M_1^{(2)} \qquad \text{and} \qquad N_1^- - 1 = N_1^{(1)} + N_1^{(2)}$$

$$M_e^- - e^{i\lambda x}\, e^{-2\delta\lambda} = M_e^{(1)} + M_e^{(2)} \qquad \text{and} \qquad N_e^- - e^{i\lambda x}\, e^{-2\delta\lambda} = N_e^{(1)} + N_e^{(2)}$$

satisfying

$$\langle x\rangle^{1+v}\left|M_{(\,\cdot\,)}^{(1)}(x)\right| \lesssim 1 \quad (\text{for } x \ll -1), \qquad \langle x\rangle^{1+v}\left|N_{(\,\cdot\,)}^{(1)}(x)\right| \lesssim 1 \quad (\text{for } x \gg 1),$$

and

$$\langle \,\cdot\, \rangle^{\tau} M^{(2)}_{(\,\cdot\,)}, \;\; \langle \,\cdot\, \rangle^{\tau} N^{(2)}_{(\,\cdot\,)} \in L^2(\mathbb{R})$$

for any $\upsilon \in \left(0, \frac{1}{2}\right)$ and $\tau \in [0, 1)$.

The direct scattering map $\mathscr{D}$ maps u to the reflection coefficient $r(\lambda) := b(\lambda)/a(\lambda)$ where a and b are determined by the following formulas involving the boundary value M_1^+ of the Jost solution M_1

$$a(\lambda) = 1 + i\alpha(\lambda) \int_{\mathbb{R}} u(x)\, M_1^+(x; \lambda, \delta, u)\, \mathrm{d}x \tag{5.3a}$$

$$b(\lambda) = i\beta(\lambda) \int_{\mathbb{R}} e^{-i\lambda x}\, u(x)\, M_1^+(x; \lambda, \delta, u)\, \mathrm{d}x, \tag{5.3b}$$

where we have written the Jost solutions as functions of u in order to explicitly highlight the dependence of the Jost solutions on the eigenfunction u. The goal of this chapter is to both prove that $\mathscr{D}$ is well-defined and Lipschitz continuous for $|\lambda| > 1$.

As we see from (5.3), proving that $\mathscr{D}$ is well defined requires the Jost solutions to exist and be unique. Further, in order to prove that $\mathscr{D}$ is Lipschitz continuous, we need to establish that the four maps from u to each of the four Jost solutions M_1, M_e, N_1, and N_e are themselves Lipschitz as maps from $B_X(0, c_0)$ to $\langle \,\cdot\, \rangle\, L^\infty(\mathbb{R})$. As is common practice in the inverse scattering world, to prove these desired results we seek to reformulate (5.1) with asymptotic conditions (5.2) as a set of integral equations that are easier to analyze. This approach is relevant in our case given the limited theory about partial differential equations involving functions analytic in a complex strip and the functions' lower and upper boundary values along the corresponding boundary values of the strip.

Following our analysis of G_L and G_R in Chapters 2 and 4 we are almost now in a position to prove (under the right hypotheses) the equivalence of the Jost solutions and solutions to the following integral equations

$$\text{(5.4a)}\qquad \begin{pmatrix} M_1^+(x;\zeta,\delta) \\ M_e^+(x;\zeta,\delta) \end{pmatrix} = \begin{pmatrix} 1 \\ e^{i\lambda x} \end{pmatrix} + \int_{\mathbb{R}} G_L(x-x';\zeta,\delta)u(x') \begin{pmatrix} M_1^+(x';\zeta,\delta) \\ M_e^+(x';\zeta,\delta) \end{pmatrix} \mathrm{d}x'$$

$$\text{(5.4b)}\qquad \begin{pmatrix} N_1^+(x;\zeta,\delta) \\ N_e^+(x;\zeta,\delta) \end{pmatrix} = \begin{pmatrix} 1 \\ e^{i\lambda x} \end{pmatrix} + \int_{\mathbb{R}} G_R(x-x';\zeta,\delta)u(x') \begin{pmatrix} N_1^+(x';\zeta,\lambda) \\ N_e^+(x';\zeta,\lambda) \end{pmatrix} \mathrm{d}x'.$$

In Section 5.3 we prove this equivalence. We begin by presenting the framework we use to prove this equivalence, followed by proving in Subsection 5.3.1 that Jost solutions solve the corresponding integral equations (5.4). We prove solutions to (5.4) are Jost solutions in the following subsection, Section 5.3.2. However, for reasons that are made obvious in Section 5.3, prior to proving the equivalence of Jost solutions and solutions to (5.4), we do need several results about both the existence of solutions of solutions to (5.4) and the continuity of maps from u to (5.4) solutions. For this reason, we begin this chapter by studying the (5.4) solutions in Section 5.2, and note that because we prove in Section 5.3 the equivalence of Jost solutions and the solutions to the integral equations (5.4), the results proven in Section 5.2 about the existence and uniqueness (5.4) solutions applies to Jost solutions.

The final section of this chapter, Section 5.4, is the *rasion d'être* of this dissertation, in that after 102 pages of diligent mathematical exploration, we arrive at our study of the ILW direct scattering map. We begin Section 5.4 by verifying the so-called "scattering equations" which are instrumental in the formulation of the ILW inverse scattering map. We then prove that as a map from $B_X(0,c_0)$ to $L^\infty_\lambda(\mathbb{R})$ the direct scattering map $\mathscr{D}$ is well-defined (Theorem 5.4.3), where $B_X(0,c_0)$ in the space X of radius c_0 and c_0 is chosen according to our work in Section 5.2 to ensure the existence of the Jost solutions. In the remainder of Section 5.4, we turn our attention towards understanding the Lipschitz continuity properties of $\mathscr{D}$. Specifically, we prove that as a map from $B_X(0,c_0)$ to $L^\infty_\lambda\big((-\infty,k]\cap[k,\infty)\big)$, the direct scattering map $\mathscr{D}$ is Lipschitz continous for all $k > 0$ (Theorem 5.4.4). As an almost immediate consequence of our proof of Theorem 5.4.4, we also prove Corollary 5.4.5, which holds that for every $u \in B_X(0, c_0)$ with the property that

$$\int_{\mathbb{R}} u\, M_1^+(x; \lambda = 0)\, \mathrm{d}x \neq 0,$$

there is a neighborhood $\mathcal{N}(u)$ in $B_X(0, c_0)$ about u for which the map $\mathscr{D} : \mathcal{N}(u) \to L^\infty_\lambda(\mathbb{R})$ is Lipschitz continuous.

While we have not yet found a proof that $\mathscr{D}$ is Lipschitz continuous uniformly in the parameter λ for all real λ (which is necessary as the scattering data are functions of λ), we do discuss the regimes under which we are currently able to prove that $\mathscr{D}$ is Lipschitz continuous.

A final remark as we set off on the ultimate leg of our mathematical peregrination within this dissertation: given the δ-dilation property satisfied by both G_L and G_R, throughout the remainder of this Chapter, we again take $\delta = 1$ noting that the more general case of $\delta > 0$ arbitrary follows from this dilation property and the results contained within this chapter.

5.2 Existence and Continuity of Jost Solutions

Proposition 5.2.1 (Existence and Uniqueness of Jost Solutions)**.** *There is a $c_0 > 0$ so that for real valued measurable functions $u \in X$ with $\|u\|_X < c_0$ and any $\lambda \in \mathbb{R}$, integral equations* (5.4) *are uniquely solvable in $\langle \cdot \rangle\, L^\infty(\mathbb{R})$. If $\lambda \neq 0$ these solutions are essentially bounded (i.e. L^∞).*

Proof. Consider the operators $T_{\star,\lambda,u}$ ($\star = L$, or R) given by

$$\big(T_{\star,\lambda,u} f\big)(x) := \int_{\mathbb{R}} G_\star^+(x - x'; \lambda)\, u(x') f(x')\, \mathrm{d}x'.$$

Unless necessary to avoid confusion, we write $T_\star$ instead of $T_{\star,\lambda,u}$. The integral equations (5.4) can be written in terms of $T_\star$ as

$$\begin{pmatrix} 1 \\ e^{ix\lambda} \end{pmatrix} = (I - T_L) \begin{pmatrix} M_1^+ \\ M_e^+ \end{pmatrix} \tag{5.5a}$$

$$\begin{pmatrix} 1 \\ e^{ix\lambda} \end{pmatrix} = (I - T_R) \begin{pmatrix} N_1^+ \\ N_e^+ \end{pmatrix} \tag{5.5b}$$

As such, our task is to invert the operators $(I - T_\star)$ on the appropriate spaces, which we do via Neumann series. Indeed, Proposition 3.2.2 implies that we may choose $c_0 > 0$ independently of $\lambda \in \mathbb{R}$ so that

$$\|T_\star\|_{\langle \cdot \rangle L^\infty \circlearrowleft} < \frac{1}{2} \tag{5.6}$$

whenever $\|u\|_X < c_0$. Since the Jost solutions are the same as the solutions to (5.5), estimate (5.6) allows us to conclude that the Jost solutions exist and are unique for $u \in B_X(0, c_0) := \{u \in X \ : \ \|u\|_X < c_0\}$.

Further, an analogous argument involving Proposition 3.2.3 also allows us to conclude that the Jost solutions are essentially bounded for each fixed real $\lambda \neq 0$. □

Lemma 5.2.2 (Continuity in u)**.** *Let $c_0 > 0$ be the same as in Proposition 5.2.1. Denote by $M_1(x; \lambda, u), \dots, N_e(x; \lambda, u)$ the Jost solutions $M_1, \dots, N_e$ corresponding to the potential u. Then the maps*

$$u \mapsto M_1^+(\,\cdot\,; \lambda, u), \qquad u \mapsto M_e^+(\,\cdot\,; \lambda, u),$$
$$u \mapsto N_1^+(\,\cdot\,; \lambda, u), \qquad u \mapsto N_e^+(\,\cdot\,; \lambda, u),$$

from the open ball $B_X(0, c_0)$ into $\langle \cdot \rangle L^\infty(\mathbb{R})$ are Lipschitz continuous with Lipschitz constant uniform in $\lambda \in \mathbb{R}$.

Proof. We prove the Lipschitz continuity for the map $u \mapsto M_1^+$ and note that analogous arguments are sufficient to prove continuity for the remaining maps.

By the second resolvent formula,

$$M_1^+(x;\lambda,u_1) - M_1^+(x;\lambda,u_2) = \left[(1-T_{L,\lambda,u_1})^{-1} - (1-T_{L,\lambda,u_2})^{-1}\right] 1$$
$$= (1-T_{L,\lambda,u_1})^{-1} G_L^+(\,\cdot\,;\lambda) * \left[(u_1-u_2)\, M_1(\,\cdot\,;\lambda,u_2)\right] \tag{5.7}$$

Thus, we see from (3.10) that

$$\begin{aligned}
&\left\|M_1^+(x;\lambda,u_1) - M_1^+(x;\lambda,u_2)\right\|_{\langle\,\cdot\,\rangle L^\infty} \\
&\quad\le \left\|\,(1-T_{L,\lambda,u_1})^{-1}\,\right\|_{\langle\,\cdot\,\rangle L^\infty \circlearrowleft} \|T_{L,\lambda,u_1-u_2}\|_{\langle\,\cdot\,\rangle L^\infty \circlearrowleft} \|M_1(\,\cdot\,;\lambda,u_2)\| \\
&\quad\lesssim \|u_1-u_2\|_X,
\end{aligned} \tag{5.8}$$

as

$$\left\|\,(1-T_{L,\lambda,u_1})^{-1}\,\right\|_{\langle\,\cdot\,\rangle L^\infty \circlearrowleft} \le \sum_{n\ge 0} \frac{1}{2^n} = 2$$

by Neumann series expansion. For $\lambda \neq 0$, the estimate

$$\left\|M_1^+(x;\lambda,u_1) - M_1^+(x;\lambda,u_2)\right\|_{L^\infty} \lesssim \|u_1-u_2\|_X$$

also follows from (5.7). □

Proposition 5.2.3. *Let $c_0 > 0$ be the same as in Proposition 5.2.1. Then the maps*

$$u \mapsto u\, M_1^+(\,\cdot\,;\lambda,u), \qquad u \mapsto u\, M_e^+(\,\cdot\,;\lambda,u),$$
$$u \mapsto u\, N_1^+(\,\cdot\,;\lambda,u), \qquad u \mapsto u\, N_e^+(\,\cdot\,;\lambda,u),$$

from the open ball $B_X(0,c_0)$ into $L^{1,1}(\mathbb{R})$ are Lipschitz continuous with Lipschitz constant uniform in $\lambda \in \mathbb{R}$.

Proof. As usual, we prove Proposition 5.2.3 for the map $u \mapsto u\, M_1^+$ and note similar arguments prove that the remaining maps are Lipschitz.

First note that $u\, M_1^+ \in L^{1,1}(\mathbb{R})$ as

$$\|u\, M_1^+\|_{L^{1,1}} = \left\|\,\left(\langle\,\cdot\,\rangle^{-1} M_1^+\right)\left(\langle\,\cdot\,\rangle^2 u\right)\right\|_{L^1} \le \|M_1^+\|_{\langle\,\cdot\,\rangle L^\infty} \|u\|_X.$$

Choose any $u_1,\, u_2 \in B_X(0, c_0)$. Since the map $B_X(0, c_0) \ni u \mapsto M_1^+ \in \langle\,\cdot\,\rangle\, L^\infty(\mathbb{R})$ is Lipschitz (uniformly in λ) by Lemma 5.2.2,

$$\left\| M_1^+(\,\cdot\,; \lambda, u_1) \right\|_{\langle\,\cdot\,\rangle L^\infty} = \left\| M_1^+(\,\cdot\,; \lambda, u_1) - 0 \right\|_{\langle\,\cdot\,\rangle L^\infty} \lesssim \|u_1 - 0\|_X < c_0$$

where the implied constant is the Lipschitz constant for the map $u \mapsto M_1^+$. We then have

$$\begin{aligned}
&\left\| u_1\, M_1(\cdot; \lambda, u_1) - u_2\, M_1(\cdot; \lambda, u_2) \right\|_{L^{1,1}} \\
&\quad = \int_{\mathbb{R}} \langle x \rangle \left| u_1(x)\, M_1^+(x; \lambda, u_1) - u_2(x)\, M_1^+(x; \lambda, u_1) \right| \mathrm{d}x \\
&\qquad + \int_{\mathbb{R}} \langle x \rangle \left| u_2(x)\, M_1^+(x; \lambda, u_1) - u_2(x)\, M_1^+(x; \lambda, u_2) \right| \mathrm{d}x \\
&\quad \leq \int_{\mathbb{R}} \left(\langle x \rangle^{-1} \left| M_1^+(x; \lambda, u_1) \right| \right) \left(\langle x \rangle^2 \left| u_1(x) - u_2(x) \right| \right) \mathrm{d}x \\
&\qquad + \int_{\mathbb{R}} \left(\langle x \rangle^{-1} \left| M_1^+(x; \lambda, u_1) - M_1^+(x; \lambda, u_2) \right| \right) \left(\langle x \rangle^2\, u_2(x) \right) \mathrm{d}x \\
&\quad \leq \| M_1^+(\,\cdot\,; \lambda, u_1) \|_{\langle\,\cdot\,\rangle L^\infty} \left\| (u_1 - u_2) \langle\,\cdot\,\rangle^2 \right\|_{L^1} \\
&\qquad + \left\| u_2 \langle\,\cdot\,\rangle^2 \right\|_{L^1} \| M_1^+(\,\cdot\,; \lambda, u_1) - M_1^+(\,\cdot\,; \lambda, u_2) \|_{\langle\,\cdot\,\rangle L^\infty} \\
&\quad \lesssim \|u_1 - u_2\|_X,
\end{aligned}$$

where the implied constant is independent of λ and u. □

Remark 17. Note that since $L^{1,1}(\mathbb{R}) \subset L^1(\mathbb{R})$, Proposition 5.2.3 also implies that the maps involved are also Lipschitz continuous, when considered as maps from $B_X(0, c_0)$ into $L^1(\mathbb{R})$.

