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ESTIMATION OF THE LOCATION AND SCALE PARAMETERS OF MOYAL DISTRIBUTION

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Abstract. In this study, we estimate the parameters of the Moyal distribution by using well-known and widely-used maximum likelihood (ML) and method of moments (MoM) methodologies. The ML estimators of the location and scale parameters of the Moyal distribution cannot be obtained in closed forms therefore iterative methods should be utilized. To make the study complete, modifed ML (MML) estimators for the location and the scale parameters of the Moyal distribution are also derived. The MML estimators are in closed forms and asymptotically equivalent to the ML estimators. Efficiencies of the MML estimators are compared with their ML and MoM counterparts using Monte-Carlo (MC) simulation study. Results of the simulation study show that the ML estimators are more efficient than the MML and MoM estimators for small sample sizes. However when the sample size increases performances of the ML and MML estimators are almost same in terms of the Defficiency (Def) criterion as expected. At the end of the study, a real data set is used to show the implementation of the methodology developed in this paper.

Keywords: Moyal Distribution, Maximum Likelihood, Modified Maximum Likelihod, Method of Moments, Efficiency.

AMS Subject Classification: 62F10, 62F35, 62N02, 62P30

1. Introduction

The Moyal distribution which is an approximation to the Landou distribution was proposed to model the energy loss by ionization for a fast charged particle and the number of ion pairs produced in this process; see Moyal [11]. It is also used in considering quantum resonance and atomic structure of the absorber. Although the Moyal distribution has a wide usage in Physics, it has not drawn enough attention in Statistics literature. To the best of our knowledge, there are limited number of studies about the Moyal distribution. For example, Cordeiro et al. [9] proposed the beta-Moyal as an extension of the Moyal

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distribution. Genc et al. [10] considered a scale-mixture extension of the Moyal distribution which is more flexible than the Moyal and beta-Moyal distributions in terms of the range of the kurtosis and skewness values. Bahdi and Ravi [7] introduced a generalized form of the Moyal distribution and used it to model the actuarial data set.

In this study, well-known and widely-used maximum likelihood (ML) methodology is used to estimate the parameters of the Moyal distribution. In estimating the parameters of the Moyal distribution, modified ML (MML) methodology proposed by Tiku [14, 15] is also used to avoid computational complexities encountered in iterative procedures such as (i) convergence to wrong root, (ii) convergence to multiple root and (iii) nonconvergence of iterations; see for example Barnett [8], Puthenpura and Sinha [12], and Vaughan [16].

In the Monte-Carlo simulation study, the ML and MML estimators are compared with respect to bias, mean squares error (MSE) and Defficiency (Def) criteria. To make the study complete, method of moments (MoM) estimators of the location and scale parametrs of the Moyal distribution are also included into the study.

The rest of the paper is organized as follows. The Moyal distribution and some statistical properties of it are provided in Section 2. Section 3 includes a brief description of the parameter estimation methodologies used in the study. The results of the Monte-Carlo (MC) simulation study are presented in Section 4. Section 5 is reserved to application in which exceedances of Wheaton River flood data set is modeled. The paper is finalized with some concluding remarks.

2. The Moyal Distribution

The probability density function (pdf) of the Moyal distribution is given by

$$f(w)=\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}w-\frac{1}{2}\exp\left(-w\right)\right];\quad w\in\mathbb{R}. \tag{1}$$

It can be extended to location-scale family by using transformation $X=\mu+W\sigma$. Resulting distribution is called as two-parameter Moyal distribution and has the following pdf:

$$f_X(x;\mu,\sigma)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)-\frac{1}{2}\exp\left(-\left(\frac{x-\mu}{\sigma}\right)\right)\right];x\in\mathbb{R},\mu\in\mathbb{R},\sigma\in\mathbb{R}^{+}. \tag{2}$$

Its cumulative distribution function (cdf) is also formulated as follows:

$$F_X(x;\mu,\sigma)=\Gamma\left[\frac{1}{2}\exp\left(-\left(\frac{x-\mu}{\sigma}\right),\quad\frac{1}{2}\right)\right], \tag{3}$$

where μ and σ are the location and the scale parameters, respectively. Here,

$$\Gamma\left[\frac{1}{2}\exp\left(-\left(\frac{x-\mu}{\sigma}\right),\frac{1}{2}\right)\right]=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{\frac{1}{2}\exp\left(-\left(\frac{x-\mu}{\sigma}\right)\right)}^{\infty}u^{0.5-1}\exp(-u)du$$

stands for the upper incomplete gamma function.

