TWMS J. App. Eng. Math. V.10, N.1, 2020, pp. 34-46

MULTIPLE POSITIVE SOLUTIONS FOR A SYSTEM OF FRACTIONAL HIGHER-ORDER INFINITE-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this article, we establish some results on the existence of multiple positive solutions for a system of nonlinear fractional order infinite-point boundary value problems. The main tool is a fixed point theorem of the cone expansion and compression of functional type due to Avery, Anderson and O'Regan for at least one positive solution. We also prove that the boundary value problems has at least three positive solutions by applying the five functional fixed point theorem. And then, we establish the existence of at least $2k-1$ positive solutions to the fractional order boundary value problems for any arbitrary positive integer k.

Keywords: Fractional differential equations, infinite-point boundary value problem, multiple positive solutions, five functional fixed point theorem, Green's function, cone.

AMS Subject Classification: 26A33, 34B15, 34B18.

1. Introduction

In this paper, we study the existence of multiple positive solutions for the following coupled of nonlinear fractional order differential equations

$$\begin{aligned} D_{0^+}^{\nu_1} u(t) + f_1(t, u(t), v(t)) = 0,\ t \in (0,1), \\ D_{0^+}^{\nu_2} v(t) + f_2(t, u(t), v(t)) = 0,\ t \in (0,1), \end{aligned} \tag{1}$$

with infinite-point boundary conditions

$$\begin{aligned} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0,\ u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j) \\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0,\ v^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j v(\xi_j) \end{aligned} \tag{2}$$

where $\nu_1, \nu_2 > 2$, $\nu_1, \nu_2 \in (n-1, n]$, $i \in [1, n-2]$ is a fixed integer, $\alpha_j \geq 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1 (j = 1, 2, \cdots)$, $D_{0^+}^{\nu_1}$, $D_{0^+}^{\nu_2}$ are the standard Riemann-Liouville fractional derivatives of order ν_1 and ν_2 respectively and the function $f_j : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous, for $j = 1, 2$.

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§ Manuscript received: January 25, 2018; accepted: September 06, 2018.

In recent years, fractional differential equations have gained importance due to their various applications in science and engineering such as rheology, dynamical processes in self-similar and porous structures, heat conduction, control theory, electroanalytical chemistry, chemical physics, economics, etc. Many researchers have shown their interest in fractional differential equations. The motivation for those works stems from both the intensive development of the theory of fractional calculus itself and the applications. Many papers and books on fractional calculus, fractional differential equations have appeared [10, 11, 14, 16].

Recently, there have been some papers dealing with the existence and multiplicity of positive solutions of nonlinear fractional differential equations with various boundary conditions. See, Bai and Sun [4], Bai, Sun and Zhang [5], Bai and Lü [6], Chai [7], Goodrich [8], Henderson and Luca [9], Liang and Zhang [12], Sun and Zhang [19], Tian and Liu [20] and Xu, Jiang and Yuan [21]. The above papers that are motivated to this work.

The rest of the paper is organized as follows. In Section 2, we present some definitions and background results. For sake of convenience, we also state the fixed point theorems. In Section 3, we construct the Green's function for the homogeneous BVP corresponding to (1)-(2) and estimate the bounds for the Green's function. In Section 4, we establish the existence and multiplicity of positive solutions of the BVP (1)-(2) by using an Avery generalization of the Leggett-Williams fixed point theorem. We also establish the existence of at least $2k-1$ positive solutions to the fractional order BVP (1)-(2). In Section 5, as an application, we demonstrate our results with some examples.

2. **Preliminaries**

In this section, we provide some background materials from theory of cones in Banach spaces. We also state a fixed point theorem of cone expansion and compression of functional type due to Avery et al.[2] and five functional fixed point theorem [3] for a cone.

Definition 2.1. *Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is said to be a cone provided the following conditions are satisfied:*

(i) *If $\mu \in P$ and $\lambda \geq 0$, then $\lambda\mu \in P$;*
(ii) *If $\mu \in P$ and $-\mu \in P$, then $\mu = 0$.*

Every cone $P \subset E$ induces an ordering in E given by $\mu \leq \vartheta$ if and only if $\vartheta - \mu \in P$.

Definition 2.2. *A map μ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\mu : P \to [0,\infty)$ is continuous and $\mu(\lambda u + (1-\lambda)v) \geq \lambda\mu(u) + (1-\lambda)\mu(v)$, for all $u, v \in P$ and $0 \leq \lambda \leq 1$.*

Similarly, we say the map ϑ is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\vartheta : P \to [0,\infty)$ is continuous and $\vartheta(\lambda u + (1-\lambda)v) \leq \lambda\vartheta(u) + (1-\lambda)\vartheta(v)$, for all $u, v \in P$ and $0 \leq \lambda \leq 1$.

We say the map ψ is sublinear functional if $\psi(\lambda u) \leq \lambda\psi(u),\ u \in P,\ 0 \leq \lambda \leq 1$.

Property H1: Let P be a cone in a real Banach space E and Ω be a bounded open subset of E with $0 \in \Omega$. Then a continuous functional $\mu : P \to [0,\infty)$ is said to satisfy Property $H1$ if one of the following conditions holds:

(a) μ is convex, $\mu(0) = 0,\ \mu(u) \neq 0$ if $u \neq 0$ and $\inf_{u \in P\cap\partial\Omega} \mu(u) > 0$;
(b) μ is sublinear, $\mu(0) = 0,\ \mu(u) \neq 0$ if $u \neq 0$ and $\inf_{u \in P\cap\partial\Omega} \mu(u) > 0$;
(c) μ is concave and unbounded.

Property H2: Let P be a cone in a real Banach space E and Ω be a bounded open subset of E with $0 \in \Omega$. Then a continuous functional $\vartheta : P \to [0,\infty)$ is said to satisfy Property $H2$ if one of the following conditions holds:

(a) ϑ is convex, $\vartheta(0)=0,\ \vartheta(u)\neq 0$ if $u\neq 0$;
(b) ϑ is sublinear, $\vartheta(0)=0,\ \vartheta(u)\neq 0$ if $u\neq 0$;
(c) $\vartheta(u+v)\geq \vartheta(u)+\vartheta(v)$ for all $u,v\in P,\ \vartheta(0)=0,\ \vartheta(u)\neq 0$ if $u\neq 0$.

The approach used in proving the existence results in this paper are the following fixed point theorems of cone expansion and compression of functional type due to Avery, Henderson and O'Regan's [2], which generalized the functional compression fixed point theorems of Anderson-Avery [1] and Sun-Zhang [18] and five functional fixed point theorem [3], which is Avery generalization of the Leggett-Williams fixed point theorem.