Lemma 5.2.4. *For fixed $x \in \mathbb{R}$, the Jost solution boundary value M_1^+ is continuous in λ.*

Proof. Define

$$M_h := M_1^+(\,\cdot\,; \lambda + h) - M_1^+(\,\cdot\,; \lambda),$$

and denote by T_λ the convolution operator given by

$$T_\lambda f := G_L^+(\,\cdot\,; \lambda) * (u\, f).$$

To show M_1^+ is continuous in λ, it suffices to show $\|M_h\|_{\langle x\rangle L_x^\infty} \to 0$ as $h \to 0$. Recalling that $M(\lambda) = 1 + T_\lambda M(\lambda)$, we see that

$$\begin{aligned} M_h &= T_{\lambda+h} M(\lambda+h) - T_\lambda M(\lambda) \\ &= \big(T_{\lambda+h} - T_\lambda\big) M(\lambda+h) - T_\lambda M_h \end{aligned}$$

which implies that

$$M_h = \big(1 - T_\lambda\big)^{-1}\big(T_{\lambda+h} - T_\lambda\big) M(\lambda+h), \tag{5.9}$$

as $I - T_\lambda$ is invertible on $\langle \cdot \rangle L^\infty(\mathbb{R})$. Since

$$\big\|\big(T_{\lambda+h} - T_\lambda\big) M(\lambda+h)\big\|_{\langle x\rangle L_x^\infty} \leq \|T_{\lambda+h} - T_\lambda\|_{\langle x\rangle L_x^\infty \circlearrowleft} \|M(\lambda+h)\|_{\langle x\rangle L_x^\infty},$$

the λ-continuity of T_λ follows from the continuity of $\big(I - T_\lambda\big)^{-1}$ and Lemma 3.3.2. □

5.3 Equivalence of Integral Equation Solutions and Jost Solutions

In order to prove the equivalence of the Jost solutions and solutions to the integral equations (5.4), we need to define what doing so actually means. To that end, we respectively define explicitly what a Jost solution is (Definition 5.3.1) or what it means for a function to solve the linear spectral problem reformulated as integral equations equations (Definition 5.3.2).

Remark 18. In this chapter we prove that the Jost solution M_1 solves (1.9) and *vise versa* in the sense of Definitions 5.3.1 and 5.3.2 and note that the proofs of the analogous results for M_e, N_1 and N_e are similar. Definitions 5.3.1 and 5.3.2 are written accordingly, and, throughout this section, we write M en lieu of M_1.

Unless stated otherwise, in the remainder of this section we take $u \in B_X(0, c_0)$ where c_0 is chosen according to Proposition 5.2.1 to ensure that the integral equations (5.4) are uniquely solvable. For $M^+ \in L^\infty(\mathbb{R})$, $u \in X$ implies $uM \in L^2(\mathbb{R})$ and the solution map $u \mapsto M$ for (1.9) is continuous from L^2 to L^∞.

Definition 5.3.1 (Analytic Weak Jost Solution)**.** Fix $\lambda \in \mathbb{R}$. We say that a function M analytic on the strip

$$\mathcal{S}_1 = \{z \in \mathbb{C} \ : \ 0 < \operatorname{Im} z < 2\}$$

with respective lower and upper boundary values M^+, M^- solves the linear spectral problem

$$L_1(M)(x) := \frac{1}{i}\frac{\partial M^+}{\partial x}(x) - \zeta\big(M^+(x) - M^-(x)\big) = u(x)M^+(x) \tag{5.10}$$

with $M^+(x) \to 1$ as $x \to -\infty$ if

(i) M satisfies the following asymptotic conditions:

(a) **Lower boundary value asymptotic condition:**
$\lim_{x\to-\infty} \langle x\rangle \left(M^+(x) - 1\right) = 0$

(b) **Upper boundary value asymptotic condition:**
There exist M_1, M_2 so that

$$M^-(x) - 1 = M_1(x) + M_2(x),$$

where

$$\langle x\rangle^{1+\upsilon} |M_1(x)| \lesssim 1$$

as $x \to -\infty$ and

$$\langle \cdot \rangle^{\tau} M_2 \in L^2(\mathbb{R})$$

for any $\upsilon \in \left(0, \frac{1}{2}\right)$ and $\tau \in [0, 1)$.

(ii) $M^+ \in \langle \cdot \rangle L^\infty(\mathbb{R})$. If $\lambda \neq 0$, then $M^+ \in L^\infty(\mathbb{R})$.

(iii) M is continuous in $0 \leq \operatorname{Im} z \leq 2 - \varepsilon$ for any $0 < \varepsilon < 2$.

(iv) There is a decomposition $M(z) = M_c(z) + M_s(z)$ for $0 < \operatorname{Im} z < 2$ so that

(a) M_c extends to a continuous function on the closure $\overline{\mathcal{S}}_1$ of $\mathcal{S}_1$ with

$$\lim_{\substack{x\to-\infty\\ x\in\mathbb{R}}} M_c(x+2i)=1$$

(b) The estimates

$$\|M_s(\,\cdot\,+iy)\|_{L^\infty}\le(2-y)^{-1/2},\qquad \sup_{0\le y<2}\|M_s(\,\cdot\,+iy)\|_{L^2}<\infty$$

hold. Moreover, M_s has an L^2 boundary value $M_s^-(x):=\lim_{\varepsilon\searrow 0}M_s\big(\,\cdot\,+i(2-\varepsilon)\big)$ on $\operatorname{Im} z=2$ with $M_s(x+iy)\to M_s^-(x)$ for almost every x.

(v) Defining $M^-(x)=M_c(x+2i)+M_s(x+2i)$, the differential equation (5.10) holds in the weak sense, testing against $\phi\in C_0^\infty(\mathbb{R})$.

Definition 5.3.2 (Associated Integral Equation Solution)**.** Fix $\lambda\in\mathbb{R}$ and $u\in X$. A function $M^+(x;\lambda,u)\in\langle\,\cdot\,\rangle\,L_x^\infty(\mathbb{R})$ solves the integral form of the linear spectral problem if the identity

$$M^+(x)=1+G_L^+*(uM^+)(x) \tag{5.11}$$

holds for almost every $x\in\mathbb{R}$.

As discussed following the statement of Theorem 4.1.1, In breaking up M into M_c and M_s in Definition 5.3.1, we are decomposing M into a piece which has a continuous upper boundary value and a piece which exists only in an L^2 sense. The inspiration for this decomposition stems from the similar way we can decompose $G_\star$ ($\star=L$, or R) hinted at in Theorem 4.1.1.

In Section 5.3.2, we use the following property of functions analytic on the complex strip $\mathcal{S}_1$.

Proposition 5.3.3. *Suppose F is analytic in the open strip $\mathcal{S}_1$ and that $|F(x+iy)|\lesssim(2-y)^{-1/2}$ for $y\in[0,2)$. Suppose further that $F=F_1+F_2$ where*

(i) F_1 is bounded and continuous on the closure $\overline{\mathcal{S}}_1$, and

(ii) for any $\varepsilon \in (0,2)$, F_2 is bounded and continuous on $\mathbb{R} \times [0, 2-\varepsilon)$, the estimate

$$\sup_{0<y<2} \|F_2(\,\cdot\, + iy)\|_{L^2} < \infty$$

holds, and there is a function $F_2(\,\cdot\, + 2i)$ so that $F(\,\cdot\, + iy) \to F_2(\,\cdot\, + 2i)$ in $L^2(\mathbb{R})$ as $y \nearrow 2$.

Denote by F^+ the boundary value $F_1(x + i0) + F_2(x + i0)$ and by F^- the boundary value $F_1\big(x + (2i - 0i)\big) + F_2\big(x + (2i - 0i)\big)$, where $2i - 0i$ is the implied limit $(2-\varepsilon)i$ as $\varepsilon \searrow 0$. Then, as distributions in $\mathcal{D}'(\mathbb{R})$,

$$\big(\mathcal{F}F^+\big)(\xi) = e^{2\xi}\big(\mathcal{F}F^-\big)(\xi).$$

Proof. Let γ denote the contour shown in red in Figure 5.1, where $R > 0$ and $0 < \varepsilon < 1$.

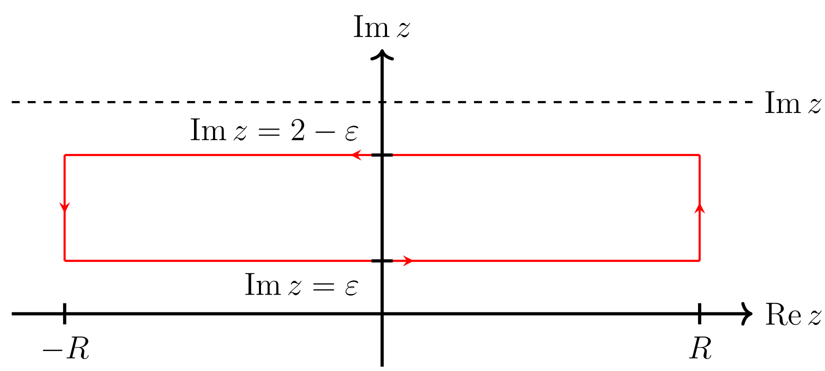


Figure 5.1: Contour of integration for proof of Proposition 5.3.3

Since $F(z)$ is analytic in the interior of the strip $\mathcal{S}_1$, the integral of $e^{-i\xi x}F(z)$ around the contour γ is zero for all appropriate R and ε. Thus, taking $\varepsilon \searrow 0$ for a fixed $R > 0$ yields

$$I_1 - I_3 = I_4 - I_2, \tag{5.12}$$

where

$$
\begin{aligned}
I_1 &:= \int_{-R}^{R} e^{-i\xi x} F^+(x)\,\mathrm{d}x \\
I_2 &:= i\int_0^2 e^{-i\xi R} e^{-t\xi} F(R+it)\,\mathrm{d}t \\
I_3 &:= e^{2\xi}\int_{-R}^{R} e^{-i\xi x} F^-(x)\,\mathrm{d}x \\
I_4 &:= i\int_0^2 e^{i\xi R} e^{-t\xi} F(-R+it)\,\mathrm{d}t
\end{aligned}
$$

correspond to the four straight segments of γ. We claim I_2 and I_4 converge to zero as distributions as $R \to \infty$. Indeed, for $\phi(\xi) \in C_0^\infty(\mathbb{R})$,

$$
\int_{\mathbb{R}} \phi(\xi) I_2(\xi)\,\mathrm{d}\xi = i\int_0^2 \left(\int_{\mathbb{R}} \phi(\xi) e^{-i\xi R} e^{-t\xi}\,\mathrm{d}\xi\right) F(R+it)\,\mathrm{d}t
$$

Since ϕ is compactly supported, using integration by parts on the interior integral above yields

$$
\int_{\mathbb{R}} \phi(\xi) e^{-i\xi R} e^{-t\xi}\,\mathrm{d}\xi = \frac{1}{iR}\int_{\mathbb{R}} e^{i\xi R}\big(\phi'(\xi)\, e^{-t\xi} - t\,\phi(\xi)\, e^{-t\xi}\big)\, d\xi,
$$

which implies

$$
\left|\int_{\mathbb{R}} \phi(\xi) I_2(\xi)\,\mathrm{d}\xi\right| \le \frac{1}{R}\int_{\mathbb{R}} e^{-t\xi}\big(|\phi'(\xi)| + 2|\phi(\xi)\big)\, d\xi \tag{5.13}
$$

Using the hypothesis $|F(x+iy)| \lesssim (2-y)^{-1/2}$ in conjunction with estimate (5.13) allows us to conclude that the integral of I_2 with ϕ is bounded (up to a positive constant) by the integral

$$
\frac{1}{R}\int_0^2 (2-R)^{-1/2} \int_{\mathbb{R}} e^{i\xi R}\big(\phi'(\xi)\, e^{-t\xi} - t\,\phi(\xi)\, e^{-t\xi}\big)\, d\xi, \tag{5.14}
$$

which vanishes in the limit $R \to \infty$ due to the compact support of ϕ. The same argument with R replaced by $-R$ also shows I_4 also vanishes (as a distribution) in the $R \to \infty$ limit. It therefore follows from (5.12) that

$$
\int_{\mathbb{R}} \phi(\xi)\left[\big(\mathcal{F}F^+\big)(\xi) - e^{2\xi}\big(\mathcal{F}F^-\big)(\xi)\right]\mathrm{d}\xi = \lim_{R\to\infty}\int_{\mathbb{R}} (I_1 - I_3)\,\phi(\xi)\,\mathrm{d}\xi = 0
$$

as claimed. □

5.3.1 Jost Solutions Solve the Integral Equations

In this subsection we prove that Jost solutions also solve the corresponding integral equation, as respectively defined in Definitions 5.3.1 and 5.3.2. We do so in two phases: first for $\lambda \neq 0$ (Lemma 5.3.4), and then for $\lambda = 0$ (Lemma 5.3.5).

Lemma 5.3.4. *A Jost solution M satisfying Definition 5.3.1 also solves the associated integral equation 5.11 in the sense of Definition 5.3.2 whenever $\lambda \in \mathbb{R}\backslash\{0\}$.*

Proof. Proposition 5.3.3 implies $\widehat{M^-} = e^{-2\xi}\widehat{M^+}$. Moreover, $\widehat{M^+}$ is a tempered distribution by the hypothesis of Definition 5.3.1 which means that $e^{2\xi}\widehat{M^-}$ is also tempered. Taking the distribution Fourier transform of both sides of (5.10) we consequently find

$$\begin{aligned} \widehat{uM^+} &= \xi\widehat{M^+} - \zeta\left(\widehat{M^+} - \widehat{M^-}\right) \\ &= \left(\xi - \zeta(1 - e^{-2\xi})\right)\widehat{M^+} \\ &= p(\xi)\widehat{M^+} \end{aligned} \tag{5.15}$$

in the sense of distributions in $\mathcal{D}'(\mathbb{R})$. To avoid the zeros of the symbol p, we rewrite (5.15) as

$$p(\xi - i\varepsilon)\widehat{M^+} = [p(\xi - i\varepsilon) - p(\xi)]\widehat{M^+} + \widehat{uM^+} \tag{5.16}$$

for $0 < \varepsilon \ll 1$ and introduce the approximate Green's function

$$G_L^\varepsilon(x;\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\frac{1}{p(\xi - i\varepsilon)}\,d\xi. \tag{5.17}$$

Using the contour shift $\mathbb{R} - i\varepsilon \mapsto \mathbb{R} + i\,\mathrm{sign}(x)\pi$ in our work from Section 2.3 to prove (2.4a) shows that

$$G_L^\varepsilon(x;\lambda) = \begin{cases} K^+(x;\lambda) + i\big[\alpha(\lambda) + \beta(\lambda)e^{i\lambda x}\big]e^{-\varepsilon x}\,\chi_L(x) & \lambda \neq 0 \\ K^+(x;\lambda) + i\left[\frac{2}{3} + ix\right]e^{-\varepsilon x}\,\chi_L(x) & \lambda = 0 \end{cases} \tag{5.18}$$

where K^+ is as defined in Theorem 2.1.1

$$K^+(x) = \frac{e^{-\pi|x|}}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\frac{1}{p(\xi) + i\,\mathrm{sign}(x)\pi}\,\mathrm{d}\xi.$$

An immediate consequence of (2.4a) and (5.18) is that

$$G_L^+(x;\lambda) - G_L^\varepsilon(x;\lambda) = i\Big[\alpha(\lambda) + \beta(\lambda)e^{ix\lambda}\Big]\big(1 - e^{-\varepsilon x}\big)\chi_{(0,\infty)}.$$

Hence, we see from (3.13) that

$$\big|\big(G_L^+ - G_L^\varepsilon\big) * f(x)\big| \lesssim \int_{-\infty}^{x} \Big[1 - e^{-\varepsilon(x-x')}\Big]\,|f(x')|\,\mathrm{d}x'. \tag{5.19}$$

for real $\lambda \neq 0$.

The distribution identity $\widehat{1} = 2\pi\delta_0$ (where δ_0 denotes a Dirac delta-function centered at $\xi = 0$) allows us "subtract 1" from both sides of (5.16) to obtain

$$\begin{aligned} &p(\xi - i\varepsilon)(\widehat{M^+ - 1})(\xi) + 2\pi\,p(\xi - i\varepsilon)\,\delta_0 \\ &\qquad = [p(\xi - i\varepsilon) - p(\xi)](\widehat{M^+ - 1})(\xi) + 2\pi\,p(\xi - i\varepsilon)\,\delta_0 + \widehat{uM^+} \end{aligned}$$

Dividing both sides by $p(\xi - i\xi)$—since it has no zeros for $\xi \in \mathbb{R}$—we have

$$(\widehat{M^+ - 1})(\xi) = \frac{p(\xi - i\varepsilon) - p(\xi)}{p(\xi - i\varepsilon)}(\widehat{M^+ - 1})(\xi) + \frac{1}{p(\xi - i\varepsilon)}\widehat{uM^+}. \tag{5.20}$$

We therefore see from (5.20) that in order to verify that M^+ satisfies definition 5.3.2, it suffices to prove that the following two limits

$$\lim_{\varepsilon \searrow 0} \mathcal{F}^{-1}\left[\frac{1}{p(\xi - i\varepsilon)}\widehat{uM^+}\right](x) = G_L^+ * (uM)(x) \tag{5.21a}$$

and

$$\lim_{\varepsilon \searrow 0} \mathcal{F}^{-1}\left[\frac{p(\xi - i\varepsilon) - p(\xi)}{p(\xi - i\varepsilon)}(\widehat{M^+ - 1})(\xi)\right](x) = 0 \tag{5.21b}$$

hold for *a.e.* x.

Since $uM^+ \in L^1(\mathbb{R})$, (5.21a) follows from estimate (5.19) and the Dominated Convergence Theorem.

