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Lorentzian Dynamics and Factorization Beyond Rationality

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Abstract

We investigate the emergence of topological defect lines in the conformal Regge limit of two-dimensional conformal field theory. We explain how a local operator can be factorized into a holomorphic and an anti-holomorphic defect operator connected through a topological defect line, and discuss implications on Lorentzian dynamics including aspects of chaos. We derive a formula for the infinite boost limit, which holographically encodes the transparency/opacity of bulk scattering, in terms of the action of topological defect lines on local operators, and argue for a unitarity bound. Factorization also gives a formula relating the local and defect operator algebras and fusion categorical data. We review factorization in rational conformal field theory from a defect perspective, and examine irrational theories. On the orbifold branch of the $c = 1$ free boson theory, a dichotomy between rationality and irrationality is found regarding the factorization of the twist field.

Contents

1 Introduction

Two-dimensional conformal field theory enjoys special kinematics that lead to holomorphically factorized continuous symmetries [1]. However, except in very special cases, the full theory is not holomorphically factorized. The local operators transform as bi-modules of the left and right-moving chiral algebras, but a generic local operator cannot be regarded as the composite of holomorphic and anti-holomorphic *local* operators. In rational conformal field theory [2] there is a weaker sense of holomorphic factorization. Loosely speaking, on an oriented manifold $\mathcal{M}_2$, the holomorphic and anti-holomorphic degrees of freedom dwell on two separate copies of $\mathcal{M}_2$, connected through a bulk topological quantum field theory [3–8]. The truly holomorphically factorized case is when the bulk theory is trivial. Extensive studies in the past have revealed that rational conformal field theory, three-dimensional topological quantum field theory, modular tensor category, and various other mathematical structures are different facets of the same underlying truth [9–18, 4–8]. In particular, the nontrivial dynamics of the conformal field theory, encoded in the three-point structure constants, can be explicitly expressed in terms of invariant data of modular tensor category, or equivalently as link invariants of the topological quantum field theory; crossing symmetry is solved by solutions to the pentagon identity.[1] This paper investigates whether some of this rich structure and insight survive when we venture beyond rationality. Since general conformal field theory has no relation to bulk topological quantum field theory, it is instructive to first reformulate holomorphic factorization in a purely two-dimensional framework. The role of line defects in the bulk topological quantum field theory is replaced by topological defect lines of the conformal field theory, and a local operator can be regarded as the composite of a holomorphic and an anti-holomorphic *defect* operator connected by a topological defect line.[2] For rational theory, this reformulation is a superficial one, obtained essentially by ignoring the third

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[1] This formulation is ignorant of the explicit form of the chiral algebra blocks, and in particular, the normalization of the blocks is a gauge ambiguity. There is no purely categorical way to decide which gauge gives the canonically normalized blocks (corresponding to normalizing the two-point function of chiral algebra primaries). Other means such as solving the null state decoupling equation [19] or the Wronskian method [20] are necessary to determine this piece of the conformal field theory data. An explicit illustration of this point will be given in Section 5.2.

[2] Topological defect lines in two-dimensional quantum field theory have been investigated in [21–23, 4–7, 24,8,25–42,68,43–46]. The modernized view of (generalized) symmetries as topological defects was developed in [33, 47–49].

dimension of the bulk, and giving a new name, Verlinde lines [50, 23, 33, 40], to the projected shadows of line operators in the bulk theory. Nonetheless, this new perspective permits the extrapolation of key ideas to theories that need not have a bulk correspondence. Mathematically, only the structure of fusion category [51, 52], and not modular tensor category, is required to describe the dynamics of topological defect lines. Less is more.

Loosely speaking, a local operator $\mathcal{O}$ on the Euclidean plane $\bar{z} = z^*$ is holomorphically-defect-factorized if

$$\odot\ \mathcal{O}(z,\bar{z}) \ \sim\ \mathcal{D}(z) \overset{\mathcal{L}}{\bullet\!\!\longrightarrow\!\!\circ} \overline{\mathcal{D}}(\bar{z}) \ , \tag{1.1}$$

where $\mathcal{L}$ is a topological defect line, and $\mathcal{D}$ and $\overline{\mathcal{D}}$ are holomorphic and anti-holomorphic defect operators. These objects are introduced in Section 2, and a precise definition of factorization is given in Definition 1. To avoid confusion with the stronger sense of holomorphic factorization (of the full theory), the factorization described above will be referred to as "holomorphic-defect-factorization" throughout this paper.

Holomorphic-defect-factorization obscures the meaning of spacetime signature. Starting from a Euclidean correlator, Lorentzian dynamics are obtained by continuing the complex coordinates z, $\bar{z}$ of local operators independently to real z and $\bar{z}$ [53–57]. However, for a holomorphically-defect-factorized local operator, a new interpretation is available: The correlator stays in the Euclidean regime, but becomes one involving defect operators and topological defect lines. This dual perspective suggests that aspects of Lorentzian dynamics are dictated by fundamental properties of topological defect lines. In particular, for a four-point function involving holomorphically-defect-factorized local operators, the conformal Regge limit [58, 59] at infinite boost is completely fixed by the action of the topological defect line on local operators. For rational theories, this connection was already explored from a bulk perspective by [60] in the context of out-of-time-ordered correlators and chaos. We show that such a connection can be reformulated in a purely two-dimensional way, and thereby reaches beyond rationality. Interestingly, in higher dimensional conformal field theory, light-ray operators [61] dominate the Regge limit of four-point functions, and explain the analyticity in spin of the Lorentzian inversion formula [62]. The central role played by line operators in the conformal Regge limit appears to be a common theme.

The connection between topological defect lines and Lorentzian dynamics is bidirectional. The Regge limit of correlators allow the discovery of topological defect lines given correlators of local operators. Traditionally, a topological defect line $\mathcal{L}$ is characterized by a topological map $\widehat{\mathcal{L}}$ on the Hilbert space $\mathcal{H}$ of local operators, subject to stringent consistency conditions, including the condition that the modular S transform of the twisted partition function $\mathrm{Tr}_{\mathcal{H}}\, \widehat{\mathcal{L}}\, q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}$ gives a sensible partition function for the defect Hilbert space [23]. A close analogy is the characterization of a consistent conformal boundary condition as a (closed string) state satisfying the Cardy condition [63–65]. At the level of principle, it would be desirable to have a direct formula for $\widehat{\mathcal{L}}$ in terms of correlators of local operators. As we will explain, assuming that a local operator is holomorphically-defect-factorized through $\mathcal{L}$, the Regge limit provides such a formula. Conversely, the Regge limit serves as a nontrivial test of whether a local operator is holomorphically-defect-factorized. We call this the Holomorphic-Defect-Factorization Criterion (Definition 3).

The Holomorphic-Defect-Factorization Criterion is put to test in the $c = 1$ free boson theory, on both the toroidal branch and the orbifold branch. On the toroidal branch, all local operators are holomorphically-defect-factorized through U(1) symmetry defect lines, regardless of rationality. On the orbifold [66, 67] branch, although the cosine operators are always factorized, for the twist fields we find a dichotomy between rational and irrational points. At rational points, the twist field correlator satisfies the Holomorphic-Defect-Factorization Criterion, and we obtain a uniform formula describing the map $\widehat{\mathcal{L}}$ for the topological defect line $\mathcal{L}$ through which the twist field factorizes; in particular, at $r^2 = u/v$ with u, v coprime, the planar loop expectation value is $\langle \mathcal{L} \rangle_{\mathbb{R}^2} = \sqrt{uv}$. At special rational points, it can be explicitly checked that our formula agrees with the Verlinde formula [50]. At irrational points, the Holomorphic-Defect-Factorization Criterion fails. More precisely, the Hilbert space on the cylinder quantized with twisted periodic boundary conditions by the hypothetical $\mathcal{L}$ exhibits a continuous spectrum.[3] Such a topological defect line cannot be described by a simple object in a fusion category; to describe it, one would need to relax the condition of semi-simplicity. It is unclear whether there is merit in regarding $\mathcal{L}$ as a physically meaningful defect. As a piece of evidence for its existential consistency, the (vacuum-normalized) defect partition function (6.37) coincides with that obtained in the limit of *any* sequence of rational r^2 converging to *any* irrational number.

This paper is organized as follows. Section 2 introduces the topological defect lines, explains the meaning of holomorphic-defect-factorization, and expresses the three-point function of local operators in terms of defect data. Section 3 studies correlators of holomorphically-defect-factorized local operators, and connects the conformal Regge limit to fundamental properties of topological defect lines. In particular, it is explained how the Regge limit provides a way to discover topological defect lines. Section 4 explores further aspects of Lorentzian dynamics, including connection to chaos. Section 5 examines holomorphic-defect-factorization in rational theories, first from a purely two-dimensional perspective, and then reviews the three-dimensional bulk perspective. Section 6 tests holomorphic-defect-factorization beyond rationality, by studying the $c = 1$ free boson theory on both the toroidal branch and the orbifold branch. Section 7 ends with a summary and further comments. Appendix A proves that the crossing symmetry of holomorphic defect operators implies the

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[3] Topological defect lines exhibiting continuous spectra were previously encountered in [28].

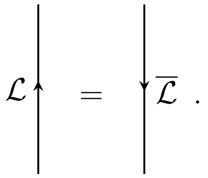
crossing symmetry of holomorphically-defect-factorized local operators. Appendix B proves a unitarity bound on the conformal Regge limit at infinite boost. Appendix C collects formulae and computations relevant for the study of the free boson orbifold theory in Section 6.2.

2 Holomorphic-defect-factorization of local operators

2.1 Topological defect lines

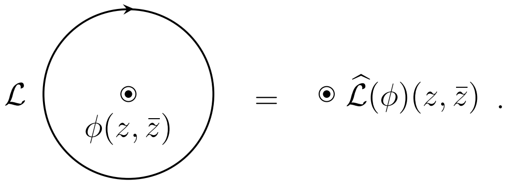
Let us first review basic properties of topological defect lines (TDLs), which encompass and generalize symmetry defect lines. The exposition here largely follows [68]; for other relevant references see Footnote 2. TDLs can reverse orientation, act on local operators by circling and shrinking, end on defect operators, join in junctions, undergo isotopic transformations without changing the correlation functionals, and different configurations of TDLs are equivalent under the so-called F-moves. The direct sum of two TDLs gives another TDL, and correlation functionals are additive under direct sums. A *simple* TDL is one that cannot be further decomposed into the direct sum of multiple TDLs. Since we are interested in both rational and irrational theories, we do not limit ourselves to having finite sets of simple lines. Nonetheless, we still assume that every TDL can be uniquely expressed as a possibly infinite sum of simple lines.

A TDL $\mathcal{L}$ has an orientation reversal $\overline{\mathcal{L}}$, meaning the equivalence of



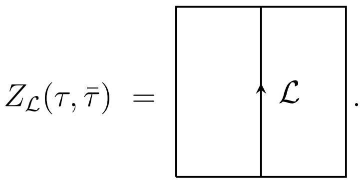
(2.1)

It acts on a local operator by circling and shrinking,



(2.2)

And it is associated with a defect Hilbert space obtained by quantizing on the cylinder with twisted (by the TDL) periodic boundary conditions. The defect partition function is



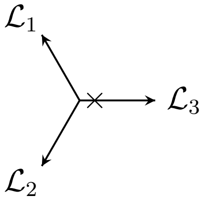
(2.3)

Via the state-operator map, states in the defect Hilbert space $\mathcal{H}_{\mathcal{L}}$ correspond to defect operators on which the TDL can end. Since the defect Hilbert space has a norm, every defect operator $\mathcal{D} \in \mathcal{H}_{\mathcal{L}}$ has a hermitian conjugate $\mathcal{D}^\dagger \in \mathcal{H}_{\overline{\mathcal{L}}}$ of the same weight,

$$(h_{\mathcal{D}},\, \bar{h}_{\mathcal{D}}) = (h_{\mathcal{D}^\dagger},\, \bar{h}_{\mathcal{D}^\dagger})\,, \tag{2.4}$$

and the two are related by charge conjugation.

A trivalent junction of TDLs is depicted as

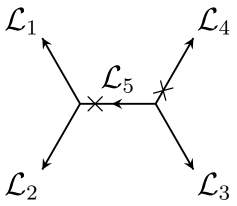


(2.5)

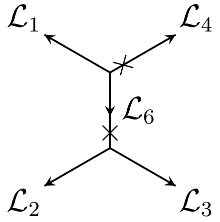
The marking × labels the ordering of edges at trivalent junctions, and can be permuted around by the *cyclic permutation map* $V_{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3} \to V_{\mathcal{L}_2,\mathcal{L}_3,\mathcal{L}_1}$. The junction vector space $V_{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}$ associated to a trivalent junction is defined as the subspace of topological weight $(0,0)$ states in the defect Hilbert space $\mathcal{H}_{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}$. The space of possible trivalent junctions is encoded in the *fusion ring* of the simple TDLs; the fusion ring coefficients correspond to the dimensions of the junction vector spaces.

There is a trivial TDL $\mathcal{I}$ that represents no TDL insertion. However, when it ends on another TDL $\mathcal{L}$ forming a trivalent junction, it introduces a map from the junction vector space $V_{\mathcal{L},\overline{\mathcal{L}},\mathcal{I}}$ (resp. other permuted orderings) to $\mathbb{C}$. Such a trivalent junction could be removed by evaluating the map on the identity junction vector $1_{\mathcal{L},\overline{\mathcal{L}},\mathcal{I}}$ (resp. other permuted orderings).

