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Existence and Qualitative Properties of Solutions to Nonlinear Schrödinger Equations on Metric Graphs

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Dedication

To my parents for all their love, sacrifice and unconditional support...

Acknowledgements

First and foremost, I wish to thank my parents, to whom this dissertation is dedicated, for all their support ever since I was born, all their sacrifice so that I would be able to pursue my dreams and above all, for their love throughout my whole life. Thank you. I also have to thank my sister, Beatriz, and the rest of my family, without whom I would not be who I am today.

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Abstract

In this dissertation we do a detailed study of the results of Adami R., Serra E., Tilli P., in the articles [NLS ground states on graphs, Calc. Var. (2015) 54:743–761] and [Threshold phenomena and existence results for NLS ground states on metric graphs, J. Funct. Anal. (2016) 271(I):201-223] concerning the existence of ground states of prescribed mass for the nonlinear Schrödinger energy functional on metric graphs. The problem under consideration in these articles is

$$\mathcal{E}_{\mathcal{G}}(\mu) = \inf\left\{E(u,\mathcal{G}) = \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})},\ u\in H^1(\mathcal{G}),\ \|u\|^2_{L^2(\mathcal{G})} = \mu\right\},$$

where $\mathcal{G}$ is a non-compact metric graph and $\mu > 0$ is prescribed. Note that through Lagrange multiplier theory we see that any solution of the above minimization problem is a weak solution to the stationary nonlinear Schrödinger equation

$$-u'' + \lambda u = |u|^{p-2}u \quad \text{on} \quad \mathcal{G},$$

for some Lagrange multiplier $\lambda > 0$. The main focus of our study is, by following the aforementioned articles, to show how does the topology (shape) of the graph influences the existence or non-existence of minimizers. We start by reviewing the theory of ground states in the classical case $\mathcal{G} = \mathbb{R}$, where existence and qualitative properties of minimizers are established. We will refer to them as *solitons*. Passing on to graphs and following the work of Adami, Serra and Tilli, we exhibit a topological assumption on the graph can rule out the existence of ground states except for very particular graphs which are fully described. This allows a general existence/non-existence result within the class of graphs that satisfy this assumption. For graphs that do not satisfy this assumption the question of existence of minimal energy solutions is more delicate. We show that in fact the condition

$$\mathcal{E}_{\mathcal{G}}(\mu) < \mathcal{E}_{\mathbb{R}}(\mu)$$

is a sufficient condition for existence of ground states. This existence result is deeply connected with a dichotomy principle which completely characterizes the behaviour of minimizing sequences for this problem.

Following the second aforementioned reference we show that not only the topology of a graph affects the existence of solutions. In fact, it is shown that a particular interplay between the mass of the solutions and certain metric properties of the graph, such as the length of bounded edges, provides the existence of a sharp threshold between existence and non-existence of ground states.

The question concerning existence of solutions to the equation

$$-u'' + \lambda u = |u|^{p-2}u$$

can be posed in different two different ways. The first one is to consider λ unknown and, by imposing a mass constraint, λ will arise as a Lagrange multiplier. The second one is to consider the value λ fixed and considering as constraint the L^p norm. In the case $\mathcal{G} = \mathbb{R}$ due to scalings the solutions can be related. However, for a fixed graph $\mathcal{G}$ scalings no longer work and therefore, we need to focus on the minimization problem

$$\min\left\{T(u,\mathcal{G}) = \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} + \frac{\lambda}{2}\|u\|^2_{L^2(\mathcal{G})},\ u\in H^1(\mathcal{G}),\ \|u\|^p_{L^p(\mathcal{G})} = p\mu\right\},$$

where, again, $\mathcal{G}$ is a non-compact metric graph and μ is prescribed. We will refer to this problem as the Pohozaev problem. We start again by studying the case $\mathcal{G} = \mathbb{R}$. After this study we investigate if on a graph the same results of Adami, Serra and Tilli are still valid. In particular, we show that the same topological assumption still allows for a complete characterization of which graphs admit and do not admit minimizers.

Key-words: Nonlinear Schrödinger equation, minimization, ground states, metric graphs.

Mathematics Subject Classification: 35R02, 35Q55, 81Q35, 49J40, 58E30

Resumo

Nesta dissertação é abordada a equação de Schrödinger não linear no contexto de grafos métricos. Em particular, são seguidos com grande detalhe os artigos de Adami R., Serra E., Tilli P., [NLS ground states on graphs, Calc. Var. (2015) 54:743–761] e [Threshold phenomena and existence results for NLS ground states on metric graphs, J. Funct. Anal. (2016) 271(I):201-223]. O foco destes artigos é a existência ou não existência de soluções de energia mínima, com massa fixa, para o funcional de energia da equação de Schrödinger com um fator não linear do tipo potência em grafos métricos.

Considere-se $\mathbb{R}^N$, $N \geq 1$. A equação de Schrödinger com este tipo de não linearidade toma a forma:

$$i\partial_t u + \Delta u + \lambda |u|^{p-2}u = 0,\ 2 < p < 2 + \frac{4}{N-2} \quad (2 < p < \infty,\ \text{if } N \leq 2), \tag{NLS}$$

onde $\lambda \in \mathbb{R}$ determina a tipologia da equação, ou seja, se é de tipo *focusing* ($\lambda > 0$), ou de tipo *defocusing* ($\lambda < 0$). Sem perda de generalidade, podemos supôr, multiplicando por uma constante conveniente, que $\lambda = \pm 1$.

É sabido que o problema de Cauchy local para esta equação localmente está bem posto em $H^1(\mathbb{R}^N)$ para uma certa condição inicial $\varphi \in H^1(\mathbb{R}^N)$. O comportamento assimptótico das soluções está depende do sinal de λ e do expoente crítico de massa L^2 dado por $\alpha = 2 + \frac{4}{N}$. Como estamos interessados em soluções globais para qualquer dado inicial em $H^1(\mathbb{R}^N)$ focamo-nos no chamado caso *subcrítico*, $\lambda > 0$ e $p < \alpha$. Destas soluções globais estamos particularmente interessados em construir soluções da forma

$$u(t,x) = e^{i\omega t}\varphi(x), \tag{1}$$

onde $\omega \in \mathbb{R}$ e $\varphi \in H^1(\mathbb{R}^N)$. Estas soluções são chamadas de *estados estacionários*. Dada a periodicidade em t, esta classe de soluções é de grande importância em física matemática. Note-se que se u é da forma (1) então, a função φ é uma solução do problema elíptico semilinear

$$\begin{cases} -\Delta\varphi + \omega\varphi = |\varphi|^{p-2}\varphi, \\ \varphi \in H^1(\mathbb{R}^N). \end{cases} \tag{2}$$

A equação (NLS) goza de algumas *leis de conservação*; são de particular interesse para nós a lei de *conservação da massa* e da *energia*. Estas duas quantidades são expressas, respetivamente, por

$$M(u(t,x)) = \int_\Omega |u(t,x)|^2 dx$$

e

$$E(u(t,x)) = \frac{1}{2}\int_\Omega |\nabla u(t,x)|^2 dx - \frac{1}{p}\int_\Omega |u(t,x)|^p dx.$$

Tendo em conta a conservação destas quantidades, podemos usar o problema de minimização

$$\min\left\{E(u) = \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{p}\|u\|^p_{L^p(\mathbb{R}^N)},\ u \in H^1(\mathbb{R}^N),\ \|u\|^2_{L^2(\mathbb{R}^N)} = \mu\right\}. \tag{3}$$

para construir soluções de (NLS) da forma (1), caso em que ω, como dado em (2), será visto como um multiplicador de Lagrange. Esta abordagem só é possível no caso subcrítico, pelas desigualdades de Gagliardo-Nirenberg.

O porquê de estudar a dinâmica da equação de Schrödinger não linear em grafos pode ser motivado de diversas formas. Aplicações comuns surgem na teoria de *Condensados de Bose-Einstein* ou em ótica não linear. Em geral as motivações são de natureza física e a ideia fundamental é usar grafos para modelar espaços mais complexos onde um certo fenómento em estudo pode ocorrer.

Recorde-se que um grafo é uma estrutura matemática composta por vértices e arestas, arestas essas que estabelecem uma ligação entre pares de vértices. Neste tipo de estrutura não é possível fazer muita análise. Para a poder fazer precisamos de um *grafo métrico*. Dito de forma simples um grafo métrico é um grafo em que cada aresta é indentificado com um intervalo da forma $[0, \ell]$, onde $\ell > 0$ é o comprimento da aresta, ou uma semi-reta, $[0, +\infty)$.

Posto isto, o problema fundamental desta tese é estudar, em grafos, o problema de minimização (3) e responder às questões:

- De que modo a topologia (forma) de um grafo métrico afeta a existência ou inexistência de soluções de energia mínima?
- Existem propriedades de um grafo, que não a sua forma, que afetem a existência ou inexistência de soluções de energia mínima?

Para este fim, a tese encontra-se estruturada da seguinte forma:

- No Capítulo 2 é feito um estudo detalhado do problema de minimização do funcional de energia da equação NLS com massa fixa em $\mathbb{R}$. Obtém-se a existência de solução através do Lema de *Concentração-Compacidade* e caracterizam-se as soluções do problema de forma explícita.
- No Capítulo 3 é feito com detalhe a construção de grafos métricos e é adaptado a este novo *setting* o problem (3). É também abordado o conceito de rearranjo de funções em grafos e deduzida uma desigualdade do tipo Pólya-Szegő.
- O Capítulo 4 é dedicado inteiramente a resultados de existência e de não existência. Mostra-se que grafos compactos (i.e. com todas as arestas de comprimento finito) admitem sempre soluções para qualquer massa. Dentro da classe de grafos não compactos é dado um argumento de natureza topológica que garante a não existência de soluções à exceção de certas topologias bem caracterizadas. Ainda dentro da classe de grafos não compactos é dada uma condição suficiente para a existência de soluções de energia mínima. É também analisado, para uma família particular de grafos, um fenómeno em que existe uma transição de existência de soluções para não existência. Este fenómeno está dependente de uma relação íntima entre a massa e o comprimento de certas arestas do grafo. Isto dá ainda uma resposta positiva à última questão feita acima, tornando evidente que a existência de solução não está dependente de forma única da topologia do grafo.
- No último capítulo da tese investiga-se o problema de minimização de Pohozaev que é dado como:

$$\min\left\{T(u) = \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^N)} - \frac{\omega}{2}\|u\|^2_{L^2(\mathbb{R}^N)},\ u \in H^1(\mathbb{R}^N),\ \|u\|^p_{L^p(\mathbb{R}^N)} = \mu\right\}, \tag{4}$$

  para $\lambda, \mu > 0$. A escolha de nomenclatura para este problema deve-se a Lions em [The concentration -compactness principle in the Calculus of Variations. The Locally compact case, part 2,Volume 1, Issue 4, 1984, Pages 223-283]. A motivação deste problema está no facto de que a equação (2) pode também ser vista como equação de Euler-Lagrange deste problema de minimização. Esta perspetiva corresponde a fixar o valor ω enquanto, na abordagem anterior, ele é desconhecido e surge como um multiplicador de Lagrange. Em $\mathbb{R}^N$ os dois problemas estão relacionados através de scalings apropriados. Dado que em grafos os scalings não se podem utilizar os resultados dos capítulos anteriores para obter soluções deste novo problema. Tendo isso em conta, é feito em detalhe o estudo do problema (4) e, em paralelo com os capítulos anteriores, são obtidos resultados no que toca a não existencia de minimizantes para o problem (4). Em particular, vemos que a mesma condição topológica continua a garantir os mesmos resultados não existência.

Palavras-Chave: Equação Schrödinger não linear, minimização, soluções de energia mínima, grafos métricos.

Classificação AMS 2010: 35R02, 35Q55, 81Q35, 49J40, 58E30

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Chapter 1

Introduction

Nonlinear Schrödinger equations, NLS, are a class of equations that arise naturally in several fields of physics, such as quantum field theory and nonlinear optics. Even though most of the applications are physical in nature this equation received a lot of attention from mathematicians, since it is one of the good equations to model dispersive phenomena. Other notable dispersive equations are for example the wave and KdV equations. We will focus on a particular class of NLS equations, namely those with a prototype power nonlinearity which can be taken as

$$i\partial_t u + \Delta u + \lambda |u|^{p-2} u = 0,\ 2 < p < 2 + \frac{4}{N-2} \quad (2 < p < \infty,\ \text{if } N \leq 2), \tag{1.1}$$

defined in $\mathbb{R}^N$ and $\lambda \in \mathbb{R}$. Note that for $\lambda = 0$ the nonlinear part of the equation degenerates, making (1.1) into the linear Schrödinger equation. For $\lambda < 0$, the equation is said to be of *defocusing* type, and for $\lambda > 0$ it is said to be of *focusing* type.

It is known, see for example [10, Corollary 4.34], that the local Cauchy Problem is well posed in $H^1(\mathbb{R}^N)$. Depending on the sign of λ and the value of the L^2-mass critical exponent for the nonlinearity, $\alpha = 2 + \frac{4}{N}$, the asymptotic behaviour of the solutions changes significantly. We summarize this as follows:

- If $\lambda < 0$, then for all initial condition $\varphi \in H^1(\mathbb{R}^N)$ solutions are global in time.
- If $\lambda > 0$ and $p \geq \alpha$ then, the asymptotic behaviour of the solution depends also on the on the "size" initial data. For small initial data solutions converge weakly to zero as $t \to \infty$, see [10, Theorem 6.2.1], and if the initial data is large, then blow-up in finite time occurs, see [10, Remark 6.8.1].

In the literature, the case $p < \alpha$ is known as the *subcritical* problem. The cases $p \geq \alpha$ correspond to the *critical* and *supercritical* problems.

In order to motivate the contents of this dissertation let us delve into the world of quantum physics. In particular, we will use as a motivation *Bose-Einstein Condensates*, BEC. For more applications and motivations, we refer for example to [27] and the references therein.

First of all, what is a Bose-Einstein Condensate? In nature there are four states in which matter can exist, solid, liquid, gas and plasma. A BEC is what is sometimes referred as a fifth state of matter. Satyendra Nath Bose and Albert Einstein, two theoretical physicists, in the years of 1924-1925 theorized that this new state of matter could be attained. Moreover, they stated that this state of matter is attained on a single quantum level when a gas of subatomic particles, called bosons (in honor of Bose), with low density is exposed to temperatures near absolute zero. What was verified experimentally in the 1990's was that under these conditions the particles of the gas occupy the lowest quantum state (energy) for the system. These experimentations confirmed, seventy years later, the theoretical results of Bose and Einstein.

It is shown in [13] that when considering a boson gas composed of N particles confined through some external potential the total energy of this system is given by the functional

$$E(\Phi) = \int -\frac{\hbar}{2m} |\nabla \Phi(\mathbf{r}, t)|^2 + V(\mathbf{r}) |\Phi(\mathbf{r}, t)|^2 + \frac{g}{2} |\Phi(\mathbf{r}, t)|^4 d\mathbf{r},$$

where

- Φ is a complex wave function that defines the probability of finding a particle in the position $\mathbf{r}$ at time t;
- m is the mass of the boson particles;
- V is the external potential acting on the system;
- g is a constant, positive or negative, relating the interactions between particles the gas.

Taking a variational approach, we can deduce that the equation that governs this system is the so called *Gross-Pitaevskii equation* which takes the form:

$$i\hbar\Phi_t(\mathbf{r},t) = \Phi(\mathbf{r},t)\left(-\frac{\hbar}{2m}\Delta + V(\mathbf{r}) + g|\Phi(\mathbf{r},t)|^2\right).$$

Note right away that this is a particular case of the NLS equation with power nonlinearity coupled with a potential. If we take $V \equiv 0$ and normalize some of the constants we reach the case $p = 4$ in the equation (1.1).

As shown in [13], under the formalism of mean-field theory, which is something that goes beyond the scope of this dissertation, a ground state that is, a minimal energy solution, exists. Under this formalism it is also proven that any ground state will give rise to a condensate wave function that has the shape of a *stationary state*, that is, the function that describes the condensate is of the form

$$\Phi(\mathbf{r},t) = \phi(\mathbf{r})e^{-i\omega t},$$

where ϕ is a real valued function renormalized in L^2 to the total number of particles and ω a constant depending on the conditions under which the condensation is taking place. Functions like Φ above represent standing waves, meaning functions that do not travel in space. Essentially, ϕ gives the amplitude of the wave and the parameter ω gives its frequency. Note that Φ is a solution of the Gross-Pitaevskii equation if and only if the function ϕ is a solution of the stationary semilinear elliptic equation:

$$\omega\phi(\mathbf{r}) = \phi(\mathbf{r})\left(-\frac{\hbar^2}{2m}\Delta + V(\mathbf{r}) + g\phi^2(\mathbf{r})\right).$$

To summarize, the mathematical treatment of BEC is as follows. Firstly, choose a spatial domain Ω and minimizer the Gross-Pitaevskii energy functional. If minimizers exist then, theoretically, it is possible that a BEC can arise in the domain Ω.

One might now ask: what kind of domains can one use to produce a BEC? In practical experiments they are often taken to be optical or magnetic traps with the shape of disks or cigar shaped traps. However, it was envisioned recently the possibility of constructing ramified traps, see [1] and the references therein. This matter of domains leads us finally into graphs. It is expected that a ramified trap Ω can be modelled by a suitable graph $\mathcal{G}$. Then, if minimizers exist in $\mathcal{G}$ this suggests the possibility of creating a BEC in the trap modelled by the graph. An absence of minimizers on the graph would then suggest that the system on the trap would be unstable and that condensation would not occur in the trap modelled by the graph.

Before any considerations are done let us quickly recall what is a graph. A graph on its own is a very interesting mathematical object and has had an intense focus from mathematicians. It is a structure composed of two key elements which we refer to as vertices and edges, the latter component also establishes a connection with the former. The study of graphs was extremely important throughout mathematics. From the viewpoint of pure mathematics graphs allowed the connection between what at first might seem very distinct fields of study, take for instance combinatorics and topology, see for example [6]. From the viewpoint of applications, they are one of the most useful tools to model very complex systems. To name a few take for example road connections, airline flights, social media networks and even the internet.

The previous motivation establishes a connection with graphs and the NLS equation, whilst also giving an important and practical motivation for studying the NLS equation on graphs. The most natural question is now: how can we study it?

The main problem of this dissertation is to find functions of prescribed mass that minimize the NLS energy functional on graphs. Taking a graph in the combinatorial sense is not enough, in these graphs not much analysis can be done. In order to do it we need a *metric graph*. Put simply, a metric graph is a graph upon which we assign to each edge an interval of the form $[0, \ell]$, where $\ell > 0$ is the length of the edge, or of the form $[0, +\infty)$. This means these graphs admit bounded and unbounded edges. The problem only becomes mathematically relevant if there exits at least one unbounded edge in the graph. In fact, a multitude of situations might happen. Take Figures 1.1 and 1.2, for instance.

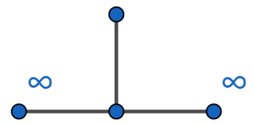

Figure 1.1: A metric graph which admits a ground state

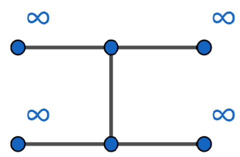

Figure 1.2: A metric graph that does not admit a ground state

We will see that the first graph admits ground states of arbitrary mass but the second one does not admit ground states at all! Note that the graphs are not that different. Simple differences like this one culminated in the question: what properties of the graphs (like their shape or length of particular edges) affect the existence or non-existence of minimal energy solutions? The answer to this question is the core of this dissertation.

We now describe the contents of this thesis. In order to translate this problem into graphs we need some tools. Given that metric graphs can have edges of infinite length it makes sense to first understand the minimization problem in $\mathbb{R}$. Chapter 2 is devoted to this. Consider the equation

$$i\partial_t u + u'' + |u|^{p-2}u = 0, \quad 2 < p < \infty,$$

where u' denotes the spatial derivative. It is known that stationary states, meaning solutions of the form

$$u(x,t) = e^{i\omega t}\varphi(x), \tag{1.2}$$

exist for $p > 2$ (in fact we construct them). We focus on the *subcritical* case, $p \in (2,6)$, in order to have solutions defined for all times. On this matter of criticality we make the remark that even though this thesis concerns the subcritical case, in graphs some advances have been made recently in the critical case, see for example [14] and [15]. The problem under study in Chapter 2 is then to find solutions of the following minimization problem,

$$\min\left\{E(u) = \frac{1}{2}\|u'\|^2_{L^2(\mathbb{R})} - \frac{1}{p}\|u\|^p_{L^p(\mathbb{R})},\ u \in H^1(\mathbb{R}),\ \|u\|^2_{L^2(\mathbb{R})} = \mu\right\}. \tag{1.3}$$

Section 2.1 is devoted to the study of existence of solutions to the above problem. Since the functional is bounded from below, see Lemma 2.2, solutions might exist. However, the lack of compactness that comes with the unboundedness of $\mathbb{R}$ prevents us from using the direct method of the Calculus of Variations. This problem is then solved by the *Concentration-Compactness Principle*, and the use of a powerful lemma which states a certain dichotomy for the behaviour of minimizing sequences, see Lemma 2.4; the lemma and the whole principle is due to P. Lions, see [23] and [24]. This principle is a fundamental tool in Calculus of Variations in order to treat problems in unbounded domains and therefore, a large part of Chapter 2 is devoted to present this technique. With this lemma proved we can finally show that minimizing sequences converge to a minimizer, see Theorem 2.7. In Section 2.2 we provide a characterization of solutions. The first natural question is: are the solutions unique? Through the differentiability of E we can apply the theory of constrained extrema problems in order to see that a solution to the minimization problem (1.3) necessarily needs to satisfy, in a weak sense, the stationary nonlinear Schrödinger equation

$$-u'' + \lambda u = |u|^{p-2}u, \tag{1.4}$$

for some Lagrange multiplier $\lambda > 0$. Solutions to this problem have been completely characterized. In particular, in dimension one, all solutions are, up to *translation* and *phase multiplication*, strictly decreasing in $(0, +\infty)$, positive and even. In fact, the main result of this section, Theorem 2.16, states that u is a complex valued solution if and only if u has the form

$$u(x) = e^{i\theta}\varphi(x - y), \quad y, \theta \in \mathbb{R}, \tag{1.5}$$

and where φ has all the properties mentioned above. Moreover, it is also seen that the mass of the function u, as well as its energy, is given precisely by those of function φ. This is related with the conservation laws of the solutions to the NLS equation, which we also discuss. Finally, in Section 2.3 it is shown that solutions scale with the mass, that is, if φ_μ is the unique positive and even minimizer of (1.3) with mass μ, then,

$$\varphi_\mu(x) = \mu^\alpha \varphi_1(\mu^\beta x), \tag{1.6}$$

where α and β are positive constants that depend on p, and φ_1 is the unique positive and even minimizer with mass 1. Moreover, we show that φ_1 takes the form of a hyperbolic secant. We will refer to these solutions as *solitons*.

In Chapter 3, we turn our attention to graphs and formalize the minimization problem (1.3) in this new setting. In Section 3.1 all the concepts related to graphs are introduced. Special attention is given to the definition of *metric graph* and to the formalization of some properties related with this structure. Function spaces such as $C(\mathcal{G})$, $H^1(\mathcal{G})$ and $L^p(\mathcal{G})$, for $p \in [1, +\infty]$, are then defined on a metric graph $\mathcal{G}$. We make sense of the differential expression

$$f \mapsto -\frac{d^2 f}{dx^2} + \lambda f$$

in graphs and through it we define an operator. In this context we introduce the *standard* or *Neumann-Kirchhoff* vertex conditions which can be seen as an analogue of the Neumann boundary conditions on an interval. We then finish the chapter by stating the new problem, which now becomes

$$\inf \left\{ E(u, \mathcal{G}) = \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})},\ u \in H^1(\mathcal{G}),\ \|u\|^2_{L^2(\mathcal{G})} = \mu \right\}, \tag{1.7}$$

and by proving some *a priori* regularity results, see Proposition 3.17, as well as necessary conditions for the existence of minimizers. To solve this problem the notion of rearrangement of a function will be fundamental and, therefore, some time is devoted to bring this concept into the setting of metric graphs. Particular attention is devoted to *decreasing rearrangement* and the *Schwarz symmetrization* or *spherically symmetric decreasing rearrangement*. The ability to know if a rearrangement increases or decreases the energy of a function will be crucial. Therefore, following the ideas of [18], we state and prove a Pólya-Szegő-type inequality in metric graphs, see Theorem 3.21, which states that for the decreasing rearrangement the kinetic term of the energy functional always decreases and that a sufficient condition for the same to happen for the Schwarz symmetrization is that the pre-level sets of the function to be rearranged to have at least 2 elements. To finish this chapter we present two results which are Gagliardo-Nirenberg-type inequalities to estimate L^p and L^∞-norms on non-compact graphs (meaning the graph has at least one unbounded edge).

Chapter 4 is devoted entirely to the matter of existence and non existence of ground states and follows closely the work done in [1] and in [2]. We begin by proving that the problem is only mathematically interesting when graphs are non-compact, since for compact graph $\mathcal{G}$, meaning a graph where all its edges have finite length, we have that $H^1(\mathcal{G})$ is compactly embedded in $L^p(\mathcal{G})$ for all $p \geq 1$ and thus the infimum in (1.7) is always attained. For non-compact graphs a multitude of situations might happen, as was mentioned before with Figures 1.1 and 1.2. As we will see in Section 4.1, it will be easier to rule out existence than to prove it and while there is no general theory concerning existence, the question of non-existence has some general results. As can be seen in [1], we will see that when a graph satisfies a certain topological assumption, which roughly states that if we disconnect the graph then all the connected components contain at least one half-line, then minimizers will not be attained except for graphs with a very particular structure (they are completely described, see Figures 3.5 and 4.3). Therefore, under the class of graphs that satisfy this assumption we have a necessary and sufficient condition for existence of ground states. Consequently, also for non-existence. This result is given in Theorem 4.9. Note that the graph in Figure 1.2 satisfies this assumption, while the one in Figure 1.1 does not.

Section 4.2 is devoted entirely to existence results. We start by proving some *a priori* estimates for minimizers and by establishing qualitative properties of ground states on the unbounded edges of graphs, see Proposition 4.15. In this section an important scaling rule for both functions and graphs is also introduced, see Proposition 4.11. This result simplifies computations significantly, since it roughly says that the minimization problem (1.7) is equivalent to minimize with a different mass constraint on a homotheticaly scaled graph. The main results of this section are Theorem 4.18 and 4.19. The first shows that the behaviour of minimizing sequences only has two possibilities: either the sequence converges to zero or to a minimizer. The second result is the one that gives us a sufficient condition to have a minimizer. Namely, let $\mathcal{E}_{\mathcal{G}}(\mu)$ be defined by the quantity in (1.7), then ground states of mass μ will exist if the energy of a solution of the same mass in $\mathbb{R}$ is a *strict* upper bound to $\mathcal{E}_{\mathcal{G}}(\mu)$. A practical corollary of this existence result is introduced straight away and this section finishes with a series of examples. Finally, Section 4.3 means to exhibit in detail what in [2] is called the threshold phenomenon. Comparing back with the other existence results we see now that not only the topology of the graph influences the (non-)existence of solution; some metric properties of the graphs, such as the lengths of bounded edges, can influence the existence of solution. This is shown in Proposition 4.23 and, after proving a stability result, it is shown that graphs like the one in Figure 4.15 admit ground states of mass μ if and only if the quantity $\mu^{\beta}\ell$ is larger than a certain threshold (hence the name). Here the constant β is exactly the one given in the rescaling (1.6) and ℓ is the length of the terminal bounded edge of these graphs.

In the previous sections we saw that if a minimizer u to the minimization problem exists, then there exists a Lagrange multiplier $\lambda > 0$ such that the minimizer solves the stationary equation (1.5). Note now that by considering the functionals

$$T : H^1(\mathcal{G}) \to \mathbb{R}; \quad T(u) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2 dx + \frac{\lambda}{2}\int_{\mathcal{G}} |u|^2 dx,$$

where $\lambda > 0$, subject to the constraint

$$R : H^1(\mathcal{G}) \to \mathbb{R}; \quad R(u) = \frac{1}{p}\int_{\mathcal{G}} |u|^p dx = \mu,$$

where $\mu > 0$ and $p > 2$, we arrive, via constrained minimization, at the Euler-Lagrange equation

$$-u'' + \lambda u = \theta |u|^{p-2} u,$$

where $\theta \in \mathbb{R}$ is a Lagrange multiplier. By multiplying by a suitable constant depending on θ and p we can reduce to the equation (1.4). This motivated an important question: given any fixed Lagrange multiplier $\lambda > 0$ and a graph $\mathcal{G}$ taking a different variational approach can we deduce the existence of solutions to the same equation? In $\mathbb{R}^N$, through scalings we can relate both problems. However, for a fixed graph $\mathcal{G}$ scalings no longer work and therefore we need to focus on the minimization problem:

$$\inf\left\{ T(u,\mathcal{G}) = \frac{1}{2}\left(\|u'\|^2_{L^2(\mathcal{G})} + \lambda \|u\|^2_{L^2(\mathcal{G})} \right) \mid u \in H^1(\mathcal{G}),\ R(u,\mathcal{G}) = \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})} = \mu \right\},$$

for $\mu > 0$. We will call this problem as the Pohozaev problem, see [24] for a justification of the nomenclature.

To finish this dissertation, in Chapter 5 we focus on studying this problem. In particular, we investigate, in parallel with the previous chapters, if the results of [1] and [2] still valid. In Section 5.1 we turn ourselves to the study of the Pohozaev problem in $\mathbb{R}$. We do not use the scalings approach to solve the problem in $\mathbb{R}$, instead we use a method that is entirely similar to that of minimizing the NLS energy functional in $\mathbb{R}$. It involves a variation of the Concentration-Compactness Lemma presented in Chapter 2 which was introduced by Lions in [23]. See Lemma 5.6. Section 5.2 is then devoted to graphs. The first non-existence results are obtained. In particular, we show that under assumption(H) the graphs for which there is existence of minimizers to the Pohozaev problem are exactly the same as the ones in Chapter 2. This again gives us a necessary and sufficient condition for non-existence within this class of graphs that satisfy this assumption.

The appendices are designed to make the reading of this dissertation as self-contained as possible. The more frequently used results are stated there. Even though the proofs are omitted, references are provided. Appendix A contains results of measure theory and Sobolev embeddings. A quick overview of rearrangements of functions in $\mathbb{R}^N$ is also presented. Appendix B contains some notions of Differentiable Calculus in Banach spaces and the Theory of Lagrange Multipliers in Banach spaces as well. Finally, in Appendix C one can find results on the one dimensional NLS equation like regularity, existence and qualitative properties.

Chapter 2

A Constrained Minimization Problem in $\mathbb{R}$

Let us begin by studying the one dimensional NLS and clarifying some of the matters said in the introduction. For now let us assume all functions are complex valued. We are concerned with finding a solution in $H^1(\mathbb{R},\mathbb{C})$ to the problem

$$\begin{cases} i\partial_t u + u'' = \lambda |u|^{p-2}u, \\ u(0,x) = \varphi(x). \end{cases} \tag{2.1}$$

where u' denotes the spatial derivative of u, $\varphi \in H^1(\mathbb{R},\mathbb{C}) \setminus \{0\}$, $p > 2$ and $\lambda \in \mathbb{R}$. Henceforth we consider only the *subcritical case*, $\lambda > 0$ and $2 < p < 6$. Note that $\alpha = 6$ is, for us, the L^2-mass critical exponent. Suppose now a solution u to the above problem exists. Setting $v = \lambda^{\frac{1}{p-2}}u$ and plugging v into (2.1) we arrive at

$$i\partial_t u + u'' = |u|^{p-2}u.$$

Hence, without loss of generality, we may assume $\lambda = 1$ in (2.1) and thus in what follows the problem to be considered is

$$\begin{cases} i\partial_t u + u'' = |u|^{p-2}u, \\ u(0,x) = \varphi(x). \end{cases} \tag{2.2}$$

Before passing to the question of existence of solutions let us state the conservation laws of mass and energy mentioned in the introduction.

Proposition 2.1: *Conservation Laws of the NLS*

If u is a solution to the Cauchy Problem (2.2) then, in its interval of definition,

$$\frac{d}{dt}\left[\int_{\mathbb{R}} |u(x,t)|^2 dx\right] = 0,$$

and,

$$\frac{d}{dt}\left[\frac{1}{2}\int_{\mathbb{R}} |u'(x,t)|^2 dx - \frac{1}{p}\int_{\mathbb{R}} |u(x,t)|^p dx\right] = 0.$$

For a proof of this result we refer to [11, Section 7.4]. For smooth solutions the main idea of the proof is to multiply the equation by $\overline{u}$ and $\partial_t \overline{u}$ and integrate over $\mathbb{R}$. By taking the imaginary part in the first and the real part in the second we get both conservation laws.

We are now interested in constructing solutions of the form

$$u(t,x) = e^{i\omega t}\varphi(x) \tag{2.3}$$

where $\omega \in \mathbb{R}$ and $\varphi \in H^1(\mathbb{R},\mathbb{C})$. These solutions have physical interest because they are periodic in time. In the literature they are referred as *standing waves* or *stationary states*. It is clear that if u as above is a solution to (2.2) then φ has to solve the following stationary semilinear elliptic equation

$$-\varphi'' + \omega\varphi = |\varphi|^{p-2}\varphi. \tag{2.4}$$

This equation was intensively studied and it is known that for $\omega \leq 0$ no solutions exist. For a justification of this fact we refer the reader to [7] or [29]. Henceforth, we assume $\omega > 0$.

Note now that if a solution to (2.2) is of the form given in (2.3) then, since both the mass and the energy are conserved this yields that the energy and mass of the solution are exactly those of the function φ. We are now interested in discovering which of the solutions of (2.4), for a certain prescribed mass, minimize the energy functional of the NLS equation. These are to be called *ground states* or minimal energy solutions.

To state this problem rigorously let $\mu > 0$, $p \in (2, 6)$ and consider the nonlinear Schrödinger energy functional, $E : H^1(\mathbb{R}, \mathbb{C}) \to \mathbb{R}$, as follows:

$$\begin{aligned} E(u) &= \frac{1}{2}\|u'\|^2_{L^2(\mathbb{R},\mathbb{C})} - \frac{1}{p}\|u\|^p_{L^p(\mathbb{R},\mathbb{C})} \\ &= \frac{1}{2}\int_{\mathbb{R}} |u'|^2 dx - \frac{1}{p}\int_{\mathbb{R}} |u|^p dx. \end{aligned} \tag{2.5}$$

The functional is well defined since, from Corollary A.10, if $u \in H^1(\mathbb{R}, \mathbb{C})$ then $u \in L^q(\mathbb{R}, \mathbb{C})$ for $q \geq 2$.

As a consequence of the negative term in (2.5) and since $p > 2$, the functional E is clearly not bounded from below. This naturally creates a problem when looking for minimizers. Therefore we focus on the problem of minimizing E in the following subset of $H^1(\mathbb{R}, \mathbb{C})$:

$$H^1_\mu(\mathbb{R}, \mathbb{C}) := \left\{ u \in H^1(\mathbb{R}, \mathbb{C}) : \|u\|^2_{L^2(\mathbb{R},\mathbb{C})} = \mu \right\}.$$

That is, we look for solutions with prescribed mass. Thus, in a more compact way, our problem can be written as:

$$\text{find } u_0 \in H^1_\mu(\mathbb{R}, \mathbb{C}) \text{ such that } E(u_0) = \inf_{u \in H^1_\mu(\mathbb{R},\mathbb{C})} E(u). \tag{2.6}$$

Furthermore, should a minimizer exist, we wish to characterize it. The goals for this chapter are as follows. In the first section we concern ourselves with the existence of ground states followed by a characterization of them in the second section. In the final part of this chapter we focus mainly on scaling properties of the solutions to problem (2.6).

Let us start by establishing some notation. We will see in the second section of this chapter that if $\varphi \in H^1_\mu(\mathbb{R}, \mathbb{C})$ is a solution to the above problem then, up to phase multiplication by a complex constant, the function φ is a real valued function. Moreover, it is also straightforward to check that $E(|u|) \leq E(u)$ for all $u \in H^1_\mu(\mathbb{R}, \mathbb{C})$, hence

$$\inf_{u \in H^1_\mu(\mathbb{R},\mathbb{C})} E(u) = \inf_{u \in H^1_\mu(\mathbb{R},\mathbb{R})} E(u).$$

Therefore, there is no loss of generality when working with real valued functions. With that in mind we establish the following notations:

$$\begin{aligned} H^1(\mathbb{R}) &:= H^1(\mathbb{R}, \mathbb{R}), \\ H^1_\mu(\mathbb{R}) &:= H^1_\mu(\mathbb{R}, \mathbb{R}). \end{aligned}$$

The same reasoning holds for Lebesgue spaces.

2.1 Existence of Solution to the Minimization Problem

In this section, via the direct method of the calculus of variations, we prove existence of solution to problem (2.6). The fundamental idea of the direct method is to consider minimizing sequences, that is, a sequence in the space $H^1_\mu(\mathbb{R})$ such that the sequence $E(u_n)$ converges to the value of (2.6). However, as a consequence of the negative term in the functional E, *a priori*, the functional does not have to be bounded from below even when considering the subset $H^1_\mu(\mathbb{R})$, thus, such a sequence might not exist. Therefore, the first step is to prove that E is in fact bounded from below in $H^1_\mu(\mathbb{R})$.

Lemma 2.2

Let $\mu > 0$, $p \in (2,6)$. Then there exists $K = K(p,\mu) > 0$ such that

$$E(u) \geq \frac{1}{2}\|u'\|^2_{L^2(\mathbb{R})} - \frac{K}{p}\|u'\|^{\frac{p-2}{2}}_{L^2(\mathbb{R})} \quad \text{for all} \quad u \in H^1_\mu(\mathbb{R}). \tag{2.7}$$

In particular, the functional E is bounded from below in $H^1_\mu(\mathbb{R})$.

Proof. Let $u \in H^1_\mu(\mathbb{R})$. By Corollary A.4 we have, for $\alpha = \frac{p-2}{2p}$, the following estimate

$$\|u\|_{L^p(\mathbb{R})} \leq C\|u'\|^{\alpha}_{L^2(\mathbb{R})}\|u\|^{1-\alpha}_{L^2(\mathbb{R})}, \quad \text{for all } u \in H^1(\mathbb{R}),$$

for some positive constant $C = C(p)$. By restricting u to $H^1_\mu(\mathbb{R})$ we get that:

$$\|u\|^p_{L^p(\mathbb{R})} \leq K\|u'\|^{\frac{p-2}{2}}_{L^2(\mathbb{R})},$$

where $K = K(\mu,p)$ is a positive constant. Plugging this estimate into (2.5) we get the estimate (2.7). We can now use this to show that E is in fact bounded from below. Consider the auxiliary function $f : \mathbb{R}^+_0 \to \mathbb{R}$ given by $f(x) = \frac{1}{2}x^2 - \frac{K}{p}x^{\frac{p-2}{2}}$. For $p \in (2,6)$ we have that f is continuous and coercive, therefore the functional E is bounded from below. ∎

To conclude general considerations we prove the following result by using the same argument as in [31, Remark II.3.2].

Proposition 2.3: *Negativity of the infimum*

Let $\mu > 0$ and $I_\mu := \inf_{u \in H^1_\mu(\mathbb{R})} E(u)$. Then $I_\mu < 0$.

Proof. By proving that there exist admissible functions whose NLS energy is negative it follows immediately that $I_\mu < 0$. By reasons that will be better understood at the end of this chapter let us look for functions of the form

$$g_{a,c}(x) = ae^{-\frac{1}{2}(\frac{x}{c})^2} \in H^1(\mathbb{R})$$

with $a, c > 0$. Given a fixed we wish to find $c > 0$ such that $g_{a,c} \in H^1_\mu(\mathbb{R})$, that is, $\int_{\mathbb{R}} |g_{a,c}|^2 dx = \mu$. By performing a simple change of variables we get the following relation

$$\mu = ca^2 \int_{\mathbb{R}} e^{-y^2} dy = ca^2\sqrt{\pi}.$$

Thus, we have in fact that $c = c(a)$ is given by

$$c = \frac{\mu}{a^2\sqrt{\pi}}.$$

Hence, the above mass constraint allows us to take the following one parameter family of functions

$$g_a(x) = ae^{-\frac{1}{2}\left(\frac{x}{c(a)}\right)^2} \in H^1_\mu(\mathbb{R}).$$

By plugging the expression for these functions into the functional E we get

$$\begin{aligned} E(g_a) &= \frac{a^2}{2c^2}\int_{\mathbb{R}}\left(\frac{x}{c}\right)^2 e^{-\left(\frac{x}{c}\right)^2}dx - \frac{1}{p}a^p\int_{\mathbb{R}} e^{-\frac{p}{2}\left(\frac{x}{c}\right)^2}dx \\ &= \frac{a^2}{2c}\int_{\mathbb{R}} z^2e^{-z^2}dz - \frac{a^pc\sqrt{2}}{p\sqrt{p}}\int_{\mathbb{R}} e^{-z^2}dz \\ &= \frac{a^2\sqrt{\pi}}{2c} - \frac{a^pc\sqrt{2}}{p\sqrt{p}}\sqrt{\pi} = a^4\frac{\pi}{4\mu} - a^{p-2}\mu\sqrt{\frac{2}{p^3}}. \end{aligned}$$

By the assumption $p \in (2,6)$ we have that for a small enough the NLS energy of the functions g_a is strictly negative. ■

2.1.1 The Concentration-Compactness Lemma

As said before, the fundamental idea underlying the direct method of the calculus of variations is the use of minimizing sequences. For problem (2.6), with $\mu > 0$ fixed, a minimizing sequence is a sequence $(u_n)_{n\in\mathbb{N}} \subset H^1_\mu(\mathbb{R})$ that satisfies $E(u_n) \to I_\mu$, whose existence is guaranteed by Lemma 2.2. The key step of the direct method comes next: through compactness results one obtains a candidate for minimizer to the problem. If the same problem was posed in the case where the domain of the functions is a bounded subset of $\mathbb{R}$, things would become significantly simpler. The main reason for that is because of *Rellich-Kondrachov* compact embedding results, in particular, for the case of dimension one, Corollary A.9. When working in $\mathbb{R}$ this result does not hold so we need to find a way to regain compactness in order to get a possible candidate for solution to our problem. The way to do so is via the *Concentration-Compactness Principle*, which we will explain ahead. The main result that makes the whole principle work is the following concentration-compactness lemma due to *Pierre-Louis Lions*, see [23, Lemma I.1,Lemma III.1], which we now state and prove.

