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ON OPERATOR EQUATION AXB - CXD = CE VIA SUBNORMALITY IN HILBERT SPACES

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Abstract. The purpose of this study is to give the necessary and sufficient conditions of the existence of solution for an operator equation of Sylvester type with subnormality of bounded operators in finite dimension complex separable Hilbert space. Our results improve and generalize some results with operators in restricted cases.

Keywords: Sylvester equation, Fuglede-Putnam property, subnormal operator, Kronecker canonical form.

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1. Introduction

The operator equations have some applications in various fields of mathematics, physics, quantum mechanic.... The equation

$$AXB - CXD = E \tag{1.1}$$

is one of the important kind of operators equations, since its applications in the study of perturbations of the generalized eigenvalue problem as in [7], in the stability problems for descriptor systems [2], and in the numerical solution of implicit ordinary differential equations [10].

Equation (1.1) can be written in general form as

$$\sum_{i=1}^{n} A_i X B_i = E,$$

where X is unknown and A_i, B_i and E are operators.

Many authors have gave some technics to prove existence of solution for equation (1.1), Rosa [14,15] introduced a method for solving this equation based on the reduction of the

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pencils $\lambda C - A$ and $\lambda B - D$ to the Kronecker canonical form. Later, Chu [8] proposed another approach for solvability of (1.1) based on the reduction of the pencils $\big((A, C), (D, B)\big)$ to the Hessenberg form or the Schur form. In [7] it has been given necessary and sufficient conditions of the existence and uniqueness of the solution, such that the pencils $\lambda C - A$ and $\lambda B - D$ are regular and the intersection of the spectra of these pencils is empty.

In this study, motivated by previous results we propose to give necessary and sufficient conditions for the existence of solution for equation (1.1) and the equation

$$AXB - XD = E \tag{1.2}$$

We will use in this approach some operators technics, subnormality of operators and generalized Fuglede-Punam property for the subnormality case.

2. Preliminaries

Let $B(H)$ be the algebra of all bounded linear operators on an infinite dimensional complex Hilbert space H.

Definition 2.1 (9)**.** *Let S be an operator in $B(H)$, S is said to be normal if and only if it commutes with its adjoint. i.e., $SS^* = S^*S$.*

Definition 2.2 (9)**.** *Let S be an operator in $B(H)$, S is said to be subnormal if there exists a space K, on which S admits an extension N_S such that*

(1) *$H \subset K$.*
(2) *N_S is normal on K.*
(3) *$N_S/H = S$.*

In general we can taking $K = H \oplus H^\perp$, so N_S is given as $N_S = \begin{pmatrix} S & Q \\ 0 & T \end{pmatrix}$, where $Q : H^\perp \to H$ and T is defined on $H^\perp$.

Lemma 2.1 (17)**.** *Let S be a subnormal operator on Hilbert space H, then $\alpha S + \beta S^*$ is subnormal, for all complex numbers α, β.*

Lemma 2.2 (12)**.** *Let S be a subnormal operator on Hilbert space H. The normal extension N_S can be written in the form*

$$N_S = \begin{pmatrix} S & (S^*S - SS^*)^{\frac{1}{2}} \\ 0 & Q^*S(Q^*)^{-1} \end{pmatrix},$$

where $Q = (S^*S - SS^*)^{\frac{1}{2}}$.

Definition 2.3 (13)**.** *Let S and T be two normal operators in $B(H)$. The pair (S, T) is said to satisfy Fuglede-Putnam property if for any operator $Q \in B(H)$ such that $SQ = QT$, then $S^\star Q = QT^\star$.*

Theorem 2.1 (11)**.** *Let A and B^* be subnormal and X an operator such that $AX = XB$, then $A^*X = XB^*$.*

From lemma 2.1, we can deduce the following result.
If A and B are two subnormal operators in $B(H)$ and $AX = XB$ for some operator X in $B(H)$, then $A^*X = XB^*$.

Lemma 2.3. *[11] Let A, B^* and C be subnormal operators such that N_A commutes with N_C and N_D* commutes with N_{B^*},where N_A, N_C, N_{D^*} and N_{B^*} denote the normal extensions of A, C, D^* and B^* respectively. If for an operator X we have $AXD = CXB$, then $A^*XD^* = C^*XB^*$.*

Proposition 2.1. *Let A and B subnormal operators in $B(H)$ and N_A, N_B their normal minimal extensions (In the sens: if the smallest closed sub- space of H containing H and reducing N_A and N_B is H itself). If N_A commutes with N_B, then A commutes with B.*

Proof. The extension N_A and N_B can be written in the form

$$N_A = \begin{pmatrix} A & A_1 \\ 0 & A_2 \end{pmatrix}, \quad N_B = \begin{pmatrix} B & B_1 \\ 0 & B_2 \end{pmatrix}.$$