Lemma 2.1. *Quantile function of the two-parameter Moyal distribution is*

$$X_p=\mu-\sigma\ln\left[2\Gamma^{-}\left(p,0.5\right)\right],\quad 0<p<1 \tag{4}$$

where $\Gamma^{-}(\cdot,\cdot)$ *is inverse of the upper incomplete gamma function.*

Proof. It is trivial since X_p is the solution of the equation $\int_{-\infty}^{X_p}f_X(x;\mu,\sigma)dx=p$. □

Remark 2.1. *The median of the two-parameter Moyal distribution is easily obtained by taking* $p=0.5$.

The moment genarating function (mgf) of the two-parameter Moyal distribution is given as follows:

$$\mathbb{M}_X(t) = \mathbb{E}\left[e^{tX}\right] = \frac{2^{-t\sigma}e^{\mu t}}{\sqrt{\pi}}\Gamma\left(\frac{1}{2} - t\sigma\right), \quad t\sigma < 0.5. \tag{5}$$

The pdf and cdf plots of the two-parameter Moyal distribution for certain values of the parameters are given in Figure 1.

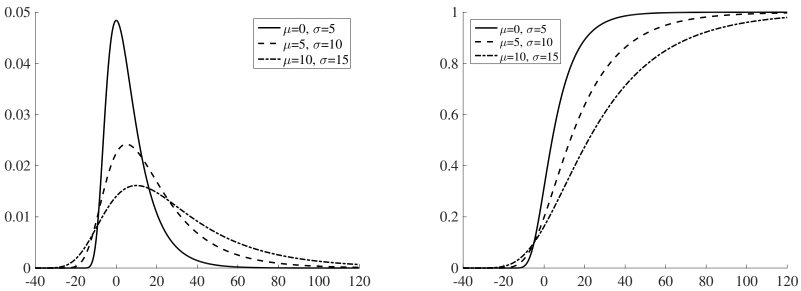


FIGURE 1. The pdf and cdf plots of the two-parameter Moyal distribution for certain values of the parameters.

Remark 2.2. *The skewness* ($\sqrt{\beta_1}$) *and kurtosis* (β_2) *values of the two-parameter Moyal distribution are calculated as* 1.54 *and* 7.00*, respectively.*

Lemma 2.2. *The cumulant generating function,* $\kappa_X(t)$*, of the two-parameter Moyal distribution is*

$$\kappa_X(t) = \mu t - t\sigma \ln 2 - 0.5 \ln \pi + \ln\left[\Gamma\left(\frac{1}{2} - t\sigma\right)\right]. \tag{6}$$

Proof. It can easily be obtained by using the transformation $\ln\left[\mathbb{E}\left(e^{tX}\right)\right]$. □

In the rest of the paper, the two-parameter Moyal distribution is called as the Moyal distribution for the sake of simplicity.

3. The Parameter Estimation

In this section, the ML methodology utilizing iterative techniques is considered to obtain the estimates of the location and scale parameters of the Moyal distribution. The MML and MoM methodologies giving the explicit estimators of the unknown model parameters are also described.

3.1. **The ML methodology.** Let $x_1, x_2, \ldots, x_n$ be a random sample from the Moyal distribution, then the log-likelihood ($\ln L$) function is expressed as follows:

$$\ln L = C - n \ln \sigma - \frac{1}{2}\sum_{i=1}^{n} z_i - \frac{1}{2}\sum_{i=1}^{n} \exp\left(-z_i\right) \tag{7}$$

where $C = -\frac{n}{2}\ln(2\pi)$ and $z_i = \left(\frac{x_i - \mu}{\sigma}\right)$.