Lemma 2.4: *Concentration-Compactness Lemma*

Take $\mu > 0$ and let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $H^1(\mathbb{R})$ with $\|u_n\|^2_{L^2(\mathbb{R})} = \mu$. Then, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ satisfying one of the following three properties:

1. (Compactness) There exists a sequence $(y_k)_{k\in\mathbb{N}}$ of real numbers with the property that for all $\epsilon > 0$, there exists $T > 0$ such that
$$\int_{y_k-T}^{y_k+T} |u_{n_k}|^2dx \geq \mu - \epsilon \text{ for all } k \in \mathbb{N}.$$
2. (Vanishing) For all $t > 0$, one has:
$$\lim_{k\to+\infty}\sup_{y\in\mathbb{R}}\int_{y-t}^{y+t} |u_{n_k}|^2dx = 0.$$
3. (Dichotomy) There exist $\alpha \in (0,\mu)$ and sequences $(u_{k,1})_{k\in\mathbb{N}}$, $(u_{k,2})_{k\in\mathbb{N}}$ bounded in $H^1(\mathbb{R})$, such that:
   (a) $\|u_{n_k} - (u_{k,1} + u_{k,2})\|_{L^q(\mathbb{R})} \to 0$ as $k \to +\infty$ for $q \in [2,+\infty)$;
   (b) $\lim_{k\to+\infty}\int_{\mathbb{R}} |u_{k,1}|^2dx - \alpha = \lim_{k\to+\infty}\int_{\mathbb{R}} |u_{k,2}|^2dx - (\mu-\alpha) = 0$;
   (c) $\mathrm{dist}(\mathrm{supp}\ u_{k,1}, \mathrm{supp}\ u_{k,2}) \to +\infty$ as $k \to +\infty$;
   (d) $\liminf_{k\to+\infty}\int_{\mathbb{R}} |u'_{n_k}|^2 - |u'_{k,1}|^2 - |u'_{k,2}|^2dx \geq 0$.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $H^1(\mathbb{R})$ with $\|u_n\|^2_{L^2(\mathbb{R})} = \mu$, for all $n \in \mathbb{N}$. Given $n \in \mathbb{N}$ and $t \geq 0$, consider the *concentration function*

$$F_n(t) := \sup_{y\in\mathbb{R}}\int_{y-t}^{y+t} |u_n|^2dx. \tag{2.8}$$

This proof, being quite long, is divided in several steps.

Step 1: Properties of the Concentration Function

Fix $n \in \mathbb{N}$. We start by showing that the supremum in (2.8) is a maximum. To prove this claim let $t \geq 0$ be fixed. Take now $(y_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ a maximizing sequence. In search of a contradiction, suppose also that $|y_k| \to +\infty$ as $k \to +\infty$, otherwise the maximum is indeed attained by Lebesgue's dominated convergence, Theorem A.1. Passing to a subsequence if necessary we can assume that

$$(y_k - t, y_k + t) \cap (y_{k+1} - t, y_{k+1} + t) = \emptyset \quad \text{for all } k \in \mathbb{N}.$$

We then have the following

$$\mu = \int_{\mathbb{R}} |u_n|^2 dx \geq \int_{\cup_k (y_k - t, y_k + t)} |u_n|^2 dx = \sum_{k\in\mathbb{N}} \int_{y_k - t}^{y_k + t} |u_n|^2 dx.$$

Since μ is finite then the above series converges. This, together with the way the sequence (y_k) was chosen, yields

$$\lim_{k\to+\infty} \int_{y_k - t}^{y_k + t} |u_n|^2 dx = F_n(t) = 0.$$

However, since the integrand function is both positive and continuous, we get that $u_n \equiv 0$ which is a contradiction since $u_n \in H^1_\mu(\mathbb{R})$.

Note now that F_n is a real valued function defined in $\mathbb{R}^+_0$ which is non-decreasing and non-negative. The non-negativity follows by definition. As for the non-decreasing property note that for $s, t \geq 0$ such that $s > t$ we have $[y - t, y + t] \subset [y - s, y + s]$ for all $y \in \mathbb{R}$ and therefore $F_n(t) \geq F_n(s)$, for all $n \in \mathbb{N}$.

The sequence of functions $(F_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}^+_0)$ since

$$0 \leq F_n(t) \leq \int_{\mathbb{R}} |u_n|^2 dx = \mu, \text{ for all } n \in \mathbb{N}, t \geq 0. \tag{2.9}$$

Moreover, for fixed $n \in \mathbb{N}$, F_n is continuous. To check this let $s, t \geq 0$ and, without loss of generality, suppose that $s \leq t$. Then

$$\begin{aligned}
|F_n(t) - F_n(s)| &= \left| \max_{y\in\mathbb{R}} \int_{y-t}^{y+t} |u_n|^2 dx - \max_{y\in\mathbb{R}} \int_{y-s}^{y+s} |u_n|^2 dx \right| \\
&\leq \max_{y\in\mathbb{R}} \left| \int_{y-t}^{y+t} |u_n|^2 dx - \int_{y-s}^{y+s} |u_n|^2 dx \right| \\
&= \max_{y\in\mathbb{R}} \left| \int_{y-t}^{y-s} |u_n|^2 dx + \int_{y+s}^{y+t} |u_n|^2 dx \right| \leq 2K|t - s|.
\end{aligned}$$

The first inequality comes from the difference of maximums being smaller then the maximum of the difference, as well as passing the absolute value inside the maximum. The constant $K > 0$ comes from the embedding of $H^1(\mathbb{R})$ in $L^\infty(\mathbb{R})$. This, however, proves more than continuity. Since the constant in the previous estimate is independent of n we also get that (F_n) is an equicontinuous family of functions.

Step 2: Existence of three Mutually Exclusive Regimes

We now deduce the existence of a subsequence of (F_n) that converges pointwise to a non-negative and non-decreasing function $F : [0, +\infty) \to \mathbb{R}$. Our hope would be that we could apply directly *Arzelà-Ascoli*, see [17, Chapter 4], to the sequence (F_n). However, we cannot apply it to sequences of functions defined in non-compact domains. We perform a diagonal argument to extract a subsequence which converges only pointwise, which for our purposes is enough. The sequence can be constructed as follows. When restricting, for all $n \in \mathbb{N}$, the functions F_n to the interval $[0,1]$, they still remain equicontinuous and uniformly bounded and we are in conditions to apply Arzelà-Ascoli. Thus, there exists a subsequence $(F_{n_k})_{k\in\mathbb{N}}$ and $F^1 : [0,1] \to \mathbb{R}$ such that

$$F_{n_k} \to F^1 \quad \text{uniformly in } [0,1].$$

Let us denote this sequence by $F_k^1 := F_{n_k}$. Applying now to F_k^1 the same result but while restricting the functions to the interval $[0,2]$ we get another subsequence, F_k^2, and a function $F^2 : [0,2] \to \mathbb{R}$ which satisfies

$$F_k^2 \to F^2 \quad \text{uniformly in } [0,2].$$

Moreover, since this is a subsequence from the previous one we also have that: $F^1 = F^2$ in $[0,1]$. By repeating this argument for a general $N \in \mathbb{N}$, we have constructed a family of subsequences $(F_k^N)_{k,N\in\mathbb{N}}$ such that $F_k^N \to F^N$ uniformly in $[0,N]$ as $k \to \infty$ for all $N \in \mathbb{N}$ and such that the limit function F^N satisfies $F^N = F^{N+1}$ in $[0,N]$ for all $N \in \mathbb{N}$. We can now find a sequence $N_k \to \infty$ for which the elements $(F_k^{N_k})_{k\in\mathbb{N}}$ define a function $F : [0,+\infty) \to \mathbb{R}$ by:

$$F(t) = \lim_{k\to\infty} F_k^{N_k}(t).$$

It is immediate by construction that F is non-negative and non-decreasing.

Let now $\alpha := \lim_{t\to+\infty} F(t)$. By (2.9) we have that $\alpha \in [0,\mu]$. We now split our study in three cases: $\alpha = 0$, $\alpha \in (0,\mu)$ and $\alpha = \mu$, which give us the possible regimes.

Step 3: Determination of the Regimes

- **Case 1:** $\alpha = 0$;
  This is the easiest case. Since F is non-negative and non-decreasing then $\alpha = 0$ implies that $F \equiv 0$. By definition of F the vanishing regime is the one that holds in this case.
- **Case 2:** $\alpha = \mu$;
  We prove that in this case compactness holds. Firstly note that there exists $t_0 > 0$ such that

  $$F(t) \geq F(t_0) > \frac{\mu}{2} \quad \text{for all } t \geq t_0.$$

  By discarding some elements of the sequence F_{n_k} we can assume that:

  $$F_{n_k}(t) \geq F_{n_k}(t_0) > \frac{\mu}{2}, \quad \text{for all } t \geq t_0 \text{ and for all } k \in \mathbb{N}.$$

  Take now $(y_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ to be such that

  $$F_{n_k}(t_0) = \int_{y_k-t_0}^{y_k+t_0} |u_{n_k}|^2 dx. \tag{2.10}$$

  We prove that this sequence satisfies the compactness property. Let now $\epsilon > 0$ be arbitrary. Then there exist $t_1 > 0$ and $k_0 \in \mathbb{N}$ such that

  $$F_{n_k}(t) \geq F_{n_k}(t_1) > \mu - \epsilon \quad \text{for all } k \geq k_0 \text{ and } t \geq t_1.$$

  Now take a sequence $(z_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ for which

  $$F_{n_k}(t_1) = \int_{z_k-t_1}^{z_k+t_1} |u_{n_k}|^2 dx > \mu - \epsilon. \tag{2.11}$$

  From the equations (2.10) and (2.11) we can get that, for ϵ small enough, say $\epsilon < \frac{\mu}{2}$,

  $$(z_k - t_1, z_k + t_1) \cap (y_k - t_0, y_k + t_0) \neq \emptyset.$$

Indeed, if they were disjoint then, for each $k \in \mathbb{N}$,

$$\mu = \int_{\mathbb{R}} |u_{n_k}|^2 dx \geq \int_{z_k - t_1}^{z_k + t_1} |u_{n_k}|^2 dx + \int_{y_k - t_0}^{y_k + t_0} |u_{n_k}|^2 dx > (\mu - \epsilon) + \frac{\mu}{2} > \mu,$$

which is a contradiction.
Taking now $T := t_0 + 2t_1$, since $[z_k - t_1, z_l + t_1] \subset [y_k - t_0, y_k + t_0]$, we get easily that, for all $k \geq k_0$,

$$\int_{y_k - T}^{y_k + T} |u_{n_k}|^2 dx \geq \int_{z_k - t_1}^{z_k + t_1} |u_{n_k}|^2 dx > \mu - \epsilon.$$

By discarding the initial elements of the sequence and re indexing it we have compactness.

- **Case 3:** $\alpha \in (0, \mu)$;
  We now prove that, in this last case, dichotomy holds. As seen in [31, Lemma II.3.5] we can choose a sequence $(t_k)_{k \in \mathbb{N}}$ such that $F_{n_k}(t_k) \to \alpha$ and $F_{n_k}(8t_k) \to \alpha$ as $k \to +\infty$. Our goal now is to construct the sequences $(u_{k,1})_{k \in \mathbb{N}}$ and $(u_{k,2})_{k \in \mathbb{N}}$. Take θ, φ smooth cutoff functions such that $0 \leq \varphi, \theta \leq 1$ and

$$\begin{cases} \theta(x) = 1, \varphi(x) = 0, \text{ if } |x| \leq 1, \\ \theta(x) = 0, \varphi(x) = 1, \text{ if } |x| \geq 2. \end{cases}$$

Define the rescaling $\theta_\lambda(x) = \theta(\frac{x}{\lambda})$ and $\varphi_\lambda(x) = \varphi(\frac{x}{\lambda})$. Take $(y_k)_{k \in \mathbb{N}}$ to be such that, for each $k \in \mathbb{N}$,

$$F_{n_k}(t_k) = \int_{y_k - t_k}^{y_k + t_k} |u_{n_k}|^2 dx.$$

With this sequence we now define as well

$$u_{k,1}(\cdot) := \theta_{t_k}(\cdot + y_k) u_{n_k}, \; u_{k,2}(\cdot) := \varphi_{4t_k}(\cdot + y_k) u_{n_k}, \quad \text{for all } k \in \mathbb{N}. \tag{2.12}$$

We now prove 3.(c). By definition of the sequences $(u_{k,1})$ and $(u_{k,2})$ and their supports that for each fixed $k \in \mathbb{N}$

$$\text{supp } u_{k,1} \subset (y_k - 2t_k, y_k + 2t_k) \tag{2.13}$$

and

$$\text{supp } u_{k,2} \subset (-\infty, y_k - 4t_k) \cup (y_k + 4t_k, +\infty). \tag{2.14}$$

Thus,

$$\text{dist}(\text{supp } u_{k,1}, \text{supp } u_{k,2}) > t_k \to \infty \text{ as } k \to \infty.$$

To prove 3.(a) start by fixing $k \in \mathbb{N}$ and noting that, by construction, $u_{n_k} - (u_{k,1} + u_{k,2})$ is uniformly bounded in $H^1(\mathbb{R})$ and hence in $L^\infty(\mathbb{R})$. As a consequence of the interpolation inequality (A.1) in Corollary A.10 we only need to prove that

$$\int_{\mathbb{R}} |u_{n_k} - (u_{k,1} + u_{k,2})|^2 dx \to 0.$$

Using the definition of the sequences $(u_{k,1})$, $(u_{k,2})$ and equations (2.13) and (2.14) we have, for every fixed $k \in \mathbb{N}$, that

$$\begin{aligned} \int_{\mathbb{R}} |u_{n_k} - (u_{k,1} + u_{k,2})|^2 dx &= \int_{y_k - 8t_k}^{y_k - 4t_k} |u_{n_k} - u_{k,2}|^2 dx + \int_{y_k - 4t_k}^{y_k - 2t_k} |u_{n_k}|^2 dx + \\ &+ \int_{y_k - 2t_k}^{y_k + 2t_k} |u_{n_k} - u_{k,1}|^2 dx + \int_{y_k + 2t_k}^{y_k + 4t_k} |u_{n_k}|^2 dx + \int_{y_k + 4t_k}^{y_k + 8t_k} |u_{n_k} - u_{k,2}|^2 dx \\ &\leq \int_{\{x \in \mathbb{R} : |x - y_k| \in (t_k, 8t_k)\}} |u_{n_k}|^2 dx = \int_{y_k - 8t_k}^{y_k + 8t_k} |u_{n_k}|^2 dx - \int_{y_k - t_k}^{y_k + t_k} |u_{n_k}|^2 dx. \end{aligned}$$

Hence,

$$\int_{\mathbb{R}} |u_{n_k} - (u_{k,1} + u_{k,2})|^2 dx \le F_{n_k}(8t_k) - F_{n_k}(t_k).$$

Taking the limit as $k \to +\infty$ concludes the proof of 3.(a).
Let us now prove 3.(b). Note that by fixing $k \in \mathbb{N}$ we can use the support of the functions $u_{k,1}$ and $u_{k,2}$ to deduce that:

$$\int_{y_k - t_k}^{y_k + t_k} |u_{n_k}|^2 dx \le \int_{\mathbb{R}} |u_{k,1}|^2 dx \le \int_{y_k - 8t_k}^{y_k + 8t_k} |u_{n_k}|^2 dx$$

and similarly we have that

$$\int_{\mathbb{R}} |u_{n_k}|^2 dx - \int_{y_k - 8t_k}^{y_k + 8t_k} |u_{n_k}|^2 dx \le \int_{\mathbb{R}} |u_{k,2}|^2 dx \le \int_{\mathbb{R}} |u_{n_k}|^2 dx - \int_{y_k - t_k}^{y_k + t_k} |u_{n_k}|^2 dx.$$

By the way the sequence (y_k) was taken and the definition of the function F we get that

$$F_{n_k}(t_k) \le \int_{\mathbb{R}} |u_{k,1}|^2 dx \le F_{n_k}(8t_k)$$

and

$$\mu - F_{n_k}(8t_k) \le \int_{\mathbb{R}} |u_{k,2}|^2 dx \le \mu - F_{n_k}(t_k).$$

Thus, taking the limit as $k \to +\infty$ in both of the above inequalities yields

$$\lim_{k\to\infty} \int_{\mathbb{R}} |u_{k,1}|^2 dx = \alpha \quad \text{and} \quad \lim_{k\to\infty} \int_{\mathbb{R}} |u_{k,2}|^2 dx = \mu - \alpha.$$

Finally, to finish the proof, we wish to show 3.(d). Let $M := \sup_n \|u_n\|_{H^1(\mathbb{R})}$, which is finite by assumption. Note now that, for all $y \in \mathbb{R}$, we have the following estimates.

$$\left| \int_{\mathbb{R}} |(\theta_t(x+y) u_{n_k}(x))'|^2 - \theta_t^2(x+y) |u_{n_k}'(x)|^2 dx \right| \le \frac{C}{t^2}, \tag{2.15}$$

where $C = C(M)$. Indeed, one has:

$$\begin{aligned}
&\left| \int_{\mathbb{R}} |(\theta_t(x+y) u_{n_k}(x))'|^2 - \theta_t^2(x+y) |u_{n_k}'(x)|^2 dx \right| = \\
&= \left| \int_{\mathbb{R}} \left| \frac{1}{t} \theta' \left(\frac{x+y}{t} \right) u_{n_k}(x) + \theta_t(x+y) u_{n_k}'(x) \right|^2 - \theta_t^2(x+y) |u_{n_k}'(x)|^2 dx \right| \\
&\le \frac{1}{t^2} \int_{\mathbb{R}} \left| \theta' \left(\frac{x+y}{t} \right) u_{n_k}(x) \right|^2 + \frac{2}{t} \theta_t(x+y) |u_{n_k}'(x)| |\theta' \left(\frac{x+y}{t} \right)| |u_{n_k}(x)| dx = \\
&\le \frac{C}{t^2} \|u_{n_k}\|_{L^2(\mathbb{R})}^2 + \frac{2}{t^2} \left(\int_{\mathbb{R}} \theta_t^2(x+y) |u_{n_k}'|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \theta'^2 \left(\frac{x-y}{t} \right) |u_{n_k}|^2 \right)^{\frac{1}{2}} \\
&\le \frac{C}{t^2} \mu + \frac{C}{t^2} \sqrt{\mu} M = \frac{C}{t^2}.
\end{aligned}$$

Similarly, one has

$$\left| \int_{\mathbb{R}} |(\varphi_t(x+y) u_{n_k}(x))'|^2 - \varphi_t^2(x+y) |u_{n_k}'(x)|^2 dx \right| \le \frac{C}{t^2}. \tag{2.16}$$

Fix now $k \in \mathbb{N}$. Given that $|\theta|, |\varphi| \leq 1$ and that the supports of θ_{t_k} and φ_{4t_k} are disjoint we deduce that

$$\int_{\mathbb{R}} \theta_{t_k}^2(x+y_k)|u'_{n_k}|^2 + \varphi_{4t_k}^2(x+y_k)|u'_{n_k}|^2 dx \leq \int_{y_k+\operatorname{supp}\theta_{t_k}} |u'_{n_k}|^2 dx + \int_{y_k+\operatorname{supp}\varphi_{4t_k}} |u'_{n_k}|^2 dx$$
$$\leq \int_{\mathbb{R}} |u'_{n_k}|^2 dx.$$

Now, using the estimates (2.15) and (2.16) with $t = t_k$ and $t = 4t_k$, repectively, we have that

$$\begin{aligned}
\int_{\mathbb{R}} |u'_{n_k}|^2 dx &\geq \int_{\mathbb{R}} \theta_{t_k}^2(x+y_k)|u'_{n_k}|^2 + \varphi_{4t_k}^2(x+y_k)|u'_{n_k}|^2 dx \\
&= \int_{\mathbb{R}} \theta_{t_k}^2(x+y_k)|u'_{n_k}|^2 + |u'_{k,1}|^2 - |u'_{k,1}|^2 + \varphi_{4t_k}^2(x+y_k)|u'_{n_k}|^2 \\
&\quad + |u'_{k,2}|^2 - |u'_{k,2}|^2 dx \\
&= \int_{\mathbb{R}} |u'_{k,1}|^2 dx - \int_{\mathbb{R}} \left(\left|(\theta_{t_k}(x+y_k)u_{n_k}(x))'\right|^2 - \theta_{t_k}^2(x+y_k)|u'_{n_k}|^2 \right) dx \\
&\quad + \int_{\mathbb{R}} |u'_{k,2}|^2 dx - \int_{\mathbb{R}} \left(\left|(\varphi_{4t_k}(x+y_k)u_{n_k}(x))'\right|^2 - \varphi_{4t_k}^2(x+y_k)|u'_{n_k}|^2 \right) dx \\
&\geq \int_{\mathbb{R}} |u'_{k,1}|^2 + |u'_{k,2}|^2 dx + O\left(\frac{1}{t_k^2}\right).
\end{aligned}$$

Passing the integrals in the last inequality to the left hand side and taking the limit as $k \to \infty$ gives us the desired inequality.

■

Remark 2.5:

1. It is important to note that this result can be adapted to higher dimensions, see for example [23], [24] and [31].
2. It is clear from the statement that only one of the three alternatives can occur each time.
3. The construction of the sequences $u_{k,1}$ and $u_{k,2}$ in (2.12) will be crucial ahead. In particular, the estimate

$$\int_{\mathbb{R}} |u'_{n_k}|^2 dx \geq \int_{\mathbb{R}} |u'_{k,1}|^2 + |u'_{k,2}|^2 dx + O\left(\frac{1}{t_k^2}\right), \tag{2.17}$$

where $t_k \to \infty$ as $k \to \infty$. This in fact can be used to show that we can split the sequence u_{n_k} in two sequences without any essential loss of energy. In fact, it follows from equation (2.17) that

$$E(u_{n_k}) \geq \frac{1}{2}\left(\int_{\mathbb{R}} |u'^1_k|^2 dx + \int_{\mathbb{R}} |u'^2_k|^2 dx + O\left(\frac{1}{t_k^2}\right)\right) - \frac{1}{p}\int_{\mathbb{R}} |u_{n_k}|^p dx.$$

Moreover, note that $\|u_{n_k} - (u_{k,1} + u_{k,2})\|_{L^q(\mathbb{R})} \to 0$ as $k \to +\infty$ for $q \in [2, +\infty)$, therefore we can choose a sequence $(a_k)_{k\in\mathbb{N}} \subset \mathbb{R}$, with $a_k \to 0$ as $k \to +\infty$, and such that, for each k,

$$\int_{\mathbb{R}} |u_{n_k}|^p dx + a_k = \int_{\mathbb{R}} |u_{k,1} + u_{k,2}|^p dx = \int_{\mathbb{R}} |u_{k,1}|^p dx + \int_{\mathbb{R}} |u_{k,2}|^p dx.$$

The last equality follows from using the disjointness of the supports. From here it follows that

$$E(u_{n_k}) \geq E(u_{k,1}) + E(u_{k,2}) + o(1), \tag{2.18}$$

as $k \to \infty$.

2.1.2 Compactness Regained

In this subsection we will illustrate how to apply the *Concentration-Compactness Principle* to regain compactness of the minimizing sequences for problem (2.6), yielding the existence of a minimizer, our main objective. This technique, as seen in [23] and [24] through numerous examples, was proven to be fundamental to solve problems in calculus of variations. The same references show that the steps which one uses when applying this technique in different contexts are essentially the same. Below we give a quick outline for simpler cases.

We begin by proving the following lemma:

Lemma 2.6: *Strict Subadditivity of I_μ.*

For all $\alpha \in (0, \mu)$, one has
$$I_\mu < I_\alpha + I_{\mu-\alpha}. \tag{2.19}$$

Proof. By symmetry, without loss of generality, one can assume that $\alpha \in [\frac{\mu}{2}, \mu)$. Now fix $\theta \in (1, \frac{\mu}{\alpha}]$. We then have that

$$\begin{aligned} I_{\theta\alpha} &= \inf_{\|u\|^2_{L^2(\mathbb{R})}=\theta\alpha} \left\{ \frac{1}{2}\int_{\mathbb{R}} |u'|^2 dx - \frac{1}{p}\int_{\mathbb{R}} |u|^p dx \right\} \\ &= \theta \inf_{\|v\|^2_{L^2(\mathbb{R})}=\alpha} \left\{ \frac{1}{2}\int_{\mathbb{R}} |v'|^2 dx - \frac{\theta^{(p-2)/2}}{p}\int_{\mathbb{R}} |v|^p dx \right\} \\ &< \theta \inf_{\|v\|^2_{L^2(\mathbb{R})}=\alpha} \left\{ \frac{1}{2}\int_{\mathbb{R}} |v'|^2 dx - \frac{1}{p}\int_{\mathbb{R}} |v|^p dx \right\} = \theta I_\alpha, \end{aligned}$$

where in the second equality the change of variable $u = \theta^{\frac{1}{2}} v$ was used. The inequality above can be justified by the fact that $\theta > 1$ and that the exponent $\frac{p-2}{2}$ is also strictly positive from our assumption on p. By choosing $\theta = \frac{\mu}{\alpha} > 1$, it follows that

$$I_\mu < \frac{\mu}{\alpha} I_\alpha = I_\alpha + \frac{\mu-\alpha}{\alpha} I_\alpha.$$

Now note that $\frac{\mu-\alpha}{\alpha} \le 1$. Thus,

$$\begin{aligned} \frac{\mu-\alpha}{\alpha} I_\alpha &= \inf_{\|u\|^2_{L^2(\mathbb{R})}=\alpha} \left\{ \frac{1}{2}\int_{\mathbb{R}} \left| \left(\frac{\mu-\alpha}{\alpha}\right)^{\frac{1}{2}} u' \right|^2 dx - \frac{(\frac{\mu-\alpha}{\alpha})^{1-\frac{p}{2}}}{p}\int_{\mathbb{R}} \left| \left(\frac{\mu-\alpha}{\alpha}\right)^{\frac{1}{2}} u \right|^p dx \right\} \\ &= \inf_{\|v\|^2_{L^2(\mathbb{R})}=\mu-\alpha} \left\{ \frac{1}{2}\int_{\mathbb{R}} |v'|^2 dx - \frac{(\frac{\mu-\alpha}{\alpha})^{1-\frac{p}{2}}}{p}\int_{\mathbb{R}} |v|^p dx \right\} \le I_{\mu-\alpha}, \end{aligned}$$

where the inequality comes from the fact that the exponent $1-\frac{p}{2}$ is negative. Hence, we obtain (2.19). ■

We now outline, in an informal way, how to apply the Concentration-Compactness Principle.

1. To begin we need to prove a sub-additivity condition like the one in (2.19).
2. Take a minimizing sequence. After confirming that the assumptions on Lemma 2.4 are satisfied, we proceed to prove that only the *compactness* property of said lemma is satisfied. The subadditivity condition proved in step 1 is crucial to rule out the *dichotomy* regime - item 3 in Lemma 2.4.
3. The last step is to conclude that there exists a minimizer. By looking at the numerous examples in [23] and [24] we see that this step is different from problem to problem. It depends not only on the functional one is minimizing but also on the space where the minimization is being done. The dimension also plays an important role due to the possible Sobolev embeddings it provides. One example where this dependence is explicit is [24, Lemma I.1], which in higher dimensions is crucial to rule out the vanishing regime, together with some interpolation results. In Lemma 2.10 we apply the ideas of this proof in dimension 1.

Finally, note that this outline is for the simpler cases. In fact, it was shown in [23] and [24] that this method can be used to obtain existence of minimizers for problems with more general nonlinearities and potentials. The main result of this section is the following:

Theorem 2.7: *Compactness of minimizing sequences*

Let $p \in (2,6)$ and $\mu > 0$. Then, for any minimizing sequence $(u_n)_{n\in\mathbb{N}}$ of problem (2.6) there exists $(y_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ and $u \in H^1(\mathbb{R})$ such that, up to a subsequence, $u_n(\cdot + y_n) \to u$ strongly in $H^1(\mathbb{R})$ and u is a minimizer.

In order to simplify the proof we will divide it in several lemmas.

Lemma 2.8: *Minimizing sequences are bounded in $H^1(\mathbb{R})$*

Let $\mu > 0$ and let $(u_n)_{n\in\mathbb{N}} \subset H^1_\mu(\mathbb{R})$ be a minimizing sequence to the problem (2.6). Then u_n is bounded in $H^1(\mathbb{R})$.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence. Suppose, in search of a contradiction, that $\|u_n\|_{H^1(\mathbb{R})} \to \infty$ as $n \to \infty$. Since $(u_n)_{n\in\mathbb{N}} \subset H^1_\mu(\mathbb{R})$ we have that $\|u_n'\|_{L^2(\mathbb{R})} \to \infty$ as $n \to \infty$. Then, from (2.7), and since $2 > \frac{p-2}{2}$, $E(u_n) \to +\infty$ which, according to Proposition 2.3, is a contradiction. ■

Lemma 2.9: *Dichotomy does not occur*

Let $\mu > 0$ and $(u_n)_{n\in\mathbb{N}} \subset H^1_\mu(\mathbb{R})$ be a minimizing sequence to the problem (2.6). Then item 3 in Lemma 2.4 does not occur.

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset H^1_\mu(\mathbb{R})$ be a minimizing sequence. By Lemma 2.8 we know that u_n is bounded in $H^1(\mathbb{R})$. Suppose, in search of a contradiction, that dichotomy occurs. Then there exist a subsequence (u_{n_k}) of (u_n), $\alpha \in (0,\mu)$ and sequences $(u_{k,1})_{k\in\mathbb{N}}$ and $(u_{k,2})_{k\in\mathbb{N}}$, defined as in (2.12), for which the properties (a) through (d) in item 3 of Lemma 2.4 are satisfied.

Take now $(\alpha_k)_{k\in\mathbb{N}}$ and $(\beta_k)_{k\in\mathbb{N}}$ sequences of positive real numbers such that

$$\|\alpha_k u_{k,1}\|^2_{L^2(\mathbb{R})} = \alpha, \quad \|\beta_k u_{k,2}\|^2_{L^2(\mathbb{R})} = \mu - \alpha, \quad \text{for all } k \in \mathbb{N}. \tag{2.20}$$

It follows easily from the above equalities that

$$\lim_{k\to+\infty} \alpha_k = \lim_{k\to+\infty} \beta_k = 1. \tag{2.21}$$

Now consider the estimate (2.18), that is,

$$E(u_{n_k}) \geq E(u_{k,1}) + E(u_{k,2}) + o(1) \ \text{ as } \ k \to \infty.$$

Multiplying and dividing, conveniently, by α_k and β_k the last term in the previous inequality yields

$$\begin{aligned} E(u_{n_k}) = &\frac{1}{2\alpha_k^2}\|\alpha_k u'_{k,1}\|^2_{L^2(\mathbb{R})} - \frac{1}{p\alpha_k^p}\|\alpha_k u_{k,1}\|^p_{L^p(\mathbb{R})} + \\ &+ \frac{1}{2\beta_k^2}\|\beta_k u'_{k,2}\|^2_{L^2(\mathbb{R})} - \frac{1}{p\beta_k^p}\|\beta_k u_{k,2}\|^p_{L^p(\mathbb{R})} + o(1). \end{aligned} \tag{2.22}$$

From (2.21) we get that $\frac{1}{2\alpha_k^2} = \frac{1}{2} + o(1)$ and $\frac{1}{p\alpha_k^p} = \frac{1}{p} + o(1)$ as $k \to \infty$ and similarly for β_k. Moreover, since all the L^2 and L^p norms in (2.22) are bounded, we have that

$$E(u_{n_k}) \geq E(\alpha_k u_{k,1}) + E(\beta_k u_{k,2}) + o(1), \ \text{ as } \ k \to \infty.$$

From (2.20) we also have that $E(\alpha_k u_{k,1}) \geq I_\alpha$ and $E(\beta_k u_{k,2}) \geq I_{\mu-\alpha}$. Thus,

$$E(u_{n_k}) \geq I_\alpha + I_{\mu-\alpha} + o(1), \quad \text{as} \quad k \to \infty. \tag{2.23}$$

Taking the limit on the previous inequality we have

$$I_\mu \geq I_\alpha + I_{\mu-\alpha},$$

which, according to Lemma 2.6, is a contradiction. ■

Lemma 2.10: *Vanishing does not occur*

Let $\mu > 0$ and $(u_n)_{n\in\mathbb{N}} \subset H^1_\mu(\mathbb{R})$ be a minimizing sequence to the problem (2.6). Then item 2 in Lemma 2.4 does not occur.

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset H^1_\mu(\mathbb{R})$ be a minimizing sequence. By Lemma 2.8 we know that u_n is bounded in $H^1(\mathbb{R})$. Suppose, in search of a contradiction, that vanishing occurs for some subsequence (u_{n_k}) of (u_n). We claim that

$$\|u_{n_k}\|^p_{L^p(\mathbb{R})} \to 0 \quad \text{as } k \to +\infty.$$

Note that if this happens then

$$I_\mu = \lim_{k\to\infty} E(u_{n_k}) \geq 0$$

which, according to Proposition 2.3, is a contradiction.

Let $z \in \mathbb{Z}$. It follows from the interpolation inequality (A.1) and Sobolev embeddings that

$$\|u\|^4_{L^4(z-1,z+1)} \leq K\|u\|^2_{H^1(z-1,z+1)}\|u\|^2_{L^2(z-1,z+1)}, \quad \text{for all} \quad u \in H^1(\mathbb{R}). \tag{2.24}$$

Then, since the union of the intervals $(z-1, z+1)$ with $z \in \mathbb{Z}$ covers $\mathbb{R}$ and any $x \in \mathbb{R}$ is in at most 2 of these intervals we have

$$\|u_{n_k}\|^4_{L^4(\mathbb{R})} \leq \sum_{z\in\mathbb{Z}} \|u_{n_k}\|^4_{L^4(z-1,z+1)} \leq K \sup_{y\in\mathbb{R}} \|u_{n_k}\|^2_{L^2(y-1,y+1)} \sum_{z\in\mathbb{Z}} \|u_{n_k}\|^2_{H^1(z-1,z+1)}.$$

From the way the intervals $(z-1, z+1)$ were chosen we have that

$$\sum_{z\in\mathbb{Z}} \|u_{n_k}\|^2_{H^1(z-1,z+1)} = 2\|u_{n_k}\|^2_{H^1(\mathbb{R})},$$

hence,

$$\|u_{n_k}\|^4_{L^4(\mathbb{R})} \leq 2K \sup_{y\in\mathbb{R}} \|u_{n_k}\|^2_{L^2(y-1,y+1)}\|u_{n_k}\|^2_{H^1(\mathbb{R})}.$$

By the vanishing assumption we then have that $\|u_{n_k}\|_{L^4(\mathbb{R})} \to 0$ as $k \to \infty$. Now, from (A.1) we know that L^p norms converge to zero for $p \geq 4$. By interpolation the same happens for $p \in [2, 4]$, therefore

$$\|u_{n_k}\|^p_{L^p(\mathbb{R})} \to 0 \quad \text{as} \quad k \to \infty, \quad \text{for all} \quad p > 2.$$

■

Remark 2.11:
This proof is, for dimension 1, a particular case of Lemma I.1 shown in [24].

Now we proceed to the proof of Theorem 2.7.

Proof of Theorem 2.7. We now follow the steps 1 through 3 mentioned before Theorem 2.7.

Step 1 follows by Lemma 2.6. The hardest step is number 2, to show that we are in the compactness regime. This problem was solved by Lemmas 2.9 and 2.10. We now conclude the proof, by showing that any minimizing sequence converges strongly in $H^1(\mathbb{R})$, up to a subsequence, to a minimizer. It follows from the first item in Lemma 2.4 that there exists $(y_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ such that, for all $\epsilon > 0$, there exists $T > 0$ such that

$$\int_{y_k-T}^{y_k+T} |u_{n_k}|^2 dx \geq \mu - \epsilon \ \text{ for all } \ k \in \mathbb{N}. \tag{2.25}$$

Moreover, we also have that

$$\int_{\mathbb{R}\setminus(y_k-T,y_k+T)} |u_{n_k}|^2 dx \leq \epsilon. \tag{2.26}$$

We now define $v_k := u_{n_k}(\cdot + y_k)$, for $k \in \mathbb{N}$, and apply a diagonal argument to extract a subsequence that converges in $L^2(\mathbb{R})$. Start by recalling that by construction we have that (v_k) is bounded in $H^1(\mathbb{R})$, therefore up to a subsequence, which we still denote by (v_k), converges weakly to some $v \in H^1(\mathbb{R})$. It follows from the above weak convergence and *Rellich-Kondrachov* that, up to a subsequence

$$v_k \to v \quad \text{in } L^2_{loc}(\mathbb{R}). \tag{2.27}$$

Let now $\epsilon_n = \frac{1}{n}$, where $n \in \mathbb{N}$. Then, from (2.25) and (2.26) we have that there exists $T_n > 0$ such that, for all $k \in \mathbb{N}$,

$$\int_{-T_n}^{T_n} |v_k|^2 dx \geq \mu - \frac{1}{n} \tag{2.28}$$

and

$$\int_{\mathbb{R}\setminus(-T_n,T_n)} |v_k|^2 dx \leq \frac{1}{n}. \tag{2.29}$$

Now note that by (2.27) we have that

$$\int_{-T_n}^{T_n} |v_k - v|^2 dx \to 0 \quad \text{as } k \to +\infty \ \text{ for all } \ n \in \mathbb{N}.$$

From this, we can now conclude that in fact $v_k \to v$ in $L^2(\mathbb{R})$. Indeed note that, for each $n \in \mathbb{N}$,

$$\|v_k - v\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |v_k - v|^2 dx = \int_{-T_n}^{T_n} |v_k - v|^2 dx + \int_{\mathbb{R}\setminus(-T_n,T_n)} |v_k - v|^2 dx$$
$$\leq \int_{-T_n}^{T_n} |v_k - v|^2 dx + 2\int_{\mathbb{R}\setminus(-T_n,T_n)} |v_{n_k}|^2 + |v|^2 dx$$

Taking the limit as $k \to \infty$ yields

$$\limsup_{k\to+\infty} \|v_k - v\|^2_{L^2(\mathbb{R})} = \limsup_{k\to+\infty} \int_{\mathbb{R}\setminus(-T_n,T_n)} |v_{n_k} - v|^2 dx \leq 2\limsup_{k\to+\infty} \int_{\mathbb{R}\setminus(-T_n,T_n)} |v_{n_k}|^2 + |v|^2 dx \leq \frac{4}{n},$$

for all $n \in \mathbb{N}$, where we also used that $\int_{\mathbb{R}\setminus(-T_n,T_n)} |v|^2 \leq \frac{1}{n}$. By letting $n \to +\infty$ we get that

$$\|v_k - v\|_{L^2(\mathbb{R})} \to 0, \quad \text{as } \ k \to \infty.$$

In particular, we have $\|v\|^2_{L^2(\mathbb{R})} = \mu$ and therefore $v \in H^1_\mu(\mathbb{R})$ and $I_\mu \leq E(v)$. From here on the proof is straight forward. Note that since $v_k \rightharpoonup v$ in $H^1(\mathbb{R})$ then it is uniformly bounded, which together with the interpolation inequality (A.1), yields

$$v_k \to v \quad \text{in } L^p(\mathbb{R}).$$

We also have from the weak convergence in $H^1(\mathbb{R})$ that

$$\int_{\mathbb{R}} |v'|^2 dx \leq \liminf_{k\to+\infty} \int_{\mathbb{R}} |v_k'|^2 dx.$$

It then follows that

$$I_\mu \leq E(v) \leq \liminf_{k\to+\infty} E(v_k) = I_\mu.$$

Hence v is a minimizer. To finish we show that $v_k \to v$ in $H^1(\mathbb{R})$. Since $E(v_k) \to E(v)$ and $\|v_k\|_{L^p(\mathbb{R})} \to \|v\|_{L^p(\mathbb{R})}$ we have that $\|v_k'\|_{L^2(\mathbb{R})} \to \|v'\|_{L^2(\mathbb{R})}$, thus $\|v_k\|_{H^1(\mathbb{R})} \to \|v\|_{H^1(\mathbb{R})}$. From the convergence of norms together with the weak convergence in $H^1(\mathbb{R})$ we get the desired strong convergence. ■

2.2 Characterization of Minimizers

In the previous section we proved that solutions to the problem (2.6) do exist. We now focus on a more delicate problem, that of characterizing them. For this purpose, concepts of differentiable calculus in Banach spaces will be required. We refer the reader to Appendix B.

Let us begin with an auxiliary result:

Lemma 2.12: *E is a functional of class C^1*

Let $p \in (2, 6)$ and consider the functional E defined by

$$E : H^1(\mathbb{R}) \to \mathbb{R};\ E(u) = \frac{1}{2}\|u'\|^2_{L^2(\mathbb{R})} - \frac{1}{p}\|u\|^p_{L^p(\mathbb{R})}.$$

Then $E \in C^1(H^1(\mathbb{R}))$. Moreover, given any point $u \in H^1(\mathbb{R})$, the differential of E at u, is given by:

$$E'(u)v = \int_{\mathbb{R}} u'v' - |u|^{p-2}uv dx, \quad \text{for all } \ v \in H^1(\mathbb{R}).$$

Proof. Let $E = J - K$, where both J and K are the functionals defined by:

$$J : H^1(\mathbb{R}) \to \mathbb{R};\ J(u) = \frac{1}{2}\|u'\|^2_{L^2(\mathbb{R})} \ \text{ and } \ K : H^1(\mathbb{R}) \to \mathbb{R};\ K(u) = \frac{1}{p}\|u\|^p_{L^p(\mathbb{R})}.$$

Step 1: Differentiability of J:

Note that J can be seen as the quadratic form of the bilinear continuous and symmetric form defined in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ by $a(u, v) = \frac{1}{2}\int_{\mathbb{R}} u'v'dx$. The continuity of a is an immediate consequence of the Cauchy-Schwartz inequality. Therefore J is differentiable in $H^1(\mathbb{R})$ and its differential at a point $u \in H^1(\mathbb{R})$ is given by:

$$J'(u)v = \int_{\mathbb{R}} u'v'dx \ \text{ for all } \ v \in H^1(\mathbb{R}).$$

The only thing left to check is that the derivative of J, that is, the map $J' : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$, is continuous. Recall that $H^{-1}(\mathbb{R})$ is by definition the dual space of $H^1(\mathbb{R})$.
Let $u, v, w \in H^1(\mathbb{R})$. Then,

$$\begin{aligned}\big|(J'(u) - J'(v))w\big| = \left|\int_{\mathbb{R}} u'w' - v'w'dx\right| \leq \int_{\mathbb{R}} |u'w' - v'w'|dx \leq \|u' - v'\|_{L^2(\mathbb{R})}\|w'\|_{L^2(\mathbb{R})} \\ \leq \|u - v\|_{H^1(\mathbb{R})}\|w\|_{H^1(\mathbb{R})},\end{aligned}$$

where the second inequality is a consequence of the Cauchy-Schwartz inequality. From this estimate we get:

$$\|J'(u)-J'(v)\|_{H^{-1}(\mathbb{R})}=\sup_{\|w\|_{H^1(\mathbb{R})}=1}\left|(J'(u)-J'(v))w\right|\leq\|u-v\|_{H^1(\mathbb{R})}.$$

Thus J' is continuous.