$N_A N_B = N_B N_A$, implies that

$$\begin{pmatrix} AB & AB_1 + A_1 B_2 \\ 0 & A_2 B_2 \end{pmatrix} = \begin{pmatrix} BA & BA_1 + B_1 A_2 \\ 0 & B_2 A_2 \end{pmatrix}.$$

This yields $AB = BA$. □

Lemma 2.4. *[16] Let Q, R, S and T be some operators in $B(H)$, if $\begin{pmatrix} Q & R \\ S & T \end{pmatrix}$ is invertible, then $SS^* + TT^*$ is invertible.*

Definition 2.4. *Two triplets (T_1, T_2, T_3) and (S_1, S_2, S_3) of operators in $B(H)$ are said to be equivalent if and only if there exist invertible operators U, V and W such that*

$$\begin{cases} UT_1 = S_1 V \\ UT_2 = S_2 W \\ WT_3 = S_3 V \end{cases}$$

3. Main results

Theorem 3.1. *Let A, B, D and E subnormal operators in $B(H)$ such that N_B commutes with N_D, where N_B and N_D the minimal normal extensions of B and D respectively. Then the equation*

$$AXB - XD = E, \tag{3.1}$$

has a solution in $B(H)$ if and only if $\left(\begin{pmatrix} A & E \\ 0 & D \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right)$ is equivalent to $\left(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right)$.

Proof. Let $U = \begin{pmatrix} I & CX \\ 0 & I \end{pmatrix}, \quad V = \begin{pmatrix} I & XB \\ 0 & I \end{pmatrix}$.
Since U and V are invertible, then

$$\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} A & E \\ 0 & D \end{pmatrix} = \begin{pmatrix} A & E + XD \\ 0 & D \end{pmatrix}$$

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & XB \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & AXB \\ 0 & D \end{pmatrix},$$

$$\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & XB \\ 0 & I \end{pmatrix},$$

which implies that

$$AXB - XD = E$$

.

Reciprocally, suppose that $\left(\begin{pmatrix} A & E \\ 0 & D \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right)$
and $\left(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right)$ are equivalent.

Let $U = \begin{pmatrix} Q & R \\ S & T \end{pmatrix}$ and $V = \begin{pmatrix} Q' & R' \\ S' & T' \end{pmatrix}$, so we get

$$\begin{pmatrix} Q & R \\ S & T \end{pmatrix}\begin{pmatrix} A & E \\ 0 & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} Q' & R' \\ S' & T' \end{pmatrix}$$

and

$$\begin{pmatrix} Q & R \\ S & T \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} Q' & R' \\ S' & T' \end{pmatrix}.$$

Thus we obtain

$$AQ' = QA, \;\; QE + RD = AR' \;\text{ and }\; SA = DS'$$
$$SE + TD = DT', \;\; Q = Q'\text{and }\; RB = R'$$
$$S = BS' \;\text{ and‘ }\; TB = BT'.$$

Since $Q = Q'$ and $QA = AQ'$, so from lemma 2.1 and theorem 2.1 (generalized Fuglede-Putnam property), we get

$$QA^* = A^*Q.$$

Passing to the adjoint we get

$$AQ^* = Q^*A.$$

Then

$$Q^*QE = Q^*AR' - Q^*RD = Q^*ARB - Q^*RD.$$

Which gives

$$Q^*QE = A(Q^*R)B - (Q^*R)D. \tag{3.2}$$

We have also

$$S^*SE = S^*DT' - S^*TD. \tag{3.3}$$

But

$$BSA = BDS' = DBS' = DS.$$

Since $N_B N_D = N_D N_B$, then from proposition 2.1 we get $BD = DB$ and so using lemma 2.3 we obtain

$$B^*SA^* = D^*S.$$

Passing to the adjoint we get

$$AS^*B = S^*D.$$

Substring in (3.3), it becomes

$$S^*SE = AS^*BT' - S^*TD = A(S^*T)B - (S^*T)D$$

. Next (3.2) and (3.3) give

$$(Q^*Q + S^*S)E = A(Q^*R + S^*T)B - (Q^*R + S^*T)D.$$

Since $(Q^*Q + S^*S)$ is invertible and commutes with A, then

$$E = A(Q^*Q + S^*)S^{-1}(Q^*R + S^*T)B - (Q^*Q + S^*)S^{-1}(Q^*R + S^*T)D.$$

Then

$$X = Q^*Q + S^*S)^{-1}(Q^*)R + S^*T.$$

□

Theorem 3.2. *Let A, B, C, D and E subnormal operators in $B(H)$. Assume that*

(1) *N_A commutes with N_C*
(2) *N_B commutes with N_D,*

where N_A, N_B, N_C and N_C are normal minimal extensions of A, B, C and D respectively. Then the equation

$$AXB - CXD = CE \tag{3.4}$$

has a solution in $B(H)$ if and only if $\left(\begin{pmatrix} A & E \\ 0 & D \end{pmatrix}, \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right)$ *is equivalent to* $\left(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right)$.

