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VARIABLE MESH DISCRETIZATION OF SYSTEM OF NONLINEAR SINGULAR BOUNDARY VALUE PROBLEMS

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Abstract. In this paper two generalized numerical schemes using variable mesh has been developed to solve the system of nonlinear two point boundary value problems. Analytical convergence using a model fourth order problem has been provided. The order of convergence of the proposed methods are two and three. The methods are applicable to singular problems as well. Comparative study of numericals are given to prove the efficiency of the schemes.

Keywords:Variable mesh, Singular, Nonlinear, System, Root mean square errors

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1. Introduction

Consider the system of M nonlinear singular boundary value problems(BVPs) :

$$y_{xx} = f, \tag{1}$$

$$y(0) = \mathrm{A}, y(1) = \mathrm{B} \tag{2}$$

where

$$\begin{aligned}
&y_{xx} = [{y_{xx}}^{(1)}, {y_{xx}}^{(2)}, ..., {y_{xx}}^{(i)}, ..., {y_{xx}}^{(M)}]^T,\\
&f = [f^{(1)}, f^{(2)}, ..., f^{(i)}, ..., f^{(M)}]^T,\\
&f^{(i)} = f^{(i)}(x, y^{(1)}, ..., y^{(i)}, ..., y^{(M)}, {y_x}^{(1)}, ..., {y_x}^{(i)}, ..., {y_x}^{(M)}),\\
&y(0) = [y^{(1)}(0), y^{(2)}(0), ..., y^{(i)}(0), ..., y^{(M)}(0)]^T,\\
&y(1) = [y^{(1)}(1), y^{(2)}(1), ..., y^{(i)}(1), ..., y^{(M)}(1)]^T,\\
&A = [A_1, A_2, ..., A_i..., A_M]^T,\\
&B = [B_1, B_2, ..., B_i, ..., B_M]^T.
\end{aligned}$$

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We assume that $y(x)$ is sufficiently smooth and required higher order derivatives exist in the solution domain $0 \leq x \leq 1$. The boundary conditions in $(1)-(2)$ are given with sufficient smoothness. Also, we ensure the existence of solutions by assuming that for $-\infty < y^{(i)}, {y_x}^{(i)} < \infty$, we have

$$(i) f^{(i)} \text{is continuous;}$$

$$(ii) \frac{\partial f^{(i)}}{\partial y^{(j)}} \text{ and } \frac{\partial f^{(i)}}{\partial {y_x}^{(j)}} \text{ exist and are continuous;}$$

$$(iii) \sum_{i,j=1}^{M} \frac{\partial f^{(i)}}{\partial y^{(j)}} > 0 \text{ and } \left|\frac{\partial f^{(i)}}{\partial {y_x}^{(j)}}\right| \leq C,$$

for some positive constant C and $i, j = 1(1)M$.

Many real life complex models [5] are simulated using system of boundary value problems. These BVPs are very useful in solving higher order problems which arise in the study of fluid dynamics, astronomy, astrophysics, hydrodynamics, beam and wave theory, engineering and other applied sciences and have been considerably explored by Glatzmaier [3], Terril [8] and Chandrasekhar [13] to name a few. Also, more recently system of nonlinear second-order BVPs was solved by using cubic B-spline scaling functions by Dehgan et.al. [1], sixth-order two-point BVPs were solved by a new method based on uniform Haar wavelet developed by Haq et. al. [4]. Ullah et. al. [10] solved fifth and sixth order BVPs using iteration method, Twizell [9] solved fourth order nonlinear BVPs by using method of extrapolation and Talwar et.al. [7] developed finite difference methods to solve fourth order ordinary differential equations.

The proposed methods solves system of nonlinear singular second order problems. The problem is first discretized at the interior points using only three consecutive nodal points. Then, in case of a higher order problem; it is decomposed into system of second order problems. Also, as nonlinear problems are solved, it results in a nonlinear system of BVPs. Newton's block method is used to solve such systems.

There are seven sections in this paper. In the section "Derivation of the Schemes", we provide complete derivation of the methods using second order BVPs and in section "Generalized Schemes", the generalization of both the schemes is discussed. In section "Application to Fourth Order Singular BVP", the application of the proposed methods is provided using a fourth order singular BVPs; in section "Convergence Analysis", we discuss the analytical convergence of the proposed scheme and in section "Numerical Illustrations", we provide five numericals and comparison to justify the efficiency of the proposed scheme. In the last section "Conclusion", we provide final remarks.

2. Derivation of the Schemes

We derive the scheme using the following second order nonlinear singular BVP :

$$y_{xx} = f(x, y, y_x), \text{such that } y(0) = \alpha, y(1) = \beta \tag{3}$$

Now, to derive the scheme we first discretize the solution interval [0,1] into N subintervals using nodal points $x_j = x_{j-1} + h_j, j = 1(1)N$, where h_j is the mesh size and $\sigma = \frac{h_{j+1}}{h_j} > 0, j = 1, 2, 3, ..., N-1$ be the mesh ratio. When $\sigma = 1$, the mesh becomes to a uniform mesh i.e., $h_{j+1} = h_j = h$. Also, at the nodal points $x_j, j = 1, ..., N$ assume y_j and Y_j be the exact and approximate solution of (3). Next, we follow the method developed by Mohanty [11] and accordingly following approximations are defined at the nodal points $x_j, j = 1(1)N-1$,

