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# PARTITION ENERGY OF SOME TREES AND THEIR GENERALIZED COMPLEMENTS

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Abstract. Let $G = (V, E)$ be a graph and $P_k = \{V_1, V_2, \ldots, V_k\}$ be a partition of $V$. The *k-partition energy* of a graph $G$ with respect to partition $P_k$ is denoted by $E_{P_k}(G)$ and is defined as the sum of the absolute values of $k$-partition eigenvalues of $G$. In this paper we obtain partition energy of some trees and their generalized complements with respect to equal degree partition. In addition, we develop a matlab program to obtain partition energy of a graph and its generalized complements with respect to a given partition.

Keywords: Trees, equal degree partition, generalized complements, partition eigenvalues, partition energy.

AMS Subject Classification: 05C70,05C05,05C50

## 1. Introduction

Let $G = (V, E)$ be a graph of order $n$ where $V = \{v_1, v_2, \ldots, v_n\}$ is the vertex set and $E$ is the edge set. The energy of a graph $G$ was defined by I. Gutman [5] as the sum of the absolute values of eigenvalues of $G$. The concept of graph energy has application in chemistry to estimate the total $\pi$-electron energy of a molecule. The adjacency matrix $A(G)$ of $G$ is a real symmetric matrix whose $(i, j)^{th}$ entry $a_{ij} = 1$ or 0 according as $\{v_i, v_j\}$ is an edge or not. The eigenvalues of this matrix represent the energy level of the electron in the molecule. In Hückel theory, the $\pi$-electron energy of a molecule is defined as the sum of the energies of all the electrons in a molecule.

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Graph partitioning problem arises in various areas of computer science, engineering, and related fields. Recently, the concept of graph partition has gained importance due to its application in route planning, clustering and detection of cliques in social, pathological and biological networks and high performance computing. The graph partition problems are defined on the data which can be represented in the form of a graph $G = (V, E)$. Let $V_1, V_2, \ldots, V_k$ be non-empty disjoint subsets of $V$ such that their union equal to $V$. Then $\{V_1, V_2, \ldots, V_k\}$ is called partition of vertex set $V$. There are many ways of partitioning a given graph. One can partition $G$ into smaller components arbitrarily or with respect to some specific properties. For example, Uniform graph partition is a type of graph partitioning problem that consists of dividing a graph into components such that the components are of about the same size and there are few connections between the components. Equal degree partition of a graph is a partition of the vertex set of the graph such that all vertices of same degree are in the same set.

If $V_1, V_2, \ldots, V_k$ is a partition of vertex set $V$, Then $\{V_1, V_2, \ldots, V_k\}$ is called a $k$-partition of $V$ denoted by $P_k$. The partition $P_k$ of a graph $G = (V, E)$ introduces two more graphs called *$k$-complement* and *$k(i)$-complement* which are defined as follows.

**Definition 1.1.** [7] *For all $V_i$ and $V_j$ in $P_k$, $i \neq j$ remove the edges between vertices of $V_i$ and $V_j$ and add the edges between the vertices of $V_i$ and $V_j$ which are not in $G$, the resulting graph is called* $k$-complement *of $G$ and is denoted by $\overline{(G)_k}$.*

**Definition 1.2.** [8] *For each set $V_r$ in $P_k$, remove the edges of $G$ joining the vertices within $V_r$ and add the edges in $G$ which are non-adjacent in $V_r$, the graph obtained is called* $k(i)$-complement *and is denoted by $\overline{(G)_{k(i)}}$.*

In [9], the $L$-matrix of $G = (V, E)$ of order $n$ with respect to a partition $P_k = \{V_1, V_2, \ldots, V_k\}$ of the vertex set $V$ is defined as a unique square symmetric matrix $P_k(G) = [a_{ij}]$ whose entries $a_{ij}$ are defined as follows:

$$a_{ij} = \begin{cases} 2 & \text{if } v_i \text{ and } v_j \text{ are adjacent where } v_i,\, v_j \in V_r,\, 1 \leq r \leq k \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent where } v_i,\, v_j \in V_r,\, 1 \leq r \leq k \\ 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent between the sets} \\ & V_r \text{ and } V_s \text{ for } r \neq s \text{ where } v_i \in V_r \text{ and } v_j \in V_s,\, 1 \leq r, s \leq k \\ 0 & \text{otherwise.} \end{cases}$$

In [10], we defined $k$-partition eigenvalues of $G$ as the eigenvalues of the matrix $P_k(G)$ and the $k$-partition energy $E_{P_k}(G)$ is defined as the sum of the absolute values of $k$-partition eigenvalues of $G$. In [10] we determined partition energy of some known graphs, their $k$-complement and $k(i)$-complement. In addition, some bounds for $E_{P_k}(G)$ are obtained.

