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SYMMETRIZED p-CONVEXITY AND RELATED SOME INTEGRAL INEQUALITIES

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Abstract. In this paper, a new concept called as the symmetrized p -convex function which is a generalization of the symmetrized convex and symmetrized harmonic convex functions is introduced and some Hermite-Hadamard type inequalities for symmetrized p-convex functions is given.

Keywords: Symmetrized convex function, symmetrized p-convex function, symmetrized harmonic convex function, Hermite-Hadamard type inequalities.

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1. Introduction

Let f be real function defined on some nonempty interval I of real line $\mathbb{R}$. The function f is said to be convex on I if the inequality

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

In [3], the author, gave the definition Harmonically convex and concave functions as follow.

Definition 1.1. *Let $I \subset \mathbb{R}\setminus\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y)+(1-t)f(x) \tag{2}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (2) is reversed, then f is said to be harmonically concave.

The following result of the Hermite-Hadamard type holds for harmonically convex functions.

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Theorem 1.1 ([3])**.** *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{3}$$

The above inequalities are sharp.

In [4], the author gave the definition of p-convex function as follow:

Definition 1.2. *Let $I \subset (0,\infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$. A function $f : I \to \mathbb{R}$ is said to be a p-convex function, if*

$$f\left([tx^p + (1-t)y^p]^{1/p}\right) \leq tf(x) + (1-t)f(y) \tag{4}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (4) is reversed, then f is said to be p-concave.

According to Definition 1.2, it can be easily seen that for $p = 1$ and $p = -1$, p-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I \subset (0,\infty)$, respectively.

Since the condition (4) can be written as

$$(f \circ g)(tx^p + (1-t)y^p) \leq t(f \circ g)(x^p) + (1-t)(f \circ g)(y^p),\ g(x) = x^{1/p},$$

then it is observed that $f : I \subseteq (0,\infty) \to \mathbb{R}$ is p-convex on I if and only if $f \circ g$ is convex on $g^{-1}(I) := \{z^p, z \in I\}$.

Example 1.1. *Let $f : (0,\infty) \to \mathbb{R},\ f(x) = x^p, p \neq 0,$ and $g : (0,\infty) \to \mathbb{R},\ g(x) = c,\ \ c \in \mathbb{R}$, then f and g are both p-convex and p-concave functions.*

In [2, Theorem 5], if it is taken $I \subset (0,\infty)$, $p \in \mathbb{R}\backslash\{0\}$ and $h(t) = t$, then the following Theorem can be given.

Theorem 1.2. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\backslash\{0\}$, and $a, b \in I$ with $a < b$. If $f \in L[a, b]$ then the following inequalities hold*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) \leq \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \leq \frac{f(a)+f(b)}{2}. \tag{5}$$

Definition 1.3 ([6])**.** *Let $p \in \mathbb{R}\backslash\{0\}$. A function $w : [a, b] \subseteq (0,\infty) \to \mathbb{R}$ is said to be p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$ if*

$$w(x) = w\left([a^p + b^p - x^p]^{1/p}\right)$$

holds for all $x \in [a, b]$.

In [5], Kunt and İşcan gave Hermite-Hadamard-Fejér type inequalities for p-convex functions as follow:

Theorem 1.3. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\backslash\{0\}$, $a, b \in I$ with $a < b$. If $f \in L[a, b]$ and $w : [a, b] \to \mathbb{R}$ is nonnegative, integrable and p-symmetric with* *respect to* $\left[\frac{a^p+b^p}{2}\right]^{1/p}$*, then the following inequalities hold:*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_a^b \frac{w(x)}{x^{1-p}}dx \leq \int_a^b \frac{f(x)w(x)}{x^{1-p}}dx$$

$$\leq \frac{f(a)+f(b)}{2}\int_a^b \frac{w(x)}{x^{1-p}}dx. \tag{6}$$

Definition 1.4. *Let* $f \in L[a,b]$*. The left-sided and right-sided Riemann Liouville fractional integrals* $J_{a+}^{\alpha}f$ *and* $J_{b-}^{\alpha}f$ *of oder* $\alpha > 0$ *with* $b > a \geq 0$ *are defined by*

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt,\ a < x < b$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt,\ a < x < b$$

respectively, where $\Gamma(\alpha)$ *is the Gamma function defined by* $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$ *(see [7]).*

In [6], the authors presented Hermite–Hadamard-Fejér inequalities for p-convex functions in fractional integral forms as follows:

Theorem 1.4. *Let* $f : I \subseteq (0,\infty) \to \mathbb{R}$ *be a* p*-convex function,* $p \in \mathbb{R}\backslash\{0\}, \alpha > 0$ *and* $a, b \in I$ *with* $a < b$*. If* $f \in L[a,b]$ *and* $w : [a,b] \to \mathbb{R}$ *is nonnegative, integrable and* p*-symmetric with respect to* $\left[\frac{a^p+b^p}{2}\right]^{1/p}$*, then the following inequalities for fractional integrals hold*

i.) If $p > 0$

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\left[J_{a^p+}^{\alpha}(w\circ g)(b^p) + J_{b^p-}^{\alpha}(w\circ g)(a^p)\right] \tag{7}$$

$$\leq \left[J_{a^p+}^{\alpha}(fw\circ g)(b^p) + J_{b^p-}^{\alpha}(fw\circ g)(a^p)\right]$$

$$\leq \frac{f(a)+f(b)}{2}\left[J_{a^p+}^{\alpha}(w\circ g)(b^p) + J_{b^p-}^{\alpha}(w\circ g)(a^p)\right]$$

with $g(x) = x^{1/p}, x \in [a^p, b^p]$.