$$S = \sigma(\sigma+1), \tag{4}$$

$$\bar{y}_{x_{j+1}} = \frac{(1+2\sigma)y_{j+1} - (1+\sigma)^2 y_j + \sigma^2 y_{j-1}}{h_j S}, \tag{5}$$

$$\bar{y}_{x_{j-1}} = \frac{-y_{j+1} + (1+\sigma)^2 y_j - \sigma(2+\sigma) y_{j-1}}{h_j S}, \tag{6}$$

$$\bar{y}_{x_j} = \frac{y_{j+1} + (\sigma^2-1)y_j - \sigma^2 y_{j-1}}{h_j S}, \tag{7}$$

$$\bar{F}_r = f(x_r, y_r, \bar{y}_{x_r}), r = j, j \pm 1, \tag{8}$$

$$\bar{\bar{y}}_{x_j} = \bar{y}_{x_j} + h_{j+1}\mu[\bar{F}_{j+1} - \bar{F}_{j-1}], \tag{9}$$

$$\bar{\bar{F}}_j = f(x_j, y_j, \bar{\bar{y}}_{x_j}), \tag{10}$$

where μ is a parameter to be determined. Also the schemes considered are evaluated at three consecutive nodal points:

$$\sigma y_{j-1} - (1+\sigma)y_j + y_{j+1} = \frac{h_j^2}{6}(AF_{j+1} + BF_{j-1}) + T_j^{(2)}, j = 1(1)N-1 \tag{11}$$

$$\sigma y_{j-1} - (1+\sigma)y_j + y_{j+1} = \frac{h_j^2}{12}(PF_{j+1} + QF_j + RF_{j-1}) + T_j^{(3)}, j = 1(1)N-1 \tag{12}$$

where $A = \sigma(2+\sigma), B = \sigma(1+2\sigma), P = \sigma^2+\sigma-1, Q = (1+\sigma)(\sigma^2+3\sigma+1)$,
$R = \sigma(1+\sigma-\sigma^2)$ and $\sigma \neq 1$.
Simplifying the approximations $(5)-(10)$, we get

$$\bar{y}_{x_j} = y_{x_j} + \frac{1}{6}\sigma h_j^2 y_{xxx_j} + O(h_j^3), \tag{13}$$

$$\bar{y}_{x_{j+1}} = y_{x_{j+1}} - \frac{1}{6}\sigma(1+\sigma) h_j^2 y_{xxx_j} + O(h_j^3), \tag{14}$$

$$\bar{y}_{x_{j-1}} = y_{x_{j-1}} - \frac{1}{6}(1+\sigma) h_j^2 y_{xxx_j} + O(h_j^3), \tag{15}$$

$$\bar{F}_{j+1} = F_{j+1} - \frac{1}{6}\sigma(1+\sigma) h_j^2 y_{xxx_j} G_j + O(h_j^3), \tag{16}$$

$$\bar{F}_{j-1} = F_{j-1} - \frac{1}{6}(1+\sigma) h_j^2 y_{xxx_j} G_j + O(h_j^3), \tag{17}$$

$$\bar{F}_j = F_j + \frac{1}{6}\sigma h_j^2 y_{xxx_j} G_j + O(h_j^3), \tag{18}$$

$$\bar{\bar{y}}_{xj} = y_{x_j} + \frac{\sigma + 6\mu(1+\sigma)}{6} h_j^2 y_{xxx_j} + O(h_j^3), \tag{19}$$

$$\bar{\bar{F}}_j = F_j + \frac{\sigma + 6\mu(1+\sigma)}{6} h_j^2 y_{xxx_j} G_j + O(h_j^3), \text{where } G_j = \frac{\partial f}{\partial y_{x_j}} \tag{20}$$

Hence, using the approximations $(5)-(7)$ in (11), we get the following equation for $j = 1(1)N-1$ and $\sigma \neq 1$:

$$\sigma y_{j-1} - (1+\sigma)y_j + y_{j+1} = \frac{h_j^2}{6}(A\bar{F}_{j+1} + B\bar{F}_{j-1}) + T_j^{(2)} \tag{21}$$

Now, using $(5)-(6)$ and (9) in (12), we obtain the following equation:

$$\sigma y_{j-1} - (1+\sigma)y_j + y_{j+1} = \frac{h_j^2}{12}(P\bar{F}_{j+1} + Q\bar{\bar{F}}_j + R\bar{F}_{j-1})$$
$$+(\frac{P}{6}\sigma(1+\sigma) + \frac{R}{6}(1+\sigma) - Q\frac{\sigma + 6\mu(1+\sigma)}{6}) h_j^2 y_{xxx_j} G_j + O(h_j^5) \tag{22}$$

To find the value of μ, we equate the coefficient of h_j^4 to zero, thus raising the local truncation error $T_j^{(3)}$ in the equation (22) to $O(h_j^5)$ i.e.,

$$\frac{P}{6}\sigma(1+\sigma)+\frac{R}{6}(1+\sigma)-Q\frac{\sigma+6\mu(1+\sigma)}{6}=0$$

Therefore,

$$\mu=-\frac{\sigma(1+\sigma+\sigma^2)}{6(1+\sigma)(\sigma^2+3\sigma+1)}.$$

Similarily, in case of the first discretized equation (21) it can be easily proved that the local truncation error is of $O(h_j^4)$ and in case of uniform mesh it is $O(h^6)$. Also, note that the coefficients A, B are positive for $\sigma > 0$ and P, Q, R are positive if $\frac{(\sqrt{5}-1)}{2} < \sigma < \frac{(\sqrt{5}+1)}{2}$, a condition required for the convergence of the schemes given by Jain[6] and Mohanty[11].