In spectral graph theory, different kinds of energy of a graph $G$ have been extensively studied by many researchers and some of them can be found in [1], [5], [6], [10] and references there on. Also energies of various trees have been studied in [2], [3], [4]. An edge independent set of $G$ has no two of its edges incident to a common vertex and the maximum cardinality of such a set is called the edge independence number of $G$. The two classes of trees with edge independence number two are,

In [11], the energy of trees with edge independence number two is discussed. In this paper we plan to obtain partition energy of the above trees and their generalized complements with respect to equal degree partition. We also develop a matlab program to

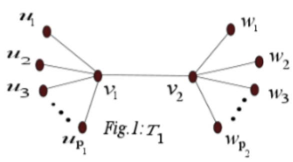

Fig.1: $T_1$

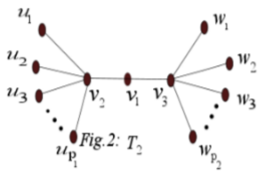

Fig.2: $T_2$

obtain partition energy of a graph and its generalized complements with respect to a given partition.

## 2. Partition energy of some trees and their generalized complements with respect to equal degree partition

In this section, we derive characteristic polynomial of trees with edge independence number two and their generalized complements with respect to equal degree partition and in some particular cases, their partition energy is obtained. We also discuss the partition eigenvalues of a graph having pendant vertices and isolated vertices with respect to equal degree partition.

In [6], equal degree partition of a graph is defined as follows.

**Definition 2.1.** *Given a graph $G$ there is a unique partition $P_k = \{V_1, V_2, \ldots, V_k\}$ with the following conditions.*
*(i) if for any $V_r \in P_k$ and $v_i, v_j \in V_r, 1 \leq r \leq k,\ d(v_i) = d(v_j)$.*
*(ii) if for any $v_i \in V_r, v_j \in V_s$ where $r \neq s$, $d(v_i) \neq d(v_j)$. This unique partition $P_k$ is called equal degree partition of a graph $G$.*

**Lemma 2.1.** [5] *Let $M, N, P, Q$ be matrices and $M$ be invertible. If*

$$S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix},$$

*then $detS = detM\ det[Q - PM^{-1}N]$.*

**Theorem 2.1.** *Let $P_3 = \{V_1, V_2, V_3\}$ be the equal degree partition of the vertex set $V$ of order $n$ of the tree $T_1$(Fig.1). Then the characteristic polynomials of $T_1$, $\overline{(T_1)_3}$, $\overline{(T_1)_{3(i)}}$ with respect to $P_3$ are respectively,*

*(i)* $(\lambda - 1)^{n-4}[\lambda^4 + (p_1 + p_2 - 2)\lambda^3 - 2(p_1 + p_2)\lambda^2 + 2(1 - p_1p_2)\lambda + (p_1 + p_2 + 3p_1p_2 - 1)]$.

*(ii)* $(\lambda - 1)^{n-4}[\lambda^4 + (p_1 + p_2 - 2)\lambda^3 + (1 - 2(p_1 + p_2))\lambda^2 + (p_1 + p_2 - 2p_1p_2)\lambda + p_1p_2]$.

*(iii)* $(\lambda+2)^{n-4}[\lambda^4 - 2(p_1+p_2-2)\lambda^3 + (3-5(p_1+p_2))\lambda^2 + 4(p_1p_2-1)\lambda + 4(p_1+p_2-1) - 3p_1p_2]$.