A configuration of TDLs is a (linear) correlation functional of junction vectors, and different configurations are equivalent under F-moves



$$= \sum_{\mathcal{L}_6}$$



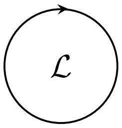
$$\circ\ (F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\overline{\mathcal{L}}_4})_{\mathcal{L}_5,\mathcal{L}_6}\,, \tag{2.6}$$

where the F-symbols are bilinear maps

$$(F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\overline{\mathcal{L}}_4})_{\mathcal{L}_5,\mathcal{L}_6} \ : \ V_{\mathcal{L}_1,\mathcal{L}_2,\overline{\mathcal{L}}_5} \otimes V_{\mathcal{L}_5,\mathcal{L}_3,\mathcal{L}_4} \to V_{\mathcal{L}_2,\mathcal{L}_3,\overline{\mathcal{L}}_6} \otimes V_{\mathcal{L}_1,\mathcal{L}_6,\mathcal{L}_4}\,. \tag{2.7}$$

The loop expectation value of a TDL $\mathcal{L}$ on the plane is[4]

$$\langle \mathcal{L} \rangle_{\mathbb{R}^2} =$$

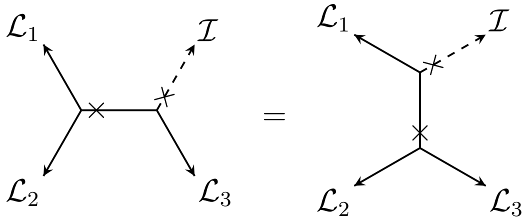


$$. \qquad (2.10)$$

It is related to an F-symbol by

$$(F_{\mathcal{L}}^{\mathcal{L},\overline{\mathcal{L}},\mathcal{L}})_{\mathcal{I},\mathcal{I}} \ : \ 1_{\mathcal{L},\overline{\mathcal{L}},\mathcal{I}} \otimes 1_{\mathcal{I},\mathcal{L},\overline{\mathcal{L}}} \ \mapsto \ \frac{1}{\langle \mathcal{L} \rangle_{\mathbb{R}^2}} \times (1_{\overline{\mathcal{L}},\mathcal{L},\mathcal{I}} \otimes 1_{\mathcal{L},\mathcal{I},\overline{\mathcal{L}}})\,. \qquad (2.11)$$

The aforementioned cyclic permutation map is related to an F-symbol the F-move



$$\circ \ (F_{\mathcal{I}}^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3})_{\overline{\mathcal{L}}_3,\overline{\mathcal{L}}_1}\,. \qquad (2.12)$$

For simplicity, the marking × will be ignored subsequently, which means that our formulae will be correct up to cyclic permutation maps.

This section assumes that the F-symbols and the planar loop expectation values are well-defined and finite. This is violated in generic irrational theories, as there exist TDLs whose fusion cannot be decomposed into a finite sum of simple TDLs. To incorporate such TDLs, the usual fusion categorical framework would have to be enlarged. The sum in the F-move (2.6) should become an integral, with the F-symbol being the integration measure. However, such a mathematical framework has not been developed. Alternatively, one can consider a sequence of theories/fusion categories that converge to the case of interest, so that formal expressions can be evaluated as limits.

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[4] The planar loop expectation value $\langle \mathcal{L} \rangle_{\mathbb{R}^2}$ is related to the quantum dimension $d_{\mathcal{L}}$ in the fusion category language by a factor of the Frobenius-Schur indicator $\chi_{\mathcal{L}}$

$$\langle \mathcal{L} \rangle_{\mathbb{R}^2} = \frac{d_{\mathcal{L}}}{\chi_{\mathcal{L}}}\,. \qquad (2.8)$$

The quantum dimension $d_{\mathcal{L}}$ is equal to the vacuum expectation value of $\mathcal{L}$ wrapping the non-contractible cycle on a cylinder, *i.e.*

$$d_{\mathcal{L}} = \langle \mathcal{L} \rangle_{\mathrm{S}^1 \times \mathbb{R}}\,. \qquad (2.9)$$

The two loop expectation values are related by at most a phase arising from the extrinsic curvature improvement term [68].

2.2 Holomorphic-Defect-Factorization Hypothesis

Definition 1 (Holomorphic-Defect-Factorization) *A local operator $\mathcal{O}$ on the Euclidean plane $\bar{z} = z^*$ with definite conformal weight $(h, \bar{h})$ is said to be holomorphically-defect-factorized if it can be obtained in the following coincidence limit:*

$$\odot\ \mathcal{O}(z, \bar{z})|_{\bar{z}=z^*} \ = \ \sqrt{\langle \mathcal{L} \rangle_{\mathbb{R}^2}} \times \lim_{\bar{z}' \to \bar{z} = z^*} \ \mathcal{D}(z) \ \bullet\!\!\xrightarrow{\ \mathcal{L}\ }\!\!\circ \ \overline{\mathcal{D}}(\bar{z}') \ , \tag{2.13}$$

where $\mathcal{L}$ is a simple topological defect line, $\mathcal{D}$ is a holomorphic defect operator of weight $(h, 0)$ in the defect Hilbert space $\mathcal{H}_{\mathcal{L}}$, and $\overline{\mathcal{D}}$ is an anti-holomorphic defect operator of weight $(0, \bar{h})$ in the dual defect Hilbert space $\mathcal{H}_{\overline{\mathcal{L}}}$.

Throughout this paper, we use solid dots to represent holomorphic defect operators, empty dots to represent anti-holomorphic defect operators, and solid-inside-empty dots to represent local operators. The limit in (2.13) is well-defined because there is no singularity. As we will see in Section 2.3 the overall factor is such that if $\mathcal{D}$ and $\overline{\mathcal{D}}$ are each properly normalized,

$$\langle\, \mathcal{D}(0) \ \bullet\!\!\xrightarrow{\ \mathcal{L}\ }\!\!\bullet \ \mathcal{D}^\dagger(1) \rangle \ = \ \langle\, \overline{\mathcal{D}}^\dagger(0) \ \circ\!\!\xrightarrow{\ \mathcal{L}\ }\!\!\circ \ \overline{\mathcal{D}}(1) \rangle \ = \ 1 \,, \tag{2.14}$$

then $\mathcal{O}$ is too,

$$\langle \mathcal{O}(0)\, \mathcal{O}^\dagger(1) \rangle \ = \ 1 \,. \tag{2.15}$$

We write

$$\mathcal{O} = \mathcal{D} \overset{\mathcal{L}}{—} \overline{\mathcal{D}} \tag{2.16}$$

for brevity.

When studying local operators in a conformal field theory, it is often natural to choose a real basis, in which the two-point function of every basis operator with itself is nonzero. However, holomorphically-defect-factorized local operators are generally complex. In fact, as we will see in Section 2.3, if a local operator is holomorphically-defect-factorized through an oriented line ($\mathcal{L} \neq \overline{\mathcal{L}}$), then its two-point function with itself vanishes, so it cannot be real.[5] In the concrete example of the free compact boson theory, the exponential operators, which are complex, are holomorphically-defect-factorized through U(1) symmetry defects. By contrast, the cosine and sine operators, which are real combinations of exponential operators, are themselves not holomorphically-defect-factorized by Definition 1.[6]

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[5] Throughout this paper, $\mathcal{L} = \mathcal{L}'$ means that they are in the same isomorphism class.

[6] One could define a relaxed notion of factorization by allowing finite sums of holomorphically-defect-factorization operators. We do not do so here.

Definition 2 (Holomorphic-Defect-Factorization Prerequisite) *A local operator $\mathcal{O}$ of weight $(h, \bar{h})$ is said to satisfy the Holomorphic-Defect-Factorization Prerequisite if there exists a simple topological defect line $\mathcal{L}$ such that the defect Hilbert space $\mathcal{H}_{\mathcal{L}}$ contains a defect operator of weight $(h, 0)$, and the dual defect Hilbert space $\mathcal{H}_{\overline{\mathcal{L}}}$ contains one of weight $(0, \bar{h})$.*

2.3 Operator product expansion

If $\mathcal{O}_1$ and $\mathcal{O}_2$ are both holomorphically-defect-factorized, $\mathcal{O}_i = \mathcal{D}_i \overset{\mathcal{L}_i}{—} \overline{\mathcal{D}}_i$, then all operators in the $\mathcal{O}_1 \times \mathcal{O}_2$ operator product expansion (OPE) are holomorphically-defect-factorized. This follows from performing an F-move on $\mathcal{L}_1$ and $\mathcal{L}_2$ and expressing the $\mathcal{O}_1 \times \mathcal{O}_2$ OPE as a sum of products of $\mathcal{D}_1 \times \mathcal{D}_2$ and $\overline{\mathcal{D}}_1 \times \overline{\mathcal{D}}_2$ OPEs,

$$
\begin{aligned}
\mathcal{O}_1(z_1, \bar{z}_1)\, \mathcal{O}_2(z_2, \bar{z}_2) &= \sqrt{\textstyle\prod_{i=1}^{2} \langle \mathcal{L}_i \rangle_{\mathbb{R}^2}}\; \begin{array}{c} \mathcal{D}_1(z_1) \xrightarrow{\mathcal{L}_1} \overline{\mathcal{D}}_1(\bar{z}_1) \\[1ex] \mathcal{D}_2(z_2) \xrightarrow{\mathcal{L}_2} \overline{\mathcal{D}}_2(\bar{z}_2) \end{array} \\
&= \sqrt{\textstyle\prod_{i=1}^{2} \langle \mathcal{L}_i \rangle_{\mathbb{R}^2}} \sum_{\mathcal{L}} \begin{tikzcd}[row sep=small, column sep=small]
\mathcal{D}_1(z_1) \arrow[dr] & & & \overline{\mathcal{D}}_1(\bar{z}_1) \\
& {} \arrow[r, "\mathcal{L}"] & {} \arrow[ur] \arrow[dr] & \\
\mathcal{D}_2(z_2) \arrow[ur] & & & \overline{\mathcal{D}}_2(\bar{z}_2)
\end{tikzcd} \circ (F^{\mathcal{L}_1, \overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2}_{\overline{\mathcal{L}}_2})_{\mathcal{I}, \overline{\mathcal{L}}}\; (1_{\mathcal{L}_1, \overline{\mathcal{L}}_1, \mathcal{I}}, 1_{\mathcal{I}, \overline{\mathcal{L}}_2, \mathcal{L}_2})\,,
\end{aligned}
\tag{2.17}
$$

Let us assume that $\mathcal{O}_2$ is the hermitian conjugate of $\mathcal{O}_1$, *i.e.* $\mathcal{O}_2 = \mathcal{O}_1^\dagger$, and take the vacuum expectation value. Holomorphy forces

$$
\mathcal{L} = \mathcal{I}\,, \quad \mathcal{L}_2 = \overline{\mathcal{L}}_1\,, \quad \mathcal{D}_2 = \mathcal{D}_1^\dagger\,, \quad \overline{\mathcal{D}}_2 = \overline{\mathcal{D}}_1^\dagger\,, \tag{2.18}
$$

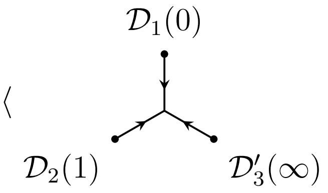
which gives

$$
\langle \mathcal{O}_1(0,0)\, \mathcal{O}_1^\dagger(1,1) \rangle \;=\; \langle\, \mathcal{D}_1(0) \xrightarrow{\mathcal{L}_1} \mathcal{D}_1^\dagger(1) \rangle \times \langle\, \overline{\mathcal{D}}_1^\dagger(0) \xrightarrow{\mathcal{L}_1} \overline{\mathcal{D}}_1(1) \rangle\,. \tag{2.19}
$$

This shows that the OPE formula (2.17) has the correct normalization factor.

Next, define the three-point defect correlation functionals[7]

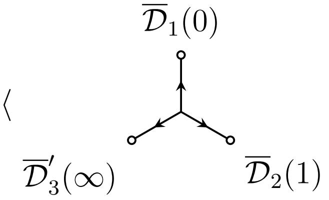
$$C_{\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3} \;=\; \langle$$



$$\rangle \;:\; V_{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3} \to \mathbb{C}\,,$$

(2.20)

$$C_{\overline{\mathcal{D}}_1,\overline{\mathcal{D}}_3,\overline{\mathcal{D}}_2} \;=\; \langle$$



$$\rangle \;:\; V_{\overline{\mathcal{L}}_1,\overline{\mathcal{L}}_3,\overline{\mathcal{L}}_2} \to \mathbb{C}\,.$$

A central formula is a relation between them and the three-point coefficient $C_{\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3}$ of local operators,

$$C_{\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3} \;=\; \sqrt{\textstyle\prod_{i=1}^{3}\langle\mathcal{L}_i\rangle_{\mathbb{R}^2}} \times (C_{\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3} \otimes C_{\overline{\mathcal{D}}_1,\overline{\mathcal{D}}_3,\overline{\mathcal{D}}_2}) \circ \Theta_{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}\,, \tag{2.21}$$

where the bi-vector $\Theta_{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}$ has multiple equivalent expressions

$$\begin{aligned}
\Theta_{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3} \;&=\; \frac{1}{\langle\mathcal{L}_3\rangle_{\mathbb{R}^2}} \times (F^{\mathcal{L}_1,\overline{\mathcal{L}}_1,\overline{\mathcal{L}}_2}_{\overline{\mathcal{L}}_2})_{\mathcal{I},\mathcal{L}_3}\,(1_{\mathcal{L}_1,\overline{\mathcal{L}}_1,\mathcal{I}},\, 1_{\mathcal{I},\overline{\mathcal{L}}_2,\mathcal{L}_2}) \\
&=\; \frac{1}{\langle\mathcal{L}_1\rangle_{\mathbb{R}^2}} \times (F^{\mathcal{L}_2,\overline{\mathcal{L}}_2,\overline{\mathcal{L}}_3}_{\overline{\mathcal{L}}_3})_{\mathcal{I},\mathcal{L}_1}\,(1_{\mathcal{L}_2,\overline{\mathcal{L}}_2,\mathcal{I}},\, 1_{\mathcal{I},\overline{\mathcal{L}}_3,\mathcal{L}_3}) \\
&=\; \frac{1}{\langle\mathcal{L}_2\rangle_{\mathbb{R}^2}} \times (F^{\mathcal{L}_3,\overline{\mathcal{L}}_3,\overline{\mathcal{L}}_1}_{\overline{\mathcal{L}}_1})_{\mathcal{I},\mathcal{L}_2}\,(1_{\mathcal{L}_3,\overline{\mathcal{L}}_3,\mathcal{I}},\, 1_{\mathcal{I},\overline{\mathcal{L}}_1,\mathcal{L}_1})\,.
\end{aligned} \tag{2.22}$$