Theorem 2.1. *Let Ω_1 and Ω_2 be two bounded open subsets in a Banach space E such that $0\in\Omega_1$ and $\Omega_1\subseteq\Omega_2$ and P is a cone in E. Suppose $T: P\cap(\overline{\Omega}_2\backslash\Omega_1)\to P$ is completely continuous operator, μ and ϑ are nonnegative continuous functional on P, and one of the two conditions;*

(i) *μ satisfies Property $H1$ with $\mu(Tu)\geq\mu(u)$, for all $u\in P\cap\partial\Omega_1$ and ϑ satisfies Property $H2$ with $\vartheta(Tu)\leq\vartheta(u)$, for all $u\in P\cap\partial\Omega_2$;*
(ii) *ϑ satisfies Property $H2$ with $\vartheta(Tu)\leq\vartheta(u)$, for all $u\in P\cap\partial\Omega_1$ and μ satisfies Property $H1$ with $\mu(Tu)\geq\mu(u)$, for all $u\in P\cap\partial\Omega_2$*

is satisfied. Then T has at least one fixed point in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$.

Let γ,β,θ be nonnegative continuous convex functional on P and α,ψ be nonnegative continuous concave functionals on P. Then for nonnegative numbers h,a,b,d and c, we define the following convex sets,

$$\begin{aligned}
P(\gamma,c)&=\{u\in P:\gamma(u)<c\},\\
P(\gamma,\alpha,a,c)&=\{u\in P: a\leq\alpha(u);\ \gamma(u)\leq c\},\\
Q(\gamma,\beta,d,c)&=\{u\in P:\beta(u)\leq d;\ \gamma(u)\leq c\},\\
P(\gamma,\theta,\alpha,a,b,c)&=\{u\in P: a\leq\alpha(u);\ \theta(u)\leq b,\ \gamma(u)\leq c\},\\
Q(\gamma,\beta,\psi,h,d,c)&=\{u\in P: h\leq\psi(u);\ \beta(u)\leq d,\ \gamma(y)\leq c\}.
\end{aligned}$$

In order to obtain multiple positive solutions of the BVP (1)-(2), the following called five functional fixed point theorem will play an important role in the proof of our main results.

Theorem 2.2. [3]. *Let P be a cone in a real Banach space E. Suppose there exist positive numbers c and M, nonnegative continuous concave functionals α and ψ on P, and nonnegative continuous convex functionals γ,β and θ are on P with $\alpha(u)\leq\beta(u)\ \|u\|\leq M\gamma(u)$ for all $u\in\overline{P(\gamma,c)}$. Suppose that $T:\overline{P(\gamma,c)}\to\overline{P(\gamma,c)}$ is completely continuous operator and there exist nonnegative numbers h,a,k,b with $0<a<b$ such that:*

(B1) *$\{u\in P(\gamma,\theta,\alpha,b,k,c):\alpha(u)>b\}\neq\emptyset$ and $\alpha(Ty)>b$ for $u\in P(\gamma,\theta,\alpha,b,k,c)$;*
(B2) *$\{u\in Q(\gamma,\beta,\psi,h,a,c):\beta(u)<a\}\neq\emptyset$ and $\beta(Tu)<a$ for $u\in Q(\gamma,\beta,\psi,h,a,c)$;*
(B3) *$\alpha(Tu)>b$ for $u\in P(\gamma,\alpha,b,c)$ with $\theta(Tu)>k$;*
(B4) *$\beta(Tu)<a$ for $u\in Q(\gamma,\beta,a,c)$ with $\psi(Tu)<h$.*

Then T has at least three fixed points $u_1,u_2,u_3\in\overline{P(\gamma,c)}$ such that $\beta(u_1)<a,\ b<\alpha(u_2)$ and $a<\beta(u_3)$ with $\alpha(u_3)<b$.

3. Green's Function and Bounds

In this section, we construct the Green's function for the homogeneous BVP corresponding (1)-(2) and estimate bounds for the Green's function which are essential to establish the main results.

Lemma 3.1. *Let $h(t) \in C[0,1]$ be given. Then the unique solution to the problem*

$$\begin{cases} D_{0+}^{\nu}u(t) + h(t) = 0, \ t \in (0,1), \ n-1 < \nu \leq n, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j), \ 1 \leq i \leq n-2 \end{cases} \tag{3}$$

can be expressed by $u(t) = \int_0^1 G(t,s)h(s)ds$, *where*

$$G(t,s) = \frac{1}{p(0)\Gamma(\nu)} \begin{cases} t^{\nu-1}p(s)(1-s)^{\nu-1-i} - p(0)(t-s)^{\nu-1}, & 0 \leq s \leq t \leq 1, \\ t^{\nu-1}p(s)(1-s)^{\nu-1-i}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{4}$$

Here $p(s) = \Delta - \sum_{s \leq \xi_j} \left(\frac{\xi_j - s}{1-s}\right)^{\nu-1}(1-s)^i$, $\Delta - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\nu-1} > 0$, $\Delta = (\nu-1)(\nu-2)\cdots(\nu-i)$. *Obviously,* $G(t,s)$ *is continuous on* $[0,1] \times [0,1]$.

Proof. The proof is similar to Lemma 2.2 in [13]; we only need to replace $\Delta - \sum_{j=1}^{m-2} \alpha_j \xi_j^{\nu-1}$ with $\Delta - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\nu-1}$. We omit the details here. □

Lemma 3.2. *Suppose that* $p(0) > 0$, *and then the Green's function* $p(s) > 0, s \in [0,1]$ *and* p *is nondecreasing on* $[0,1]$.

Proof. By simple computation, we have that $p'(s) = \sum_{s \leq \xi_j} \alpha_j (\xi_j - s)^{\nu-2}(1-s)^{-\nu+i}[(\nu-1)(1-\xi_j) + i(\xi_j - s)] > 0$. This implies that p is nondecreasing on $[0,1]$ and $p(s) \geq p(0) > 0, s \in [0,1]$. □

Lemma 3.3. *Let* $I = \left[\frac{1}{4}, \frac{3}{4}\right]$. *Then the function* $G(t,s)$ *defined by* (4) *admits the following properties:*
(i) $G(t,s) > 0, \ \frac{\partial G(t,s)}{\partial t} > 0, \ 0 < t, s < 1$;
(ii) $G(t,s) \leq G(1,s), 0 \leq t, s \leq 1$;
(iii) $G(t,s) \geq \eta G(1,s), \ t \in I, s \in [0,1]$, *where* $0 < \eta = \frac{1}{4^{\nu-1}} < 1$.