ii.) If $p > 0$

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\left[J_{b^p+}^{\alpha}(w\circ g)(a^p) + J_{a^p-}^{\alpha}(w\circ g)(b^p)\right]$$

$$\leq \left[J_{b^p+}^{\alpha}(fw\circ g)(a^p) + J_{a^p-}^{\alpha}(fw\circ g)(b^p)\right]$$

$$\leq \frac{f(a)+f(b)}{2}\left[J_{b^p+}^{\alpha}(w\circ g)(a^p) + J_{a^p-}^{\alpha}(w\circ g)(b^p)\right]$$

with $g(x) = x^{1/p}, x \in [b^p, a^p]$.

For a function $f:[a,b]\rightarrow\mathbb{R}$, the symmetrical transform of f on the interval $[a,b]$ is denoted by $\breve{f}_{[a,b]}$ or simply $\breve{f}$, when the interval $[a,b]$ is implicit, and is defined by

$$\breve{f}(x):=\frac{1}{2}\left[f(x)+f(a+b-x)\right],\ x\in[a,b].$$

The anti symmetrical transform of f on the interval $[a,b]$ is denoted by $\tilde{f}_{[a,b]}$or simply $\tilde{f}$ and is defined by

$$\tilde{f}(x):=\frac{1}{2}\left[f(x)-f(a+b-x)\right],\ x\in[a,b].$$

It is obvious that $\breve{f}+\tilde{f}=f$ for any function f .

If f is convex on $[a,b]$, then $\breve{f}$ is also convex on $[a,b]$. But, when $\breve{f}$ is onvex on $[a,b]$, f may not be convex on $[a,b]$ ([1]).

In [1], Dragomir introduced symmetrized convexity concept as follow:

Definition 1.5. *A function $f:[a,b]\rightarrow\mathbb{R}$ is said to be symmetrized convex (concave)on $[a,b]$ if symmetrical transform $\breve{f}$ is convex (concave) on $[a,b]$.*

Dragomir extends the Hermite-Hadamard inequality to the class of symmetrized convex functions as follow:

Theorem 1.5 ([1])**.** *Assume that $f:[a,b]\rightarrow\mathbb{R}$ is symmetrized convex on the interval $[a,b]$, then the following Hermite-Hadamard inequalities hold*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{8}$$

Theorem 1.6 ([1])**.** *Assume that $f:[a,b]\rightarrow\mathbb{R}$ is symmetrized convex on the interval $[a,b]$. Then the following bounds hold*

$$f\left(\frac{a+b}{2}\right)\leq\breve{f}(x)=\frac{1}{2}\left[f(x)+f(a+b-x)\right]\leq\frac{f(a)+f(b)}{2} \tag{9}$$

for any $x\in[a,b]$.

Corollary 1.1. *If $f:[a,b]\rightarrow\mathbb{R}$ is symmetrized convex on the interval $[a,b]$ and $w:[a,b]\rightarrow[0,\infty)$is integrable on $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\leq\int_a^b f(x)w(x)dx\leq\frac{f(a)+f(b)}{2}\int_a^b w(x)dx.$$

Theorem 1.7 ([1])**.** *Assume that $f:[a,b]\subseteq(0,\infty)\rightarrow\mathbb{R}$ is symmetrized convex on the interval $[a,b]$. Then for any $x,y\in[a,b]$ with $x\neq y$ the Hermite-Hadamard inequalities hold*

$$\begin{aligned}
&\frac{1}{2}\left[f\left(\frac{x+y}{2}\right)+f\left(a+b-\frac{x+y}{2}\right)\right] \qquad (10)\\
\leq\ &\frac{1}{2(y-x)}\left[\int_x^y f(t)dt+\int_{a+b-y}^{a+b-x}f(t)dt\right]\\
\leq\ &\frac{1}{4}\left[f(x)+f(y)+f(a+b-x)+f(a+b-y)\right].
\end{aligned}$$

For a function $f:[a,b]\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{C}$, the symmetrical transform of f on the interval $[a,b]$, is denoted by $\breve{H}f_{[a,b]}$ or simply $\breve{H}f$ and is defined by

$$\breve{H}f(x):=\frac{1}{2}\left[f(x)+f\left(\frac{abx}{(a+b)\,x-ab}\right)\right],\ x\in[a,b]\,.$$

Definition 1.6 ([8])**.** *A function $f:I\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ is said to be symmetrized harmonic convex (concave)on $[a,b]$ if $\breve{H}f$ is harmonic convex (concave) on I.*

The results, similar to those mentioned above for the symmetrized convex function class were found by Wu et. al., in [8], in the case of harmonic symmetrized convex functions class.