The ML estimates of the parameters μ and σ are the solutions of the following likelihood equations:

$$\frac{\partial \ln L}{\partial \mu} = \frac{1}{2\sigma}\sum_{i=1}^{n} g\left(z_i\right) = 0 \tag{8}$$

and

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{2\sigma}\sum_{i=1}^{n} z_i g(z_i) = 0 \tag{9}$$

where $g(z) = 1 - \exp(-z)$.

Remark 3.1. *The solutions of the likelihood equations are obtained iteratively due to the nonlineer functions of the parameters. Here, Newton-Raphson (NR) method is preferred among various different iterative techniques to solve the equations in* (8) *and* (9)*, simulatenously. In the NR method, we first need the Hessian matrix* $\mathbf{H}$ *defined below*

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 \ln L}{\partial \mu^2} & \frac{\partial^2 \ln L}{\partial \mu \partial \sigma} \\ \frac{\partial^2 \ln L}{\partial \sigma \partial \mu} & \frac{\partial^2 \ln L}{\partial \sigma^2} \end{bmatrix}. \tag{10}$$

The elements of the $\mathbf{H}$ *matrix and the Fisher information matrix denoted by* $\mathbf{I}$ *are given in the Appendix; see Arslan and Senoglu* [6] *in the context of the Jones and Faddy's skew t distribution. The Rao-Cramer lower bounds (RCLB) for the parameters* μ *and* σ *are also given in the Appendix. Obviously,* $RCLB(\mu) = \left(\mathbf{I}^{-1}\right)_{11}$ *and* $RCLB(\sigma) = \left(\mathbf{I}^{-1}\right)_{22}$.

3.2. **The MML methodology.** This methodology originated by Tiku [14, 15] is utilized to avoid the computational complexities encountered in the ML methodology. It results in closed form estimators called as the MML estimators. The MML methodology is asymptotically equivalent to the ML methodology and therefore has all the attractive asymptotic properties of it.

The steps of the MML methodology are explained as follows:

Step 1: Standardized observations z_i are ordered in ascending way, i.e.

$$z_{(1)} \leq z_{(2)} \leq \ldots \leq z_{(n)}$$

.

Step 2: Nonlinear function $g(z_{(i)}) = 1 - \exp(-z_{(i)})$ is linearized using the first two terms of Taylor series expansion around the expected values of the standardized order statistics $t_{(i)} = E(z_{(i)})$ $(i = 1, 2, \ldots, n)$. This results in

$$g\left(z_{(i)}\right) \cong \alpha_i - \beta_i z_{(i)} \tag{11}$$

where

$$\alpha_i = 1 - exp(-t_{(i)}) - t_{(i)}\exp\left(-t_{(i)}\right), \quad \beta_i = \exp\left(-t_{(i)}\right), \quad i = 1, 2, \ldots, n.$$

Remark 3.2. *It should be noted that* $t_{(i)} = E\left(z_{(i)}\right)$ *values cannot be obtained exactly. We therefore use their approximate values using the following equality:*

$$t_{(i)} = F^{-1}\left(\frac{i}{n+1}\right), \quad i = 1, 2, \ldots, n$$

where $F^{-1}(\cdot)$ *is inverse of the cdf of standard Moyal distribution.*

After incorporating equation (11) into the likelihood equations in (8) and (9), we obtain the following modified likelihood equations:

$$\frac{\partial \ln L^*}{\partial \mu} = \frac{1}{2\sigma}\sum_{i=1}^{n}\left(\alpha_i + \beta_i z_{(i)}\right) = 0 \tag{12}$$

and

$$\frac{\partial \ln L^*}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{2\sigma}\sum_{i=1}^{n} z_{(i)}\left(\alpha_i + \beta_i z_{(i)}\right) = 0. \tag{13}$$