To verify (5.21b), first note that

$$\frac{p(\xi - i\varepsilon) - p(\xi)}{p(\xi - i\varepsilon)} = \frac{-i\varepsilon + \zeta e^{-2\xi}\left(1 - e^{2i\varepsilon}\right)}{p(\xi - i\varepsilon)},$$

which means it suffices to prove

$$\lim_{\varepsilon\searrow 0} \varepsilon \mathcal{F}^{-1}\left[\frac{1}{p(\xi - i\varepsilon)}(\widehat{M^+ - 1})(\xi)\right](x) = 0 \tag{5.22a}$$

and

$$\lim_{\varepsilon\searrow 0}(1 - e^{2i\varepsilon})\mathcal{F}^{-1}\left[\frac{e^{-2\xi}}{p(\xi - i\varepsilon)}(\widehat{M^+ - 1})(\xi)\right](x) = 0. \tag{5.22b}$$

From the definition of G_L^ε and (5.18) we see that

$$\mathcal{F}^{-1}\left[\frac{1}{p(\xi - i\varepsilon)}(\widehat{M^+ - 1})(\xi)\right](x) = G_L^\varepsilon * (M^+ - 1)$$

and

$$\begin{aligned} G_L^\varepsilon * (M^+ - 1)(x) = i\alpha(\lambda) &\int_{-\infty}^{x} e^{-\varepsilon(x-x')}\big(M^+(x') - 1\big)\,\mathrm{d}x' \\ &+ i\beta(\lambda)\int_{-\infty}^{x} e^{i\lambda(x-x')}e^{-\varepsilon(x-x')}\big(M^+(x') - 1\big)\,\mathrm{d}x' \\ &+ \left(\int_{-\infty}^{x} K^+(x - x') + \int_{x}^{\infty} K^+(x - x')\right)\big(M^+(x') - 1\big)\,\mathrm{d}x' \end{aligned}$$

Thus, since $e^{i\lambda(x-x')}$ is a unitary phase (*i.e.* has complex modulus 1), in order to verify (5.22a), we need to show that

$$\lim_{\varepsilon\searrow 0} \varepsilon \int_{-\infty}^{x} e^{-\varepsilon(x-x')}|M^+(x') - 1|\,\mathrm{d}x' = 0 \tag{5.23a}$$

and

$$\lim_{\varepsilon\searrow 0} \varepsilon \int_{\mathbb{R}} K^+(x - x')\big(M^+(x') - 1\big)\,\mathrm{d}x' = 0. \tag{5.23b}$$

To prove (5.23a), we choose an arbitrary $\varepsilon' > 0$, split the integral $\int_{-\infty}^{x}$ into $\int_{-\infty}^{x-L} + \int_{x-L}^{x}$, and use the fact that $M^+(x) \to 0$ as $x \to -\infty$ to choose $L > 0$ sufficiently large that $|M^+(x') - 1| < \varepsilon'/2$ for $x' < x - L$. Then, since

$$\int_{-\infty}^{x-L} e^{-\varepsilon(x-x')}\,dx' = \int_{-\infty}^{-L} e^{\varepsilon t}\,\mathrm{d}t < \int_{-\infty}^{0} e^{\varepsilon t}\,\mathrm{d}t = \frac{1}{\varepsilon},$$

where $t = x' - x$, we have

$$\varepsilon \int_{-\infty}^{x-L} e^{-\varepsilon(x-x')}|M(x') - 1|\,dx' < \frac{\varepsilon'}{2}\,\varepsilon \int_{-\infty}^{x-L} e^{-\varepsilon(x-x')}\,\mathrm{d}x' < \frac{\varepsilon'}{2}. \tag{5.24}$$

Now, since M^+ is continuous, $M^+ - 1$ is bounded by $\varepsilon'/2$ on $(\infty, x - L)$, and $[x - L, x]$ is compact, there exists a constant $C_x > 0$ depending only on x so that $\sup_{x' \leq x-L} |M^+(x') - 1| \leq C_x$. Set $\delta' := \frac{\varepsilon'}{2LC_x}$. For all $\varepsilon < \delta'$, we have

$$\varepsilon \int_{x-L}^{x} e^{-\varepsilon(x-x')} |M(x') - 1| \, dx' \leq \varepsilon \, C_x \int_{-L}^{0} e^{\varepsilon x'} \, \mathrm{d}x' < \frac{\varepsilon'}{2}, \tag{5.25}$$

as $e^{\varepsilon x'} \leq 1$ for $x' \leq 0$ implies $\int_{-L}^{0} e^{\varepsilon x'} \, \mathrm{d}x' \leq L$. Limit (5.23a) follows from (5.24) and (5.25).

Since we proved that $K^+ \in \mathcal{S}(\mathbb{R})$ in Section 2.4 and therefore in $L^1(\mathbb{R})$, limit (5.22b) is an immediate consequence of Definition 5.3.1(ii) which states that $M^+ \in L^\infty(\mathbb{R})$.

Lastly, to complete the proof that M^+ satisfies Definition 5.3.2, we now verify limit (5.22b). To do so, observe that it suffices by the Taylor expansion of $1 - e^{2i\varepsilon}$ to verify the (slightly) simpler limit

$$\lim_{\varepsilon \searrow 0} \varepsilon \mathcal{F}^{-1} \left[\frac{e^{-2\xi}}{p(\xi - i\varepsilon)} (\widehat{M^+ - 1})(\xi) \right] (x) = 0, \tag{5.26}$$

We use Proposition 5.3.3 and Definition 5.3.1(v) to rewrite (5.26) as

$$\begin{aligned} &\lim_{\varepsilon \searrow 0} \varepsilon \, \mathcal{F}^{-1} \left[\frac{1}{p(\xi - i\varepsilon)} (\widehat{M^- - 1})(\xi) \right] (x) \\ &\qquad = \lim_{\varepsilon \searrow 0} \varepsilon \, \mathcal{F}^{-1} \left[\frac{1}{p(\xi - i\varepsilon)} (\widehat{M_c^- - 1})(\xi) \right] (x) \\ &\qquad\quad + \lim_{\varepsilon \searrow 0} \varepsilon \, \mathcal{F}^{-1} \left[\frac{1}{p(\xi - i\varepsilon)} (\widehat{M_s^-})(\xi) \right] (x) \end{aligned} \tag{5.27}$$

An analogous argument to the one employed to verify (5.22a) shows

$$\lim_{\varepsilon \searrow 0} \varepsilon \, \mathcal{F}^{-1} \left[\frac{1}{p(\xi - i\varepsilon)} (\widehat{M_c^- - 1})(\xi) \right] (x) = 0.$$

To analyze the second right-hand term, we again appeal to the representation (5.18). The "pole terms" in (5.18) give two terms which can be estimated by

$$\varepsilon \int_{-\infty}^{x} e^{-\varepsilon(x-x')} |M_s^-(x')| \, \mathrm{d}x'$$

which is $\mathcal{O}\left(\varepsilon^{1/2}\right)$ by the Schwartz inequality. To control the integrals involving $K^{\pm}$, we again use the L^2 bound on M_s^- to show that the integrals

$$\int_{\mathbb{R}} |K_{\pm}(x-x')||M_s^-(x')|\,\mathrm{d}x$$

converge, and hence the corresponding terms are $\mathcal{O}(\varepsilon)$. ☐

We now finish our proof that Jost solutions solve the associated integral equation by considering the case where $\lambda = 0$.

Lemma 5.3.5. *Let $\lambda = 0$ and suppose M is a Jost solution in accordance with Definition 5.3.1. Then M is a solution for 5.11 as specified in Definition 5.3.2.*

Proof. As in the proof of Lemma 5.3.4, we begin with the distribution identity $p(\xi;\lambda)\widehat{M^+} = \widehat{u\,M^+}$, which may be rewritten as

$$p(\xi,\lambda)\widehat{M^+ - 1} = \widehat{u\,M^+},$$

since $p(0;\lambda) = 0$. Mimicking our proof of Lemma 5.3.4, we write

$$p(\xi - i\varepsilon)\widehat{M^+ - 1} = \left[p(\xi - i\varepsilon) - p(\xi)\right]\widehat{M^+ - 1} + \widehat{u\,M^+},$$

where here and in what follows we write $p(\xi)$ for $p(\xi;\lambda = 0)$ since $\lambda = 0$ is fixed throughout. Dividing we get

$$\widehat{M^+ - 1} = \frac{p(\xi - i\varepsilon) - p(\xi)}{p(\xi - i\varepsilon)}\widehat{M^+ - 1} + \frac{\widehat{u\,M^+}}{p(\xi - i\varepsilon)}. \tag{5.28}$$

We wish to show that, on taking inverse Fourier transforms and taking $\varepsilon \searrow 0$, we obtain

$$M^+(x) - 1 = G_L * (u\,M^+).$$

Recalling the definition the approximate Green's function

$$G_L^{\varepsilon}(x) = \frac{1}{2\pi}\int \frac{e^{ix\xi}}{p(\xi - i\varepsilon)}\,\mathrm{d}\xi$$

from Equation 5.17 in the proof of Lemma 5.3.4, we note that

$$G_L^\varepsilon(x) = i\left(\frac{2}{3} + ix\right) e^{-\varepsilon x}\chi_L(x) + e^{-\pi|x|}k(x), \tag{5.29}$$

where $k(x)$ is as defined in Remark 1.

Thus the inverse Fourier transform of the second right-hand term in (5.28) is given by

$$\begin{aligned}\mathcal{F}^{-1}\left(\frac{\widehat{uM}}{p(\xi - i\varepsilon)}\right)(x) = i\int_{-\infty}^{x}\left(\frac{2}{3} + i(x-x')\right) e^{-\varepsilon(x-x')}u(x')M(x')\,\mathrm{d}x' \\ + \int_{\mathbb{R}} e^{-\pi|x-x'|}k(x-x')u(x')M(x')\,\mathrm{d}x'.\end{aligned}$$

It follows by dominated convergence that this expression approaches $G_L * (uM)(x)$ pointwise as $\varepsilon \searrow 0$ as $u \in L^{2,4}(\mathbb{R})$ implies $u \in L^{1,1}(\mathbb{R}) \cap L^2(\mathbb{R})$.

It remains to show that the first term vanishes pointwise as $\varepsilon \searrow 0$. We write

$$\begin{aligned}\mathcal{F}^{-1}\left(\frac{p(\xi - i\varepsilon) - p(\xi)}{p(\xi - i\varepsilon)}\widehat{M^+ - 1}\right) = i\varepsilon G_L^\varepsilon * (M^+ - 1) \\ - \frac{1}{2}\left(e^{2i\varepsilon} - 1\right) G_L^\varepsilon * (M^- - 1)\end{aligned}$$

where we used $\widehat{M^-} = e^{-2\xi}\widehat{M^+}$. The goal is to use the asymptotic behavior of M^+ and M^- as $x \to -\infty$ to show that these terms vanish as $\varepsilon \searrow 0$. Due to the linear growth of the Green's function we need a more stringent rate of decay for $M^+ - 1$ and $M^- - 1$ as $x \to -\infty$ to control convolution with the pole term in G_L^ε.

First we consider

$$\begin{aligned}i\varepsilon\, G_L^\varepsilon * (M^+ - 1)(x) = -\varepsilon\int_{-\infty}^{x}\left(\frac{2}{3} + i(x-x')\right) e^{-\varepsilon x}\left(M^+(x') - 1\right)\,\mathrm{d}x' \\ + i\varepsilon\int e^{-\pi|x-x'|}k(x-x')\big(M^+(x') - 1\big)\,\mathrm{d}x'\end{aligned}$$

We use equation (5.29) and asymptotic condition (a) from property 5.3.1 of Definition 5.3.1. The second right-hand integral is bounded by ε times

$$\int_{\mathbb{R}} \langle x - x'\rangle^{-2}\,|k(x-x')|\,\langle x'\rangle\,\mathrm{d}x' \lesssim \langle x\rangle \int_{\mathbb{R}} \langle x - x'\rangle^{-1}\,|k(x-x')|\,dx' \lesssim \langle x\rangle\,\|k\|_{L^2}$$

and so goes to zero pointwise as $\varepsilon \searrow 0$. Let $H(x) = \langle x \rangle \big(M^+(x) - 1\big)$. The first right-hand integral is bounded by

$$\begin{aligned}\varepsilon \int_{-\infty}^{x} \langle x - x' \rangle e^{-\varepsilon(x-x')} \langle x' \rangle^{-1} |H(x')| \, \mathrm{d}x' &\lesssim \varepsilon \int_{-\infty}^{x} e^{-\varepsilon(x-x')} |H(x')| \, \mathrm{d}x' \\ &= \int_{0}^{\infty} e^{-\Xi} |H(x - \Xi/\varepsilon)| \, \mathrm{d}\Xi\end{aligned}$$

which goes to 0 as $\varepsilon \searrow 0$ by dominated convergence since $\lim_{x\to-\infty} H(x) = 0$, where we used the substitution $\Xi = \varepsilon(x - x')$ in the above integral.

We seek to carry out an analogous estimate for the term involving M^-. We will use equation (5.29) and asymptotic condition (b) from property 5.3.1 of Definition 5.3.1. Since $e^{2i\varepsilon} - 1$ is of order ε, it suffices to show that $\varepsilon \left|G_L^{\varepsilon} * (M^- - 1)(x)\right| = o(1)$ as $\varepsilon \searrow 0$, where we use the "little oh" notation $f = o(g)$ to indicate that $\lim y \to a \frac{f(y)}{g(y)} = 0$ (in this case $y = \varepsilon$ and $a = 0$). We have

$$\begin{aligned}\varepsilon \left|(G_L^{\varepsilon} * (M_1))(x)\right| &\lesssim \varepsilon \int_{-\infty}^{x} \langle x - x' \rangle e^{-\varepsilon x'} \langle x' \rangle^{-1-\upsilon} \, \mathrm{d}x' \\ &\lesssim \int_{-\infty}^{x} \varepsilon e^{-\varepsilon(x-x')} \langle x' \rangle^{-\upsilon} \, \mathrm{d}x' \\ &= \int_{0} \infty e^{-\Xi} \langle x - \Xi/\varepsilon \rangle^{-\upsilon} \, \mathrm{d}\Xi\end{aligned}$$

which goes to zero as $\varepsilon \searrow 0$ by dominated convergence. We leave the second term, involving $k(x - x')$, as an exercise to the reader.

Finally

$$\varepsilon \left|(G_L^{\varepsilon}(M_2))(x)\right| \lesssim \varepsilon \int_{-\infty}^{x} e^{-\varepsilon(x-x')} \langle x - x' \rangle \langle x' \rangle^{-1-\upsilon} g(x') \, \mathrm{d}x',$$

where $g \in L^2$. By the Cauchy-Schwarz inequality and the fact that $\langle x - x' \rangle \langle x' \rangle^{-1}$ is bounded for $x' < x < 0$ we again get an $\varepsilon^{\frac{1}{2}}$ estimate which suffices for the purpose.

We conclude that, for $u \in L^{1,2+\upsilon} \cap L^{2,2} \supset X$, the Jost solution M the satisfying asymptotic conditions 5.3.1 from Definition 5.3.1 solves the corresponding integral equation. □

5.3.2 Integral Equation Solutions are Jost Solutions

We continue in this subsection with our discussion of the equivalence of the linear spectral problem with prescribed asymptotics its corresponding integral equation reformulation by turning our attention to showing that the analytic continuation M to the strip $\mathcal{S}_1$ of a function M^+ satisfying Definition 5.3.2 satisfies Definition 5.3.1.

From Proposition 3.2.4 and our work in Chapter 4, we already know that any function M^+ satisfying Definition 5.3.2 has an analytic continuation to the complex strip $\mathcal{S}_1$ which satisfies part (a) of property 5.3.1, and property 5.3.1 and property 5.3.1 of Definition 5.3.1. As such, our task in this section is to first show that the boundary value M^- satisfies property 5.3.1, which we do in Lemma 5.3.6. We then show in Lemma 5.3.7 M has decomposition satisfying property 5.3.1, and prove in 5.3.8 that M satisfies property 5.3.1—that is, M weakly solves (5.10).

Lemma 5.3.6. *Suppose then M^+ satisfies Definition 5.3.2 and denote by M the analytic continuation of M^+ to the complex strip $\mathcal{S}_1$. The upper boundary value M^- of M satisfies asymptotic condition (b) of property 5.3.1 from Definition 5.3.1. That is, there exist M_1, M_2 so that*

$$M^-(x) - 1 = M_1(x) + M_2(x),$$

where

$$\langle x \rangle^{1+\upsilon} \left| M_1(x) \right| \lesssim 1$$

as $x \to -\infty$ and

$$\langle \,\cdot\, \rangle^{\tau} M_2 \in L^2(\mathbb{R})$$

for any $\upsilon \in (0, 1]$ and $\tau \in [0, 1)$.