Proof. (i) For $0 < s \leq t < 1$, noticing that $1 \leq i \leq n-2$, $p(0) = \Delta - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\nu-1}$, by Lemma 3.2, is easy to see that

$$\begin{aligned} G(t,s) &= \frac{1}{p(0)\Gamma(\nu)} \left[t^{\alpha-1}p(s)(1-s)^{\nu-1-i} - p(0)(t-s)^{\nu-1}\right] \\ &= \frac{1}{p(0)\Gamma(\nu)} t^{\nu-1} \left[p(s)(1-s)^{\nu-1-i} - p(0)\left(1 - \frac{s}{t}\right)^{\nu-1}\right] \\ &\geq \frac{p(s)t^{\nu-1}}{p(0)\Gamma(\nu)} \left[(1-s)^{\nu-1-i} - \left(1 - \frac{s}{t}\right)^{\nu-1}\right] \\ &> \frac{p(s)t^{\nu-1}}{p(0)\Gamma(\nu)} \left[(1-s)^{\nu-1} - \left(1 - \frac{s}{t}\right)^{\nu-1}\right] \geq 0 \end{aligned}$$

It is clear that for $0 < t \leq s < 1$, $G(t,s) > 0$.

By direct computation, we have

$$\frac{\partial G(t,s)}{\partial t} = \frac{1}{p(0)\Gamma(\nu)} \begin{cases} (\nu-1)p(s)(1-s)^{\nu-1-i}t^{\nu-2} - (\nu-1)p(0)(t-s)^{\nu-2}, & 0 \leq s \leq t \leq 1, \\ (\nu-1)p(s)(1-s)^{\nu-1-i}t^{\nu-2}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Obviously, $\frac{\partial G(t,s)}{\partial t}$ is continuous on $[0,1]\times[0,1]$. In a similar manner, for $0<s\le t<1$, noticing that $1\le i\le n-2$, by Lemma 3.2, we get that

$$\begin{aligned}
\frac{\partial G(t,s)}{\partial t} &= \frac{1}{p(0)\Gamma(\nu)}\Big[(\nu-1)p(s)(1-s)^{\nu-1-i}t^{\nu-2}-(\nu-1)p(0)(t-s)^{\nu-2}\Big]\\
&= \frac{(\alpha-1)}{p(0)\Gamma(\nu)}t^{\nu-2}\Big[p(s)(1-s)^{\nu-1-i}-p(0)\Big(1-\frac{s}{t}\Big)^{\nu-2}\Big]\\
&\ge \frac{(\nu-1)p(s)t^{\nu-2}}{p(0)\Gamma(\nu)}\Big[(1-s)^{\nu-1-i}-\Big(1-\frac{s}{t}\Big)^{\nu-2}\Big]\\
&\ge \frac{(\nu-1)p(s)t^{\nu-2}}{p(0)\Gamma(\nu)}\Big[(1-s)^{\nu-2}-\Big(1-\frac{s}{t}\Big)^{\nu-2}\Big]\ge 0
\end{aligned}$$

(ii) By (i), we know that $G(t,s)$ is increasing with respect to t; thus we have that $G(t,s)\le G(1,s)=\frac{1}{p(0)\Gamma(\nu)}[p(s)(1-s)^{\nu-i-1}-p(0)(1-s)^{\nu-1}],\ 0\le s\le 1.$
(iii). For $t\in I$ and $s\in[0,1]$, we get that

$$\begin{aligned}
G(t,s) &= \frac{1}{p(0)\Gamma(\nu)}\Big[t^{\nu-1}p(s)(1-s)^{\nu-1-i}-p(0)(t-s)^{\nu-1}\Big]\\
&= \frac{1}{p(0)\Gamma(\nu)}t^{\nu-1}\Big[p(s)(1-s)^{\nu-1-i}-p(0)\Big(1-\frac{s}{t}\Big)^{\nu-1}\Big]\\
&\ge \frac{1}{p(0)\Gamma(\nu)}t^{\nu-1}\Big[p(s)(1-s)^{\nu-1-i}-p(0)(1-s)^{\nu-1}\Big]\\
&= t^{\nu-1}G(1,s)\\
&\ge \frac{1}{4^{\nu-1}}G(1,s).
\end{aligned}$$

For $t\in I$ and $s\in[0,1]$, we have that

$$G(t,s)=\frac{1}{p(0)\Gamma(\nu)}t^{\nu-1}p(s)(1-s)^{\nu-1-i}=t^{\nu-1}\cdot\frac{1}{p(0)\Gamma(\nu)}p(s)(1-s)^{\nu-1-i}\ge t^{\nu-1}G(1,s)\ge\frac{1}{4^{\nu-1}}G(1,s).$$

□

For use in the sequel, let η_1 and η_2 the constants given by Lemma 3.3 associated, respectively, to the Greens' functions G_1 and G_2 and define η by $\eta=\min\{\eta_1,\eta_2\}$, and notice that $\eta\in(0,1)$.

By using the Green's function G_1 and G_2, our problem (1)-(2) can be written equivalently as the following nonlinear system of integral equations

$$u(t)=\int_0^1 G_1(t,s)f_1(s,u(s),v(s))ds,\ t\in[0,1],\ v(t)=\int_0^1 G_2(t,s)f_2(s,u(s),v(s))ds,\ t\in[0,1].$$

We consider the Banach space $B=E\times E$, where $E=\big\{u:u\in C[0,1]\big\}$ equipped with the norm $\|(u,v)\|=\|u\|+\|v\|$, for $(u,v)\in B$ and the norm is defined as $\|u\|=\max_{t\in[0,1]}|u(t)|$.
We define the cone $P\subset B$ by

$$P=\left\{(u,v)\in B: u(t)\ge 0,\ v(t)\ge 0,\ t\in[0,1],\ \text{and}\ \min_{t\in I}[u(t)+v(t)]\ge\eta\|(u,v)\|\right\}.$$

Define the operators $T_1,T_2:P\to E$ by

$$T_1(u,v)(t)=\int_0^1 G_1(t,s)f_1(s,u(s),v(s))ds,\ T_2(u,v)(t)=\int_0^1 G_2(t,s)f_2(s,u(s),v(s))ds,$$

where $G_1(t,s)$ is the Green's function of Lemma 3.1 with ν replaced by ν_1 and, like wise, $G_2(t,s)$ is the Green's function of Lemma 3.1 with ν replaced by ν_2. Using the above operators, define an operator $T : P \to B$ by

$$(T(u,v))(t) = \big((T_1(u,v))(t), (T_2(u,v))(t)\big), \ (u,v) \in P. \tag{5}$$

Thus the existence and multiplicity of positive solutions the system (1)-(2) are equivalent to the existence and multiplicity of fixed points of the operator T.