Proof. Putting

$$U = \begin{pmatrix} C & CX \\ 0 & I \end{pmatrix}, \quad V = \begin{pmatrix} C & XB \\ 0 & I \end{pmatrix}, \quad W = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}.$$

Since U, V and W are invertible, then

$$\begin{pmatrix} C & CX \\ 0 & I \end{pmatrix}\begin{pmatrix} A & E \\ 0 & D \end{pmatrix} = \begin{pmatrix} A & CE + CXD \\ 0 & D \end{pmatrix}$$

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} C & XB \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & AXB \\ 0 & D \end{pmatrix}$$

$$\begin{pmatrix} I & CX \\ 0 & I \end{pmatrix}\begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$$

$$\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & XB \\ 0 & I \end{pmatrix}$$

which implies that

$$AXB - CXD = CE$$

.

Reciprocally, suppose that $\left(\begin{pmatrix} A & E \\ 0 & D \end{pmatrix}, \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right)$ and $\left(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right)$ are equivalent.

Let $U = \begin{pmatrix} Q & R \\ S & T \end{pmatrix}$, $V = \begin{pmatrix} Q' & R' \\ S' & T' \end{pmatrix}$ and $W = \begin{pmatrix} Q'' & R'' \\ S'' & T'' \end{pmatrix}$, then we have

$$\begin{pmatrix} Q & R \\ S & T \end{pmatrix}\begin{pmatrix} A & E \\ 0 & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} Q' & R' \\ S' & T' \end{pmatrix}$$

$$\begin{pmatrix} Q & R \\ S & T \end{pmatrix}\begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} Q'' & R'' \\ S'' & T'' \end{pmatrix}$$

$$\begin{pmatrix} Q'' & R'' \\ S'' & T'' \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} Q' & R' \\ S' & T' \end{pmatrix}.$$

Which implies that

$$\begin{pmatrix} QA & QE + RD \\ SA & SE + TD \end{pmatrix} = \begin{pmatrix} AQ' & AR' \\ DS' & DT' \end{pmatrix}$$

$$\begin{pmatrix} QC & R \\ SC & T \end{pmatrix} = \begin{pmatrix} CQ'' & CR'' \\ S'' & T'' \end{pmatrix}$$

$$\begin{pmatrix} Q'' & R''B \\ S'' & T''B \end{pmatrix} = \begin{pmatrix} Q' & R' \\ BS' & BT' \end{pmatrix}.$$

Hence we get

$$AQ' = QA, \quad QE + RD = AR'$$
$$SA = DS', \quad SE + TD = DT'$$

$$QC = CQ'', \quad R = CR''$$
$$SC = S'', \quad T = T''$$
$$Q'' = Q', \quad R''B = R'$$
$$S'' = BS', \quad T''B = BT'.$$

We have also

$$QE + RD = AR'.$$

Then

$$QE = AR' - RD.$$

Multiplying by Q^*, we get

$$Q^*QE = Q^*AR' - Q^*RD, . \tag{3.5}$$

On the other hand we have

$$QA = AQ'.$$

Multiplying by C, it becomes

$$CQA = CAQ'.$$

Since N_A commutes with N_C, so from proposition 2.1, we get $AC = CA$. Hence

$$CQA = ACQ' = AQC.$$

From lemma 2.1 and theorem 2.1 (generalized Fuglede-Putnam property), we get

$$C^*QA^* = A^*QC^*.$$

Taking the adjoint we get

$$AQ^*C = CQ^*A.$$

Returns to (3.5), we get

$$Q^*QE = Q^*AR' - Q^*RD.$$
$$CQ^*QE = CQ^*AR' - CQ^*RD = AQ^*CR' - CQ^*RD.$$

Since $CR' = RB$, then

$$CQ^*QE = A(Q^*R)B - C(Q^*R)D. \tag{3.6}$$

Since $SA = DS'$ and $BD = DB$ (from proposition 2.1 and hypothesis (2)), we get

$$BSA = BDS' = DBS' = DS'' = DSC$$

.