The formula (2.21) can be derived by starting with

$$\langle \mathcal{O}_1(z_1,\bar z_1)\,\mathcal{O}_2(z_2,\bar z_2)\,\mathcal{O}_3(z_3,\bar z_3)\rangle \;=\; \sqrt{\textstyle\prod_{i=1}^{3}\langle\mathcal{L}_i\rangle_{\mathbb{R}^2}} \;\times\; \left\langle\; \begin{array}{c} \mathcal{D}_1(z_1) \;\bullet\!\xrightarrow{\mathcal{L}_1}\!\circ\; \overline{\mathcal{D}}_1(\bar z_1) \\ \mathcal{D}_2(z_2) \;\bullet\!\xrightarrow{\mathcal{L}_2}\!\circ\; \overline{\mathcal{D}}_2(\bar z_2) \\ \mathcal{D}_3(z_3) \;\bullet\!\xrightarrow{\mathcal{L}_3}\!\circ\; \overline{\mathcal{D}}_3(\bar z_3) \end{array} \;\right\rangle\,, \tag{2.23}$$

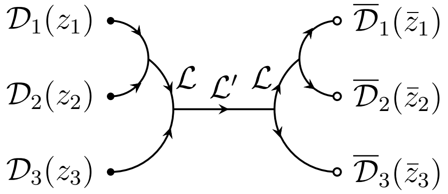
performing an OPE via (2.17), and then performing an F-move on a trivial line connecting

---

[7] The notation $'$ means moving an operators to the other patch of the sphere while taking into account the conformal factors.

$\mathcal{L}$ and $\mathcal{L}_3$ to arrive at

$$\sqrt{\prod_{i=1}^{3} \langle \mathcal{L}_i \rangle_{\mathbb{R}^2}} \times \sum_{\mathcal{L},\mathcal{L}'} \langle$$



$$\rangle \tag{2.24}$$

$$\circ \, (F^{\mathcal{L},\overline{\mathcal{L}},\overline{\mathcal{L}}_3}_{\overline{\mathcal{L}}_3})_{\mathcal{I},\overline{\mathcal{L}}'} \, (1_{\mathcal{L},\overline{\mathcal{L}},\mathcal{I}}, 1_{\mathcal{I},\overline{\mathcal{L}}_3,\mathcal{L}_3}) \otimes (F^{\mathcal{L}_1,\overline{\mathcal{L}}_1,\overline{\mathcal{L}}_2}_{\overline{\mathcal{L}}_2})_{\mathcal{I},\overline{\mathcal{L}}} \, (1_{\mathcal{L}_1,\overline{\mathcal{L}}_1,\mathcal{I}}, 1_{\mathcal{I},\overline{\mathcal{L}}_2,\mathcal{L}_2}) \, .$$

If we take the vacuum expectation value, then holomorphy forces $\mathcal{L}' = \mathcal{I}$ and $\mathcal{L} = \overline{\mathcal{L}}_3$, and gives (2.21) with $\Theta_{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}$ written in its first expression in (2.22). Analogous derivations by first taking the $\mathcal{O}_2 \times \mathcal{O}_3$ or the $\mathcal{O}_1 \times \mathcal{O}_3$ OPE arrive at the other two expressions for $\Theta_{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}$. Note that the equivalence of the three expressions for $\Theta_{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}$ is a purely fusion categorical property.

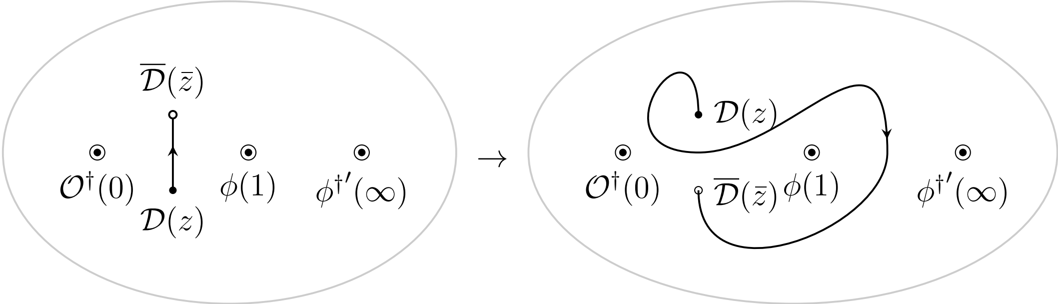
In Appendix A, we show that given (2.21), the crossing symmetry of holomorphically-defect-factorized local operators follows from the crossing symmetry of holomorphic defect operators.

3 Holomorphically-defect-factorized local operators in the Regge limit

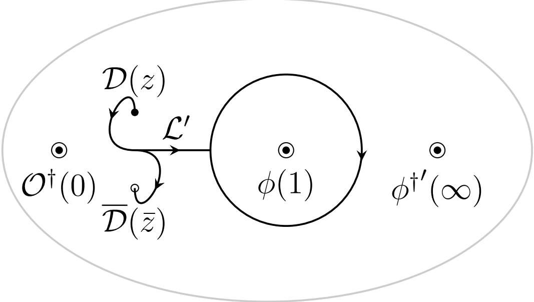
3.1 Action on local operators in the Regge limit

Suppose a local operator is holomorphically-defect-factorized, $\mathcal{O} = \mathcal{D} \overset{\mathcal{L}}{—} \overline{\mathcal{D}}$, then to study the action (2.2) of $\mathcal{L}$ on a particular local operator ϕ, we can take the four-point function $\langle \mathcal{O}^\dagger(0)\mathcal{O}(z,\bar{z})\phi(1)\phi^{\dagger\prime}(\infty)\rangle$ and send z around 1 while keeping $\bar{z}$ fixed. This wraps $\mathcal{L}$ around $\phi(1)$. By then sending z, $\bar{z} \to 0$ with $z/\bar{z}$ fixed and removing the leading singularity, we obtain $\langle \widehat{\mathcal{L}}(\phi)\, \phi' \rangle$ divided by $\langle \mathcal{L} \rangle_{\mathbb{R}^2}$ up to a sign. This limit is none other than the **conformal**

Regge limit [58, 59] of the four-point function. The following is a visual:

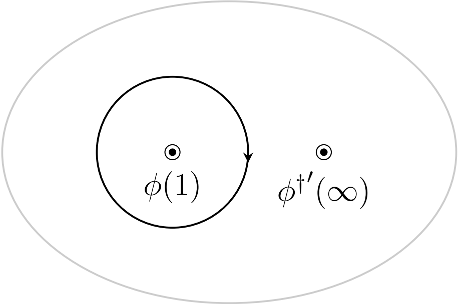


$$= \sum_{\mathcal{L}'}$$



$$\circ \; (F_{\mathcal{L}}^{\mathcal{L},\overline{\mathcal{L}},\mathcal{L}})_{\mathcal{I},\mathcal{L}'} \; (1_{\mathcal{L},\overline{\mathcal{L}},\mathcal{I}}, 1_{\mathcal{I},\mathcal{L},\overline{\mathcal{L}}}) \quad (3.1)$$

$$\sim \frac{1}{\langle \mathcal{L} \rangle_{\mathbb{R}^2}} \frac{1}{z^{2h} \bar{z}^{2\bar{h}}} \times$$



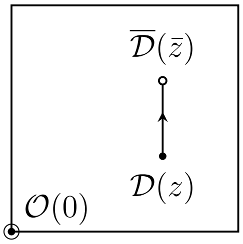
.

In the last line, we kept the leading term in the z, $\bar{z} \to 0$ limit corresponding to the domination of $\mathcal{L}' = \mathcal{I}$, and used (2.11) to rewrite the F-symbol as an inverse planar loop expectation value.

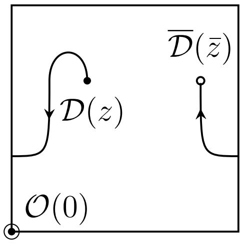
Normally, continuing z and $\bar{z}$ independently takes a correlator off the Euclidean plane. However, if one of the operators is holomorphically-defect-factorized, then the correlator has a new interpretation as a Euclidean correlator involving not only local operators, but also defect operators joined by topological defect lines.

3.2 Holomorphic-Defect-Factorization Criterion in the torus Regge limit

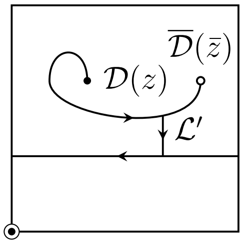
To study the action of $\mathcal{L}$ on all local operators at once, one can consider the torus two-point function $\langle \mathcal{O}(z, \bar{z}) \mathcal{O}(0) \rangle_{T^2(\tau,\bar{\tau})}$. By sending $z \to z + 1$ (spatial translation) with $\bar{z}$ fixed and then z, $\bar{z} \to 0$ with $z/\bar{z}$ fixed while removing the leading singularity, one obtains the torus partition function $Z^{\mathcal{L}}(\tau, \bar{\tau})$ with $\mathcal{L}$ wrapped along the spatial direction. The following is a visual:



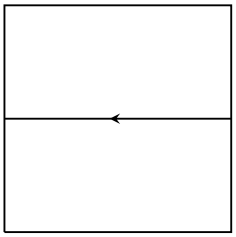
$$\to$$



$$=$$



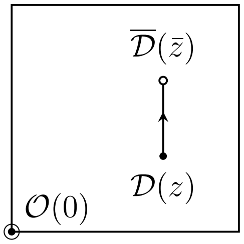
$$\sim \frac{1}{\langle \mathcal{L} \rangle_{\mathbb{R}^2}} \frac{e^{2i\pi h}}{z^{2h} \bar{z}^{2\bar{h}}} \times \tag{3.2}$$



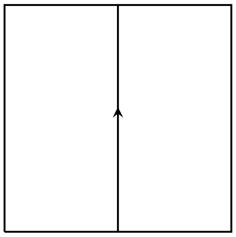
.

In the last step, we kept the dominant $\mathcal{L}' = \mathcal{I}$ contribution, and performed a 2π angle rotation of $\mathcal{L}$ at $\mathcal{D}$ to reproduce to the original configuration, thereby creating the extra phase of $e^{2i\pi h}$. With the $e^{2i\pi h}$ phase stripped off, we call this the **spatial torus Regge limit**.

The modular S transform of $Z^{\mathcal{L}}(\tau, \bar{\tau})$ gives the defect partition function $Z_{\mathcal{L}}(\tau, \bar{\tau})$, *i.e.* the torus partition function with $\mathcal{L}$ wrapped along the temporal direction. The latter could be obtained directly from $\langle \mathcal{O}(z, \bar{z}) \mathcal{O}(0) \rangle_{T^2(\tau, \bar{\tau})}$ by sending $z \to z - \tau$ (temporal translation) with $\bar{z}$ fixed and then z, $\bar{z} \to 0$ with $z/\bar{z}$ fixed while removing the leading singularity. The following is a visual:

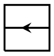
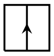
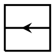


$$\to \quad \cdots \quad \sim \frac{1}{\langle \mathcal{L} \rangle_{\mathbb{R}^2}} \frac{e^{2i\pi h}}{z^{2h} \bar{z}^{2\bar{h}}} \times \tag{3.3}$$



.

With the $e^{2i\pi h}$ stripped off, we call this the **temporal torus Regge limit**.

We call  and  *Cardy-normalized* torus partition functions, since the latter of the two is expected to have a q, $\bar{q}$ expansion with positive integer coefficients. And we call the result of divided by $\langle \mathcal{L} \rangle$ *vacuum-normalized* torus partition functions, since  $/\langle \mathcal{L} \rangle$ has unit coefficient in its leading term in the q, $\bar{q}$ expansion.

Definition 3 (Holomorphic-Defect-Factorization Criterion) *For a given local operator $\mathcal{O}$, if the torus two-point function $\langle \mathcal{O}(z,\bar{z})\, \mathcal{O}(0)\rangle_{T^2(\tau,\bar{\tau})}$ in the temporal torus Regge limit has a q, $\bar{q}$ expansion with positive integer coefficients up to some overall real number, then $\mathcal{O}$ is said to satisfy the Holomorphic-Defect-Factorization Criterion.*

It may be possible to extend the notion of holomorphic-defect-factorization by relaxing the integer requirement or by allowing the q, $\bar{q}$ expansion to be a continuous integral over the exponents. This situation will arise in the $S^1/\mathbb{Z}_2$ free boson orbifold theory at irrational points.

4 Lorentzian dynamics

As discussed in Section 3, the conformal Regge limit [58, 59] of the four-point function of a pair of holomorphically-defect-factorized local operators $\mathcal{O} = \mathcal{D} \overset{\mathcal{L}}{—} \overline{\mathcal{D}}$ computes the matrix element of the map $\widehat{\mathcal{L}}$ on the Hilbert space of local operators. Traditionally, the conformal Regge limit is interpreted as a limit of Lorentzian correlators, since analytically continuing z around 1 while fixing $\bar{z}$ moves the local operator off the Euclidean plane onto the Lorentzian sheet. In holographic theories, the conformal Regge limit corresponds to the Regge limit of the bulk S-matrix — the high energy limit with a fixed impact parameter. There is also a close connection to chaos [69–72], as the conformal Regge limit is equivalent to the late time limit of the out-of-time-ordered-correlator (OTOC) at finite temperature [73].