Step 2: Differentiability of K:

To study the differentiability of K we apply a standard argument. We first compute the Gâteaux derivative of K and then prove its continuity. These conditions, as seen in Theorem B.5, are sufficient to deduce differentiability in the sense of Fréchet.

Consider the real function of a real variable $f(t)=p|t|^{p-2}t$. It is clear that f is continuous. Moreover, we have that K is a functional of the form $K(u)=\frac{1}{p}\int_{\mathbb{R}}F(u(x))dx$ where F is the continuously differentiable real function defined by $F(t)=|t|^p=\int_0^t f(s)ds$. Now we compute the first variation of K. We claim that the following equality holds:

$$\frac{1}{p}\lim_{t\to0}\int_{\mathbb{R}}\frac{F(u+tv)-F(u)}{t}dx=\frac{1}{p}\int_{\mathbb{R}}\lim_{t\to0}\frac{F(u+tv)-F(u)}{t}dx=\int_{\mathbb{R}}|u|^{p-2}uvdx. \tag{2.30}$$

It is, as expected, a matter of applying Lebesgue's dominated convergence theorem, Theorem A.1. In order to do so take $u,v\in H^1(\mathbb{R})$ and note that, almost everywhere in $\mathbb{R}$, we have that

$$\begin{aligned}\frac{1}{p}\lim_{t\to0}\frac{|u(x)+tv(x)|^p-|u(x)|^p}{t}&=\frac{1}{p}\lim_{t\to0}\frac{F(u(x)+tv(x))-F(u(x))}{t}=\frac{1}{p}F'\left(u(x)\right)v(x)\\&=|u(x)|^{p-2}u(x)v(x).\end{aligned}$$

Moreover, for each fixed $x\in\mathbb{R}$, the C^1 map defined by

$$\varphi(t)=F(u(x)+tv(x))$$

is a real function of a real variable. It follows now from the mean value theorem for real functions that, for some real number θ such that $|\theta|\leq|t|$ we have

$$|\varphi(t)-\varphi(0)|=|\varphi'(\theta)||t|=|f(u+\theta v)v||t|.$$

Hence, from the definition of φ, we get:

$$\left|\frac{F(u+tv)-F(u)}{t}\right|=\left|\frac{\varphi(t)-\varphi(0)}{t}\right|=|f(u+\theta v)v|=p\left|(u+\theta v)\left|u+\theta v\right|^{p-2}v\right|.$$

Recalling that we are working with fixed x, it follows from Lemma A.11 that

$$\left|(u+\theta v)\left|u+\theta v\right|^{p-2}v\right|=\left|u+\theta v\right|^{p-1}|v|\leq K\left(|u|^{p-1}|v|+|\theta|^{p-1}|v|^p\right)\leq K\left(|u|^{p-1}|v|+|v|^p\right)$$

where the last inequality comes from taking $|t|\leq1$. Now, integrating the last term in the above inequalities over $\mathbb{R}$ and taking into account the Sobolev embedding of $H^1(\mathbb{R})$ in $L^\infty(\mathbb{R})$, Cauchy-Schwarz inequality and (A.1), we get that

$$\begin{aligned}\int_{\mathbb{R}}|u|^{p-1}|v|+|v|^pdx=\int_{\mathbb{R}}|u|^{p-1}|v|dx+\|v\|^p_{L^p(\mathbb{R})}&\leq\|u\|^{p-2}_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|u||v|dx+\|v\|^p_{L^p(\mathbb{R})}\\&\leq\|u\|^{p-2}_{L^\infty(\mathbb{R})}\|u\|_{L^2(\mathbb{R})}\|v\|_{L^2(\mathbb{R})}+\|v\|^p_{L^p(\mathbb{R})}<+\infty\end{aligned}$$

Therefore, from the Dominated Convergence Theorem we have the validity of the equality in (2.30), thus proving the claim. From this equality and the continuity in v, we conclude that for all $u \in H^1(\mathbb{R})$ the Gâteaux differential at u is given by the linear map defined in $H^1(\mathbb{R})$ by

$$K'_G(u)v = \int_{\mathbb{R}} |u|^{p-2} uv dx.$$

To finish the proof the only thing that is left to see is that the Gâteux derivative of K, $K'_G : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$, is continuous. To do so take a sequence (u_n) such that $u_n \to u$ in $H^1(\mathbb{R})$ and any $v \in H^1(\mathbb{R})$. Let us estimate the quantity

$$\left| K'_G(u_n)v - K'_G(u)v \right|.$$

We have by linearity that

$$\left| K'_G(u_n)v - K'_G(u)v \right| = \left| (K'_G(u_n) - K'_G(u))v \right| \leq \int_{\mathbb{R}} \left| |u_n|^{p-2} u_n - |u|^{p-2} u \right| |v| dx. \tag{2.31}$$

Applying the mean value theorem to the function $s \mapsto s|s|^{p-2}$ we get, for fixed $n \in \mathbb{N}$, that

$$\left| K'_G(u_n)v - K'_G(u)v \right| \leq (p-1) \int_{\mathbb{R}} |u_n(x) - u(x)| |\xi_n(x)|^{p-2} |v(x)| dx,$$

where $\xi_n(x)$ is between $u(x)$ and $u_n(x)$ for every $x \in \mathbb{R}$, since u_n and u are continuous. From this we conclude that ξ_n is measurable, $|\xi_n(x)| \leq \max\left\{ \|u\|_{L^\infty(\mathbb{R})}, \|u_n\|_{L^\infty(\mathbb{R})} \right\}$, whence $\xi_n \in L^\infty(\mathbb{R})$, and $\xi_n \to u$ uniformly as $n \to \infty$ by Sobolev embeddings.

We can now proceed with the estimate in (2.31):

$$\begin{aligned}
\left| K'_G(u_n)v - K'_G(u)v \right| &= (p-1) \int_{\mathbb{R}} |u_n(x) - u(x)| |\xi_n(x)|^{p-2} |v(x)| dx \\
&\leq (p-1) \|\xi_n\|^{p-2}_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |u_n(x) - u(x)| |v(x)| dx \\
&\leq K(p) \|u_n - u\|_{H^1(\mathbb{R})} \|v\|_{H^1(\mathbb{R})},
\end{aligned}$$

where the second inequality comes by the Cauchy-Schwartz inequality, the inclusion of $H^1(\mathbb{R})$ in $L^2(\mathbb{R})$ and the fact that $\|\xi_n\|_{L^\infty(\mathbb{R})}$ is uniformly bounded. To finish we only need to compute the operator norm of K'_G:

$$\|K'_G(u_n) - K'_G(u)\|_{H^{-1}(\mathbb{R})} = \sup_{\|v\|_{H^1(\mathbb{R})}=1} \left| K'_G(u_n)v - K'_G(u)v \right| \leq K \|u_n - u\|_{H^1(\mathbb{R})} \underset{n \to +\infty}{\longrightarrow} 0,$$

which concludes the proof. ■

Remark 2.13:
An important result to keep in mind for the following chapter, is that even in bounded subsets of $\mathbb{R}$ this functional still remains of class C^1.

We have that E is a C^1 functional in $H^1(\mathbb{R})$, but the space where we want to minimize E is $H^1_\mu(\mathbb{R})$. Let us take a closer look at this space. We begin by defining the map:

$$G : H^1(\mathbb{R}) \to \mathbb{R}; \; G(u) = \|u\|^2_{L^2(\mathbb{R})}.$$

We have straight away that $H^1_\mu(\mathbb{R}) = G^{-1}\{\mu\}$. This means that the minimization taking place can be seen as minimizing E in $H^1(\mathbb{R})$ subject to the constraint $G(u) = \mu$. Should this restriction map G have a surjective differential at any point of $H^1_\mu(\mathbb{R})$ then we would be in the setting of constrained extremum problems and the results in Appendix B can be applied.

The first verification that we must do is that G is a C^1 map in $H^1(\mathbb{R})$. Indeed note that G can be seen as the quadratic form of the continuous symmetric bilinear form defined by

$$\begin{aligned} a : H^1(\mathbb{R}) \times H^1(\mathbb{R}) &\to \mathbb{R} \\ (u, v) &\mapsto \int_{\mathbb{R}} uv dx. \end{aligned}$$

Then, G is of class $C^1(H^1(\mathbb{R}))$ and its differential at a point $u \in H^1(\mathbb{R})$ is given by

$$G'(u)v = 2 \int_{\mathbb{R}} uv dx, \text{ for all } v \in H^1(\mathbb{R}). \tag{2.32}$$

Finally, we need to check that $G'(u)$ is a surjective map for all $u \in H^1_\mu(\mathbb{R})$; equivalently, note that

$$G'(u)u = 2\mu \neq 0.$$

We can now prove, via constrained extrema problems, the following

Proposition 2.14: *Constrained Euler-Lagrange equation*

Let $\mu > 0$, $p \in (2, 6)$ and $u \in H^1_\mu(\mathbb{R})$ be a solution to the minimization problem

$$\inf_{u \in H^1_\mu(\mathbb{R})} E(u).$$

Then,there exists $\lambda \in \mathbb{R}$ such that u satisfies

$$\int_{\mathbb{R}} u'v' + \lambda uv dx = \int_{\mathbb{R}} |u|^{p-2} uv dx dx \text{ for all } v \in H^1(\mathbb{R}).$$

In other words, u is a weak solution of the equation

$$-u'' + \lambda u = |u|^{p-2} u. \tag{2.33}$$

Proof. From the previous section we know that there exists a solution $u \in H^1_\mu(\mathbb{R})$ to problem (2.6). Moreover, according to the discussion made prior to Proposition 2.14, we know that we can apply the results from Appendix B. In particular, according to Proposition B.8 we know that u is a critical point of $E|_{G^{-1}\{\mu\}}$, that is, there exists a Lagrange multiplier, $\theta \in \mathbb{R}$, such that:

$$E'(u) = \theta G'(u).$$

From Lemma 2.12 and equation (2.32) the previous equation becomes:

$$\int_{\mathbb{R}} u'v' - |u|^{p-2} uv dx = 2\theta \int_{\mathbb{R}} uv dx \text{ for all } v \in H^1(\mathbb{R}).$$

This means exactly that, by choosing $\lambda = -2\theta$, u is a weak solution to the equation (2.33). ■

This result gives us a fundamental tool to characterize the ground states. In fact we deduced just now that if $u \in H^1_\mu(\mathbb{R})$ is a ground state, then u is a non-trivial solution to an equation of the form given in (2.4). This then leads us into the setting of Appendix C. Moreover, this last result also gives a different way to prove existence of solution to the stationary equation in (2.4) in a completely different way from the one in [10, Chapter 8], see the Appendix C for details. In the setting of this appendix we can, *a priori*, assume that $\lambda > 0$ and even some extra regularity on the solution, namely it will be of class $C^2(\mathbb{R})$, according to Lemma C.1.

To start the characterization of the solutions consider complex valued solutions as well. All the results presented so far still hold. In particular, in this case the constrained Euler-Lagrange equation is still the same. The following result describes how the solutions look like:

Lemma 2.15

Suppose $u \in H^1_\mu(\mathbb{R}, \mathbb{C})$ is a solution to problem (2.6). Then there exist y, $\theta \in \mathbb{R}$ and an $H^1(\mathbb{R})$ function φ that is positive, even and strictly decreasing on the interval $(0, +\infty)$, such that:

$$u(x) = e^{i\theta}\varphi(x - y), \quad \forall x \in \mathbb{R}. \tag{2.34}$$

Moreover, $E(u) = E(\varphi)$, $\|u\|^2_{L^2(\mathbb{R},\mathbb{C})} = \|\varphi\|^2_{L^2(\mathbb{R})}$; consequently φ is also a minimizer for problem (2.6).

Proof. Take $u \in H^1_\mu(\mathbb{R}, \mathbb{C})$ a solution to the problem (2.6). Then, u is a weak solution to the equation (2.33) for some positive Lagrange multiplier λ. As a consequence of Theorem C.3 we have that there exist real numbers θ and y and an $H^1(\mathbb{R})$ function φ that satisfy the following properties: φ is positive, even, it is strictly decreasing on the non-negative real numbers and

$$u(x) = e^{i\theta}\varphi(x - y), \ \text{ for all } \ x \in \mathbb{R}.$$

Then, $\|u\|_{L^2(\mathbb{R},\mathbb{C})} = \|\varphi\|_{L^2(\mathbb{R})}$ and

$$\begin{aligned}
\min_{v \in H^1_\mu(\mathbb{R},\mathbb{C})} E(v) = E(u) &= \frac{1}{2}\|e^{i\theta}\varphi'\|^2_{L^2(\mathbb{R},\mathbb{C})} - \frac{1}{p}\|e^{i\theta}\varphi\|^p_{L^p(\mathbb{R},\mathbb{C})} \\
&= \frac{1}{2}\|\varphi'\|^2_{L^2(\mathbb{R})} - \frac{1}{p}\|\varphi\|^p_{L^p(\mathbb{R})} \\
&= E(\varphi),
\end{aligned}$$

which concludes the proof of the lemma. ■

The natural question now is: is the solution unique? By the previous lemma, we have that if u is a solution to our problem then it is of the form given by (2.34). By choosing different constants y and θ we still have a solution to the problem. So in a way we do not have uniqueness of solution. However, our main concern should be the function φ. Since it arises as a consequence of the existence of a Lagrange multiplier it could also depend on it. This would lead to as many different solutions as there are Lagrange multipliers. However, should we be able to prove that the function φ does not depend on the Lagrange multiplier then, up to *phase multiplication* and *translation*, we do have a unique solution to the problem. The result that follows will clarify this matter.

Theorem 2.16: *Uniqueness of Solution to the Problem* (2.6).

Let $\mu > 0$ and $p \in (2, 6)$ and consider the problem of

$$\text{finding } \ u_0 \in H^1_\mu(\mathbb{R}, \mathbb{C}) \ \text{ such that } \ E(u_0) = \min_{u \in H^1_\mu(\mathbb{R},\mathbb{C})} E(u).$$

Then there exists a unique function $\varphi \in H^1_\mu(\mathbb{R})$, depending on both μ and p, which is a positive, even and strictly decreasing function on the interval $[0, +\infty)$, such that every minimizer of the functional E is given, up to *phase multiplication* and *translation*, by the function φ. In other words, $u \in H^1_\mu(\mathbb{R}, \mathbb{C})$ is a minimizer for problem (2.6) if and only if

$$u(x) = e^{i\theta}\varphi(x - y), \quad \forall x \in \mathbb{R},$$

for some θ, $y \in \mathbb{R}$.

Proof. The necessary condition is an immediate consequence of Theorem C.3. We prove the sufficient condition by showing that φ is unique and independent of the Lagrange multiplier. Taking Lemma 2.15 into account, let $\varphi, \psi \in H^1(\mathbb{R})$ be two solutions of (2.6). Then, there exist two Lagrange multipliers, λ_1 and λ_2 which are positive real numbers, such that φ and ψ satisfy the equations:

$$\begin{cases} -\varphi'' + \lambda_1\varphi = |\varphi|^{p-2}\varphi, & \text{(2.35a)} \\ -\psi'' + \lambda_2\psi = |\psi|^{p-2}\psi. & \text{(2.35b)} \end{cases}$$

Our goal now is to relate these two functions and their respective Lagrange multipliers. To do so we resort to standard scaling techniques. Let us make the following *ansatz*:

$$\overline{\psi}(x) := k\varphi(hx)$$

where φ is the solution to equation (2.35a) above and

$$k = \left(\frac{\lambda_1}{\lambda_2}\right)^{-\frac{1}{p-2}} \quad \text{and} \quad h = \left(\frac{\lambda_1}{\lambda_2}\right)^{-\frac{1}{2}}.$$

For these values we have that, for all $x \in \mathbb{R}$,

$$\begin{aligned} -\overline{\psi}''(x) + \lambda_2\overline{\psi}(x) &= -\left(\frac{\lambda_1}{\lambda_2}\right)^{-1-\frac{1}{p-2}} \varphi''(hx) + \lambda_1 \left(\frac{\lambda_1}{\lambda_2}\right)^{-1-\frac{1}{p-2}} \varphi(hx) \\ &= \left(\frac{\lambda_1}{\lambda_2}\right)^{-1-\frac{1}{p-2}} \left[-\varphi''(hx) + \lambda_1\varphi(hx)\right] \\ &= |\overline{\psi}(x)|^{p-2}\overline{\psi}(x). \end{aligned}$$

Thus, from Theorem C.3, we get that up to phase multiplication and translation, $\overline{\psi} = \psi$.

Now, by taking $\dfrac{\lambda_1}{\lambda_2} = \varpi$ we get the following relation

$$\psi(x) = \varpi^{-\frac{1}{p-2}}\varphi\left(\varpi^{-\frac{1}{2}}x\right). \tag{2.36}$$

It is now clear that we have $\varphi = \psi$ if and only if $\lambda_1 = \lambda_2$, or equivalently $\varpi = 1$, which we now prove. Since $\varphi, \psi \in H^1_\mu(\mathbb{R})$ we have from (2.36) that:

$$\mu = \int_{\mathbb{R}} \psi^2(x)dx = \varpi^{-\frac{2}{p-2}} \int_{\mathbb{R}} \varphi^2\left(\varpi^{-\frac{1}{2}}x\right)dx = \varpi^{-\frac{2}{p-2}+\frac{1}{2}} \int_{\mathbb{R}} \varphi^2(z)dz = \varpi^{-\frac{2}{p-2}+\frac{1}{2}}\mu.$$

In turn this yields $\varpi^{-\frac{6-p}{2(p-2)}} = 1$, whence $\varpi = 1$. ■

Remark 2.17:
Note that Lemma 2.15 and Theorem 2.16 are stated for complex valued functions in order to make it consistent with the notation in appendix C. However, we mentioned that there is no loss of generality in assuming that solutions are real valued therefore we can, *a priori*, assume that the expression $e^{i\theta}$ reduces to ± 1. Put differently, since the solutions are real, then up to a *change of sign*, they are given by the translation of the function φ.

2.3 Some Scaling Properties

In this final section we focus on scaling properties of the solutions to the problem (2.6). By themselves the properties we will consider are interesting; however, in the developments that are to follow, they will also have great importance.

Theorem 2.18: Minimizers scaling properties.

Let $\mu > 0$ and $p \in (2, 6)$. Consider the problem of

$$\text{finding } \ u_0 \in H^1_\mu(\mathbb{R}) \ \text{ such that } \ E(u_0) = \min_{u \in H^1_\mu(\mathbb{R})} E(u),$$

and let $\varphi_\mu \in H^1(\mathbb{R})$ be the unique positive and even minimizer. Then, the minimizer satisfies the following scaling rule:

$$\varphi_\mu(x) = \mu^\alpha \varphi_1(\mu^\beta x), \tag{2.37}$$

where the constants α and β are given by $\alpha = \frac{2}{6-p}$ and $\beta = \frac{p-2}{6-p}$. Moreover, there exist positive constants C_p and c_p such that

$$\varphi_1(x) = C_p \operatorname{sech}(c_p x)^{\frac{\alpha}{\beta}} \ \ \forall x \in \mathbb{R}.$$

Proof. Let $\varphi_\mu \in H^1_\mu(\mathbb{R})$ be the unique positive and even minimizer to problem (2.6). We know that there exists $\lambda > 0$ such that φ_μ satisfies the equation:

$$-\varphi_\mu'' + \lambda\varphi_\mu = |\varphi_\mu|^{p-2}\varphi_\mu.$$

We make the following *ansatz*:

$$\varphi_\mu(x) = k\varphi(hx)$$

where k, $h > 0$ and φ is a function in the unit circle of $L^2(\mathbb{R})$. Our goal is now to find under which conditions will φ also be a solution to an equation of the type given in (2.33), with $\|\varphi\|_{L^2(\mathbb{R})} = 1$. Computing the derivatives of φ_μ in this new form and using the fact that it is also a solution to the above problem it follows that

$$-\varphi_\mu''(x) + \lambda\varphi_\mu(x) = |\varphi_\mu(x)|^{p-2}\varphi_\mu(x) \Leftrightarrow -kh^2\varphi''(hx) + k\lambda\varphi(hx) = k^{p-1}|\varphi(hx)|^{p-2}\varphi(hx).$$

Note that due to regularity of the soliton the above equivalence holds in $\mathbb{R}$, therefore to ease the notation we forget the points on which the functions are being evaluated. It now follows that dividing the last equation above by kh^2 we have:

$$-\varphi_\mu'' + \lambda\varphi_\mu = |\varphi_\mu|^{p-2}\varphi_\mu \Leftrightarrow -\varphi'' + \frac{\lambda}{h^2}\varphi = \frac{k^{p-2}}{h^2}|\varphi|^{p-2}\varphi.$$

Now since we want φ to solve an equation of the form (2.33) this gives us a relation between the constants k and h, that is $\frac{k^{p-2}}{h^2} = 1$. Moreover, for φ with $\int_\mathbb{R} \varphi^2 = 1$, we can get another relation between these constants through the following equation:

$$\mu = \int_\mathbb{R} \varphi_\mu^2(x)dx = k^2 \int_\mathbb{R} \varphi^2(hx)dx.$$

By performing the change of variable $y = hx$ and using the fact that φ is in the unit circle of $L^2(\mathbb{R})$ yields us the following necessary condition:

$$h\mu = k^2.$$

We thus have the following system in the variables k and h:

$$\begin{cases} \dfrac{k^{p-2}}{h^2} = 1 \\ h\mu = k^2 \end{cases} \Leftrightarrow \begin{cases} k = h^{\frac{2}{p-2}} \\ h = \dfrac{k^2}{\mu} \end{cases} \Leftrightarrow \begin{cases} k = \mu^{\frac{2}{6-p}} \\ h = \mu^{\frac{p-2}{6-p}} \end{cases}.$$

From this it follows that

$$\varphi_\mu(x) = \mu^\alpha \varphi(\mu^\beta x),$$

where α and β are as in the statement. We claim that $\varphi = \varphi_1$. With the computations above we have shown that φ is the unique positive and even solution to the equation:

$$-\varphi'' + \frac{\lambda}{\mu^{2\beta}}\varphi = |\varphi|^{p-2}\varphi$$

We also know that there exist $\theta > 0$ a Lagrange multiplier and $\varphi_1 \in H^1(\mathbb{R})$ with $\|\varphi_1\|_{L^2(\mathbb{R})} = 1$ that is the unique positive and even solution to the equation

$$-\varphi_1'' + \theta\varphi_1 = |\varphi_1|^{p-2}\varphi_1.$$

Recurring to the equation (2.36) we get the following equality:

$$\varphi_1(x) = \left(\frac{\lambda}{\theta\mu^{2\beta}}\right)^{-\frac{1}{p-2}} \varphi\left(\left(\frac{\lambda}{\theta\mu^{2\beta}}\right)^{-\frac{1}{2}} x\right)$$

Now computing the square of the L^2-norm of ξ yields

$$1 = \int_{\mathbb{R}} \varphi_1^2 dx = \left(\frac{\lambda}{\mu^{2\beta}}\right)^{-\frac{2}{p-2}+\frac{1}{2}} \int_{\mathbb{R}} \varphi^2 dx.$$

From here it follows easily that

$$\theta = \frac{\lambda}{\mu^{2\beta}}.$$

In turn, this gives that φ_1 is also a solution to the equation $-\varphi_1'' + \frac{\lambda}{\mu^{2\beta}}\varphi_1 = |\varphi_1|^{p-2}\varphi_1$. It follows now from Theorem C.3 that $\varphi = \varphi_1$. This gives us the desired scaling:

$$\varphi_\mu(x) = \mu^\alpha \varphi_1(\mu^\beta x).$$

To finish the proof we wish to check that, up to a choice of positive constants C_p and c_p, the function φ_1 can be taken as:

$$\psi(x) = C_p \operatorname{sech}(c_p x)^{\frac{\alpha}{\beta}}.$$

Now we need to find for which values C_p and c_p, do we have

$$-\psi'' + \theta\psi = |\psi|^{p-2}\psi \tag{2.38}$$

and

$$\int_{\mathbb{R}} \psi^2 dx = 1, \tag{2.39}$$

for if this happens then Theorem C.3 asserts that $\psi = \varphi_1$. By differentiation we have that

$$\begin{aligned}
\psi'(x) &= C_p \left[\frac{\alpha}{\beta} \operatorname{sech}(c_p x)^{\frac{\alpha}{\beta}-1} \tanh(c_p x)(-\operatorname{sech}(c_p x))c_p\right] \\
&= -C_p c_p \frac{\alpha}{\beta} \operatorname{sech}(c_p x)^{\frac{\alpha}{\beta}} \tanh(c_p x) \\
&= -c_p \frac{\alpha}{\beta} \psi(x) \tanh(c_p x)
\end{aligned}$$

Hence,

$$\begin{aligned}\psi''(x) &= -c_p\frac{\alpha}{\beta}\left[\psi'(x)\tanh(c_px)+\psi(x)\operatorname{sech}^2(c_px)c_p\right]\\ &= -c_p\frac{\alpha}{\beta}\left[-C_pc_p\frac{\alpha}{\beta}\operatorname{sech}(c_px)^{\frac{\alpha}{\beta}}\tanh^2(c_px)+C_pc_p\operatorname{sech}^{\frac{\alpha}{\beta}+2}(c_px)\right]\\ &= -C_pc_p^2\frac{\alpha}{\beta}\operatorname{sech}^{\frac{\alpha}{\beta}}(c_px)\left[-\frac{\alpha}{\beta}\tanh^2(c_px)+\operatorname{sech}^2(c_px)\right]\\ &= -C_pc_p^2\frac{\alpha}{\beta}\operatorname{sech}^{\frac{\alpha}{\beta}}(c_px)\left[-\frac{\alpha}{\beta}+(1+\frac{\alpha}{\beta})\operatorname{sech}^2(c_px)\right],\end{aligned}$$

where in the last equality we used the following hyperbolic trigonometric identity:

$$\operatorname{sech}^2(x)+\tanh^2(x)=1.$$

Note also that since ψ is a positive function,

$$|\psi(x)|^{p-2}\psi(x)=C_p^{p-1}\operatorname{sech}(c_px)^{\frac{\alpha(p-2)}{\beta}+\frac{\alpha}{\beta}}=C_p^{p-1}\operatorname{sech}(c_px)^{2+\frac{\alpha}{\beta}}.$$

Since we want ψ to solve equation (2.38) we require that:

$$\begin{aligned}&-\psi''(x)+\theta\psi(x)=|\psi(x)|^{p-2}\psi(x)\\ &\Leftrightarrow C_pc_p^2\frac{\alpha}{\beta}\operatorname{sech}^{\frac{\alpha}{\beta}}(c_px)\left[-\frac{\alpha}{\beta}+(1+\frac{\alpha}{\beta})\operatorname{sech}^2(c_px)\right]+\theta C_p\operatorname{sech}(c_px)^{\frac{\alpha}{\beta}}=\\ &=C_p^{p-1}\operatorname{sech}(c_px)^{2+\frac{\alpha}{\beta}}.\end{aligned}$$

Dividing the last equation above by $C_p\operatorname{sech}(c_px)^{\frac{\alpha}{\beta}}>0$, we arrive at the following identity:

$$c_p^2\frac{\alpha}{\beta}\left[-\frac{\alpha}{\beta}+(1+\frac{\alpha}{\beta})\operatorname{sech}^2(c_px)\right]+\theta=C_p^{p-2}\operatorname{sech}^2(c_px),$$

which in turn is equivalent to having

$$-c_p^2(\frac{\alpha}{\beta})^2+\left[c_p^2\frac{\alpha}{\beta}(1+\frac{\alpha}{\beta})-C_p^{p-2}\right]\operatorname{sech}^2(c_px)=-\theta.$$

Now, given that the first term on the left hand side and the term on the right hand side of the equation are constant, this forces that the quantity in brackets to be zero, thus leading to the following system of equations:

$$\begin{cases}c_p^2\frac{\alpha}{\beta}(1+\frac{\alpha}{\beta})-C_p^{p-2}=0,\\ -c_p^2(\frac{\alpha}{\beta})^2=-\theta.\end{cases}$$

Solving now for the values of C_p and c_p we have:

$$\begin{cases}c_p^2\frac{\alpha}{\beta}(1+\frac{\alpha}{\beta})-C_p^{p-2}=0\\ -c_p^2(\frac{\alpha}{\beta})^2=-\theta\end{cases}\Leftrightarrow\begin{cases}c_p^2\frac{\alpha}{\beta}(1+\frac{\alpha}{\beta})-C_p^{p-2}=0\\ c_p=\pm\frac{\beta}{\alpha}\theta^{\frac{1}{2}}\end{cases}\Leftrightarrow\begin{cases}C_p=\left(\frac{\beta+\alpha}{\alpha}\right)^{\frac{1}{p-2}},\\ c_p=\pm\frac{\beta}{\alpha}\theta^{\frac{1}{2}}.\end{cases}$$

Moreover, since ψ is in the unit circle of $L^2(\mathbb{R})$ we can deduce that

$$1=\int_{\mathbb{R}}\psi^2(x)dx=\int_{\mathbb{R}}C_p^2\operatorname{sech}(c_px)^{\frac{2\alpha}{\beta}}dx=\frac{C_p^2}{c_p}\int_{\mathbb{R}}\operatorname{sech}(z)^{\frac{2\alpha}{\beta}}dz.$$

From this $c_p > 0$ and therefore the constants are determined:

$$\begin{cases} C_p = \left(\frac{\theta p}{2}\right)^{\frac{1}{p-2}}, \\ c_p = \frac{p-2}{2}\theta^{\frac{1}{2}}. \end{cases}$$

■

We conclude this chapter with some remarks. In the proofs of several previous results we made use of the *ansatz* $u(x) = kv(hx)$ for constants k and h. We can make this ansatz due to the homogeneity of the terms in the nonlinear Schrödinger equation. Its utility, as it was seen, is what allows us to do algebraic manipulations to obtain either new solutions with different mass or to relate solutions with the same mass. Secondly, note that in this last result we confirm the remarks done prior to Lemma 2.15. Finally, we depict in the Figure 2.1 below the graph of an hyperbolic secant. Observe that hyperbolic secants have the shape of a wave and, by scaling, any solution of arbitrary mass will have this shape. We will refer to these solutions as *solitons*.

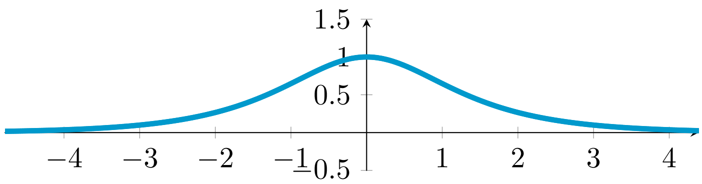

Figure 2.1: The graph of an hyperbolic secant

Chapter 3

Minimization of the NLS Energy Functional on Metric Graphs

3.1 Metric Graphs

The main goal of this section is to introduce the necessary notions from graph theory in order to create a good setting for the problem to which this chapter is devoted to. With this objective in mind we will use reference [6] for the graph theoretical notions, whereas [8] will be the main reference for the aspects related directly with *metric graphs*. As a combinatorial structure a graph can be defined as follows:

Definition 3.1: *Graph*

A *graph*, $\mathcal{G}$, is a triple $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}), I_{\mathcal{G}})$ composed of two non-empty disjoint and countable sets $E(\mathcal{G})$ and $V(\mathcal{G})$ and where $I_{\mathcal{G}}$ is a map that associates to each element of $E(\mathcal{G})$ an unordered pair of elements of $V(\mathcal{G})$ (possibly the same). We refer to $V(\mathcal{G})$ as the set of *vertices* and to $E(\mathcal{G})$ as the set of *edges* of the graph.

Even though a formal definition is required, when working with graphs one always envisions a diagrammatic representation of them in the plane. As an example see Figures 3.1 and 3.2 below. From this, the relation $I_{\mathcal{G}}$ is typically well understood and therefore we refer to a graph simply as a pair $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$.

Definition 3.2: *Finite and Infinite Graphs*

We say that a graph $\mathcal{G}$ is *finite* if $|V(\mathcal{G})| < \infty$ and $|E(\mathcal{G})| < \infty$, where $|\cdot|$ denotes the cardinality of a set. If the graph is not finite we say that it is *infinite*.

Henceforth, we consider only finite graphs. Another concept we will require from graph theory is that of *degree* of a vertex.

Definition 3.3: *Degree of a Vertex*

Let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a graph.

1. We say that an edge is $e \in E(\mathcal{G})$ is *incident* to a vertex $v \in V(\mathcal{G})$ if there is a vertex $w \in V(\mathcal{G})$ such that $I_{\mathcal{G}}(e) = \{v, w\}$. The notation $e \prec v$ is then used to say that the edge e is incident to the vertex v;
2. The *degree* of a vertex as the number of times the vertex v appears in the sets $I_{\mathcal{G}}(e)$ for all $e \in E(\mathcal{G})$. If $I_{\mathcal{G}}(e)$ is a singleton then it is to be counted twice.

An easy way to envision the degree at a vertex is simply to count how many edges are incident to the vertex. Loops are to be counted twice. As an example, in Figure 3.1, the degree of both the vertices v_1 and v_2 is 1. As for Figure 3.2 the vertex v_1 as degree 3 and v_2 has degree 4. As a final remark on this definition, we assume throughout this dissertation that the degree of a vertex is always *finite* and *positive*. This rules out the existence of graphs with isolated vertices.

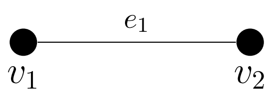

Figure 3.1: A line segment as a graph. According to the definition this is the representation of the graph $V(\mathcal{G}) = \{v_1, v_2\}$, $E(\mathcal{G}) = \{e_1\}$ and $I_{\mathcal{G}}(e_1) = \{v_1, v_2\}$.

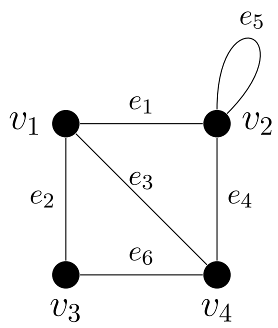

Figure 3.2: The graph depicted in this figure is $V(\mathcal{G}) = \{v_1, v_2, v_3, v_4\}$, $E(\mathcal{G}) = \{e_1, \cdots, e_6\}$ and $I_{\mathcal{G}}(e_1) = \{v_1, v_2\}$, $I_{\mathcal{G}}(e_2) = \{v_1, v_3\}$, ..., $I_{\mathcal{G}}(e_5) = \{v_2, v_2\} = \{v_2\}$ and $I_{\mathcal{G}}(e_6) = \{v_3, v_4\}$.

Definition 3.4: *Digraph*

A *digraph*, $\mathcal{G}$, is a triple $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}), D_{\mathcal{G}})$ composed of two non-empty disjoint and countable sets $E(\mathcal{G})$ and $V(\mathcal{G})$ and where $D_{\mathcal{G}}$ is a map that associates to each element of $E(\mathcal{G})$ an ordered pair of elements of $V(\mathcal{G})$ (possibly the same). We refer to $V(\mathcal{G})$ as the set of *vertices* and to $E(\mathcal{G})$ as the set of *directed edges*, or *bonds*, of the graph.

Remark 3.5:

1. When working with a bond $b \in E(\mathcal{G})$ we can specify its origin and terminal vertices via the maps: $o : E(\mathcal{G}) \to V(\mathcal{G})$ and $t : E(\mathcal{G}) \to V(\mathcal{G})$ which are defined, respectively, by $o(b) :=$ the first component of the pair $D_{\mathcal{G}}(b)$ and $t(b) :=$ the second component of the pair $D_{\mathcal{G}}(b)$;
2. For a fixed vertex $v \in V(\mathcal{G})$ we now have *outgoing* bonds, those that satisfy $o(b) = v$, and *incoming* bonds, which satisfy $t(b) = v$;
3. It is clear that if $\mathcal{G}$ is a digraph then the degree of a vertex $v \in V(\mathcal{G})$ is the sum of all the incoming and all the outgoing bonds;

Figures 3.3 and 3.4 depict the diagrammatic representation of directed graphs.

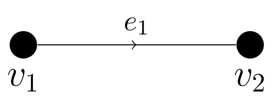

Figure 3.3: According to the definition this is the representation of the digraph $V(\mathcal{G}) = \{v_1, v_2\}$, $E(\mathcal{G}) = \{e_1\}$ and $D_{\mathcal{G}}(e_1) = (v_1, v_2)$.

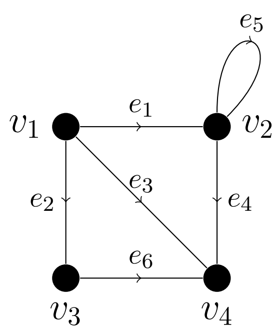

Figure 3.4: The digraph depicted in this figure is $V(\mathcal{G}) = \{v_1, v_2, v_3, v_4\}$, $E(\mathcal{G}) = \{e_1, \cdots, e_6\}$ and $D_{\mathcal{G}}(e_1) = (v_1, v_2)$, $D_{\mathcal{G}}(e_2) = (v_1, v_3)$, ... , $D_{\mathcal{G}}(e_5) = (v_2, v_2)$ and $D_{\mathcal{G}}(e_6) = (v_3, v_4)$.

When one takes a combinatorial approach to study a graph, the main focus are the vertices, while the edges only exhibit how the vertices are related with one another. Our approach will be exactly the opposite. The main players for us will be the edges of the graph, in particular, the way they are connected to each other through the vertices and the length of the edges themselves. This viewpoint of graphs leads to the following definition:

Definition 3.6: *Metric Graph*

A *metric graph* is a pair $(\mathcal{G}, \ell)$ where $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ is a graph and $\ell : E(\mathcal{G}) \to (0, +\infty]$ is a map that to each edge $e \in E(\mathcal{G})$ assigns a positive length $l_e \in (0, +\infty]$, which we call the length of the edge e. We are then assigning to each edge an interval I_e that is either a closed interval $[0, l_e]$ or a half-line, which is naturally associated to $[0, +\infty)$. To ease the notation a metric graph will always be denoted by $\mathcal{G}$ unless it is specified otherwise.

Through the above identifications we can now consider points inside the edges as elements of the metric graph. In fact if e is an edge of the metric graph then it is identified with an interval I_e which is either a compact interval or a half-line. In this interval we have a natural coordinate x_e that allows to identify each point in the edge of the graph. That is, we say that $x \in e$ if $x = x_e$ for some element $x_e \in I_e$. Now there is the matter of vertices. For a given vertex v, if e is any edge such that $e \prec v$ then, either $v = 0$ or $v = l_e$, if e is identified with the interval $[0, l_e]$, otherwise $v = 0$ or $v = \infty$ if e is identified with a half-line. Consequently, we can now make sense of the expression $x \in \mathcal{G}$, it is either a vertex that is not identified with ∞ or some point $x_e \in I_e$ for some edge e. Note that by doing this we are also giving a direction to the edge e, which is the one that increases with the coordinate x_e in I_e. Therefore we have that a metric graph is a digraph. Later in this thesis we will in fact realize that all the results will be independent of the direction of the edges. Figures 3.3 and 3.4 are also examples of metric graphs on which all the edges have assigned finite length. We make a the following structural assumption on metric graphs: all the vertices of the form $v = \infty$ have *degree one*.

Contrary to what the name might suggest, by itself a metric graph is not a metric space. Nonetheless we can define on it a natural metric, thus endowing the graph with the extra structure of a metric space. To do so we need to define the concept of a path in a metric graph.

Definition 3.7: *Path between Vertices*

Let $\mathcal{G}$ be a metric graph. A *path between vertices* is a finite alternating sequence of elements of $V(\mathcal{G})$ and $E(\mathcal{G})$, $\gamma = v_1 e_1 v_2 e_2 \cdots v_{k-1} e_{k-1} v_k$, that starts and finishes with an element of $V(\mathcal{G})$ and such that for each $i \in \{1, \cdots, k\}$, the edge e_i is incident to both the vertices v_i and v_{i+1}. We define the length of this path as $\mathcal{L}_1(\gamma) = \sum_{i=1}^{k-1} l_{e_i}$

For example, in Figure 3.2 the alternating sequence $v_1 e_3 v_4 e_4 v_2 e_1 v_1$ defines a path on the graph while $v_1 e_5 v_2$ does not. The following definition will also be required ahead.

Definition 3.8: *Trails and Cycles in Graphs*

Let $\mathcal{G}$ be a metric graph. We call *trail* a path between vertices in which no edge is repeated. We say that a path between vertices is closed if the terminal vertex is the same as the original one. We call cycle to a closed trail in which all the vertices are also distinct, with the exception of the first and last ones, which are equal.

In Figure 3.4 the sequence $v_1 e_1 v_2 e_5 v_2 e_4 v_4$ defines a trail. In this figure a cycle is for example $v_2 e_5 v_2$. Note that there are no other cycles, however if we reverse the orientation of the edge e_3 then cycles do exist. One such cycle is, for example, the sequence $v_1 e_1 v_2 e_4 v_4 e_3 v_1$.