4. Existence of Multiple Positive Solutions

In this section, we establish the existence of at least one and at least three positive solutions for the fractional order BVP (1)-(2) by using fixed point theorems of cone expansion and compression of functional type due to Avery, Henderson and O'Regan's [2] and five functional fixed point theorem [3]. And, then we establish the existence of at least $2k-1$ positive solutions to the fractional order BVP (1)-(2) for an arbitrary positive integer k.

Now we discuss the existence of at least one positive solution by using fixed point theorems of cone expansion and compression of functional type due to Avery, Henderson and O'Regan's [2].

For notational convenience, we define

$$L = \min\Big\{ \int_0^1 G_1(1,s)ds, \ \int_0^1 G_2(1,s)ds\Big\}, \ M = \max\Big\{ \int_{s\in I} \eta G_1(1,s)ds, \ \int_{s\in I} \eta G_2(1,s)ds\Big\}.$$

Let us define two continuous functionals μ and ϑ on the cone P by

$$\mu(u,v) = \min_{t\in I}\{u(t)+v(t)\} \text{ and } \vartheta(u,v) = \max_{t\in[0,1]}\{u(t)+v(t)\} = u(1)+v(1) = \|(u,v)\|.$$

It is clear that $\mu(u,v) \le \vartheta(u,v)$ for all $(u,v) \in P$.

Theorem 4.1. *Suppose that there exist positive numbers r, R with $r < \eta R$ such that the following conditions are satisfied:*

(C1) $f_j(t,u(t),v(t)) \ge \frac{r}{2M}$ *for all* $t \in I$ and $u, v \in [r, R]$

(C2) $f_j(t,u(t),v(t)) \le \frac{R}{2L}$ *for all* $t \in [0,1]$ and $u, v \in [0, R]$.

Then fractional order BVP (1)-(2) has at least one positive and nondecreasing solution (u^*, v^*) satisfying $r \le \mu(u^*,v^*)$ and $\vartheta(u^*,v^*) \le R$.

Proof. Let $\Omega_1 = \{(u,v) : \mu(u,v) < r\}$ and $\Omega_2 = \{(u,v) : \vartheta(u,v) < R\}$. It is easy to see that $0 \in \Omega_1$, Ω_1 and Ω_2 are bounded open subsets of E. Let $(u,v) \in \Omega_1$, then we have $r > \mu(u,v) = \min_{t\in I}\{u(t)+v(t)\} \ge \eta[\|u\|+\|v\|] = \eta\vartheta(u,v)$. Thus $R > \frac{r}{\eta} > \vartheta(u,v)$, i.e $(u,v) \in \Omega_2$, so $\Omega_1 \subseteq \Omega_2$.

Claim 1: If $(u,v) \in P \cap \partial\Omega_1$, then $\mu(T(u,v)) \ge \mu(u,v)$.

To see this let $(u,v) \in P \cap \partial\Omega_1$, then $R = \vartheta(u,v) \ge (u(s)+v(s)) \ge \mu(u,v) = r, \ s \in I$. Thus it follows from $(C1)$ and Lemma 3.2 that

$$\begin{aligned}\mu(T(u,v)(t)) &= \min_{t\in I}\Big[\int_0^1 G_1(t,s)f_1(s,u(s),v(s))ds + \int_0^1 G_2(t,s)f_2(s,u(s),v(s))ds\Big]\\ &\ge \int_{s\in I}\eta G_1(1,s)f_1(s,u(s),v(s))ds + \int_{s\in I}\eta G_2(1,s)f_2(s,u(s),v(s))ds\\ &\ge \frac{r}{2M}\int_{s\in I}\eta G_1(1,s)ds + \frac{r}{2M}\int_{s\in I}\eta G_2(1,s)ds = r = \mu(u,v).\end{aligned}$$

Claim 2: If $(u,v) \in P \cap \partial\Omega_2$, then $\vartheta(T(u,v)) \leq \vartheta(u,v)$.
To see this let $(u,v) \in P \cap \partial\Omega_2$, then $u(s)+v(s) \leq \vartheta(u,v) = R,\ s \in [0,1]$. Thus it follows from $(C2)$ and Lemma 3.2 yield

$$\begin{aligned}\vartheta(T(u,v)(t)) &= \max_{t\in[0,1]}\Big[\int_0^1 G_1(t,s)f_1(s,u(s),v(s))ds + \int_0^1 G_2(t,s)f_2(s,u(s),v(s))ds\Big]\\ &\leq \int_0^1 G_1(1,s)f_1(s,u(s),v(s))ds + \int_0^1 G_2(1,s)f_2(s,u(s),v(s))ds\\ &\leq \frac{R}{2L}\int_0^1 G_1(1,s)ds + \frac{R}{2L}\int_0^1 G_2(1,s)ds = R = \vartheta(u,v).\end{aligned}$$

Clearly μ satisfies Property $H1(c)$ and ϑ satisfies Property $H2(a)$. Therefore the condition (i) of Theorem 2.1 is satisfied and hence T has at least one fixed point $(u^*,v^*) \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$ i.e FBVP (1)-(2) has at least one positive and nondecreasing solution $(u^*,v^*) \in P$ such that $r \leq \mu(u^*,v^*)$ and $\vartheta(u^*,v^*) \leq R$. This completes the proof. □

Theorem 4.2. *Suppose that there exist positive numbers r, R with $r < R$ such that the following conditions are satisfied:*

(C3) $f_j(t,u(t),v(t)) \leq \frac{r}{2L}$, *for all* $t \in [0,1]$ and $u,v \in [0,r]$
(C4) $f_j(t,u(t),v(t)) \geq \frac{R}{2M}$, *for all* $t \in I$ and $u,v \in [R,\frac{R}{\eta}]$.

Then fractional order BVP (1)-(2) *has at least one positive and nondecreasing solution* $(u^*,v^*) \in P$ *such that* $r \leq \vartheta(u^*,v^*)$ *and* $\mu(u^*,v^*) \leq R$.