To be concrete, let us consider the Euclidean four-point function of a pair of hermitian conjugate operators W, $W^\dagger$ with another pair of hermitian conjugate operators V, $V^\dagger$ on the complex plane

$$G(z,\bar{z}) = \frac{\langle W^\dagger(z_1,\bar{z}_1) W(z_2,\bar{z}_2) V^\dagger(z_3,\bar{z}_3) V(z_4,\bar{z}_4)\rangle}{\langle W^\dagger(z_1,\bar{z}_1) W(z_2,\bar{z}_2)\rangle \langle V^\dagger(z_3,\bar{z}_3) V(z_4,\bar{z}_4)\rangle}\,, \tag{4.1}$$

where the cross ratios are

$$z = \frac{z_{12} z_{34}}{z_{13} z_{24}}\,, \quad \bar{z} = \frac{\bar{z}_{12} \bar{z}_{34}}{\bar{z}_{13} \bar{z}_{24}}\,. \tag{4.2}$$

By conformal symmetry, the positions of the operators can be fixed to

$$z_1 = -\rho\,, \quad z_2 = \rho\,, \quad z_3 = 1\,, \quad z_4 = -1\,. \tag{4.3}$$

Then the cross ratios are related to the global variables ρ, $\bar{\rho}$ by [74]

$$\rho = \frac{z}{(1+\sqrt{1-z})^2}\,, \quad \bar{\rho} = \frac{\bar{z}}{(1+\sqrt{1-\bar{z}})^2}\,. \tag{4.4}$$

Under the analytic continuation sending z around 1 while fixing $\bar{z}$, the cross ratios become independent variables; on the Lorentzian sheet, they are both real. In the conformal Regge limit z, $\bar{z} \to 0$ with $z/\bar{z}$ fixed, ρ and $\bar{\rho}$ scale as

$$\rho = \frac{4}{z} + \mathcal{O}(z^0)\,, \quad \bar{\rho} = \frac{\bar{z}}{4} + \mathcal{O}(\bar{z}^2)\,. \tag{4.5}$$

The analytic continuation and the conformal Regge limit could be equivalently described in the ρ-coordinate. One first write ρ and $\bar{\rho}$ as

$$\rho = re^{i\theta}\,, \quad \rho = re^{-i\theta}\,. \tag{4.6}$$

In Euclidean signature, the distance from the origin r and the angle θ are real. One then analytic continues the angle θ as $\theta = -iv - \epsilon$, and arrives at the Rindler coordinates

$$\rho = re^{v-i\epsilon}\,, \quad \rho = re^{-v+i\epsilon\theta}\,, \tag{4.7}$$

where the v is the boost parameter of the W, $W^\dagger$ operators relative to the V, $V^\dagger$ operators.

The conformal Regge limit [58,59] corresponds to the large boost limit where the pair of W, $W^\dagger$ operators become time-like separated from the pair $V^\dagger$, V, respectively, and W and $W^\dagger$ approach the light-cone of each other. Under the holographic duality, this limit can be interpreted as the high energy scattering of particles created by the operators V and W with a fixed finite impact parameter.

4.1 Probe of transparency/opacity

The four-point function in the conformal Regge limit is expected to behave as [59][8]

$$G(z,\bar{z})^{\circlearrowleft} \sim 1 - \#\left(\frac{\rho}{\bar{\rho}}\right)^{\frac{1}{2}(j_0-1)}\,, \tag{4.9}$$

where $G(z,\bar{z})^{\circlearrowleft}$ denotes the four-point function after the continuation of z around 1, and j_0 is the Regge intercept, *i.e.* the analytic continuation of the leading Regge trajectory $j(\Delta)$ to $\Delta = 1$. In unitary theories, the Regge intercept is bounded by [62]

$$j_0 \leq 1\,. \tag{4.10}$$

---

[8] In the conformal Regge limit, our variables ρ and $\bar{\rho}$ are related to the variables σ and ρ in (55) and (56) of [59] by

$$16\frac{\bar{\rho}}{\rho} = \sigma^2\,, \quad \rho\bar{\rho} = e^{2\rho} \tag{4.8}$$

where the variables σ and ρ on the RHSs are the ones defined in [59].

The Regge behavior of the Lorentzian four-point function can be separated into two distinct classes, transparent $j_0 < 1$ and opaque $j_0 = 1$ [75]. When $j_0 < 1$, the Lorentzian four-point function factorizes into a product of two-point functions in the conformal Regge limit. Holographically, the particle created by the operator V and that by W pass through each other without interacting in the high energy fixed impact parameter limit. By contrast, when $j_0 = 1$, the Lorentzian four-point function does not factorize, and the bulk scattering is nontrivial.

If the operator W is holomorphically-defect-factorized through a topological defect line $\mathcal{L}$, then according to the formula (3.1) for the conformal Regge limit at infinite boost, the four-point function approaches

$$\lim_{z \to 0,\, z/\bar{z}\,\text{fixed}} G(z,\bar{z})^{\circlearrowleft} \;=\; r[W,V] \;\equiv\; \frac{1}{\langle \mathcal{L} \rangle_{\mathbb{R}^2}} \frac{\big\langle V^\dagger(0)\, \widehat{\mathcal{L}}(V)(1) \big\rangle}{\big\langle V^\dagger(0)\, V(1) \big\rangle}\,. \tag{4.11}$$

The bulk scattering is transparent if $r[W,V] = 1$, and opaque otherwise. In Appendix B, we prove that certain special conditions imply the unitarity bound $|r[W,V]| \leq 1$. We conjecture that $|r[W,V]| \leq 1$ is always true.

Note that the spatial torus Regge limit (3.2) of the torus two-point function of W conveniently packages the infinitely-boosted conformal Regge limit for all possible V.

4.2 Aspects of chaos

The relation between the conformal Regge limit and the chaos limit of the Lorentzian four-point function at finite temperature $T = \beta^{-1}$ could be seen by conformally mapping the complex plane to the cylinder $\mathrm{S}^1 \times \mathbb{R}$ by $z = e^{\frac{2i\pi}{\beta}(\tau + ix)}$, where the the S^1 is the thermal circle with periodicity β [70, 73]. The Euclidean time τ could be further analytically continued to Lorentzian time t by

$$\begin{aligned} &z_1 = e^{\frac{2\pi}{\beta}(t+i\epsilon_1)}\,, \quad z_2 = e^{\frac{2\pi}{\beta}(t+i\epsilon_2)}\,, \quad z_3 = e^{\frac{2\pi}{\beta}(x+i\epsilon_3)}\,, \quad z_4 = e^{\frac{2\pi}{\beta}(x+i\epsilon_4)}\,, \\ &\bar{z}_1 = e^{-\frac{2\pi}{\beta}(t+i\epsilon_1)}\,, \quad \bar{z}_2 = e^{-\frac{2\pi}{\beta}(t+i\epsilon_2)}\,, \quad \bar{z}_3 = e^{\frac{2\pi}{\beta}(x-i\epsilon_3)}\,, \quad \bar{z}_4 = e^{\frac{2\pi}{\beta}(x-i\epsilon_4)}\,. \end{aligned} \tag{4.12}$$

The ordering of the operators in the correlator is specified by choosing $\epsilon_1 < \epsilon_3 < \epsilon_2 < \epsilon_4$. At $t = 0$, the operators are space-like separated and $\bar{z}_i = z_i^*$. When t increases from $t = 0$ to $t > |x|$, the cross ratio z moves across the branch cut at $[1, \infty)$ onto the second sheet, while $\bar{z}$ remains on the first sheet. In the late time limit $t \to \infty$, both z and $\bar{z}$ approach 0 with their ratio $z/\bar{z} = e^{\frac{4\pi}{\beta}x} + \mathcal{O}(e^{-\frac{2\pi}{\beta}t})$ fixed, which is precisely the conformal Regge limit.

The out-of-time-ordered correlator (OTOC) captures the perturbation caused by the operators V on the later measurements W. The behavior of the four-point function in the conformal Regge limit (4.9) translates to the exponential time dependence of the OTOC at late time

$$G(z,\bar{z})^{\circlearrowleft} \sim 1 - \# e^{\frac{2\pi}{\beta}\lambda t}\,. \tag{4.13}$$

The exponent λ is related to the Regge intercept j_0 by $\lambda = j_0 - 1$, and bounded according to (4.10) by $\lambda \leq 0$. When $\lambda < 0$, the OTOC approaches the product of two-point functions signifying that the effect of the operators V's on the the measurements W's exponentially decays at late time. When $\lambda = 0$, the effect of the operator V could have finite imprint on the measurement W at infinite time.

In a chaotic system, the effect of the operator V on the measurement W could grow exponentially during some intermediate time scale. At large central charge c and the time scale $t \sim \beta \log c$, the OTOC is expected to behave as [70, 76–80]

$$G(z,\bar{z})^{\circlearrowleft} \sim 1 - \frac{\#}{c} e^{\frac{2\pi}{\beta}\lambda_L t}\,. \tag{4.14}$$

The chaos exponent λ_L could take positive values and bounded in unitary theories by [72]

$$\lambda_L \leq 1\,. \tag{4.15}$$

Probing the chaotic behavior (4.14) of the OTOC requires taking the limit $z \to 0$ while fixing $\bar{z}/z$ and $c \times z$. Such a limit could be similarly studied by applying the manipulations in (3.1) to large c theories. One would need to include subleading terms that involve lasso diagrams [68].

5 Rational conformal field theory

5.1 Holomorphic-defect-factorization and Lorentzian dynamics

The Holomorphic-Defect-Factorization Prerequisite (Definition 2) is the existence of holomorphic and anti-holomorphic defect operators of suitable weights in some defect Hilbert spaces, so that holomorphic-defect-factorization is at all possible.

The local operators transform as bi-modules of the holomorphic and anti-holomorphic chiral algebras. The highest-weight operators in the bi-modules are labeled by $\mathcal{O}_{i,j}$, where the indices i and j label the irreducible modules of the holomorphic and anti-holomorphic chiral algebras. Modular invariance further constrains the set of $\mathcal{O}_{i,j}$ that appear in the theory, and the Holomorphic-Defect-Factorization Prerequisite is satisfied by the existence of the Verlinde lines [50, 23, 33, 40].

In a diagonal modular invariant rational conformal field theory, the partition function of local operators is

$$Z(\tau,\bar\tau)=\sum_i \chi_i(\tau)\bar\chi_i(\bar\tau)\,. \tag{5.1}$$

The Verlinde line $\mathcal{L}_k$ acts the local operator $\mathcal{O}_{i,i}$ by

$$\widehat{\mathcal{L}}_k(\mathcal{O}_{i,i})=\frac{S_{ki}}{S_{0i}}\,\mathcal{O}_{i,i}\,, \tag{5.2}$$

where S_{ki} is the modular S matrix, and $i=0$ denotes the vacuum module. The partition function twisted by the Verlinde line $\mathcal{L}_k$ is

$$Z^{\mathcal{L}_k}(\tau,\bar\tau)=\mathrm{Tr}_{\mathcal{H}}\,\widehat{\mathcal{L}}\,q^{L_0-\frac{c}{24}}\bar q^{\bar L_0-\frac{\bar c}{24}}=\sum_i\frac{S_{ki}}{S_{0i}}\,\chi_i(\tau)\bar\chi_i(\bar\tau)\,. \tag{5.3}$$

The partition function for the defect Hilbert space $\mathcal{H}_{\mathcal{L}_k}$ is obtained by a modular S transform. The result is

$$Z_{\mathcal{L}_k}(\tau,\bar\tau)=\sum_{i,j}N_{ki}^{\;j}\,\chi_i(\tau)\bar\chi_j(\bar\tau)\,, \tag{5.4}$$

where the fusion coefficients $N_{ki}^{\;j}$ are non-negative integers given by the Verlinde formula [50],

$$N_{ki}^{\;j}=\sum_\ell\frac{S_{k\ell}S_{i\ell}S^*_{j\ell}}{S_{0\ell}}\,. \tag{5.5}$$

The Holomorphic-Defect-Factorization Prerequisite is satisfied because $N_{ki}^{\;0}=\delta_{ki}$ and $N_{k0}^{\;j}=\delta_k^j$. In other words, for any admissible highest-weight operator $\mathcal{O}_{i,i}$ with weight (h_i,h_i), the defect Hilbert space of the Verlinde line $\mathcal{L}_i$ contains one defect highest-weight operator of weight $(h_i,0)$ and another one of weight $(0,h_i)$.

When there exists a permutation automorphism ζ of the irreducible modules of the chiral algebra, satisfying

$$\zeta(0)=0\,,\quad S_{\zeta(i)\zeta(j)}=S_{ij}\,,\quad T_{\zeta(i)\zeta(j)}=T_{ij}\,, \tag{5.6}$$

there is a modular invariant partition function

$$Z(\tau,\bar\tau)=\sum_i\chi_i(\tau)\bar\chi_{\zeta(i)}(\bar\tau)\,. \tag{5.7}$$

The topological defects lines in such theories were classified by Petkova and Zuber [23]. The Verlinde line $\mathcal{L}_k$ acts on the local operator $\mathcal{O}_{i,\zeta(i)}$ by

$$\widehat{\mathcal{L}}_k(\mathcal{O}_{i,\zeta(i)})=\frac{S_{ki}}{S_{0i}}\,\mathcal{O}_{i,\zeta(i)}\,. \tag{5.8}$$

After similar manipulations as before, we find the partition function for the defect Hilbert space $\mathcal{H}_{\mathcal{L}_k}$,

$$Z_{\mathcal{L}_k}(\tau,\bar{\tau}) = \sum_{i,j} N_{ki}^{\zeta^{-1}(j)}\, \chi_i(\tau)\bar{\chi}_j(\bar{\tau})\,. \tag{5.9}$$

The Holomorphic-Defect-Factorization Prerequisite in this case follows from $N_{ki}^{0} = \delta_{ki}$ and $N_{k0}^{\zeta^{-1}(j)} = \delta_{\zeta(k)}^{j}$. In other words, for any admissible highest-weight operator $\mathcal{O}_{i,\zeta(i)}$ with weight $(h_i, h_{\zeta(i)})$, the defect Hilbert space of the Verlinde line $\mathcal{L}_i$ contains one defect highest-weight operator of weight $(h_i, 0)$ and another one of weight $(0, h_{\zeta(i)})$.