We now wish to extend this notion of path not only between vertices but between arbitrary points of the graph. Take then $x, y \in \mathcal{G}$. If they belong to the same edge $e \in E(\mathcal{G})$ we say that the portion between them forms a path γ of length $\mathcal{L}(\gamma) = |x_e - y_e|$. If $x \in e_1 \in E(\mathcal{G})$ and $y \in e_2 \in E(\mathcal{G})$ then we connect

x to a vertex v_1 and y to v_2 such that $e_1 \prec v_1$ and $e_2 \prec v_2$. Finally take a path γ_1 between the vertices v_1 and v_2, thus obtaining, by concatenation, a path γ whose total length is:

$$\mathcal{L}(\gamma) = |x_e - v_1| + \mathcal{L}_1(\gamma_1) + |y_e - v_2|.$$

Note that the usage of v_1 and v_2 in the above expression is an abuse of notation. In fact both v_1 and v_2 are being used as the coordinate that they represent on the respective edges.

It is through this notion of path that we can define the concept of connectedness in a graph.

Definition 3.9: *Connected Graph*

Let $\mathcal{G}$ be a metric graph. We say that $\mathcal{G}$ is a *connected graph* if for any $x, y \in \mathcal{G}$ there is a path connecting them. If a graph is not connected we say that it is disconnected.

We can now define the following map in $\mathcal{G} \times \mathcal{G}$:

$$\begin{aligned} \rho : \mathcal{G} \times \mathcal{G} &\longrightarrow \mathbb{R}_0^+ \\ (x, y) &\mapsto \min\{\mathcal{L}(\gamma) : \gamma \text{ is a path between } x \text{ and } y\}. \end{aligned}$$

as the canonic metric in $\mathcal{G}$. Note that if $\mathcal{G}$ is disconnected then there exist points which cannot be joined by a path and in those cases the above metric would not make sense.

Note that $(\mathcal{G}, \ell)$ is not the same as $(\mathcal{G}, \rho)$. However, if all the edges of the graph are identified with compact intervals of the real line then they can be identified. Should edges of infinite length be present then we would need to be a bit more careful with the identifications. In $(\mathcal{G}, \ell)$ such an edge will have an original vertex, the one corresponding to $x_e = 0$, and a terminal one that we will refer to as a *vertex at infinity*. However, vertices at infinity are not points of $(\mathcal{G}, \rho)$, since the coordinate $x_e \to +\infty$. In the Figure 3.5 below is depicted a representation of $\mathbb{R}$ as a metric graph.

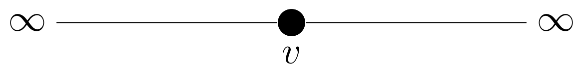

Figure 3.5: The real line as metric graph

The symbol ∞ represents a vertex at infinity. Thus, as a metric graph, $\mathbb{R}$ is composed of one vertex v (the origin) on which two half-lines meet.

Finally, the last concepts we will require are related with the notion of compactness.

Definition 3.10: *Compact Graph*

Let $\mathcal{G}$ be a finite metric graph. We say that $\mathcal{G}$ is *compact* if all its edges have finite length.

Remark 3.11:
It is easy to check that if $(\mathcal{G}, \ell)$ is compact then so is $(\mathcal{G}, \rho)$ for the topology induced from the metric. Let $(U_\alpha)_{\alpha \in I}$ be an open cover of $\mathcal{G}$. Take now any edge, e. By considering $U_\alpha \cap e$ we get an induced open cover of the compact interval I_e. By compactness of I_e in $\mathbb{R}$ we can extract a finite number of indices, $\alpha_1^e, \cdots, \alpha_p^e$, $p \in \mathbb{N}$, such that $e \subset \cup_{i=1}^p U_{\alpha_i^e}$. By performing this for each edge of the graph, which by assumption has a finite number of edges, we get a finite subcover of $\mathcal{G}$.

The relation between the compact part of the graph (bounded edges) and its non-compact part (unbounded edges) will play an important role ahead. Therefore we introduce the following definition.

Definition 3.12: *Compact core of a metric graph*

If $\mathcal{G}$ is a metric graph we call *compact core* the metric graph obtained from $\mathcal{G}$ by removing every unbounded edge. We denote this graph by $\mathcal{K}$.

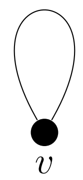

Figure 3.6: This represents the compact core of the graph in Figure 4.1

Of course this graph is composed solely by bounded edges and therefore, as a metric space, it is compact, as the name suggests. From the viewpoint of metric spaces what we formally do is to remove the *interior* of the unbounded edges so that the original vertex of these edges still remains in $\mathcal{K}$. Finally, note that since all unbounded edges of $\mathcal{G}$ are terminal edges of the graph, we have that if $\mathcal{G}$ is connected then so is $\mathcal{K}$. For a more concrete example, Figure 3.6 represents the compact core of the metric graph depicted in Figure 4.1 ahead.

3.1.1 Function Spaces defined on Metric Graphs

Before tackling the equivalent problem to (2.6) on graphs we need to know how to define functions and functions spaces on them. On a metric graph all the points inside the edges also belong to the graph and thus we can define a function f on the whole graph as a family of functions, $(f_e)_{e\in E(\mathcal{G})}$, defined on each edge. Naturally, we can now speak of continuity of a function. We say that a function f is continuous if it is continuous on each edge and if for any vertex $v \in V(\mathcal{G})$; whenever $e, e' \in E(\mathcal{G})$ are such that $e \prec v$ and $e' \prec v$ then $f_e(v) = f_{e'}(v)$. This allows us to define the space of continuous functions, which we denote by $C(\mathcal{G})$.

Take now $\mathcal{G}$ to be a metric graph. Let I_e be the interval or half-line that represents the edge e. Consider the Lebesgue measure dx_e associated to the coordinate x_e. This allows to define a Lebesgue-type measure, dx, on the whole graph. With the existence of this measure we can now define more useful spaces of functions on graphs such as Lebesgue and Sobolev spaces. Henceforth, we assume that a metric graph $\mathcal{G}$ is imbued with a canonic metric, ρ, and a measure dx without any reference.

Definition 3.13: *Lebesgue Spaces*

Let $p \in [1, +\infty]$, $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a finite metric graph and $u : \mathcal{G} \to \mathbb{R}$, $u = (u_e)_{e\in E(\mathcal{G})}$, a function. The *Lebesgue* spaces on graphs are defined as

$$L^p(\mathcal{G}) = \{u : \mathcal{G} \to \mathbb{R} \mid u_e \in L^p(I_e),\ \forall e \in E(\mathcal{G})\}.$$

These are Banach spaces for the norms

$$\|u\|^p_{L^p(\mathcal{G})} = \sum_{e\in E(\mathcal{G})} \|u_e\|^p_{L^p(I_e)}, \quad \text{for } p < \infty \quad \textit{and} \quad \|u\|_{L^\infty(\mathcal{G})} = \sup_{e\in E(\mathcal{G})} \|u_e\|_{L^\infty(I_e)}.$$

The above notion of Lebesgue space is quite natural given the structure that a metric graph possesses. In fact, we have that

$$L^p(\mathcal{G}) = \bigoplus_{e\in E(\mathcal{G})} L^p(I_e). \tag{3.1}$$

For general Sobolev spaces it is not so clear how one can define them and that is because of how the functions might behave at the vertices. Recall that if I is an open interval of $\mathbb{R}$ we can define distributional derivatives of a given function defined over the set I, see for example [9], [22]. We then define the Sobolev space $H^1(I)$ as

$$H^1(I) := \left\{u : I \to \mathbb{R} : u, u' \in L^2(I)\right\},$$

which is a Hilbert space for the norm $\|u\|^2_{H^1(I)} := \|u\|^2_{L^2(I)} + \|u'\|^2_{L^2(I)}$. Since in a metric graph, functions are, for all purposes, defined in one dimension through the identifications with intervals, we define the notion of a distributional derivative in the same way on a metric graph. Moreover recall that, in one dimension, H^1-functions admit a continuous representative. This gives us a natural way to define the Sobolev space $H^1(\mathcal{G})$.

Definition 3.14: *Sobolev Space $H^1(\mathcal{G})$*

Let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a finite metric graph. The *Sobolev* space $H^1(\mathcal{G})$ is defined as:

$$H^1(\mathcal{G}) := \left\{u : \mathcal{G} \to \mathbb{R} \mid u \in C(\mathcal{G}) \text{ and } u_e \in H^1(I_e), \forall e \in E(\mathcal{G})\right\}.$$

Moreover, we endow it with the norm

$$\|u\|^2_{H^1(\mathcal{G})} := \sum_{e \in E(\mathcal{G})} \|u_e\|^2_{H^1(I_e)}$$

which makes it into a Hilbert space.

Remark 3.15:
Note that for finite graphs the above norms are always finite. When working with infinite graphs a finiteness condition on the sums needs to be imposed for both $L^p(\mathcal{G})$ and $H^1(\mathcal{G})$.

3.1.2 Differential Operators on Metric Graphs

Let $\mathcal{G}$ be a metric graph. Consider the operator that acts of functions defined in $\mathcal{G}$ as

$$(Af)(x) = -\frac{d^2 f}{dx^2}(x) + \lambda f(x), \tag{3.2}$$

where λ is a positive real number and x is to be understood as the coordinate along each edge.

The case where the graph is simply a line segment is included in this one. In this classical case, to completely define the above operator, we need to state the domain in which it is defined, this includes the smoothness inside the interval and eventually boundary conditions. In a graph, $\mathcal{G}$, the idea is the same. We need to specify the smoothness along the edges of the graph and eventually vertex conditions, which are the boundary conditions analogue for an interval. Recall now that if $I \subset \mathbb{R}$ is an open interval, the natural domain for the operator A is $H^2(I)$, that is:

$$H^2(I) := \left\{u : I \to \mathbb{R} : u, u', u'' \in L^2(I)\right\}.$$

Remember also that this is a Hilbert space when endowed norm:

$$\|u\|^2_{H^2(I)} := \|u\|^2_{L^2(I)} + \|u\|^2_{L^2(I)} + \|u''\|^2_{L^2(I)}.$$

Then, in a metric graph, it is natural to consider

$$D(A) = \widetilde{H}^2(\mathcal{G}) := \bigoplus_{e \in E(\mathcal{G})} H^2(I_e),$$

as the domain of A. In other words, to require that f is in H^2 on the interval that identifies each edge e. Of course the natural norm to be considered in this space is

$$\|f\|^2_{\widetilde{H}^2(\mathcal{G})} := \sum_{e \in E(\mathcal{G})} \|f_e\|^2_{H^2} < \infty.$$

Again, the finiteness condition is only relevant in infinite graphs.

As for vertex conditions there are many conditions one may take, however, for the variational approach we will be undertaking, the natural ones are the homogeneous *Neumann-Kirchhoff conditions*, or *standard conditions*, which are given by

$$\begin{cases} f \text{ continuous on } \mathcal{G}, \text{ and} \\ \sum\limits_{e \in E_v} \dfrac{df}{dx_e}(v) = 0, \text{ for all } v \in V(\mathcal{G}), \end{cases} \tag{N-K}$$

where the above derivative is taken from the vertex into the edge, and $E_v := \{e \in E(\mathcal{G}) : e \prec v\}$. Recall that by Sobolev embeddings, $H^2(I)$ is embedded in $C^{1,\frac{1}{2}}(\overline{I})$, see [16, Section 5.6.3] and [22, Theorem 12.55], therefore the second equation in (N-K) makes sense.

Remark 3.16:

- Note that for vertices of degree one the above conditions are in fact the homogeneous Neumann conditions. Moreover, these conditions will ensure that the quadratic form associated with the above operator will be defined in $H^1(\mathcal{G})$. Indeed, let $\langle \cdot, \cdot \rangle$ denote the inner product in $L^2(\mathcal{G})$, then

$$\langle Au, u \rangle_{L^2(\mathcal{G}) \times L^2(\mathcal{G})} = \sum_{e \in E(\mathcal{G})} \langle Au_e, u_e \rangle_{L^2(I_e) \times L^2(I_e)} = \sum_{e \in E(\mathcal{G})} \int_{I_e} -\frac{d^2 u_e}{dx_e^2} u_e + \lambda u_e^2 dx_e.$$

Now using the Neumann-Kirchhoff conditions we can integrate by parts without worrying about the boundary terms, whence

$$\langle Au, u \rangle_{L^2(\mathcal{G}) \times L^2(\mathcal{G})} = \sum_{e \in E(\mathcal{G})} \int_{I_e} |u_e'|^2 + \lambda |u_e|^2 dx_e = \int_{\mathcal{G}} |u'|^2 + \lambda |u|^2 dx.$$

- A final remark on the space $H^1(\mathcal{G})$. The space $\widetilde{H}^1(\mathcal{G})$ can be constructed just as it was done above for $\widetilde{H}^2$. Since we want functions to be continuous this is in fact not the correct definition for us. However, since both H^1 and $\widetilde{H}^1$ are endowed with the same norm, and convergence in this norm implies uniform convergence (by Sobolev embeddings), we have that $H^1(\mathcal{G})$ is in fact a closed subspace of $\widetilde{H}^1(\mathcal{G})$.
- For vertices of degree 2 the Neumann-Kirchhoff conditions allow the removal or creation of vertices. The first condition in (N-K) ensures continuity of the function while the second ensures the continuity of the first derivative at the vertex, therefore we can glue the two H^2 -pieces of the function in a single H^2-function defined on the longer edge. Such a vertex is often called a *dummy vertex*.

3.2 A New Setting for an Old Problem

In what follows, $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ is always a connected metric graph. Let again $\mu > 0$ and $p \in (2, 6)$. We now have the tools to reconstruct problem (2.6) in the new setting of graphs.

Let $u \in H^1(\mathcal{G})$. The NLS energy functional of u is defined as

$$\begin{aligned} E(u, \mathcal{G}) &:= \frac{1}{2} \int_{\mathcal{G}} |u'|^2 dx - \frac{1}{p} \int_{\mathcal{G}} |u|^p dx \\ &= \sum_{e \in E(\mathcal{G})} \left\{ \frac{1}{2} \int_{I_e} |u'_e|^2 dx_e - \frac{1}{p} \int_{I_e} |u_e|^p dx_e \right\}. \end{aligned} \tag{3.3}$$

Notation: We will consider now the NLS functional dependent also on the graph, therefore we slightly change the notation from Chapter 2 in order to make this dependence more evident.

For precisely the same reason as in Chapter 2, for $\mathcal{G} = \mathbb{R}$, this functional is well defined. Defining

$$H^1_\mu(\mathcal{G}) := \left\{ u \in H^1(\mathcal{G}) : \|u\|^2_{L^2(\mathcal{G})} = \mu \right\}$$

the analogue of problem (2.6) takes the form:

$$\text{finding } u_0 \in H^1_\mu(\mathcal{G}) \text{ such that } E(u_0, \mathcal{G}) = \inf_{u \in H^1_\mu(\mathcal{G})} E(u, \mathcal{G}). \tag{3.4}$$

Let us now take a moment to make some connections with the previous section. Consider on a metric graph $\mathcal{G}$ the equation

$$-u'' + \lambda u = |u|^{p-2}u,$$

A function $u \in H^1(\mathcal{G})$ is called a weak solution to this equation if

$$\int_{\mathcal{G}} u'\eta' + \lambda u\eta - |u|^{p-2}u\eta dx = 0 \text{ for all } \eta \in H^1(\mathcal{G}). \tag{3.5}$$

The following result will give us some necessary conditions for the existence of ground states to the problem (3.4).

Proposition 3.17

Let $\mathcal{G}$ be a connected metric graph, $\mu > 0$ and $p \in (2, 6)$. Let also $u \in H^1_\mu(\mathcal{G})$ be a solution to problem (3.4). Then:

(*i*) there exists $\lambda > 0$ such that u is a weak solution to the equation $-u'' + \lambda u = |u|^{p-2}u$ in $\mathcal{G}$ and, on each edge e, u_e is a classical solution of the equation:

$$-u_e'' + \lambda u_e = u_e|u_e|^{p-2} \text{ for all } e \in E(\mathcal{G}). \tag{3.6}$$

Consequently, $u \in \tilde{H}^2(\mathcal{G})$.

(*ii*) For every vertex $v \in V(\mathcal{G})$ that is not a vertex at infinity the conditions (N-K) are satisfied.

(*iii*) Up to a change of sign, $u > 0$ on $\mathcal{G}$.

Proof. Let us begin by proving (*i*). The functional defined in (3.3) is a class C^1 functional in $H^1(\mathcal{G})$. This can be seen as a consequence of Lemma 2.12 and Remark 2.13. Since for each edge e the functionals are $C^1(I_e)$ then the sum over all the edges is in $C^1(H^1(\mathcal{G}))$. The same happens to the functional that defines the restriction. Then if $u \in H^1_\mu(\mathcal{G})$ is a solution to the problem we have that u is a critical point in the sense of constrained extrema problems. Therefore, there exists a Lagrange multiplier $\theta \in \mathbb{R}$ such that

$$\int_{\mathcal{G}} u'\eta' - |u|^{p-2}u\eta dx = \theta \int_{\mathcal{G}} u\eta dx, \text{ for all } \eta \in H^1(\mathcal{G}). \tag{3.7}$$

Passing all to the left hand side and taking $\lambda := -\theta$ gives that u is a weak solution to the equation. By fixing now an edge $e \in E(\mathcal{G})$, taking $\eta \in C_0^\infty(I_e)$ we get that u_e is also a weak solution to the equation on the interval I_e. By the regularity Lemma C.1 we have that $u_e \in H^2(I_e)$ and, by Sobolev embedding, it is a classical solution. Consequently, $u \in \tilde{H}^2(\mathcal{G})$.

For (*ii*) fix $v \in V(\mathcal{G})$ that is not a vertex at infinity. Let e be an edge such that $e \prec v$. In this edge we can construct a smooth cutoff function φ such that $|\varphi| \leq 1$ and $\varphi(v) \neq 0$, and $\varphi = 0$ on the other vertex of e. Performing this construction in each edge incident to v and extending to the rest of the graph by zero we construct a function $\eta \in H^1(\mathcal{G})$. Moreover, the value of η at v can be made arbitrary. Note also that this function is zero at every vertex except v, then testing (3.5) with η and using the fact that u is C^2 on each edge yields

$$\begin{aligned} 0 &= \sum_{e\in E(\mathcal{G})} \left\{ \left[u_e'\eta_e\right]_0^{l_e} + \int_0^{l_e} \left(-u_e'' + \lambda u_e - |u_e|^{p-2}u_e\right)\eta_e dx \right\} \\ &= \sum_{e\in E(\mathcal{G})} \left[u_e'\eta_e\right]_0^{l_e} + \int_{\mathcal{G}} \left(-u'' + \lambda u - |u|^{p-2}u\right)\eta dx. \end{aligned}$$

Taking into account (*i*), the construction of η and that the derivatives in the Neumann-Kirchhoff conditions are taken in the direction of the edges we have

$$\sum_{e\in E_v} \frac{du_e}{dx_e}(v)\eta_e(v) = 0.$$

Using now the fact that $\eta \in H^1(\mathcal{G})$ and that $\eta(v)$ can be made arbitrary by construction we get

$$\sum_{e \in E_v} \frac{du_e}{dx_e}(v) = 0 \tag{3.8}$$

where $E_v = \{e \in E(\mathcal{G}) : e \prec v\}$. Thus the second item is proved.

To prove (*iii*) start by noticing that since we assume that solutions are real valued and that for these we have that $E(|u|, \mathcal{G}) = E(u, \mathcal{G})$, there is no loss of generality in assuming that solutions are non-negative. Suppose that for some vertex v we have that $u(v) = 0$. Since $u \geq 0$ we have that all the derivatives in (3.8) are non-negative. Hence, the Neumann-Kirchhoff conditions give us that

$$\frac{du_e}{dx_e}(v) = 0, \quad \text{for all } e \prec v.$$

Since for all theses edges we have that $u_e(v) = \frac{du_e}{dx_e}(v) = 0$ and that for each edge, u_e satisfies the ODE (3.6); then, by uniqueness of solution we have that $u_e \equiv 0$ for all edges incident to v. By repeating this argument to any neighbouring vertices to v and using the connectedness of $\mathcal{G}$ we get that $u \equiv 0$ on $\mathcal{G}$, which is a clear contradiction since u, as a ground state, has strictly positive mass. Finally, if there exists an edge e such that $u_e(x) = 0$ for some $x \in (0, l_e)$ then, from the non-negativity of u_e, we deduce again that $u_e(x) = \frac{du_e}{dx_e}(x) = 0$. From here, uniqueness of solution to the ODE (3.6) gives us that $u_e \equiv 0$ the edge e. In particular, $u = 0$ at the vertices that are incident with the edge e. From here the previous reasoning follows. ■

We finish this section by introducing the notion of rearrangement of functions on metric graphs. We refer the reader to Kawohl [20, Chapter II] and Kesavan [21, Chapter 1] as well as to the section A.4 of the Appendix A for rearrangements in subsets of $\mathbb{R}^N$. Since we assume solutions to be non-negative we introduce the notion of rearrangement on graphs only for these functions. Fix now a graph $\mathcal{G}$ and let $\omega = m(\mathcal{G}) := \sum_{e \in E(\mathcal{G})} m(I_e)$ be the length of $\mathcal{G}$, where m denotes the one dimensional Lebesgue measure.

Definition 3.18: *Distribution Function*

Let $\mathcal{G}$ be a metric graph, $u \in H^1(\mathcal{G})$, $u \geq 0$. The *distribution function* of u is defined as $\rho_u(t) : [0, +\infty) \to [0, \omega]$ by

$$\rho_u(t) := \sum_{e \in E(\mathcal{G})} m\left(\{x_e \in I_e : u_e(x_e) > t\}\right), \quad t \geq 0.$$

Note that for each $t > 0$ this function is finite. First, recall that all the graphs are assumed to be finite. Now, if $\mathcal{G}$ is compact then by the continuity of the function u we have that all the measures involved are finite. When $\mathcal{G}$ is non compact, that is $\omega = +\infty$, we need a bit more than the continuity alone, we need the function to vanish at infinity on the unbounded edges of the graph, see section A.4. This is automatically satisfied by $H^1(\mathcal{G})$ functions.

Definition 3.19: *Rearrangement*

We say that two functions are equimeasurable if they have the same distribution function. In this case we say that they are rearrangements of one another.

We can now define two particular types of rearrangements of a function $u \in H^1(\mathcal{G})$. We define:

(*i*) the *decreasing rearrangement* $u^\# : I^\# := [0, \omega) \to \mathbb{R}$ as the function

$$u^\#(x) := \inf\{t > 0 : \rho_u(t) < x\}. \tag{3.9}$$

(*ii*) the symmetric decreasing rearrangement, or Schwarz symmetrization, $u^* : I^* := \left(-\frac{\omega}{2}, \frac{\omega}{2}\right) \to \mathbb{R}$ as the function

$$u^*(x) := \inf\{t > 0 : \rho_u(t) < 2|x|\}. \tag{3.10}$$

Also, since no special adaptation to the proofs are required, just as in the Appendix A we refer the reader to [21] for the proof that these functions are equimeasurable. Consequently, from the definition of the L^p norms on graphs and Corollaries A.16 and A.18 we have that

$$\int_{\mathcal{G}} |u|^q dx = \int_{I^{\#}} |u^{\#}|^q dx = \int_{I^*} |u^*|^q dx \ \text{ for all } \ q \geq 1. \tag{3.11}$$

Remark 3.20:
Taking $q = 2$ in the above equalities says that the above rearrangements do not change the mass of a function.

In what follows we will need to compare energies of functions with the energy of their rearrangements. Holding the previous remark into account we see immediately that Polya-Szegő inequality, see Theorems A.19 and A.20 will be a fundamental tool to determine whether the energy is maintained or decreases, by providing information on the kinetic part of the energy functional. We now state and prove a Pólya-Szegő type inequality for metric graphs. Let us start by establishing some notation. Let $u \in H^1(\mathcal{G})$. For simplicity assume $u \geq 0$ and let $M := \sup_{x\in\mathcal{G}} u > 0$ and $m := \inf_{x\in\mathcal{G}} u \geq 0$. For $t \in (m, M)$ define

$$n(t) := \# \{x \in \mathcal{G} : u(x) = t\}.$$

Theorem 3.21: *A Pólya-Szegő type Inequality*

Let $\mathcal{G}$ be a connected metric graph and let $u \in H^1(\mathcal{G})$ be a non-negative function. Then

$$\int_{I^{\#}} |(u^{\#})'|^2 dx \leq \int_{\mathcal{G}} |u'|^2 dx, \tag{3.12}$$

with strict inequality unless $n(t) = 1$ for almost every $t \in (m, M)$. Also,

$$n(t) \geq 2 \ \text{ for a.e. } \ t \in (m, M) \Rightarrow \int_{I^*} |u^{*\prime}|^2 dx \leq \int_{\mathcal{G}} |u'|^2 dx, \tag{3.13}$$

where the equality implies that $n(t) = 2$ for almost every $t \in (m, M)$.

Before proceeding with the proof of the above result let us prove the following Lemma

Lemma 3.22

Let $\mathcal{G}$ be a metric graph and define

$$\mathcal{U}(\mathcal{G}) := \{u \in C(\mathcal{G}) : u_e \in C^\infty(I_e) \cap H^1(I_e), \ \ \forall e \in E(\mathcal{G})\}.$$

Then, $\mathcal{U}(\mathcal{G})$ is dense in $H^1(\mathcal{G})$.

Proof. Let $u \in H^1(\mathcal{G})$. We will split the proof by construction the approximation in two cases:

1. In finite edges. Let e be a bounded edge of the graph identified with the compact interval $I_e = [0, l_e]$. Consider the function $u_e \in I_e$. Since u_e is continuous take now the following affine function

$$f(x) = \frac{u(l_e) - u(0)}{l_e} x + u(0) \in H^1(0, l_e).$$

   Note now that $u_e - f \in H_0^1(0, l_e)$ Therefore, we can take a sequence $u_n \in C_c^\infty([0, l_e])$ such that $u_n \to u_e - f$ strongly in $H^1(0, l_e)$. Consequently, $v_n := u_n + f$ converges strongly to u_e in $H^1(0, l_e)$. Moreover, we have $v_n \in C^\infty([0, l_e]) \cap H^1(0, l_e)$, $u_e(0) = v_n(0)$ and that $u_e(l_e) = v_n(l_e)$ for all $n \in \mathbb{N}$.

2. In the half-lines. Fix any $x_0 > 0$ and the values $u_e(0)$ and $u_e(x_0)$. Take now ρ a $C^\infty(\mathbb{R}^+)$ cutoff function such that $\operatorname{supp}\rho \subset [0, x_0]$ and such that $\rho(0) = u_e(0)$. Again, $u_e - \rho \in H_0^1(\mathbb{R}^+)$ and therefore we can take a sequence $u_n \in C_c^\infty(\mathbb{R}^+)$ such that $u_n \to u_e - \rho$ strongly in $H^1(\mathbb{R}^+)$. Consequently, $v_v := u_n + \rho$ converges strongly to u_e in $H^1(\mathbb{R}^+)$. Moreover, by the way the function ρ was chosen we have that $v_n \in C^\infty(\mathbb{R}^+) \cap H^1(\mathbb{R}^+)$ and that $v_n(0) = u_e(0)$ for all $n \in \mathbb{N}$.

From items 1 and 2 we can now take a sequence $(w_n)_{n\in\mathbb{N}}$ continuous on the whole graph and that it is $C^\infty(I_e) \cap H^1(I_e)$ on each edge and that converges strongly in $H^1(\mathcal{G})$ to u. ■

With this lemma we can now prove Pólya-Szegő inequality on graphs.

Proof of Theorem 3.21. According to the previous lemma we apply a density argument:

1. **Proof for u** $\in \mathcal{U}(\mathcal{G})$. Let $u \in \mathcal{U}(\mathcal{G})$. Given the regularity of u on the edges we can speak of *critical points*. We define *critical points* as a point $x \in I_e$, where I_e identifies the edge e, for which $u_e'(x) = 0$ or as being a vertex. Now it follows from Sard's Theorem, see [19, Section 1.7], that the set of critical values of u has measure zero (note that even if a vertex is a critical point we only have a finite number of them), hence almost every $t \in (m, M)$ is a regular value of u . We now claim that $n(t)$ is finite for almost every $t \in (m, M)$. In order to see this we resort to a result from manifold theory. Consider $u = (u_e)_{e\in E(\mathcal{G})}$, where $u_e : I_e \to \mathbb{R}$. By definition, we have that $n(t) = \sum_{e\in E(\mathcal{G})} n_e(t)$, where $n_e(t) := \#\{x \in I_e : u_e(x) = t\}$. If I_e is compact, since u_e is also C^∞ and t is a regular value, then, from the Regular Level Set Theorem, see [25, Theorem 9.9], $u_e^{-1}\{t\}$ defines a 0 dimensional submanifold of I_e. Moreover, it is compact. Recalling that 0-dimensional compact manifolds are finite we get that $n_e(t) < \infty$ for each finite edge of the graph. For the unbounded edges recall that since $u \in H^1(I_e)$, $\lim_{|x|\to\infty} u_e(x) = 0$. This, together with the fact that t is a regular value, yields that also in the unbounded case $n_e(t) < \infty$, whence $n(t)$ is also finite.
   Applying the Coarea Formula, see [21, Section 2.2], we get that
$$\int_{\mathcal{G}} |u'|^2 dx = \int_m^M \int_{u^{-1}\{t\}} |u'| d\sigma dt,$$
   where $d\sigma$ is the counting measure. With this in mind we can rewrite the kinetic term as:
$$\int_{\mathcal{G}} |u'|^2 dx = \int_m^M \sum_{x\in u^{-1}\{t\}} |u'(x)| dt.$$
   It also follows by the definition of $n(t)$ and Cauchy-Schwarz inequality that
$$\begin{aligned} n(t)^2 = \left(\sum_{x\in u^{-1}\{t\}} 1\right)^2 &= \left(\sum_{x\in u^{-1}\{t\}} |u'(x)|^{\frac{1}{2}} |u'(x)|^{-\frac{1}{2}}\right)^2 \\ &\le \sum_{x\in u^{-1}\{t\}} |u'(x)| \sum_{x\in u^{-1}\{t\}} |u'(x)|^{-1}. \end{aligned}$$
   Therefore,
$$\left(\sum_{x\in u^{-1}\{t\}} |u'(x)|^{-1}\right)^{-1} n(t)^2 \le \sum_{x\in u^{-1}\{t\}} |u'(x)|. \tag{3.14}$$
   Using (3.14) and the fact that $n(t) \ge 1$ we have the following estimate for the L^2 norm of the derivative:
$$\int_{\mathcal{G}} |u'|^2 dx \ge \int_m^M \left(\sum_{x\in u^{-1}\{t\}} |u'(x)|^{-1}\right)^{-1} = -\int_m^M \frac{dt}{\rho_u'(t)}. \tag{3.15}$$

In the last equality we used [21, Theorem 2.2.3]. If we now consider the function $u^{\#}$ the same precise computations hold for the rearrangement too. However, since $u^{\#}$ is strictly decreasing $n_{u^{\#}}(t) = 1$ for almost every $t \in (m, M)$, where $n_{u^{\#}}(t)$ is the number of elements in the pre-image of the function $u^{\#}$ at the level $t \in (m, M)$, and all the computations above hold with equalities. The conclusion now follows from the equimeasurability between u and $u^{\#}$, which means they have the same distribution function, whence

$$\int_{\mathcal{G}} |u^{\#\prime}|^2 dx = -\int_m^M \frac{dt}{\rho'_{u^{\#}}(t)} = -\int_m^M \frac{dt}{\rho'_u(t)} \leq \int_{\mathcal{G}} |u'|^2 dx.$$

This proves (3.12). To prove (3.13) the computations are exactly the same. The only thing to notice is that since u^* is an even function then $n_{u^*}(t) = 2$ for almost every $t \in (m, M)$, where $n_{u^*}(t)$ is the number of elements in the pre-image of the function u^* at the level $t \in (m, M)$, thus the computations hold as long as $n(t) \geq 2$. If we suppose that an equality holds, then carrying out the computations backwards yields that $n(t) = 2$.

2. **Passing to the limit**
Let now $u \in H^1(\mathcal{G})$ and $(u_n)_{n\in\mathbb{N}} \subset \mathcal{U}(\mathcal{G})$ be such that $u_n \to u$ in $H^1(\mathcal{G})$. It then follows by the convergence in L^2 of both the function and its derivative that

$$\int_{\mathcal{G}} |u'|^2 dx = \lim \int_{\mathcal{G}} |u_n'|^2 dx \geq \lim \int_0^{m(\mathcal{G})} |u_n^{\#\prime}|^2 dx = \int_0^{m(\mathcal{G})} |u^{\#\prime}|^2 dx.$$

We just point out that the last equality holds because the operator $\# : L^2(\mathcal{G}) \to L^2(0, m(\mathcal{G}))$ is continuous, see [21, Theorem 1.2.3]. Similarly we have the same argument for the Schwarz symmetrization.

■

Remark 3.23:
Note that Theorem 3.21 and the equalities in (3.11) allows one to conclude that for any graph $\mathcal{G}$, $u^{\#} \in H^1(0, \omega)$. It also gives a sufficient condition for the Schwarz symmetrization u^* to be in $H^1\left(-\frac{\omega}{2}, \frac{\omega}{2}\right)$.

To finish this chapter, now that the notion of rearrangement was already introduced in graphs, we show some universal Gagliardo-Nirenberg-type inequalities in non-compact graphs.

Proposition 3.24: *Universal Gagliardo-Nirenberg Inequality*

Let $\mathcal{G}$ be a non-compact metric graph. Then, there exists C_p, depending only on p, such that

$$\|u\|_{L^p(\mathcal{G})}^p \leq C_p \|u\|_{L^2(\mathcal{G})}^{\frac{p}{2}+1} \|u'\|_{L^2(\mathcal{G})}^{\frac{p}{2}-1} \quad \text{for all } u \in H^1_\mu(\mathcal{G}). \tag{3.16}$$

Proof. We know, from Corollary A.4, that the same inequality holds when the domain is $\mathbb{R}$. Then, it also holds when the domain is $\mathbb{R}^+$ by simply taking the even extension to the whole $\mathbb{R}$.

Take now $u \in H^1(\mathcal{G})$. Given that $\mathcal{G}$ is non-compact we can consider the decreasing rearrangement of u, $u^{\#} : \mathbb{R}^+ \to \mathbb{R}$. The equimeasurability of $u^{\#}$ and u it follows that

$$\|u\|_{L^p(\mathcal{G})}^p = \|u^{\#}\|_{L^p(\mathbb{R}^+)}^p.$$

Applying the Gagliardo-Nirenberg inequality for $\mathbb{R}^+$ yields

$$\|u\|_{L^p(\mathcal{G})}^p \leq C_p \|u^{\#}\|_{L^2(\mathbb{R}^+)}^{\frac{p}{2}+1} \|u^{\#\prime}\|_{L^2(\mathbb{R}^+)}^{\frac{p}{2}-1} \leq C_p \|u\|_{L^2(\mathcal{G})}^{\frac{p}{2}+1} \|u'\|_{L^2(\mathcal{G})}^{\frac{p}{2}-1},$$

where the last inequality holds by the preservation of mass and Polya-Szegő's inequality (3.12). ■

Also, using Corollary A.5, we prove in the exact same way

Proposition 3.25: *Universal Gagliardo-Nirenberg Inequality for L^∞ norm*

Let $\mathcal{G}$ be a non-compact metric graph. Then, there exists C_p, dependig only on p, such that

$$\|u\|^2_{L^\infty(\mathcal{G})} \leq C_p \|u\|_{L^2(\mathcal{G})} \|u'\|_{L^2(\mathcal{G})} \quad \text{for all } u \in H^1_\mu(\mathcal{G}). \tag{3.17}$$

Remark 3.26:
The term universal here is used to express that the constants in both the embeddings are independent of the graph. Finally note that these inequalities are valid only for non-compact graphs. We will see in the next chapter that these inequalities will only be required in the non-compact case.

Chapter 4

Existence and Non-existence Results

Our interest in this chapter is to study the existence or non existence of ground states of prescribed mass for the NLS energy functional on metric graphs. Recall that the problem at hand is to

$$\text{find } u_0 \in H^1_\mu(\mathcal{G}) \text{ such that } E(u_0, \mathcal{G}) = \inf_{u \in H^1_\mu(\mathcal{G})} E(u, \mathcal{G}). \tag{4.1}$$

Throughout this chapter $\mathcal{G}$ is a connected metric graph. We will follow closely the approach taken [1] and in [2]. The focus of these references is only on non-compact graphs; here we present a detailed description of what happens in the compact case.

Lemma 4.1: *A compact embedding in graphs*

Let $p \geq 1$. If $\mathcal{G}$ is a *compact* metric graph then $H^1(\mathcal{G})$ is compactly embedded in $L^p(\mathcal{G})$.

Proof. Let $p \geq 1$, $\mathcal{G}$ a compact metric graph and $u \in H^1(\mathcal{G})$. Firstly notice that

$$\|u\|^p_{L^p(\mathcal{G})} = \sum_{e \in E(\mathcal{G})} \|u_e\|^p_{L^p(I_e)}.$$

Since $\mathcal{G}$ is compact, all its edges are bounded. Thus, from Corollary A.9 it follows that,

$$\|u\|^p_{L^p(\mathcal{G})} \leq C(p, \mathcal{G}) \|u\|^p_{H^1(\mathcal{G})}.$$

with compact embedding because we can write the inclusion operator on each edge as

$$\iota : H^1(I_e) \xrightarrow{\iota_1} C(I_e) \xrightarrow{\iota_2} L^p(I_e),$$

with both ι_1, ι_2 linear and continuous and, ι_1 compact. Moreover, we have a finite number of edges. ■

Thus, the direct method of the calculus of variations can be carried out in these graphs

Proposition 4.2: *Existence of solution in a compact graph*

Let $\mathcal{G}$ be a compact metric graph, $\mu > 0$ and $p \in (2, 6)$. Then, problem (4.1) admits a solution.

Proof. Fix $\mathcal{G}$ a compact metric graph. We show that in this case E still remains bounded from below. From applying Corollary A.7 on each edge we get that:

$$\|u\|^p_{L^p(\mathcal{G})} \leq C \|u\|^{\frac{p}{2}+1}_{L^2(\mathcal{G})} \|u\|^{\frac{p}{2}-1}_{H^1(\mathcal{G})}, \quad \text{for all } u \in H^1(\mathcal{G}),$$

where the constant depends also on the graph. Plugging this estimate in the functional we get a lower bound for E as in (2.7). Let then $(u_n)_{n \in \mathbb{N}}$ be a minimizing sequence. This new estimate on the graph gives us that (u_n) is bounded in $H^1(\mathcal{G})$, just as in Lemma 2.8; whence there exists $u \in H^1(\mathcal{G})$ such that $u_n \rightharpoonup u$ in $H^1(\mathcal{G})$ up to a subsequence. From here, invoking the compactness of the previous lemma, the proof is straightforward. ■

Having this in mind the search for ground states for the NLS energy functional in graphs is only non standard for non-compact graphs. We will also see that the existence of ground states depends strongly on the topology and metric properties of the graphs.

As said before, the case $\mathcal{G} = \mathbb{R}$, that is, for a graph like the one in Figure 3.5, falls under the scope of this new problem. The following *a priori* estimates show that in fact ground states to the minimization problem in $\mathbb{R}$, that is, solitons, provide upper and lower bounds for the energy of the solutions (should they exist) to the problem (4.1).

Theorem 4.3

Let $\mathcal{G}$ be a non-compact metric graph. Then we have that

$$\frac{1}{2}E(\varphi_{2\mu}, \mathbb{R}) = \min_{u\in H^1_\mu(\mathbb{R}^+)} E(u, \mathbb{R}^+) \le \inf_{u\in H^1_\mu(\mathcal{G})} E(u, \mathcal{G}) \le \min_{u\in H^1_\mu(\mathbb{R})} E(u, \mathbb{R}) = E(\varphi_\mu, \mathbb{R}),$$

where φ_μ and $\varphi_{2\mu}$ are the unique positive and even solutions to the minimization problem in $\mathbb{R}$. Moreover, these inequalities hold even if the infimum is not attained.

Before proving the above result let us prove a lemma that justifies the first equality.

Lemma 4.4: *Minimization Problem on a half-line.*

Let $\mu > 0$ and $p \in (2, 6)$. Consider now problem (4.1) where $\mathcal{G} = \mathbb{R}^+$. Then the infimum is attained at $\varphi_{2\mu}|_{\mathbb{R}^+}$ and, moreover,

$$\min_{u\in H^1_\mu(\mathbb{R}^+)} E(u, \mathbb{R}^+) = \frac{1}{2}E(\varphi_{2\mu}, \mathbb{R}).$$

Proof. Note that if $\varphi_{2\mu}$ is a ground state to the problem

$$\min_{u\in H^1_{2\mu}(\mathbb{R})} E(u, \mathbb{R})$$

then the restriction of $\varphi_{2\mu}$ to $\mathbb{R}^+$ is an admissible solution to

$$\inf_{u\in H^1_\mu(\mathbb{R}^+)} E(u, \mathbb{R}^+).$$

Indeed, since $\varphi_{2\mu}$ is even, the mass constraint is automatically satisfied and moreover,

$$E(\varphi_{2\mu}, \mathbb{R}^-) = E(\varphi_{2\mu}, \mathbb{R}^+),$$

whence

$$E(\varphi_{2\mu}, \mathbb{R}^+) = \frac{1}{2}E(\varphi_{2\mu}, \mathbb{R}).$$

We now claim that the above function is in fact the solution. By way of contradiction, suppose that there exists $v \in H^1_\mu(\mathbb{R}^+)$ such that $E(v, \mathbb{R}^+) < E(\varphi_{2\mu}, \mathbb{R}^+)$. Let now w be the even extension of v to the whole real line. Clearly, $w \in H^1(\mathbb{R})$ with mass 2μ, that is, $w \in H^1_{2\mu}(\mathbb{R})$. However,

$$\begin{aligned} E(w, \mathbb{R}) &= \frac{1}{2}\int_0^{+\infty} |v'(x)|^2 dx - \frac{1}{p}\int_0^{+\infty} |v(x)|^p dx + \frac{1}{2}\int_{-\infty}^0 |v'(-x)|^2 dx - \frac{1}{p}\int_{-\infty}^0 |v(-x)|^p dx \\ &= 2E(v, \mathbb{R}^+) < 2E(\varphi_{2\mu}, \mathbb{R}^+) = E(\varphi_{2\mu}, \mathbb{R}), \end{aligned}$$

which is a contradiction. ∎

We now conclude this initial section with the proof of Theorem 4.3.