Theorem 1.8 ([8])**.** *Assume that $f:[a,b]\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ is symmetrized harmonic convex and integrable on the interval $[a,b]$. Then the Hermite-Hadamard type inequalities hold*

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int\limits_a^b\frac{f(x)}{x^2}dx\leq\frac{f(a)+f(b)}{2}. \tag{11}$$

Theorem 1.9 ([8])**.** *Assume that $f:[a,b]\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ is symmetrized harmonic convex on the interval $[a,b]$. Then the following bounds hold*

$$f\left(\frac{2ab}{a+b}\right)\leq\breve{H}f\,(x)=\frac{1}{2}\left[f(x)+f\left(\frac{abx}{(a+b)\,x-ab}\right)\right]\leq\frac{f(a)+f(b)}{2} \tag{12}$$

for any $x\in[a,b]$.

Theorem 1.10 ([8])**.** *Assume that $f:[a,b]\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ is symmetrized harmonic convex on the interval $[a,b]$. Then the following Hermite-Hadamard inequalities hold*

$$\begin{aligned}&\frac{1}{2}\left[f\left(\frac{2xy}{x+y}\right)+f\left(\frac{2abxy}{2xy\,(a+b)-ab(x+y)}\right)\right] \\ \leq\ &\frac{xy}{2\,(y-x)}\left[\int_x^y\frac{f(t)}{t^2}dt+\int_{\frac{aby}{(a+b)y-ab}}^{\frac{abx}{(a+b)x-ab}}\frac{f(t)}{t^2}dt\right] \\ \leq\ &\frac{1}{4}\left[f(x)+f(y)+f\left(\frac{abx}{(a+b)\,x-ab}\right)+f\left(\frac{aby}{(a+b)\,y-ab}\right)\right].\end{aligned} \tag{13}$$

for any $x,y\in[a,b]$ with $x\neq y$.

Motivated by the above results, in this paper the concept of the symmetrized p-convex function is introduced and some Hermite-Hadamard type inequalities is established. Some examples of interest are provided as well.

2. Symmetrized p-Convexity

For a function $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$, the p-symmetrical transform of f on the interval $[a,b]$ is denoted by $P_{(f;p),[a,b]}$ or simply $P_{(f;p)}$ and is defined by

$$P_{(f;p)}(x):=\frac{1}{2}\left[f(x)+f\left([a^p+b^p-x^p]^{1/p}\right)\right],\ x\in[a,b]\,.$$

The anti p-symmetrical transform of f on the interval $[a,b]$ is denoted by $AP_{(f;p),[a,b]}$ or simply $AP_{(f;p)}$ and is defined by

$$AP_{(f;p)}(x):=\frac{1}{2}\left[f(x)-f\left([a^p+b^p-x^p]^{1/p}\right)\right],\ x\in[a,b]\,.$$

It is obvious that $P_{(f;p)} + \smile{A}P_{(f;p)} = f$ for any function f. Also, it is seen that $P_{(f;1)}(x) = \frac{1}{2}\left[f(x) + f(a+b-x)\right] = \breve{f}(x)$ and $P_{(f;-1)}(x) = \frac{1}{2}\left[f(x) + f\left(\frac{abx}{(a+b)x-ab}\right)\right] = \breve{H}f(x)$.

If f is p-convex on $[a,b]$, then $P_{(f;p)}$ is also p-convex on $[a,b]$. Indeed, for any $x,y \in [a,b]$ and $t \in [0,1]$ it is easily seen that

$$\begin{aligned}
& P_{(f;p)}([tx^p + (1-t)y^p]^{1/p}) \\
= \;& \frac{1}{2}\left[f([tx^p + (1-t)y^p]^{1/p}) + f\left([a^p + b^p - tx^p - (1-t)y^p]^{1/p}\right)\right] \\
= \;& \frac{1}{2}\left[f([tx^p + (1-t)y^p]^{1/p}) + f\left([t(a^p + b^p - x^p) + (1-t)(a^p + b^p - y^p)]^{1/p}\right)\right] \\
\leq \;& t\frac{1}{2}\left[f(x) + f\left([a^p + b^p - x^p]^{1/p}\right)\right] + (1-t)\frac{1}{2}\left[f(y) + f\left([a^p + b^p - y^p]^{1/p}\right)\right] \\
= \;& tP_{(f;p)}(x) + (1-t)P_{(f;p)}(y).
\end{aligned}$$

Remark 2.1. *If $P_{(f;p)}$ is p-convex on $[a,b]$ for a function $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$, then the function f is nor necessary p-convex on $[a,b]$. For example, let $p = -1$, consider the function $f(x) = -\ln x, x \in (0,\infty)$. The function f is not -1-convex (or harmonically convex), but $P_{(f;-1)}$ is -1-convex* [8].