*Proof.* *(i)* In tree $T_1$, let $V_1 = \{v_1\}$, $V_2 = \{v_2\}$ and $V_3 = \{u_1, u_2, \cdots, u_{p_1}, w_1, w_2, \cdots, w_{p_2}\}$ where $p_1 \neq p_2$. Then $P_3 = \{V_1, V_2, V_3\}$ is an equal degree partition of the vertex set $V$ of $T_1$. The vertices in $V_1$ are of degree $p_1 + 1$, vertices in $V_2$ are of degree $p_2 + 1$ and vertices in $V_3$ are of degree one.
The L-matrix of $T_1$ (partition matrix of $T_1$) with respect to $P_3$ is

$$P_3(T_1) = \begin{array}{c} - \\ v_1 \\ v_2 \\ u_1 \\ u_2 \\ \vdots \\ u_{p_1} \\ w_1 \\ w_2 \\ \vdots \\ w_{p_2} \end{array} \begin{array}{c} \begin{array}{cccccccccc} v_1 & v_2 & u_1 & u_2 & \cdots & u_{p_1} & \cdots w_1 & w_2 & \cdots & w_{p_2} \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & -1 & \dots & -1 & -1 & -1 & \dots & -1 \\ 1 & 0 & -1 & 0 & \dots & -1 & -1 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & -1 & -1 & \dots & 0 & -1 & -1 & \dots & -1 \\ 0 & 1 & -1 & -1 & \dots & -1 & 0 & -1 & \dots & -1 \\ 0 & 1 & -1 & -1 & \dots & -1 & -1 & 0 & \dots & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & -1 & -1 & \dots & -1 & -1 & -1 & \dots & 0 \end{pmatrix} \end{array}.$$

In $det[\lambda I - P_3(T_1)]$, we carry out the following transformations.
This determinant has $p_1 + p_2 + 2$ rows and columns. Let the rows and columns be denoted by $R_i$ and $C_i$ respectively, $i = 1, 2, 3 \dots, p_1 + p_2 + 2$.

Step.1: Subtract the row $R_3$ from the rows $R_4, R_5, \dots, R_{p_1+2}$ and subtract the row $R_{p_1+3}$ from the rows $R_{p_1+4}, R_{p_1+5}, \dots, R_{p_1+p_2+2}$.
Step.2: Add the columns $C_4, C_5, \dots, C_{p_1+2}$ to the column $C_3$ and add the columns $C_{p_1+4}$, $C_{p_1+5}, \dots, C_{p_1+p_2+2}$ to the column $C_{p_1+3}$.
Step.3: Take $(\lambda - 1)^{n-4}$ as common factor.

Further simplification gives

$$\begin{vmatrix} \lambda & -1 & -p_1 & 0 \\ -1 & \lambda & 0 & -p_2 \\ -1 & 0 & \lambda + (p_1 - 1) & p_2 \\ 0 & -1 & p_1 & \lambda + (p_2 - 1) \end{vmatrix} \tag{1}$$

which is of the form $\begin{vmatrix} M & N \\ P & Q \end{vmatrix}$ where $M = \begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix}$, $N = \begin{bmatrix} -p_1 & 0 \\ 0 & -p_2 \end{bmatrix}$, $P = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $Q = \begin{bmatrix} \lambda + (p_1 - 1) & p_2 \\ p_1 & \lambda + (p_2 - 1) \end{bmatrix}$.

By using Lemma 2.1,
we get $[\lambda - 1 + p_1(1 - \lambda X)][\lambda - 1 + p_2(1 - \lambda X)] - p_1 p_2 (1 - X)^2$ where $X = \frac{1}{\lambda^2 - 1}$.

This on expansion gives
$[\lambda^4 + (p_1 + p_2 - 2)\lambda^3 - 2(p_1 + p_2)\lambda^2 + 2(1 - p_1 p_2)\lambda + (p_1 + p_2 + 3p_1 p_2 - 1)]$.

Hence, the characteristic polynomial of $P_3(T_1)$ is

$$(\lambda - 1)^{n-4}[\lambda^4 + (p_1 + p_2 - 2)\lambda^3 - 2(p_1 + p_2)\lambda^2 + 2(1 - p_1 p_2)\lambda + (p_1 + p_2 + 3p_1 p_2 - 1)].$$

The proof of $(ii)$ and $(iii)$ is similar to that of $(i)$. Hence we omit the proof. □

**Corollary 2.1.** *Let $T_3$ be the tree obtained from $T_1$ (Fig.1) by taking $p_1 = p_2 = p$ and $P_2 = \{V_1, V_2\}$ be the equal degree partition of its vertex set. Then*

$(i)$ $E_{P_2}(T_3) = E_{P_2}\overline{(T_3)_2} = 2p - 2 + \sqrt{(2p-3)^2 + 4(5p-2)} + \sqrt{9 + 4p}$.