Proof. Recall from Equation (4.2a) of Theorem 4.2 in Section 4.1 that

$$\begin{aligned} M^-(x) - 1 &= G_L(\,\cdot\,;\lambda)^- * \left[u\, M^+\right](x) \\ &= \left[\mathfrak{C}(\,\cdot\,, 2) + \mathcal{R}_L(\,\cdot\, + i2; \lambda)\right] * \left[u\, M^+\right](x) + \left[E - \frac{1}{2}\right] \left[u\, M^+\right](x) \end{aligned}$$

Define

$$I_1(x;\lambda) := \mathfrak{C}(\,\cdot\,,2) * \big[u\,M^+\big](x)$$
$$I_2(x;\lambda) := \mathcal{R}_L(\,\cdot\,+i2;\lambda) * \big[u\,M^+\big](x)$$
$$I_3(x;\lambda) := \left[E - \frac{1}{2}\right]\big[u\,M^+\big](x),$$

and set

$$M_1(x;\lambda) := I_1(x;\lambda) + I_2(x;\lambda), \qquad \text{and} \qquad M_2(x;\lambda) := \left[E - \frac{1}{2}\right]\big[u\,M^+\big](x).$$

A result of Fefferman-Stein from the theory of Calderón-Zygmund operators with Muckenhaupt A_p weights holds that Calderón-Zygmund operators are bounded between $L^p(\omega\,dx)$ if $1 < p < \infty$ and $\omega \in A_p$. Since $|\,\cdot\,|^\iota$ is an A_2-weight in $\mathbb{R}^1$ for any $\iota \in [0,1)$ and the exponentially weighted Hilbert trasnform is a Calderón-Zygmund operator, we have that

$$\int_{\mathbb{R}} (1+|x|)^{2\iota}\,|(Ef)(x)|^2\;\mathrm{d}x \lesssim_\iota \int_{\mathbb{R}} (1+|x|)^{2\iota}|f(x)|^2\,\mathrm{d}x$$

for any $\iota \in [0,1)$. Hence, since $u \in L^{2,4}(\mathbb{R})$ implies $u\,M^+ \in L^{2,\tau}(\mathbb{R})$ for $\tau \in [0,3]$, we conclude that $E(uM^+) \in L^{2,\tau}(\mathbb{R})$ and thus $M_2 \in L^{2,\tau}(\mathbb{R})$ for $\tau \in [0,1)$.

Since

$$\mathfrak{C}(x,2) = \frac{e^{-\pi|x|}}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\rho\big(\xi,2,\mathrm{sign}(x)\big)\,\mathrm{d}x,$$

where ρ is absolutely integrable in ξ, it is easy to see that $\mathfrak{C}(\,\cdot\,,2)$ is bounded by a Schwartz class function. As such, a simple application of Dominated Convergence implies $\langle x\rangle^{1+\upsilon}\,I_1(x;\lambda) \to 0$ as $x \to -\infty$ for each $\lambda \in \mathbb{R}$.

For I_2, we note that $\mathcal{R}_L$ grows linearly only when $\lambda = 0$ and is otherwise bounded. As such, we verify that $\langle x\rangle^{1+\upsilon}\,I_2(x;\lambda=0)$ is bounded for $\upsilon \in (0,1]$ and $x < 0$ and note that the corresponding result for $\lambda \neq 0$ follows. Now

$$I_2(x;\lambda=0) = \int_{-\infty}^{x}\left(i\frac{2}{3} - (x-x')\right)u(x')\,M^+(x')\,dx.$$

Assuming $x < 0$, then $\langle x' \rangle \geq \langle x \rangle$, $\langle x \rangle^{-1} \geq \langle x' \rangle^{-1}$, and

$$\langle x - x' \rangle \leq \langle x \rangle + \langle x' \rangle \lesssim \langle x' \rangle$$

whenever $x' \leq x$. Consequently,

$$\begin{aligned}
|I_2(x; \lambda = 0)| &\lesssim \int_{-\infty}^{x} \langle x - x' \rangle \, |u(x')| \left| M^+(x') \right| \mathrm{d}x' \\
&\lesssim \langle x \rangle^{-1-\upsilon} \int_{-\infty}^{x} \langle x' \rangle^{2+\upsilon} |u(x')| \langle x' \rangle \left(\langle x' \rangle^{-1} \left| M^+(x') \right| \right) \mathrm{d}x' \\
&\lesssim \langle x \rangle^{-1-\upsilon} \left\| M^+ \right\|_{\langle \, \cdot \, \rangle L^\infty} \, \|u\|_{\langle \, \cdot \, \rangle^{3+\upsilon} L^1} ,
\end{aligned}$$

where $\|u\|_{\langle \, \cdot \, \rangle^{3+\upsilon} L^1} < \infty$ as

$$\|u\|_{\langle \, \cdot \, \rangle^{3+\upsilon} L^1} = \left| \int_{\mathbb{R}} \langle x \rangle^{-1+\upsilon} \langle x \rangle^4 u(x) \, \mathrm{d}x \right| \leq \left\| \langle \, \cdot \, \rangle^{-1+\upsilon} \right\|_{L^2} \left\| \langle \, \cdot \, \rangle^4 u \right\|_{L^2} ,$$

and $-1+\upsilon < -\frac{1}{2}$ implies $\langle \, \cdot \, \rangle^{-1+\upsilon} \in L^2(\mathbb{R})$. It therefore follows that $\langle x \rangle^{1+\upsilon} |M_1(x)| \lesssim 1$ for $x < 0$. $\square$

Lemma 5.3.7. *Suppose then M^+ satisfies Definition 5.3.2. Then its analytic continuation M to the complex strip $\mathcal{S}_1$ satisfies property 5.3.1 of Definition 5.3.1. That is, there is a decomposition $M(z) = M_c(z) + M_s(z)$ for $0 < \operatorname{Im} z < 2$ so that*

(a) M_c extends to a continuous function on the closure $\overline{\mathcal{S}}_1$ of $\mathcal{S}_1$ with

$$\lim_{\substack{x \to -\infty \\ x \in \mathbb{R}}} M_c(x + 2i) = 1$$

(b) The estimates

$$\|M_s(\, \cdot \, + iy)\|_{L^\infty} \leq (2 - y)^{-1/2}, \qquad \sup_{0 \leq y < 2} \|M_s(\, \cdot \, + iy)\|_{L^2} < \infty$$

hold. Moreover, M_s has an L^2 boundary value $M_s^-(x) := \lim_{\varepsilon \searrow 0} M_s\big(\, \cdot \, + i(2-\varepsilon)\big)$ on $\operatorname{Im} z = 2$ with $M_s(x + iy) \to M_s^-(x)$ for almost every x.

Proof. Recalling the discussion following Theorem 4.1.1, we note that

$$M(z) = M_c(z) + M_s(z),$$

where

$$M_c(x+iy) := 1 + \big[\mathfrak{C}(\,\cdot\,,y) + \mathcal{R}_L(\,\cdot\,+iy;\lambda)\big] * \big[u\,M^+\big](x)$$

and

$$M_s(x+iy) := \mathcal{E}_{2-y}\big[u\,M^+\big](x).$$

That $M_c(x+i2)$ is continous for $x \in \mathbb{R}$ follows from our work in Sections 4.2 and 4.3. Further, since $\mathfrak{C}(x,2)$ is bounded by Schwartz class function, it is easy to show through direct computation that $\mathfrak{C}(\,\cdot\,,2) * [u\,M^+](x) \to 0$ as $|x| \to \infty$. By following the proof of Proposition 3.2.4, one can also easily show that $\mathcal{R}_L(\,\cdot\,+i2;\lambda)*[u\,M^+] \to 0$ as $x \to -\infty$ for every real λ, which implies $M_c(x+i2) \to 1$ as $x \to -\infty$. Lastly, the convergence of $M_r(x+iy)$ to $M_r^-(x)$ and the estimates on $\|M_s(\,\cdot\,+iy)\|_{L^\infty}$ and $\sup_{0\le y<2}\|M_s(\,\cdot\,+iy)\|_{L^2}$ are all consequences of our analyses in Sections 4.4 and 4.5. □

Lemma 5.3.8. *The analytic continuation M of a solution M^+ of integral equation form of the linear spectral problem as defined by Definition 5.3.2 also satisfies property 5.3.1 of Definition 5.3.1 and is therefore a Jost solution.*

Proof. To prove that the solution M of (5.3.2) solves (5.10) in the sense of Definition 5.3.1 we first consider the case $u \in C_0^\infty(\mathbb{R})$. It is not difficult to see that if $u \in C_0^\infty(\mathbb{R})$, M is also C^∞. By Laurent Schwartz's formulation of the Paley-Wiener Theorem [15], the function $\widehat{u\,M^+}$ is entire and rapidly decaying in ξ for $\operatorname{Im}\xi$ bounded. Thus we may compute

$$M^+(x) - 1 = \int_{\Gamma_L} \frac{e^{i\xi x}}{p(\xi)} \widehat{u\,M^+}(\xi)\,\mathrm{d}\xi$$

where the right-hand side makes sense owing to the analyticity of $\widehat{uM}$. We also have

$$M(x+iy) - 1 = \int_{\Gamma_L} \frac{e^{i\xi x}e^{-y\xi}}{p(\xi)} \widehat{u\,M^+}(\xi)\,\mathrm{d}\xi$$

from which it follows that

$$\frac{1}{i}\frac{\partial M^+}{\partial x}(x) - \zeta(M^+(x) - M^-(x)) = \int_{\Gamma_L} e^{i\xi x}\widehat{u\,M^+}(\xi)\,\mathrm{d}\xi = u(x)M^+(x)$$

for each x, where we used analyticity of $\widehat{u\,M^+}$ to deform the contour from Γ_L to $\mathbb{R}$.

Now let $u \in X$ and let $\{u_n\}$ be a sequence from $C_0^\infty(\mathbb{R})$ with $u_n \to u$ in X. Let M_n be the corresponding solution of (5.11) for $u = u_n$. For any $\varphi \in C_0^\infty(\mathbb{R})$ we have

$$\frac{1}{i}\left\langle \varphi',\, M_n^+\right\rangle - \zeta\left\langle \varphi,\, M_n^+ - M_n^-\right\rangle - \left\langle \varphi,\, u_n M_n^+\right\rangle = 0. \tag{5.30}$$

As $u_n \to u$ in X it follows that $M_n \to M$ in L^∞, hence $u_n M_n \to uM$ in $L^{1,2}$. To show that M is a weak solution, that is, M satisfies (5.30), it suffices to show that for any $\varphi \in C_0^\infty(\mathbb{R})$, the differences

$$\left\langle \varphi',\, M^+ - M_n^+\right\rangle, \quad \left\langle \varphi,\, (M^\pm - M_n^\pm)\right\rangle, \quad \left\langle \varphi,\, u\,M^+ - u_n\,M_n^+\right\rangle$$

all converge to zero as $n \to \infty$. Note that

$$\begin{aligned}\left|\left\langle \varphi',\, M^+ - M_n^+\right\rangle\right| &= \left|\int_{\mathbb{R}} \phi'\left(M^+ - M_n^+\right)\mathrm{d}x\right| \\ &\le \left\|M^+ - M_n^+\right\|_{\langle\,\cdot\,\rangle L^\infty} \left\|\langle\,\cdot\,\rangle\,\phi'\right\|_{L^1}.\end{aligned}$$

Since $\phi' \in C_0^\infty$ implies $\|\langle\,\cdot\,\rangle\,\phi'\|_{L^1}$ and the map $B_X(0, c_0) \ni u \mapsto M \in \langle\,\cdot\,\rangle\, L^\infty(\mathbb{R})$ is Lipschitz by Lemma 5.2.2, it follows that

$$\lim_{n\to\infty}\left\langle \varphi',\, M^+ - M_n^+\right\rangle = 0.$$

A similar argument also shows that

$$\lim_{n\to\infty}\left\langle \varphi,\, (M^+ - M_n^+)\right\rangle = 0$$

The Lipschitz continuity of the map $B_X(0, c_0) \ni u \mapsto u\,M \in L^1(\mathbb{R})$ (Lemma 5.2.3) in conjunction with the fact that ϕ is bounded again implies

$$\lim_{n\to\infty}\left\langle \varphi,\, u\,M^+ - u_n\,M_n^+\right\rangle = 0.$$

To prove $\langle \varphi, (M^- - M_n^-)\rangle \to 0$ as $n \to \infty$, we recall that

$$\left[M^-(x;\lambda) - M_n^-(x;\lambda)\right] = \left[M_c^-(x;\lambda) - \left(M_n^-\right)_c(x;\lambda)\right] + \left[M_s^-(x;\lambda) - \left(M_n^-\right)_s(x;\lambda)\right]$$

where

$$M_c^-(x;\lambda) - \left(M_n^-\right)_c(x;\lambda) = \left[\mathfrak{C}(\,\cdot\,,2) + \mathcal{R}_L(\,\cdot\,+i2,\lambda)\right] * \left[u\,M^+ - u_n\,M_n^+\right](x) \tag{5.31}$$

and

$$M_s^-(x;\lambda) - \left(M_n^-\right)_s(x;\lambda) \tag{5.32}$$

$$= E\left[u\,M^+(\,\cdot\,;\lambda) - u_n\,M_n^+(\,\cdot\,;\lambda)\right](x) - \frac{1}{2}\left[u\,M^+(x;\lambda) - u_n\,M_n^+(x;\lambda)\right] \tag{5.33}$$

Hence, using the fact that $\phi \in C_0^\infty(\mathbb{R})$ implies $\langle\,\cdot\,\rangle\,\phi \in L^1$ and estimate (3.22) from Technical Lemma 3.3.1 we find for all $\lambda \in \mathbb{R}$

$$\begin{aligned}\left\langle \phi,\, M_c^- - \left(M_n^-\right)_c\right\rangle &\leq \|\langle\,\cdot\,\rangle\,\phi\|_{L^1}\left\|M_c^- - \left(M_n^-\right)_c\right\|_{\langle\,\cdot\,\rangle L^\infty}\\ &\leq \|\langle\,\cdot\,\rangle\,\phi\|_{L^1}\left\|M^+ - M_n^+\right\|_{L^{1,1}}\|\mathfrak{C}(\,\cdot\,,2) + \mathcal{R}_L(\,\cdot\,+i2)\|_{\langle\,\cdot\,\rangle L^\infty}\end{aligned}$$

Again appealing to Lemma 5.2.3 we conclude,

$$\lim_{n\to\infty}\left\langle \phi,\, M_c^- - \left(M_n^-\right)_c\right\rangle = 0.$$

Lastly, to prove

$$\lim_{n\to\infty}\left\langle \phi,\, M_s^- - \left(M_n^-\right)_s\right\rangle = 0,$$

we can use the $\langle\,\cdot\,\rangle\,L^\infty$ convergence of M_n^+ to M^+, the X-convergence of u_n to u, and dominated convergence to show that the right-hand side of (5.32) goes to zero in L^2 as $n \to \infty$, and, by passing to a subsequence, goes to zero almost everywhere. □

5.4 The Direct Scattering Map

We now have the tools we need to prove that the direct scattering map is well-defined as a map from $B_X(0, c_0)$ to $L^\infty(\mathbb{R})$ and Lipschitz continuous as a map from $B_X(0, c_0)$ to $L^\infty\big(\mathbb{R}\backslash(-k, k)\big)$ (for any $k > 0$). As a warm-up exercise, we first verify in Proposition 5.4.1 the validity of the scattering equations first presented (without proof) in [6], which are key to the construction of the inverse scattering map for the ILW. Following Proposition 5.4.1, we turn our attention to proving that the direct scattering map is well-defined (Lemma 5.4.2 and Theorem 5.4.3), and wrap up this dissertation with results on the Lipschitz continuity of the direct scattering map (Theorem 5.2.1 and Corollary 5.4.5).

Proposition 5.4.1 (Scattering Equations)**.** *Suppose that u satisfies the hypotheses of Proposition 5.2.1. Let*

$$a(\lambda) := 1 + i\alpha(\lambda) \int_{\mathbb{R}} u(x)\, M_1^+(x;\lambda,u)\,\mathrm{d}x \tag{5.34a}$$

$$b(\lambda) = i\beta(\lambda) \int_{\mathbb{R}} e^{-ix\lambda}\, u(x)\, M_1^+(x;\lambda,u)\,\mathrm{d}x \tag{5.34b}$$

$$\breve{a}(\lambda) := 1 + \alpha(\lambda) \int_{\mathbb{R}} u(x)\, N_1(x;\lambda,u)\,\mathrm{d}x \tag{5.35a}$$

$$\breve{b}(\lambda) = i\beta(\lambda) \int_{\mathbb{R}} e^{-ix\lambda}\, u(x)\, N_1(x;\lambda,u)\,\mathrm{d}x \tag{5.35b}$$

For $\lambda \in \mathbb{R}\backslash\{0\}$,

$$M_1(x;\lambda) = a(\lambda)\, N_1(x;\lambda) + b(\lambda)\, N_e(x;\lambda) \tag{5.36}$$

$$N_1(x;\lambda) = \breve{a}(\lambda)\, M_1(x;\lambda) + \breve{b}(\lambda)\, M_e(x;\lambda) \tag{5.37}$$

Proof. An immediate consequence of formulas (2.4) is the jump relation[1]

$$G_L^+ - G_R^+ = i\alpha(\lambda) + i\beta(\lambda)e^{i\lambda x}. \tag{5.38}$$

[1] This jump relation can also be proven by taking a contour around the real axis. For details, please see Appendix 3: Jump Relation.