Proof of Theorem 4.3. Let $\mathcal{G}$ be any non-compact metric graph. We begin by showing that the energy of a soliton of mass μ in $\mathbb{R}$ gives an upper bound for $\inf_{u\in H^1_\mu(\mathcal{G})} E(u,\mathcal{G})$.

Let $\varphi_\mu \in H^1_\mu(\mathbb{R})$ denote the unique positive and even ground state for problem (2.6). We will now construct a family of functions of compact support converging strongly in $H^1(\mathbb{R})$ to φ_μ. To do so take $\eta \in C^\infty(\mathbb{R})$ a cutoff function such that $\eta(x) = 0$ if $|x| \geq 2$ and $\eta(x) = 1$ if $|x| \leq 1$ and such that $|\eta| \leq 1$. Now, let $\epsilon > 0$ and consider $\eta_\epsilon(x) = \eta(\epsilon x)$. For

$$c_\epsilon = \left(\frac{\mu}{\int_{\operatorname{supp}\eta_\epsilon} |\varphi_\mu|^2 \eta_\epsilon^2 dx} \right)^{\frac{1}{2}} \searrow 1 \ \text{ as } \ \epsilon \to 0 \tag{4.2}$$

we define

$$u_\epsilon(x) := c_\epsilon \varphi_\mu(x) \eta_\epsilon(x). \tag{4.3}$$

For this choice of c_ϵ note that we have $u_\epsilon \in H^1_\mu(\mathbb{R})$, for each $\epsilon > 0$. We now claim that $u_\epsilon \to \varphi_\mu$ as $\epsilon \to 0$ in $H^1(\mathbb{R})$. Recall that φ_μ, by regularity of the solitons, is C^∞ then, since η is smooth and $c_\epsilon \to 1$ we have that

$$u_\epsilon \to \varphi_\mu \ \text{ and } u'_\epsilon \to \varphi'_\mu \ \text{ as } \ \epsilon \to 0$$

pointwise in $\mathbb{R}$. Moreover,

$$|u_\epsilon(x)|^2 \leq c_\epsilon^2 |\varphi_\mu(x)|^2 |\eta_\epsilon(x)|^2 \leq K |\varphi_\mu(x)|^2,$$

which is integrable, and also, taking ϵ small

$$|u'_\epsilon(x)|^2 \leq |\varphi'_\mu(x)|^2 + |\varphi_\mu(x)|^2 \epsilon^2 |\eta'(\epsilon x)|^2 \leq K \left(|\varphi'_\mu(x)|^2 + |\varphi_\mu(x)|^2 \right).$$

From Lebesgue's dominated convergence theorem we have that

$$u_\epsilon \to \varphi_\mu \ \text{ and } u'_\epsilon \to \varphi'_\mu \ \text{ in } \ L^2(\mathbb{R}),$$

whence $u_\epsilon \to \varphi_\mu$ in $H^1(\mathbb{R})$ as $\epsilon \to 0$, and in particular $\lim_{\epsilon\to 0} E(u_\epsilon, \mathbb{R}) = E(\varphi_\mu, \mathbb{R})$.

Assume now that $\operatorname{supp}\ u_\epsilon$ is contained in the interval $[0, +\infty)$ (taking $v_\epsilon(x) = u_\epsilon(x + \frac{2}{\epsilon})$, which satisfies $\operatorname{supp} v_\epsilon \subset [0, +\infty)$, does the job). Even though the convergence to φ_μ is lost, we still have that $\lim_{\epsilon\to 0} E(v_\epsilon, \mathbb{R}) = E(\varphi_\mu, \mathbb{R})$. Since $\mathcal{G}$ is non-compact it has an unbounded edge, thus by identifying one such edge with the above interval we can also consider $v_\epsilon \in H^1_\mu(\mathcal{G})$ simply extending by zero to the remaining edges. Then

$$\inf_{u\in H^1_\mu(\mathcal{G})} E(u,\mathcal{G}) \leq \lim_{\epsilon\to 0} E(v_\epsilon,\mathcal{G}) = \lim_{\epsilon\to 0} E(v_\epsilon,\mathbb{R}) = E(\varphi_\mu,\mathbb{R}) = \min_{u\in H^1_\mu(\mathbb{R})} E(u,\mathbb{R}),$$

which establishes the second inequality in the statement.

Let us now prove the first inequality. Note that for all $u \in H^1_\mu(\mathcal{G})$ we can define its decreasing rearrangement $u^\# \in H^1(\mathbb{R}^+)$. Since the mass is preserved by this rearrangement, see (3.11), and, by Theorem 3.21, the kinetic part of the energy functional does not increase, we conclude that

$$\inf_{\varphi\in H^1_\mu(\mathbb{R}^+)} E(\varphi,\mathbb{R}^+) \leq \inf_{u\in H^1_\mu(\mathcal{G})} E(u,\mathcal{G}).$$

Recalling Lemma 4.4 the proof is finished. ■

4.1 Nonexistence Results for Non-Compact Graphs

For non-compact graphs there is no general theory in what concerns the existence of ground states for the NLS energy functional (3.3). However, in [1] it was shown that the topology of a graph has an important effect in what comes to the existence or non-existence of ground states. In [1] a topological assumption that rules out the existence of minimizers was introduced and it is given by:

Let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a metric graph. After the removal of the interior of any edge $e \in E(\mathcal{G})$, every connected component of the subgraph $(V(\mathcal{G}), E(\mathcal{G}) \setminus \{e\})$ contains at least one vertex at infinity. (H)

In the figures below we can see one example where the assumption (H) is satisfied and another where it is not.

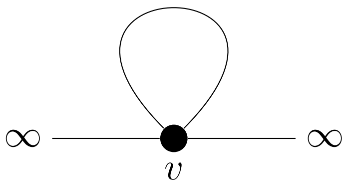

Figure 4.1: A metric graph which satisfies (H)

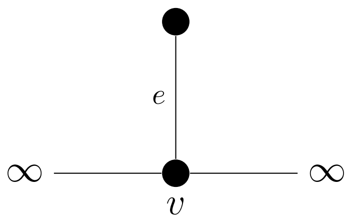

Figure 4.2: A metric graph for which (H) is not satisfied. The removal of the interior of the edge e segregates all the infinity vertices in one of the two connected component.

Remark 4.5:

1. A *cut-edge* of a metric graph is an edge which upon the removal of its interior disconnects the graph. Note that condition (H) is only relevant for these edges, otherwise the condition is trivially satisfied. This makes in practice the verification of condition (H) simpler;
2. Note that (H) implies the existence of at least two vertices at infinity in $\mathcal{G}$. This follows by noticing that every vertex at infinity, having degree one, the edge connecting it to the graph is necessarily a cut-edge.

The goal of this whole section is outlined in the following. Note that from assumption (H) we can injectively immerse $\mathbb{R}$, as a metric graph, inside $\mathcal{G}$. This immersion will then allow us to deduce that for all the graphs that satisfy condition (H), $\inf_{u \in H^1_\mu(\mathcal{G})} E(u, \mathcal{G}) = E(\varphi_\mu, \mathbb{R})$. The assumption is crucial to deduce that $n(t) = \#\{x \in \mathcal{G} : u(x) = t\} = 2$ for almost every $t \in (0, \sup_{\mathcal{G}} u)$. Then, Theorem 3.21 gives us the equality and that no solutions exist except if the graph $\mathcal{G}$ has certain specific topologies that we can fully describe. This means that under (H) the question of existence and non-existence of ground states can be completely classified.

The next key step is to make clear how we can immerse $\mathbb{R}$ inside the graph. For this we need the following lemma.

Lemma 4.6: Necessary condition for the assumption (H)

Let $\mathcal{G}$ be a metric graph. If $\mathcal{G}$ satisfies (H) then, as a metric space, $\mathcal{G}$ satisfies the following condition:

For every $x_0 \in \mathcal{G}$, there exist two injective curves $\gamma_1, \gamma_2 : [0, +\infty) \to \mathcal{G}$ parametrized by arclength, with disjoint images except for finitely many points, and such that $\gamma_1(0) = x_0 = \gamma_2(0)$. (H')

Proof. Let us define $\mathcal{T}$ to be the set of trails in $\mathcal{G}$ such that the original and terminal vertices are vertices at infinity. Since $\mathcal{G}$ is connected and satisfies (H), it has at least two vertices at infinity, therefore, $\mathcal{T}$ is non-empty.

Suppose now that $x_0 \in \mathcal{G}$ is covered by at least one trail $T \in \mathcal{T}$. Such trail allows us to define $\gamma : \mathbb{R} \to \mathcal{G}$ an immersion between $\mathbb{R}$ and $\mathcal{G}$. Due to possible loops this immersion does not need to be injective, however, by removing all the loops we assume it is injective (up to a finite number of points). We can now explicitly construct two curves α_1 and α_2 in the following way:

$$\alpha_1(x) := \gamma(x_0 + x) \text{ and } \alpha_2(x) := \gamma(x_0 - x), \text{ for } x \geq 0.$$

Since $T \in \mathcal{T}$ both α_1 and α_2 will be trapped in a half-line at some point, therefore α_1 and α_2 are indeed defined on $[0, +\infty)$. We may re-parametrize by arclength both these curves to create $\gamma_1, \gamma_2 : [0, +\infty) \to \mathcal{G}$ as in the statement. Moreover, since T is a trail we have that no edge is repeated and therefore if both curves intersect it will only be at vertices, which are in finite number.

We now prove that we can cover the whole graph by trails of $\mathcal{T}$. Let $E_0 \subset E(\mathcal{G})$ be the set of edges that are not covered by any trail $T \in \mathcal{T}$. Suppose, by way of contradiction, that $E_0 \neq \emptyset$. Since $\mathcal{G}$ is connected there exists $e \in E_0$ such that one of its terminal or original vertices will be in some trail T; call it w. Moreover, since vertices at infinity have degree one we have that w is not a vertex at infinity, otherwise $e \notin E_0$. We now split into two cases:

1. e is a cut-edge. Then, by (H) there exists v_1 a vertex at infinity in the connected component that is disjoint from T. Now take a trail that connects v_1 and w while passing through e. Following the rest of the trail T from w to another vertex at infinity (which is different from v_1, because e is a cut-edge) we have constructed, by concatenating both trails, a trail that uses the edge e, which is a contradiction.
2. e is not a cut-edge. Then there exists a cycle C in $\mathcal{G}$ such that e is an edge of the cycle. Since e is connected to T through w we can construct a trail T' from T by going through the cycle C, which is again a contradiction.

■

Remark 4.7:
It is easy to check that the conditions (H) and (H') are equivalent. The previous proof shows the sufficient condition. As for the necessary one, take $\mathcal{G}$ satisfying (H') but not (H). Take an edge $e \in E(\mathcal{G})$ that disconnects the graph. Let $\mathcal{G}'$ denote the connected component that, as a subgraph of $(V(\mathcal{G}), E(\mathcal{G}) \setminus \{e\})$, is compact. For any $x_0 \in \mathcal{G}'$ applying (H') implies that $\mathcal{G}'$ contains at least two vertices at infinity, since the curves γ_1 and γ_2 have infinite length and are injective except in a finite number of points. This is a contradiction.

The next step is to prove:

Theorem 4.8

If $\mathcal{G}$ satisfies (H), then

$$\inf_{u \in H^1_\mu(\mathcal{G})} E(u, \mathcal{G}) = \min_{u \in H^1_\mu(\mathbb{R})} E(u, \mathbb{R}) = E(\varphi_\mu, \mathbb{R}). \tag{4.4}$$

Proof. By Theorem 4.3 we have that

$$\inf_{u \in H^1_\mu(\mathcal{G})} E(u, \mathcal{G}) \leq \min_{u \in H^1_\mu(\mathbb{R})} E(u, \mathbb{R}) = E(\varphi_\mu, \mathbb{R}). \tag{4.5}$$

We now claim that

$$\min_{v \in H^1_\mu(\mathbb{R})} E(v, \mathbb{R}) = E(\varphi_\mu, \mathbb{R}) \leq E(u, \mathcal{G}), \text{ for all } u \in H^1_\mu(\mathcal{G}). \tag{4.6}$$

From here, taking the infimum over the set $H^1_\mu(\mathcal{G})$ in (4.6), together with the inequality in (4.5) gives the intended equality. Let us prove the claim. Fix $u \in H^1_\mu(\mathcal{G})$. Without loss of generality, assume that $u \geq 0$. Let now $M := \sup_{\mathcal{G}} u > 0$, which is attained since $u \to 0$ in the unbounded edges, and note that $\inf_{\mathcal{G}} u = 0$ for the same reason. Take $u^* \in H^1_\mu(\mathbb{R})$ to be the Schwarz symmetrization of u, as in (3.10), then

$$\min_{u \in H^1_\mu(\mathbb{R})} E(u, \mathbb{R}) \leq E(u^*, \mathbb{R}).$$

Provided we can prove that $n(t) = \#u^{-1}\{t\} \geq 2$ for almost every $t \in (0, M)$, then it follows from Polya-Szegő inequality, Theorem 3.21, that indeed $E(u^*, \mathbb{R}) \leq E(u, \mathcal{G})$.

By assumption we have that $\mathcal{G}$ satisfies (H), and therefore, by Lemma 4.6, also (H'). Let $x_0 \in \mathcal{G}$ be such that $u(x_0) = M$ and γ_1 and γ_2 be as in Lemma 4.6. Define the function $v : \mathbb{R} \to \mathbb{R}$, by

$$v(z) = \begin{cases} u(\gamma_1(z)), \ z \geq 0, \\ u(\gamma_2(-z)), \ z < 0. \end{cases}$$

Since v is continuous and for z large enough γ_1 and γ_2 parametrize half lines, we then have that

$$\lim_{|z| \to +\infty} v(z) = 0.$$

Thus, $\#v^{-1}\{t\} \geq 2$ for almost every $t \in (0, M)$. Moreover, since the curves only intersect at most at a finite number of points (which have measure 0), we conclude that $n(t) \geq 2$ for almost every $t \in (0, M)$ and the conclusion follows. ∎

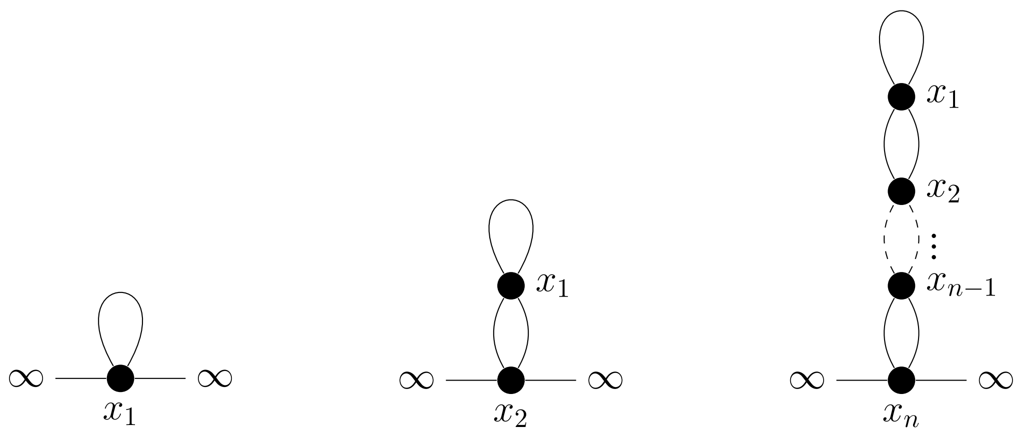

Figure 4.3: The family of graphs for which assumption (H) is satisfied and ground states exist.

We mentioned before that under (H) specific topologies give rise to the existence of ground states. Take the following examples:

- The simplest graphs that satisfy (H) are those that are isometric to $\mathbb{R}$ like the one in Figure 3.5 above. The unique positive and even solution for problem (4.1) is the soliton φ_μ, since it can now be seen as an element of $H^1_\mu(\mathcal{G})$. Note that any other better competitor on the graph would lead to the existence of a function in $\mathbb{R}$ with less energy than that of φ_μ, which is a contradiction.
- Now take $x_1 > 0$ and let $\mathcal{G}$ be identified with the quotient space $\mathbb{R}/\{\pm x_1\}$. This creates a graph whose topology is the same as in Figure 4.3 (on the left). We can exploit the symmetry of a soliton of mass μ in $\mathbb{R}$ in order to construct a ground state here. Gluing the soliton φ_μ at the points $\varphi_\mu(-x_1) = \varphi_\mu(x_1)$ gives rise to a function $u \in H^1_\mu(\mathcal{G})$ with the same energy than that of φ_μ in $\mathbb{R}$. Consequently we have a ground state. This procedure is illustrated in Figure 4.7.

- The previous argument can be generalized to any finite number of points $x_n > \cdots > x_1 > 0$. Considering $\mathcal{G}$ the graph which we obtain from $\mathbb{R}$ by gluing together the pairs of points $\pm x_n, \cdots, \pm x_1$, which can be seen as in Figure 4.3 (on the right), we can use the same procedure to construct a ground state. The procedure is illustrated in Figure 4.8.

To finish this section we prove the result which states that under assumption (H) no ground states exist with the exception of graphs isometric to the ones in Figures 3.5 and 4.3. The main idea of the proof is to show, using the equality in Theorem 4.8, that if a ground state is present then $\mathcal{G}$ needs to have the topologies described above.

Theorem 4.9: *Under (H) ground states exist if and only if $\mathcal{G}$ has specific topologies*

Let $\mathcal{G}$ be a metric graph. If $\mathcal{G}$ satisfies (H) then

$$\inf_{u\in H^1_\mu(\mathcal{G})} E(u,\mathcal{G}) = E(\varphi_\mu,\mathbb{R})$$

is never attained unless $\mathcal{G}$ is isometric to the graphs depicted in Figures 3.5 and 4.3.

Proof. Let $\mathcal{G}$ be a metric graph for which condition (H) is satisfied. Let $u \in H^1_\mu(\mathcal{G})$ be a solution to problem (4.1). Without loss of generality assume $u > 0$ and let $0 < M := \sup_{\mathcal{G}} u$. Take $u^* \in H^1_\mu(\mathbb{R})$ to be the Schwarz symmetrization of u, and recall from the proof of Theorem 4.8 that $n(t) \geq 2$ for almost every $t \in (0, M)$. From equality (4.4) we deduce immediately the following equalities:

$$E(\varphi_\mu,\mathbb{R}) = E(u^*,\mathbb{R}) = E(u,\mathcal{G}).$$

It follows from here the following:

1. the first equality together with Theorem 2.16 gives us $u^* = \varphi_\mu$;
2. the second equality together with Theorem 3.21 yields $n(t) = 2$ for almost every $t \in (0, M)$.

Take again $x_0 \in \mathcal{G}$ such that $u(x_0) = M$ and curves $\gamma_1, \gamma_2 : [0, +\infty) \to \mathcal{G}$ parametrized by arclength with $\gamma_1(0) = \gamma_2(0) = x_0$, as in (H'). Let Γ_i denote the image of the curve γ_i, $i = 1, 2$.

We claim that

$$\mathcal{G} = \Gamma := \Gamma_1 \cup \Gamma_2.$$

In other words, we claim that we can cover the graph with a trail.

Define the function $v : \mathbb{R} \to \mathbb{R}$, by

$$v(x) = \begin{cases} u(\gamma_1(z)), & z \geq 0 \\ u(\gamma_2(-z)), & z < 0 \end{cases}.$$

Since v is continuous we conclude that $n(t) \geq 2$ for almost every $t \in (0, M)$, in fact the previous argument shows that $\#\left(u^{-1}\{t\} \cap \Gamma\right) \geq 2$ for almost every $t \in (0, M)$.

From item 2. above we can deduce that the values of $t \in (0, M)$ attained by u on $\mathcal{G} \setminus \Gamma$ forms a set of measure 0. Indeed, notice that $\mathcal{G}$ can be written as the disjoint union of Γ and $\mathcal{G} \setminus \Gamma$. Then,

$$n(t) = \#\left(u^{-1}\{t\} \cap \Gamma\right) + \#\left(u^{-1}\{t\} \cap (\mathcal{G} \setminus \Gamma)\right),$$

whence

$$2 \geq 2 + \#\left(u^{-1}\{t\} \cap (\mathcal{G} \setminus \Gamma)\right),$$

which in turn implies that $u^{-1}\{t\} \cap (\mathcal{G} \setminus \Gamma) = \emptyset$, for almost every $t \in (0, M)$. Thus,

$$m\left(\left\{t \in (0, M) : u^{-1}\{t\} \cap \mathcal{G} \setminus \Gamma \neq \emptyset\right\}\right) = 0.$$

Since Γ is a trail we have that Γ is sequentially closed in $\mathcal{G}$, and since $\mathcal{G}$ is a metric space, Γ is closed in $\mathcal{G}$. Therefore, $\mathcal{G} \setminus \Gamma$ is an open set. From the continuity of u we infer that u is constant on any edge

$e \in \mathcal{G} \setminus \Gamma$. From the equimeasurability between $u^* = \varphi_\mu$ and u we can in fact deduce that $u \equiv 0$ in $\mathcal{G} \setminus \Gamma$, because every level set of φ_μ has measure 0 and therefore so do the ones of u. However, from Proposition 3.17, $u > 0$ on $\mathcal{G}$. If $\mathcal{G} \setminus \Gamma$ is not empty then this leads to a contradiction, whence $G = \Gamma$.

To finish the proof we show that $\mathcal{G}$ has to be precisely one of the graphs in the statement. From (H') we know that the curves γ_1 and γ_2 intersect at a finite number of points $x_0, x_1, \cdots, x_n \in \mathcal{G}$, where x_0 is their common origin. If $n = 0$, meaning they only intersect at the starting point, then the graph $\mathcal{G} = \Gamma$ is isometric to the real line; if $n = 1$ then $\mathcal{G} = \Gamma$ is isometric to the second graph in the Figure 4.3. For the case $n > 1$ we need to be more detailed. Notice that since γ_1 and γ_2 are injective curves, $E(v, \mathbb{R}) = E(u, \mathcal{G}) = E(u^*, \mathbb{R}) = E(\varphi_\mu, \mathbb{R})$. Moreover, since $\mathcal{G} = \Gamma$ we have also that $v \in H^1_\mu(\mathbb{R})$. Thus, by uniqueness, $v = \varphi_\mu$. Now take $z_1, z_2 > 0$ such that $\gamma_1(z_1) = \gamma_2(z_2)$; the definition and parity of v imply that $z_1 = z_2$. Since the curves are parametrized by arclength, then both curves have the same length between any two intersection points. This yields that the graph has to be isometric to the one in Figure 4.3 (the one on the right). ∎

4.2 Existence Results for Non-compact Graphs

Note that in the previous section only topological arguments were used to rule out the existence of ground states. In what comes, following closely reference [2], we show that topological arguments alone are not enough to deduce existence of minimal energy solutions of fixed mass, $\mu > 0$. We will in fact see that there exists an intimate connection between μ and some metric properties of the graphs, such as lengths of bounded edges.

4.2.1 *A priori* Estimates for Minimizers

Before progressing to questions of existence we want to show some new *a priori* estimates for ground states of problem (4.1).

Let us establish some notation for the remainder of this section. The important part of the approach taken in [2] was to consider the ground state energy level as a function of the mass, $\mu \geq 0$, which is defined, for a fixed graph $\mathcal{G}$, by

$$\mathcal{E}_{\mathcal{G}}(\mu) := \inf_{u \in H^1_\mu(\mathcal{G})} E(u, \mathcal{G}). \tag{4.7}$$

In the previous section we have indeed shown that for any non-compact graph,

$$E(\varphi_\mu, \mathbb{R}^+) = \frac{1}{2} E(\varphi_{2\mu}, \mathbb{R}) \leq \mathcal{E}_{\mathcal{G}}(\mu) \leq E(\varphi_\mu, \mathbb{R}). \tag{4.8}$$

With this notation, and the notation in Theorem 2.18, we can explicitly give upper and lower bounds for the energy of a ground state.

Lemma 4.10

Let $\mu > 0$ and $\mathcal{G}$ be a non-compact graph. Then if $\theta_p := -E(\varphi_1, \mathbb{R}) > 0$ we have that

$$-2^{2\beta} \theta_p \mu^{2\beta+1} \leq \mathcal{E}_{\mathcal{G}}(\mu) \leq -\theta_p \mu^{2\beta+1}, \tag{4.9}$$

for $\beta = \frac{p-2}{6-p}$.

Proof. Let $\mu > 0$ and $\mathcal{G}$ be a non-compact graph. From Theorem 2.18 we have the following scaling rule for solitons

$$\varphi_\mu(x) = \mu^\alpha \varphi_1(\mu^\beta x),$$

where $\alpha = \frac{2}{6-p}$ and $\beta = \frac{p-2}{6-p}$. It then follows that

$$E(\varphi_\mu, \mathbb{R}) = \frac{\mu^{2\alpha+2\beta}}{2} \int_{\mathbb{R}} |\varphi_1'(\mu^\beta x)|^2 dx - \frac{\mu^{\alpha p}}{p} \int_{\mathbb{R}} |\varphi_1(\mu^\beta x)|^p dx.$$

By taking $z = \mu^\beta x$ in the above integrals we have

$$E(\varphi_\mu, \mathbb{R}) = \frac{\mu^{2\alpha+\beta}}{2} \int_{\mathbb{R}} |\varphi_1'(z)|^2 dz - \frac{\mu^{\alpha p-\beta}}{p} \int_{\mathbb{R}} |\varphi_1(z)|^p dz.$$

Upon evaluation of the exponents we arrive at:

$$\mathcal{E}_{\mathbb{R}}(\mu) = E(\varphi_\mu, \mathbb{R}) = \mu^{2\beta+1} E(\varphi_1, \mathbb{R}) = -\theta_p \mu^{2\beta+1}. \tag{4.10}$$

Finally, from Lemma 4.4 we get that

$$\mathcal{E}_{\mathbb{R}^+}(\mu) = \frac{1}{2} E(\varphi_{2\mu}, \mathbb{R}) = -2^{2\beta} \theta_p \mu^{2\beta+1}. \tag{4.11}$$

By plugging (4.10) and (4.11) into (4.8) we get (4.9). ■

It was seen in [2] that it was not only useful to scale the solitons but also to scale the graphs themselves, this shown that certain quantities were preserved. Note that in metric graphs we have attributed a length to each edge, by scaling the length of each edge by a certain factor $t > 0$, we produce a scaled version of the same graph. We say that $\mathcal{G}$ is homothetic to $\mathcal{G}'$ if there exists a constant $t > 0$ such that $\mathcal{G}' = t\mathcal{G}$. That is, for each edge e, we perform the scaling $I_e' = tI_e = t[0, \ell_e] = [0, t\ell_e]$.

Proposition 4.11: *Scaling preserved quantities*

Let $\mathcal{G}$ be a metric graph and $u \in H^1_\mu(\mathcal{G})$, which implies that $\mu = \|u\|^2_{L^2(\mathcal{G})}$. Under the following homothetic scaling of $\mathcal{G}$ and rescaling of u,

$$\mathcal{G} \to t^{-\beta}\mathcal{G}, \quad u \mapsto t^\alpha u(t^\beta \cdot) \;\text{ for }\; t > 0$$

the following quantities are preserved:

$$\mu^{-2\beta-1} \|u'\|^2_{L^2(\mathcal{G})}, \tag{4.12}$$

$$\mu^{-2\beta-1} \|u\|^p_{L^p(\mathcal{G})}, \tag{4.13}$$

$$\mu^{-\beta-1} \|u\|^2_{L^\infty(\mathcal{G})}. \tag{4.14}$$

Proof. Let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a metric graph, $u \in H^1_\mu(\mathcal{G})$, $t > 0$ and consider both rescales in the statement.

We begin by proving (4.13). Let $\mathcal{G}' = t^{-\beta}\mathcal{G}$. Denote $v(\cdot) = t^\alpha u(t^\beta \cdot)$. Denote by $\mathcal{K} \subset E(\mathcal{G})$ be the compact core of the graph, recall Definition 3.12. We have that

$$\|v\|^p_{L^p(\mathcal{G}')} = t^{\alpha p} \left[\sum_{e \in \mathcal{K}} \int_0^{t^{-\beta}\ell(I_e)} |u_e(t^\beta x_e)|^p dx_e + \sum_{e \in \mathcal{G} \setminus \mathcal{K}} \int_0^{+\infty} |u_e(t^\beta x_e)|^p dx_e \right]$$

$$= t^{\alpha p-\beta} \left[\sum_{e \in \mathcal{K}} \int_0^{\ell(I_e)} |u_e(z_e)|^p dz_e + \sum_{e \in \mathcal{G} \setminus \mathcal{K}} \int_0^{+\infty} |u_e(z_e)|^p dz_e \right] = t^{\alpha p-\beta} \|u\|^p_{L^p(\mathcal{G})}.$$

Since $\alpha p - \beta = 2\beta + 1$ we have that

$$\|u\|^p_{L^p(\mathcal{G})} = t^{-2\beta-1} \|v\|^p_{L^p(\mathcal{G}')},$$

which upon multiplying by $\mu^{-2\beta-1}$ yields

$$\mu^{-2\beta-1} \|u\|^p_{L^p(\mathcal{G})} = (t\mu)^{-2\beta-1} \|v\|^p_{L^p(\mathcal{G}')}. \tag{4.15}$$

Finally note that $\|u\|^2_{L^2(\mathcal{G})} = \mu$. By performing the same homothety and scaling similar computations yield

$$\|v\|^2_{L^2(\mathcal{G}')} = t^{2\alpha-\beta}\|u\|^2_{L^2(\mathcal{G})}.$$

Observing that $2\alpha - \beta = 1$ we get that while scaling both the graph and the functions the mass transforms like: $\mu \mapsto t\mu$. Therefore, going back to equation (4.15) we have the desired invariance:

$$\|u\|^{2(-2\beta-1)}_{L^2(\mathcal{G})}\|u\|^p_{L^p(\mathcal{G})} = \|v\|^{2(-2\beta-1)}_{L^2(\mathcal{G}')}\|v\|^p_{L^p(\mathcal{G}')}.$$

Let us now focus on (4.12). Taking into account what was done in the previous case the only thing left to prove is that

$$\|u'\|^2_{L^2(\mathcal{G})} = t^{-2\beta-1}\|v'\|^2_{L^2(\mathcal{G}')}. \tag{4.16}$$

Indeed, $v'(x) = t^{\alpha+\beta}u'(t^\beta x)$. Thus,

$$\|v'\|^2_{L^2(\mathcal{G}')} = t^{2(\alpha+\beta)}\int_{\mathcal{G}'}|u'(t^\beta x)|^2 dx,$$

which upon changing variables and using that $2\alpha + \beta = 2\beta + 1$ yields (4.16).

Finally, to prove (4.14) we use the same argument. Let us just prove then that

$$\|u\|^2_{L^\infty(\mathcal{G})} = t^{-\beta-1}\|v\|^2_{L^\infty(\mathcal{G}')}. \tag{4.17}$$

In fact,

$$\|v\|^2_{L^\infty(\mathcal{G}')} = \left(\sup_{e\in E(\mathcal{G}')}\|v_e\|_{L^\infty(I'_e)}\right) = t^{2\alpha}\left(\sup_{e\in E(\mathcal{G}')}\ \operatorname*{ess\,sup}_{x_e\in I'_e}|u_e(t^\beta x_e)|\right)^2 = t^{2\alpha} =$$
$$= t^{2\alpha}\left(\sup_{e\in E(\mathcal{G})}\ \operatorname*{ess\,sup}_{z_e\in I_e}|u_e(z_e)|\right)^2 = t^{2\alpha}\|u\|^2_{L^\infty(\mathcal{G})},$$

whence (4.17) follows from $2\alpha = \beta + 1$. ■

Remark 4.12:

1. From equations (4.12) and (4.13) we have that the energy, under the same rescaling and homothety, satisfies the same invariance, meaning that

$$\mu^{-2\beta-1}\mathcal{E}_{\mathcal{G}}(\mu) = (t\mu)^{-2\beta-1}\mathcal{E}_{\mathcal{G}'}(t\mu).$$

   This means that for a fixed graph $\mathcal{G}$, the problem of minimizing the functional E under a certain mass constraint becomes equivalent to minimizing E on a homothetically scaled graph with a different mass. The constant $\mu^{-2\beta-1}$ is then acting as a normalizing constant.
2. Similarly, we have that for some characteristic length of a graph, for example the length of a bounded edge, the homothety transforms $\ell \mapsto t^{-\beta}\ell$. In the same way we see that the quantity $\mu^\beta\ell$ is also preserved. This remark will be of particular use in the next chapter.

The next result will provide us upper and lower bounds for both the kinetic and potential terms of the functional E, as well as bounds for the L^∞ norms of ground states.

Lemma 4.13

Let $\mathcal{G}$ be a non-compact graph, and $u \in H^1_\mu(\mathcal{G})$ be such that

$$E(u,\mathcal{G}) \leq \frac{1}{2} \inf_{v \in H^1_\mu(\mathcal{G})} E(v,\mathcal{G}) < 0. \tag{4.18}$$

Then,

$$C_p^{-1}\mu^{2\beta+1} \leq \|u'\|^2_{L^2(\mathcal{G})} \leq C_p\mu^{2\beta+1}, \tag{4.19}$$

$$C_p^{-1}\mu^{2\beta+1} \leq \|u\|^p_{L^p(\mathcal{G})} \leq C_p\mu^{2\beta+1}, \tag{4.20}$$

$$C_p^{-1}\mu^{\beta+1} \leq \|u\|^2_{L^\infty(\mathcal{G})} \leq C_p\mu^{\beta+1}, \tag{4.21}$$

for some constant $C_p > 0$ depending only on p.

Proof. Let $\mathcal{G}$ be a non-compact graph and $u \in H^1_\mu(\mathcal{G})$ be such that (4.18) holds. As a consequence of the first item in Remark 4.12 we assume that $\mu = 1$. We begin by proving the upper estimates in (4.19)-(4.21). Let $V := \|u\|^p_{L^p(\mathcal{G})}$ and $T := \|u'\|^2_{L^2(\mathcal{G})}$, then (3.16) becomes

$$V \leq C_p T^{\frac{p-2}{4}}. \tag{4.22}$$

Notice now that the assumption satisfied by u leads to

$$\frac{1}{2}T - \frac{1}{p}V = E(u,\mathcal{G}) \leq -\frac{\theta_p}{2} < 0 \tag{4.23}$$

where in the first inequality (4.9) was used. Thus, $T < \frac{2}{p}V$. From here and (4.22) it follows that

$$T \leq \frac{2}{p} C_p T^{\frac{p-2}{4}}.$$

Solving for T, yields simultaneously $V, T \leq C_p'$. Recurring to (3.17) we also have $\|u\|^2_{L^\infty(\mathcal{G})} \leq CT^{\frac{1}{2}} \leq C_p''$. To finish we show the lower estimates. Notice that from (4.23) we can deduce that

$$V \geq \frac{p}{2}\theta_p. \tag{4.24}$$

Therefore we can take $D_p > 0$ such that $D_p^{-1} = \frac{p}{2}\theta_p$. This yields the lower bound in (4.19). Returning to (4.22) we now have

$$D_p^{-1} \leq V \leq C_p T^{\frac{p-2}{4}}. \tag{4.25}$$

Solving again for T yields the lower bound in (4.20). Finally, the interpolation inequality in (A.1) yields

$$D_p^{-1} \leq V \leq \|u\|^{p-2}_{L^\infty(\mathcal{G})}.$$

From here taking the power $\frac{2}{p-2}$ in the first inequality in (4.21) follows. ∎

Remark 4.14:
One might wonder what kind of functions satisfy the assumption in equation (4.18). Recalling that the infimum is negative, any ground state, for example, satisfies this assumption. Also, if u is an element of a minimizing sequence $(u_n)_{n\in\mathbb{N}}$, the assumption is also satisfied for n large.

The next result gives us an idea of the qualitative properties of minimizers on unbounded edges of a non-compact graph, while also providing some upper and lower bounds to the value of the Lagrange multiplier, λ, that arose in Proposition 3.17.

Proposition 4.15: *On unbounded edges solutions are portions of solitons*

Let $\mathcal{G}$ be a non-compact graph and let $u \in H^1_\mu(\mathcal{G})$ be a ground state. Then,

$$C_p^{-1}\mu^{2\beta} \leq \lambda \leq C_p\mu^{2\beta}. \tag{4.26}$$

Moreover, the restriction of u to any half-line of $\mathcal{G}$ takes the form

$$u(x) = \varphi_m(x+y),\ x \geq 0, \tag{4.27}$$

where $y \in \mathbb{R}$ depends on the half-line, while m is the same for all the half-lines, and satisfies

$$C_p^{-1}\mu \leq m \leq C_p\mu. \tag{4.28}$$

In order to prove this result we will need to take a few steps back and focus on the following *double-constrained* minimization problem:

$$\min\left\{E(\varphi,\mathbb{R}^+)\ |\ \varphi \in H^1_{\frac{m}{2}}(\mathbb{R}^+),\ \varphi(0) = a\right\}, \quad \text{for } m, a > 0. \tag{4.29}$$

We now show that solutions to this problem exist and describe them. Following this we present a proof for Proposition 4.15.

Lemma 4.16: *Double-constrained Minimization Problem*

For every $m, a > 0$ there exists a unique $M > 0$ and $y \in \mathbb{R}$ such that the soliton φ_M satisfies

$$\varphi_M(y) = a \ \text{ and } \int_0^{+\infty} \varphi_M(x+y)^2 dx = \frac{m}{2}. \tag{4.30}$$

The function $\varphi_M(\cdot + y)$ is the unique solution to (4.29) and we have that $y > 0$ if and only if $a > \varphi_m(0)$.

Proof. We divide the proof in 3 steps.

1. **Existence of y and M;**
   Recall that solitons satisfy the scaling rule given in (2.37). We then set $z = M^\beta y$, where M and y are to be determined. Plugging it into (4.30) yields

$$M^\alpha \varphi_1(z) = a \tag{4.31}$$

   and also, making the change of variable $x = M^{-\beta}t$ in the integral gives us:

$$\frac{m}{2} = \int_0^{+\infty} \varphi_M(x+y)^2 dx = M^{2\alpha-\beta}\int_0^{+\infty} \varphi_1(z+t)^2 dt = M\int_0^{+\infty} \varphi_1(z+t)^2 dt \tag{4.32}$$

   Provided we can determine z, and consequently y, equation (4.31) gives us the value of M, in fact $M = a^{\frac{1}{\alpha}}\varphi_1(z)^{-\frac{1}{\alpha}}$. By putting this value of M in (4.32) we can reduce the problem to studying a real valued function of a real variable defined by

$$g(z) := \varphi_1(z)^{-\frac{1}{\alpha}}\int_0^{+\infty} \varphi_1(z+t)^2 dt = \frac{ma^{-\frac{1}{\alpha}}}{2}. \tag{4.33}$$

The function g is clearly continuous, in fact it belongs to $C^\infty(\mathbb{R})$ by the regularity of solitons. We claim that the range of this function is $\mathbb{R}^+$ and that it is invertible. By taking limits going to infinity, we have

$$\lim_{z\to-\infty} g(z) = +\infty \tag{4.34}$$

since the soliton φ_1 goes to zero and

$$\lim_{z\to-\infty}\int_0^{+\infty}\varphi_1(z+t)^2dt = \lim_{z\to-\infty}\int_z^{+\infty}\varphi_1(s)^2ds = 1.$$

Now notice that since φ_1 is decreasing in $(0,+\infty)$, for $t\geq 0$

$$\varphi_1(z+t)^2 = \varphi_1(z+t)^{\frac{1}{\alpha}}\varphi(z+t)^{2-\frac{1}{\alpha}} \leq \varphi_1(z)^{\frac{1}{\alpha}}\varphi(z+t)^{2-\frac{1}{\alpha}}. \tag{4.35}$$

Therefore,

$$0 \leq \lim_{z\to+\infty} g(z) \leq \lim_{z\to+\infty}\int_0^{+\infty}\varphi(z+t)^{2-\frac{1}{\alpha}} = \lim_{z\to+\infty}\int_z^{+\infty}\varphi_1(s)^{2-\frac{1}{\alpha}}ds = 0.$$

Then, we have indeed that $g(\mathbb{R}) = \mathbb{R}^+$. We now show that g is strictly decreasing. If $z<0$ then it is the product of two strictly decreasing functions, hence g is strictly decreasing for $z<0$. For $z\geq 0$ we differentiate

$$\begin{aligned}
g'(z) &= -\frac{1}{\alpha}\varphi_1'(z)\varphi_1(z)^{-\frac{1}{\alpha}-1}\int_0^{+\infty}\varphi_1(z+t)^2dt + \varphi(z)^{-\frac{1}{\alpha}}\frac{d}{dz}\int_0^{+\infty}\varphi_1(z+t)^2dt\\
&= -\frac{2}{2\alpha}\varphi_1'(z)\varphi_1(z)^{-\frac{1}{\alpha}-1}\int_0^{+\infty}\varphi_1(z+t)^2dt + \varphi(z)^{-\frac{1}{\alpha}}\int_0^{+\infty}2\varphi_1(z+t)\varphi_1'(z+t)dt\\
&= 2\int_0^{+\infty}\frac{\varphi_1(z+t)^2}{\varphi_1(z)^{\frac{1}{\alpha}}}\left[\frac{\varphi_1'(z+t)}{\varphi_1(z+t)} - \frac{1}{2\alpha}\frac{\varphi_1'(z)}{\varphi_1(z)}\right]dt.
\end{aligned}$$

Since $\frac{1}{2\alpha}<1$ and $\varphi_1'<0$ in $(0,+\infty)$, continuing the above computations yields

$$g'(z) < 2\int_0^{+\infty}\frac{\varphi_1(z+t)^2}{\varphi_1(z)^{\frac{1}{\alpha}}}\left[\frac{\varphi_1'(z+t)}{\varphi_1(z+t)} - \frac{\varphi_1'(z)}{\varphi_1(z)}\right]dt.$$

Recalling that from the explicit form of φ_1 we have that

$$\varphi_1'(x) = -c_p\left(\frac{\alpha}{\beta}\right)\varphi_1(x)\tanh(c_px),$$

it follows that

$$g'(z) < 2c_p\left(\frac{\alpha}{\beta}\right)\varphi(z)^{-\frac{1}{\alpha}}\int_0^{+\infty}\varphi_1(z+t)^2\left[\tanh(c_pz) - \tanh(c_p(z+t))\right]dt < 0.$$

Finally since $\varphi_1,\alpha,\beta,c_p$ are positive and $\tanh$ is a strictly increasing function we get that $g'<0$ and therefore, g is invertible. Given now any $a,m>0$ we know that there exists a unique z (hence a unique y and M) such that the equality in (4.33) (hence 4.30) is satisfied.