(*ii*) $E_{P_2}\overline{(T_3)_{2(i)}} = 4p - 4 + \sqrt{4p+9} + \sqrt{(4p-3)^2 + 4(5p-2)}$.

*Proof.* (*i*) In Theorem 2.1, choose $p_1 = p_2 = p$. Then the partition changes to $P_2 = \{V_1, V_2\}$ where $V_1 = \{v_1, v_2\}$, $V_2 = \{u_1, u_2, \cdots, u_p, w_1, w_2, \cdots, w_p\}$. Consequently (1) changes to

$$\begin{vmatrix} \lambda & -2 & -p & 0 \\ -2 & \lambda & 0 & -p \\ -1 & 0 & \lambda + (p-1) & p \\ 0 & -1 & p & \lambda + (p-1) \end{vmatrix}.$$

By using Lemma 2.1, we get $[\lambda^2 + (2p-3)\lambda + (2-5p)][\lambda^2 + \lambda - (2+p)]$.
Hence the eigenvalues of $P_2(T_1)$ are

$$\begin{cases} 1 & 2p-2 \ times \\ \dfrac{-(2p-3) + \sqrt{[(2p-3]^2 + 4(5p-2)}}{2} & once \\ \dfrac{-(2p-3) - \sqrt{[(2p-3]^2 + 4(5p-2)}}{2} & once \\ \dfrac{-1 + \sqrt{9+4p}}{2} & once \\ \dfrac{-1 - \sqrt{9+4p}}{2} & once \end{cases}$$

Thus, $E_{P_2}(T_3) = 2p - 2 + \sqrt{[(2p-3]^2 + 4(5p-2)} + \sqrt{9+4p}$.

Since $T_3$ and $\overline{(T_3)_2}$ are isomorphic, $E_{P_2}(T_3) = E_{P_2}\overline{(T_3)_2}$.

(*ii*) In $P_2(T_3)$, interchange 2 and $-1$ to get $P_2\overline{(T_3)_{2(i)}}$.
With similar simplification, we get the eigenvalues of $P_2\overline{(T_3)_{2(i)}}$ as follows.

$$\begin{cases} -2 & 2p-2 \ times \\ \dfrac{(4p-3) + \sqrt{[4p-3]^2 + 4(5p-2)}}{2} & once \\ \dfrac{-(4p-3) - \sqrt{[4p-3]^2 + 4(5p-2)}}{2} & once \\ \dfrac{-1 + \sqrt{9+4p}}{2} & once \\ \dfrac{-1 - \sqrt{9+4p}}{2} & once \end{cases}$$

Hence, $E_{P_2}\overline{(T_3)_{2(i)}} = 2(2p-2) + \sqrt{[(4p-3]^2 + 4(5p-2)} + \sqrt{9+4p}$.

□

**Theorem 2.2.** *Let $P_4 = \{V_1, V_2, V_3, V_4\}$ be the equal degree partition of the vertex set $V$ of order $n$ of the tree $T_2$ (Fig.2). Then the characteristic polynomial of $T_2$, $\overline{(T_2)_4}$, $\overline{(T_2)_{4(i)}}$ are*

(*i*) $(\lambda - 1)^{n-5}\{\lambda^5 + (p_1 + p_2 - 2)\lambda^4 - [1 + 2(p_1 + p_2)]\lambda^3 + [4 - (p_1 + p_2) - 2p_1p_2]\lambda^2$
$+ [3(p_1 + p_2) + p_1p_2 - 2]\lambda - (p_1 + p_2) + 4p_1p_2\}$.

(*ii*) $(\lambda - 1)^{n-5}\{\lambda^5 + (p_1 + p_2 - 2)\lambda^4 - 3(p_1 + p_2)\lambda^3 + [(p_1 + p_2) - 2p_1p_2 + 2]\lambda^2$
$+ [2(p_1 + p_2) + 5p_1p_2 - 1]\lambda - (p_1 + p_2) - 2p_1p_2\}$.