Under the jump relation (5.38), the integral equation (5.4a) for M_1^+ becomes

$$(5.39)\qquad \begin{aligned} M_1^+(x;\lambda) &= 1 + \int_{\mathbb{R}} G_L^+(x-x';\lambda)u(x')M_1^+(x')\,dx' \\ &= 1 + \int_{\mathbb{R}} \big(i\alpha(\lambda) + i\beta(\lambda)\,e^{i\lambda(x-x')}\big)u(x')M_1^+(x')\,\mathrm{d}x \\ &= \left(1 + i\alpha(\lambda)\int_{\mathbb{R}} u(x')M_1^+(x';\lambda)\,dx\right) \\ &\qquad + i\beta(\lambda)\,e^{i\lambda x}\int_{\mathbb{R}} e^{-i\lambda x'}(x')M_1^+(x';\lambda)\,dx \\ &\qquad + \int_{\mathbb{R}} G_R^+(x-x';\lambda)u(x')M_1^+(x')\,dx' \\ &= a(\lambda) + b(\lambda)\,e^{i\lambda x} + G_R^+(\,\cdot\,;\lambda) * \big[uM_1^+(\,\cdot\,;\lambda)\big]. \end{aligned}$$

Recalling from Proposition 3.2.4 that $\lim_{x\to\infty} G_R^+ * uM_1^+(x) = 0$, we see that M_1^+ satisfies the asymptotic condition

$$(5.40)\qquad \lim_{x\to+\infty}\left|M_1^+(x;\lambda) - a(\lambda) - b(\lambda)e^{i\lambda x}\right| = 0.$$

The simple computation

$$\begin{aligned} G_R^+ * \big[u(a\,N_1 + b\,N_e)\big] &= a\,G_R^+ * (u\,N_1) + b\,G_R^+ * (u\,N_e) \\ &= a(N_1 - 1) + b(N_e - e^{i\lambda x}) \\ &= a\,N_1 + b\,N_e - a - b\,e^{i\lambda x} \end{aligned}$$

shows that $a\,N_1 + b\,N_e$ is a solution to (5.39). Further, since $a\,N_1 + b\,N_e$ also satisfies (5.40) as $\lim_{x\to+\infty}|N_1 - 1| = \lim_{x\to+\infty}|N_e - e^{ix\lambda}| = 0$, equation (5.36) follows from the uniqueness of Jost solutions. Equation (5.37) is verified analogously. □

While perhaps not immediately apparent, the significance of the following lemma, Lemma 5.4.2, is that it allows us to conclude that the reflection coefficient $r(\lambda) = b(\lambda)/a(\lambda)$ is bounded in λ. That is, $r \in L^\infty_\lambda(\mathbb{R})$, which is the final piece we need to prove that the ILW direct scattering map $\mathscr{D} : B_X(0, c_0) \ni u \mapsto r \in L^\infty_\lambda(\mathbb{R})$ is well-defined.

Lemma 5.4.2. *For $\lambda \neq 0$, the functions a and b defined in Proposition 5.4.1 satisfy the equation*

$$|a(\lambda)|^2 = 1 + \frac{2\zeta(-\lambda) - 1}{1 - 2\zeta(\lambda)} |b(\lambda)|^2. \tag{5.41}$$

The following proof is taken from the unpublished notes of Professor Allen Wu.

Proof. In this proof, we use the identity

$$\begin{aligned}\left\langle G_L^+(\,\cdot\,;\lambda) * f,\, g\right\rangle = {} & \left\langle f,\, G_L^+(\,\cdot\,;\lambda) * g\right\rangle \\ & + i\alpha(\lambda)\,\langle f, 1\rangle\,\langle 1, g\rangle + i\beta(\lambda)\left\langle f,\, e^{i(\,\cdot\,)\lambda}\right\rangle \left\langle e^{i(\,\cdot\,)\lambda},\, g\right\rangle\end{aligned}$$

which follows from Proposition 2.3.1 identity (ii) and the jump relation (5.38) as

$$\overline{G_L^+(x;\lambda)} = G_R^+(-x;\lambda) = G_L^+(-x;\lambda) - i\alpha(\lambda) - i\beta(\lambda)\, e^{-i\lambda x}.$$

Using $M_1^+ = 1 + G_L^+(\,\cdot\,;\lambda) * uM_1$ we compute

$$\begin{aligned}\left\langle M_1^+,\, uM_1^+\right\rangle &= \left\langle 1 + G_L^+(\,\cdot\,;\lambda) * uM_1^+,\, uM_1^+\right\rangle \\ &= \left\langle 1,\, uM_1^+\right\rangle + \left\langle uM_1^+,\, G_L^+(\,\cdot\,;\lambda) * uM_1^+\right\rangle \\ &\quad + i\alpha(\lambda)|\left\langle uM_1^+,\, 1\right\rangle|^2 + i\beta(\lambda)|\left\langle uM_1^+,\, e^{i(\,\cdot\,)\lambda}\right\rangle|^2 \\ &= \left\langle 1,\, uM_1^+\right\rangle + \left\langle uM_1^+,\, M_1^+ - 1\right\rangle \\ &\quad + i\alpha(\lambda)|\left\langle uM_1^+,\, 1\right\rangle|^2 + i\beta(\lambda)|\left\langle uM_1^+,\, e^{i(\,\cdot\,)\lambda}\right\rangle|^2\end{aligned}$$

Since u is real, we have $\left\langle M_1^+,\, uM_1^+\right\rangle = \left\langle uM_1^+,\, M_1^+\right\rangle$ and

$$0 = \overline{\left\langle uM_1^+,\, 1\right\rangle} - \left\langle uM_1^+,\, 1\right\rangle + i\alpha(\lambda)\left|\left\langle uM_1^+,\, 1\right\rangle\right|^2 + i\beta(\lambda)\left|\left\langle uM_1^+,\, e^{i(\,\cdot\,)\lambda}\right\rangle\right|^2.$$

Identity 5.41 then follows, as $\left\langle uM_1^+,\, 1\right\rangle = \frac{1}{i}\frac{a-1}{\alpha(\lambda)}$, $\left\langle uM_1^+,\, e^{i(\,\cdot\,)\lambda}\right\rangle = \frac{1}{i}\frac{b}{\beta(\lambda)}$, $\alpha(\lambda) = \frac{1}{1-2\zeta(\lambda)}$, and $\beta(\lambda) = \frac{1}{1-2\zeta(-\lambda)}$. □

While Lemma 5.4.2 holds only for $\lambda \neq 0$, we can nonetheless use Lemma 5.4.2 to show that the r remains bounded near $\lambda = 0$ and is therefore at least essentially bounded and hence in L^∞_λ.

Theorem 5.4.3. *Let $r(\lambda) := b(\lambda)/a(\lambda)$. The direct scattering map $\mathscr{D}$ given by*

$$\begin{aligned} \mathscr{D} : B_X(0, c_0) &\to L^\infty_\lambda(\mathbb{R}) \\ u &\mapsto r \end{aligned}$$

is well-defined.

Proof. Since M_1^+ and N_1^+ exist and are unique for each $u \in B_X(0, c_0)$, the map $u \mapsto r$ is well-defined as a function. Moreover, it is easy to check that $\frac{1-2\zeta(\lambda)}{2\zeta(-\lambda)-1}$ is both positive and uniformly bounded in λ for all real $\lambda \neq 0$, which means that (5.41) and Lemma 5.4.2 implies both that $|a(\lambda)| \geq 1$ for all $\lambda \neq 0$ and, as a consequence

$$\left| \frac{b(\lambda)}{a(\lambda)} \right|^2 = \frac{1 - 2\zeta(\lambda)}{2\zeta(-\lambda) - 1} \left[1 - \left| \frac{1}{a(\lambda)} \right|^2 \right].$$

Since $|a(\lambda)| \to \infty$ as $\lambda \to 0$ and a simple computation shows that

$$\lim_{\lambda \to 0} \frac{1 - 2\zeta(\lambda)}{2\zeta(-\lambda) - 1} = 1,$$

$r = b/a$ is at least essentially bounded near $\lambda = 0$. Moreover, since

$$\lim_{\lambda \to -\infty} \frac{1 - 2\zeta(\lambda)}{2\zeta(-\lambda) - 1} = 0, \tag{5.42}$$

we need only prove $r(\lambda)$ stays bounded for large *positive* λ. Indeed, a straight forward computation shows

$$\lim_{\lambda \to +\infty} |\beta(\lambda)| = 1, \qquad \text{and} \qquad \lim_{\lambda \to +\infty} \alpha(\lambda) = 0. \tag{5.43}$$

Recalling that

$$\begin{aligned} a(\lambda) &:= 1 + i\alpha(\lambda) \int_{\mathbb{R}} u(x)\, M_1^+(x; \lambda, u)\, \mathrm{d}x \\ b(\lambda) &= i\beta(\lambda) \int_{\mathbb{R}} e^{-ix\lambda}\, u(x)\, M_1^+(x; \lambda, u)\, \mathrm{d}x, \end{aligned}$$

we see that

$$\left| \frac{b(\lambda)}{a(\lambda)} \right| \lesssim \left\| u\, M_1^+(\,\cdot\,; \lambda) \right\|_{L^1} \leq \left\| M_1^+(\,\cdot\,; \lambda) \right\|_{\langle\,\cdot\,\rangle L^\infty} \|u\|_{L^{1,1}}$$

for $\lambda \gg 1$. Given $M_1^+ = (1 - T_{L,\lambda,u})^{-1} 1$ and the operator $T_{L,\lambda,u}$ is bounded uniformly in λ—in fact, $\|T_{L,\lambda,u}\|_{\langle \cdot \rangle L^\infty \circlearrowleft} < \frac{1}{2}$—we conclude by Neumann series that $\left\|M_1^+(\cdot \,; \lambda)\right\|_{\langle \cdot \rangle L^\infty}$ is also uniformly bounded in λ. The result therefore follows. □

While, we do not yet have a proof that the ILW direct scattering map is Lipschitz as a map from $B_X(0, c_0)$ into L^∞_λ, we are able to prove that it is Lipschitz in more restrictive regimes.

Theorem 5.4.4. *For $c_0 > 0$ from Proposition 5.2.1 and for all fixed $k > 0$, the ILW direct scattering map*

$$\mathscr{D} : B_X(0, c_0) \ni u \mapsto r \in L^\infty_\lambda\big((-\infty, -k] \cup [k, \infty)\big)$$

is Lipschitz continuous with Lipschitz constant depending on k.

Proof. Let $u_1, u_2 \in B_X(0, c_0)$ be arbitrary and respectively denote by $r_1 = b_1/a_1$, $r_2 = b_2/a_2$ the corresponding ILW direct scattering map $\mathscr{D}$ outputs. Since $|a_1(\lambda)| \geq 1$ for all $\lambda \in \mathbb{R}$ by Lemma 5.4.2, we find

$$\begin{aligned} \left|\frac{b_1}{a_1} - \frac{b_2}{a_2}\right| &\leq \left|\frac{b_1}{a_1} - \frac{b_2}{a_1}\right| + \left|\frac{b_2}{a_1} - \frac{b_2}{a_2}\right| \\ &= \frac{1}{|a_1|}|b_1 - b_2| + \frac{1}{|a_1|}\left|\frac{b_2}{a_2}\right| |a_1 - a_2| \,. \end{aligned} \tag{5.44}$$

Proposition 5.2.3 implies the map $u \mapsto uM_1^+$ is Lipschitz as a map into L^1_x. As such,

$$\begin{aligned} \frac{1}{|a_1|}|b_1 - b_2| &= \frac{|\beta(\lambda)|}{|a_1(\lambda)|}\left|\int_{\mathbb{R}} e^{ix\lambda}\big(u_1(x)\, M_1^+(x; \lambda, u_1) - u_2(x)\, M_1^+(x; \lambda, u_2)\big)\, \mathrm{d}x\right| \\ &\leq \frac{|\beta(\lambda)|}{|a_1(\lambda)|}\left\|u_1\, M_1^+(\cdot \,; \lambda, u_1) - u_2\, M_1^+(\cdot \,; \lambda, u_2)\right\|_{L^1} \\ &\lesssim \frac{|\beta(\lambda)|}{|a_1(\lambda)|}\left\|u_1 - u_2\right\|_X , \end{aligned} \tag{5.45}$$

where the implied constant is uniform in λ. Similarly,

$$\begin{aligned} |a_1 - a_2| &\leq |\alpha(\lambda)|\left\|u_1\, M_1^+(\cdot \,; \lambda, u_1) - u_2\, M_1^+(\cdot \,; \lambda, u_2)\right\|_{L^1} \\ &\lesssim |\alpha(\lambda)|\left\|u_1 - u_2\right\|_X , \end{aligned} \tag{5.46}$$

where the implied constant is again uniform in λ. Now, the proof of Theorem 5.4.3 implies that the term

$$\frac{1}{|a_1|}\left|\frac{b_2}{a_2}\right||a_1 - a_2|$$

is bounded for $|\lambda| > 0$. Through direct computation, it is straightforward to show

$$\lim_{\lambda\to-\infty} |\alpha(\lambda)| = \lim_{\lambda\to+\infty} |\beta(\lambda)| = 1,$$

$$\lim_{\lambda\to+\infty} |\alpha(\lambda)| = \lim_{\lambda\to-\infty} |\beta(\lambda)| = 0,$$

which implies $|\alpha(\lambda)|$ and $|\beta(\lambda)|$ are bounded for $|\lambda| \gg 1$. Further, since α and β have exactly one singularity, namely $\lambda = 0$, we conclude by estimates (5.44) through (5.46) that

$$\|r_1 - r_2\|_{L^\infty_\lambda} \lesssim_k \|u_1 - u_2\|_X \tag{5.47}$$

as $|a_1| \geq 1$ for $\lambda \neq 0$, where the implied constant depends on k but is otherwise independent of λ. □

Remark 19. The difficulty in extending Theorem 5.4.4 to all values of real λ is due to the possibility of the implied constant in (5.47) "blowing-up" as $k \to 0$—especially when either $\int_{\mathbb{R}} u_1\, M_1^+(x; \lambda = 0, u_1)\, \mathrm{d}x$ or $\int_{\mathbb{R}} u_2\, M_1^+(x; \lambda = 0, u_2)\, \mathrm{d}x$ are zero. To see why this is so, note that

$$\frac{\alpha(\lambda)}{a(\lambda)} = \frac{\alpha(\lambda)}{1 + \alpha(\lambda)\int_{\mathbb{R}} u\, M_1^+(x;\lambda)\,\mathrm{d}x} = \frac{1}{\frac{1}{\alpha(\lambda)} + \int_{\mathbb{R}} u\, M_1^+(x;\lambda)\,\mathrm{d}x,}$$

and

$$\frac{\beta(\lambda)}{a(\lambda)} = \frac{\beta(\lambda)}{1 + \alpha(\lambda)\int_{\mathbb{R}} u\, M_1^+(x;\lambda)\,\mathrm{d}x} = \frac{1}{\frac{1}{\beta(\lambda)} + \frac{\alpha(\lambda)}{\beta(\lambda)}\int_{\mathbb{R}} u\, M_1^+(x;\lambda)\,\mathrm{d}x.}$$

which means

$$\frac{\alpha(\lambda)}{a(\lambda)},\ \frac{\beta(\lambda)}{a(\lambda)} \sim \mathcal{O}\left(\frac{1}{\int_{\mathbb{R}} u\, M_1^+(x;\lambda)\,\mathrm{d}x}\right)$$

for $|\lambda| \ll 1$, as $\lim_{\lambda\to 0} \alpha(\lambda)/\beta(\lambda) = 1$. Thus, if either $\int_{\mathbb{R}} u_1\, M_1^+(x; \lambda = 0, u_1)\, \mathrm{d}x$ or $\int_{\mathbb{R}} u_2\, M_1^+(x; \lambda = 0, u_2)\, \mathrm{d}x$ are zero, then the approach in the proof of Theorem 5.4.4 fails miserably for $|\lambda|$ that is not controlled below.

In light of Remark 19, we obtain the following easy "extension" of Theorem 5.4.4, which emphasizes the challenge in actually extending Theorem 5.4.4 to all real values of λ.

Corollary 5.4.5. *For every $u \in B_X(0, c_0)$ with the property that*

$$\int_{\mathbb{R}} u\, M_1^+(x; \lambda = 0, u)\, \mathrm{d}x \neq 0,$$

there is a neighborhood $\mathcal{N}(u)$ in $B_X(0, c_0)$ about u for which the map $\mathscr{D} : \mathcal{N}(u) \mapsto L^\infty_\lambda(\mathbb{R})$ is Lipschitz continuous.