Finally, since Y_j is the approximate solution of (3), using(21) − (22) we write the discretization schemes as follows:

$$\sigma Y_{j-1}-(1+\sigma)Y_j+Y_{j+1}=\frac{h_j^2}{6}(A\bar{F}_{j+1}+B\bar{F}_{j-1}), j=1(1)N-1 \quad (23)$$

$$\sigma Y_{j-1}-(1+\sigma)Y_j+Y_{j+1}=\frac{h_j^2}{12}(PF_{j+1}+Q\bar{\bar{F}}_j+R\bar{F}_{j-1}), j=1(1)N-1 \quad (24)$$

3. Generalized Schemes

We generalize the proposed schemes. At the grid point $x_j, j = 1(1)N-1$, the following approximations and schemes are used:

$$S=\sigma(\sigma+1), \quad (25)$$

$$\bar{Y}^{(i)}_{x_{j+1}}=\frac{(1+2\sigma)Y^{(i)}_{j+1}-(1+\sigma)^2Y^{(i)}_j+\sigma^2Y^{(i)}_{j-1}}{h_jS}, \quad (26)$$

$$\bar{Y}^{(i)}_{x_{j-1}}=\frac{-Y^{(i)}_{j+1}+(1+\sigma)^2Y^{(i)}_j-\sigma(2+\sigma)Y^{(i)}_{j-1}}{h_jS}, \quad (27)$$

$$\bar{Y}^{(i)}_{x_j}=\frac{Y^{(i)}_{j+1}+(\sigma^2-1)Y^{(i)}_j-\sigma^2Y^{(i)}_{j-1}}{h_jS}, \quad (28)$$

$$\bar{F}^{(i)}_r=f^{(i)}(x_r, {Y_r}^{(1)}, {Y_r}^{(2)}, ..., {Y_r}^{(i)}, ..., {Y_r}^{(M)}, \bar{Y}^{(1)}_{x_r}, \bar{Y}^{(2)}_{x_r}, ..., \bar{Y}^{(i)}_{x_r}, ..., \bar{Y}^{(M)}_{x_r}), \quad (29)$$

$$\bar{\bar{Y}}^{(i)}_{x_j}=\bar{Y}^{(i)}_{x_j}+h_{j+1}\mu_i(\bar{F}^{(i)}_{j+1}-\bar{F}^{(i)}_{j-1}), \quad (30)$$

$$\bar{\bar{F}}^{(i)}_j=f^{(i)}(x_j, Y^{(1)}_j, Y^{(2)}_j, ..., Y^{(i)}_j, ..., Y^{(M)}_j, \bar{\bar{Y}}^{(1)}_{x_j}, \bar{\bar{Y}}^{(2)}_{x_j}, ..., \bar{\bar{Y}}^{(i)}_{x_j}, ..., \bar{\bar{Y}}^{(M)}_{x_j}), \quad (31)$$

$$\text{where } i=1(1)M,\ r=j, j\pm1 \text{ and } \mu_i=\frac{\sigma(1+\sigma+\sigma^2)}{6Q}. \quad (32)$$

Then the discretization schemes are:

$$Y^{(i)}_{j+1}-(1+\sigma)Y^{(i)}_j+\sigma Y^{(i)}_{j-1}=\frac{h_j^2}{6}\big(A\bar{F}^{(i)}_{j+1}+B\bar{F}^{(i)}_{j-1}\big), \quad (33)$$

$$\sigma Y^{(i)}_{j-1}-(1+\sigma)Y^{(i)}_j+Y^{(i)}_{j+1}=\frac{h_j^2}{12}(P\bar{F}^{(i)}_{j+1}+Q\bar{\bar{F}}_j^{(i)}+R\bar{F}^{(i)}_{j-1}), \quad (34)$$

where $A=\sigma(2+\sigma), B=\sigma(1+2\sigma), P=\sigma^2+\sigma-1, Q=(1+\sigma)(\sigma^2+3\sigma+1)$, $R=\sigma(1+\sigma-\sigma^2), \sigma\neq1$.

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4. Application to Fourth Order Singular BVP

All even ordered boundary value problems can be solved by the method developed in the paper. Moreover, the same concept is applicable in odd ordered BVPs. The only difference is being that instead of second order we get system of first order BVPs. However, we consider only a fourth order singular BVPs to exhibit the application of the method and accordingly the method can be generalised for all even ordered BVPs. The fourth order problem is decomposed into system of second order BVPs:

$$y_{xxxx}(x) = c(x)y_x(x) + d(x)y(x) + f(x), c(x) \neq 0, \tag{35}$$

such that:

$$y(0) = \alpha_1, y_{xx}(0) = \alpha_2, y(1) = \beta_1, y_{xx}(1) = \beta_2 \tag{36}$$

where $c(x) = -\frac{1}{x}, x \neq 0$ and $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real constants. We write the problem $(35) - (36)$ as follows:

$$y_{xx}(x) = v(x), \tag{37}$$
$$v_{xx}(x) = c(x)y_x(x) + d(x)y(x) + f(x), \tag{38}$$

such that

$$y(0) = \alpha_1, v(0) = \alpha_2, y(1) = \beta_1, v(1) = \beta_2 \tag{39}$$

Now, using the scheme (34) we discretize the problem $(37) - (38)$ as follows:

$$\sigma Y_{j-1} - (1+\sigma)Y_j + Y_{j+1} = \frac{h_j^2}{6}(PV_{j+1} + QV_j + RV_{j-1}), \tag{40}$$

$$\begin{aligned}\sigma V_{j-1} - (1+\sigma)V_j + V_{j+1} = \frac{h_j^2}{6}\big(P(c_{j+1}\ \bar{Y}_{x_{j+1}} + d_{j+1}Y_{j+1} + f_{j+1})\\ +Q(c_j\ \bar{\bar{Y}}_{x_j} + d_jY_j + f_j) + R(c_{j-1}\bar{Y}_{x_{j-1}} + d_{j-1}Y_{j-1} + f_{j-1})\big),\end{aligned} \tag{41}$$

where $Y(x_j), V(x_j)$ are approximate solutions of $(35) - (36)$. Also , we notice that the derived schemes fails when $j = 1$ due to presence of singularity at $x = 0$. Therefore, we define the following approximations for $c_{j\pm1}$ at $x = 0$ respectively:

$$c^{**}_{j-1} = c_j - h_jc_{xj} + \frac{(h_j)^2}{2}c_{xxj} + O(h_j^3), \tag{42}$$

$$c^{**}_{j+1} = c_j + \sigma h_jc_{xj} + \frac{(\sigma h_j)^2}{2}c_{xxj} + O(h_j^3) \tag{43}$$

Similar approximations can be defined for $d_{j\pm1}, f_{j\pm1}$. Applying the approximations (42)-(43) to the coupled second order scheme $(40) - (41)$, we obtain the following equations:

$$\sigma Y_{j-1} - (1+\sigma)Y_j + Y_{j+1} = \frac{h_j^2}{6}(PV_{j+1} + QV_j + RV_{j-1}) \tag{44}$$

$$\begin{aligned}\sigma V_{j-1} - (1+\sigma)V_j + V_{j+1} = \frac{h_j^2}{6}\big(P(c^{**}_{j+1}\ \bar{Y}_{x_{j+1}} + d^{**}_{j+1}Y_{j+1} + f^{**}_{j+1})\\ +Q(c_j\ \bar{\bar{Y}}_{x_j} + d_jY_j + f_j) + R(c^{**}_{j-1}\bar{Y}_{x_{j-1}} + d^{**}_{j-1}Y_{j-1} + f^{**}_{j-1})\big).\end{aligned} \tag{45}$$

The equations are simplified upto $O(h_j^5)$ terms. Then, the matrix form of the derived scheme $(44) - (45)$ is written as:

$$L\hat{Y} + \hat{\phi} = \begin{bmatrix} sub_j & diag_j & sup_j \end{bmatrix} \begin{bmatrix} \hat{Y}_{j-1} \\ \hat{Y}_j \\ \hat{Y}_{j+1} \end{bmatrix} + \hat{\phi}_j = \hat{0}, \tag{46}$$

where L is a block tridiagonal matrix of order $N-1$; $sub_j, sup_j, diag_j$ are block matrices of order 2×2 in L. Also

$$\begin{aligned}\hat{Y} &= [\hat{Y}_1, \hat{Y}_2, ..., \hat{Y}_j, ...\hat{Y}_{N-1}]^T, where \quad \hat{Y}_j = [Y_j, V_j]^T\\ \hat{\phi} &= [\hat{\phi}_1 + sub_1[\alpha_1,\alpha_2]^T, \hat{\phi}_2, ..., \hat{\phi}_j, ...\hat{\phi}_{N-1} + sup_{N-1}[\beta_1,\beta_2]^T]^T,\ where\ \hat{\phi}_j = [\phi_j^1, \phi_j^2]^T\\ \hat{0} &= [[0,0]^T, [0,0]^T, ..., [0,0]^T]^T.\end{aligned}$$

Simplifying (46), we obtain the vector difference equation as follows:

$$\begin{bmatrix} sub^{11} & sub^{12}\\ sub^{21} & sub^{22}\end{bmatrix}\begin{bmatrix}Y_{j-1}\\ V_{j-1}\end{bmatrix} + \begin{bmatrix} diag_j^{11} & diag_j^{12}\\ diag_j^{21} & diag_j^{22}\end{bmatrix}\begin{bmatrix}Y_j\\ V_j\end{bmatrix} + \begin{bmatrix} sup^{11} & sup^{12}\\ sup^{21} & sup^{22}\end{bmatrix}\begin{bmatrix}Y_{j+1}\\ V_{j+1}\end{bmatrix} = \begin{bmatrix}\phi_j^1\\ \phi_j^2\end{bmatrix}, \tag{47}$$

where

$$sub^{11} = -\sigma, sub^{12} = \frac{h_j^2}{12}R, \tag{48}$$

$$\begin{aligned}sub^{21} &= \frac{h_j}{12S}\Big(c_j((P-R-Q)\sigma^2 - 2R\sigma) + h_j(c_{x_j}(P\sigma^3 + R\sigma^2 + 2R\sigma) + RSd_j)\\ &\quad + h_j^2(c_{xx_j}(\frac{P\sigma^4 - R\sigma(2+\sigma)}{2}) - RSd_{x_j}) + h_j^3\frac{RS}{2}d_{xx_j}\Big),\end{aligned}$$