Proof. Let $\Omega_3 = \{(u,v) : \vartheta(u,v) < r\}$ and $\Omega_4 = \{(u,v) : \mu(u,v) < R\}$, we have $0 \in \Omega_3$, and $\Omega_3 \subseteq \Omega_4$, with Ω_3 and Ω_4 being bounded open subsets of E.
Claim 1: If $(u,v) \in P \cap \partial\Omega_3$, then $\vartheta(T(u,v)) \leq \vartheta(u,v)$.
To see this let $(u,v) \in P \cap \partial\Omega_3$, then $u(s)+v(s) \leq \vartheta(u,v) = r, s \in [0,1]$. Thus condition $(C3)$ and Lemma 3.2 yield

$$\begin{aligned}\vartheta(T(u,v)(t)) &= \max_{t\in[0,1]}\Big[\int_0^1 G_1(t,s)f_1(s,u(s),v(s))ds + \int_0^1 G_2(t,s)f_2(s,u(s),v(s))ds\Big]\\ &\leq \int_0^1 G_1(1,s)f_1(s,u(s),v(s))ds + \int_0^1 G_2(1,s)f_2(s,u(s),v(s))ds\\ &\leq \frac{r}{2L}\int_0^1 G_1(1,s)ds + \frac{r}{2L}\int_0^1 G_2(1,s)ds = r = \vartheta(u,v).\end{aligned}$$

Claim 2: If $(u,v) \in P \cap \partial\Omega_4$, then $\mu(T(u,v)) \geq \mu(u,v)$. To see this let $(u,v) \in P \cap \partial\Omega_4$, then $\frac{R}{\eta} = \frac{\mu(u,v)}{\eta} \geq \vartheta(u,v) \geq u(s)+v(s) \geq \mu(u,v) = R, s \in I$. Thus it follows from $(C4)$ and Lemma 3.2 that one has

$$\begin{aligned}\mu(T(u,v)(t)) &= \min_{t\in I}\Big[\int_0^1 G_1(t,s)f_1(s,u(s),v(s))ds + \int_0^1 G_2(t,s)f_2(s,u(s),v(s))ds\Big]\\ &\geq \int_{s\in I} \eta G_1(1,s)f_1(s,u(s),v(s))ds + \int_{s\in I} \eta G_2(1,s)f_2(s,u(s),v(s))ds\\ &\geq \frac{R}{2M}\int_{s\in I} \eta G_1(1,s)ds + \frac{R}{2M}\int_{s\in I} \eta G_2(1,s)ds = R = \mu(u,v).\end{aligned}$$

Clearly μ satisfies Property $H1(c)$ and ϑ satisfies Property $H2(a)$. Therefore the condition (ii) of Theorem 2.1 is satisfied and hence T has at least one fixed point $(u^*,v^*) \in P \cap (\overline{\Omega}_4 \backslash \Omega_3)$ i.e FBVP (1)-(2) has at least one positive and nondecreasing solution $(u^*,v^*) \in P$ such that $r \leq \vartheta(u^*,v^*)$ and $\mu(u^*,v^*) \leq R$. This completes the proof. □

Now we discuss the existence of at least three positive solutions by using five functional fixed point theorem [3].

Now, let $I_1 = \left[\frac{1}{3}, \frac{2}{3}\right]$ and define the nonnegative continuous concave functionals $\alpha,\ \psi$ and the nonnegative continuous convex functionals β, θ, γ on P by

$$\begin{aligned}
\alpha(u,v) &= \min_{t\in I}\{|u|+|v|\},\ \psi(u,v) = \min_{t\in I_1}\{|u|+|v|\},\ \gamma(u,v) \quad = \max_{t\in[0,1]}\{|u|+|v|\},\\
\beta(u,v) &= \max_{t\in I_1}\{|u|+|v|\},\ \theta(u,v) = \max_{t\in I}\{|u|+|v|\}.
\end{aligned}$$

We observe that for any $(u,v)\in P$, we have

$$\begin{aligned}
\alpha(u,v) &= \min_{t\in I}\{|u|+|v|\} \le \max_{t\in I_1}\{|u|+|v|\} = \beta(u,v),\ \text{and}\\
\|(u,v)\| &= \frac{1}{\eta}\min_{t\in I}\{|u|+|v|\} \le \frac{1}{\eta}\max_{t\in[0,1]}\{|u|+|v|\} = \frac{1}{\eta}\gamma(u,v).
\end{aligned} \tag{6}$$

Theorem 4.3. *Suppose there exist $0<a<b<\frac{b}{\eta}\le c$ such that the following conditions are satisfied:*

(D1) $f_j(t,u(t),v(t)) < \frac{a}{2L}$ *for all* $t\in[0,1]$ *and* $|u|+|v|\in[\eta a, a]$

(D2) $f_j(t,u(t),v(t)) > \frac{b}{2M}$ *for all* $t\in I$ *and* $|u|+|v|\in[b,\frac{b}{\eta}]$

(D3) $f_j(t,u(t),v(t)) < \frac{c}{2L}$ *for all* $t\in[0,1]$ *and* $|u|+|v|\in[0,c]$.

Then the fractional order BVP (1)-(2) has at least three positive solutions $(x_1,x_2),\ (y_1,y_2)$ and (z_1,z_2) such that $\beta(x_1,x_2)<a,\ b<\alpha(y_1,y_2)$ and $a<\beta(z_1,z_2)$ with $\alpha(z_1,z_2)<b$.

Proof. We seek three fixed points $(x_1,x_2),\ (y_1,y_2)$ and $(z_1,z_2)\in P$ of T defined by (5), for each $(u,v)\in P,\ \alpha(u,v)\le\beta(u,v)$ and $\|(u,v)\|\le\frac{1}{\eta}\gamma(u,v)$. We show that $T:\overline{P(\gamma,c)}\to\overline{P(\gamma,c)}$. Let $(u,v)\in\overline{P(\gamma,c)}$. Then $0\le|u|+|v|\le c$. We may use the condition $(D3)$ to obtain

$$\begin{aligned}
\gamma(T(u,v)(t)) &= \max_{t\in[0,1]}\Big[\int_0^1 G_1(t,s)f_1(s,u(s),v(s))ds + \int_0^1 G_2(t,s)f_2(s,u(s),v(s))ds\Big]\\
&\le \int_0^1 G_1(1,s)f_1(s,u(s),v(s))ds + \int_0^1 G_2(1,s)f_2(s,u(s),v(s))ds\\
&< \frac{c}{2L}\int_0^1 G_1(1,s)ds + \frac{c}{2L}\int_0^1 G_2(1,s)ds < \frac{c}{2}+\frac{c}{2} = c.
\end{aligned}$$