Solutions of these equations are the following MML estimators:

$$\hat{\mu}_{MML} = \bar{x}_w - \frac{\Delta}{m}\hat{\sigma}_{MML} \quad \text{and} \quad \hat{\sigma}_{MML} = \frac{B + \sqrt{B^2 + 4nC}}{2\sqrt{n(n-1)}} \tag{14}$$

where

$$\bar{x}_w = \frac{\sum_{i=1}^{n} \beta_i x_{(i)}}{m}, \quad m = \sum_{i=1}^{n} \beta_i, \quad \Delta = \sum_{i=1}^{n} \alpha_i,$$

$$B = \frac{1}{2}\sum_{i=1}^{n} \alpha_i (x_{(i)} - \bar{x}_w) \quad \text{and} \quad C = \frac{1}{2}\sum_{i=1}^{n} \beta_i (x_{(i)} - \bar{x}_w)^2.$$

Remark 3.3. *The denominator of the $\hat{\sigma}_{MML}$ is replaced by $2\sqrt{n\,(n-1)}$ for bias correction.*

The MML methodology gives small weights to the outlying observations to deplete their dominant effects. This property makes the MML estimators insensitive to the outlying observations; see e.g. Acitas et al. [1] and references therein for further information.

3.3. **The MoM methodology.** The MoM estimators of the location and scale parameters of the Moyal distribution are obtained by equating the theoretical moments to the corresponding sample moments and solving them with respect to the parameters of interest. By using the definition of the MoM methodology, estimators of the parameters μ and σ are obtained as follows:

$$\hat{\mu}_{MoM} = \bar{x} + [\ln 2 + \Psi(0, 0.5)]\,\hat{\sigma}_{MoM} \tag{15}$$

and

$$\hat{\sigma}_{MoM} = s\big/\sqrt{\Psi(1, 0.5)} \tag{16}$$

respectively. Here,

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad s = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}, \quad \text{and} \quad \Psi(r-1, k) = \frac{d^r \ln \Gamma(k)}{dk^r}$$

are sample mean, sample standart deviation and well-known polygamma function, respectively.

4. The Simulation Study

In this section, performances of the ML, MML and MoM estimators are compared via the MC simulation study. In the MC simulation, four different sample sizes $n = 10, 20$ (small), $n = 50$ (moderate) and $n = 100$ (large) are considered. Without loss of generality, the location parameter μ and scale parameter σ are taken to be 0 and 1, respectively. All the simulations are conducted for $\lfloor 100,000/n \rfloor$ MC runs where $\lfloor \cdot \rfloor$ denotes the integer value function. MATLAB2017a software is utilized for all computations. In the ML estimation procedure, initial values for $\hat{\mu}$ and $\hat{\sigma}$ are taken as $\mu^0 = \hat{\mu}_{MML}$ and $\sigma^0 = \hat{\sigma}_{MML}$, respectively.

The performances of the ML, MML and MoM estimators are compared by using bias and mean squares error (MSE) criteria. Def criterion defined as the natural measure of the joint efficiency of the estimators of unknown parameters is also used in the comparisons. It is formulated as follows:

$$Def(\hat{\mu}, \hat{\sigma}) = MSE(\hat{\mu}) + MSE(\hat{\sigma}), \tag{17}$$

see for example Akgul et al. [2, 3]. The results of the simulation study are tabulated in Table 1.

Table 1. The simulated bias, variance, MSE and Def values of the ML, MML and MoM estimators.

		$\mu = 0$			$\sigma = 1$			
	Estimator	Bias	Variance	MSE	Bias	Variance	MSE	Def
	ML	-0.078	0.252	0.258	0.079	0.067	0.073	0.332
$n = 10$	MML	-0.194	0.265	0.302	0.022	0.076	0.077	0.379
	MoM	-0.074	0.278	0.283	0.057	0.116	0.119	0.403
	ML	-0.035	0.118	0.119	0.041	0.034	0.035	0.154
$n = 20$	MML	-0.095	0.121	0.130	0.015	0.036	0.036	0.166
	MoM	-0.035	0.140	0.142	0.030	0.063	0.063	0.205
	ML	-0.020	0.043	0.044	0.014	0.013	0.013	0.057
$n = 50$	MML	-0.045	0.044	0.046	0.005	0.013	0.013	0.059
	MoM	-0.018	0.053	0.053	0.007	0.026	0.026	0.079
	ML	-0.010	0.023	0.023	0.008	0.007	0.007	0.030
$n = 100$	MML	-0.023	0.024	0.024	0.004	0.007	0.007	0.031
	MoM	-0.009	0.030	0.030	0.005	0.014	0.014	0.044

Following conclusions are drawn from Table 1 using the bias criterion. The ML, MML and MoM estimators of μ have negligible biases. It should be noted that the MML estimator of μ has the largest bias for all sample sizes. On the other hand, the MML estimator of σ has the lowest bias.