$$sub^{22} = -\sigma + \frac{h_j^3 Q c_j \sigma\mu}{12}, \tag{49}$$

$$diag_j^{11} = (1+\sigma), diag_j^{12} = \frac{h_j^2}{12}Q, \tag{50}$$

$$\begin{aligned}diag_j^{21} &= \frac{h_j}{12S}\Big(c_j((-P+R)(1+\sigma)^2 + Q(\sigma^2-1)) + h_j(c_{x_j}(-P\sigma - R)(1+\sigma)^2 + QSd_j)\\ &\quad + h_j^2 c_{xx_j}(-\frac{P\sigma^2}{2} + \frac{R}{2})(1+\sigma)^2\Big),\end{aligned}$$

$$diag_j^{22} = 1+\sigma, \tag{51}$$

$$sup^{11} = -1, sup^{12} = \frac{h_j^2}{12}P, \tag{52}$$

$$\begin{aligned}sup^{21} &= \frac{h_j}{12S}\Big(c_j(P(1+2\sigma) - R + Q) + h_j(c_{x_j}(P\sigma(1+2\sigma) + R) + PSd_j)\\ &\quad + h_j^2(c_{xx_j}(\frac{P(1+2\sigma)\sigma^2 - R}{2}) + PS\sigma d_{x_j}) + h_j^3\frac{PS\sigma^2}{2}d_{xx_j}\Big),\end{aligned}$$

$$sup^{22} = -1 - \frac{h_j^3 Q c_j \sigma\mu}{12}, \tag{53}$$

$$\phi_j^1 = 0, \phi_j^2 = -\frac{h_j^2}{12}\Big(f_j(P+Q+R) + h_j f_{x_j}(P\sigma - R) + \frac{h_j^2}{2} f_{xx_j}(P\sigma^2 + R)\Big). \tag{54}$$

5. Convergence Analysis

Vector convergence i.e., for $M = 2$ a coupled second order BVP is provided. Consider a coupled nonlinear singular second order boundary value problem (37) − (38). As $y(x_j), v(x_j)$ is the exact solution, we let:

$$\begin{aligned} y = [[y_1, v_1]^T \quad &, \quad [y_2, v_2]^T ......, [y_j, v_j]^T, ..., [y_{N-1}, v_{N-1}]^T]^T \cong \hat{y} \text{ satisfy}\\ Ly \quad &+ \quad \hat{\phi} + \hat{T}_j^3 = 0, \text{ where L is defined in (46) .}\end{aligned} \tag{55}$$

Let $\hat{e_j} = [Y_j - y_j, V_j - v_j]^T \equiv [e_{j_y}, e_{j_v}]^T$ be the discretization error

then $Y - \hat{y} = E = [\hat{e_1}, \hat{e_2}, ..., \hat{e_{N-1}}]^T$.

where $\hat{T}_j^3 = [\ (T_1^3(h_1), T_1^3(h_1)),\ (T_2^3(h_2), T_2^3(h_2)), ...,\ (T_{N-1}^3(h_{N-1}), T_{N-1}^3(h_{N-1})\ )^T]^T$.

We subtract (46) from (55) and obtain the error equation as follows

$$LE = \hat{T}_j^3. \tag{56}$$

Let $|d_j| \leq K_1$ and $|d_{x_j}| \leq K_2$ for some $K_1, K_2 > 0$, then using $(48) - (49), (52) - (53)$ we get,

$$\|sup_j\|_\infty \leq \max_{1\leq j\leq N-2} \begin{cases} 1 + \frac{h_j^2 P}{12} + O(h_j^4), \\ 1 + \frac{h_j^2\sigma^2 P}{2}(K_1 + \frac{h_j\sigma}{2}K_2) + O(h_j^4), \end{cases} \tag{57}$$

$$\|sub_j\|_\infty \leq \max_{2\leq j\leq N-1} \begin{cases} \sigma + \frac{h_j^2 R}{12} + O(h_j^4), \\ \sigma + \frac{h_j^2\sigma^2 R}{2}(K_1 + \frac{h_j}{2}K_2) + O(h_j^4), \end{cases} \tag{58}$$

Thus for sufficiently small h_j, we get $\|sub_j\|_\infty \leq \sigma$ and $\|sup_j\|_\infty \leq 1$. Hence, L is irreducible [12]. Let the sum of elements of row_j of L be $sum_{row\,j}$

$$sum_{row\,j} = \begin{cases} \sigma + \frac{h_j^2}{12}(P+Q) + O(h_j^4), j = 1 \\ \sigma + \frac{h_j}{12S}(-P+R+Q+2R\sigma)c_j + \frac{h_j^2}{12S}\Big((c_{x_j}(-P\sigma^3 - R\sigma^2 - 2R\sigma) + d_jS(P+Q)) \\ +\frac{h_j}{2}(c_{xx_j}(-P\sigma^4 + R(\sigma^2+2\sigma)) + 2S(P\sigma d_{x_j} - Qc_j\mu\sigma))\Big) + O(h_j^4),\ j = 2 \end{cases} \tag{59}$$

$$sum_{row\,j} = \begin{cases} \frac{h_j^2}{12}(P+Q+R) + O(h_j^4),\ j = 3, 5, ..., N-4 \\ \frac{h_j^2}{12}\big((P+Q+R)d_j + h_jd_{x_j}(P\sigma - R) + \frac{h_j^2}{2}d_{xx_j}(R+\sigma^2P)\big),\ j = 4, 6, ..., N-3 \end{cases} \tag{60}$$

$$sum_{row\,j} = \begin{cases} 1 + \frac{h_j^2}{12}(R+Q) + O(h_j^4),\ j = N-2 \\ 1 + \frac{h_j}{12S}(-P\sigma - P + R - Q)c_j + \frac{h_j^2}{12S}\Big((c_{x_j}(-P\sigma(1+\sigma) - R) + d_jS(R+Q)) \\ +\frac{h_j}{2}(c_{xx_j}(-P\sigma^2(1+\sigma) + R) + 2S(-d_{x_j}R + Qc_j\mu\sigma))\Big) + O(h_j^4),\ j = N-1 \end{cases} \tag{61}$$