Therefore $T:\overline{P(\gamma,c)}\to\overline{P(\gamma,c)}$. Now the conditions $(B1)$ and $(B2)$ of Theorem 2.2 are to be verified. It is obvious that

$$\frac{b+(\frac{b}{\eta})}{2} \in \Big\{(u,v)\in P\big(\gamma,\theta,\alpha,b,\frac{b}{\eta},c\big) : \alpha(u,v)>b\Big\}\ne\emptyset,$$

and

$$\frac{\eta a + a}{2} \in \Big\{(u,v)\in Q\big(\gamma,\beta,\psi,\eta a,a,c\big) : \beta(u,v)<a\Big\}\ne\emptyset.$$

Next, let $(u,v)\in P\big(\gamma,\theta,\alpha,b,\frac{b}{\eta},c\big)$ or $(u,v)\in Q\big(\gamma,\beta,\psi,\eta a,a,c\big)$. Then $b\le |u(t)|+|v(t)|\le \frac{b}{\eta}$ and $\eta a\le |u(t)|+|v(t)|\le a$. Now, we may apply the condition $(D2)$ to get

$$\begin{aligned}\alpha(T(u,v)(t)) &= \min_{t\in I}\Big[\int_0^1 G_1(t,s)f_1(s,u(s),v(s))ds+\int_0^1 G_2(t,s)f_2(s,u(s),v(s))ds\Big]\\ &\ge \int_{s\in I}\eta G_1(1,s)f_1(s,u(s),v(s))ds+\int_{s\in I}\eta G_2(1,s)f_2(s,u(s),v(s))ds\\ &> \frac{b}{2M}\int_{s\in I}\eta G_1(1,s)ds+\frac{b}{2M}\int_{s\in I}\eta G_2(1,s)ds>\frac{b}{2}+\frac{b}{2}=b.\end{aligned}$$

Clearly, by the condition $(D1)$, we have

$$\begin{aligned}\beta(T(u,v)(t)) &= \max_{t\in I}\Big[\int_0^1 G_1(t,s)f_1(s,u(s),v(s))ds+\int_0^1 G_2(t,s)f_2(s,u(s),v(s))ds\Big]\\ &\le \int_0^1 G_1(1,s)f_1(s,u(s),v(s))ds+\int_0^1 G_2(1,s)f_2(s,u(s),v(s))ds\\ &< \frac{a}{2L}\int_0^1 G_1(1,s)ds+\frac{a}{2L}\int_0^1 G_2(1,s)ds<\frac{a}{2}+\frac{a}{2}=a.\end{aligned}$$

To see that $(B3)$ is satisfied, let $(u,v)\in P(\gamma,\alpha,b,c)$ with $\theta(T(u,v))(t)>\frac{b}{\eta}$. Then, we have

$$\begin{aligned}\alpha(T(u,v)(t)) &= \min_{t\in I}\Big[\int_0^1 G_1(t,s)f_1(s,u(s),v(s))ds+\int_0^1 G_2(t,s)f_2(s,u(s),v(s))ds\Big]\\ &\ge \eta\Big[\int_0^1 G_1(1,s)f_1(s,u(s),v(s))ds+\int_0^1 G_2(1,s)f_2(s,u(s),v(s))ds\Big]\\ &\ge \eta\max_{t\in[0,1]}\Big[\int_0^1 G_1(t,s)f_1(s,u(s),v(s))ds+\int_0^1 G_2(t,s)f_2(s,u(s),v(s))ds\Big]\\ &> \eta\max_{t\in I}\Big[\int_0^1 G_1(t,s)f_1(s,u(s),v(s))ds+\int_0^1 G_2(t,s)f_2(s,u(s),v(s))ds\Big]\\ &= \eta\theta(T(u,v))(t)\\ &> b.\end{aligned}$$

Finally, it is shown that $(B4)$ holds. Let $(u,v)\in Q(\gamma,\beta,a,c)$ with $\psi(T(u,v))<\eta a$. Then, we have

$$\begin{aligned}\beta(T(u,v)(t)) &= \max_{t\in I_1}\Big[\int_0^1 G_1(t,s)f_1(s,u(s),v(s))ds+\int_0^1 G_2(t,s)f_2(s,u(s),v(s))ds\Big]\\ &\le \max_{t\in[0,1]}\Big[\int_0^1 G_1(t,s)f_1(s,u(s),v(s))ds+\int_0^1 G_2(t,s)f_2(s,u(s),v(s))ds\Big]\\ &\le \frac{1}{\eta}\Big[\eta\int_0^1 G_1(1,s)f_1(s,u(s),v(s))ds+\eta\int_0^1 G_2(1,s)f_2(s,u(s),v(s))ds\Big]\\ &\le \frac{1}{\eta}\min_{t\in I}\Big[\int_0^1 G_1(t,s)f_1(s,u(s),v(s))ds+\int_0^1 G_2(t,s)f_2(s,u(s),v(s))ds\Big]\\ &\le \frac{1}{\eta}\min_{t\in I_1}\Big[\int_0^1 G_1(t,s)f_1(s,u(s),v(s))ds+\int_0^1 G_2(t,s)f_2(s,u(s),v(s))ds\Big]\\ &= \frac{1}{\eta}\psi(T(u,v))(t)<a.\end{aligned}$$

It has been proved that all the conditions of Theorem 2.2 are satisfied. Therefore, the fractional order BVP (1)-(2) has at least three positive solutions, (x_1, x_2), (y_1, y_2) and (z_1, z_2) such that $\beta(x_1, x_2) < a$, $b < \alpha(y_1, y_2)$ and $a < \beta(z_1, z_2)$ with $\alpha(z_1, z_2) < b$. This completes the proof of the theorem. □

Now, we establish the existence of at least $2k-1$ positive solutions for the FBVP (1)-(2), by using induction on k.

Theorem 4.4. *Let k be an arbitrary positive integer. Assume that there exist numbers $a_r (r = 1, 2, 3, \cdots, k)$ and $b_s (s = 1, 2, 3, \cdots, k-1)$ with $0 < a_1 < b_1 < \frac{b_1}{\eta} < a_2 < b_2 < \frac{b_2}{\eta} < \cdots < a_{k-1} < b_{k-1} < \frac{b_{k-1}}{\eta} < a_k$ such that f_j, for $j = 1, 2$ satisfies the following conditions:*

(D4) $f_j(t, u, v) < \frac{a_r}{2L}$, $t \in [0, 1]$ *and* $|u| + |v| \in [\eta a_r, a_r]$, $r = 1, 2, 3, \cdots, k$,
(D5) $f_j(t, u, v) > \frac{b_s}{2M}$, $t \in I$ *and* $|u| + |v| \in [b_s, \frac{b_s}{\eta}]$, $s = 1, 2, 3, \cdots, k-1$.