*(iii)* $(\lambda+2)^{n-5}\{\lambda^5+2[2-(p_1+p_2)]\lambda^4+[2-5(p_1+p_2)]\lambda^3+2[2p_1p_2+(p_1+p_2)-4]\lambda^2$ $+[9(p_1+p_2)+p_1p_2-8]\lambda+2(p_1+p_2)-8p_1p_2\}$.

*Proof.* Given that $P_4=\{V_1,V_2,V_3,V_4\}$ is the equal degree partition of the vertex set $V$ of order $n$ of the tree $T_2$. Let us suppose that $V_1=\{v_1\}$, $V_2=\{v_2\}$, $V_3=\{v_3\}$ and $V_4=\{u_1,u_2,\cdots,u_{p_1},w_1,w_2,\cdots,w_{p_2}\}$. Then vertices in $V_1$, $V_2$, $V_3$ and $V_4$ are of degree 2, $p_1+1$, $p_2+1$ and 1 respectively. The $L$-matrix of $T_2$ with respect to $P_4$ is

$$P_4(T_2)=\begin{array}{c} - \\ v_1\\ v_2\\ v_3\\ u_1\\ u_2\\ \vdots\\ u_{p_1}\\ w_1\\ w_2\\ \vdots\\ w_{p_2}\end{array}\begin{array}{c} \begin{array}{ccccccccccc} v_1 & v_2 & v_3 & u_1 & u_2 & \cdots & u_{p_1} & \cdots w_1 & w_2 & \cdots & w_{p_2}\end{array}\\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0\\ 1 & 0 & 0 & 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0\\ 1 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1\\ 0 & 1 & 0 & 0 & -1 & \dots & -1 & -1 & -1 & \dots & -1\\ 0 & 1 & 0 & -1 & 0 & \dots & -1 & -1 & -1 & \dots & -1\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 0 & 0 & -1 & -1 & \dots & 0 & -1 & -1 & \dots & -1\\ 0 & 0 & 1 & -1 & -1 & \dots & -1 & 0 & -1 & \dots & -1\\ 0 & 0 & 1 & -1 & -1 & \dots & -1 & -1 & 0 & \dots & -1\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 1 & -1 & -1 & \dots & -1 & -1 & -1 & \dots & 0 \end{pmatrix}\end{array}.$$

With the same operations on $det[\lambda I-P_4(T_2)]$ as in Theorem 2.1, we get $(\lambda-1)^{n-5}$ and

$$\begin{vmatrix} \lambda & -1 & -1 & 0 & 0\\ -1 & \lambda & 0 & -p_1 & 0\\ -1 & 0 & \lambda & 0 & -p_2\\ 0 & -1 & 0 & \lambda+(p_1-1) & p_2\\ 0 & 0 & -1 & p_1 & \lambda+(p_2-1)\end{vmatrix} \tag{2}$$

which is of the form $\begin{vmatrix} M & N\\ P & Q\end{vmatrix}$ where $M=\begin{bmatrix}\lambda & -1 & -1\\ -1 & \lambda & 0\\ -1 & 0 & \lambda\end{bmatrix}$, $N=\begin{bmatrix}0 & 0\\ -p_1 & 0\\ 0 & -p_2\end{bmatrix}$, $P=\begin{bmatrix}0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$ and $Q=\begin{bmatrix}\lambda+(p_1-1) & p_2\\ p_1 & \lambda+(p_2-1)\end{bmatrix}$.

By using Lemma 2.1,
we get $[\lambda-1+p_1(1-(\lambda^2-1)Y)][\lambda-1+p_2(1-(\lambda^2-1)Y)]-p_1p_2(1-Y)^2$
where $Y=\frac{1}{\lambda^3-2\lambda}$.