Definition 2.1. *A function $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ is said to be symmetrized p-convex (p-concave)on $[a,b]$, if p-symmetrical transform $P_{(f;p)}$ is p-convex (p-concave) on $[a,b]$.*

Example 2.1. *Let $a,b \in \mathbb{R}$ with $0 < a < b$ and $\alpha \geq 2$. Then the function $f : [a,b] \to \mathbb{R}$, $f(x) = (x^p - a^p)^{\alpha-1}$, $p > 0$, (or $f(x) = (a^p - x^p)^{\alpha-1}$, $p < 0$) is p-convex on $[a,b]$. Indeed, for any $u,v \in [a,b]$ and $t \in [0,1]$ by convexity of the function $g(\varsigma) = \varsigma^{\alpha-1}, \zeta \geq 0$, it is easily seen that*

$$\begin{aligned}
f([tu^p + (1-t)v^p]^{1/p}) &= (tu^p + (1-t)v^p - a^p)^{\alpha-1} \\
&= (t[u^p - a^p] + (1-t)[v^p - a^p])^{\alpha-1} \\
&\leq t(u^p - a^p)^{\alpha-1} + (1-t)(v^p - a^p)^{\alpha-1} \\
&= tf(u) + (1-t)f(v).
\end{aligned}$$

Thus $P_{(f;p)}$ is also p-convex on $[a,b]$. Therefore, f is symmetrized p-convex function.

Example 2.2. *Let $\alpha \geq 2$. Then the function $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$, $f(x) = (b^p - x^p)^{\alpha-1}$, $p > 0$, (or $f(x) = (x^p - b^p)^{\alpha-1}$, $p < 0$) is p-convex on $[a,b]$. Therefore f is symmetrized p-convex function.*

Example 2.3. *Let $\alpha \geq 2$. Then the function $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$, $f(x) = (x^p - a^p)^{\alpha-1} + (b^p - x^p)^{\alpha-1}$, $p > 0$, (or $f(x) = (a^p - x^p)^{\alpha-1} + (x^p - b^p)^{\alpha-1}$, $p < 0$) is symmetrized p-convex function.*

Now if $PC[a,b]$ is the class of p-convex functions defined on I and $SPC[a,b]$ is the class of symmetrized p-convex functions on $[a,b]$ then

$$PC[a,b] \subseteq SPC[a,b].$$

Also, if $[c,d] \subset [a,b]$ and $f \in SPC[a,b]$, then this does not imply in general that $f \in SPC[c,d]$.

Proposition 2.1. *Let $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ be a function and $g(x) = x^{1/p}, x > 0, p \neq 0$. f is symmetrized p-convex on the interval $[a,b]$ if and only if $f \circ g$ is symmetrized convex on the interval $g^{-1}([a,b]) = [a^p, b^p]$ (or $[b^p, a^p]$).*

Proof. Let f be a symmetrized p-convex function on the interval $[a, b]$ and $x, y \in g^{-1}([a, b])$ be arbitrary, then there exist $u, v \in [a, b]$ such that $x = u^p$ and $y = g(v) = v^p$

$$\begin{aligned}
& (f \breve{\circ} g)(tx + (1-t)y) \qquad (14)\\
= \; & \frac{1}{2}\left[(f \circ g)(tx + (1-t)y) + (f \circ g)(a^p + b^p - tx - (1-t)y)\right] \\
= \; & \frac{1}{2}\left[(f \circ g)(tu^p + (1-t)v^p) + (f \circ g)(a^p + b^p - [tu^p + (1-t)v^p])\right] \\
= \; & P_{(f;p)}([tu^p + (1-t)v^p]^{1/p}).
\end{aligned}$$

Since f is a symmetrized p-convex function on the interval $[a, b]$, it is easily seen that

$$P_{(f;p)}([tu^p + (1-t)v^p]^{1/p}) \leq tP_{(f;p)}(u) + (1-t)P_{(f;p)}(v) \qquad (15)$$

$$\begin{aligned}
= \; & t\frac{1}{2}\left[f(u) + f\left([a^p + b^p - u^p]^{1/p}\right)\right] + (1-t)\frac{1}{2}\left[f(v) + f\left([a^p + b^p - v^p]^{1/p}\right)\right] \\
= \; & t\frac{1}{2}\left[(f \circ g)(x) + (f \circ g)(a^p + b^p - x)\right] + (1-t)\frac{1}{2}\left[(f \circ g)(y) + (f \circ g)(a^p + b^p - v^p)\right] \\
= \; & t\,(f \breve{\circ} g)(x) + (1-t)(f \breve{\circ} g)(y).
\end{aligned}$$

By (14) and (15), the function $f \circ g$ is symmetrized convex on the interval $[a^p, b^p]$ (or $[b^p, a^p]$) is obtained.