Let

$$0 < K_{min} \leq min(K_1, K_2) \leq K_{max} \tag{62}$$

Using $(59)-(61)$ and for sufficiently small h_j, Monotonicity of L can be easily proved. Therefore, L^{-1} exist and $L^{-1} \geq 0$[12]. Hence by (56) we have,

$$||E|| = ||L^{-1}||||\hat{T}_j^3|| \tag{63}$$

Now by $(59) - (61)$ and for sufficiently small h_j we can say that:

$$sum_{row\,j} > \begin{cases} \frac{h_j^2}{12}(P+Q),\ j = 1 \\ \frac{h_j^2}{12}(P+Q)K_{min}, j = 2 \end{cases} \tag{64}$$

$$sum_{row\,j} \geq \begin{cases} \frac{h_j^2}{12}(P+Q+R),\ j = 3, 5, ..., N-4 \\ \frac{h_j^2}{12}(P+Q+R)K_{min},\ j = 4, 6, ..., N-3 \end{cases} \tag{65}$$

$$sum_{row\,j} > \begin{cases} \frac{h_j^2}{12}(R+Q),\ j = N-2 \\ \frac{h_j^2}{12}(R+Q)K_{min}, j = N-1 \end{cases} \tag{66}$$

Since $\sigma \neq 0$ we can say that:

$$\begin{aligned} sum_{row\,j} &> \max[\frac{h_j^2}{12}(P+Q)\ , \frac{h_j^2}{12}(P+Q)K_{min}] \\ &= \frac{h_j^2}{12}(P+Q)K_{min}, \text{for } j = 1, 2 \end{aligned} \tag{67}$$

$$\begin{aligned} sum_{row\,j} &\geq \max[\frac{h_j^2}{12}(P+Q+R)\ ,\frac{h_j^2}{12}(P+Q+R)K_{min}] \\ &= \frac{h_j^2}{12}(P+Q+R)K_{min}, \quad \text{for } j=3,4,...,N-3 \end{aligned} \tag{68}$$

$$\begin{aligned} sum_{row\,j} &> \max[\frac{h_j^2}{12}(R+Q)\ ,\frac{h_j^2}{12}(R+Q)K_{min}] \\ &= \frac{h_j^2}{12}(R+Q)K_{min}, \text{for } j=N-2,N-1 \end{aligned} \tag{69}$$

Let ${L_{i,j}}^{-1}$ be the $(i,j)^{th}$ element of L^{-1} then, for $i=1,2,...,N-1$

$$ {L_{i,j}}^{-1} \leq \frac{1}{sum_{row\,j}} \tag{70}$$

By using $(67)-(69)$, we have

$$\frac{1}{sum_{row\,j}} \leq \begin{cases} \frac{12}{h_j^2(P+Q)K_{min}}, & j=1,2 \\ \frac{12}{h_j^2(P+Q+R)K_{min}}, & j=3,4,5,...,N-3 \\ \frac{12}{h_j^2(Q+R)K_{min}}, & j=N-2,N-1 \end{cases} \tag{71}$$

Now let us define,

$$\parallel {L_{i,j}}^{-1} \parallel = \max_{1\leq i\leq N-1} \sum_{j=1}^{N-1} \mid {L_{i,j}}^{-1} \mid \textit{and} \parallel T \parallel = \max_{1\leq j\leq N-1} \mid \hat{T}_j^3 \mid \tag{72}$$

Therefore,using (63) and $(70)-(72)$ we get,

$$\parallel E \parallel \leq \frac{12}{h_j^2 K_{min}\sigma}(\frac{1}{P+Q}+\frac{1}{R+Q}+\frac{1}{(P+R+Q)})O(h_j^5)=O(h_j^3). \tag{73}$$

Hence, scheme (34) has third order convergence for fourth order singular BVPs. On the similar lines without loss of generality, third order vector convergence for system of second order BVPs of the type $(1)-(2)$ can be proved. Similarly, second order convergence of the difference scheme (33) can also be proved.

Theorem 1. Let the solution of boundary value problems $(1)-(2)$ be sufficiently smooth such that the required higher order derivatives of $y(x)$ exist in the solution domain. Then, the scheme derived in (34) with sufficiently small $h_j, 0<\sigma<1$ and $\frac{(\sqrt{5}-1)}{2}<\sigma<\frac{(\sqrt{5}+1)}{2}$ has third order convergence.