Diagonal or not, the defect Hilbert space $\mathcal{H}_{\mathcal{L}_k}$ projected onto the subspace of holomorphic operators (resp. anti-holomorphic operators) is an irreducible module of the holomorphic (resp. anti-holomorphic) chiral algebra, encapsulated in the equations

$$\lim_{\bar{q}\to 0} \bar{q}^{\,-\frac{c}{24}} Z_{\mathcal{L}_k}(\tau,\bar{\tau}) = \chi_k(\tau)\,, \quad \lim_{q\to 0} q^{-\frac{c}{24}} Z_{\mathcal{L}_k}(\tau,\bar{\tau}) = \bar{\chi}_{\zeta(k)}(\bar{\tau})\,. \tag{5.10}$$

The diagonal case is when the permutation map ζ is the identity map.

As proven by Moore and Seiberg [10], every rational theory has a maximally extended chiral algebra with respect to which the theory is either diagonal or permutation modular invariant. And since all operators in the same chiral algebra module can be factorized through the same topological defect line, the preceding discussion covers all possibilities.

Let us discuss the Lorentzian dynamics in rational conformal field theory. Using (5.2) and (5.8) for the action of $\mathcal{L}_k$ on local operators, the infinite boost limit (4.11) is given by the modular S matrix as

$$r[\mathcal{O}_{k,\zeta(k)}, \mathcal{O}_{i,\zeta(i)}] \;=\; \frac{S_{00}S_{ki}}{S_{0k}S_{0i}}\,. \tag{5.11}$$

The diagonal case (ζ being the trivial permutation) reproduces the result of [60] derived from the monodromy properties of the chiral algebra blocks, or equivalently from a bulk perspective (reviewed in Section 5.3) by use of the braiding of anyons. However, we emphasize that the derivation of our formula (4.11) only involves the F-symbols alone, and hence applies beyond rationality.

5.2 Example: Ising conformal field theory

The Ising conformal field theory has three local operators, the identity 1, the energy operator ε, and the spin operator σ. It has three topological defect lines, the trivial $\mathcal{I}$, the $\mathbb{Z}_2$ symmetry defect line η, and the non-invertible Kramers-Wannier duality line $\mathcal{N}$ [24,25,31]. The fusion rule is

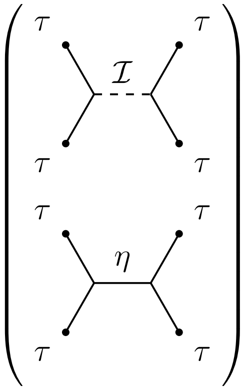
$$\eta^2 = \mathcal{I}\,, \quad \mathcal{N}^2 = \mathcal{I} + \eta\,, \quad \eta\mathcal{N} = \mathcal{N}\,. \tag{5.12}$$

The local operators are holomorphically-defect-factorized as follows:

$$\varepsilon = \psi \overset{\eta}{—} \bar{\psi}\,, \quad \sigma = \tau \overset{\mathcal{N}}{—} \bar{\tau}\,, \tag{5.13}$$

where ψ is a weight $(\frac{1}{2}, 0)$ free fermion, and τ is a weight $(\frac{1}{16}, 0)$ defect operator. Consider the vector of holomorphic-defect four-point functions

$$\mathbf{f}(z) \;=$$



$$= \begin{pmatrix} \mathcal{F}\left[\begin{smallmatrix}\frac{1}{16} & \frac{1}{16}\\ \frac{1}{16} & \frac{1}{16}\end{smallmatrix}\right]^{\frac{1}{2}}_{0}(z) \\[2em] C^2_{\tau,\tau,\psi} \times \mathcal{F}\left[\begin{smallmatrix}\frac{1}{16} & \frac{1}{16}\\ \frac{1}{16} & \frac{1}{16}\end{smallmatrix}\right]^{\frac{1}{2}}_{\frac{1}{2}}(z) \end{pmatrix}. \tag{5.14}$$

Under crossing, the (properly normalized) Virasoro blocks transform as

$$\begin{pmatrix} \mathcal{F}\left[\begin{smallmatrix}\frac{1}{16} & \frac{1}{16}\\ \frac{1}{16} & \frac{1}{16}\end{smallmatrix}\right]^{\frac{1}{2}}_{0}(1-z) \\[2em] \mathcal{F}\left[\begin{smallmatrix}\frac{1}{16} & \frac{1}{16}\\ \frac{1}{16} & \frac{1}{16}\end{smallmatrix}\right]^{\frac{1}{2}}_{\frac{1}{2}}(1-z) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \frac{1}{2} \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \mathcal{F}\left[\begin{smallmatrix}\frac{1}{16} & \frac{1}{16}\\ \frac{1}{16} & \frac{1}{16}\end{smallmatrix}\right]^{\frac{1}{2}}_{0}(z) \\[2em] \mathcal{F}\left[\begin{smallmatrix}\frac{1}{16} & \frac{1}{16}\\ \frac{1}{16} & \frac{1}{16}\end{smallmatrix}\right]^{\frac{1}{2}}_{\frac{1}{2}}(z) \end{pmatrix}. \tag{5.15}$$

A gauge choice means that canonical junction vectors have been chosen, so all correlation functionals can be turned into correlation functions by the implicit insertion of canonical junction vectors. Henceforth defect thee-point correlation functionals become simply defect three-point coefficients. Suppose we adopt the gauge choice of [68] where the nontrivial F-symbols are

$$(F^{\eta,\mathcal{N},\eta}_{\mathcal{N}})_{\mathcal{N},\mathcal{N}} = -1\,, \quad F^{\mathcal{N},\mathcal{N},\mathcal{N}}_{\mathcal{N}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{5.16}$$

The crossing equation

$$\mathbf{f}(1-z) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{f}(z) \tag{5.17}$$

becomes simply

$$\begin{pmatrix} 1 & \\ & C^2_{\tau,\tau,\psi} \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \frac{1}{2} \\ 2 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & \\ & C^2_{\tau,\tau,\psi} \end{pmatrix}, \tag{5.18}$$

which gives $C^2_{\tau,\tau,\psi} = \frac{1}{2}$. The formula (2.21) and (2.22) give the three-point coefficient

$$C_{\sigma,\sigma,\varepsilon} = \langle\mathcal{N}\rangle_{\mathbb{R}^2} \sqrt{\langle\eta\rangle_{\mathbb{R}^2}}\, C_{\tau,\tau,\psi}\, C_{\bar{\tau},\bar{\tau},\bar{\psi}}\, \Theta_{\eta,\eta,\mathcal{N}} = \sqrt{2} \times 1 \times \sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2}} = \frac{1}{2}\,, \tag{5.19}$$

up to a sign that can be absorbed into a redefinition of ψ and $\bar{\psi}$.

Alternatively, one may choose a gauge in which the F-symbols are identical to the crossing matrix of Virasoro blocks,

$$F^{\mathcal{N},\mathcal{N},\mathcal{N}}_{\mathcal{N}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \frac{1}{2} \\ 2 & -1 \end{pmatrix} , \tag{5.20}$$

trivializing the defect three-point coefficients. The formula (2.21) becomes

$$C_{\sigma,\sigma,\varepsilon} = \langle \mathcal{N} \rangle_{\mathbb{R}^2} \sqrt{\langle \eta \rangle_{\mathbb{R}^2}} \, \Theta_{\mathcal{N},\mathcal{N},\eta} = \sqrt{2} \, \Theta_{\mathcal{N},\mathcal{N},\eta} \, , \tag{5.21}$$

and in this gauge (2.22) is computed to be (using the first expression)

$$\Theta_{\mathcal{N},\mathcal{N},\eta} \; = \; \frac{1}{\langle \eta \rangle_{\mathbb{R}^2}} \times (F^{\mathcal{N},\mathcal{N},\mathcal{N}}_{\mathcal{N}})_{\mathcal{I},\eta} = \frac{1}{2\sqrt{2}} \, , \tag{5.22}$$

giving the same result $C_{\sigma,\sigma,\varepsilon} = \frac{1}{2}$. However, in this gauge, many previously trivial (= 1) F-symbols have become nontrivial. For instance,

$$(F^{\mathcal{N},\mathcal{N},\eta}_{\eta})_{\mathcal{I},\mathcal{N}} = \frac{1}{2} \, . \tag{5.23}$$

The trivialization of defect three-point coefficients is at the cost of complicating the F-symbols.

Next let us study the emergence of the Kramers-Wannier duality line $\mathcal{N}$ from Lorentzian dynamics. The torus two-point function of the spin operator σ is [81]

$$\langle \sigma(z,\bar{z}) \sigma(0) \rangle_{T^2} = \left| \frac{\partial_z \theta_1(0|\tau)}{\theta_1(z|\tau)} \right|^{\frac{1}{4}} \sum_{\nu=2}^{4} \left| \frac{\theta_\nu(\frac{z}{2}|\tau)}{\eta(\tau)} \right| \, , \tag{5.24}$$

normalized such that in the limit z, $\bar{z} \to 0$,

$$\langle \sigma(z,\bar{z}) \sigma(0) \rangle_{T^2} \to |z|^{-\frac{1}{4}} Z(\tau,\bar{\tau}) \, , \tag{5.25}$$

where $Z(\tau,\bar{\tau})$ is the torus partition function

$$Z(\tau,\bar{\tau}) = \sum_{\nu=2}^{4} |\theta_\nu(0|\tau)| \, . \tag{5.26}$$

Consider the torus Regge limits.

Spatial torus Regge limit. Under $z \to z+1$,

$$\begin{aligned}\langle \sigma(z,\bar z)\sigma(0)\rangle_{T^2} &\to \langle \sigma(z+1,\bar z)\sigma(0)\rangle_{T^2} \\ &= e^{\frac{i\pi}{8}} \left| \frac{\partial_z \theta_1(0|\tau)}{\theta_1(z|\tau)} \right|^{\frac{1}{4}} \frac{-\theta_1(\frac{z}{2}|\tau)\theta_2(\frac{\bar z}{2}|\bar\tau) + \theta_4(\frac{z}{2}|\tau)\theta_3(\frac{\bar z}{2}|\bar\tau) + \theta_3(\frac{z}{2}|\tau)\theta_4(\frac{\bar z}{2}|\bar\tau)}{|\eta(\tau)|}\,.\end{aligned} \tag{5.27}$$

Then

$$\lim_{z,\bar z\to 0} e^{-\frac{i\pi}{8}} |z|^{\frac{1}{4}} \langle \sigma(z+1,\bar z)\sigma(0)\rangle_{T^2} = \frac{\theta_4(\frac{z}{2}|\tau)\theta_3(\frac{\bar z}{2}|\bar\tau) + \theta_3(\frac{z}{2}|\tau)\theta_4(\frac{\bar z}{2}|\bar\tau)}{|\eta(\tau)|} = \frac{Z^N(\tau,\bar\tau)}{\sqrt{2}}\,. \tag{5.28}$$

Temporal torus Regge limit. Under $z \to z+\tau$,

$$\begin{aligned}\langle \sigma(z,\bar z)\sigma(0)\rangle_{T^2} &\to \langle \sigma(z+\tau,\bar z)\sigma(0)\rangle_{T^2} \\ &= e^{\frac{i\pi}{8}} \left| \frac{\partial_z \theta_1(0|\tau)}{\theta_1(z|\tau)} \right|^{\frac{1}{4}} \frac{\theta_3(\frac{z}{2}|\tau)\theta_2(\frac{\bar z}{2}|\bar\tau) + \theta_2(\frac{z}{2}|\tau)\theta_3(\frac{\bar z}{2}|\bar\tau) + i\,\theta_1(\frac{z}{2}|\tau)\theta_4(\frac{\bar z}{2}|\bar\tau)}{|\eta(\tau)|}\,.\end{aligned} \tag{5.29}$$

Then

$$\lim_{z,\bar z\to 0} e^{-\frac{i\pi}{8}} |z|^{\frac{1}{4}} \langle \sigma(z+1,\bar z)\sigma(0)\rangle_{T^2} = \frac{\theta_3(\frac{z}{2}|\tau)\theta_2(\frac{\bar z}{2}|\bar\tau) + \theta_2(\frac{z}{2}|\tau)\theta_3(\frac{\bar z}{2}|\bar\tau)}{|\eta(\tau)|} = \frac{Z_N(\tau,\bar\tau)}{\sqrt{2}}\,. \tag{5.30}$$

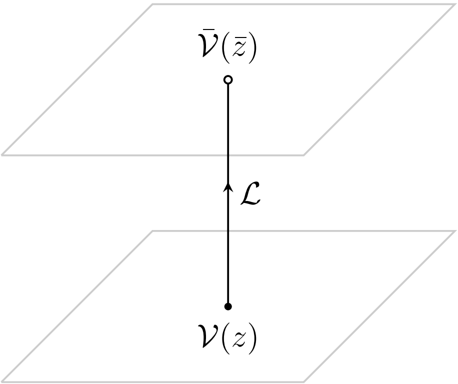
In the above we used some identities (C.5) for Jacobi theta functions. Noting that $\langle \mathcal{L}\rangle_{\mathbb{R}^2} = \sqrt{2}$, we recover the expected twisted torus partition functions $Z^{\mathcal{L}}(\tau,\bar\tau)$ and $Z_{\mathcal{L}}(\tau,\bar\tau)$.