2. **Existence of Unique Solution to** (4.29).
By translation, we know that $\varphi_M(\cdot+y)$ minimizes E among all functions that satisfy $\|\varphi\|^2_{L^2(\mathbb{R})} = M$ and $\varphi(0)=a$. Suppose, by way of contradiction, that there exists φ a minimizer for (4.29) better than $\varphi_M(\cdot+y)$. Then the function $v:\mathbb{R}\to\mathbb{R}$ defined by

$$v(x) = \begin{cases}\varphi_M(x+y), & x<0;\\ \varphi(x), & x\geq 0,\end{cases}$$

satisfies $v(0) = a$ and

$$\|v\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{0} |\varphi_M(x+y)|^2 dx + \int_0^{+\infty} |\varphi(x)|^2 dx = M - \frac{m}{2} + \frac{m}{2} = M.$$

Moreover, $E(v, \mathbb{R}) < E(\varphi_M(\cdot + y), \mathbb{R})$, which is a contradiction. The uniqueness follows then from the uniqueness of M and y.

3. $y > 0$ **if and only if** $a > \varphi_m(0)$.
Again, note that by rescaling having $a > \varphi_m(0)$ is equivalent to having $a > m^\alpha \varphi_1(0)$, and since

$$g(0) = \varphi_1(0)^{-\frac{1}{\alpha}} \int_0^{+\infty} \varphi_1(t)^2 dt = \frac{\varphi_1(0)^{-\frac{1}{\alpha}}}{2}$$

we have $a > m^\alpha \varphi_1(0)$, equivalent to $g(0) > \frac{ma^{-\frac{1}{\alpha}}}{2}$. The conclusion follows now from the construction of z and g. Since g is strictly decreasing and $g(z) = \frac{ma^{-\frac{1}{\alpha}}}{2}$, the last inequality is equivalent to having $z > 0$; equivalently $y > 0$.

■

In other words, this result gives us that by fixing a certain height at $x = 0$ the resulting solution to the minimization problem (4.29) is always a portion of a soliton.

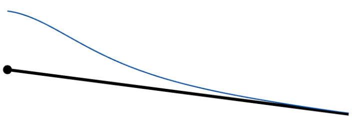

Figure 4.4: The case $y = 0$.

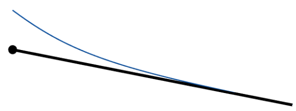

Figure 4.5: The case $y > 0$.

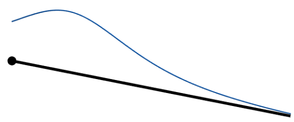

Figure 4.6: The case $y < 0$.

We are now in a position to prove the Proposition 4.15.

Proof of Proposition 4.15. Let $\mathcal{G}$ be a non-compact graph and $u \in H^1_\mu(\mathcal{G})$ be a ground state. Being a ground state, u satisfies

$$\int_{\mathcal{G}} -u'\eta' + u|u|^{p-2}\eta dx = \lambda \int_{\mathcal{G}} u\eta dx, \quad \text{for all } \eta \in H^1(\mathcal{G}).$$

Taking $\eta = u$ we have that

$$-\int_{\mathcal{G}} |u'|^2 dx + \int_{\mathcal{G}} |u|^p = \lambda\mu.$$

We have immediately from (4.20) that

$$\lambda\mu \leq \|u\|^p_{L^p(\mathcal{G})} \leq C_p \mu^{2\beta+1}. \tag{4.36}$$

Moreover, because the energy of a ground state is negative, we have that

$$-\frac{2}{p}\int_{\mathcal{G}} |u|^p dx < -\int_{\mathcal{G}} |u'|^2 dx,$$

whence, using again (4.20) and the fact that $\frac{2}{p} < 1$, yields

$$\lambda\mu > (1 - \frac{2}{p}) \int_{\mathcal{G}} |u|^p dx \geq D_p \mu^{2\beta+1}. \tag{4.37}$$

From (4.37) and (4.36) it follows indeed that

$$D_p \mu^{2\beta} < \lambda \leq C_p \mu^{2\beta}.$$

It follows now from Lemma 4.16 that the restriction of u to any half-line is indeed a portion of a soliton. For if it was not we could apply the lemma and construct a better competitor than u, which is a contradiction since u is a ground state. Let now E_1 be the set of unbounded edges of $\mathcal{G}$. For each $e \in E_1$ we know that there exist y_e and M_e such that $u_e(x) = \varphi_{M_e}(x + y_e)$. We now show that M_e is in fact determined by the Lagrange multiplier, independently of each edge. Recall that φ_1 solves in $\mathbb{R}$ the equation

$$-\varphi_1'' + \lambda_p \varphi_1 = |\varphi_1|^{p-2}\varphi_1 \tag{4.38}$$

for some Lagrange multiplier λ_p. Since φ_{M_e} satisfies the equation

$$-\varphi_{M_e}''(x + y_e) + \lambda\varphi_{M_e}(x + y_e) = \varphi_{M_e}(x + y_e)^{p-1},$$

through the scaling given in Theorem 2.18 $\varphi_{M_e}(x) = M_e^\alpha \varphi_1(M_e^\beta(x + y_e))$ we have that φ_1 has to solve

$$-\varphi_1''(M_e^\beta(x + y_e)) + \lambda M_e^{-2\beta}\varphi_1(M_e^\beta(x + y_e)) = \varphi_1(M_e^\beta(x + y_e))^{p-1}.$$

By subtracting the previous equation to the equation (4.38) computed at the points $M_e^\beta(x + y_e)$ yields

$$\lambda_p = \frac{\lambda}{M_e^{2\beta}}. \tag{4.39}$$

Therefore, the mass of a ground state in the unbounded edges comes uniquely determined by the Lagrange multiplier. Letting $m := M_e$, for each edge we obtain $\lambda = \lambda_p m^{2\beta}$. Now, from (4.26) the estimates in (4.28) follow. ∎

The last result of this section provides two important properties of the energy level function when $\mathcal{G}$ is non-compact..

Theorem 4.17: *$\mathcal{E}_\mathcal{G}$ is strictly concave and subadditive*

The function $\mathcal{E}_\mathcal{G} : [0, +\infty) \to (-\infty, 0]$ defined as in (4.8) is strictly concave and subadditive, that is $\mathcal{E}_\mathcal{G}(\mu + \mu') \leq \mathcal{E}_\mathcal{G}(\mu) + \mathcal{E}_\mathcal{G}(\mu')$ for all $\mu, \mu' \in [0, +\infty)$. Moreover, the inequality is strict for $\mu, \mu' > 0$.

Proof. Let $\mathcal{G}$ be a non-compact metric graph. It follows from Lemma 4.10 that $\mathcal{E}_\mathcal{G}$ is a real function of a real variable with domain $[0, +\infty)$ and range $(-\infty, 0]$. Recall that any concave function f defined in $[0, +\infty)$ that satisfies $f(0) \geq 0$ is subadditive. Since $\mathcal{E}_\mathcal{G}(0) = 0$ we only need to prove that $\mathcal{E}_\mathcal{G}$ is concave. To do so we consider the following set:

$$A := \left\{ u \in H^1_\mu(\mathcal{G}) : \|u\|^p_{L^p(\mathcal{G})} \geq C_p^{-1}\mu^{2\beta+1} \right\},$$

where C_p^{-1} is given in (4.20). As a consequence of Remark 4.14 we know that $A \neq \emptyset$. Moreover, from the same remark we see that if we take a minimizing sequence $(u_n)_{n\in\mathbb{N}}$ there exist $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have $u_n \in A$, therefore we can conclude that minimizing in $H^1_\mu(\mathcal{G})$ is the same as minimizing in A. It is easy to see now that $A = \sqrt{\mu}U$ where U is the set given by

$$U := \left\{ u \in H^1_1(\mathcal{G}) : \mu^{\frac{p}{2}}\|u\|^p_{L^p(\mathcal{G})} \geq C_p^{-1}\mu^{2\beta+1} \right\}.$$

Therefore we have that

$$\mathcal{E}_\mathcal{G}(\mu) = \inf_{v\in A} E(v, \mathcal{G}) = \inf_{u\in U} \left\{ \frac{\mu}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{\mu^{\frac{p}{2}}}{p}\|u\|^p_{L^p(\mathcal{G})} \right\} = \inf_{u\in U} f_u(\mu) \tag{4.40}$$

where, for each fixed $u \in U$ f_u is defined as $\mu \mapsto f_u(\mu) := \frac{\mu}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{\mu^{\frac{p}{2}}}{p}\|u\|^p_{L^p(\mathcal{G})}$. Notice now that $f_u \in C^2((0,+\infty))$, moreover,

$$f_u''(\mu) = -\frac{p-2}{4}\mu^{\frac{p}{2}-2}\|u\|^p_{L^p(\mathcal{G})} \leq -C_p\mu^{2\beta-1} < 0, \quad \text{for all } \ u \in U,$$

whence f_u is strictly concave in compact subsets of $(0,+\infty)$ independently of u. The independence of u of the strict concavity estimate of the function f_u gives us the desired strict concavity of $\mathcal{E}_{\mathcal{G}}$. ■

We close this section with some remarks on the last proof. The key part of the proof is to consider the set U for two reasons. Firstly, it allowed us to reduce the problem to study a family of functions of a real variable. Secondly, without considering this set it would have been harder to prove that $\mathcal{E}_{\mathcal{G}}$ is in fact *strictly* concave since it was this set that provided us the independence of u when we computed the derivatives f_u''.

4.2.2 Existence Results for Non-compact graphs

We have seen that if a graph satisfies (H) then no ground states will be present except when $\mathcal{G}$ is isometric to one of the graphs described in Theorem 4.9 . One of the most simple examples where this assumption fails is when the graph is of the form given in Figure 4.2. In [1], by *ad hoc* techniques that relied heavily on the topology of the graph, it was shown that ground states do exist for arbitrary mass $\mu > 0$. In particular, it was seen that for this particular graph the condition $\mathcal{E}_{\mathcal{G}}(\mu) < E(\varphi_\mu, \mathbb{R})$ was a sufficient condition for a minimizer to exist! In [2] this was generalized to any non-compact graph, see Theorem 4.19, through a dichotomy principle which we now state and prove.

Theorem 4.18: *Dichotomy of minimizing sequences*

Let $\mathcal{G}$ be a non-compact metric graph and $\mu > 0$. Let also $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence for problem (3.4). Then u_n is weakly compact in $H^1(\mathcal{G})$. Moreover, if $u_n \rightharpoonup u$ in $H^1(\mathcal{G})$, then either, up to a subsequence,

1. $u_n \to 0$ in $L^\infty_{loc}(\mathcal{G})$ and $u \equiv 0$, or
2. $u \in H^1_\mu(\mathcal{G})$, u is a minimizer and $u_n \to u$ strongly in $H^1(\mathcal{G}) \cap L^p(\mathcal{G})$, for all $p \in (2,6)$.

Proof. Let $\mathcal{G}$ be a non-compact metric graph and $\mu > 0$. Let also $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence for problem (3.4). Since every minimizing sequence satisfies the assumption in Lemma 4.13 we have that u_n is bounded in $H^1(\mathcal{G}) \cap L^p(\mathcal{G})$, indeed if not then $E(u_n,\mathcal{G}) \to +\infty$ which is a contradiction. Being bounded in $H^1(\mathcal{G})$ we know that there exists $u \in H^1(\mathcal{G})$ such that $u_n \rightharpoonup u$ in $H^1(\mathcal{G})$. Moreover, due to the compact embedding in Corollary A.9 we have that

$$u_n \to u \ \text{ in } \ L^\infty_{loc}(\mathcal{G}).$$

Define now $m := \|u\|^2_{L^2(\mathcal{G})}$. From the weak convergence in $L^2(\mathcal{G})$ we have that

$$m = \|u\|^2_{L^2(\mathcal{G})} \leq \liminf_{n\to\infty} \|u_n\|^2_{L^2(\mathcal{G})} = \mu.$$

We now claim that m can only have two possible values: $m = 0$ or $m = \mu$; each of which give rise to the cases 1. and 2. in the statement, respectively.

We start by ruling out any other possible value for m. To do so notice that the L^∞_{loc} convergence gives us pointwise convergence and also that the L^p norm of the sequence is uniformly bounded. Then we are in conditions to apply the Brézis-Lieb Lemma, see Lemma A.2, from which we get that

$$\frac{1}{p}\int_{\mathcal{G}} |u_n|^p dx - \frac{1}{p}\int_{\mathcal{G}} |u_n - u|^p dx - \frac{1}{p}\int_{\mathcal{G}} |u|^p dx = o(1), \quad \text{as } \ n \to \infty. \tag{4.41}$$

Since $H^1(\mathcal{G})$ is a Hilbert space the weak convergence gives the same result for the L^2 norms of the derivatives. Indeed, recalling that we are working with real valued functions,

$$\int_{\mathcal{G}} |u_n' - u'|^2 dx = \langle u_n' - u', u_n' - u' \rangle_{L^2 \times L^2} = \int_{\mathcal{G}} |u_n'|^2 dx + \int_{\mathcal{G}} |u'|^2 dx - 2 \int_{\mathcal{G}} u_n' u' dx. \tag{4.42}$$

Passing everything to the right side, dividing by $\frac{1}{2}$ and taking the limit as $n \to \infty$ we get that

$$\frac{1}{2} \int_{\mathcal{G}} |u_n'|^2 dx - \frac{1}{2} \int_{\mathcal{G}} |u_n' - u'|^2 dx - \frac{1}{2} \int_{\mathcal{G}} |u'|^2 dx = o(1), \text{ as } n \to \infty. \tag{4.43}$$

Taking the difference between the equations (4.41) and (4.43) we get that

$$E(u_n, \mathcal{G}) = E(u_n - u, \mathcal{G}) + E(u, \mathcal{G}) + o(1), \text{ as } n \to \infty.$$

Denoting by $\nu_n := \|u_n - u\|^2_{L^2(\mathcal{G})}$ we get that

$$E(u_n, \mathcal{G}) \geq \mathcal{E}_{\mathcal{G}}(\nu_n) + E(u, \mathcal{G}) + o(1), \text{ as } n \to \infty, \tag{4.44}$$

since $\mathcal{E}_{\mathcal{G}}(\nu_n) \leq E(u_n - u, \mathcal{G})$ for every $n \in \mathbb{N}$, by definition of energy level function. Similarly, from the weak convergence in L^2 we deduce, just as in (4.42), that

$$\nu_n = \mu + m - 2 \int_{\mathcal{G}} u_n u dx.$$

Taking the limit yields $\nu_n \to \mu - 2m + m = \mu - m$. Therefore, taking the limit in (4.44) and using the continuity of $\mathcal{E}_{\mathcal{G}}$ we get that

$$\mathcal{E}_{\mathcal{G}}(\mu) \geq \mathcal{E}_{\mathcal{G}}(\mu - m) + E(u, \mathcal{G}) \geq \mathcal{E}_{\mathcal{G}}(\mu - m) + \mathcal{E}_{\mathcal{G}}(m). \tag{4.45}$$

Suppose now, by contradiction, that $m \in (0, \mu)$. Then, since $\mathcal{E}_{\mathcal{G}}$ is subadditive we would have $\mathcal{E}_{\mathcal{G}}(\mu) < \mathcal{E}_{\mathcal{G}}(\mu - m) + \mathcal{E}_{\mathcal{G}}(m)$, which is a contradiction. We then have two cases. If $m = 0$ then $u \equiv 0$ and we have proved item 1. in the statement. If $m = \mu$ then $u \in H^1_\mu(\mathcal{G})$ and putting the new value for m in (4.45) yields that u is also a minimizer. Finally, it follows now from the definition of m that $u_n \to u$ in $L^2(\mathcal{G})$; consequently, it also converges strongly in $L^p(\mathcal{G})$ since $p > 2$. This convergence in turn implies the convergence of the L^2 norms of the derivatives because

$$\frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})} = E(u, \mathcal{G}) = \mathcal{E}_{\mathcal{G}}(\mu) = \lim_{n \to \infty} \left(\frac{1}{2}\|u_n'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})} \right),$$

whence the desired strong convergence in $H^1(\mathcal{G})$ follows. ■

The main result of this section is the following

Theorem 4.19: *Sufficient condition for the existence of minimizers*

Let $\mathcal{G}$ be a non-compact graph. If

$$\mathcal{E}_{\mathcal{G}}(\mu) = \inf_{v \in H^1_\mu(\mathcal{G})} E(v, \mathcal{G}) < E(\varphi_\mu, \mathbb{R}), \tag{4.46}$$

then, the infimum is attained and a minimizer exists.

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset H^1_\mu(\mathcal{G})$ be a minimizing sequence. By Theorem 4.18 it is enough to prove that u_n cannot converge to zero in L^∞_{loc}. By way of contradiction let us suppose so. The goal is to construct a minimizing sequence from u_n for which the inequality of the statement is not satisfied, yielding the contradiction. To do so let $\mathcal{K}$ be the compact core of $\mathcal{G}$. Note that this set is always non-empty. In fact, we only remove the interior of unbounded edges in order to obtain the compact core. Since by assumption $\mathcal{G}$ contains at least one unbounded edge, its original vertex remains in the compact core. Let now $M_n := \max_{x\in\mathcal{K}} u_n(x)$, for each $n \in \mathbb{N}$. Since we assume that $u_n \to 0$ in L^∞_{loc} we have that $M_n \to 0$ as $n \to \infty$. We define the following sequence

$$v_n := \max\{0, u_n - M_n\} = (u_n - M_n)^+, \quad \text{for all } \ n \in \mathbb{N} \quad x \in \mathcal{G}.$$

and we claim that v_n is also a minimizing sequence. Start by noticing that

$$|u_n(x) - v_n(x)| = |u_n(x) - \max\{0, u_n - M_n\}| \leq M_n$$

Therefore, $\|u_n - v_n\|_{L^\infty(\mathcal{G})} \to 0$ as $n \to \infty$. Since $p > 2$ it follows that

$$\|u_n - v_n\|^p_{L^p(\mathcal{G})} \leq \|u_n - v_n\|^{p-2}_{L^\infty(\mathcal{G})}\|u_n - v_n\|^2_{L^2(\mathcal{G})} = o(1), \tag{4.47}$$

as $n \to \infty$ since the L^2 norm is uniformly bounded by μ. Consequently, we have that

$$\|v_n\|^p_{L^p(\mathcal{G})} = \|u_n\|^p_{L^p(\mathcal{G})} + o(1), \qquad \text{as } n \to \infty. \tag{4.48}$$

Note also that by construction of v_n we have that

$$\int_{\mathcal{G}} |v_n'|^2 dx \leq \int_{\mathcal{G}} |u_n'|^2 dx. \tag{4.49}$$

By joining the equations (4.48) and (4.49) we get that

$$E(v_n, \mathcal{G}) = \frac{1}{2}\int_{\mathcal{G}} |v_n'|^2 dx - \frac{1}{p}\int_{\mathcal{G}} |v_n|^p dx \leq \frac{1}{2}\int_{\mathcal{G}} |u_n'|^2 dx - \frac{1}{p}\int_{\mathcal{G}} |u_n|^p dx + o(1) = E(u_n, \mathcal{G}) + o(1),$$

as $n \to \infty$. Hence, v_n is a minimizing sequence. Note that by construction $\|v_n\|^2_{L^2(\mathcal{G})} \leq \mu$. It follows by the subbadditivity and strict negativity of $\mathcal{E}_{\mathcal{G}}$ that $\mathcal{E}_{\mathcal{G}}(\mu) \leq \mathcal{E}_{\mathcal{G}}(m)$ for all $m \in [0, \mu]$, and since $H^1_\mu(\mathcal{G}) \subset H^1_{\leq\mu}(\mathcal{G}) := \{u \in H^1(\mathcal{G}) : \|u\|^2_{L^2(\mathcal{G})} \leq \mu\}$ it follows that for any $u \in H^1_{\leq\mu}(\mathcal{G})$ there exists $m \in [0, \mu]$ such that:

$$\mathcal{E}_{\mathcal{G}}(\mu) \leq \mathcal{E}_{\mathcal{G}}(m) \leq E(u, \mathcal{G}).$$

Now taking the infimum yields that minimizing in $H^1_\mu(\mathcal{G})$ is the same as minimizing in $H^1_{\leq\mu}(\mathcal{G})$, since $H^1_\mu(\mathcal{G}) \subset H^1_{\leq\mu}(\mathcal{G})$.

We can now reach a contradiction. Note that since $u_n \to 0$ as $x \to \infty$ in unbounded edges then $\#v_n^{-1}\{t\} \geq 2$ for almost every $t \in (0, \max_{\mathcal{G}} v_n)$ and for each n. Thus by taking the Schwarz symmetrization v_n^* we get, as a consequence of Polya-Szegő inequality, that

$$E(v_n, \mathcal{G}) \geq E(v_n^*, \mathbb{R}) \geq \mathcal{E}_{\mathbb{R}}(\mu). \tag{4.50}$$

Now taking the limit in the above inequality contradicts the assumption in the statement. ■

The following corollary gives us a very practical criterion to deduce the existence of a ground state.

Corollary 4.20

Let $\mathcal{G}$ be a non-compact graph. If there exists $u_0 \in H^1_\mu(\mathcal{G})$ such that

$$E(u_0, \mathcal{G}) \leq E(\varphi_\mu, \mathbb{R}), \tag{4.51}$$

then $\mathcal{G}$ admits a ground state of mass μ.

Proof. If u_0 is indeed a ground state then we are done. If not

$$\inf_{u \in H^1_\mu(\mathcal{G})} E(u, \mathcal{G}) < E(u_0, \mathcal{G}) \leq E(\varphi_\mu, \mathbb{R}) = \min_{\varphi \in H^1_\mu(\mathbb{R})} E(\varphi, \mathbb{R}).$$

Then the conclusion follows from Theorem 4.19. ■

Therefore, if one can construct or deduce the existence of a function with less or equal energy than that of a soliton the above corollary guarantees the existence of a ground state. Note that in general the constructed function is not a ground state. We will see this in the following examples.

4.2.3 Examples and Qualitative Properties of Minimizers

The general idea is to start from a soliton of fixed mass in $\mathbb{R}$ and by cutting, gluing and rearranging it conveniently to produce a function of the same mass on the graph with lower or equal energy. Note that the lengths of the edges of the graphs will play a crucial role here! We need them to do the correct rearrangements, since we are not always interested in rearranging a whole function but rather some particular sections.

Example 1: Graphs as Quotients of $\mathbb{R}$

Note that what was done in the graphs of Figures 3.5 and 4.3 already falls under the scope of Corollary 4.20. Moreover, the process which we used to create the ground states in these graphs is precisely the one we just described. For clarification, Figures 4.7 and 4.8 below illustrate such construction, where the function in blue represents the gluing of a soliton on the real line of mass μ. Since this construction does not alter the energy, joining Theorem 4.8 and Corollary 4.20, yields that these are in fact ground states.

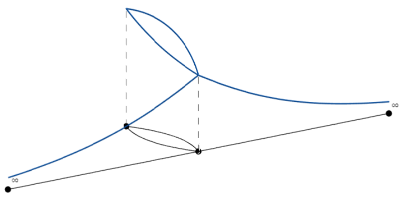

Figure 4.7: A ground state, for the graph in Figure 4.3 depicted in blue.

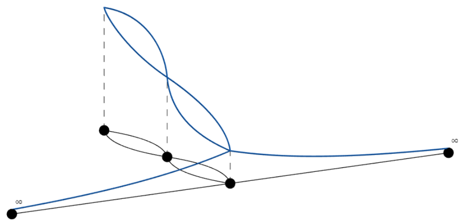

Figure 4.8: A ground state, for the graph in Figure 4.3 (in the middle) depicted in blue.

Example 2: Signpost graph

Sign post graphs are graphs composed of two half-lines joined and one edge of length ℓ_2, all joined at the same vertex, and on the end of the edge we attach a loop of length ℓ_1. See for example Figure 4.11. It is clear also that this graph does not satisfy assumption (H) so ground states might indeed exist. We now illustrate how one can obtain in this graph an $H^1_\mu(\mathcal{G})$ function with less energy than a soliton of mass μ. Consider the folded soliton in Figure 4.8. Suppose that the middle loop of this graph has a total length of ℓ_2 (each edge will have a length of $\frac{\ell_2}{2}$) and that the other loop has length ℓ_1. Performing a decreasing rearrangement on the middle loop into a single $H^1(0, \ell_2)$ function, we can place the rearranged soliton on top of the signpost graph. Consequently, we get an $H^1_\mu(\mathcal{G})$ function with energy strictly less than that of the soliton. This is a consequence of Polya-Szegő inequality for the decreasing rearrangement, see Theorem 3.21, since in this section the sets $\varphi_\mu^{-1}\{t\}$ have two elements. We have now constructed a function with stricly less energy on the graph than the soliton on the real line. An illustration of this construction is depicted in Figure 4.9. As a consequence of Corollary 4.20 a ground state of mass μ exists.

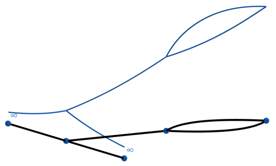

Figure 4.9: Application of Corollary 4.20 on a sign post graph.

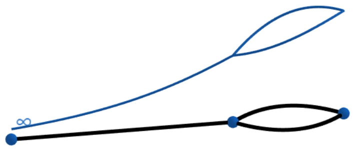

Figure 4.10: Application of Corollary 4.20 on a tadpole graph.

Example 3: Tadpole graph

A tadpole graph is a graph composed of one unbounded edge and a loop of length ℓ_1. An example is depicted in Figure 4.12. In what concerns existence, grounds states of prescribed mass may exist since this graph, containing only one half-line, does not satisfy assumption (H). We now use Corollary 4.20 to deduce the existence of ground states of arbitrary prescribed mass. We construct an $H^1_\mu(\mathcal{G})$ function with energy strictly less than that of a soliton as follows. Consider the construction made for the signpost graph above. We can further do a decreasing rearrangement of the function on both the half lines into a single $H^1(\mathbb{R}^+)$ function. Since the soliton goes to zero in the unbounded edges, this rearrangement further decreases the energy of the function constructed in the previous example. In view of the last item in Remark 3.16, since the middle vertex is a *dummy vertex*, we can remove it whilst still having a function in $H^1(\mathbb{R}^+)$. The resulting function is clearly in $H^1_\mu(\mathcal{G})$ and has energy strictly less than a soliton. By performing again a decreasing rearrangement on the unbounded edge one produces a function like the one depicted on Figure 4.10.

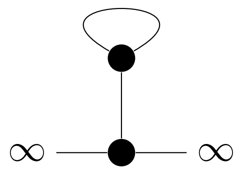

Figure 4.11: A *sign post* Graph

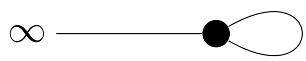

Figure 4.12: A *tadpole* Graph

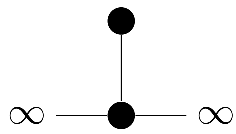

Figure 4.13: Line with a terminal edge

Example 4: Line with a terminal edge

Let $\mathcal{G}$ be the graph depicted in Figure 4.13, which is composed of two half-lines and one bounded edge of length ℓ_3 all joined together at the same vertex. This particular graph has an interesting history on the references [1] and [2]. We now know that in any non-compact graph existence of ground states can be verified by Theorem 4.19. In [1] this result was not yet known. In fact, in this reference, the topology of these particular graphs was heavily exploited to prove that in fact a condition like the one on Theorem 4.19 holds. This was without doubt a stepping stone for the existence results in the latter reference. Additionally, they also characterized ground states for these graphs. We refer the reader for [1] above, in particular for Theorems 2.6 and 2.7 for this approach. We now give two different ways to apply Corollary 4.20 in order to deduce existence of ground states of arbitrary mass. The first one is to consider the folded soliton in Figure 4.7. Suppose that the loop in this figure has length ℓ_3. By performing a decreasing rearrangement of the folded solition in the section of the loop we can produce an $H^1(0, \ell_3)$ function that we can now place on top of the graph $\mathcal{G}$. As before, by Pólya-Szegő inequality, the energy decreases strictly with this rearrangement and Corollary 4.20 can now be applied. The constructed function will look like the function in Figure 4.14 The second one has its origin on the construction done for the sign post graph. Start by considering the function produced in the second example. If one unfolds the soliton we get an $H^1_\mu(0, \ell_1 + \ell_2)$ function. Since in the construction of the signpost graph the values of ℓ_1 and ℓ_2 can be made arbitrary, we can choose them so that $\ell_1 + \ell_2 = \ell_3$. This way we can now place the produced graph on top of the graph $\mathcal{G}$. By further decreasing rearrangement on the terminal edge we get a function like the one depicted in Figure 4.14. Again, note that the constructed function does not have to be, in general, the ground state. However, it was shown in [1], through *ad hoc* techniques that the ground states for this graph will be exactly like the one depicted in Figure 4.14.

The following auxiliary result will allow us to understand what ground states look like and this will be consistent with the *ad hoc* approach taken in [1].

Theorem 4.21

Assume that $\mathcal{G}$ is not homeomorphic to $\mathbb{R}$ and that N half-lines, $N \geq 2$, emanate from the same vertex, v. Then, along each of the half-lines, any ground state u, if it exists, takes the form (4.27) with the same non-negative value y.

Proof. Suppose $\mathcal{G}$ contains at least $N \geq 2$ half-lines and that it is not homomorphic to $\mathbb{R}$. Let also $u \in H^1_\mu(\mathcal{G})$ be a ground state. Let $\mathcal{H}_i$, $i = 1, \cdots, N$ denote the half-lines with the origin at the same vertex. From Proposition 4.15 we know that there exists $m > 0$ and $y_1, \cdots, y_N \in \mathbb{R}$ such that

$$u\,|_{\mathcal{H}_i} = \varphi_m(x + y_i), \text{ for all } x \geq 0, i = 1, \cdots, N.$$

Since u is continuous, we know that $u(v) = u\,|_{\mathcal{H}_i}(0)$ for all $i = 1, \cdots, N$. Moreover, given that solitons are even and decreasing functions then have that the values y_i are determined by $\varphi_m(\pm y_i) = u(v)$. This means that $|y_i|$ is independent of i. We then need to check that $y_i \geq 0$ for all i. Assuming, by way of contradiction, that $y_1 < 0$ we will construct functions with lower energy level than u.

Let us decompose $\mathcal{G}$ in the following way, $\mathcal{G}_1 := \mathcal{H}_1 \cup \mathcal{H}_2$, isometric to $\mathbb{R}$, and $\mathcal{G}_2 := (\mathcal{G} \setminus \mathcal{G}_1) \cup \{v\}$. Note straight away the following, $\mathcal{G}_1 \cap \mathcal{G}_2 = \{v\}$ and $\mathcal{G}_2$ has at least one edge, otherwise $\mathcal{G}$ would be isometric to $\mathbb{R}$. We now split the ground state u as $(u_1, u_2) \in H^1(\mathcal{G}_1) \times H^1(\mathcal{G}_2)$. Let also $\mu_1 := \|u_1\|^2_{L^2(\mathcal{G}_1)}$, $\mu_2 := \|u_2\|^2_{L^2(\mathcal{G}_2)}$. Clearly, $\mu = \mu_1 + \mu_2$ and by having $u > 0$ in G it follows that $\mu_1, \mu_2 > 0$. Finally let us remark that by having $y_1 < 0$,

$$0 = \inf_{\mathcal{G}_1} u_1 < u_1(v) < \max_{\mathcal{G}_1} u_1. \tag{4.52}$$

Let now $w_i^\epsilon \in H^1(\mathcal{G}_i)$, $i = 1, 2$, be given by

$$w_1^\epsilon(x) := (1+\epsilon)^{\frac{1}{2}} u_1(x) \quad \text{and} \quad w_2^\epsilon(x) := \left(1 - \epsilon\frac{\mu_1}{\mu_2}\right)^{\frac{1}{2}} u_2(x) \tag{4.53}$$

For $\epsilon \geq \min\{-1, \frac{\mu_2}{\mu_1}\}$ we have in fact that $w_i^\epsilon \in H^1(\mathcal{G}_i)$. Moreover, $\|w_1^\epsilon\|^2_{L^2(\mathcal{G}_1)} + \|w_2^\epsilon\|^2_{L^2(\mathcal{G}_2)} = \mu$. We now claim that for all ϵ we can shift the function w_1^ϵ by an amount so that the translated function $\tilde{w}_1^\epsilon$ makes the function $\tilde{w}^\epsilon := (\tilde{w}_1^\epsilon, w_2^\epsilon) \in H^1_\mu(\mathcal{G})$. It is clear that if $\epsilon \neq 0$ then w^ϵ is not continuous at this vertex. Since the perturbations of u_1 and u_2 are made continuously, condition (4.52) gives us that for $|\epsilon|$ small enough

$$0 = \inf_{\mathcal{G}_1} u_1 < w_2^\epsilon(v) < \max_{\mathcal{G}_1} u_1.$$

This, followed by the construction of w_1^ϵ, gives us

$$0 = \inf_{\mathcal{G}_1} w_1^\epsilon < w_2^\epsilon(v) < \max_{\mathcal{G}_1} w_1^\epsilon.$$

Therefore, since $\mathcal{G}_1$ is isometric to $\mathbb{R}$, we can shift w_1^ϵ, i.e $\tilde{w}_1^\epsilon(x) = w_1^\epsilon(x + \delta_\epsilon)$ for some $\delta_\epsilon \in \mathbb{R}$ such that $\delta_\epsilon \to 0$ as $\epsilon \to 0$, by an amount such that the continuity condition at v is satisfied, whence $\tilde{w}^\epsilon = (\tilde{w}_1^\epsilon, w_2^\epsilon)$ becomes a function in $H^1_\mu(\mathcal{G})$. Now let

$$f(\epsilon) := E(\tilde{w}^\epsilon, \mathcal{G}) = E(\tilde{w}_1^\epsilon, \mathcal{G}_1) + E(w_2^\epsilon, \mathcal{G}_2) = E(w_1^\epsilon, \mathcal{G}_1) + E(w_2^\epsilon, \mathcal{G}_2).$$

Note that at $\epsilon = 0$, since u is a ground state, we know that f has a minimum at this point; consequently $f''(0) \geq 0$, since f is of class C^2 because the ground state $u = (u_1, u_2)$ is fixed. However,

$$f'(\epsilon) = \frac{1}{2}\int_{\mathcal{G}_1} |u_1'|^2 dx - \frac{(1+\epsilon)^{\frac{p}{2}-1}}{2}\int_{\mathcal{G}_1} |u_1|^p dx - \frac{\mu_1}{2\mu_2}\int_{\mathcal{G}_2} |u_2'|^2 dx - \frac{(1-\epsilon\frac{\mu_1}{\mu_2})^{\frac{p}{2}-1}}{2\mu_2}\int_{\mathcal{G}_2} |u_2|^p dx,$$

whence, since $\frac{p}{2} - 1 > 0$, making $|\epsilon|$ smaller if necessary, yields

$$f''(\epsilon) = -\left(\frac{p}{2} - 1\right)\frac{(1+\epsilon)^{\frac{p}{2}-2}}{2}\int_{\mathcal{G}_1} |u_1|^p dx - \frac{\mu_1^2}{2\mu_2^2}\left(\frac{p}{2} - 1\right)\left(1 - \epsilon\frac{\mu_1}{\mu_2}\right)^{\frac{p}{2}-2}\int_{\mathcal{G}_2} |u_2|^p dx < 0.$$

This in turn implies that f cannot have a minimum at $\epsilon = 0$, which is a contradiction, since u is a ground state. ∎

Remark 4.22:
One of the simplest cases of a non-compact graph that is not homeomorphic to $\mathbb{R}$ is a graph like the one in Figure 4.13. Figure 4.14 below represents what a ground state looks like for these graphs. For completeness Figures 4.15-4.16 depict an example of a star graph with a terminal edge and what ground states on the unbounded edges of these graphs look like, should they exist. In the following section we prove a necessary and sufficient condition for the existence of ground states in these graphs.

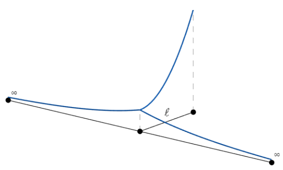

Figure 4.14: A ground state for a graph composed of a line with a terminal edge (depicted in blue).

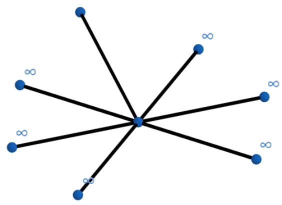

Figure 4.15: A star graph with one terminal edge.

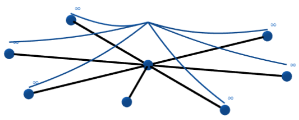

Figure 4.16: Ground state on the half-lines of a star graph with a terminal edge (depicted in blue).

4.3 The Threshold Phenomenon

We will now describe what in [2] is called the threshold phenomenon. The phenomenon shows how for some particular graphs, the interplay between some metric properties of the graph (such as length of bounded edges) and the prescribed mass, influence the existence or non-existence of ground states. Moreover, we will see that this interplay between the mass and metric properties will allow, for certain graphs, the existence of a sharp phase transition from non-existence to existence of ground states.

Firstly recall that for a ground state to exist assumption (H) needs to be violated with the exception of the family of graphs described by Theorem 4.9. The simplest case where this holds is when the graph has a terminal edge. Henceforth, we will identify such edge with $[0, \ell]$, $\ell > 0$ being its length, without further notice. Similarly, recall the value of the constant β and the solitons scaling rule given in Theorem 2.18.

As an application of Corollary 4.20 we get the following result that gives existence of solution when a non-compact graph contains a terminal edge.

Proposition 4.23

Let $\mathcal{G}$ be a non-compact graph with a terminal edge of length ℓ. There exists C_p^* such that if $\mu^\beta \ell \geq C_p^*$, then $\mathcal{G}$ admits a ground state.

Proof. By rescaling we can assume that $\mu = 1$, recall Remark 4.12. Note now that since $2^{2\beta} > 1$, it follows from (4.10) and (4.11) that

$$\mathcal{E}_{\mathbb{R}^+}(1) = E(\varphi_2, \mathbb{R}^+) = 2^{2\beta} E(\varphi_1, \mathbb{R}) < E(\varphi_1, \mathbb{R}).$$

Thus,

$$E(\varphi_2, \mathbb{R}^+) < E(\varphi_1, \mathbb{R}). \tag{4.54}$$

We now construct a function $u \in H^1_\mu(\mathcal{G})$ that satisfies the assumption of Corollary 4.20 in the following way. For $\epsilon > 0$, let

$$u_\epsilon(x) := \frac{(\varphi_2 - \epsilon)^+(x)}{\|(\varphi_2 - \epsilon)^+\|_{L^2(\mathbb{R}^+)}} \in H^1(\mathbb{R}^+).$$

Since as $\epsilon \to 0$ we have $u_\epsilon \to \varphi_2$ in $H_1(\mathbb{R}^+)$ we can fix ϵ_0 small so that $E(u_{\epsilon_0}, \mathbb{R}^+) \leq E(\varphi_1, \mathbb{R})$. Moreover, by construction, $\|u_{\epsilon_0}\|^2_{L^2(\mathbb{R}^+)} = 1$ and there exists $C_p^* > 0$ for which $u_{\epsilon_0}(x) = 0$ for all $x \geq C_p^*$.

Suppose now that $\ell \geq C_p^*$. By attaching the terminal edge to $\mathcal{G}$ at the coordinate $x = \ell$ we can extend u_{ϵ_0} by zero on the rest of $\mathcal{G}$. This way, $u_{\epsilon_0} \in H^1_1(\mathcal{G})$ has energy less or equal to that of a soliton of mass 1 in $\mathbb{R}$. The conclusion now follows from Corollary 4.20. ■

The following result can be seen as a stability result for the existence of ground states.

Theorem 4.24: *Stability of existence of ground states*

Let $\mu > 0$, $\mathcal{G}$ be any non-compact graph and, for $n \in \mathbb{N}$, let $\mathcal{K}_n$ be a connected compact graph of total length $m(\mathcal{K}_n)$. Denote $\mathcal{G}_n$ the graph obtained from $\mathcal{G}$ by attaching, to a fixed vertex $v \in \mathcal{G}$, the graph $\mathcal{K}_n$. If every $\mathcal{G}_n$ admits a ground state u_n of mass μ and $m(\mathcal{K}_n) \to 0$ as $n \to \infty$, then $\mathcal{G}$ has a ground state of mass μ.

Proof. By scaling let us assume that $\mu = 1$. Suppose, by way of contradiction, that $\mathcal{G}$ does not admit a ground state of mass 1.

Note that for each $n \in \mathbb{N}$ it follows from (4.9) that

$$\mathcal{E}_{\mathcal{G}_n}(1) = E(u_n, \mathcal{G}_n) \leq -\theta_p.$$

Moreover, since $\mathcal{G}$ does not admit a ground state of mass 1 then, by Theorem 4.19, we have that

$$\mathcal{E}_{\mathcal{G}}(1) \geq -\theta_p := E(\varphi_1, \mathbb{R}). \tag{4.55}$$

Consequently, $\mathcal{E}_{\mathcal{G}}(1) = -\theta_p$.