6. Numerical Illustrations

In this section we have solved five nonlinear BVPs and compared the numerical results with existing methods. The root mean square errors(e_{rms}) for variable mesh, maximum absolute error (e_{max}) for uniform mesh and computational order of convergence (COC) are tabulated in the Tables 1-5:

$$e_{rms} = (\frac{1}{N-1}\sum_{j=1}^{N-1}|y_j - Y_j|^2)^{\frac{1}{2}},$$

$$e_{max} = \max_{1\leq j\leq N-1}|y_j - Y_j|,$$

$$COC = log_2\frac{e_{max} \text{ at } j}{e_{max} \text{ at } j+1}.$$

We have considered $h_1 = \frac{(\sigma-1)}{(\sigma^N-1)}, \sigma \neq 1$. Therefore, the rest of the h_j 's has been obtained as $h_{j+1} = \sigma h_j, j = 1(1)N-1$. The nonlinear system of difference equations is solved by Newton's Block method. Thus we consider $y_0 = 0$ as the initial approximation. Also all calculations are done in Matlab 07. In the following questions $y^{(i)}(x)$ means i^{th} derivative of $y(x)$.

Example 5.1 Consider fourth order nonlinear BVP [9] :

$$y^{(iv)}(x) = 6e^{-4\ y(x)} - \frac{12}{(1+x)^4}, 0 < x < 1$$

with

$$y(0) = 0, y^{(ii)}(0) = -1,$$
$$y(1) = 0.6931, y^{(ii)}(1) = -0.25.$$

The exact solution is given by $y(x) = \log(1+x)$. The e_{rms} and e_{max} errors are tabulated in Table 1.

Table 1. e_{rms} errors for $\sigma = 0.9$ and e_{max} errors for $\sigma = 1$

	e_{rms}		e_{max}	
N	$O({h_j}^2)$method	$O({h_j}^3)$ method	Twizell[9]	$O(h^4)$method
8	4.2831e-03	8.7695e-05	3.7e-04	1.4817e-05
16	1.8422e-03	2.2401e-05	2.9e-05	9.6743e-07
32	1.1828e-03	1.1389e-05	1.9e-06	6.0883e-08

Example 5.2 Consider a sixth order nonlinear BVP([4],[10]):

$$y^{(vi)}(x) = e^{-x} y(x), 0 < x < 1$$

with

$$y(0) = y^{(ii)}(0) = y^{(iv)}(0) = 1, y(1) = y^{(ii)}(1) = y^{(iv)}(1) = e.$$

The test solution is e^x. The e_{rms} and e_{max} errors are tabulated in Table 2.

Table 2. e_{rms} errors for $\sigma = 0.8$ and e_{max} errors for $\sigma = 1$

	e_{rms}				e_{max}
N	$O({h_j}^2)$ method	$O({h_j}^3)$method	Haq et.al.[4]	Ullah et.al.[10]	$O(h^4)$method
.1	6.3259e-02	5.3858e-06	-1.2e-04	1.1106e-07	2.5257e-08
.2	9.2757e-02	6.1146e-06	-2.3e-04	2.1138e-07	4.6154e-08
.3	1.0035e-01	4.2164e-06	-3.2e-04	2.9128e-07	6.2412e-08
.4	9.4516e-02	3.1467e-06	-3.8e-04	3.4229e-07	7.3657e-08
.5	8.1200e-02	2.2326e-06	-4.0e-04	3.6143e-07	7.9407e-08
.6	6.4401e-02	3.1467e-06	-3.9e-04	3.4461e-07	7.9065e-08
.7	4.6677e-02	1.4826e-06	-3.3e-04	2.9390e-07	7.1906e-08
.8	2.9559e-02	8.7666e-07	-2.4e-04	2.1404e-07	5.7058e-08
.9	1.3877e-02	3.9002e-07	-1.2e-04	1.1271e-07	3.3493e-08

Example 5.3 Consider a coupled second order nonlinear problem of the form([1],[2]):

$$y^{(ii)}(x) + y^{(i)}(x) + xy(x) + 2xz(x) + xy^2(x) = f_1(x)$$
$$z^{(ii)}(x) + z(x) + x^2y(x) + \sin(x)z^2(x) = f_2(x)$$

with

$$y(0) = y(1) = 0,\ z(0) = z(1) = 1,$$

where $f_1(x) = -2x\sin(x) - 2 + x^2 - 2x^4 + x^5$, $f_2(x) = (1-x)x^3 + (1-\pi^2)\sin(\pi x) + \sin(x)\sin^2(\pi x)$ and $0 \le x \le 1$. The exact solution is $y(x) = x - x^2, z(x) = \sin(\pi x)$. The e_{max} errors are tabulated in Table 3.

TABLE 3. e_{max} errors for $\sigma = 1$

	$z(x)$			$y(x)$		
N	Geng[2]	Dehgan et.al.[1]	$O(h^4)$method	Geng[2]	Dehgan et.al.[1]	$O(h^4)$method
.08	8.1e-04	1.3e-08	6.2649e-10	2.1e-04	5.4e-10	7.7420e-11
.24	8.3e-04	9.9e-09	1.7075e-09	1.6e-04	1.2e-09	6.5540e-11
.40	7.0e-04	3.5e-08	2.3610e-09	8.5e-05	2.2e-09	4.6350e-10
.56	3.5e-04	1.2e-07	2.4373e-09	1.3e-04	2.4e-09	8.5945e-10
.72	1.7e-04	1.0e-07	1.9188e-09	8.8e-05	5.8e-10	9.7664e-10
.88	7.4e-04	4.9e-08	9.2456e-10	2.3e-04	3.4e-10	6.1619e-10
.96	4.6e-04	5.8e-09	3.1683e-10	1.3e-04	1.6e-10	2.3631e-10

Example 5.4 Consider the fourth order nonlinear singular problem of the form :

$$y^{(iv)}(x) + \frac{4}{x}y^{(iii)}(x) - e^y = e^x(\frac{4+x}{x}) - e^{e^x}, 0 < x < 1$$

with

$$y(0) = y^{(ii)}(0) = 1,$$
$$y(1) = y^{(ii)}(1) = 2.7183.$$

The test solution is $y(x) = e^x$. The e_{rms} and e_{max} errors are tabulated in Table 4.