Proof. Fix $\varepsilon > 0$ so that $\left|\int_{\mathbb{R}} u\, M_1^+(x; \lambda = 0, u)\, \mathrm{d}x\right| > 2\varepsilon$ Using the Lipschitz continuity of the map $B_X(0, c_0) \ni w(x) \to w(x)\, M_1^+(x; \lambda, w) \in L^1_x(\mathbb{R})$, we may choose $\mathcal{N}_\varepsilon(u)$ so that every w in $\mathcal{N}(u)$ satisfies

$$\left|\int_{\mathbb{R}} w(x)\, M_1^+(x; \lambda = 0, w)\, dx\right| \geq \varepsilon.$$

Then, Corollary 5.4.5 follows from the proof of Theorem 5.4.4, Remark 19, and the Dominated Convergence Theorem. □

The following lemma, Lemma 5.4.6, was developed as part of an, as yet, unsuccessful bid to extend Theorem 5.4.4 to all real r. We include this lemma here in the hopes that it may eventually be useful in completing the proof that the ILW direct scattering map is Lipschitz continuous.

Lemma 5.4.6. *For $u \in X \cap \langle x \rangle^{-5} L^\infty_x(\mathbb{R})$, the Jost solution boundary value M_1^+ has the following $\langle x \rangle^4 L^\infty_x(\mathbb{R})$ linear approximation in λ centered at $\lambda = 0$*

$$M_1^+(x; \lambda, u) = M^{(0)}(x; u) + \lambda\, M^{(1)}(x; u) + o(\lambda), \tag{5.48}$$

where $M^{(0)}(x; u) = M_1^+(x; \lambda = 0, u)$ and $M^{(1)}$ is $(\partial M_1^+/\partial\lambda)(x; 0)$

Proof. It suffices to prove that M_1^+ has an $\langle x \rangle^4 L^\infty(\mathbb{R})$ derivative in λ at $\lambda = 0$. To do so, we define M_h as the difference quotient

$$M_h(\lambda; x) := \frac{M_1^+(x; \lambda + h) - M_1^+(x; \lambda)}{h}.$$

In order to simplify notation, throughout the rest of this proof, we denote G_L^+ by G, M_1^+ by M, and suppress x dependency. That is, $G(\lambda) := G_L^+(x;\lambda)$ and $M(\lambda) := M_1^+(x;\lambda)$. Please note that while not explicitly indicated by the notation in this proof, all convolutions are with respect to the variable x.

By linearity of convolution operators

$$\begin{aligned} M_h(\lambda;x) &= \frac{G(\lambda+h)*\big[u\,M(\lambda+h)\big]-G(\lambda)*\big[u\,M(\lambda)\big]}{h} \\ &= \frac{\big[G(\lambda+h)-G(\lambda)\big]*\big[u\,M(\lambda+h)\big]-G(\lambda)*\big\{u\big[M(\lambda+h)-M(\lambda)\big]\big\}}{h} \\ &= \left(\frac{G(\lambda+h)-G(\lambda)}{h}\right)*\big[u\,M(\lambda+h)\big]-G(\lambda)*\big[u\,M_h(\lambda)\big] \end{aligned}$$

Define G_h to be the difference quotient $G_h = \frac{1}{h}\big[G(\lambda+h)-G(\lambda)\big]$ and let T_λ denote the operator given by $T_\lambda f = G(\lambda)*(u\,f)$. Since, as we see in the proof of Proposition 5.2.1, $I+T_\lambda$ is invertible, the following formula for M_h follows from the above computation:

$$M_h(\lambda) := (I+T_\lambda)^{-1}\big(G_h * u\,M(\lambda+h)\big). \tag{5.49}$$

Given the continuity of $(I+T_\lambda)^{-1}$, equation (5.49) implies that M_1^+ is differentiable in λ ($\lambda\in\mathbb{R}$) if and only if the limit

$$\lim_{h\to 0} G_h * u\,M(\lambda+h)$$

holds pointwise for each $x\in\mathbb{R}$.

Since a natural candidate for the limit of $G_h * u\,M(\lambda+h)$ as $h\to 0$ is $\big(\frac{\partial}{\partial\lambda}G\big)*(uM)$, note that

$$\begin{aligned} G_h * \big(uM(\lambda+h)\big) - \left(\frac{\partial}{\partial\lambda}G\right) * \big(uM(\lambda)\big) \\ = \left(G_h - \frac{\partial}{\partial\lambda}G\right) * \big(uM(\lambda+h)\big) + \left(\frac{\partial}{\partial\lambda}G\right) * u\big(M(\lambda+h)-M(\lambda)\big). \end{aligned} \tag{5.50}$$

By Technical Lemma 3.3.1,

$$\left\|\left(G_h - \frac{\partial}{\partial\lambda}G\right) * \big(uM(\lambda+h)\big)\right\|_{\langle x\rangle^4 L_x^\infty} \le \left\|G_h - \frac{\partial}{\partial\lambda}G\right\|_{\langle x\rangle^4 L_x^1} \big\|\langle\,\cdot\,\rangle^4\,u\,M(\lambda+h)\big\|_{L_x^\infty}.$$

Hence

$$\lim_{h\to\infty}\left\|\left(G_h-\frac{\partial}{\partial\lambda}G\right)*\big(uM(\lambda+h)\big)\right\|_{\langle x\rangle^4L^\infty_x}=0, \tag{5.51}$$

as

$$\big\|\langle\,\cdot\,\rangle^4\,u\,M(\lambda+h)\big\|_{L^\infty_x}=\operatorname*{ess\,sup}_{x\in\mathbb{R}}\big(\,\langle x\rangle^{-1}\,M\big)\big(\,\langle x\rangle^5\,u\big)\le\|M\|_{\langle x\rangle L^\infty_x}\|u\|_{\langle x\rangle^{-5}L^\infty_x}.$$

Similarly,

$$\begin{aligned}&\left\|\left(\frac{\partial}{\partial\lambda}G\right)*u\big(M(\lambda+h)-M(\lambda)\big)\right\|_{\langle x\rangle^4L^\infty_x}\\&\quad\le\left\|\frac{\partial}{\partial\lambda}G\right\|_{\langle x\rangle^4L^1}\big\|\langle\,\cdot\,\rangle^4\,u\big[M(\lambda+h)-M(\lambda)\big]\big\|_{L^\infty_x}\\&\quad\le\left\|\frac{\partial}{\partial\lambda}G\right\|_{\langle x\rangle^4L^1}\|u\|_{\langle x\rangle^{-5}L^\infty_x}\|M(\lambda+h)-M(\lambda)\|_{\langle x\rangle L^\infty_x}\end{aligned}$$

Hence, by Lemma 5.2.4, we also find

$$\lim_{h\to\infty}\left\|\left(\frac{\partial}{\partial\lambda}G\right)*u\big(M(\lambda+h)-M(\lambda)\big)\right\|_{\langle x\rangle^4L^\infty_x}=0, \tag{5.52}$$

from which the result follows. □

APPENDICES

APPENDIX 1. LAX REPRESENTATION

The observant reader may notice that our choice of Lax pair for the ILW differs from Lax pair typically given in the literature. Indeed, the Lax Pair for the ILW given in [13] and [6] is as follows:

$$\frac{1}{i}\frac{\partial}{\partial x}\psi^{+} + \mu_1\psi^{+} + \mu_2\psi^{-} = u\psi^{+} \tag{A.1a}$$

$$\frac{1}{i}\frac{\partial}{\partial t}\psi_t^{\pm} - 2i(\mu_1 + 1/2\delta)\psi_x^{\pm} - \psi_{xx}^{\pm} = \left[\mp iu_x - Tu_x + \nu\right]\psi^{\pm}, \tag{A.1b}$$

where, in our notation, μ_1 and μ_2 are given in terms of the the spectral parameter λ as

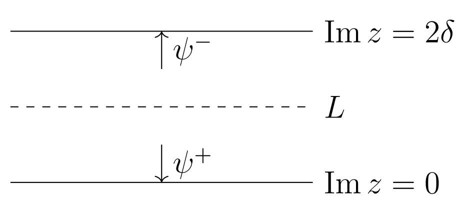
$$\mu_1 = -\frac{1}{2}\lambda\coth(\lambda\delta) \qquad \text{and} \qquad \mu_2 = \frac{1}{2}\lambda\,\mathrm{csch}(\lambda\delta),$$

and ν is an arbitrary constant.

In order to see how the above Lax pair from the liturature compares with (1.5), we consider the function ψ as a function defined along the line $L := \{z \in \mathbb{C} : \operatorname{Im} z = \delta\}$ whose analytic extension to the complex strip $\mathcal{S} := \{z \in \mathbb{C} : 0 < \operatorname{Im} z < 2\delta\}$ has boundary values $\psi^{\pm}$. That is, if Ψ denotes the analytic extension of ψ to $\mathcal{S}$, then we define $\psi(x) := \Psi(x + i\delta)$ $(x \in \mathbb{R})$ and note that

$$\psi^{+} = \lim_{y\searrow 0}\Psi(x + iy) \qquad \text{and} \qquad \psi^{-} = \lim_{y\nearrow 2\delta}\Psi(x + iy)$$

In order to simplify notation, throughout the remainder of these notes, we shall associate ψ with its analytic extension Ψ centered along the line L, so that $\psi(x) = \Psi(x + i\delta)$. We further use the notation $\psi^{\pm}(x) := \psi(x \mp i\delta) := \lim_{y\nearrow\delta}\psi(x \mp iy)$, as demonstrated below in the following diagram:



Using ψ, we define a new function w by $w(z) = e^{-i\lambda z/2}\psi(z)$, where λ is the parameter for μ_1 and μ_2. Plugging $\psi^{\pm}(x) = e^{\pm\delta\lambda/2}e^{i\lambda x/2}w^{\pm}(x)$ into (A.1a) yields

$$iw_x^+ - \frac{\lambda}{2}w^+ + (u-\mu_1)w^+ = \mu_2 e^{-\lambda\delta}w^-$$

which, after rearrangement, becomes

$$iw_x^+ + (-\lambda/2 - \mu_1)w^+ - \mu_2 e^{-\lambda\delta}w^- = -uw^+.$$

Now

$$\begin{aligned}
-\lambda/2 - \mu_1 &= \lambda/2\coth(\lambda\delta) - \lambda \\
&= \frac{\lambda}{2}\left(\frac{e^{\lambda\delta}+e^{-k\delta}}{e^{\lambda\delta}-e^{\lambda\delta}} - 1\right) \\
&= \frac{\lambda}{2}\left(\frac{2e^{-\lambda\delta}}{e^{\lambda\delta}-e^{-\lambda\delta}}\right) \\
&= \frac{1}{2}e^{-\lambda\delta}\lambda\,\mathrm{csch}(\lambda\delta) \\
&= e^{-\lambda\delta}\mu_2 \\
&= \frac{\lambda}{1-e^{-2\lambda\delta}},
\end{aligned}$$

which implies

$$iw_x^+ + \zeta\left(w^+ - w^-\right) = -uw^+, \tag{A.2}$$

where $\zeta := \zeta(\lambda) = \frac{\lambda}{1-e^{-2\lambda\delta}}$ is as defined earlier in theses notes. Given that the steps from (A.1a) to (A.2) are reversible, it follows that (A.1a) and (A.2) are equivalent.

Now, since

$$\widehat{w}(\xi) = \int_{\mathbb{R}} e^{-i\xi x}w(x)\,\mathrm{d}x = \int_{\mathbb{R}} e^{-i\xi x}e^{i\lambda x/2}\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}} e^{-i(\xi-\lambda/2)x}\psi(x)\,\mathrm{d}x = \widehat{\psi}(\xi-\lambda/2),$$

we see from the Fourier Inversion Theorem that

$$w(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}\widehat{\psi}(\xi-\lambda/2)\,\mathrm{d}\xi = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}\widehat{w}(\xi)\,\mathrm{d}\xi, \tag{A.3}$$

where we use $w(x)$ (as opposed to $w(z)$) to denote the restriction of w to the line L. Thus, the boundary values for w. The boundary values $w^\pm$ of w are therefore given by

$$w^\pm(x) = w(x \mp i\delta) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} e^{\pm\delta\xi}\, \widehat{w}(\xi)\, \mathrm{d}\xi.$$

That is,

$$w^\pm(x) = \mathcal{F}^{-1}\left(e^{\pm\delta(\,\cdot\,)}\widehat{w}(\,\cdot\,)\right)(x),$$

which implies

$$\widehat{w}^+(\xi) = e^{2\delta\xi}\, \widehat{w}^-(\xi). \tag{A.4}$$

We further see from (A.3) that

$$\mathcal{F}\big(w(\,\cdot\, - i\delta + iy)\big)(\xi) = e^{-(y-\delta)\xi}\, \widehat{w}(\xi) = e^{-y\xi}\, \widehat{w}^+(\xi).$$

In particular, if $\Psi(z) := w(z + i\delta)$, then

$$\mathcal{F}\big(\Psi(\,\cdot\, + iy)\big)(\xi) = e^{-y\xi}\, \widehat{\Psi}^+(\xi), \tag{A.5}$$

where $\Psi^+(x) := \lim_{y\searrow 0} \Psi(x+iy)$. Using an analogous argument for A.1b, we rewrite (A.1) as

$$L_\delta(\Psi) := \frac{1}{i}\frac{\partial}{\partial x}\Psi^+ - \zeta(\Psi^+ - \Psi^-) = u\Psi^+ \tag{A.6a}$$

$$\frac{1}{i}\frac{\partial}{\partial t}\Psi^\pm + 2i\left(\zeta - \frac{1}{2\delta}\right) + \Psi_{xx} = \left[\pm i u_x - T u_x + \eta\right]\Psi^\pm, \tag{A.6b}$$

where $\eta_\delta(\lambda, \delta, \nu) = \lambda\left(\zeta - \frac{1}{2\delta}\right) + \left(\frac{\lambda}{2}\right)^2 + \nu$, and ζ can be thought of as another spectral parameter parametrized by $\lambda \in \mathbb{R}$ so that

$$\zeta(\lambda) = \frac{\lambda}{1 - e^{-2\delta\lambda}}.$$

Stated in another way, the ILW is an isospectral flow for the linear spectral problem (A.6a).

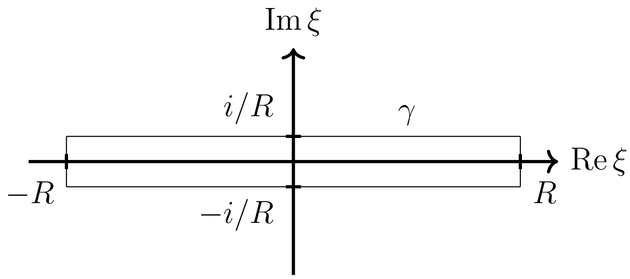
APPENDIX 2. JUMP RELATION

In this appendix we present an alternate proof of the jump relation (5.38) which does not utlize the formulas (2.4).

We begin by considering the integral

$$\int_{\gamma} \frac{e^{ix\xi}}{\xi - \zeta(\lambda;\delta)\left(1 - e^{-2\delta\xi}\right)}\,\mathrm{d}\xi$$

taken over the counterclockwise contour γ shown below



where

$$\zeta(\lambda;\delta) = \frac{\lambda}{1 - e^{-\lambda\delta}} = \lambda\frac{e^{\lambda\delta}}{e^{\lambda\delta} - e^{-\lambda\delta}} = \frac{1}{2}\lambda e^{\lambda\delta}\operatorname{csch}\lambda\delta.$$

To simplify notation, define

$$f(\xi) := f(\xi, x; \lambda, \delta) := \frac{e^{ix\xi}}{\xi - \zeta(\lambda)\left(1 - e^{-2\delta\xi}\right)}.$$

Since

$$|f(\pm R + iy)| = \left|\frac{e^{\pm ixR}e^{-xy}}{\pm R + iy + \zeta(\lambda)\, e^{\mp 2\delta R}e^{-2iy\delta} - \zeta(\lambda)}\right| \le \frac{e^{-xy}}{|R^2 + y^2 + \zeta(\lambda)\, e^{\mp 2\delta R} - \zeta(\lambda)|} \to 0,$$

as $R \to \infty$, it follows that

$$\lim_{R\to\infty}\int_{\gamma} f(\xi)\,\mathrm{d}\xi = \lim_{R\to\infty}\int_{-R+i/R}^{R+i/R} f(\xi)\,\mathrm{d}\xi + \lim_{R\to\infty}\int_{-R-i/R}^{R-i/R} f(\xi)\,\mathrm{d}\xi = 2\pi\big(G_+(x) - G_-(x)\big).$$

Thus, since

$$\operatorname{Res}_{\xi=0} f = -\frac{e^{2\delta\lambda} - 1}{2\delta\lambda e^{2\delta\lambda} - e^{2\delta\lambda} + 1},$$

and

$$\operatorname{Res}_{\xi=\lambda} f = \frac{\left(e^{2\delta\lambda} - 1\right)e^{i\lambda x}}{e^{2\delta\lambda} - 2\delta\lambda - 1},$$

we see from Cauchy's Residue Theorem that

$$G_L^+(x) - G_R^+(x) = \frac{i\left(e^{2\delta\lambda} - 1\right)e^{i\lambda x}}{e^{2\delta\lambda} - 2\delta\lambda - 1} - \frac{ie^{2\delta\lambda} - 1}{2\delta\lambda e^{2\delta\lambda} - e^{2\delta\lambda} + 1} \tag{A.7}$$
$$= i\alpha(\lambda;\delta) + i\beta(\lambda;\delta)e^{i\lambda x},$$

where

$$\alpha(\lambda;\delta) := \frac{1 - e^{-2\delta\lambda}}{1 - e^{-2\delta\lambda} - 2\delta\lambda}, \qquad \text{and} \qquad \beta(\lambda;\delta) := \frac{1 - e^{-2\delta\lambda}}{1 - 2\delta - e^{-2\delta\lambda}}.$$

In the limit $\lambda \to 0$ (*i.e.* the pole at $\xi = \lambda$ collapses to the pole at $\xi = 0$), we find

$$\lim_{\lambda\to 0} G_L^+(x) - G_R^+(x) = -\frac{x}{\delta} + i\frac{2}{3}.$$

APPENDIX 3. HARMONIC ANALYSIS RESULTS

In Sections 4.4 and 4.5 we use multiple results from Grafakos' book *Classical Fourier Analysis*. Statements of these results are given without proof in this appendix. The theorem, corollary and definition numbering within this appendix is consistent with the numbering found in [3]. Unless otherwise stated, throughout this appendix G is assumed to be a locally compact group, and λ an invarniant Haar measure on G.