On further simplification, we get
$\lambda^5+(p_1+p_2-2)\lambda^4-(1+2(p_1+p_2))\lambda^3+[4-(p_1+p_2)-2p_1p_2]\lambda^2+[3(p_1+p_2)+p_1p_2-2]\lambda-(p_1+p_2)+4p_1p_2$.
Hence, the characteristic polynomial of $P_4(T_2)$ is

$(\lambda-1)^{n-5}\{\lambda^5+(p_1+p_2-2)\lambda^4-[1+2(p_1+p_2)]\lambda^3+[4-(p_1+p_2)-2p_1p_2]\lambda^2$ $+[3(p_1+p_2)+p_1p_2-2]\lambda-(p_1+p_2)+4p_1p_2\}$.
The techniques used in proof of *(ii)* and *(iii)* are similar to that of *(i)*.
Hence we omit the proof. □

**Corollary 2.2.** *Let $T_4$ be the tree obtained from $T_2$ by taking $p_1=p_2=p$ and $P_3=\{V_1,V_2,V_3\}$ be the equal degree partition of its vertex set. Then,*

*(i)* $E_{P_3}(T_4) = 2p - 2 + 2\sqrt{p} + \sum \mid \lambda_t \mid$
*where* $\lambda_t$ *are roots of* $\lambda^3 + 2p\lambda^2 + (p-3)\lambda - (4p-2) = 0$.

*(ii)* $E_{P_3}\overline{(T_4)_3} = 2p - 2 + 2\sqrt{p} + \sum \mid \lambda_t \mid$
*where* $\lambda_t$ *are roots of* $\lambda^3 + 2p\lambda^2 - (p+1)\lambda - 2p = 0$.

*(iii)* $E_{P_3}\overline{(T_4)_{3(i)}} = 4p - 4 + 2\sqrt{p} + \sum \mid \lambda_t \mid$ *for* $p \geq 4$
*where* $\lambda_t$ *are roots of* $\lambda^3 - 4p\lambda^2 + (7p-6)\lambda + (8p-4) = 0$.

*Proof.* *(i)* In Theorem 2.2, choose $p_1 = p_2 = p$. Then the partition changes to $P_3 = \{V_1, V_2, V_3\}$ where $V_1 = \{v_1\}$, $V_2 = \{v_2, v_3\}$, $V_3 = \{u_1, u_2, \ldots, u_p, w_1, w_2, \ldots, w_p\}$. Consequently (2) changes to

$$\begin{vmatrix} \lambda & 0 & 0 & -p & -p \\ 0 & \lambda & 1 & 0 & -p \\ 0 & 1 & \lambda & -p & 0 \\ -1 & 0 & -1 & \lambda + (p-1) & p \\ -1 & -1 & 0 & p & \lambda + (p-1) \end{vmatrix}$$

By using Lemma 2.1, we get the eigenvalues of $P_3(T_4)$ as 1 repeated $2p-2$ times, $1 \pm \sqrt{p}$ once and the roots of

$$\lambda^3 + 2p\lambda^2 + (p-3)\lambda - (4p-2) = 0. \tag{3}$$

Hence, $E_{P_3}(T_4) = 2p - 2 + 2\sqrt{p} + \sum \mid \lambda_t \mid$, where $\lambda_t$ are roots of (3).
*(ii)* With similar operations on $det[\lambda I - P_3\overline{(T_4)_3}]$ as in *(i)*, we get the eigenvalues of $P_3\overline{(T_4)_3}$ as 1 repeated $2p-2$ times, $1 \pm \sqrt{p}$ once and roots of

$$\lambda^3 + 2p\lambda^2 - (p+1)\lambda - 2p = 0. \tag{4}$$

Hence, $E_{P_3}\overline{(T_4)_3} = 2p - 2 + 2\sqrt{p} + \sum \mid \lambda_t \mid$, where $\lambda_t$ are roots of (4).

*(iii)* Applying similar operations as in *(i)* for $det[\lambda I - P_3\overline{(T_4)_{3(i)}}]$, we get the eigenvalues of $P_3\overline{(T_4)_{3(i)}}$ as $-2$ repeated $2p-2$ times, $-2 \pm \sqrt{p}$ once and roots of

$$\lambda^3 - 4p\lambda^2 + (7p-6)\lambda + (8p-4) = 0. \tag{5}$$

Hence, $E_{P_3}\overline{(T_4)_{3(i)}} = 4p - 4 + 2\sqrt{p} + \sum \mid \lambda_t \mid$ for $p \geq 4$, where $\lambda_t$ are roots of (5). □