Let $\sigma_n := \int_{\mathcal{G}} |u_n|^2 dx$. By construction $\mathcal{G}$ can be seen as a subgraph of $\mathcal{G}_n$ for all $n \in \mathbb{N}$. Let us define

$$v_n(x) := \sigma_n^{-\frac{1}{2}} u_n(x), \quad x \in \mathcal{G}.$$

This is now a sequence of functions in $H^1_1(\mathcal{G})$. Note now that $\sigma_n < 1$ for all $n \in \mathbb{N}$. Indeed, if there exists n_0 such that $\sigma_{n_0} = 1$ then the restriction of u_{n_0} to $\mathcal{G}$ would give rise to a ground state by application of Corollary 4.20. It then follows that

$$\begin{aligned}
-\theta_p \leq E(v_n, \mathcal{G}) &= \frac{1}{2}\int_{\mathcal{G}} |v_n'|^2 dx - \frac{1}{p}\int_{\mathcal{G}} |v_n|^p dx = \frac{\sigma_n^{-1}}{2}\int_{\mathcal{G}} |u_n'|^2 dx - \frac{\sigma_n^{-\frac{p}{2}}}{p}\int_{\mathcal{G}} |u_n|^p dx \\
&= \sigma_n^{-1}\left(\frac{1}{2}\int_{\mathcal{G}} |u_n'|^2 dx - \frac{1}{p\sigma_n^{\frac{p}{2}-1}}\int_{\mathcal{G}} |u_n|^p dx\right) \leq \sigma_n^{-1} E(u_n, \mathcal{G}) \\
&= \sigma_n^{-1}(E(u_n, \mathcal{G}_n) - E(u_n, \mathcal{K}_n)) \leq \sigma_n^{-1}(-\theta_p - E(u_n, \mathcal{K}_n)) \\
&= \sigma_n^{-1}\left(-\theta_p - \frac{1}{2}\int_{\mathcal{K}_n} |u_n'|^2 dx + \frac{1}{p}\int_{\mathcal{K}_n} |u_n|^p dx\right) \\
&\leq \sigma_n^{-1}\left(-\theta_p + \frac{1}{p}\int_{\mathcal{K}_n} |u_n|^p dx\right).
\end{aligned}$$

Note that $\mathcal{K}_n$ is compact and u_n, being ground states, are uniformly bounded, see (4.21), and Remark 4.14. Hence,

$$\int_{\mathcal{K}_n} |u_n|^2 dx \le C_p m(\mathcal{K}_n)^2 \to 0, \quad \text{as} \quad n \to \infty,$$

and

$$\int_{\mathcal{K}_n} |u_n|^p dx \le C_p m(\mathcal{K}_n)^p \to 0, \quad \text{as} \quad n \to \infty.$$

In particular,

$$1 = \int_{\mathcal{G}_n} |u_n|^2 dx = \int_{\mathcal{G}} |u_n|^2 + \int_{\mathcal{K}_n} |u_n|^2 dx,$$

from which follows that $\sigma_n \to 1$ as $n \to \infty$ and also that

$$E(v_n, \mathcal{G}) \to -\theta_p = \mathcal{E}_{\mathcal{G}}(1). \tag{4.56}$$

In other words, v_n is a minimizing sequence. Now since we have a minimizing sequence, and $\mathcal{G}$ does not admit a ground state, it follows from the first case in Theorem 4.18 that $v_n \to 0$ in $L^\infty_{loc}(\mathcal{G})$. Given that $\sigma_n \to 1$, by definition of v_n, it follows that also $u_n \to 0$ in $L^\infty_{loc}(\mathcal{G})$. In particular, $u_n(v) \to 0$. We now prove that $u_n \to 0$ uniformly in $\mathcal{K}_n$ as well. Since $\mathcal{K}_n$ is connected, let $x \in \mathcal{K}_n$ and $\gamma : \mathbb{R} \to \mathcal{G}$ a curve such that γ connects x and v in $\mathcal{K}_n$. Then,

$$u_n(x) = u_n(v) + \int_\gamma u_n'(t)dt \le u_n(v) + \int_{\mathcal{K}_n} |u_n'(x)|dx.$$

It follows now from Cauchy-Schwarz and the uniform bound of the derivatives of u_n given in (4.19) that

$$u_n(x) \le u_n(v) + C_p m(\mathcal{K}_n)^{\frac{1}{2}} \to 0, \quad \text{as} \quad n \to \infty.$$

Letting $M_n := \|u_n\|_{L^\infty(\mathcal{K}_n)}$ we have that $M_n \to 0$ as $n \to \infty$. Recall the estimate

$$-\theta_p \le \sigma_n^{-1} \left(-\theta_p + \frac{1}{p} \int_{\mathcal{K}_n} |u_n|^p dx \right);$$

or equivalently,

$$(1 - \sigma_n)\theta_p < \frac{1}{p} \int_{\mathcal{K}_n} |u_n|^p dx \le \frac{M_n^{p-2}}{p} \int_{\mathcal{K}_n} |u_n|^2 dx = \frac{M_n^{p-2}}{p} (1 - \sigma_n).$$

Dividing by $(1 - \sigma_n)$ and letting $n \to \infty$ yields $\theta_p \le 0$, which is a contradiction. ■

We remark that this stability result exploits heavily the topological properties of the graph in question, in particular the connectedness and compactness of the $\mathcal{K}_n$.

Following [2] we now show the threshold phenomenon for a very particular family of graphs, namely, star graphs on which we attach a terminal edge. See Figure 4.15 for an example of such graphs.

Theorem 4.25: *Threshold Phenomenon*

Let $\mathcal{G}_\ell$ denote the graph with N ($N \ge 3$) half-lines and a terminal edge of length $\ell > 0$, all emanating from the same vertex. Then there exists $C^* > 0$, depending only on N and p, such that the $\mathcal{G}_\ell$ admits a ground state of mass μ if and only if $\mu^\beta \ell \ge C^*$.

Proof. By scaling, without loss of generality, we can assume that $\mu = 1$. We define the constant C^* as follows

$$C^* := \inf\{\ell > 0 : \mathcal{G}_\ell \text{ admits a ground state of mass } 1\}. \tag{4.57}$$

By Proposition 4.23 we know that the above set is non empty, since for large enough ℓ, $\mathcal{G}_\ell$ will admit a ground state of mass 1. Moreover, $C^* > 0$. Indeed if $C^* = 0$ then taking a sequence $\ell_n \to 0$ and applying Theorem 4.24 we would deduce that the graph $\mathcal{G}_0$ has a ground state of mass 1. However, note that $\mathcal{G}_0$ is a star graph. Since star graphs satisfy condition (H) and are not isometric to any of the cases discussed above in Figures 3.5 or to any of the ones in 4.3, then we can rule out existence of ground states of any mass, and this is a contradiction.

Take now a sequence $\ell_n \searrow C^*$ such that, for each $n \in \mathbb{N}$, $\mathcal{G}_{\ell_n}$ admits a ground state of mass 1. Again applying Theorem 4.24 we deduce the existence of a ground state of mass 1 for $\mathcal{G}_{C^*}$. We complete the proof by showing that if $\ell' > \ell$ and $\mathcal{G}_\ell$ carries a ground state u of mass 1, then also $\mathcal{G}_{\ell'}$ admits a ground state. To do this we rely again on Corollary 4.20 to construct $v \in H^1_1(\mathcal{G}_{\ell'})$ such that

$$E(v, \mathcal{G}_{\ell'}) \leq E(u, \mathcal{G}_\ell) = \mathcal{E}_{\mathcal{G}_\ell}(1) \leq \mathcal{E}_{\mathbb{R}}(1) = E(\varphi_1, \mathbb{R}).$$

Recall that $u = (u_{e_1}, \cdots, u_{e_N}, u_\ell) \in H^1_1(\mathcal{G}_\ell)$. The following steps describe the construction of v:

1. Let $e_1, \cdots, e_N$ be the half-lines of $\mathcal{G}_{\ell'}$ and $e_{\ell'}$ it's terminal edge of length ℓ'. Now, recall that, from Theorem 4.21, there exist $m, y > 0$ such that the restriction of u to any half-line is given by $u_{e_i}(\cdot) = \varphi_m(\cdot + y)$, for all $i \in \{1, \cdots, N\}$. Let $\delta := \frac{\ell' - \ell}{N}$. On each half-line we "cut the piece" of the function u defined in the interval $[0, \delta)$ and do the translation $v_{e_i}(x) := u_{e_i}(x + \delta)$, for all $i \in \{1, \cdots, N\}$ and $x \in [0, +\infty)$, so that they all match at the original vertex. Note that since these functions are all the same and on each we are removing the exact same portion the continuity at the joining vertex of the half-lines remains assured.
2. We now rearrange manually the portion removed from the solitons in the previous step. Fix any $i \in \{1, \cdots, N\}$, say $i = 1$. Let $I = [0, N\delta] = [0, \ell' - \ell]$. By reflecting, scaling and stretching this portion horizontally by a factor of N we can define

$$v_1(x) := u_{e_1}\left(\delta - \frac{x}{N}\right), \; x \in I. \tag{4.58}$$

   This is clearly an $H^1(I)$ function. Note also that, $v_1(0) = u_{e_1}(\delta)$, and therefore, continuity is assured at the joining vertex. Also, $v_1(N\delta) = u_{e_1}(0)$. Since the origin vertex of all the edges can be identified with the coordinate $x = 0$ for each edge, we can now glue the portion of u on the terminal edge of length ℓ to v_1 on the terminal edge of length ℓ', thus constructing $v_{\ell'} \in H^1(0, \ell')$.; consequently we have $v = (v_{e_1}, \cdots, v_{e_N}, v_{\ell'}) \in H^1(\mathcal{G}_{\ell'})$.

For clarification we have v defined as follows:

$$v(x) = \begin{cases} v_{e_i}(x) = u_{e_i}(x + \delta), \; x \in [0, +\infty), \text{ for all } i \in \{1, \cdots, N\}; \\ v_{\ell'}(x) = \begin{cases} u_{e_1}(\delta - \frac{x}{N}), \; x \in [0, \ell' - \ell]; \\ u_\ell(x - (\ell' - \ell)), \; x \in [\ell' - \ell, \ell']. \end{cases} \end{cases}$$

Note now that, with this construction of v:

$$\begin{aligned} \int_{\mathcal{G}_{\ell'}} |v|^2 dx = \sum_{i=1}^{N} \int_0^{+\infty} |v_{e_i}(x)|^2 dx + \int_0^{\ell'} |v_{\ell'}(x)|^2 dx \\ N \int_\delta^{+\infty} |u_{e_1}(y)|^2 \, dy + \int_0^{\ell' - \ell} \left| u_{e_1}\left(\delta - \frac{x}{N}\right)\right|^2 dx \\ + \int_{\ell' - \ell}^{\ell'} \left| u_\ell(x - (\ell' - \ell))\right|^2 dx = N \int_\delta^{+\infty} |u_{e_1}(y)|^2 \, dy \\ + N \int_0^\delta |u_{e_1}(y)|^2 \, dy + \int_0^\ell |u_\ell(y)|^2 \, dy = \int_{\mathcal{G}_\ell} |u|^2 dx = 1. \end{aligned}$$

The same argument allows one to compute the L^p norm, and see that it is also left invariant. We now have that the derivative of v is given by:

$$v'(x) = \begin{cases} v'_{e_i}(x) = u'_{e_i}(x+\delta),\ x \in [0,+\infty),\ \text{for all } i \in \{1,\cdots,N\}; \\ v'_{\ell'}(x) = \begin{cases} -\frac{1}{N}u'_{e_1}(\delta - \frac{x}{N}),\ x \in [0,\ell'-\ell]; \\ u'_{\ell}(x-(\ell'-\ell)),\ x \in [\ell'-\ell,\ell']. \end{cases} \end{cases}$$

Computing the kinetic term of the energy functional for the function v yields

$$\begin{aligned} \int_{\mathcal{G}_{\ell'}} |v'|^2 dx &= \sum_{i=1}^{N} \int_0^{+\infty} |v'_{e_i}(x)|^2 dx + \int_0^{\ell'} |v'_{\ell'}(x)|^2 dx \\ & N \int_\delta^{+\infty} \left|u'_{e_1}(y)\right|^2 dy + \frac{1}{N^2} \int_0^{\ell'-\ell} \left|u'_{e_1}\left(\delta - \frac{x}{N}\right)\right|^2 dx + \int_{\ell'-\ell}^{\ell'} \left|u'_{\ell}(x-(\ell'-\ell))\right|^2 dx \\ &= N \left(\int_\delta^{+\infty} \left|u'_{e_1}(y)\right|^2 dy + \frac{1}{N^2} \int_0^{\delta} \left|u'_{e_1}(y)\right|^2 dy \right) + \int_0^{\ell} \left|u'_{\ell}(y)\right|^2 dy \\ &\leq N \left(\int_\delta^{+\infty} \left|u'_{e_1}(y)\right|^2 dy + \int_0^{\delta} \left|u'_{e_1}(y)\right|^2 dy \right) \\ &\quad + \int_0^{\ell} \left|u'_{\ell}(y)\right|^2 dy = \int_{\mathcal{G}_\ell} |u'|^2 dx. \end{aligned}$$

It follows from the last two computations that indeed $E(v,\mathcal{G}_{\ell'}) \leq E(u,\mathcal{G}_\ell)$. ■

Chapter 5

Pohozaev Minimization Problem on Metric Graphs

In the previous chapters we saw that if u is a solution to the minimization problem (3.4) then for some Lagrange multiplier $\lambda > 0$ it solves the equation

$$-u'' + \lambda u = |u|^{p-2}u,$$

not only in the weak, but also in the classical sense on each edge. Could it be possible that for any given Lagrange multiplier $\lambda > 0$ a solution of the same equation also exists? In $\mathbb{R}$ we know that we can relate both problems through scalings, however, for a fixed graph, the used scalings no longer work. Therefore, in order to answer this question we rewrite the problem in a different way.

Suppose now $\lambda > 0$ is fixed and consider the following functionals:

$$T : H^1(\mathbb{R}) \to \mathbb{R};\ T(u) = \frac{1}{2}\int_{\mathbb{R}} |u'|^2 dx + \frac{\lambda}{2}\int_{\mathbb{R}} |u|^2 dx, \quad R : H^1(\mathbb{R}) \to \mathbb{R};\ R(u) = \frac{1}{p}\int_{\mathbb{R}} |u|^p dx,$$

where $p \geq 1$. We will see the constrained Euler-Lagrange equation to this problem is exactly the stationary one dimensional Schrödinger equation with the parameter $\lambda > 0$. A new Lagrange Multiplier will appear but we can multiply solutions by a convenient constant so that the new Lagrange multiplier disappears. Therefore we focus on the problem which consists of minimizing the functional $T(u)$ with the constraint $R(u) = \mu$, for $\mu > 0$. This is, according to Lions, the Pohozaev problem, see [24]. Taking into account the strategy of the previous chapters, in Section 5.1 we start by solving the problem in $\mathbb{R}$ using the Concentration-Compactness Principle and not scalings. Moreover, in the context of this dissertation, it is natural to ask if the results of the previous chapters still hold for this problem. Therefore, in the second section some original work is done in formalizing this problem in the setting of metric graphs and investigating these questions.

5.1 The Pohozaev Minimization Problem in $\mathbb{R}$

In this section, following arguments of [24], we focus on the following minimization problem. Consider $p \geq 1$, $\mu > 0$ and let $R_\mu := R^{-1}\{\mu\}$. We are concerned on the problem of

$$\text{finding } u_0 \in R_\mu \text{ such that } T(u_0) = \inf_{u \in R_\mu} \frac{1}{2}\left\{\int_{\mathbb{R}} |u'|^2 + \lambda |u|^2 dx\right\} = \inf_{u \in R_\mu} T(u). \tag{5.1}$$

Remark 5.1:

- Taking into account Lemma 2.12 we immediately have that $T, R \in C^1(H^1(\mathbb{R}))$ and, moreover,

$$T'(u)v = \int_{\mathbb{R}} u'v' + \lambda uv dx, \quad \text{and} \quad R'(u)v = \int_{\mathbb{R}} |u|^{p-2}uv dx, \quad \text{for all } v \in H^1(\mathbb{R}).$$

- Observe that the functional T is clearly non-negative, moreover, it satisfies

$$\min\{1,\lambda\}\|u\|^2_{H^1(\mathbb{R})} \le T(u) \le \max\{1,\lambda\}\|u\|^2_{H^1(\mathbb{R})},$$

  which makes $\sqrt{T}$ an equivalent norm to the standard H^1 norm, and also a coercive functional;
- Note that $T(|u|) = T(u)$, so there is no loss of generality when working with real valued functions;
- The problem only becomes interesting for $p > 2$. Note that for $p \le 2$ the embedding of H^1 in L^p fails.

Proposition 5.2: *The case $p \in [1,2]$*

For $p \in [1,2]$, problem (5.1) admits no solutions in $H^1(\mathbb{R})$.

Proof. Let us start with the case $p = 2$. Suppose that problem (5.1) admits a solution. Since T and R are functionals of class $C^1(H^1(\mathbb{R}))$, there exists a Lagrange multiplier $\theta \in \mathbb{R}$ such that the constrained Euler-Lagrange equation to the problem is:

$$-u'' + \lambda u = \theta u.$$

Taking $\lambda_1 = (\theta - \lambda)$ the equation becomes

$$-u'' - \lambda_1 u = 0.$$

In one dimension, the explicit solutions to this equation are know. If $\lambda_1 < 0$ the non-trivial solutions are linear combinations of exponential functions and therefore not in $H^1(\mathbb{R})$. For $\lambda_1 > 0$ non-trivial solutions are linear combinations of the sine and cosine functions, which again are not in $H^1(\mathbb{R})$, since they do not converge to zero at infinity. For $\lambda_1 = 0$ solutions are of the form $u(x) = ax + b$, $a, b \in \mathbb{R}$. Clearly, any non-trivial solution of this form is not in $H^1(\mathbb{R})$. This then yields a contradiction.

For the case $p \in [1,2)$ we claim that the value of the infimum in (5.1) is zero. Note that setting $u = \mu^{\frac{1}{p}} v$ yields

$$I_\mu = \inf_{u \in R_\mu} \frac{1}{2}\left\{\int_{\mathbb{R}} |u'|^2 + \lambda |u|^2 dx\right\} = \inf_{v \in R_1} \frac{\mu^{\frac{2}{p}}}{2}\left\{\int_{\mathbb{R}} |v'| + \lambda |v|^2 dx\right\} = \mu^{\frac{2}{p}} I_1.$$

Therefore, we assume that $\mu = 1$. Let $u \in R_1$. Taking $v_h(x) = h^{\frac{1}{p}} u(hx)$ we have that $v_h \in R_1$ and that

$$\lim_{h\to 0} T(v_h) = \lim_{h \to 0} \frac{h^{\frac{2+p}{p}}}{2} \int_{\mathbb{R}} |u'|^2 dx + \frac{\lambda h^{\frac{2-p}{p}}}{2} \int_{\mathbb{R}} |u|^2 dx = 0,$$

Thus we have that $I_1 = 0$ and, consequently, for every $\mu > 0$ we have $I_\mu = 0$. Suppose now that a minimizer $w \in R_1$ exists. It follow that $w \equiv 0$, which is a contradiction. ∎

Let us now focus on the case $p > 2$. By analogy with what was done in Chapter 2, we start by noticing that T is a non-negative functional and thus bounded from below. Due to similarities with Chapter 2 we will only highlight the differences.

Proposition 5.3: *Minimizing sequences are uniformly bounded*

Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence for the problem (5.1). Then, u_n is uniformly bounded in $H^1(\mathbb{R})$.

Proof. This result follows immediately from the fact that $\sqrt{T}$ defines an equivalent norm to the H^1 norm. ∎

Proposition 5.4: *Positivity of the infimum*

Let $p > 2$, $\mu > 0$ and $I_\mu := \inf_{u\in R_\mu} T(u)$. Then $I_\mu > 0$.

Proof. By way of contradiction suppose that $I_\mu = 0$. Since T is coercive we then have that

$$0 = I_\mu = \inf_{u\in R_\mu} T(u) \geq \inf_{u\in R_\mu} \min\{1, \lambda\}\|u\|^2_{H^1(\mathbb{R})} \geq 0.$$

Now take a minimizing sequence $(u_n)_{n\in\mathbb{N}}$. Then we have that $\|u_n\|_{H^1(\mathbb{R})} \to 0$. Since, by Sobolev embedding, this implies that u_n converges uniformly to zero then. Hence, by (A.1), we have that $\|u_n\|^p_{L^p(\mathbb{R})} \to 0$ as $n \to \infty$, which is a contradiction, since $R(u_n) = \mu$, for all $n \in \mathbb{N}$. ■

We will now apply the direct method to deduce the existence of solution for problem (5.1). Similarly to what happened in Chapter 2 we need to apply the concentration-compactness principle.

Let us start with the strict subadditivity.

Lemma 5.5: *Strict Subadditivity*

Let $\mu > 0$. Then,

$$I_\mu < I_\alpha + I_{\mu-\alpha}, \quad \text{for all} \quad \alpha \in (0, \mu). \tag{5.2}$$

Proof. Recall that $I_\mu = \mu^{\frac{p}{2}} I_1$. Consider now the function $f : \mathbb{R}^+_0 \to \mathbb{R}^+$ defined by $f(\mu) = \mu^{\frac{2}{p}} I_1$. Since $p > 2$ we have that f is concave. Recall now that any real valued concave function g defined in $[0, +\infty)$ such that $g(0) \geq 0$ is subadditive. Then, since $f(0) = 0$, we have that f is subadditive, that is

$$I_\mu = f(\mu) \leq f(\alpha) + f(\mu - \alpha) = I_\alpha + I_{\mu-\alpha}$$

for all $\alpha \in [0, \mu]$. Using the definition of f it is easy to see that equality only holds for $\alpha = 0$ and $\alpha = \mu$, and we deduce (5.2). ■

The more general way of stating the concentration-compactness lemma, see [23, Lemma I.1], gives us some freedom in the choice of functional used for the restriction. For Lemma 2.4 we have chosen the fixed L^2 norm, but we could have also fixed the L^p norm. Taking this last remark into account we state without proof the following result.

Lemma 5.6: *Concentration-Compactness Lemma*

Let $\mu > 0$ and $p > 2$. Let also $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $H^1(\mathbb{R})$ with $\|u_n\|^p_{L^p(\mathbb{R})} = \mu$. Then, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ satisfying one of the following properties:

1. (Compactness) There exists a sequence $(y_k)_{k\in\mathbb{N}}$ of real numbers with the property that for all $\epsilon > 0$, there exists $T > 0$ such that

$$\int_{y_k - T}^{y_k + T} |u_{n_k}|^p dx \geq \mu - \epsilon \text{ for all } k \in \mathbb{N}.$$

2. (Vanishing) For all $t > 0$, one has

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}} \int_{y-t}^{y+t} |u_{n_k}|^p dx = 0.$$

3. (Dichotomy) There exist $\alpha \in (0, \mu)$ and sequences $(u_{k,1})_{k\in\mathbb{N}}$ and $(u_{k,2})_{k\in\mathbb{N}}$ bounded in $H^1(\mathbb{R})$, such that
   (a) $\|u_{n_k} - (u_{k,1} + u_{k,2})\|_{L^q(\mathbb{R})} \to 0$ as $k \to \infty$ for $q \in [p, +\infty)$;
   (b) $\lim_{k\to\infty} |u_{k,1}|^p - \alpha = \lim_{k\to\infty} |u_{k,2}|^p - (\mu - \alpha) = 0$;
   (c) $\operatorname{dist}(\operatorname{supp} u_{k,1}, \operatorname{supp} u_{k,2}) \to +\infty$ as $k \to \infty$.

Remark 5.7:
It is fundamental for what follows that the reader keeps in mind that in the proof of this result the choice of the sequences $(u_{k,1})_{k\in\mathbb{N}}$ and $(u_{k,2})_{k\in\mathbb{N}}$ is done in the exact same way as mentioned in Remark 2.5. Since $(u_k)_{k\in\mathbb{N}}$ is uniformly bounded in $H^1(\mathbb{R})$ the estimate (2.17) still holds and, moreover, by construction of these sequences we have that

$$\lambda\int_{\mathbb{R}}|u_{n_k}|^2dx \geq \lambda\int_{\operatorname{supp}u_{k,1}}|u_{n_k}|^2dx + \lambda\int_{\operatorname{supp}u_{k,2}}|u_{n_k}|^2dx \geq \lambda\int_{\mathbb{R}}|u_{k,1}|^2dx + \lambda\int_{\mathbb{R}}|u_{k,2}|^2dx.$$

Consequently, we have that

$$T(u_{n_k}) \geq T(u_{k,1}) + T(u_{k,2}) + o(1) \text{ as } k\to\infty. \tag{5.3}$$

The main result of this section is the following

Theorem 5.8: *Compactness of minimizing sequences and existence of minimizer*

Let $p>2$, $\lambda>0$ and $\mu>0$. Then, for any minimizing sequence $(u_n)_{n\in\mathbb{N}}$ of problem (5.1) there exist $(y_n)_{n\in\mathbb{N}}$ and $u\in H^1(\mathbb{R})$ such that, up to a subsequence, $u_n(\cdot+y_n)\to u$ strongly in $H^1(\mathbb{R})$ and u is a minimizer.

For the proof, in parallel to Chapter 2, we prove the following lemmas.

Lemma 5.9: *Dichotomy does not occur*

Let $\mu>0$ and $p>2$. Let also $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence to the problem (5.1). Then item 3 in Lemma 5.6 does not occur.

Proof. Suppose that the dichotomy regime holds. Then, there exist $\alpha\in(0,\mu)$ and $(u_{k,1})$, $(u_{k,2})$ subsequences of (u_{n_k}) such that the properties (a)-(c) of Lemma 5.6 hold. Let now $(\alpha_k)_{k\in\mathbb{N}}$ and $(\beta_k)_{k\in\mathbb{N}}$ be sequences of positive real numbers such that

$$\|\alpha_k u_{k,1}\|^p_{L^p(\mathbb{R})}=\alpha \quad\text{and}\quad \|\beta_k u_{k,2}\|^p_{L^p(\mathbb{R})}=\mu-\alpha \quad\text{for all } k\in\mathbb{N}. \tag{5.4}$$

Clearly, $\alpha_k,\beta_k\to 1$ as $k\to\infty$ and thus,

$$T(u_{n_k}) \geq \frac{1}{\alpha_k^2}T(\alpha_k u_{k,1}) + \frac{1}{\beta_k^2}T(\beta_k u_{k,2}) + o(1)$$

as $k\to\infty$. Since by (5.4) we have that $\alpha_k = 1+o(1)$ and $\beta_k = 1+o(1)$ as $k\to\infty$ it follows that

$$T(u_{n_k}) \geq T(\alpha_k u_{k,1}) + T(\beta_k u_{k,2}) + o(1) \geq I_\alpha + I_{\mu-\alpha} + o(1), \tag{5.5}$$

as $k\to\infty$, by the way the sequences $(\alpha_k)_{k\in\mathbb{N}}$ and $(\beta_k)_{k\in\mathbb{N}}$ were chosen. Taking the limit as $k\to\infty$ in (5.5) yields

$$I_\mu \geq I_\alpha + I_{\mu-\alpha}, \tag{5.6}$$

which by Lemma 5.5 is a contradiction. ∎

In order to rule out the vanishing regime the following auxiliary lemma will be required

Lemma 5.10

Let $p>2$. Suppose that $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^p(\mathbb{R})$ and that (u_n') is bounded in $L^2(\mathbb{R})$ and that there exists $T>0$ such that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}}\int_{y-T}^{y+T}|u_n|^p dx = 0.$$

Then, $u_n\to 0$ in $L^r(\mathbb{R})$ for any $r>p$.

Remark 5.11:
This result is stated for one dimension. However, in [24, Lemma I.1], it is stated and proved in higher dimensions. For dimension one, the argument used in the proof of Lemma 2.10, can be used to prove this result assuming $(u_n)_{n\in\mathbb{N}}$ to be bounded in $H^1(\mathbb{R})$.

Lemma 5.12: *Vanishing does not occur*

Let $\mu > 0$ and $p > 2$. Let also $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence to the problem (5.1). Then item 2 in Lemma 5.6 does not occur.

Proof. We claim that in this case, the L^p norm of the minimizing sequence converges to zero, which by definition of minimizing sequence for problem (5.1) is a contradiction. Note that by the Cauchy-Schwarz inequality,

$$\int_{\mathbb{R}} |u_{n_k}|^p dx \leq \left(\int_{\mathbb{R}} |u_{n_k}|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |u_{n_k}|^{2(p-1)} dx \right)^{\frac{1}{2}}.$$

Taking $r = 2(p-1) > p$ in the previous lemma and since (u_{n_k}) is uniformly bounded in $H^1(\mathbb{R})$ yields

$$\|u_{n_k}\|^p_{L^p(\mathbb{R})} \to 0, \quad \text{as} \quad k \to \infty.$$

■

With these two previous lemmas we can prove Theorem 5.8.

Proof of Theorem 5.8. Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence for problem (5.1). Then it follows from Proposition 5.3 together with Lemmas 5.9 and 5.12 that we are in the first case (compactness) of Lemma 5.6. Then, there exists a sequence $(y_k)_{k\in\mathbb{N}}$ such that for all $\epsilon > 0$ there exists $T > 0$ such that

$$\int_{y_k - T}^{y_k + T} |u_{n_k}|^p dx \geq \mu - \epsilon$$

and

$$\int_{\mathbb{R}\setminus(y_k - T, y_k + T)} |u_{n_k}|^p dx \leq \epsilon.$$

Let now $v_k := u_{n_k}(\cdot + y_k)$. Since v_k is bounded in $H^1(\mathbb{R})$, we know that there exists $v \in H^1(\mathbb{R})$ such that, up to a subsequence, $v_k \rightharpoonup v$ in $H^1(\mathbb{R})$. From here, by *Rellich-Kondrachov*, see Theorem A.8, we have that

$$v_k \to v \quad \text{in} \quad L^p_{loc}(\mathbb{R}). \tag{5.7}$$

Consequently, by the same argument as the one used in the proof of Theorem 2.7, we have that $v_k \to v$ strongly in $L^p(\mathbb{R})$. In particular, we have that $\|v\|^p_{L^p(\mathbb{R})} = \mu$. Moreover, from the weak convergence in $H^1(\mathbb{R})$ it follows that

$$I_\mu \leq T(v) \leq \liminf_{k\to\infty} T(v_k) = I_\mu, \tag{5.8}$$

whence v is a minimizer. To finish we only need to check that $v_k \to v$ strongly in $H^1(\mathbb{R})$. Since T defines an equivalent norm to the standard H^1 norm we have that $T(v_k) \to T(v)$ implies that $\|v_k\|_{H^1(\mathbb{R})} \to \|v\|_{H^1(\mathbb{R})}$. This together with the weak convergence in $H^1(\mathbb{R})$ yields the desired strong convergence, since $H^1(\mathbb{R})$ is a Hilbert space. ■

Now that we have existence of solutions the question follows: are the solutions unique? To answer this question we state and prove the following proposition.

Proposition 5.13: *Constrained Euler-Lagrange equation*

Let $p > 2$, $\lambda > 0$ and $\mu > 0$. Let also $u \in H^1(\mathbb{R})$ with $\|u\|_{L^p(\mathbb{R})}^p = \mu$ be a solution to the problem (5.1). Then, there exists a Lagrange multiplier $\theta > 0$ such that u satisfies,

$$\int_{\mathbb{R}} u'v' + \lambda uvdx = \theta \int_{\mathbb{R}} |u|^{p-2}uvdx, \quad \text{for all} \quad v \in H^1(\mathbb{R}). \tag{5.9}$$

In other words, u is a weak solution to the equation $-u'' + \lambda u = \theta|u|^{p-2}u$.

Proof. Since $R \in C^1(H^1(\mathbb{R}))$, note that

$$R'(u)u = \int_{\mathbb{R}} |u|^p dx = p\mu \neq 0.$$

Moreover, from the differentiability of T we can apply the theory of Lagrange multipliers. Thus, there exists a Lagrange multiplier $\theta \in \mathbb{R}$ such that

$$T'(u) = \theta R'(u),$$

or equivalently,

$$\int_{\mathbb{R}} u'v' + \lambda uvdx = \theta \int_{\mathbb{R}} |u|^{p-2}uvdx, \quad \text{for all } v \in H^1(\mathbb{R}). \tag{5.10}$$

Note that since $\lambda > 0$, testing (5.10) with u yields that $\theta > 0$. In fact,

$$\theta = \frac{2I_\mu}{p\mu} = \frac{2}{p}\mu^{\frac{p}{2}-1}I_1 > 0.$$

■

We already mentioned that if solutions exist then they are going to be real valued. To easily make the connection with the results of Appendix C suppose that we are considering problem (5.1) to be defined over complex valued functions. Then we have some immediate consequences.

Remark 5.14:

1. In fact, it follows from Lemma C.1 that u being a weak solution to the equation $-u'' + \lambda u = \theta|u|^{p-2}u$ then $u \in W^{3,q}(\mathbb{R}, \mathbb{C})$ for all $q \geq 2$, consequently, $u \in C^2(\mathbb{R}, \mathbb{C})$ and thus it is a solution of the equation in the classical sense.
2. Note that to simplify things we can get rid of the Lagrange multiplier. If u is a solution to $-u'' + \lambda u = \theta|u|^{p-2}u$ then, $v = \theta^{\frac{1}{p-2}}u$ solves the equation

$$-v'' + \lambda v = |v|^{p-2}v.$$

   Note that this answers the question in the beginning of this Chapter.

This last remark establishes an important connection with the results in Chapter 2. In particular, we now have, together with Theorem C.3, the answer to our question of uniqueness.

Theorem 5.15: *Uniqueness of Solution to the Problem* (5.1)

Let $\mu > 0$, $\lambda > 0$ and $p > 2$ and consider the minimization problem (5.1). Then, there exists a unique function $\varphi \in R_\mu$, depending on p, μ and λ, which is positive even and strictly decreasing on $[0, +\infty)$ such that every minimizer of T constrained to R_μ is given, up to *phase multiplication* and *translation*, by the function φ. In other words, $u \in H^1(\mathbb{R}, \mathbb{C})$ is a minimizer for the problem (5.1) if and only if

$$u(x) = e^{i\gamma}\varphi(x - y), \quad x \in \mathbb{R}.$$

for some $\gamma, y \in \mathbb{R}$.

Proof. Given Theorem C.3 and the the second item in Remark 5.14 we have that the necessary condition is immediate. We then focus on proving the sufficient condition. Let $u \in H^1(\mathbb{R}, \mathbb{C})$ be a minimizer for problem (5.1). Again by Remark 5.14 we know that there exists a Lagrange multiplier $\theta > 0$ such that $v = \theta^{\frac{1}{p-2}} u$ solves the equation

$$-v'' + \lambda v = |v|^{p-2} v.$$

Applying now Theorem C.3 we have that there exist a unique function $\psi \in H^1(\mathbb{R})$, with all the properties of the statement, and $y, \gamma \in \mathbb{R}$ such that

$$v(x) = e^{i\gamma} \psi(x - y) \quad x \in \mathbb{R}.$$

Therefore,

$$u(x) = e^{i\gamma} \varphi(x - y) \quad x \in \mathbb{R},$$

for $\varphi = \theta^{-\frac{1}{p-2}} \psi$ with $\|\varphi\|^p_{L^p(\mathbb{R})} = p\mu$. Moreover, since the value of θ is determined uniquely by λ and μ we have together with the uniqueness of ψ that the function φ is also unique. ■

Remark 5.16:
Since we already know that solutions are real valued we can, *a priori*, assume that the factor $e^{i\gamma}$ reduces to ± 1, which means solutions are unique up to translation and change of sign.

5.2 The Pohozaev Problem in Graphs

5.2.1 Statement of the Problem and the Compact Case

We will now investigate the existence of solutions to the equation

$$-u'' + \lambda u = |u|^{p-2} u,$$

for $p > 2$ and $\lambda > 0$ in metric graphs. Since scaling arguments in graphs do not work, we need to formalize and study the problem directly in graphs.

Let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a metric graph. Let $\mu, \lambda > 0$ and $p > 2$. We define the functionals T and R on $\mathcal{G}$ as:

$$T(\cdot, \mathcal{G}) : H^1(\mathcal{G}) \to \mathbb{R}; \quad T(u, \mathcal{G}) = \frac{1}{2} \int_{\mathcal{G}} |u'|^2 dx + \frac{\lambda}{2} \int_{\mathcal{G}} |u|^2 dx$$

and

$$R(\cdot, \mathcal{G}) : H^1(\mathcal{G}) \to \mathbb{R}; \quad R(u, \mathcal{G}) = \frac{1}{p} \int_{\mathcal{G}} |u|^p dx.$$

The problem in question is to find $u \in H^1(\mathbb{R})$ that minimizes T under the constraint $R(u, \mathcal{G}) = \mu$. Again, letting $R^\mu_{\mathcal{G}} := R^{-1}(\cdot, \mathcal{G})\{\mu\}$, in a compact way the problem is written as

$$\inf_{u \in R^\mu_{\mathcal{G}}} T(u, \mathcal{G}) = \inf \left\{ \frac{1}{2} \int_{\mathcal{G}} |u'|^2 dx + \frac{\lambda}{2} \int_{\mathcal{G}} |u|^2 dx \mid u \in R^\mu_{\mathcal{G}} \right\} \tag{5.11}$$

We are concerned with existence or non-existence of minimizers. Similarly to what was done in Chapter 3 we start by restating and proving some *a priori* necessary conditions for the existence of minimizers.

Proposition 5.17

Let $\mathcal{G}$ be a metric graph and assume $u \in R^{\mu}_{\mathcal{G}}$ is a solution to problem (5.11). Then:

(*i*) there exists $\theta > 0$ such that u is a weak solution to the equation $-u'' + \lambda u = \theta |u|^{p-2}u$, which is given by

$$\int_{\mathcal{G}} u'v' + \lambda uv dx = \theta \int_{\mathcal{G}} |u|^{p-2}uv dx, \quad \text{for all} \quad v \in H^1(\mathcal{G}).$$

Moreover, on each edge e, u_e is a classical solution of the equation:

$$-u_e'' + \lambda u_e = \theta |u_e|^{p-2}u_e \text{ for all } e \in E(\mathcal{G}); \tag{5.12}$$

Consequently, $u \in \tilde{H}^2(\mathcal{G})$.

(*ii*) For every vertex $v \in V(\mathcal{G})$ that is not a vertex at infinity the conditions (N-K) are satisfied;

(*iii*) Up to a change of sign, $u > 0$ on $\mathcal{G}$.

Proof. The proofs of items *(i)* and *(ii)* follow the exact same steps as in Proposition 3.17 so we omit them. We focus then on proving *(iii)*. Take $u \in H^1(\mathcal{G})$ a minimizer of problem (5.11). By item (i) and Remark 5.14 we know $u(x) = \theta^{-\frac{1}{p-2}} v(x)$ where $v \in H^1(\mathcal{G})$ solves the equation

$$-v'' + \lambda v = |v|^{p-2}v.$$

We can now apply the exact same argument as the one used in item *(iii)* from Proposition 3.17 to prove that, up to a change of sign, $v > 0$. Consequently, up to a change of sign, $u > 0$. ■

We now state the first existence result.

Proposition 5.18: *Existence of Solution on a Compact Graph*

Let $\mathcal{G}$ be a compact metric graph, $\mu, \lambda > 0$ and $p > 2$. Then problem (5.11) admits a solution.

Proof. Take $(u_n)_{n\in\mathbb{N}}$ a minimizing sequence for problem (5.11). Again, from the fact that $T(\cdot, \mathcal{G})$ defines an equivalent norm to the standard $H^1(\mathcal{G})$ norm, the uniform boundedness of the minimizing sequence is immediate. Therefore, there exists $u \in H^1(\mathcal{G})$ such that $u_n \rightharpoonup u$ in $H^1(\mathcal{G})$ up to a subsequence. By the compactness that follows from Lemma 4.1 the conclusion follows easily. ■

Therefore, just as in Chapter 4, the question of existence or non-existence of minimizers for problem (5.11) only becomes mathematically interesting in non-compact graphs. Henceforth, we only consider this type of graphs.

5.2.2 The Non-compact Case. Non-existence Results

Again, let us start by comparing to the particular case of $\mathcal{G} = \mathbb{R}$. Just as in Theorem 4.3, denoting by φ_μ the unique positive and even solution of problem (5.11) we have that

Theorem 5.19

Let $\mathcal{G}$ be a non-compact metric graph. Then we have that

$$\frac{1}{2}T(\varphi_{2\mu}, \mathbb{R}) = \min_{u \in R^{\mu}_{\mathbb{R}+}} T(u, \mathbb{R}^+) \le \inf_{u \in R^{\mu}_{\mathcal{G}}} T(u, \mathcal{G}) \le \min_{u \in R^{\mu}_{\mathbb{R}}} T(u, \mathbb{R}) = T(\varphi_\mu, \mathbb{R}),$$

where φ_μ and $\varphi_{2\mu}$ are the unique positive and even solutions to the minimization problem (5.11) with $R(\varphi_\mu) = \mu$ and $R(\varphi_{2\mu}) = 2\mu$. Moreover, these inequalities hold even if the infimum is not attained.

Proof. Let us start with the first equality. Its proof is done in the exact same way as the proof of Lemma 4.4. For the first inequality, note that by Pólya-Szegő inequality and the preservation of L^p norms by the decreasing rearrangement we have, for every $u \in R^\mu_{\mathcal{G}}$,

$$T(u, \mathcal{G}) \geq T(u^\#, \mathbb{R}^+) \geq \inf_{R^\mu_{\mathbb{R}^+}} T(u, \mathbb{R}^+).$$

the first inequality then follows by taking the infimum over $R^\mu_{\mathcal{G}}$. For the second inequality we construct a sequence of smooth compactly supported functions that converges to φ_μ strongly in $H^1(\mathbb{R})$. Let $\varphi_\mu \in R^\mu_{\mathcal{G}}$ denote the unique positive and even minimizer for problem (5.11). Take $\eta \in C^\infty(\mathbb{R})$ a cutoff function such that $\eta(x) = 0$ if $|x| \geq 2$ and $\eta(x) = 1$ if $|x| \leq 1$ and such that $|\eta| \leq 1$. Now, let $\epsilon > 0$ and consider $\eta_\epsilon(x) = \eta(\epsilon x)$. For

$$c_\epsilon = \left(\frac{\mu}{\int_{\operatorname{supp} \eta_\epsilon} |\varphi_\mu|^p \eta_\epsilon^p dx} \right)^{\frac{1}{p}} \searrow 1 \text{ as } \epsilon \to 0 \tag{5.13}$$

we define

$$u_\epsilon(x) := c_\epsilon \varphi_\mu(x) \eta_\epsilon(x). \tag{5.14}$$

For this choice of c_ϵ note that we have $u_\epsilon \in R^\mu_{\mathcal{G}}$, for each $\epsilon > 0$. From here the proof follows exactly like proof of Theorem 4.3. ■

We now investigate if the assumption (H) still allows to characterize all the graphs for which existence of minimizers occurs. Let us start by recalling assumption (H).

> Let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a metric graph. After the removal of the interior of any edge $e \in E(\mathcal{G})$, every connected component of the subgraph $(V(\mathcal{G}), E(\mathcal{G}) \setminus \{e\})$ contains at least one vertex at infinity.