Conversely, if $f \circ g$ is symmetrized convex on the interval $[a^p, b^p]$ (or $[b^p, a^p]$) then it is easily seen that f is symmetrized p-convex on the interval $[a, b]$ by a similar procedure. The details are omitted. □

Theorem 2.1. *If $f : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ is symmetrized p-convex on the interval $[a, b]$, then the following Hermite-Hadamard inequalities hold*

$$f\left(\left[\frac{a^p + b^p}{2}\right]^{1/p}\right) \leq \frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \leq \frac{f(a) + f(b)}{2}. \qquad (16)$$

Proof. Since $f : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ is symmetrized p-convex on the interval $[a, b]$, then by writing the Hermite-Hadamard inequality for the function $P_{(f;p)}(x)$ it is obtained that

$$P_{(f;p)}\left(\left[\frac{a^p + b^p}{2}\right]^{1/p}\right) \leq \frac{p}{b^p - a^p}\int_a^b \frac{P_{(f;p)}(x)}{x^{1-p}}dx \leq \frac{P_{(f;p)}(a) + P_{(f;p)}(b)}{2}, \qquad (17)$$

where, it is easily seen that

$$P_{(f;p)}\left(\left[\frac{a^p + b^p}{2}\right]^{1/p}\right) = f\left(\left[\frac{a^p + b^p}{2}\right]^{1/p}\right), \quad \frac{P_{(f;p)}(a) + P_{(f;p)}(b)}{2} = \frac{f(a) + f(b)}{2},$$

and

$$\frac{p}{b^p - a^p}\int_a^b \frac{P_{(f;p)}(x)}{x^{1-p}}dx = \frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx$$

Then by (17) required inequalities are got. □

Remark 2.2. *In Theorem 2.1,*

i.) if it is choosen as $p = 1$, then the inequalities (16) reduces to the inequalities (8) in Theorem (1.5).

ii.) if it is choosen as $p = -1$, then the inequalities (16) reduces to the inequalities (11) in Theorem (1.8).

Remark 2.3. *By helping Theorem 1.5 and Proposition 2.1, the proof of Theorem 2.1 can also be given as follows :*

Since $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ is symmetrized p-convex on the interval $[a,b]$, $f\circ g$ is symmetrized convex on the interval $[a^p,b^p]$ (or $[b^p,a^p]$) with $g(x)=x^{1/p}, x>0, p\neq 0$. So, by Theorem 1.5 it is obtained that

$$(f\circ g)\left(\frac{a^p+b^p}{2}\right)\leq\frac{1}{b^p-a^p}\int_{a^p}^{b^p}(f\circ g)(x)dx\leq\frac{(f\circ g)(a^p)+(f\circ g)(b^p)}{2},$$

i.e.

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\leq\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\leq\frac{f(a)+f(b)}{2}.$$

Theorem 2.2. *If $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ is symmetrized p-convex on the interval $[a,b]$. Then for any $x\in[a,b]$ the following bounds hold*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\leq P_{(f;p)}(x)\leq\frac{f(a)+f(b)}{2}. \tag{18}$$

Proof. Since $P_{(f;p)}$ is p-convex on $[a,b]$, the inequality

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)=P_{(f;p)}\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\leq\frac{P_{(f;p)}(x)+P_{(f;p)}([a^p+b^p-x^p]^{1/p})}{2}=P_{(f;p)}(x)$$

holds for any $x\in[a,b]$. This give us the first inequality in (18).

Also, for any $x\in[a,b]$, there exist a number $t_0\in[0,1]$ such that $x=[t_0a^p+(1-t_0)b^p]^{1/p}$. Thus, by the p-convexity of $P_{(f;p)}$ the following inequality hold

$$\begin{aligned}P_{(f;p)}(x) &\leq t_0P_{(f;p)}(a)+(1-t_0)P_{(f;p)}(b)\\ &= P_{(f;p)}(a)=\frac{f(a)+f(b)}{2}\end{aligned}$$

which gives the second inequality in (18). □

Remark 2.4. *In Theorem 2.2,*

i.) if it is choosen as $p=1$, then the inequalities (18) reduces to the inequalities (9) in Theorem (1.6).

ii.) if it is choosen as $p=-1$, then the inequalities (18) reduces to the inequalities (12) in Theorem (1.9).

Remark 2.5. *By helping Theorem 1.6 and Proposition 2.1, the proof of Theorem 2.2 can also be given as follows :*

Since $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ is symmetrized p-convex on the interval $[a,b]$, $f\circ g$ is symmetrized convex on the interval $[a^p,b^p]$ with $g(x)=x^{1/p}, x>0, p\neq 0$. So, by Theorem 1.6, it is obtained that

$$(f\circ g)\left(\frac{a^p+b^p}{2}\right)\leq(\breve{f\circ g})(x^p)\leq\frac{(f\circ g)(a^p)+(f\circ g)(b^p)}{2},$$

i.e.

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\leq P_{(f;p)}(x)\leq\frac{f(a)+f(b)}{2}$$

for any $x\in[a,b]$.