**Theorem 2.3.** *Let* $G = (V, E)$ *be a graph of order* $n$ *with* $p$ *pendant vertices and* $P_k = \{V_1, V_2, \cdots, V_k\}$ *be the equal degree partition of* $V$. *Suppose that the pendant vertices are in* $V_k$ *and the pendant vertices are such that for* $(l \leq k-1)$ $p_1, p_2, \cdots, p_l$ $(P_i \geq 2)$ *number of these are adjacent to the vertices* $v_1, v_2, \cdots, v_l$ *respectively. Then*

*(i)* 1 *is an eigenvalue of* $P_k(G)$ *and* $P_k\overline{(G)_k}$ *repeated* $p - l$ *times.*

*(ii)* $-2$ *is an eigenvalue of* $P_k\overline{(G)_{k(i)}}$ *repeated* $p - l$ *times.*

*Proof.* In $det[\lambda I - P_k(G)] = 0$, let us suppose that for $l \leq k-1$, the pendant vertices are such that $p_1, p_2, \cdots, p_l$ number of these are adjacent to the vertices $v_1, v_2, \cdots, v_l$ respectively. Let the first $p_1$ pendant vertices be denoted by $u_1, u_2, \cdots, u_{p_1}$. Then they are adjacent to the same vertex $v_1 \in V_r$ for some $r \in \{1, 2, \cdots, k-1\}$. In the horizontal strip corresponding to the vertices $u_1, u_2, \cdots, u_{p_1}$, the column of $v_1$ will have the entries $-1$, the

columns of all other $v_i's$ is 0 and the columns of $u_i's$ take the value 1 except the principal diagonal entries which are $0's$. Then by subtracting the row corresponding to the vertex $u_1$ from the rows corresponding to the vertices $u_2, u_3, \cdots, u_{p_1}$, we get $\lambda - 1$ as common factor in each of these $p_1 - 1$ rows. Since this is true for all strips corresponding to the remaining pendant vertices, it follows that 1 is an eigenvalue of $P_k(G)$ repeated $p - l$ times where $p = p_1 + p_2 + \cdots + p_l$.
Consider $det[\lambda I - P_k\overline{(G)_k}]$. In the horizontal strip corresponding to the vertices $u_1, u_2, \cdots, u_{p_1}$, the column of $v_1$ will have the entries 0, the columns of all other $v_i's$ is $-1$ and the columns of $u_i's$ remain unaltered. Hence by repeating the above operations, we get 1 as an eigenvalue of $P_k\overline{(G)_k}$ repeated $p - l$ times.

$(ii)$ We know that $det[\lambda I - P_k\overline{(G)_{k(i)}}] = 0$ is obtained by inter changing 1 and $-2$ in $det[\lambda I - P_k(G)]$. Hence with the same operations as in $(i)$ we get $\lambda + 2$ as common factor $p - l$ times. Hence $-2$ is an eigenvalue of $P_k\overline{(G)_{k(i)}}$ repeated $p - l$ times. □

**Theorem 2.4.** *Let $G = (V, E)$ be a graph of order $n$ with $l$ isolated vertices and $P_k = \{V_1, V_2, \cdots, V_k\}$ be the equal degree partition of $V$. Then*
$(i)$ $-1$ *is an eigenvalue of $P_k(G)$ and $P_k\overline{(G)_k}$ repeated $l - 1$ times.*
$(ii)$ 2 *is an eigenvalue of $P_k\overline{(G)_{k(i)}}$ repeated $l - 1$ times.*

We omit the proof of this theorem as the techniques used here are similar to that of Theorem 2.3.