Definition 1.1.1 (Distribution function)**.** Let X be a measurable space and let μ be a positive, not necessarily finite, measure on X. For f a measurable function on X, the *distribution function* of f is the function d_f defined on $[0, \infty)$ as follows:

$$d_f(\alpha) = \mu\big(\{x \in X \,:\, |f(x)| > \alpha\}\big)$$

Definition 1.1.5 (Weak L^p)**.** Let X be a measurable space and let μ be a positive, not necessarily finite, measure on X. For $0 < p < \infty$, the space *weak* $L^p(X, \mu)$ is defined as the set of all μ-measurable functions f such that

$$\begin{aligned}\|f\|_{L^{p,\infty}} &= \inf\left\{C > 0 \,:\, \mathrm{d}_f(\alpha) \leq \frac{C^p}{\alpha^p} \qquad \text{for all} \quad \alpha > 0\right\} \\ &= \sup\{\gamma\, d_f(\gamma)^{1/p} \,:\, \gamma > 0\}\end{aligned}$$

is finite. The space *weak*-$L^\infty(X, \mu)$ is by definition $L^\infty(X, \mu)$. The weak L^p spaces are denoted by $L^{p,\infty}(X, \mu)$.

Theorem 1.2.10 (Minkowski's inequality)**.** *Let* $1 \leq p \leq \infty$*. For* f *in* $L^p(G)$ *and* g *in* $L^1(G)$ *we have that* $g * f$ *exists* λ*-a.e. and satisfies*

$$\|g * f\|_{L^p(G)} \leq \|g\|_{L^1(G)} \|f\|_{L^p(G)}.$$

Definition 1.2.15 (Approximate Identity)**.** An approximate identity (as $\varepsilon \to 0$) is a family of $L^1(G)$ function k_ε with the following three properties:

(i) There exists a constant $c > 0$ such that $\|k_\varepsilon\|_{L^1(G)} \leq c$ for all $\varepsilon > 0$.

(ii) $\int_G k_\varepsilon \, \mathrm{d}\lambda(x) = 1$ for all $\varepsilon > 0$.

(iii) For any neighborhood V of the identity element e of the group G we have $\int_{V^c} |k_\varepsilon(x)| \, \mathrm{d}\lambda(x) \to 0$ as $\varepsilon \to 0$.

Theorem 1.2.19. *Let k_ε be an approximate identity on a locally compact group G with left Haar measure λ.*

(1) If f lies in $L^p(G)$ for $1 \leq p < \infty$, then $\|k_\varepsilon * f - f\| \to 0$ as $\varepsilon \to 0$.

(2) Let f be a function in $L^\infty(G)$ that is uniformly continuous on a subset K of G, in the sense that for all $\delta > 0$, there is a neighborhood V of the identity element such that for all $x \in K$ and $y \in V$ whe have $|f(y^{-1}x) - f(x)| < \delta$. Then we have that $\|k_\varepsilon * f - f\|_{L^\infty(K)} \to 0$ as $\varepsilon \to 0$. In particular, if f is bounded and continous at a point $x_0 \in G$, then $(k_\varepsilon * f)(x_0) \to f(x_0)$ as $\varepsilon \to 0$.

Theorem 1.2.21. *Let k_ε be a family of functions on a locally compact group G that satisfies properties (i) and (iii) of Definition 1.2.15 and also*

$$\int_G k_\varepsilon(x) \, \mathrm{d}\lambda(x) = a$$

for some fixed $a \in \mathbb{C}$ for all $\varepsilon > 0$. Let $f \in L^p(G)$ for some $1 \leq p \leq \infty$.

(a) If $1 \leq p < \infty$, then $\|k_\varepsilon * f - af\|_{L^p(G)} \to 0$ as $\varepsilon \to 0$.

(b) If $p = \infty$ and f is uniformly continuous on a subset K of G, in the sense that for any $\delta > 0$ there is a neighborhood V of the identity element in G such that $\sup_{x\in G} \sup_{y \in V} |f(y^{-1}x) - f(x)| \leq \delta$, then we have that $\|k_\varepsilon * f - af\|_{L^\infty(K)} \to 0$ as $\varepsilon \to 0$.

Definition 2.1.9. Given a function g on $\mathbb{R}^n$ and $\varepsilon > 0$, we denote by g_ε the fullowing function:

$$g_\varepsilon(x) = \varepsilon^{-n} g(\varepsilon^{-1}x).$$

The following theorem involves the Hardy-Littlewood maximal function $\mathcal{M}(f)$ which Grafakos defines as

$$\begin{aligned}\mathcal{M}(f) &:= \sup_{\varepsilon>0} \frac{1}{v_n\,\varepsilon^n} \int_{\mathbb{R}^n} |f(x-y)| \chi_{B(0,1)}\left(\frac{y}{\varepsilon}\right) \mathrm{d}y \\ &= \sup_{\varepsilon>0} (|f| * k_\varepsilon)(x),\end{aligned}$$

where v_n denotes the volume of the unit ball $B(0,1)$ and $k := v_n^{-1}\,\chi_{B(0,1)}$.

Theorem 2.1.10. *Let $k \geq 0$ be a function $[0,\infty)$ that is continuous except at a finite number of points. Suppose that $K(x) = k(|x|)$ is an integrable function on $\mathbb{R}^n$ that satisfies*

$$K(x) \geq K(y), \qquad \textit{whenever } |x| \leq |y|,$$

i.e., k is decreasing. Then the following estimate is true:

$$\sup_{\varepsilon>0} (|f| * K_\varepsilon)(x) \leq \|K\|_{L^1} \mathcal{M}(f)(x) \tag{A.8}$$

for all locally integrable functions f on $\mathbb{R}^n$.

Grafakos defines the term **radially decreasing majorant** in the following remark.

Remark 2.1.11. Theorem 2.1.10 can be generalized as follows. If K is an L^1 function on $\mathbb{R}^n$ such that $|K(x)| \leq k_0(|x|) = K_0(x)$, where k_0 is nonnegative decreasing function on $[0,\infty)$ that is continuous except at a finiate number of points, then (A.8) holds with $\|K\|_{L^1}$ replaced by $\|K_0\|_{L^1}$. Such K_0 is called a *radially decreasing majorant* of K.

Theorem 2.1.14. *Let (X,μ), (Y,ν) be measurable spaces and let $0 < p < \infty$, $0 < q < \infty$. Suppose that D is a dense subspace of $L^p(X,\mu)$, T_ε is a linear operator that maps $L^p(X,\mu)$ into a subspace of measurable functions, which are defined everywhere on Y. For $y \in Y$, define a sublinear operator*

$$T_*(f)(y) = \sup_{\varepsilon>0} |T_\varepsilon(f)(y)|$$

and assume that $T_*(f)$ *is* μ*-measurable for any* $f \in L^p(X,\mu)$*. Suppose that for some* $B > 0$ *and for al* $f \in L^p(X)$ *we have*

$$\|T_*(f)\|_{L^{q,\infty}} \leq B\|f\|_{L^p}$$

and that for all $f \in D$,

$$\lim_{\varepsilon \to 0} T_\varepsilon(f) = T(f)$$

exists and is finite ν*-a.e. (and defines a linear operator on* D*). Then for all functions* f *in* $L^p(X,\mu)$ *the limit above exists and is finite* ν*-a.e., and defines a linear operator* T *on* $L^p(X)$ *(uniquely extending* T *defined on* D*) that satisfies*

$$\|T(f)\|_{L^{q,\infty}} \leq B\|f\|_{L^p}$$

for all functions f *in* $L^p(X)$*.*

Corollary 2.1.19 (Differentiation theorem for approximate identities)**.** *Let* K *be a function on* $\mathbb{R}^n$ *that has an integrable radially decreasing majorant. Let* $c = \int_{\mathbb{R}^n} K(x)\,\mathrm{d}x$*. Then for all* $f \in L^p(\mathbb{R}^n)$ *and* $1 \leq p < \infty$,

$$\big(f * K_\varepsilon\big)(x) \to cf(x)$$

for almost all $x \in \mathbb{R}^n$ *as* $\varepsilon \to 0$*.*

Proposition 2.2.16 (Hausedorff-Young inequality)**.** *For every function* f *in* $L^p(\mathbb{R}^n)$ *we have the estimate*

$$\left\|\widehat{f}\right\|_{L^q} \leq \|f\|_{L^p}$$

whenever $1 \leq p \leq 2$ *and* $q := \frac{p}{p-1}$ *denotes the Hölder conjugate of* p*.*

Theorem 5.1.5. *Let* $1 \leq p < \infty$*. For any* $f \in L^p(\mathbb{R})$ *we have*

$$f * Q_\varepsilon - H^{(\varepsilon)}(f) \to 0$$

in L^p *and almost everywhere as* $\varepsilon \to 0$*. Moreover, for* $\phi \in \mathscr{S}(\mathbb{R})$ *we have*

$$F_\phi(x+iy) = \frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{\phi(t)}{x+iy-t}\,\mathrm{d}t \to \phi(x) + iH(\phi)(x)$$

as $y \nearrow 0$ *for all* $x \in \mathbb{R}$.

Theorem 5.1.12. *There exists a constant* C *such that for all* $1 < p < \infty$ *we have*

$$\|H^{(*)}(f)\|_{L^p} \leq C \max\big(p, (p-1)^{-2}\big)\|f\|_{L^p}.$$

APPENDIX 4. NOTATION INDEX

Fundamental Notation

Symbol	Meaning	Ref.
$:=$	defined to be; for example $a := b$ means "a is defined to be b"	p.1
p. v.	Cauchy principle value; given by $$\text{p. v.} \int_{\mathbb{R}} f(x)\,\mathrm{d}x := \lim_{\varepsilon \searrow 0} \int_{\lvert x\rvert > \varepsilon} f(x)\,\mathrm{d}x$$	p.1
a.e.	abbreviation for almost everywhere	p.51
$\mathbb{R}$	the set of all real numbers	p.12
$\mathbb{C}$	the set of all complex numbers	p.14
$\mathbb{Z}$	the set of all integers	
χ_A	characteristic function on a set A; given by $$\chi_A(x) := \begin{cases} 1, & x \in A \\ 0, & \text{otherwise} \end{cases}$$	p.13
$m(\,\cdot\,)$	Lebesgue measure on $\mathbb{R}$	p.61
$\mathcal{F}$, $\hat{(\,\cdot\,)}$	Fourier transform defined as $$\hat{f}(\xi) := (\mathcal{F}f)(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\,\mathrm{d}x$$	p.16
$\mathcal{F}^{-1}$, $\check{(\,\cdot\,)}$	inverse Fourier transform defined as $$\check{g}(x) := \left(\mathcal{F}^{-1}g\right)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi} g(\xi)\,\mathrm{d}\xi$$	p.16
$\mathscr{S}(\mathbb{R})$	space of all Schwartz class functions on $\mathbb{R}$	p.64

Continued on next page

Symbol	Meaning	Ref.
$\operatorname{Res}_{z=c} f$	complex residue of a function f at the pole $z=c$	p.25
$(\,\cdot\, \pm i0)$	implied limit of $(\,\cdot\, \pm i\varepsilon)$ as $\varepsilon \searrow 0$	p.22
f^+	lower boundary $f^+(x) := \lim_{y\searrow 0} f(x+iy)$ of a function f analytic on S_δ, where $x, y \in \mathbb{R}$	p.6
f^-	upper boundary $f^-(x) := \lim_{y\nearrow 0} f(x+i2y)$ of a function f analytic on S_δ, where $x, y \in \mathbb{R}$	p.6
$\lesssim$	$q \lesssim s$ means there exists some fixed constant C so that $q \leq C\, s$; the constant C is commonly referred to as "the implied constant"	p.8
$\lesssim_k$	$q \lesssim s$ means there exists some constant $C := C(k)$ depending only on the parameter k so that $q \leq C\, s$; the constant C is commonly referred as "the implied constant"	p.39
$\log_+ t$	the function given by $\max\big\{0, \log(t)\big\}$	p.14
$\langle x \rangle$	short-hand notation for $\big(1+\lvert x\rvert^2\big)^{1/2}$	p.7
$L^{p,s}(\mathbb{R})$	space of measurable functions with $$\lVert f\rVert_{L^{p,s}} := \left(\int_{\mathbb{R}} \langle x\rangle^{sp} \lvert f(x)\rvert^p\right)^{1/p} < \infty$$	p.36
$\langle\,\cdot\,\rangle L^\infty(\mathbb{R})$	space of measurable functions with $$\lVert f\rVert_{\langle\,\cdot\,\rangle L^\infty} := \operatorname*{ess\,sup}_{x\in\mathbb{R}} \big\lvert \langle x\rangle^{-1} f(x)\big\rvert < \infty$$	p.37
$B_Y(y_0, r)$	the open ball $\{y \in Y \;:\; \lVert y - y_0\rVert_Y < r\}$ in the metric space Y with radius r centered at $y_0 \in Y$	p.8
$Y \to Z$	a map from a space Y to a space Z	p.41

Continued on next page

Symbol	Meaning	Ref.
$Y \circlearrowleft$	a map from a space Y into itself	p.41
$\| \cdot \|_{Y \to Z}$	the induced operator norm for an operator with domain Y and co-domain Z	p.41

Chapter 1 Notation

Symbol	Meaning	Ref.
δ	depth of stratified fluids—typically taken to be $\delta = 1$	p.1
$\mathcal{S}_\delta$	the complex strip $\{z \in \mathbb{C} \ : \ 0 < \operatorname{Im} z < 2\delta\}$	p.6
f^+	lower boundary $f^+(x) := \lim_{y \searrow 0} f(x + iy)$ of a function f analytic on $\mathcal{S}_\delta$, where $x, y \in \mathbb{R}$	p.6
f^-	upper boundary $f^-(x) := \lim_{y \nearrow 0} f(x + i2y)$ of a function f analytic on $\mathcal{S}_\delta$, where $x, y \in \mathbb{R}$	p.6
L_δ	operator on functions analytic in the complex strip $\mathcal{S}_\delta$; given by $$L_\delta(\Psi) := \frac{1}{i}\frac{\partial}{\partial x}\Psi^+ - \zeta\left(\Psi^+ - \Psi^-\right) = u\Psi^+$$	p.6
λ	a spectral parameter for the linear spectral problem (1.4)	p.7
ζ	a spectral parameter for (1.4) commonly parameterized by λ as $\zeta(\lambda; \delta) = \dfrac{\lambda}{1 - e^{-2\delta\lambda}}$	p.7
$\lambda(\zeta)$	inverse of the map $\lambda \to \zeta(\lambda)$	p.7
$\langle x \rangle$	short-hand notation for $\left(1 + \lvert x\rvert^2\right)^{1/2}$	p.7
$B_Y(y_0, r)$	the open ball $\{y \in Y \ : \ \lVert y - y_0 \rVert_Y < r\}$ in the metric space Y with radius r centered at $y_0 \in Y$	p.8
M_1, M_e, N_1, N_e	depending on context, either Jost solutions or analytic extensions of solutions to the integral equations (1.9)	p.7, p.9
M_1^+, M_e^+, N_1^+, N_e^+	depending on context, either solutions to the integral equations (1.9) or lower boundary values of the Jost solutions	p.9, p.7