We conclude that the ML estimators of the location and scale parameters are more efficient than their rivals in terms of the MSE criteria for all sample sizes. The performances of the MML estimators are also promosing since the MSEs of them are very similar to those obtained for the ML estimators for small sample sizes.

It is clear that the ML estimators are more preferable than the MML and MoM estimators according to the Def criterion. The MML estimators gain efficiency when the sample size increases. Therefore, we suggest to use the ML and MML estimators for estimating the unknown parameters of the Moyal distribution.

As it is indicated previously, obtaining the ML estimates of the parameters of the Moyal distribution requires iterative methods and this may causes some problems. On the other hand, the MML estimators are easily obtained from the sample observations without any iterative computations. As a result, the MML estimators can be preferred if our focus is to avoid the computational complexities besides having efficient estimators.

5. Application

In this section, the Wheaton River flood data which corresponds to the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada are modelled using the Moyal distribution to show the implementation of the methodology developed in this paper; see Table 2. Akinsete et al. [5] model this data set using the beta-Pareto distribution and later on Cordeiro et al. [9] consider the beta-Moyal distribution for modeling mentioned data set.

TABLE 2. The Exceedances of Wheaton River flood data (1958–1984), $n = 72$.

1.7	1.4	0.6	9.0	5.6	1.5	2.2	18.7	2.2	1.7	30.8	2.5
14.4	1.1	0.4	20.6	5.3	8.5	25.5	11.6	14.1	22.1	39.0	0.3
15.0	11.0	7.3	7.0	20.1	0.4	2.8	14.1	13.3	4.2	25.5	3.4
11.9	27.4	1.0	7.1	20.2	16.8	0.7	1.9	1.1	2.5	22.9	1.7
9.9	10.4	21.5	27.6	5.3	9.7	13.0	14.4	0.1	10.7	36.4	27.5
12.0	9.3	1.7	37.6	1.1	0.6	30.0	3.6	2.7	64.0	2.5	27.0

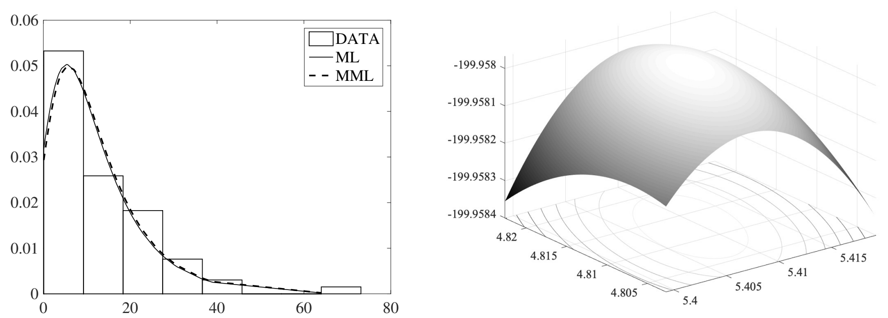
The location and scale parameters of the Moyal distribution are estimated using the ML and MML methods since the MoM method fails to exhibit a good performance; see Table 1.

The estimated values of the location and scale parameters of the Moyal distribution along with the $\ln L$, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) values are given in Table 3. See Akaike [4] and Schwarz [13] for detailed information about the AIC and BIC, respectively.

TABLE 3. The parameter estimates of the Moyal distribution.

		$\hat{\mu}$	$\hat{\sigma}$	$\ln L$	AIC	BIC
Moyal Distribution	ML	5.4092	4.8127	-199.9580	403.9519	408.4693
	MML	5.7979	4.8502	-200.0661	404.1322	408.6855

The histogram of the Wheaton River flood data along with the fitted densities based on the ML and MML estimates and the likelihood surface plot based on the ML estimates are given in Figure 2.