5.3 Bulk perspective

The holomorphic part of a rational conformal field theory (RCFT) is the boundary edge theory of a bulk topological quantum field theory (TQFT) [3,12–14]. A celebrated example is Witten's correspondence between Wess-Zumino-Witten (WZW) models and Chern-Simons theory [3]. The states of the latter quantized on any spatial slice $\mathcal{M}_2$ correspond to the chiral algebra blocks of the WZW on $\mathcal{M}_2$. General RCFTs are dual to more general topological orders, such as Dijkgraaf-Witten theories, or abstract sets of anyons described by modular tensor categories.

A TQFT on $\mathcal{M}_2 \times [0,1]$ corresponds to a diagonal RCFT on $\mathcal{M}_2$ [4–8]. The holomorphic degrees of freedom live on one boundary, the anti-holomorphic ones live on the other, connected through the bulk by anyons. From this point of view, the Verlinde lines in a diagonal RCFT are the two-dimensional avatars of anyons in the TQFT, and the holomorphic-defect-factorization of local operators becomes evident,

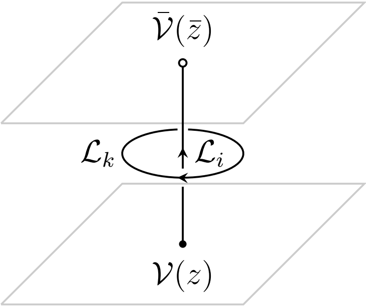
$$\mathcal{O}(z,\bar{z}) \;=$$



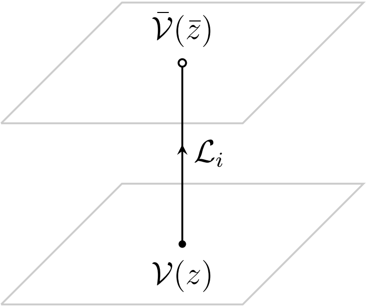
$$, \qquad (5.31)$$

where we abused the notation by labeling the anyon also by $\mathcal{L}$. The meaning of treating z, $\bar{z}$ as independent complex variables is also clear, and the nontrivial monodromies of blocks correspond to the braiding of anyons. The action of a Verlinde line $\mathcal{L}'$ on a holomorphically-defect-factorized local operator $\mathcal{O} = \mathcal{D} \overset{\mathcal{L}}{—} \overline{\mathcal{D}}$ could be realized as the linking of the anyon lines $\mathcal{L}'$ and $\mathcal{L}$ in the three-dimensional bulk. For example, the action of the Verlinde line $\mathcal{L}_k$ on local operator $\mathcal{O}_{i,i}$ is realized as

$$\widehat{\mathcal{L}}_k(\mathcal{O}_{i,i;\alpha}) \;=$$



$$= \frac{S_{ki}}{S_{0i}} \times$$



$$, \qquad (5.32)$$

where we apply braiding to unlink the $\mathcal{L}$ and $\mathcal{L}'$ and use the relation between braiding the the modular S matrix. The result agrees with the action (5.2). Since topological defect lines in general conformal field theory need not admit braiding, we refrain from using braiding in the following.

Consider $\mathrm{S}^2 \times [0,1]$, and insert four anyon lines $\mathcal{L}_1, \ldots, \mathcal{L}_4$ at $z_1, \ldots, z_4 \in \mathrm{S}^2$ extended from one S^2 boundary to the other, as shown in the upper left picture of Figure 1. This configuration gives a state in the Hilbert space $\widehat{\mathcal{H}}_{\mathrm{S}^2;z_i,\mathcal{L}_i} \times \widehat{\mathcal{H}}_{\mathrm{S}^2;\bar{z}_i,\bar{\mathcal{L}}_i}$, where $\widehat{\mathcal{H}}_{\mathrm{S}^2;z_i,\mathcal{L}_i}$ is the Hilbert space of the holomorphic chiral algebra blocks of the RCFT, and $\widehat{\mathcal{H}}_{\mathrm{S}^2;\bar{z}_i,\bar{\mathcal{L}}_i}$ the anti-holomorphic [3].

We now argue that this state corresponds to a crossing symmetric four-point function of local operators $\mathcal{O}_1, \ldots, \mathcal{O}_4$. First, we apply a sequence of F-moves on the anyons, to achieve the configuration on the upper right of Figure 1. Next, we cut the space along the spherical

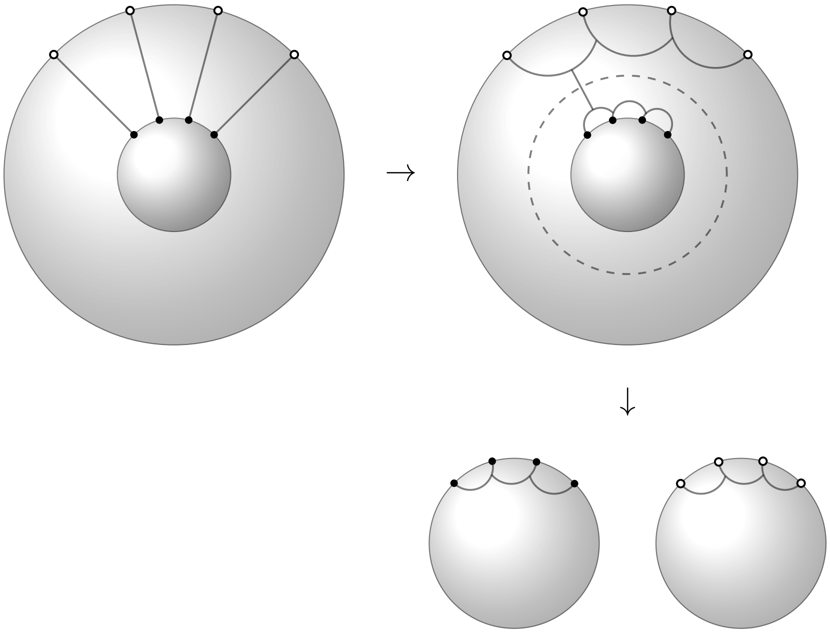


Figure 1: The conformal block decomposition of the four-point function of the holomorphically-defect-factorized local operator (5.31).

surface represented by the dashed line. The cutting generates two new boundaries with opposite orientations that could be either S^2 with one marked point or no marked point, which has a zero-dimensional or one-dimensional Hilbert space, respectively. Hence, the anyon that crosses the cutting surface must be a trivial line. By gluing this configuration with two solid B^3 balls with opposite orientations along the cutting surface, we obtain the configuration on the bottom right of Figure 1, where the left (resp. right) connected component gives a state in the Hilbert space $\widehat{\mathcal{H}}_{\mathrm{S}^2;z_i,\mathcal{L}_i}$ (resp. $\widehat{\mathcal{H}}_{\mathrm{S}^2;\bar{z}_i,\bar{\mathcal{L}}_i}$). They correspond to the holomorphic and anti-holomorphic blocks of the chiral algebra. The total configuration is a finite sum over the holomorphically factorized products and gives the conformal block decomposition.

6 Free boson theory

Are operators holomorphically-defect-factorized in irrational theories? This section examines the $c = 1$ free boson theory whose moduli space contains both rational and irrational points.

6.1 Toroidal branch

As we presently explain, all local operators in the compact boson theory are holomorphically-defect-factorized through the U(1) symmetry defect lines, which are Wilson lines of the background U(1) gauge field.

The $\mathrm{U}(1)_m \times \mathrm{U}(1)_w$ momentum and winding symmetry Wilson lines can be explicitly represented by

$$\mathcal{L}_{(\theta_m,\theta_w)} \;=\; : \exp\left[\frac{i}{2\pi}(\theta_m r + \frac{\theta_w}{r}) \int dz\, \partial X_L(z) - \frac{i}{2\pi}(\theta_m r - \frac{\theta_w}{r}) \int d\bar{z}\, \bar{\partial} X_R(\bar{z})\right] : \,. \tag{6.1}$$

Integer spectral flow gives an equivalence relation

$$\mathcal{L}_{(\theta_m,\theta_w)} \sim \mathcal{L}_{(\theta'_m,\theta'_w)}\,, \quad \theta'_m - \theta_m\,,\ \theta'_w - \theta_w \in 2\pi\mathbb{Z}\,. \tag{6.2}$$

The flavored torus partition function of $\mathcal{L}_{(\theta_m,\theta_w)}$ is

$$Z^{\mathcal{L}_{(\theta_m,\theta_w)}}(\tau,\bar{\tau}) = \frac{1}{|\eta(\tau)|^2} \sum_{m,w\in\mathbb{Z}} e^{i\theta_m m + i\theta_w w} q^{\frac{p_L^2}{4}} \bar{q}^{\frac{p_R^2}{4}}\,, \quad p_{L,R} = \frac{m}{r} \pm wr\,, \tag{6.3}$$

whose modular S transform gives the defect partition function

$$Z_{\mathcal{L}_{(\theta_m,\theta_w)}}(\tau,\bar{\tau}) = \frac{1}{|\eta(\tau)|^2} \sum_{m,w\in\mathbb{Z}} q^{\frac{p_L^2}{4}} \bar{q}^{\frac{p_R^2}{4}}\,, \quad p_{L,R} = \frac{m + \theta_w/2\pi}{r} \pm (w + \theta_m/2\pi) r\,. \tag{6.4}$$

In fact, a defect operator can be explicitly identified by taking the representation (6.1) of $\mathcal{L}_{(\theta_m,\theta_w)}$ and integrating by parts. Doing so in different spectral flow frames gives different defect operators that belong to the same defect Hilbert space of $\mathcal{L}_{(\theta_m,\theta_w)}$. We will see an example momentarily.

An exponential local operator

$$\mathcal{O}_{m,w}(z,\bar{z}) \;=: e^{ip_L X_L(z) + ip_R X_R(\bar{z})} :\,, \quad p_L = \frac{m}{r} + wr\,, \quad p_R = \frac{m}{r} - wr \tag{6.5}$$

is holomorphically-defect-factorized through a particular symmetry Wilson line $\mathcal{L}$, which has two useful representations (among infinitely many)

$$\mathcal{L}_{-\pi(\frac{m}{r^2}+w,m+wr^2)} \sim \mathcal{L}_{\pi(-\frac{m}{r^2}+w,m-wr^2)}\,, \tag{6.6}$$

which are equivalent under (w,m) units of spectral flow. Using the first representation, the defect partition function (6.4) involves the sum

$$\sum_{m',w'\in\mathbb{Z}} q^{\frac{1}{4}\left(\frac{m'-m}{r}+(w'-w)r\right)^2} \bar{q}^{\frac{1}{4}\left(\frac{m'}{r}-w'r\right)^2}\,. \tag{6.7}$$

The term $m'=w'=0$ corresponds to the unique holomorphic-defect current algebra primary $\mathcal{D}_{m,w}$, whereas the term $m'=m$, $w'=w$ corresponds to the unique anti-holomorphic one $\overline{\mathcal{D}}_{m,w}$. These two defect defect current algebra primaries can be explicitly obtained via integration by parts. Using the first representation

$$\mathcal{L}_{-\pi(\frac{m}{r^2}+w,m+wr^2)} \;=:\exp\left[-i(\frac{m}{r}+wr)\int_{z_1}^{z_2} dz\,\partial X(z)\right]:\,, \tag{6.8}$$

integration by parts gives a holomorphic defect operator on one end

$$\mathcal{D}_{m,w}(z_1) =: e^{i(\frac{m}{r}+wr)X_L(z_1)}:\,, \quad h = \frac{m^2}{r^2}+w^2r^2+2mw\,, \quad \bar{h}=0\,. \tag{6.9}$$

Using the second representation

$$\mathcal{L}_{\pi(-\frac{m}{r^2}+w,m-wr^2)} \;=:\exp\left[i(\frac{m}{r}-wr)\int_{\bar{z}_1}^{\bar{z}_2} d\bar{z}\,\bar{\partial} X(\bar{z})\right]:\,, \tag{6.10}$$

integration by parts gives an anti-holomorphic defect operator on the other end

$$\overline{\mathcal{D}}_{m,w}(\bar{z}_2) =: e^{i(\frac{m}{r}-wr)X_R(\bar{z}_2)}:\,, \quad h=0\,, \quad \bar{h} = \frac{m^2}{r^2}+w^2r^2-2mw\,. \tag{6.11}$$

The exponential local operator is holomorphically-defect-factorized as

$$\mathcal{O}_{m,w} = \mathcal{D}_{m,w} \overset{\mathcal{L}}{\text{---}} \overline{\mathcal{D}}_{m,w}\,. \tag{6.12}$$

Let us check that the torus Regge limits are consistent with the above analysis. The torus two-point function of the exponential local operator (6.5) with its conjugate is

$$\begin{aligned}\langle \mathcal{O}_{m,w}(z,\bar{z})\mathcal{O}_{-m,-w}(0)\rangle_{T^2} = &\left(\frac{\partial_z\theta_1(0|\tau)}{\theta_1(z|\tau)}\right)^{\frac{1}{2}p^2_{L,m,w}}\left(\frac{\partial_{\bar{z}}\theta_1(0|\bar{\tau})}{\theta_1(\bar{z}|\bar{\tau})}\right)^{\frac{1}{2}p^2_{R,m,w}}\\ &\times\frac{1}{|\eta(\tau)|^2}\sum_{m',w'}\Theta_{m,m',w,w'}(\tau,\bar{\tau},z,\bar{z})\,,\end{aligned} \tag{6.13}$$

where

$$\Theta_{m,m',w,w'}(\tau,\bar{\tau},z,\bar{z}) = q^{\frac{1}{4}p^2_{L,m',w'}}\bar{q}^{\frac{1}{4}p^2_{R,m',w'}}e^{i\pi\left(p_{L,m',w'}p_{L,m,w}z-p_{R,m',w'}p_{R,m,w}\bar{z}\right)}\,. \tag{6.14}$$