TABLE 4. e_{rms} errors for $\sigma = 0.9$ and e_{max} errors for $\sigma = 1$

	e_{rms}		e_{max}	
N	$O({h_j}^2)$ method	$O({h_j}^3)$ method	$O(h^4)$ method	COC
16	9.7848e-02	5.8113e-05	1.2702e-05	-
32	9.8618e-02	3.0508e-05	1.1248e-06	3.4972
64	9.9103e-02	2.6797e-05	9.2148e-08	3.6096

Example 5.5 Consider the sixth order nonlinear singular problem of the form :

$$y^{(vi)}(x) + \frac{6}{x}y^{(v)}(x) - y^2 = 6\frac{\cos(x)}{x} - \sin(x) - \sin^2(x),$$

with

$$y(0) = y^{(ii)}(0) = y^{(iv)}(0) = 0,$$
$$y(1) = -y^{(ii)}(1) = y^{(iv)}(1) = 0.8415.$$

The exact solution is $y(x) = \sin(x)$. The e_{rms} and e_{max} errors are tabulated in Table 5. We observe that, in case of second order method the error overflows.

TABLE 5. e_{rms} errors for $\sigma = 0.8$ and e_{max} errors for $\sigma = 1$

	e_{rms}		e_{max}	
N	$O({h_j}^2)$ method	$O({h_j}^3)$method	$O(h^4)$method	COC
8	-	3.0349e-06	5.2269e-06	-
16	-	1.6912e-06	2.8506e-07	4.1966
32	-	1.5162e-06	1.5987e-08	4.1563
64	-	1.5115e-06	9.3622e-10	4.0939

7. Conclusion

In this paper, second and third order variable mesh schemes have been derived for solving nonlinear higher order(mainly even ordered) and system of second order singular BVPs. Table $1, 2, 3$ proves refinement in results when compared with other nonlinear BVPs which are solved by computational methods using extrapolation, collocation and iterative method (Daftardar Jafari method) and finally using cubic B-spline scaling functions. We have compared our own results in Table 4, 5 due to inadequacy of any prior results. Thus, we have provided COC for the uniform mesh method. The proposed schemes are more computationally efficient due to use of only three consecutive nodal points at a time which leads to solving of a tri-diagonal matrix. Our methods with minor modifications are applicable to higher even order singularly perturbed BVPs and problems in polar as well as cartesian coordinates.

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References

[1] Dehghan, M. and Lakestani, M., (2008), Numerical solution of nonlinear system of second-order boundary value problems using cubic B-spline scaling functions, Int. J. Comput. Math., 85, pp. 1455-1461.
[2] Geng, F. and Cui, M., (2007), Solving a nonlinear system of second order boundary value problems, J. Math. Anal. App., 327, pp. 1167-1181.
[3] Glatzmaier, G. A., (1985), Numerical simulations of stellar convection dynamics at the base of the convection zone , Geophysics. Astro. Fluid., 31, pp. 137-150.
[4] Haq, F., Ali, A. and Hussain, I.,(2012), Solution of sixth-order boundary value problems by collocation method Haar wavelets, Int. J. Phy. Sci., 43, pp. 5729-5735.
[5] Jack, H., (2006), Dynamic System Modeling and Control (http://claymore.engineer.gvsu.edu/jackh/books.html), pp. 110-158.
[6] Jain, M. K.,(2014), Numerical Solution of Differential Equations (third edition)(New Age International(P) Ltd., New Delhi), pp.235-237.
[7] Talwar, J. and Mohanty, R. K.,(2012), A class of numerical methods for the solution of fourth-order ordinary differential equations in polar coordinates, Adv. Num. Anal., 2012, 20 page, Article ID 626419.
[8] Terril, R. M., (1964), Laminar flow in a uniformly porous channel, Aeronaut. Quart., 15(3), pp. 299-310.
[9] Twizell, E. H., (1986), A fourth-order extrapolation method for special nonlinear fourth-order boundary value problems, Commun. Appl. Numer. M., 2 , pp. 593-602.
[10] Ullah, I. ,Khan, H. and Rahim, M. T., (2014), Numerical Solutions of Fifth and Sixth Order Nonlinear Boundary Value Problems by Daftardar Jafari Method, J. Comput. Engg., Vol. 2014, Article ID 286039.
[11] Mohanty, R. K. ,(2005), A Family of Variable Mesh Methods for the Estimates of (du/dr) and the Solution of Non-linear Two Point Boundary Value Problems with Singularity, J. Comput. App. Maths., 182, pp. 173-187.
[12] Varga, R. S., (1962), Matrix Iterative Analysis (Prentice-Hall, Englewood Cliffs, NJ), 16-24, pp. 85-87.
[13] Chandrasekhar, S., (1981), Hydrodynamic and Hydromagnetic stability (Clarendon Press,Oxford, Reprinted: Dover Books, New York).

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