Continued on next page

Symbol	Meaning	Ref.
$r(\lambda;\delta)$	reflection coefficent; given by $$r(\lambda;\delta) = \frac{b(\lambda;\delta)}{a(\lambda;\delta)},$$ where $$b(\lambda) := \frac{i}{1-2\delta\zeta(-\lambda)} \int_{\mathbb{R}} e^{-i\lambda x} u(x)\, M_1^+(x;\lambda,\delta)\,\mathrm{d}x$$ $$a(\lambda) := 1 + \frac{i}{1-2\delta\zeta(\lambda)} \int_{\mathbb{R}} u(x)\, M_1^+(x;\lambda,\delta)\,\mathrm{d}x$$	p.8
$\mathscr{D}$	the direct scattering map for the Intermediate Long Wave (ILW) equation; maps ILW initial data u to the corresponding reflection coefficient r	p.8
G_L, G_R	formal Green's functions corresponding to the linear spectral problem (1.4)	p.8, p.9
$\alpha(\lambda;\delta)$	residue of $e^{iz\xi}/p(\xi)$ at the $\xi = 0$ pole; given by $$\alpha(\lambda;\delta) = \frac{1}{1-2\delta\zeta(\lambda;\delta)}$$	p.10
$\beta(\lambda;\delta)$	$e^{iz\lambda}$ times the residue of $e^{iz\xi}/p(\xi)$ at the $\xi = \lambda$ pole; given by $$\alpha(\lambda;\delta) = \frac{1}{1-2\delta\zeta(-\lambda;\delta)}$$	p.10
K^+	non-residue term resulting from shifting the integration contours of G_L^+ and G_R^+	p.10

Continued on next page

Symbol	Meaning	Ref.
$T_{\star,\lambda,u}$	bounded operators on $\langle \cdot \rangle\, L^{\infty}(\mathbb{R})$ given by $$T_{\star,\lambda,u} f(x) := \big[G_{\star}^{+}(\,\cdot\,;\lambda)\big] * (u\, f)(x),$$ where $\star = L$ or R	p.10

Chapter 2 Notation

Symbol	Meaning	Ref.
$\mathscr{D}$	the direct scattering map for the Intermediate Long Wave (ILW) equation; maps ILW initial data u to the corresponding reflection coefficient r	p.8
$\star$	used as a placeholder for both L and R; for example, if a statement contains the notation "$G_\star$ ($\star = L$, or R)," then it is equally true (or not true) for both G_L and G_R	p.15
f^+	lower boundary $f^+(x) := \lim_{y \searrow 0} f(x+iy)$ of a function f analytic on S_δ, where $x, y \in \mathbb{R}$	p.6
f^-	upper boundary $f^-(x) := \lim_{y \nearrow 0} f(x+i2y)$ of a function f analytic on S_δ, where $x, y \in \mathbb{R}$	p.6
δ	depth of stratified fluids—typically taken to be $\delta = 1$	p.1
λ	a spectral parameter for the linear spectral problem (1.4)	p.7
ζ	a spectral parameter for (1.4) commonly parameterized by λ as $\zeta(\lambda;\delta) = \dfrac{\lambda}{1-e^{-2\delta\lambda}}$	p.7
$\lambda(\zeta)$	inverse of the map $\lambda \to \zeta(\lambda)$	p.7
ζ^*	nonlinear reflection $\zeta\big(-\lambda(\zeta)\big)$	p.13
$G_\star^+$	lower boundary value of the Greens' function whose contour of integration is $\Gamma_\star$; given by $$G_\star^+(x;\lambda,\delta) := \frac{1}{2\pi}\int_{\Gamma_\star} e^{ix\xi}\,\frac{1}{p(\xi;\lambda,\delta)}\,\mathrm{d}\xi$$	p.15
Γ_L	contour along the real line which bypasses the roots of p from below	p.15

Continued on next page

Symbol	Meaning	Ref.
Γ_R	contour along the real line which bypasses the roots of p from above	p.15
p	Fourier symbol of the Green's functions; given by $$p(\xi;\lambda,\delta) = \xi - \zeta(\lambda)\left(1 - e^{-2\delta\xi}\right)$$ and commonly denoted as $p(\xi;\lambda,\delta)$, $p(\xi;\zeta,\delta)$, $p(\xi;\lambda)$, $p(\xi;\zeta)$, and $p(\xi)$.	p.14
$\mathrm{Res}_{z=c} f$	complex residue of a function f at the pole $z = c$	p.25
$\alpha(\lambda;\delta)$	residue of $e^{iz\xi}/p(\xi)$ at the $\xi = 0$ pole; given by $$\alpha(\lambda;\delta) = \frac{1}{1 - 2\delta\zeta(\lambda;\delta)}$$	p.13
$\beta(\lambda;\delta)$	$e^{iz\lambda}$ times the residue of $e^{iz\xi}/p(\xi)$ at the $\xi = \lambda$ pole; given by $$\beta(\lambda;\delta) = \frac{1}{1 - 2\delta\zeta(-\lambda;\delta)}$$	p.13
K^+	non-residue term resulting from shifting the integration contours of G_L^+ and G_R^+	p.13
$\log_+ t$	the function given by $\max\left\{0, \log(t)\right\}$	p.14
$\mathcal{R}_\delta$	the complex strip $\{z \in \mathbb{C} \ : \ -\pi/\delta \leq \mathrm{Im}\, z \leq \pi/\delta\}$ about the real line	p.15
$\mathcal{R}_\star$	sum residues of $e^{iz\xi}/p(\xi)$ at the $\xi = 0$ and $\xi = \lambda$ poles	p.16
δ_c	Dirac delta-function centered at $x = c$	p.16
W_k	k^{th} branch ($k \in \mathbb{Z}$) of the complex Lambert W function	p.18
Σ_c	the integration contour $\mathbb{R} + ic\pi$	p.26

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Symbol	Meaning	Ref.
$\Sigma(R,\ c)$	the integration contour $(-R,R)+ic\,\pi$, where $R>0$ and $(-R,R):=\{x\in\mathbb{R}\ :\ -R<x<R\}$	p.26
K_ζ	the integral function given by $$K_\zeta(x):=\int_{\mathbb{R}}\frac{e^{ix\xi}}{\xi-\zeta\left(1-e^{-2\xi}\right)+i\pi}\,\mathrm{d}\xi$$	p.29
K_q	the integral function given by $$K_q(x):=\int_{\mathbb{R}}\frac{e^{ix\xi}\chi\left(2^{-q}x\xi\right)}{\xi-\zeta\left(1-e^{-2\xi}\right)+i\pi}\,\mathrm{d}\xi$$	p.29

Chapter 3 Notation

Symbol	Meaning	Ref.
$\langle x\rangle$	short-hand notation for $\left(1+\lvert x\rvert^2\right)^{1/2}$	p.36
$L^{p,s}(\mathbb{R})$	space of measurable functions with $$\lVert f\rVert_{L^{p,s}} := \left(\int_{\mathbb{R}} \langle x\rangle^{sp}\,\lvert f(x)\rvert^p\right)^{1/p} < \infty$$	p.36
$\langle\,\cdot\,\rangle L^\infty(\mathbb{R})$	space of measurable functions with $$\lVert f\rVert_{\langle\,\cdot\,\rangle L^\infty} := \operatorname*{ess\,sup}_{x\in\mathbb{R}} \left\lvert \langle x\rangle^{-1} f(x)\right\rvert$$ finite	p.37
$L^p_\xi(\mathbb{R})$	space of measurable functions which are L^p integrable with respect to the variable ξ; similar subscript notation is used for other function spaces	p.37
$\star$	used as a placeholder for both L and R; for example, if a statement contains the notation "$G_\star$ ($\star = L$, or R)," then it is equally true (or not true) for both G_L and G_R	p.15
$T_{\star,\lambda,u}$	bounded operator on $\langle\,\cdot\,\rangle\, L^\infty(\mathbb{R})$ given by $$T_{\star,\lambda,u} f(x) := \left[G^+_\star(\,\cdot\,;\lambda)\right] * (u\,f)(x)$$ based on context, $T_{\star,\lambda,u}$ is sometimes denoted by $T_\star$, $T_{\star,\lambda}$, or T_λ	p.37
X	space of potentials u with $\left\lVert\langle\,\cdot\,\rangle^4\,u\right\rVert_{L^2} < \infty$	p.38
$\lesssim_k$	$q \lesssim s$ means there exists some constant $C := C(k)$ depending only on the parameter k so that $q \le C\,s$; the constant C is commonly referred as "the implied constant"	p.39

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Symbol	**Meaning**	**Ref.**
$\chi_{\pm}$	the characteristic functions $\chi_- := \chi_{(-\infty,0)}$, $\chi_+ := \chi_{(0,\infty)}$ on the respective intervals $(-\infty,0)$ and $(0,\infty)$	p.40
G	as specified in Remark 11, $G(x,\lambda)$ and $G(\lambda)$ are occasionally used to as shorthand notations for $G^+_\star(x;\lambda)$	p.47
$G_h(\lambda)$	the difference quotient of $G^+_\star$ with respect to λ; given by $$G_h(\lambda) := \frac{G(\lambda+h)-G(\lambda)}{h}$$	p.48
$\left(\frac{1}{p_\lambda(\xi)}\right)_h$	the difference quotient of $1/p$ with respect to λ; $$\left(\frac{1}{p_\lambda(\xi)}\right)_h := \frac{1}{h}\left[\frac{1}{p(\xi;\lambda+h)} - \frac{1}{p(\xi;\lambda)}\right]$$	p.48
$\operatorname{Res}_{z=c} f$	complex residue of a function f at the pole $z=c$	p.49

Chapter 4 Notation

Symbol	Meaning	Ref.
$\mathcal{S}_\delta$	the complex strip about the real axis defined by $\mathcal{S}_1 := \{z \in \mathbb{C} \;:\; 0 < \operatorname{Im} z < 2\}$	p.51
$G_\star$	analytic continuation of $G_\star^+$ to the analytic strip $\mathcal{S}_1$	p.51
$G_\star^-$	the upper boundary value of $G_\star$ defined as a distribution in that $$G_\star^- * f = \lim_{y \nearrow 2} G_\star^+(\,\cdot\, + iy) * f$$ for $f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ $(1 < p \leq 2)$	p.51
K	analytic continuation of K^+ to the $\mathcal{S}_1$	p.51
$\mathfrak{C}$	a portion of K whose limit as $y \nearrow 2$ is a continuous limit operator; given by $$\mathfrak{C}(x,y) := \frac{1}{2\pi} e^{-\pi\lvert x\rvert}\, e^{-\operatorname{sign}(x)\, i\pi y} \int_{\mathbb{R}} e^{ix\xi} \rho\big(\xi, y, \operatorname{sign}(x)\big)\, \mathrm{d}\xi,$$ where $x \in \mathbb{R}$ and $y \in [0,2]$	p.52
ρ	a function given by $$\rho\big(\xi, y, c; \lambda\big) := \begin{cases} \dfrac{e^{-y\xi}}{p(\xi;\lambda) + i\,c\,\pi}, & \xi > 0 \\ \dfrac{1}{\zeta(\lambda)} \dfrac{\big(\zeta(\lambda) - \xi - c\,i\pi\big)e^{(2-y)\xi}}{p(\xi;\lambda) + i\,c\,\pi}, & \xi < 0. \end{cases}$$	p.52
$\mathcal{R}_\star$	sum residues of $e^{iz\xi}/p(\xi)$ at the $\xi = 0$ and $\xi = \lambda$ poles	p.52

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Symbol	Meaning	Ref.
$\mathcal{E}_\varepsilon$	a family of convolution operators given by $$\left(\mathcal{E}_\varepsilon f\right)(x) := \frac{e^{-i\pi(2-\varepsilon)}}{2\pi i}\int_{-\infty}^{x}\frac{e^{-\pi\|x-x'\|}}{(x-x')-i\varepsilon}f(x')\,\mathrm{d}x' + \frac{e^{i\pi(2-\varepsilon)}}{2\pi i}\int_{x}^{\infty}\frac{e^{-\pi\|x-x'\|}}{(x-x')-i\varepsilon}f(x')\,\mathrm{d}x'$$	p.57
E_ε	exponentially weighted Cauchy transform; given by $$E_\varepsilon f(x) := \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{e^{-\pi\|x-x'\|}}{(x-x')-i\varepsilon}f(x')\,\mathrm{d}x'$$	p.61
E	exponentially weighted Hilbert Transform; given by $$Ef(x) := \frac{1}{2\pi i}\,\mathrm{p.\,v.}\int_{\mathbb{R}}\frac{e^{-\pi\|x-x'\|}}{x-x'}f(x')\,\mathrm{d}x',$$ where $\mathrm{p.\,v.}\int(\,\cdot\,)\,\mathrm{d}\mu$ denotes a principle value integral.	p.52
$\mathscr{S}(\mathbb{R})$	space of all Schwartz class functions on $\mathbb{R}$	p.64
$\mathcal{E}_\varepsilon,\ \mathcal{P}_\varepsilon$	two families of convolution operators given by $$\mathcal{E}_\varepsilon(y) := \frac{1}{2\pi i}\frac{y}{y^2+\varepsilon^2}e^{-\pi\|y\|},\quad \text{and}\quad \mathcal{P}_\varepsilon(y) := \frac{1}{\pi}\frac{\varepsilon}{y^2+\varepsilon^2}e^{-\pi\|y\|}$$	p.65
$E^{(\varepsilon)}$	truncated exponentially weighted Hilbert Transform; given by $$E^{(\varepsilon)}f(x) := \frac{1}{2\pi i}\int_{\|x'\|\geq\varepsilon}\frac{e^{-\pi\|x'\|}}{x'}f(x-x')\,\mathrm{d}x'$$ by definition, $(Ef)(x) = \lim_{\varepsilon\searrow 0}E^{(\varepsilon)}f(x)$	p.65
P_ε	Poisson kernel; given by $P_\varepsilon(y) = \frac{1}{\pi}\frac{\varepsilon}{y^2+\varepsilon^2}$	p.66

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Symbol	Meaning	Ref.
E^*	the maximal operator associated with the Cauchy transform E_ε; given by $$E^* f(x) := (E_\varepsilon)^* f(x) := \sup_{\varepsilon>0} \{|E_\varepsilon f(x)|\}.$$	p.68
M	The Hardy-Littlewood maximal operator; given by $$Mf(x) = \sup_{r>0} \left\{ \frac{1}{B(0,r)} \int_{B(0,r)} |f(x-x')| \, \mathrm{d}x' \right\}.$$	p.68
m_E	Fourier multiplier for the exponentially weighted Hilbert transform; given by $$m_E(\xi) = \frac{1}{\pi} \arctan(\xi/\pi)$$	p.73
$L^{p,\infty}$	weak L^p space; also denoted *weak*-L^p	p.118

Chapter 5 Notation

Symbol	Meaning	Ref.
M_1, M_e, N_1, N_e	depending on context, either Jost solutions or analytic extensions of solutions to the integral equations (5.4)	p.78, p.80
X	space of potentials u with $\left\| \langle \cdot \rangle^4 u \right\|_{L^2} < \infty$	p.38
c_0	a strictly positive constant chosen in Proposition 5.2.1 to ensure the existence and uniqueness of Jost solutions for every potential $u \in X$ with $\|u\|_X < c_0$	p.81
$T_{\star,\lambda,u}$	bounded operator on $\langle \cdot \rangle L^\infty(\mathbb{R})$ given by $$T_{\star,\lambda,u} f(x) := \left[G_\star^+(\,\cdot\,;\lambda)\right] * (u\,f)(x)$$ based on context, $T_{\star,\lambda,u}$ is sometimes denoted by $T_\star$, $T_{\star,\lambda}$, or T_λ	p.81
a, b, $\breve{a}$, $\breve{b}$	coefficients for the scattering equations (5.36) and (5.37); given by $$a(\lambda) := 1 + i\alpha(\lambda) \int_{\mathbb{R}} u(x)\, M_1^+(x;\lambda,u)\,\mathrm{d}x$$ $$b(\lambda) = i\beta(\lambda) \int_{\mathbb{R}} e^{-ix\lambda}\, u(x)\, M_1^+(x;\lambda,u)\,\mathrm{d}x$$ $$\breve{a}(\lambda) := 1 + \alpha(\lambda) \int_{\mathbb{R}} u(x)\, N_1(x;\lambda,u)\,\mathrm{d}x$$ $$\breve{b}(\lambda) = i\beta(\lambda) \int_{\mathbb{R}} e^{-ix\lambda}\, u(x)\, N_1(x;\lambda,u)\,\mathrm{d}x$$	p.103
$\alpha(\lambda;\delta)$	residue of $e^{iz\xi}/p(\xi)$ at the $\xi = 0$ pole; given by $$\alpha(\lambda;\delta) = \frac{1}{1 - 2\delta\zeta(\lambda;\delta)}$$	p.13

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Symbol	Meaning	Ref.
$\beta(\lambda;\delta)$	$e^{iz\lambda}$ times the residue of $e^{iz\xi}/p(\xi)$ at the $\xi=\lambda$ pole; given by $$\alpha(\lambda;\delta)=\frac{1}{1-2\delta\zeta(-\lambda;\delta)}$$	p.13
$\mathscr{D}$	direct scattering map for the ILW; given by $\mathscr{D}: B_X(0,c_0)\ni u\mapsto r\in L^\infty_\lambda(\mathbb{R})$	p.104
r	reflection coefficient; given by $r(\lambda)=b(\lambda)/a(\lambda)$	p.104

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