The first thing to notice is that if $\mathcal{G}$ satisfies (H), then the second inequality in Theorem 5.19 becomes an equality.

Theorem 5.20

If $\mathcal{G}$ satisfies (H), then

$$\inf_{u \in R^\mu_{\mathcal{G}}} T(u, \mathcal{G}) = \min_{u \in R^\mu_{\mathbb{R}}} T(u, \mathbb{R}) = T(\varphi_\mu, \mathbb{R}). \tag{5.15}$$

Remark 5.21:
The proof of this result follows exactly like the proof of Theorem 4.8. Note that this proof is of topological nature, meaning that the conclusion is immediate provided we know that (H) implies the Pólya-Szegő inequality for the Schwarz symmetrization. This was seen to be true exactly in Theorem 4.8.

We finish this dissertation with a necessary and sufficient condition for the non-existence of minimizers for problem (5.11).

Theorem 5.22: *Under (H) ground states exist if and only if $\mathcal{G}$ has specific topologies*

Let $\mathcal{G}$ be a metric graph. If $\mathcal{G}$ satisfies (H) then

$$\inf_{u \in R^\mu_{\mathcal{G}}} T(u, \mathcal{G}) = T(\varphi_\mu, \mathbb{R})$$

but the infimum is never attained unless $\mathcal{G}$ is isometric to the graphs depicted in Figures 3.5 and 4.3.

Remark 5.23:
The proof of this result follows exactly like the proof of Theorem 4.9 by replacing E with T and $H^1_\mu(\mathcal{G})$ with $R^\mu_\mathcal{G}$. It is important to notice this happens because of the qualitative properties of the solutions to the minimization problem (5.11) and (4.1) being the same. This is a consequence of the constrained Euler-Lagrange equation to both problems to either be of the form, or to be able to be reduced to the form

$$-u'' + \lambda u = |u|^{p-2}u.$$

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Appendix A

Important Results and Estimates

Our goal with this appendix is to make the reading of this dissertation as self contained as possible by assuring the reader that all the main results which are used continuously throughout the text are stated. Even though most results are not proved, references to the proofs are provided.

A.1 Measure Theory and Integration

The following is an fundamental result from measure and integration theory and a proof can be seen, for example, in [17, Theorem 2.24].

Theorem A.1: *Dominated Convergence*

Let Ω be an open subset of $\mathbb{R}^N$ and let $(u_k)_k$ be a sequence of measurable functions such that:

- $u_k(x) \to u(x)$ a.e. in Ω as $k \to \infty$;
- there exists $v \in L^1(\Omega)$ such that for all k, $|u_k(x)| \leq v(x)$ a.e. in Ω;

Then $u \in L^1(\Omega)$ and $u_k \to u$ in $L^1(\Omega)$.

The following result is a refined version of Fatou's Lemma. A proof of this result can be seen in [30, Lemma 1.32].

Lemma A.2: *Brézis-Lieb Lemma*

Let Ω be an open subset of $\mathbb{R}^N$, $1 \leq p < \infty$ and let $(u_n)_{n\in\mathbb{N}} \subset L^p(\Omega)$. If $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^p(\Omega)$ and there exists some function u such that $u_n \to u$ almost everywhere in Ω, then $u \in L^p(\Omega)$ and

$$\lim_{n\to\infty} \left(\|u_n\|^p_{L^p(\Omega)} - \|u_n - u\|^p_{L^p(\Omega)} \right) = \|u\|^p_{L^p(\Omega)}$$

A.2 Embeddings and Interpolation Results

We begin with an interpolation result:

Theorem A.3: *Gagliardo-Nirenberg Inequality*

Let $1 \leq q, r \leq +\infty$ and let m, j be two integers, $0 < j \leq m$. Take $\alpha \in [\frac{j}{m}, 1]$ and let

$$\frac{1}{p} = \frac{j}{N} + \alpha\left(\frac{1}{r} - \frac{m}{N}\right) + \frac{1-\alpha}{q}.$$

Then there exists $C = C(N, m, j, \alpha, q, r)$ such that

$$\|D^j u\|_{L^p(\mathbb{R}^N)} \leq C\|D^m u\|^{\alpha}_{L^r(\mathbb{R}^N)}\|u\|^{1-\alpha}_{L^q(\mathbb{R}^N)} \text{ for all } u \in L^q(\mathbb{R}^N) \cap W^{k,r}(\mathbb{R}^N).$$

For a proof we refer the reader to [26]. Also we present the following two corollaries of Theorem A.3 which are used throughout this dissertation.

Corollary A.4

Let $p \geq 2$. Taking $N = 1$, $q = r = 2$, $j = 0$ and $m = 1$ we have, for $\alpha = \frac{p-2}{2p}$, the following estimate

$$\|u\|_{L^p(\mathbb{R})} \leq C\|u'\|^{\alpha}_{L^2(\mathbb{R})}\|u\|^{1-\alpha}_{L^2(\mathbb{R})}, \quad \text{for all } u \in H^1(\mathbb{R}).$$

Corollary A.5

Taking $N = 1$, $q = r = 2$, $j = 0$ and $m = 1$ we have, for $\alpha = 1$, the following estimate

$$\|u\|_{L^\infty(\mathbb{R})} \leq C\|u'\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\|u\|^{\frac{1}{2}}_{L^2(\mathbb{R})}, \quad \text{for all } u \in H^1(\mathbb{R}).$$

The next interpolation result is similar to the one above but for the bounded case.

Theorem A.6: *Gagliardo-Nirnberg Interpolation for bounded intervals*

Let I be a bounded interval and let $1 \leq r \leq \infty$, $1 \leq q \leq p \leq \infty$. Then, there exists a constant $C > 0$ such that

$$\|u\|^p_{L^p(I)} \leq C\|u\|^{p(1-\alpha)}_{L^q(I)}\|u\|^{p\alpha}_{W^{1,r}(I)}, \quad \text{for all } u \in W^{1,r}(I),$$

where $\alpha \in [0, 1]$ is defined by $\alpha(\frac{1}{q} - \frac{1}{r} + 1) = \frac{1}{q} + \frac{1}{p}$.

For a proof we refer the reader to Brezis, [9, Chapter 8]. The following direct consequence of this result is useful in this thesis.

Corollary A.7

Let I be an open bounded interval. For $r = q = 2$ and $p > 2$ we have, for $\alpha = \frac{p-2}{2p}$, that

$$\|u\|^p_{L^p(I)} \leq C\|u\|^{\frac{p}{2}+1}_{L^2(I)}\|u\|^{\frac{p}{2}-1}_{H^1(I)}, \quad \text{for all } u \in H^1(I),$$

where $\alpha \in [0, 1]$ is defined by $\alpha(\frac{1}{q} - \frac{1}{r} + 1) = \frac{1}{q} + \frac{1}{p}$.

We now present two compact embeddings of Sobolev spaces in dimension one. The first can be seen as a one dimensional form of the Rellich-Kondrachov compact embedding theorem.

Theorem A.8: *$W^{1,p}(I)$ is compactly embedded in $C(\overline{I})$*

Let $I \subset \mathbb{R}$ be an open interval and $1 \leq p \leq \infty$. Then there exists $C = C(I) > 0$ such that

$$\|u\|_{L^\infty(I)} \leq C\|u\|_{W^{1,p}(I)}, \quad \text{for all} \ \ u \in W^{1,p}(I).$$

Furthermore, if I is bounded and $1 < p \leq \infty$ then $W^{1,p}(I)$ is compactly embedded in $C(\overline{I})$.

A proof of this result can be seen in [9, Theorem 8.8]. As a consequence we have that

Corollary A.9

Let $I \subset \mathbb{R}$ be an open and bounded interval and $1 < p \leq \infty$. Then $W^{1,p}(I)$ is compactly embedded in $L^q(I)$ for all $q \geq 1$.

Proof. Let $p > 1$ and $q \geq 1$. To prove the above result we need to show that the inclusion operator $\iota : W^{1,p}(I) \to L^q(I)$ is a compact. From the above result we can write the above inclusion operator as the composition of the following inclusion operators

$$\iota : W^{1,p}(I) \xrightarrow{\iota_1} C(\overline{I}) \xrightarrow{\iota_2} L^q(I).$$

Since both ι_1 and ι_2 are linear and continuous and ι_1 is also compact, then ι is also compact. ■

Moreover, in the unbounded case we have the following interpolation estimate

Corollary A.10

Let $I \subset \mathbb{R}$ be unbounded and $u \in H^1(I)$. Then $u \in L^q(I)$ for all $q \geq 2$ and we have:

$$\|u\|^q_{L^q(I)} \leq \|u\|^{q-2}_{L^\infty(I)}\|u\|^2_{L^2(I)}. \tag{A.1}$$

Proof. Let $u \in H^1(\mathbb{R})$ and $q \geq 2$. Since by Theorem A.8 we have $u \in L^\infty(I)$, then

$$\|u\|^q_{L^q(I)} = \int_I |u|^{q-2}|u|^2 dx \leq \|u\|^{q-2}_{L^\infty(I)}\|u\|^2_{L^2(I)} < \infty. \qquad ■$$

A.3 Estimates

For the following useful inequality we provide the reader with a proof:

Lemma A.11

For every $q > 0$ there exists C_q such that

$$|a + b|^q \leq C_q \left(|a|^q + |b|^q\right), \quad \forall a, b \in \mathbb{R}.$$

Proof. Fix $q > 0$ and let $a, b \in \mathbb{R}$.
If $b = 0$ the conclusion holds for $C_q = 1$. Suppose now that $b \neq 0$. In this case we can rewrite the inequality as

$$\left|\frac{a}{b} + 1\right|^q \leq C_q\left(\left|\frac{a}{b}\right|^q + 1\right).$$

Since $\left|\frac{a}{b}\right|^q + 1 \neq 0$ in $\mathbb{R}$, the above inequality can be rewritten as

$$\frac{\left|\frac{a}{b}+1\right|^q}{\left|\frac{a}{b}\right|^q+1} \leq C_q.$$

Taking $x = \frac{a}{b}$ we define in $\mathbb{R}$ the auxiliary function: $f(x) = \frac{|x+1|^q}{|x|^q+1}$. If this function is bounded then the conclusion follows. Note straight away that f is a strictly positive function in $\mathbb{R}$. Clearly f is continuous being the quotient of continuous functions whose denominator is never zero. Moreover, when approaching infinity, $|x+1|^q \backsim |x|^q$ and therefore

$$\lim_{|x|\to+\infty} \frac{|x+1|^q}{|x|^q+1} = 1.$$

From the continuity and the asymptotic behaviour shown above the function is bounded by a strictly positive constant C_q. ■

A.4 Rearrangements of Functions

The book of Kawhol [20] is a classical reference regarding this topic, however in this appendix we follow the approach by Kesavan [21] to introduce the notion of rearrangement of functions, some particular rearrangements and their elementary properties.

The concept or rearrangement of functions is closely related to the concept of rearranging subsets of $\mathbb{R}^N$, $N \geq 1$. For a measurable subset $\Omega \subset \mathbb{R}^N$ we denote the N-dimensional Lebesgue measure by $m(\Omega)$.

Definition A.12: *Distribution Function*

Let Ω be measurable subset of $\mathbb{R}^N$ with finite measure. Given a measurable function $u : \Omega \to \mathbb{R}$ we define the superlevel sets of u as $\{u > t\} := \{x \in \Omega : u(x) > t\}$, $t \in \mathbb{R}$. The *distribution function* of u is then defined as the function $\rho_u : \mathbb{R} \to [0, m(\Omega)]$ such that

$$\rho_u(t) := m(\{u > t\}).$$

Remark A.13:
The above definition makes sense in bounded sets. In unbounded sets the same definition will work but we need to impose more conditions on the function u. Let Ω be an unbounded subset of $\mathbb{R}^N$. We say that a measurable function $u : \Omega \to \mathbb{R}$ *vanishes at infinity* if the superlevel sets of u have finite measure for all $t > 0$, that is, $m(\{u > t\}) < \infty$ for all $t > 0$. If this assumption on u holds then we can use the above definition for its distribution function.

The above notion of distribution functions takes the sign of the functions into account. For simplicity we assume that all the functions in this section are non-negative functions.

Definition A.14: *Rearrangement*

We say that two real valued functions are equimeasurable if they have the same distribution function. In this case we say that they are rearrangements of one another.

We now define the first notion of rearrangement.

Definition A.15: *Decreasing Rearrangement*

Let Ω be a subset of $\mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$ a measurable non-negative function that vanishes at infinity. Then, its *decreasing rearrangement* is defined as the the function $u^{\#} : [0, m(\Omega)] \to \mathbb{R}$ such that $u^{\#}(0) = \operatorname{ess\,sup}(u)$ and $u^{\#}(s) = \inf\{t > 0 : \rho_u(t) < s\}$ for $s > 0$.

Proposition 1.1.3 in [21], shows that $u^{\#}$ is in fact a rearrangement of u. As a consequence of the equimeasurability between u and $u^{\#}$ we have the following corollary:

Corollary A.16: *Deacreasing Rearrangement preserves L^p- norms*

Let $1 \leq p \leq \infty$. If $u \in L^p(\Omega)$ is non-negative then $u^{\#} \in L^p(0, m(\Omega))$. Moreover,

$$\|u\|_{L^p(\Omega)} = \|u^{\#}\|_{L^p(0,m(\Omega))}.$$

The proof of this corollary can be seen in Corollary 1.1.2 of [21]. We now introduce the other notion of rearrangement that we require, the *Schwarz symmetrization*, also known as the *spherically symmetric decreasing rearrangement*. To do so we need to define a rearrangement of subsets of $\mathbb{R}^N$. For Ω a subset of $\mathbb{R}^N$ with finite measure, we define Ω^* as the ball centred at the origin with the same measure as Ω.

Definition A.17: *Schwarz Symmetrization*

Let $u : \Omega \to \mathbb{R}$ be a measurable function vanishing at infinity. We define its *Schwarz symmetrization* on $\mathbb{R}^N$ as

$$u^*(x) = \int_0^\infty \chi_{\{u>t\}^*}(x)dt.$$

This definition is as in section 1.4 of [21]. Another definition, for bounded subsets of $\mathbb{R}^N$, is provided in section 1.3 and takes the form

$$u^*(x) = u^{\#}\left(\omega_N |x|^N\right), \; x \in \Omega^*,$$

where ω_N is the volume of the N-dimensional unit ball in $\mathbb{R}^N$. In this section the most elementary properties of this rearrangement are stated and proved. Our choice of definition turns out to be equivalent to this latter one for bounded sets, however, we will need to adapt the concept of rearrangement in the unbounded case, which justifies our choice for the definition used here. Finally, the equimeasurability of u and u^* is also proved in section 1.4 of [21]. As a consequence we also have the following Corollary:

Corollary A.18: *Schwarz Symmetrization preserves L^p- norms*

Let $1 \leq p \leq \infty$. If $u \in L^p(\Omega)$ is a non-negative function then $u^* \in L^p(\Omega^*)$. Moreover,

$$\|u\|_{L^p(\Omega)} = \|u^*\|_{L^p(\mathbb{R})}.$$

We now know that the L^p- norms are preserved by the above rearrangements. What happens if we now take the rearrangement of a function $u \in W^{1,p}(\Omega)$? What happens to the generalized derivative of the function? The answer to this question is given by Pólya-Szegő-type Inequality which we now state for both rearrangements.

Theorem A.19: *Pólya-Szegő Inequality for $u^{\#}$*

Let Ω be a subset of $\mathbb{R}^N$, $1 \leq p < \infty$ and $u \in W_0^{1,p}(\Omega)$ be a non-negative function. Then

$$\int_0^{m(\Omega)} |u^{\#'}(s)|^p ds \leq \int_\Omega |\nabla u(x)|^p dx.$$

A proof for this result in dimension one can be seen in the first edition of [22, Theorem 6.28]. Finally the same inequality holds for the Schwarz Symmetrization:

Theorem A.20: *Pólya-Szegő Inequality for u^**

Let Ω be a subset of $\mathbb{R}^N$, $1 \leq p < \infty$ and $u \in W_0^{1,p}(\Omega)$ be a non-negative function. Then,

$$\int_{\Omega*} |\nabla u^*(x)|^p dx \leq \int_{\Omega} |\nabla u(x)|^p dx.$$

A proof of this result can be seen in section 2.3 of [21] for the definition given in section 1.3, that is, for the bounded case. The unbounded case follows easily with the same proof provided the function we are rearranging vanishes at infinity, which is always true for functions $W^{1,p}(\Omega)$ when Ω is not bounded.

In particular, these last two results, together with the corollaries A.16 and A.18 say that both the Schwarz and the decreasing rearrangements of $W^{1,p}$ functions are also $W^{1,p}$ functions in their respective domains.

Appendix B

Differential Calculus and Constrained Extrema Problems

B.1 Differentiable Calculus in Banach Spaces

In this section we introduce the concept of differentiability in Banach Spaces. This will be required a few times along the thesis and also to treat formally the question of Lagrange Multipliers on Banach spaces in the next section. In order to introduce this topic we follow [5].

Firstly, recall that given a Banach space X, we denote its dual space by $X' := \mathcal{L}(X, \mathbb{R})$. When endowed with the norm

$$\|A\|_{X'} = \sup_{\|u\|_X = 1} |Au|$$

this space becomes a Banach Space.

Definition B.1: *Fréchet Differentiability*

Let X be a Banach space, Y an open subset of X and let $f : Y \to \mathbb{R}$ be a functional. We say that f is (Fréchet) differentiable at a point $u \in Y$ if there exists $A \in X'$ such that:

$$\lim_{\|v\| \to 0} \frac{f(u + v) - f(u) - Av}{\|v\|} = 0.$$

Such element A, if it exists, is unique and we denote it by $f'(u)$ and call it the *differential* of f at u.

Definition B.2: *Continuously Differentiable Functional*

Let X be a Banach space, Y an open subset of X and let $f : Y \to \mathbb{R}$ be a functional. If f is differentiable at every $u \in Y$ we say that it is differentiable in Y. Moreover, if the *derivative*, that is, the map $f' : Y \to X'$ is continuous, then we say that f is *continuously differentiable* and we write $f \in C^1(Y)$.

Another concept of differentiation used in Banach spaces is one that resembles directional derivatives in $\mathbb{R}^N$ and is as follows:

Definition B.3: *Gâteaux Differentiability*

Let X be a Banach space, Y an open subset of X and let $f : Y \to \mathbb{R}$ be a functional. We say that f is (Gâteaux) differentiable at a point $u \in Y$ if the following limit exists for all $v \in X$:

$$\lim_{t \to 0} \frac{f(u + tv) - f(u)}{t}$$

Again, if such an element A exists it is unique and we denote it by $f'_G(u)$ and call it the Gâteaux differential of f at u.

Remark B.4:

1. It is important in both definitions to know the difference between differential and derivative. For example, the differential is always a linear map and defined in the whole space X, even if the derivative is only defined on a subset of X;
2. Independently of the definition used, if a functional is differentiable at a point then it is also continuous at that point;
3. These types of differentiability are not equivalent. However, Fréchet differentiability implies Gâteaux differentiability. The reverse is not true just as differentiation is not equivalent to taking directional derivatives in the case of $X = \mathbb{R}^N$.

This last remark leaves the question: under which conditions are both concepts of differentiability equivalent? The answer is provided by the following theorem:

Theorem B.5

Let $Y \subset X$ be an open set and $f : Y \to \mathbb{R}$. Assume f is Gâteaux differentiable and f'_G is linear and continuous at $u \in Y$. Then, f is also Fréchet differentiable and $f'_G(u) = f'(u)$.

The proof of this result can be seen for example in [4, Theorem 1.9]. Elementary algebraic properties of the differentials can also be seen in [5, Section 1.3].

B.2 Constrained Extrema Problems

In this section our goal is to present the Theory of Lagrange Multipliers on Banach spaces, while following closely [28, Section 1.3]. The main result of this section gives a necessary condition for solutions of problems of the type:

$$\min\{f(u) : u \in Y,\ \varphi(u) = 0\}, \quad \text{for } c \in \mathbb{R}, \tag{B.1}$$

where Y is an open subset of a Banach space X and $f, \varphi \in C^1(Y)$. Denote $M := \varphi^{-1}\{0\}$. Then the above problem is the same as finding the minimum of $f|_M$. The following conditions are assumed:
1. $M \neq \emptyset$;
2. $\varphi'(u) \neq 0$ for all $u \in M$.

Lemma B.6

Let $u_0 \in M$. Under condition 2 above, for all $z \in \text{Null}(\varphi'(u_0))$ there exists $\epsilon > 0$ and a curve of class C^1, $\alpha : (-\epsilon, \epsilon) \to M$, such that $\alpha(0) = u_0$ and $\alpha'(0) = z$.

Proof. Let $u_0 \in M$. From the condition $\varphi'(u_0) \neq 0$, since the range of the operator has dimension 1, we get that $\varphi'(u_0)$ is surjective. Thus, we can take $e \in X$ such that $\varphi'(u_0)e = 1$ which allows the construction of the following decomposition of X:

$$X = \text{Null}(\varphi'(u_0)) \oplus \mathbb{R}e.$$

Decompose $u_0 = v_0 + s_0 e$, for some $v_0 \in Null(\varphi'(u_0))$ and $s_0 \in \mathbb{R}$, and define the C^1 functional: $F : \text{Null}(\varphi'(u_0)) \times \mathbb{R} \to \mathbb{R}$ by $F(v, s) = \varphi(v+se)$ it is easy to check that F is under the hypothesis of the Implicit Function Theorem, see [4, Theorem 2.3]. Indeed by composition F is a class $C^1(\text{Null}(\varphi'(u_0)) \times \mathbb{R})$. At the point $u_0 \simeq (v_0, s_0) \in M$ we have that $F(v_0, s_0) = \varphi(u_0) = 0$. Finally note that the partial derivative of F in order to the variable s, which we denote by F'_s, at (v_0, s_0) is given by the chain rule through

$$F'_s(v_0, s_0)t = \varphi'(u_0)te = t\varphi'(u_0)e = t.$$

This means that the partial derivative is the identity and therefore an invertible map. Thus, the Implicit Function Theorem can be applied and there exist open neighbourhoods $V \subset \text{Null}(\varphi'(u_0))$ of v_0 and $I \subset \mathbb{R}$ of s_0 and a C^1 function $\phi : V \to I$ such that:

$$F(v, s) = 0 \Leftrightarrow s = \phi(v)$$

for all $v \in V$, $s \in I$. Take $z \in \text{Null}(\varphi'(u_0))$. By implicit differentiation we have that $\phi'(v_0) = 0$.

Take now $\epsilon > 0$ so that $v_0 + tz + \phi(v_0 + tz)e \in M$ for $|t| < \epsilon$. We can now define the curve $\alpha : (-\epsilon, \epsilon) \to M$ given by $\alpha(t) = (v_0 + tz) + \phi(v_0 + tz)e$ which is of class C^1 and, moreover, verifies $\alpha(0) = u_0$ and $\alpha'(0) = z$. ■

Definition B.7: Critical Point

We call $u \in M$ a *critical point* of $f|_M$ if there exists $\lambda \in \mathbb{R}$ such that $f'(u) = \lambda\varphi'(u)$. We call the scalar λ of *Lagrange Multiplier*.

The following result gives a necessary condition that solutions to problem (B.1) need to satisfy:

Proposition B.8

Suppose $\varphi'(u)$ is a surjective map for all $u \in M$. If $u_0 \in M$ is a maximum (or minimum) of $f|_M$, then u_0 is a critical point of $f|_M$.

Proof. Let M be as above and suppose $u_0 \in M$ is a maximum of $f|_M$. Let $z \in \text{Null}(\varphi'(u_0))$. It follows from Lemma B.6 that there exists $\epsilon > 0$ and $\alpha : (-\epsilon, \epsilon) \to M$ a C^1 curve in M such that $\alpha(0) = u_0$ and $\alpha'(0) = z$. Since u_0 is a maximum of $f|_M$ then the map $f \circ g : (-\epsilon, \epsilon) \to \mathbb{R}$ has a maximum at $t = 0$. Then,

$$f'(u_0)z = 0, \quad \text{for all } \; z \in \text{Null}(\varphi'(u_0)).$$

We now want to evaluate $f'(u_0)x$ for any $x \in X$. Since we can take $e \in X$ such that $\varphi'(u_0)e = 1$. This allows us to make the decomposition $X = \text{Null}(\varphi'(u_0)) \oplus \mathbb{R}e$. Hence, there exist $x_0 \in \text{Null}(\varphi'(u_0))$ and $s_0 \in \mathbb{R}$ such that $x = x_0 + s_0 e$. Now, on one hand,

$$f'(u_0)x = s_0 f'(u_0)e.$$

On another hand

$$\varphi'(u_0)x = s_0,$$

and the conclusion follows by taking $\lambda := f'(u_0)e$. ■

Appendix C

The One Dimensional NLS equation

In this appendix we will provide a characterization of the solutions of the one dimensional non linear Schrödinger equation with a general power nonlinearity, following closely reference [10]. Throughout this appendix we will consider only complex valued function. For that reason we remark that the Sobolev space $H^1(\mathbb{R}, \mathbb{C})$ is endowed with the norm:

$$\|u\|^2_{H^1(\mathbb{R},\mathbb{C})} = \int_{\mathbb{R}} |u'|^2 dx + \int_{\mathbb{R}} |u|^2 dx, \quad \text{for all } u \in H^1(\mathbb{R}, \mathbb{C}),$$

which is the norm induced by the inner product given by"

$$\langle u, v \rangle_{H^1(\mathbb{R},\mathbb{C})} = \mathfrak{Re} \int_{\mathbb{R}} u'\overline{v'} dx + \mathfrak{Re} \int_{\mathbb{R}} u\overline{v} dx, \quad \text{for all } u, v \in H^1(\mathbb{R}, \mathbb{C}),$$

We start by considering the following Cauchy Problem in $H^1(\mathbb{R}, \mathbb{C})$:

$$\begin{cases} i\partial_t u + u'' + |u|^{p-2}u = 0, \\ u(0, x) = \varphi(x), \end{cases} \tag{C.1}$$

where u' denotes the spatial derivative of u, $\varphi \in H^1(\mathbb{R}, \mathbb{C}) \setminus \{0\}$, $p > 2$.

We focus ourselves in the following family of solutions:

$$u(t, x) = e^{i\omega t}\varphi(x)$$

for some $\omega > 0$, $\varphi \in H^1(\mathbb{R}, \mathbb{C})$. In the literature these solutions are the so called *stationary states* or *bound states*. If u is in fact of this form then elementary computations show that φ has to solve the stationary equation

$$-\varphi'' + \omega\varphi = |\varphi|^{p-2}\varphi$$

Therefore we start by considering the problem

$$\begin{cases} -\varphi'' + \omega\varphi = |\varphi|^{p-2}\varphi \\ \varphi \in H^1(\mathbb{R}, \mathbb{C}) \setminus \{0\}. \end{cases} \tag{C.2}$$

To start with we state a regularity result.

Lemma C.1: Regularity of Solutions

Let $\omega > 0$, $\theta \in \mathbb{R}$ and $p > 2$. If $u \in H^1(\mathbb{R}, \mathbb{C})$ satisfies the equation $-u'' + \omega u = \theta|u|^{p-2}u$ in $H^{-1}(\mathbb{R}, \mathbb{C})$ then the following properties hold:

1. $u \in W^{3,p}(\mathbb{R}, \mathbb{C})$ for every $p \geq 2$. In particular, $u \in C^2(\mathbb{R}, \mathbb{C})$;
2. $\int_{\mathbb{R}} |u'|^2 dx + \omega \int_{\mathbb{R}} |u|^2 dx = \theta \int_{\mathbb{R}} |u|^{p+1}$.

For a proof of this result we refer the reader to [10, Theorems 8.1.1,8.1.2]. This lemma will only be used to justify why we can perform certain computations in the result that follows. Moreover, this provides a good justification as to why the defocusing case does not give rise to stationary states, since a contradiction would arise from item 2.

The first result is concerned with existence of solution.

Theorem C.2: Existence of Solution

If $\omega > 0$ and $p > 2$, then problem (C.2) admits a real valued solution that is positive, even and decreasing in $(0, +\infty)$.

Given that in the first chapter we deduce the existence of solutions to this equation via Lagrange multipliers we omit this proof. Indeed note that if $u \in H^1(\mathbb{R}, \mathbb{C})$ solves, for some positive λ, $-u'' + \lambda u = |u|^{p-2}u$ then,

$$v(x) = \left(\frac{\lambda}{\omega}\right)^{-\frac{1}{p-2}} u\left(\left(\frac{\lambda}{\omega}\right)^{-\frac{1}{2}} x\right)$$

solves the problem (C.2). In this appendix however, we recall a different proof, given in [10, Chapter 8]. This proof provides us not only an existence result but also a few qualitative properties of the solutions, such as being positive, even and strictly decreasing in $(0, +\infty)$. We now sketch a proof of these facts.

Sketch of the proof of Theorem C.2. The main idea of this proof is to study the following initial value problem:

$$\begin{cases} -\varphi'' + \omega\varphi = |\varphi|^{p-2}\varphi \\ \varphi(0) = c := \left(\frac{\omega p}{2}\right)^{\frac{1}{p-2}} \\ \varphi'(0) = 0. \end{cases} \tag{C.3}$$

Note that the normal form of the first equation in (C.3) allows the use of the Theorem of Picard-Lindelöf, see [3, II-7.4]. From here we deduce local existence of a real valued solution for the above IVP. Global existence is then deduced by showing that the solution is bounded, see [10, Chapter 8]. Now defining $\psi(x) := \varphi(-x)$ we get that ψ also solves (C.3) and therefore, by uniqueness, the solution is even. The positivity of the solution is also a consequence of the existence and uniqueness of solution to the IVP. For the decreasing property note that upon multiplying the first equation in (C.3) by φ' and using the initial conditions we deduce that

$$\frac{1}{2}\varphi'^2 - \frac{\omega}{2}\varphi^2 + \frac{1}{p}|\varphi|^p = 0. \tag{C.4}$$

By regularity we have that

$$\varphi''(0) = -\frac{\omega c(p-2)}{2} < 0.$$

Hence,

$$\varphi'' < 0 \text{ in } (-a, a),$$

for some $a > 0$. Then, the first derivative in this interval is strictly decreasing. Moreover, since $\varphi'(0) = 0$, have that

$$\varphi'(0) < 0 \text{ in } (0, a).$$

We now can extend this up to infinity. Indeed suppose, by way of a contradiction, that there exists $b > a > 0$ such that $\varphi'(b) = 0$ and that $\varphi' < 0$ in the interval $(0, b)$. By evaluating equation (C.4) at the point $x = b$ we get

$$\frac{1}{2}\varphi'(b)^2 - \frac{\omega}{2}\varphi^2(b) + \frac{1}{p}|\varphi(b)|^p = \varphi^2(b)\left(-\frac{\omega}{2} + \frac{|\varphi(b)|^{p-2}}{p}\right) = 0.$$

Since we already shown that φ is positive we only need to show that if $|\varphi(b)| = c$ we arrive to a contradiction. Again, $\varphi(b) = -c$ is impossible. If $\varphi(b) = c$ we also get to a contradiction because φ is strictly decreasing in the interval $(0, b)$, continuous, and $\varphi(0) = c$. In conclusion, we have that

$$\varphi' < 0 \text{ in } (0, +\infty). \qquad \blacksquare$$

The following results will show us that in fact any solution of problem (C.2) will be this one up to *phase multiplication* and *translation*. Let us introduce some notation. Let

$$A := \{\varphi \in H^1(\mathbb{R}, \mathbb{C}) \setminus \{0\} : -\varphi'' + \omega\varphi = |\varphi|^{p-2}\varphi\}.$$

From the previous result we already know that A is a non empty set. Let us also define the *action functional* of the stationary equation by

$$S : H^1(\mathbb{R}, \mathbb{C}) \to \mathbb{R}; \quad S(u) = \frac{1}{2}\int_{\mathbb{R}} |u'|^2 dx - \frac{1}{p}\int_{\mathbb{R}} |u|^p dx + \frac{\omega}{2}\int_{\mathbb{R}} |u|^2 dx \tag{C.5}$$

Just as it was done in Chapter 2, we can see that this functional is also of class $C^1(H^1(\mathbb{R}, \mathbb{C}))$. Additionally, it is easily seen that S is a coercive functional and, by computing its differential at any point $u \in H^1(\mathbb{R}, \mathbb{C})$, we see that its critical points are precisely the weak solutions to the stationary NLS equation. Note that

$$S'(u)v = \mathfrak{Re}\int_{\mathbb{R}} u'\overline{v'} - |u|^{p-2}u\overline{v} + \omega u\overline{v}dx, \text{ for all } v \in H^1(\mathbb{R}, \mathbb{C}).$$

Let now

$$G := \{u \in A : S(u) \leq S(v), \text{ for all } v \in A\}.$$

The following result shows, that in dimension one, all the non-trivial solutions of (C.2) are solutions that minimize the functional S. Moreover, the coming result also provides a characterization the solutions.

Theorem C.3: Characterization of Solutions

If $\omega > 0$ and $p > 1$. We then have

1. A and G are non empty;
2. $A = G$;
3. there exists a real valued, positive, even and decreasing function φ on $(0, +\infty)$ such that:

$$A = \{e^{i\theta}\varphi(\cdot - y) : \theta, y \in \mathbb{R}\}.$$

Proof. From Theorem C.2 we have the first item established. Note now that if there exist $\varphi \in H^1(\mathbb{R})$ and $\theta, y \in \mathbb{R}$ with the properties of the statement, $e^{i\theta}\varphi(\cdot - y)$ is in fact a solution to the problem (C.2). Take now $v \in A$. Recall that v is a complex valued function of class $C^2(\mathbb{R})$ as a consequence of Lemma C.1. Then v solves pointwise in $\mathbb{R}$ the equation

$$-v'' + \omega v = |v|^{p-2}v. \tag{C.6}$$

Upon multiplying (C.6) by $\overline{v}'$ and taking the real part we get that

$$\begin{aligned}
0 &= -\mathfrak{Re}(v''\overline{v}') + \omega\mathfrak{Re}(v\overline{v}') - |v|^{p-2}\mathfrak{Re}(v\overline{v}') \\
&= -\frac{1}{2}\left[v''\overline{v}' + \overline{v}''v'\right] + \frac{\omega}{2}\left[v\overline{v}' + \overline{v}v'\right] - \frac{|v|^{p-2}}{2}\left[v\overline{v}' + \overline{v}v'\right] \\
&= -\frac{1}{2}\frac{d}{dx}(v'\overline{v}') + \frac{\omega}{2}\frac{d}{dx}(v\overline{v}) - \frac{|v|^{p-2}}{2}\frac{d}{dx}(v\overline{v}) \\
&= -\frac{1}{2}\frac{d}{dx}|v'|^2 + \frac{\omega}{2}\frac{d}{dx}|v|^2 - \frac{|v|^{p-2}}{2}\frac{d}{dx}|v|^2.
\end{aligned}$$

From this we get, by integration over $\mathbb{R}$, that there exists some $K \in \mathbb{R}$ such that

$$\begin{aligned} K &= -\frac{1}{2}|v'|^2 + \frac{\omega}{2}|v|^2 - \int_{\mathbb{R}} \frac{|v|^{p-2}}{2} \frac{d}{dx}|v|^2 dx \\ &= -\frac{1}{2}|v'|^2 + \frac{\omega}{2}|v|^2 - \int_{\mathbb{R}} \frac{|v|^{p-2}}{2} 2|v|v' dx \\ &= -\frac{1}{2}|v'|^2 + \frac{\omega}{2}|v|^2 - \int_{\mathbb{R}} |v|^{p-2}|v|v' dx, \end{aligned}$$

which in turn implies that there exists $K_1 \in \mathbb{R}$ such that

$$-\frac{1}{2}|v'|^2 + \frac{\omega}{2}|v|^2 - \frac{|v|^p}{p} = K_1.$$

Since $v \in H^1(\mathbb{R})$ we have, as $|x| \to \infty$ that $v(x) \to 0$. Taking the limit in the equation (C.6) we conclude that also $v''(x) \to 0$ as $|x| \to \infty$. We now want to obtain that $K_1 = 0$. In order to do so we claim that $v'(x) \to 0$ as $|x| \to \infty$. To check this, let $x \in \mathbb{R}$. Given that v is of class $C^2(\mathbb{R})$ let us compute the first order Taylor expansion of v centred at the point $x+1$ with Lagrange remainder.

$$\begin{aligned} v(x) &= v(x+1) + v'(x+1)(x-(x+1)) + \frac{v''(\xi_x)}{2}(x-(x+1)) \\ &= v(x+1) - v'(x+1) - \frac{v''(\xi_x)}{2}, \end{aligned}$$

where $\xi_x \in (x, x+1)$. The conclusion is now obvious taking the limit in the previous equation since $x \to +\infty$ implies that $\xi_x \to +\infty$. For $x \to -\infty$ it is analogous.

We now have that pointwise in $\mathbb{R}$ the following equation is satisfied:

$$\frac{1}{2}|v'|^2 - \frac{\omega}{2}|v|^2 + \frac{1}{p}|v|^p = 0. \tag{C.7}$$

It follows now immediately from the previous equation that $|v| > 0$. Should $v(x) = 0$ at some point then we would also have from the equation that $v'(x) = 0$ and thus $v \equiv 0$ by existence and uniqueness of solution.

Having $|v| > 0$ allows us to look for solutions of the form:

$$v(x) = \rho(x)e^{i\theta(x)}$$

where $\rho > 0$ and θ are class $C^2(\mathbb{R})$. Differentiating the new form of v twice we obtain that

$$v''(x) = \left(\rho''(x) - \theta'(x)^2\rho(x)\right)e^{i\theta(x)} + i\left(2\rho'(x)\theta'(x) + \theta''(x)\rho(x)\right)e^{i\theta(x)}.$$

Since v solves equation (C.6) we obtain that ρ and θ need to satisfy pointwise in $\mathbb{R}$ the equality

$$-\left(\rho'' - \theta'^2\rho\right) - i\left(2\rho'\theta' + \theta''\rho\right) + \omega\rho = \rho^p,$$

whence

$$\rho\theta'' + 2\rho'\theta' = 0;$$

or, equivalently,

$$\rho^2\theta'' + 2\rho\rho'\theta' = 0.$$

Thus we can conclude that there exists $K \in \mathbb{R}$ such that

$$\rho^2(x)\theta'(x) = K \text{ for all } x \in \mathbb{R}.$$

From here it follows that $\theta'(x) = \frac{K}{\rho^2(x)}$. Our first goal now is to prove that $\theta \equiv \theta_0$ for some $\theta_0 \in \mathbb{R}$. Since v' is continuous and converges to zero as $|x| \to \infty$, v' is bounded. Moreover,

$$|v'|^2 = |\rho' + i\theta'\rho|^2 = \rho'^2 + \theta'^2\rho^2.$$

Thus we also have that $\rho^2\theta'^2$ is also bounded. Putting the expression for θ' in the quantity above we have that

$$\rho^2\theta'^2 = \rho^2\frac{K^2}{\rho^4} = \frac{K^2}{\rho^2}$$

and therefore the quantity $\frac{K^2}{\rho^2}$ is bounded as well. From this the conclusion follows easily. Note that since $\rho(x) \to 0$ as $|x| \to \infty$, if $K \neq 0$ then $\frac{K^2}{\rho^2}$ is unbounded which is a contradiction. Hence, $K = 0$. Again, from the strict positivity of ρ we deduce that

$$\theta'(x) = 0 \text{ for all } x \in \mathbb{R}.$$

Thus, by integration, such a θ_0 exists. We have so far that

$$v(x) = \rho(x)e^{i\theta_0}. \tag{C.8}$$

At this point necessarily we have that ρ is a function in $H^1(\mathbb{R})$ and therefore, since it converges to zero as $|x|$ approaches infinity, there exists $x_0 \in \mathbb{R}$ such that $\rho'(x_0) = 0$. It now follows from equation (C.7) that $\rho(x_0) = \left(\frac{\omega p}{2}\right)^{\frac{1}{p-2}} = c$.

Define now, in $\mathbb{R}$, the following function:

$$w(x) = \rho(x + x_0).$$

We have that $w(0) = c$ and $w'(0) = 0$. If we prove that w solves the equation (C.6) then it is immediately a solution to the IVP defined in (C.3). Taking into account the shape of the solution v given by (C.8) we have that:

$$\begin{aligned}
-w''(x) + \omega w(x) &= \frac{e^{i\theta_0}\left(-\rho''(x+x_0) + \omega\rho(x+x_0)\right)}{e^{i\theta_0}} \\
&= \frac{-v''(x+x_0) + \omega v(x+x_0)}{e^{i\theta_0}} \\
&= \frac{|v(x+x_0)|^{p-2}v(x+x_0)}{e^{i\theta_0}} \\
&= \frac{|\rho(x+x_0)e^{i\theta_0}|^{p-2}\rho(x+x_0)e^{i\theta_0}}{e^{i\theta_0}} \\
&= |\rho(x+x_0)|^{p-2}\rho(x+x_0) \\
&= |w(x)|^{p-2}w(x),
\end{aligned}$$

where in the first equality we simply used the definition of w and we multiplied and divided the equation by $e^{i\theta_0}$ and in the third equality we used the fact that v is a solution to the problem. Adopting the notation from the proof of Theorem C.2 we have by uniqueness of solution of the aforementioned IVP that:

$$w(x) = \rho(x + x_0) = \varphi(x) \text{ for all } x \in \mathbb{R}.$$

Or putting differently, $\rho(x) = \varphi(x - x_0)$. This gives us the desired conclusion. We have proved that indeed if v is a solution then it has the form $v(x) = e^{i\theta_0}\varphi(x - x_0)$. Moreover, φ is the unique positive and even solution to the equation (C.6). To finish the proof we remark that in fact $v \in G$. Note that if v takes the above form, computing the differential of S at v yields that in fact it is a critical point. From the coercivity of S we get that $v \in G$. ■

To finish with a remark.

Remark C.4:

1. Note that
$$E(u) = \frac{1}{2}\int_{\mathbb{R}} |u'|^2 dx - \frac{1}{p}\int_{\mathbb{R}} |u|^p dx = S(u) - \frac{\omega}{2}\int_{\mathbb{R}} |u|^2 dx.$$
Recalling that in Chapter 2 we were concerned about minimizing E with fixed L^2 norm, the above result gives that any non-trivial solution of the stationary NLS equation will also be a ground state for the NLS energy functional when minimizing on spheres.