## 3. Matlab Program to find partition energy of a graph

In this section we present a program in Matlab to find partition energy of any graph and its generalized complements with respect to the given partition.
In this program, for a given graph, we give input for number of vertices, partition of the vertex set and edge input is given in the form of adjacency matrix. The outputs are partition matrix of the given graph, its generalized complements with respect to the given partition and their respective spectrum and energies.

```
clc
clear
n=input('Enter size of the matrix: ');
for i = 1 : n;
ti=['Enter partition number of vertex-',num2str(i),':'];
p(i)=input(ti);
end;
disp('Enter adjacency matrix:');
for i = 1 : n;
for j = 1 : i;
Xij = ['Enter a',num2str(i),num2str(j),' :'] ;
a(i,j)=input(Xij);
a(j,i) = a(i,j);
end;
end;
disp('Adjacency Matrix of the graph is')
disp(a)
```

```
weight = [ 2 -1; 1 0];
invertedWeight = [ -1 2; 1 0];
exvertedWeight = [2 -1; 0 1];
M1 = zeros(n, n);
M2 = zeros(n, n);
M3 = zeros(n, n);
for i = 1 : n
for j = i + 1 : n
if(a(i, j) == 1)
connected = 1;
else
connected = 2;
end
if p(i) == p(j)
sameGroup = 1;
else
sameGroup = 2;
end
M1(i, j) = weight(sameGroup,connected);
M1(j, i) = weight(sameGroup,connected);
M2(i, j) = invertedWeight(sameGroup,connected);
M2(j, i) = invertedWeight(sameGroup,connected);
M3(i, j) = exvertedWeight(sameGroup,connected);
M3(j, i) = exvertedWeight(sameGroup,connected);
end
end disp('L-Matrix of the graph is')
disp(M1);
disp('Matrix of k(i)-complement of graph is')
disp(M2);
disp('Matrix of k-complement of graph is')
disp(M3);
E1 = eig(M1);
E2 = eig(M2);
E3 = eig(M3);
disp('Eigen values of L-Matrix of the graph:')
disp(E1);
disp('Eigen values of L-Matrix of k(i)-complement of the graph:')
disp(E2);
disp('Eigen values of L-Matrix of k-complement of the graph:')
disp(E3);
energy1 = 0;
energy2 = 0;
energy3 = 0;
for i = 1 : i energy1 = energy1 + abs(E1(i));
energy2 = energy2 + abs(E2(i));
energy3 = energy3 + abs(E3(i));
end disp('Partition energy of the graph is')
disp(energy1);
disp('Partition energy of k(i)-complement of the graph is')
```

```
disp(energy2);
disp('Partition energy of k-complement of the graph is')
disp(energy3);
end
```

## References


[1] Adiga C., Sampathkumar E., Sriraj M. A., Shrikanth A. S, (2013), Color energy of a graph, Proc. Jangjeon Math. Soc., 16(3), pp.335-351.

[2] Del Vecchio R. R., Gutman I., Trevisan V., Vinagre C. T. M.,(2009), On the spectra and energies of doublebroomlike trees, Kragujevac Journal of Science 31, pp.45-58.

[3] Gutman I., Furtula B., Andriantiana E. O. D.,Cvetic M.,(2012), More trees with large energy and small size, MATCH Communications in Mathematical and in Computer Chemistry 68, pp.697-702.

[4] Gutman I., (2014), Kragujevac trees and their energy, Scientific Publications of the State University of Novi Pazar. Series A:Applied Mathematics, Informatics and Mechanics 6, pp.71-79.

[5] Li X. ,Shi Y., Gutman I., (2012), Graph Energy, Springer Newyork Heidelberg Dordrecht, London.

[6] Roopa S. V, Vidya K. A., Sriraj M. A., (2016), Partition energy of Amalgamation of Complete graphs and their Generalized Complements, Indian J. Discrete Mathematics., 2, (1), pp.18-36.

[7] Sampathkumar E., Pushpalatha L., (1998), Complement of a Graph: A Generalization J. of Graphs and Combinatorics, 14, (4), pp.377-392.

[8] Sampathkumar E., Pushpalatha L., Venkatachalam C. V., Pradeep Bhat, (1998), Generalized complements of a graph, Indian J. Pure Appl. Math., 29, (6), pp.625-639.

[9] Sampathkumar E., Sriraj M. A., (2013), Vertex labeled / colored graphs, matrices and signed Graphs, J. of Combinatorics, Information and System Sciences, 38, pp.113-120.

[10] Sampathkumar E. , Roopa S. V, Vidya K. A., Sriraj M. A., (2015), Partition Energy of a graph, Proc. Jangjeon Math. Soc., 18, (4), pp.473-493.

[11] Walikar H. B., Ramane H. S., (2005), Energy of trees with edge independence number two, Proc.National Academy of Sciences, 75, (A), II, pp.137-140.

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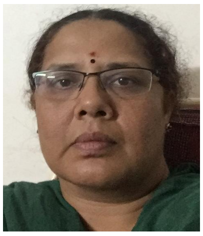

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