Remark 2.6. *If $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ is symmetrized p-convex on the interval $[a,b]$, then the following bounds hold*

$$\inf_{x\in[a,b]} P_{(f;p)}(x) = f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)$$

and

$$\sup_{x\in[a,b]} P_{(f;p)}(x) = \frac{f(a)+f(b)}{2}.$$

Corollary 2.1. *If $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ is symmetrized p-convex on the interval $[a,b]$ and $w : [a,b] \to [0,\infty)$is integrable on $[a,b]$, then*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_a^b \frac{w(x)}{x^{1-p}}dx \le \int_a^b \frac{w(x)P_{(f;p)}(x)}{x^{1-p}}dx \le \frac{f(a)+f(b)}{2}\int_a^b \frac{w(x)}{x^{1-p}}dx. \quad (19)$$

Moreover, if w is p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$ on $[a,b]$, i.e. $w(x) = w([a^p+b^p-x^p]^{1/p})$ for all $x\in[a,b]$, then

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_a^b \frac{w(x)}{x^{1-p}}dx \le \int_a^b \frac{w(x)f(x)}{x^{1-p}}dx \le \frac{f(a)+f(b)}{2}\int_a^b \frac{w(x)}{x^{1-p}}dx. \quad (20)$$

Proof. The inequality (19) follows by (18) multiplying by $w(x)/x^{1-p} \ge 0$ and integrating over x on $[a,b]$.

By changing the variable, it is seen that

$$\int_a^b \frac{w(x)f\left([a^p+b^p-x^p]^{1/p}\right)}{x^{1-p}}dx = \int_a^b \frac{w([a^p+b^p-x^p]^{1/p})f(x)}{x^{1-p}}dx.$$

Since w is p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, then

$$\int_a^b \frac{w([a^p+b^p-x^p]^{1/p})f(x)}{x^{1-p}}dx = \int_a^b \frac{w(x)f(x)}{x^{1-p}}dx.$$

Thus

$$\begin{aligned}\int_a^b \frac{w(x)P_{(f;p)}(x)}{x^{1-p}}dx &= \frac{1}{2}\left[\int_a^b \frac{w(x)f(x)}{x^{1-p}}dx + \int_a^b \frac{w(x)f\left([a^p+b^p-x^p]^{1/p}\right)}{x^{1-p}}dx\right]\\ &= \int_a^b \frac{w(x)f(x)}{x^{1-p}}dx\end{aligned}$$

and by (19), the inequality (20) is got. □

Remark 2.7. *The inequality (20) is known as weighted generalization of Hermite-Hadamard inequality for p-convex functions (it is also given in Theorem 1.3). It has been shown now that this inequality remains valid for the larger class of symmetrized p-convex functions f on the interval $[a,b]$.*

Remark 2.8. *By helping Corollary 1.1 and Proposition 2.1, the proof of Corollary 2.1 can also be given. The details is omitted.*

Remark 2.9. *Let $a, b, \alpha \in \mathbb{R}$ with $0 < a < b$ and $\alpha \geq 2$. Then the function $f : [a, b] \to \mathbb{R}$, $f(x) = (x^p - a^p)^{\alpha-1}$, $p > 0$, is symmetrized p-convex on $[a, b]$*

i.) If the function

$$f(x) = (x^p - a^p)^{\alpha-1}$$

which is symmetrized p-convex on $[a, b]$ in the inequality (19) is considered, then

$$\frac{1}{2^{\alpha-1}} \int_a^b \frac{w(x)}{x^{1-p}} dx \leq \frac{\Gamma(\alpha)}{2p\,(b^p - a^p)^{\alpha-1}} \left[J_{a^p+}^{\alpha} (w \circ g)(b^p) + J_{b^p-}^{\alpha} (w \circ g)(a^p) \right] \leq \frac{1}{2} \int_a^b \frac{w(x)}{x^{1-p}} dx$$

for any $w : [a, b] \to [0, \infty)$ is integrable on $[a, b]$ with $g(x) = x^{1/p}, x \in [a^p, b^p]$.

ii.) If the function

$$w(x) = (x^p - a^p)^{\alpha-1} + (b^p - x^p)^{\alpha-1}$$

which is p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$ in the inequality (20) is considered, then

$$f\left(\left[\frac{a^p + b^p}{2} \right]^{1/p} \right) \leq \frac{\Gamma(\alpha+1)}{2\,(b^p - a^p)^{\alpha}} \left[J_{a^p+}^{\alpha} (f \circ g)(b^p) + J_{b^p-}^{\alpha} (f \circ g)(a^p) \right] \leq \frac{f(a) + f(b)}{2},$$

where $g(x) = x^{1/p}, x \in [a^p, b^p]$.

iii.) Let φ be p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$. If the function

$$w(x) = \left[(x^p - a^p)^{\alpha-1} + (b^p - x^p)^{\alpha-1} \right] \varphi(x)$$

which is p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$ in the inequality (20) is considered, then the following inequalities hold

$$\begin{aligned} & f\left(\left[\frac{a^p + b^p}{2} \right]^{1/p} \right) \left[J_{a^p+}^{\alpha} (\varphi \circ g)(b^p) + J_{b^p-}^{\alpha} (\varphi \circ g)(a^p) \right] \\ \leq \; & \left[J_{a^p+}^{\alpha} (f\varphi \circ g)(b^p) + J_{b^p-}^{\alpha} (f\varphi \circ g)(a^p) \right] \\ \leq \; & \frac{f(a) + f(b)}{2} \left[J_{a^p+}^{\alpha} (\varphi \circ g)(b^p) + J_{b^p-}^{\alpha} (\varphi \circ g)(a^p) \right] \end{aligned}$$

which are the same of inequalities in (7). Where $g(x) = x^{1/p}, x \in [a^p, b^p]$.