From lemma 2.1 and theorem 2.1 (generalized Fuglede-Putnam property), we obtain

$$B^*SA^* = D^*SC^*.$$

Taking the adjoint, we get

$$AS^*B = CS^*D.$$

We have

$$SE = DT' - TD.$$

Multiplying by CS^*, it becomes

$$CS^*SE = CS^*DT' - CS^*TD = AS^*BT' - CS^*TD.$$

On the other hand we have

$$T''B = BT' = TB.$$

Hence

$$CS^*SE = A(S^*T)B - C(S^*T)D. \tag{3.7}$$

Adding (3.6) and (3.7), we obtain

$$C(Q^*Q + S^*S)E = A(Q^*R + S^*T)B - C(Q^*R + S^*T)D.$$

Since $(Q^*Q + S^*S)$ is invertible and commutes with A and C, then

$$CE = A(Q^*Q + S^*S)^{-1}(Q^*R + S^*T)B - C(Q^*Q + S^*S)^{-1}(Q^*R + S^*T)D,$$

which implies that

$$X = Q^*Q + S^*S)^{-1}(Q^*)R + S^*T.$$

□

4. Conclusion

The subject of this paper deals with the resolution of operator equations in the $B(H)$ algebra of linear operators bounded on a Hilbert space H. We studied those associated with generalized derivations. Also we explore much more general equations such as $AXB - CXD = E$ where A, B, C, D and E belong to $B(H)$. More precisely, it is a question of giving a description of the solutions of these equations for E belonging to a specific family (self-adjoint, normal, rank one, rank finite, compact, Fuglède Putnam couple) and for operators A, B, C and D belonging to good classes of operators (those involved in applications, especially in physics) such as subnormal operators.

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References

[1] Bender, D. J.,(1987), Lyapunov-like equations and reachability/observability gramians for descriptor systems, IEEE Trans. Auto. Control, 32 (1987), pp. 343-348.

[2] Ben Mabrouk, A., Ayadi, M., (2008), Lyapunov type operators for Numerical solutions of PDEs. Appl. Math. Comput., 204, pp. 395-407.

[3] Ben Mabrouk, A., (2018), Lyapunov-Sylvester operators for Generalized Nonlinear Euler-Poisson-Darboux System, arxiv:1705.00237, pp. 1-23.

[4] Bezia, A., Ben Mabrouk, A., (2018), Finite Difference Method for (2+1)-Kuramoto-Sivashinsky Equation. J. Part. Diff. Eq., Vol. 31, No.3, pp. 193-213.

[5] Bezia, A., Ben Mabrouk, A., Betina, K., (2016), Lyapunov-Sylvester Operators For (2+1)-Boussinesq Equa- tion, Electron. J. Differential Equations, 268, pp. 1-19.

[6] Chteoui, R., Ben Mabrouk, A., (2017), A Generalized Lyapunov-Syslvester Computational Method for Numerical Solutions of NLS Equation With Singular Potential, Anal. Theory Appl., 33, pp. 333-354.

[7] Chu, K. E., (1985), Exclusion theorems for the generalized eigenvalue problem, Numerical Analysis Rpt. NA/11/85, Dept of Mathematics, Univ. of Reading, 1985.

[8] Chu, K. E., (1985), The Solution of the Matrix Equations AXB-CXD=E and (YA- DZ, YC- BZ) = (E, F), Numerical Analysis Rpt. NA/10/85, Dept. of Mathematics, Univ. of Reading.

[9] Conway, J. B., (1991), The theory of subnormal operators, Math. Surveys and Mongraphs, Vol.36, Amer. Math. Soc. providence.

[10] Epton, M. A., (1980), Methods for the solution of AXB-CXD= E and its application in the numerical solution of implicit ordinary differential equations, BIT 20 (1980), pp. 341-345.

[11] Furuta, T., (1979), On relaxation of normality in the fugledePutnam theorem, Proc. Amer. Math. Soc. 3, 77.

[12] Lee, M. S., (1988), A note on the subnormal operators Comm. Korean Math. Soc. 3 No 1, pp. 51-58.

[13] Mecheri, S., Mansour, A., (2009), On the operator equation AXB- XD = E, Lobachevskii journal of mathematics 30 (NTA3), pp. 224-228.

[14] Resenblum, M., (1958), On a theorem of Fuglede and Putnam, J. Lond. Math. Soc. 33, pp. 376-377.

[15] Rosa, P., (1978), Lineare Matrizengleichungen und Kroneckersche Produkte, Z. Angew. ath. Mech., pp. 395-397.

[16] Rosa, P., (1978), Linear matrix equations and Kronecker products, in Proceedings of the 4th Symposium Basic Problems of Numerical Mathematics. Plzen, pp. 153-162.
[17] Schweinsberg, A., (1982), The operator equation AX-XB =C with normal A and B, Pacific Journal of Mathematics Vol. 102, No. 2.

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