Consider the spatial torus Regge limit. Under $z\to z+1$, we find

$$\begin{aligned}\langle \mathcal{O}_{m,w}(z+1,\bar{z})\mathcal{O}_{-m,-w}(0)\rangle = e^{\frac{1}{2}i\pi p^2_{L,m,w}} &\left(\frac{\partial_z\theta_1(0|\tau)}{\theta_1(z|\tau)}\right)^{\frac{1}{2}p^2_{L,m,w}}\left(\frac{\partial_{\bar{z}}\theta_1(0|\bar{\tau})}{\theta_1(\bar{z}|\bar{\tau})}\right)^{\frac{1}{2}p^2_{R,m,w}}\\ &\times\frac{1}{|\eta(\tau)|^2}\sum_{m',w'}e^{i\pi p_{L,m',w'}p_{L,m,w}}\,\Theta_{m,m',w,w'}(\tau,z)\,,\end{aligned} \tag{6.15}$$

where we have used

$$\theta_1(z+1|\tau) = -\theta_1(z|\tau)\,. \tag{6.16}$$

In the further $z, \bar{z} \to 0$ limit, we find

$$\begin{aligned}&\langle \mathcal{O}_{m,w}(z+1,\bar{z})\mathcal{O}_{-m,-w}(0)\rangle_{T^2}\\ &\to e^{\frac{1}{2}i\pi p_{L,m,w}^2} z^{-\frac{1}{2}p_{L,m,w}^2} \bar{z}^{-\frac{1}{2}p_{R,m,w}^2} \sum_{m',w'} e^{i\pi p_{L,m',w'}p_{L,m,w}} \frac{q^{\frac{1}{4}p_{L,m,w}^2}\bar{q}^{\frac{1}{4}p_{R,m,w}^2}}{|\eta(\tau)|^2}\,.\end{aligned} \tag{6.17}$$

Stripping off the leading z, $\bar{z}$ divergence and the overall $e^{\frac{1}{2}i\pi p_{L,m,w}^2}$ factor corresponding to $e^{-2i\pi h}$ phase in (3.2), the exponential operators $\mathcal{O}_{m',w'}$ are transformed by the phases

$$e^{i\pi p_{L,m',w'}p_{L,m,w}} = e^{i\pi\left(\frac{m'}{r}+w'r\right)\left(\frac{m}{r}+wr\right)} = e^{i\pi\left[m'\left(\frac{m}{r^2}+w\right)+w'\left(m+wr^2\right)\right]} = e^{i(m'\theta_m+w'\theta_w)}\,. \tag{6.18}$$

A modular S transform recovers the expected defect partition function (6.4). We could have also directly taken the temporal torus Regge limit to arrive at (6.4).

Let us comment on the Lorentzian dynamics. For the four-point function of exponential operators, (4.11) gives the conformal Regge limit at infinite boost

$$r[\mathcal{O}_{m,w}, \mathcal{O}_{m',w'}] = e^{i\pi\left[m'\left(\frac{m}{r^2}+w\right)+w'\left(m+wr^2\right)\right]}\,. \tag{6.19}$$

Suppose one is interested in the four-point function of real operators, *i.e.* the cosine and sine operators

$$\begin{aligned}\mathcal{O}^{\cos}_{m,n}(z,\bar{z}) &= \frac{1}{\sqrt{2}}(\mathcal{O}_{m,w}(z,\bar{z}) + \mathcal{O}_{-m,-w}(z,\bar{z})) = \sqrt{2}\cos(p_L X_L(z) + p_R X_R(\bar{z}))\,,\\ \mathcal{O}^{\sin}_{m,n}(z,\bar{z}) &= \frac{1}{\sqrt{2}i}(\mathcal{O}_{m,w}(z,\bar{z}) - \mathcal{O}_{-m,-w}(z,\bar{z})) = \sqrt{2}\sin(p_L X_L(z) + p_R X_R(\bar{z}))\,,\end{aligned} \tag{6.20}$$

suitable combinations of (6.19) give

$$\begin{aligned}r[\mathcal{O}^{\cos}_{m,w}, \mathcal{O}^{\cos}_{m',w'}] &= -r[\mathcal{O}^{\cos}_{m,w}, \mathcal{O}^{\sin}_{m',w'}] = r[\mathcal{O}^{\sin}_{m,w}, \mathcal{O}^{\sin}_{m',w'}]\\ &= \cos\pi\left[m'\left(\frac{m}{r^2}+w\right) + w'\left(m+wr^2\right)\right]\,.\end{aligned} \tag{6.21}$$

6.2 Orbifold branch: a dichotomy

The $S^1/\mathbb{Z}_2$ partition function is

$$\begin{aligned}Z_{S^1_r/\mathbb{Z}_2}(\tau,\bar{\tau}) &= \frac{1}{2}\left(Z_{S^1_r}(\tau,\bar{\tau}) + \frac{|\theta_3(\tau)\theta_4(\tau)|}{|\eta(\tau)|^2} + \frac{|\theta_2(\tau)\theta_4(\tau)| + |\theta_2(\tau)\theta_3(\tau)|}{|\eta(\tau)|^2}\right)\\ &= \frac{1}{2}Z_{S^1_r}(\tau,\bar{\tau}) + \left|\frac{\eta(\tau)}{\theta_2(\tau)}\right| + \left|\frac{\eta(\tau)}{\theta_3(\tau)}\right| + \left|\frac{\eta(\tau)}{\theta_4(\tau)}\right|\,.\end{aligned} \tag{6.22}$$

The first two terms enumerate the untwisted sector, and the latter two enumerate the twisted sector which is universal and independent of the radius r. At $c = 1$, an irreducible module with primary weight $h = n^2$ has a null state at level $2n + 1$, so the degenerate character is

$$\chi_{h=n^2}(\tau) = \frac{q^{n^2} - q^{(n+1)^2}}{\eta(\tau)} \,. \tag{6.23}$$

The untwisted sector can be written as

$$Z^{\text{untwisted}}_{S^1_r/\mathbb{Z}_2}(\tau, \bar\tau) = \frac{1}{2}\left(Z_{S^1_r}(\tau, \bar\tau) - \frac{1}{|\eta(\tau)|^2} \right) + \sum_{\substack{n,\bar n \in \mathbb{Z}_{\geq 0} \\ n - \bar n \in 2\mathbb{Z}}} \chi_{h=n^2}(\tau) \bar\chi_{\bar h = \bar n^2}(\bar\tau) \,, \tag{6.24}$$

where the first piece enumerates the nontrivial cosine operators

$$\mathcal{O}_{m,w}(z, \bar z) = \cos(p_L X + p_R \bar X) \,, \quad p_{L,R} = \frac{m}{r} \pm wr \,, \quad (m, w) \neq (0, 0) \tag{6.25}$$

and their Virasoro descendants (which form non-degenerate Verma modules at generic r), and the second piece enumerates the degenerate Verma modules.

At arbitrary radius r, there is a continuous family of unoriented topological defect lines with $\langle \mathcal{L} \rangle_{\mathbb{R}^2} = 2$, which are the orientation-reversal invariant combinations of the $\mathrm{U}(1)_m \times \mathrm{U}(1)_w$ symmetry Wilson lines in the S^1 theory, [9]

$$\mathcal{L}^{S^1/\mathbb{Z}_2}_{(\theta_m, \theta_w)} = \mathcal{L}^{S^1}_{(\theta_m, \theta_w)} + \mathcal{L}^{S^1}_{-(\theta_m, \theta_w)} \,. \tag{6.26}$$

For

$$(\theta_m, \theta_w) = (0, 0) \,, \, (0, \pi) \,, \, (\pi, 0) \,, \, (\pi, \pi) \,, \tag{6.27}$$

the original $\mathcal{L}^{S^1}_{(\theta_m, \theta_w)}$ is already unoriented, so $\mathcal{L}^{S^1/\mathbb{Z}_2}_{(\theta_m, \theta_w)}$ is a non-simple topological defect line consisting of two copies of the trivial or $\mathbb{Z}_2$ symmetry line. In the rest of this section, the superscript $S^1/\mathbb{Z}_2$ will be suppressed. These cosine lines act on the nontrivial cosine operators (and descendants) by

$$\widehat{\mathcal{L}}_{(\theta_m, \theta_w)}(\mathcal{O}_{m,w}) = 2\cos(m\theta_m + w\theta_w)\, \mathcal{O}_{m,w} \,, \tag{6.28}$$

on the degenerate Verma modules by a factor of 2, and annihilate the twisted sector states. They are labeled by a pair of quantum numbers (θ_m, θ_w), which not only have periodicity $(2\pi, 0)$ and $(0, 2\pi)$ due to integer spectral flow, but are also identified under $(\theta_m, \theta_w) \to -(\theta_m, \theta_w)$. The fusion rule is

$$\mathcal{L}_{(\theta_m, \theta_w)} \mathcal{L}_{(\theta'_m, \theta'_w)} = \mathcal{L}_{(\theta_m + \theta'_m, \theta_w + \theta'_w)} + \mathcal{L}_{(\theta_m + \theta'_m, \theta_w - \theta'_w)} \,. \tag{6.29}$$

---

[9] This combination is not simple before orbifold, but becomes simple after except for (6.27).

For any pair of positive integers (N_m, N_w), there is a fusion subalgebra of finitely many objects

$$\left\{\mathcal{L}_{(\theta_m,\theta_w)} \mid \theta_m \in \mathbb{Z}/N_m \,,\, \theta_w \in \mathbb{Z}/N_w\right\} \,. \tag{6.30}$$

The torus partition function twisted by $\mathcal{L}_{(\theta_m,\theta_w)}$ (in the temporal direction) is

$$\begin{aligned} Z^{\mathcal{L}_{(\theta_m,\theta_w)}}(\tau,\bar\tau) &= \frac{1}{|\eta(\tau)|^2}\left(\sum_{\substack{m\in\mathbb{Z}\\ w\in\mathbb{Z}_{>0}}} + \sum_{\substack{m\in\mathbb{Z}_{>0}\\ w=0}}\right) 2\cos(\theta_m m + \theta_w w)\, q^{\frac{p_L^2}{4}} \bar q^{\frac{p_R^2}{4}} \\ &\qquad\qquad + \sum_{\substack{n,\bar n\in\mathbb{Z}_{\geq 0}\\ n-\bar n\in 2\mathbb{Z}}} \chi_{h=n^2}(\tau)\bar\chi_{\bar h=\bar n^2}(\bar\tau) \qquad\qquad (6.31) \\ &= \frac{1}{|\eta(\tau)|^2}\sum_{m,w\in\mathbb{Z}} e^{i\theta_m m + i\theta_w w}\, q^{\frac{p_L^2}{4}} \bar q^{\frac{p_R^2}{4}} + \frac{|\theta_3(\tau)\theta_4(\tau)|}{2|\eta(\tau)|^2}\,, \quad p_{L,R} = \frac{m}{r} \pm wr\,. \end{aligned}$$

The defect partition function of $\mathcal{L}_{(\theta_m,\theta_w)}$ is obtained by a modular S transform to be

$$Z_{\mathcal{L}_{(\theta_m,\theta_w)}}(\tau,\bar\tau) = \sum_{m,w\in\mathbb{Z}} \frac{q^{\frac{p_L^2}{4}} \bar q^{\frac{p_R^2}{4}}}{|\eta(\tau)|^2} + \frac{|\theta_2(\tau)\theta_3(\tau)|}{2|\eta(\tau)|^2}\,, \quad p_{L,R} = \frac{m+\frac{\theta_w}{2\pi}}{r} \pm (w + \tfrac{\theta_m}{2\pi})r\,, \tag{6.32}$$

where

$$\sqrt{\frac{\theta_2(\tau)\theta_3(\tau)}{2}} = \sum_{n\in\mathbb{Z}_{>0}} q^{\frac{n^2}{16}} \tag{6.33}$$

is a q-series with positive integer coefficients.

Consider a cosine line $\mathcal{L}_{\pi(\frac{m}{r^2}+w, m+wr^2)}$ with $m, w \in \mathbb{Z}$. Its defect partition function involves the sum

$$\sum_{m',w'\in\mathbb{Z}} q^{\frac14\left(\frac{m'+m}{r}+(w'+w)r\right)^2} \bar q^{\frac14\left(\frac{m'}{r}-w'r\right)^2}\,. \tag{6.34}$$

The $m' = w' = 0$ term corresponds to a holomorphic defect primary $\mathcal{D}_{m,w}$ of weight $h = \frac14\left(\frac{m}{r}+wr\right)^2$, and the $m' = -m$, $w' = -w$ terms corresponds to an anti-holomorphic defect primary $\overline{\mathcal{D}}_{m,w}$ of weight $\bar h = \frac14\left(\frac{m}{r}-wr\right)^2$. The cosine operator is holomorphically-defect-factorized as

$$\mathcal{O}_{m,w} = \mathcal{D}_{m,w} \overset{\mathcal{L}}{-\!\!-} \overline{\mathcal{D}}_{m,w}\,, \quad \mathcal{L} = \mathcal{L}_{\pi(\frac{m}{r^2}+w, m+wr^2)}\,. \tag{6.35}$$

In particular, $\mathcal{O}_{m,w}$ has charge $2\cos(\pi\frac{m^2}{r^2}+\pi w^2r^2)$ under the line $\mathcal{L}_{\pi(\frac{m}{r^2}+w,m+wr^2)}$ it factorizes through.