(a) Fitted densities

(b) Likelihood surface plot based on the ML estimates

FIGURE 2. The histogram of the Wheaton River flood data along with the fitted densities and the likelikood surface plot based on the ML estimates.

It is clear from the $\ln L$, AIC and BIC values given in Table 3 that the ML estimates are more preferable than the MML estimates. It should be noted that likelihood function attains its maximum under the ML estimates. While the ML estimates are obtained iteratively, the MML estimators are obtained in closed forms. Furthermore, the $\ln L$, AIC and BIC values based on the ML and MML estimates are very close to each other. Therefore, the MML estimates can also be preferred for this data.

6. Conclusions

In this study, estimation of the location parameter μ and scale parameter σ of the Moyal distribution is considered using the ML, MML and MoM methodologies. MC simulation study is conducted to compare the performances of these estimators. Simulation results show that the performances of the ML and MML estimators of the location and scale parameters of the Moyal distribution are better than the MoM estimators based on the Def criteria; see Table 1. Furthermore, the MML and ML estimators have more or less the same performances as the sample size increases. However, ML estimators are obtained using iterative methods. It is well known that using iterative methods causes some problems as mentioned in the text. On the other hand, the MML estimators are easily obtained from the sample observations without any iterative computations. Therefore, we suggest to use the MML estimators as an alternative to the ML estimators for estimating the location and scale parameters of the Moyal distribution.

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Appendix

The elements of the **H** and **I** matrices and the RCLB values for the location parameter μ and scale parameter σ of the Moyal distribution are given as shown below.

A1-Elements of the H matrix:

$$\frac{\partial^2 \ln L}{\partial \mu^2} = -\frac{1}{2\sigma^2}\sum_{i=1}^{n} \exp\left[-\left(\frac{x_i-\mu}{\sigma}\right)\right],$$

$$\frac{\partial^2 \ln L}{\partial \mu \partial \sigma} = \frac{\partial^2 \ln L}{\partial \sigma \partial \mu} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^2}\sum_{i=1}^{n} \exp\left[-\left(\frac{x_i-\mu}{\sigma}\right)\right] - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)\exp\left[-\left(\frac{x_i-\mu}{\sigma}\right)\right]$$

and

$$\frac{\partial^2 \ln L}{\partial \sigma^2} = \frac{n}{\sigma^2} - \frac{1}{\sigma^2}\sum_{i=1}^{n}\frac{x_i-\mu}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)\exp\left[-\left(\frac{x_i-\mu}{\sigma}\right)\right] - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2\exp\left[-\left(\frac{x_i-\mu}{\sigma}\right)\right].$$

A2-Elements of the I matrix:

$$\mathbf{I} = \begin{bmatrix} -\mathbb{E}\left(\dfrac{\partial^2 \ln L}{\partial \mu^2}\right) & -\mathbb{E}\left(\dfrac{\partial^2 \ln L}{\partial \mu \partial \sigma}\right) \\ -\mathbb{E}\left(\dfrac{\partial^2 \ln L}{\partial \sigma \partial \mu}\right) & -\mathbb{E}\left(\dfrac{\partial^2 \ln L}{\partial \sigma^2}\right) \end{bmatrix} = \frac{n}{2\sigma^2}\begin{bmatrix} 1 & 1.27036 \\ 1.27036 & 4.54862 \end{bmatrix}$$

A3-RCLB values for the location parameter μ and scale parameter σ:

$$RCLB(\mu) = \left(\mathbf{I}^{-1}\right)_{11} = 1.5459\frac{2\sigma^2}{n}$$

and

$$RCLB(\sigma) = \left(\mathbf{I}^{-1}\right)_{22} = 0.3407\frac{2\sigma^2}{n}.$$

Covariance between the estimators of the parameters μ and σ is equal to the off-diagonal element of the matrix **I**, i.e. $\left(\mathbf{I}^{-1}\right)_{12} = \left(\mathbf{I}^{-1}\right)_{21} = -0.4329\frac{2\sigma^2}{n}$.

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