Theorem 2.3. *Assume that $f : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ is symmetrized p-convex on the interval $[a, b]$ with $p \in \mathbb{R} \setminus \{0\}$. Then the following Hermite-Hadamard inequalities hold*

$$\begin{aligned} & \frac{1}{2} \left[f(\left[\frac{x^p + y^p}{2} \right]^{1/p}) + f\left(\left[a^p + b^p - \frac{x^p + y^p}{2} \right]^{1/p} \right) \right] && (21) \\ \leq \; & \frac{p}{2\,(y^p - x^p)} \left[\int_x^y \frac{f(t)}{t^{1-p}} dt + \int_{[a^p+b^p-y^p]^{1/p}}^{[a^p+b^p-x^p]^{1/p}} \frac{f(t)}{t^{1-p}} dt \right] \\ \leq \; & \frac{1}{4} \left[f(x) + f(y) + f\left([a^p + b^p - x^p]^{1/p} \right) + f\left([a^p + b^p - y^p]^{1/p} \right) \right]. \end{aligned}$$

for any $x, y \in [a, b]$ with $x \neq y$.

Proof. Since $P_{(f;p),[a,b]}$ is p-convex on $[a, b]$, then $P_{(f;p),[a,b]}$ is also p-convex on any subinterval $[x, y]$ (or $[y, x]$) where $x, y \in [a, b]$.

By Hermite-Hadamard inequalities for p-convex functions, the inequalities

$$P_{(f;p),[a,b]}\left(\left[\frac{x^p+y^p}{2}\right]^{1/p}\right) \le \frac{p}{y^p-x^p}\int_x^y \frac{P_{(f;p),[a,b]}(t)}{t^{1-p}}dt \le \frac{P_{(f;p),[a,b]}(x)+P_{(f;p),[a,b]}(y)}{2} \tag{22}$$

hold for any $x, y \in [a,b]$ with $x \neq y$.

By definition of $P_{(f;p)}$, it is easily seen that

$$P_{(f;p),[a,b]}\left(\left[\frac{x^p+y^p}{2}\right]^{1/p}\right) = \frac{1}{2}\left[f(\left[\frac{x^p+y^p}{2}\right]^{1/p}) + f\left(\left[a^p+b^p-\frac{x^p+y^p}{2}\right]^{1/p}\right)\right],$$

$$\begin{aligned}
\int_x^y \frac{P_{(f;p),[a,b]}(t)}{t^{1-p}}dt &= \frac{1}{2}\int_x^y \frac{1}{t^{1-p}}\left[f(t)+f\left(\left[a^p+b^p-t^p\right]^{1/p}\right)\right]dt \\
&= \frac{1}{2}\int_x^y \frac{f(t)}{t^{1-p}}dt + \frac{1}{2}\int_x^y \frac{f\left(\left[a^p+b^p-t^p\right]^{1/p}\right)}{t^{1-p}}dt \\
&= \frac{1}{2}\int_x^y \frac{f(t)}{t^{1-p}}dt + \frac{1}{2}\int_{[a^p+b^p-y^p]^{1/p}}^{[a^p+b^p-x^p]^{1/p}} \frac{f(t)}{t^{1-p}}dt
\end{aligned}$$

and

$$\begin{aligned}
&\frac{P_{(f;p),[a,b]}(x)+P_{(f;p),[a,b]}(y)}{2} \\
=\ & \frac{1}{4}\left[f(x)+f(y)+f\left(\left[a^p+b^p-x^p\right]^{1/p}\right)+f\left(\left[a^p+b^p-y^p\right]^{1/p}\right)\right].
\end{aligned}$$

Thus, by (22) the desired the inequalities (21) are obtained. □

Remark 2.10. *By helping Theorem 1.7 and Proposition 2.1, the proof of Theorem 2.3 can also be given. The details is omitted.*

Remark 2.11. *If $x = a$ and $y = b$ are choosen in (21), then the inequalities (16) are obtained. If $y = [a^p + b^p - x^p]^{1/p}$ is choosen for a given $x \in [a,b]$, $x \neq \left[\frac{a^p+b^p}{2}\right]^{1/p}$, then from (21) the inequality*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) \le \frac{p}{a^p+b^p-2x^p}\int_x^{[a^p+b^p-x^p]^{1/p}} \frac{f(t)}{t^{1-p}}dt \le \frac{1}{2}\left[f(x)+f\left(\left[a^p+b^p-x^p\right]^{1/p}\right)\right], \tag{23}$$

hold provided that $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ is symmetrized p-convex on the interval $[a,b]$.