Then the fractional order BVP (1)-(2) has at least $2k-1$ positive solutions in $\overline{P}_{a_k}$.

Proof. We use induction on k. First, for $k = 1$, we know from the condition (D4) that $T : \overline{P}_{a_1} \to P_{a_1}$, then it follows from the Schauder fixed point theorem that the fractional order BVP (1)-(2) has at least one positive solution in $\overline{P}_{a_1}$. Next, we assume that this conclusion holds for $k = l$. In order to prove that this conclusion holds for $k = l+1$, we suppose that there exist numbers $a_r (r = 1, 2, 3, \cdots, l+1)$ and $b_s (s = 1, 2, 3, \cdots, l)$ with $0 < a_1 < b_1 < \frac{b_1}{\eta} < a_2 < b_2 < \frac{b_2}{\eta} < \cdots < a_l < b_l < \frac{b_l}{\eta} < a_{l+1}$ such that f_j, for $j = 1, 2$ satisfies the following conditions:

$$f_j(t, u, v) < \frac{a_r}{2L}, t \in [0, 1] \text{ and } |u| + |v| \in [\eta a_r, a_r],\ r = 1, 2, 3, \cdots, l+1, \tag{7}$$

$$f_j(t, u, v) > \frac{b_s}{2M}, t \in I \text{ and } |u| + |v| \in [b_s, \frac{b_s}{\eta}],\ s = 1, 2, 3, \cdots, l. \tag{8}$$

By assumption, the fractional order BVP (1)-(2) has at least $2l-1$ positive solutions $(x_i, x_i^*), i = 1, 2, 3, \cdots, 2l-1$ in $\overline{P}_{a_1}$. At the same time, it follows from Theorem 4.3, (7) and (8) that the fractional order BVP (1)-(2) has at least three positive solutions $(x_1, x_2), (y_1, y_2)$ and (z_1, z_2) in $\overline{P}_{a_{l+1}}$ such that $\beta(x_1, x_2) < a_l, b_l < \alpha(y_1, y_2)$ and $a_l < \beta(z_1, z_2)$ with $\alpha(z_1, z_2) < b_l$. Obviously, (y_1, y_2) and (z_1, z_2) are distinct from $(x_i, x_i^*), i = 1, 2, 3, \cdots, 2l-1$ in $\overline{P}_{a_1}$. Therefore, the fractional order BVP (1)-(2) has at least $2l+1$ positive solutions in $\overline{P}_{a_{l+1}}$ which shows that this conclusion also holds for $k = l+1$. This completes the proof of theorem. □

5. Examples

In this section, we demonstrate our main result with some examples.
Let $\nu_1 = \frac{7}{2}, \nu_2 = \frac{9}{2}, i = 1, \alpha_j = \frac{2}{j^2}, \xi_j = \frac{1}{j}$

We consider the system of fractional order differential equations

$$\begin{aligned} D_{0^+}^{\frac{7}{2}} u(t) + f_1(t, u(t), v(t)) = 0,\ t \in (0, 1) \\ D_{0^+}^{\frac{9}{2}} v(t) + f_2(t, u(t), v(t)) = 0,\ t \in (0, 1) \end{aligned} \tag{9}$$

with the infinite-point boundary conditions

$$\begin{aligned} &u(0) = u'(0) = u''(0) = 0, u'(1) = \sum_{j=1}^{\infty} \frac{2}{j^2} u\left(\frac{1}{j}\right) \\ &v(0) = v'(0) = v''(0) = 0, v'(1) = \sum_{j=1}^{\infty} \frac{2}{j^2} v\left(\frac{1}{j}\right). \end{aligned} \tag{10}$$

The Green's functions $G_1(t,s)$ and $G_2(t,s)$ of corresponding homogeneous BVPs are given by

$$G_1(t,s) = \frac{1}{1.3207} \begin{cases} t^{2.5} p(s)(1-s)^{1.5} - 0.3974(t-s)^{2.5}, & 0 \le s \le t \le 1, \\ t^{2.5} p(s)(1-s)^{1.5}, & 0 \le t \le s \le 1, \end{cases}$$

and

$$G_2(t,s) = \frac{1}{16.8788} \begin{cases} t^{3.5} p(s)(1-s)^{2.5} - 1.4511(t-s)^{3.5}, & 0 \le s \le t \le 1, \\ t^{2.5} p(s)(1-s)^{1.5}, & 0 \le t \le s \le 1. \end{cases}$$

We deduce that $\eta = 0.0078125,\ L \approx 0.040139$ and $M \approx 0.003244$.

5.1. **Example.** Let $f_1(t,u,v) = \frac{t}{12}(u+v) + t^2 + 1,\ f_2(t,u,v) = \frac{t^2}{4}(u+v) + e^{-(u^2+v^2)}$ and $r = 1, R = 500$, then $r < \eta R$ and f_j for $j = 1,2$ satisfies

$$(i)\ f_j(t,u,v) \ge 154.1307 = \frac{r}{2M}, \text{ for } t \in \left[\frac{1}{4}, \frac{3}{4}\right] \text{ and } u, v \in [1, 500], \text{and}$$

$$(ii)\ f_j(t,u,v) \le 6228.35 = \frac{R}{2L} \text{ for } t \in [0,1] \text{ and } u, v \in [0, 500].$$

Hence, all of the conditions of Theorem 4.1 guarantees that fractional order BVP (9)-(10) has at least one positive solution (u^*, v^*) such that $1 \le \min_{t \in [\frac{1}{4}, \frac{3}{4}]} |(u^*, v^*)|$ and $\max_{t \in [0,1]} |(u^*, v^*)| \le 500$.

5.2. **Example.** Let $f_1(t,u,v) = (1-t)[e^{-(u+v)}(u+v)],\ f_2(t,u,v) = \frac{2}{1+t^2}(u+v)$ and $r = 1,\ R = 100$, then $r < R$ and f_j for $j = 1,2$ satisfies

$$(i)\ f_j(t,u,v) \le 12.4567 = \frac{r}{2L}, \text{ for } t \in [0,1] \text{ and } u, v \in [0,1], \text{ and}$$

$$(ii)\ f_j(t,u,v) \ge 15413.07 = \frac{R}{2M}, \text{ for } t \in \left[\frac{1}{4}, \frac{3}{4}\right] \text{ and } u, v \in [100, 12800].$$

Hence, all of the conditions of Theorem 4.2 guarantees that fractional order BVP (9)-(10) has at least one positive solution (u^*, v^*) such that $1 \le \max_{t \in [0,1]} |(u^*, v^*)|$ and $\min_{t \in [\frac{1}{4}, \frac{3}{4}]} |(u^*, v^*)| \le 100$.

5.3. **Example.** Let

$$f_1(t,u,v) = \begin{cases} \dfrac{e^{u+v}}{165} + \dfrac{19(u+v)^4}{5}, & 0 \le u+v \le 5, \\ \dfrac{e^{u+v}}{165} + \dfrac{(u+v)}{2} + \dfrac{4745}{2}, & u+v \ge 5, \end{cases}$$

$$f_2(t,u,v) = \begin{cases} \dfrac{e^{u+v}}{143} + \dfrac{18(u+v)^4}{5} + \dfrac{t}{50}, & 0 \le u+v \le 5, \\ \dfrac{e^{u+v}}{143} + \dfrac{t}{50} + \dfrac{(u+v)}{2} + \dfrac{4495}{2}, & u+v \ge 5. \end{cases}$$

If we choose $a = 1.5, b = 5$ and $c = 500$, then $0 < a < b < \frac{b}{\eta} < c$ and f_j for $j = 1, 2$ satisfies

$$(i)\ f_j(t, u(t), v(t)) < 18.685 = \frac{a}{2L},\ t \in [0, 1], u + v \in [0.0117, 1.5]$$

$$(ii)\ f_j(t, u(t), v(t)) > 770.6535 = \frac{b}{2M},\ t \in I, u + v \in [5, 640]$$

$$(iii)\ f_j(t, u(t), v(t)) < 6228.3565 = \frac{c}{2L},\ t \in [0, 1], u + v \in [0, 500].$$

Then, all the conditions of Theorem 4.3 are satisfied. Therefore, by Theorem 4.3, the fractional order BVP (9)-(10) has at least three positive solutions.

References

[1] Anderson, D. R., Avery, R. I., (2002). Fixed point theorem of cone expansion and compression of functional type, J. Differ. Equn. Appl., 8 pp. 1073-1083.

[2] Avery, R.I., Henderson, J., O'Regan, D., (2008). Functional compression expansion fixed point theorem, Electron. J. Differ. Eqns., Ariticle ID 22.

[3] Avery, R. I., (1999). A generalization of the Leggett-Williams fixed point theorem, Math. Sci. Res. Hot-Line., 3, pp. 9-14.

[4] Bai, C., Sun, W., (2012). Existence and multiplicity of positive solutions for singular fractional boundary value problems, Comput. Math. Appl., 63 pp. 1369-1381.

[5] Bai, C., Sun, W., Zhang, W., (2013). Positive solutions for boundary value problems of a singular fractional differential equations, Abstr. Appl. Anal., article ID 129640.

[6] Bai, Z., Lü, (2005). Positive solutions for boundary value problem of nonlinear fractional differential equation, J. Math. Anal. Appl., 311 pp. 495-505.

[7] Chai, G., (2013). Existence results of positive solutions for boundary value problems of fractional differential equations, Boundary Value Problems., Article ID 109.

[8] Goodrich, C. S., (2012). On a fractional boundary value problem with fractional boundary conditions, Appl. Math. Lett., pp. 1101-1105.

[9] Henderson, J., Luca, R., (2013). Existence and multiplicity for positive solutions of a system of higher-order multi-point boundary value problems, Nonlinear Differ. Equn. Appl., 3 pp. 1035-1054.

[10] Lakshmikanthan, V., (2008). Theory of fractional differential equations, Nonlinear Anal., TMA 69, pp. 3337-3343.

[11] Lakshmikanthan, V., Leela, S., Vasundhara Devi, J., (2009). Theory of Fractional Dynamic Systems., Cambridge Scientific Publishers, Cambridge.

[12] Liang, S., Zhang, J., (2009). Positive solutions for boundary value problems of nonlinear fractional differential equations, Nonlinear Anal., 71, pp. 5545-5550.

[13] Lu, X., Zhang, X., Wang, L., (2014). Existence of positive solutions for a class of fractional differential equations with m-point boundary value conditions, J. Sys. Sci. & Math. Scis., 34(2) pp. 1-13.

[14] Miller, K. S., Ross, B., (1993). An introduction to the Fractional Calculus and Fractional Differential Equations., Wiley, New York.

[15] Prasad, K. R., Kameswararao, A., Nageswararao, S., (2012). Existence of positive solutions for the system of higher order two-point boundary value problems, Proc. Indian Acad. Sci., 122(1) pp. 139-152.

[16] Podlubny, I., (1999). Fractional Differential Equations., Academic Press, San Diego.

[17] Kilbas, A. A., Srivastava, H. M., Trujillo, j. J., (2006). Theory and Applications of Fractional Differential Equations., North-Holland Mathematics Studies, 204, Elsevier Science B. V., Amsterdam.

[18] Sun, J., Zhang, G., (2007). A generalization of the cone expansion and compression fixed point theorem and applications, Nonlinear Anal., 67 pp. 579-586.

[19] Sun, Y., Zhang, X., (2014). Existence and nonexistence of positive solutions for fractional order two point boundary value problems, Advances in Diff. Equn., Article ID 53.

[20] Tian, C., Liu, Y., (2012). Multiple positive solutions for a class of fractional singular boundary value problem, Mem. Differ. Equn. Math. Phys., 56 pp. 115-131.

[21] Xu, X., Jiang, D., Yuan, C., (2009). Multiple positive solutions for the boundary value problems of a nonlinear fractional differential equation, Nonlinear Anal., 71 pp. 4676-4688.

[22] Zhang, X., Zhong, Q., (2017). Uniqueness of solution for higher-order fractional differential equations with conjugate type integral conditions, Fract. Calc. Appl. Anal., 20(6) pp. 1471-1484.

[23] Zhang, X., (2015). Positive solutions for a class of singular fractional differential equation with infinite-point boundary value conditions, Appl. Math. Lett., 39 pp. 22-27.
[24] Zhang, X., Zhong, Q., (2018). Triple positive solutions for nonlocal fractional differential equations with singularities both on time and space variables, Appl. Math. Lett., 80 pp. 12-19 .
[25] Zhang, X., Wang, L., Sun, Q., (2014). Existence of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions and a parameter, Appl. Math. Comput., 226 pp. 708-718.

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