Multiplying the inequalities (23) by $\frac{1}{x^{1-p}}$, then integrating the resulting inequality over x, the following refinement of the first part of (16)

$$\begin{aligned}
&f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) \\
\le\ & \frac{p^2}{(b^p-a^p)}\int_a^b \left[\frac{1}{x^{1-p}\left(a^p+b^p-2x^p\right)}\int_x^{[a^p+b^p-x^p]^{1/p}} \frac{f(t)}{t^{1-p}}dt\right]dx \\
\le\ & \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx,
\end{aligned}$$

hold provided that $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ is symmetrized p-convex on the interval $[a,b]$.

When the function is p-convex, the following inequalities are obtained as well:

Remark 2.12. *If $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ is p-convex, then from (21) the following inequalities*

$$
\begin{aligned}
& f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) \qquad (24)\\
\leq\ & \frac{1}{2}\left[f\left(\left[\frac{x^p+y^p}{2}\right]^{1/p}\right)+f\left(\left[a^p+b^p-\frac{x^p+y^p}{2}\right]^{1/p}\right)\right]\\
\leq\ & \frac{p}{2\left(y^p-x^p\right)}\left[\int_x^y \frac{f(t)}{t^{1-p}}dt+\int_{\left[a^p+b^p-y^p\right]^{1/p}}^{\left[a^p+b^p-x^p\right]^{1/p}} \frac{f(t)}{t^{1-p}}dt\right]\\
\leq\ & \frac{1}{4}\left[f(x)+f(y)+f\left(\left[a^p+b^p-x^p\right]^{1/p}\right)+f\left(\left[a^p+b^p-y^p\right]^{1/p}\right)\right],
\end{aligned}
$$

hold for any $x,y\in[a,b]$ with $x\neq y$.

By multiplying the inequalities (24) by $\frac{1}{(xy)^{1-p}}$ and integrating (24) over (x,y) on the square $[a,b]^2$ and dividing by $\frac{p^2}{(b^p-a^p)^2}$, the following refinement of the first Hermite-Hadamard inequality for p-convex functions holds

$$
\begin{aligned}
& f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\\
\leq\ & \frac{p^2}{2\left(b^p-a^p\right)^2}\left[\int_a^b\int_a^b \frac{f\left(\left[\frac{x^p+y^p}{2}\right]^{1/p}\right)}{(xy)^{1-p}}dxdy+\int_a^b\int_a^b \frac{f\left(\left[a^p+b^p-\frac{x^p+y^p}{2}\right]^{1/p}\right)}{(xy)^{1-p}}dxdy\right]\\
\leq\ & \frac{p^2}{2\left(b^p-a^p\right)^2}\int_a^b\int_a^b \frac{1}{(xy)^{1-p}\left(y^p-x^p\right)}\left[\int_x^y \frac{f(t)}{t^{1-p}}dt+\int_{\left[a^p+b^p-y^p\right]^{1/p}}^{\left[a^p+b^p-x^p\right]^{1/p}} \frac{f(t)}{t^{1-p}}dt\right]dxdy\\
\leq\ & \frac{p}{\left(b^p-a^p\right)}\int_a^b \frac{f(x)}{x^{1-p}}dx.
\end{aligned}
$$

Remark 2.13. *In Theorem 2.3,*

i.) if it is choosen as $p=1$, then the inequalities (21) reduces to the inequalities (10) in Theorem (1.7).

ii.) if it is choosen as $p=-1$, then the inequalities (21) reduces to the inequalities (13) in Theorem (1.10).

3. Conclusions

The aim of this paper is to introduce the definition of symmetrized p-convexity for the first time and to give some interesting examples and comparisons. Also, some new Hermite-Hadamard type inequalities for this class of functions are obtained. Morever, The obtained results are reduced to another obtained the results previously in the literature in special cases.

References

[1] Dragomir, S. S., (2016), Symmetrized convexity and Hermite-Hadamard type inequalities, Journal of Mathematical Inequalities, 10 (4), pp. 901-918.

[2] Fang, Z. B. and Shi, R., (2014), On the(p, h)-convex function and some integral inequalities, J. Inequal. Appl., 2014 (45), 16 pages.

[3] İşcan, İ., (2014), Hermite-Hadamard type inequalities for harmonically convex functions, Hacettepe Journal of Mathematics and Statistics, 43 (6), pp. 935-942.

[4] İşcan, İ., (2016), Ostrowski type inequalities for p-convex functions, New Trends in Mathematical Sciences, 4(3), pp. 140-150.

[5] Kunt, M. and İşcan, İ., (2017), Hermite-Hadamard-Fejér type inequalities for p-convex functions, Arab Journal of Mathematical Sciences, 23 (2), pp. 215-230.

[6] Kunt, M. and İşcan, İ., (2018), Hermite-Hadamard-Fejér type inequalities for p-convex functions via fractional integrals, Iran J Sci Technol Trans Sci., 42, pp. 2079-2089.

[7] Kilbas, A. A., Srivastava, H.M. and Trujillo, J. J., (2006), Theory and applications of fractional differential equations. Elsevier, Amsterdam.

[8] Wu, S., Ali, B. R., Baloch, I. A. and Haq, A. U., (2017), Inequalities related to symmetrized harmonic convex functions, arxiv:1711.08051v1 [math.CA].

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