What about operators in the twisted sector? Consider the ground state twist fields $\mathcal{E}_i$ of weight $(\frac{1}{16}, \frac{1}{16})$, where $i = 1, 2$ label the two fixed points. Let $\mathcal{E}$ denote one of the twist field ground states. When rational, by the discussion in Section 5. $\mathcal{E}$ must be holomorphically-defect-factorized through some defect $\mathcal{L}_{\mathcal{E}}$, *i.e.*

$$\mathcal{E} = \mathcal{D} \overset{\mathcal{L}_{\mathcal{E}}}{-} \overline{\mathcal{D}}\,, \tag{6.36}$$

and the defect partition function of $\mathcal{L}_{\mathcal{E}}L$ is obtainable from a limit of the twist field two-point function on the torus. In Appendix C.1, we examine special rational points on the orbifold branch to identify $\mathcal{L}_{\mathcal{E}}$ as Verlinde lines. However, we can characterize $\mathcal{L}_{\mathcal{E}}$ in a more universal manner by computing the torus two-point function of twist fields in the temporal torus Regge limit. This computation is carried out in Appendix C.3, based on the formulae for general correlators in orbifolds of [82–85].

Interestingly, we find a clear distinction between rational and irrational theories:

1. If $r^2 = u/v$ is rational with u, v coprime, then the Holomorphic-Defect-Factorization Criterion can be satisfied, and the planar loop expectation value of $\mathcal{L}_{\mathcal{E}}$ is $\langle \mathcal{L}_{\mathcal{E}} \rangle_{\mathbb{R}^2} = \sqrt{uv}$. The vacuum-normalized defect partition function is given in (C.28).

2. If r^2 is irrational, then the Holomorphic-Defect-Factorization Criterion fails. More precisely, the so-obtained vacuum-normalized defect partition function is

$$\widehat{Z}_{\mathcal{L}_{\mathcal{E}}}(\tau, \bar\tau) = \frac{1}{|\eta(\tau)|^2} \sum_{n=0}^{\infty} \int_0^{\infty} dp \left(q^{\frac{(2n+1)^2}{16}} \bar q^{\frac{p^2}{4}} + q^{\frac{p^2}{4}} \bar q^{\frac{(2n+1)^2}{16}} \right), \tag{6.37}$$

   which in fact does not depend on r. The defect spectrum $\mathcal{H}_{\mathcal{L}_{\mathcal{E}}}$ is continuous, and the planar loop expectation value $\langle \mathcal{L}_{\mathcal{E}} \rangle$ is not well-defined.

Does a continuous defect spectrum make sense in a compact CFT? Assuming so, then the torus partition function with $\mathcal{L}_{\mathcal{E}}$ wrapped along the spatial direction is

$$\widehat{Z}^{\mathcal{L}_{\mathcal{E}}}(\tau, \bar\tau) = \sum_{\substack{n, \bar n = 0 \\ n - \bar n \in 2\mathbb{Z}}}^{\infty} (-)^n \chi_{h = n^2}(\tau) \bar\chi_{\bar h = \bar n^2}(\bar\tau)\,, \tag{6.38}$$

indicating that $\widehat{\mathcal{L}}_{\mathcal{E}}$ annihilates all non-degenerate modules and act on the degenerate modules by a sign.

For any irrational r^2, one can consider a sequence of coprime integers (u_n, v_n) for $n = 1, 2, \cdots$, such that in the $n \to \infty$ limit, u_n/v_n converges to r^2. One expects that the topological defect line $\mathcal{L}_{\mathcal{E}}$ in the irrational theory can be obtained by taking the $n \to \infty$ limit of $\mathcal{L}_{\mathcal{E}}$ in the rational theory. In Appendix C.3, we find that the sequence of defect partition functions $Z_{\mathcal{L}_\mathcal{E}, S^1_r/\mathbb{Z}_2}$ for $r^2 = u_n/v_n$ in the $n \to \infty$ limit reproduces the defect partition function (6.37) at irrational points. Note that there are infinitely many different sequences of coprime integers (u_n, v_n) whose ratio u_n/v_n converge to the same irrational number. At first sight, it is not obvious that the corresponding sequences of the defect partition functions all converge to the same result. However, as we find in (C.28), the defect partition function $Z_{\mathcal{L}_\mathcal{E}, S^1_r/\mathbb{Z}_2}$ depends only on the product uv. Hence, the $n \to \infty$ limit coincides with the $uv \to \infty$ limit, and the limits of all possible sequences agree. Furthermore, the result does not depend on r.

To end, let us remark on the Lorentzian dynamics of twist fields. According to (6.38), at irrational points, the Lorentzian four-point function exhibits transparent behavior for degenerate primaries with even n, and opaque behavior for degenerate primaries with odd n and all non-degenerate primary V (we have $r[W, V] = 1, -1, 0$ in the three cases, respectively).

7 Summary and discussion

In this paper, we explicated the following aspects of two-dimensional conformal field theory.

1. We presented a purely Euclidean portrayal of treating the coordinates z, $\bar{z}$ of a local operator as independent complex variables. The local operator can often be factorized into a pair of holomorphic and anti-holomophic defect operators, connected by a topological defect line. The validity of this interpretation was formulated into a Holomorphic-Defect-Factorization Criterion (Definition 3).

2. Based on factorization, we derived relations among correlation functions of local operators, correlation functionals of defect operators, and the F-symbols characterizing the splitting and joining of topological defect lines.

3. We proposed a procedure for discovering topological defect lines. This point warrants some further remarks. A topological defect line is traditionally characterized by a map on local operators satisfying stringent conditions — including but certainly not limited to the commutativity with the Virasoro algebra and the consistency of the defect partition function obtained by a modular S transform. From this perspective, a topological defect line is a solution to a set of consistency conditions, rather than something computed directly from the data of local operators. By considering the conformal Regge limit, we have shown how the four-point function or torus two-point function can directly generates the defining data for topological defect lines.

4. We characterized aspects of the conformal Regge limit by fundamental properties of

topological defect lines. In particular, whether the bulk scattering is transparent or opaque [75] is dictated by the action of topological defect lines on local operators.

5. Applying our procedure for discovering topological defect lines, we obtained a uniform description of the topological defect line through which the twist field factorizes in the $S^1/\mathbb{Z}_2$ free boson orbifold theory. The result at irrational points suggests that perhaps the defining conditions for topological defect lines need to be relaxed.

Suppose a topological defect line $\mathcal{L}$ hosts a set of holomorphic defect highest-weight operators (with respect to the maximally extended chiral algebra) $\mathcal{D}_i$, chosen to be orthonormal, then the holomorphic defect OPE gives

$$\mathcal{D}_i(z) \overset{\mathcal{L}}{\bullet\!\!\longrightarrow\!\!\bullet} \mathcal{D}_j^\dagger(0) \quad = \sum_\Omega z^{h_\Omega - h_{\mathcal{D}_i} - h_{\mathcal{D}_j}} \, C_{\mathcal{D}_i, \mathcal{D}_j^\dagger, \Omega} \, \Omega(0) \, . \tag{7.1}$$

where Ω are holomorphic local operators. All Ω must be chiral algebra descendants of the vacuum, because otherwise the chiral algebra would have been further extended. Then by associativity, $\mathcal{D}_i$ and $\mathcal{D}_j$ appear in each other's OPE with Ω, *i.e.* they are in the same chiral algebra module. Thus, every topological defect line hosts at most one holomorphic defect highest-weight operator, and only the vacuum module appears in the holomorphic defect OPE. Not every topological defect line hosts a holomorphic defect operator in its defect Hilbert space. A simple example is given by the charge conjugation symmetry defect line in the three-state Potts model.

Consider a local operator $\mathcal{O}$ that is holomorphically-defect-factorized through a topological defect line $\mathcal{L}$. There are three logical possibilities regarding the finiteness of highest-weight operators (with respect to the maximally extended chiral algebra) in the $\mathcal{O} \times \mathcal{O}$ OPE and the finiteness of simple topological defect lines in $\mathcal{L} \otimes \mathcal{L}$, as shown below:

	$\mathcal{L} \otimes \mathcal{L}$ finite	$\mathcal{L} \otimes \mathcal{L}$ not finite
$\mathcal{O} \times \mathcal{O}$ finite	(a)	(b)
$\mathcal{O} \times \mathcal{O}$ not finite	impossible	(c)

Most of our examples, including all local operators in rational theories and the exponential or cosine operators in the $c = 1$ free boson theory, fall into Scenario (a). The twist field in the free boson orbifold theory falls into scenario (c). We are not aware of any realization of Scenarios (b).

[] In a unitary compact theory, Scenario (c) implies that $\langle \mathcal{L} \rangle_{\mathbb{R}^2}$ is ill-defined, as we presently explain. According to [68], the cylinder loop expectation value $\langle\ \rangle_{S^1 \times \mathbb{R}}$ solves the abelianized fusion ring, and is related to the planar one $\langle\ \rangle_{\mathbb{R}^2}$ by at most a phase. In a unitary compact theory, the $\langle\ \rangle_{S^1\times\mathbb{R}}$ of every topological defect line is lower-bounded by 1. An infinite sum of numbers lower-bounded by 1 produces infinity, and $\langle\mathcal{L}\rangle_{\mathbb{R}^2}\langle\mathcal{L}\rangle_{\mathbb{R}^2} = N^{\mathcal{L}}_{\mathcal{L},\mathcal{L}}\langle\mathcal{L}\rangle_{\mathbb{R}^2} + \infty$ cannot have a finite solution.

The degree of prevalence of holomorphic-defect-factorization in general irrational conformal field theory is unclear to the authors. It seems delusional to expect that all local operators are holomorphically-defect-factorized. In particular, at irrational points in the free boson orbifold theory, we found that the topological defect line through which the twist field hypothetically factorizes exhibits a continuous spectrum in the defect Hilbert space. However, It could be that such a topological defect line is still physical, despite the lack of a mathematical framework to describe it. For irrational theories embedded in a conformal moduli space with "dense enough" rational points, one might be able to regard it as a limit of a sequence of Verlinde lines.[10] Then it remains a logical possibility that every local operator in any conformal field theory is holomorphically-defect-factorized in a generalized sense.

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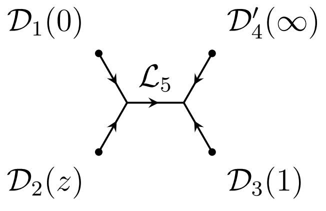
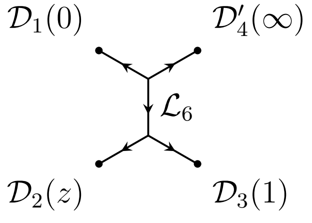
A Defect crossing implies local crossing

The crossing symmetry of a four-point function of holomorphic defect operators is the F-move





$$\langle \qquad \rangle = \langle \qquad \rangle \circ (F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\overline{\mathcal{L}_4}})_{\mathcal{L}_5,\mathcal{L}_6} \tag{A.1}$$

---

[10] See [86] and references within for discussions on the degree of prevalence of rational points in conformal moduli spaces.

decomposed into properly normalized s- and t-channel Virasoro blocks times defect three-point correlation functionals (bi-covectors),[11]

$$
\begin{aligned}
&\sum_{\mathcal{D}_5} \mathcal{F}\begin{bmatrix} h_{\mathcal{D}_1} & h_{\mathcal{D}_4} \\ h_{\mathcal{D}_2} & h_{\mathcal{D}_3} \end{bmatrix}^c_{h_{\mathcal{D}_5}}(z) \times C_{\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_5^\dagger} \otimes C_{\mathcal{D}_5,\mathcal{D}_3,\mathcal{D}_4} \\
&\qquad = \sum_{\mathcal{L}_6} \sum_{\mathcal{D}_6} \mathcal{F}\begin{bmatrix} h_{\mathcal{D}_2} & h_{\mathcal{D}_1} \\ h_{\mathcal{D}_3} & h_{\mathcal{D}_4} \end{bmatrix}^c_{h_{\mathcal{D}_6}}(1-z) \times C_{\mathcal{D}_2,\mathcal{D}_3,\mathcal{D}_6^\dagger} \otimes C_{\mathcal{D}_1,\mathcal{D}_6,\mathcal{D}_4} \circ (F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\overline{\mathcal{L}}_4})_{\mathcal{L}_5,\mathcal{L}_6} \,,
\end{aligned}
\tag{A.3}
$$

where c is the holomorphic central charge. The sums $\sum_{\mathcal{D}_5}$ and $\sum_{\mathcal{D}_6}$ are over holomorphic Virasoro primaries in the defect Hilbert spaces $\mathcal{H}_{\mathcal{L}_5}$ and $\mathcal{H}_{\mathcal{L}_6}$. When the theory has an extended chiral algebra, one could decompose the defect four-point function with respect to the extended chiral algebra. The crossing equation takes the same form as (A.3), but with $\mathcal{F}$ representing the chiral algebra blocks that may depend on other quantum numbers beside h, and the sums $\sum_{\mathcal{D}_5}$ and $\sum_{\mathcal{D}_6}$ are over holomorphic highest-weight operators of the chiral algebra in the defect Hilbert spaces $\mathcal{H}_{\mathcal{L}_5}$ and $\mathcal{H}_{\mathcal{L}_6}$.

In rational conformal field theory, the defect Hilbert space of a simple topological defect line $\mathcal{L}$ projected onto the subspace of holomorphic operators is an irreducible module of the maximally extended chiral algebra. In other words, there is a single highest-weight defect operator $\mathcal{D}_i$ for each $\mathcal{H}_{\mathcal{L}_i}$, and hence, each defect four-point correlation functional is equal to a single chiral algebra block composed with the appropriate three-point defect correlation functionals. One can always trivialize the defect three-point correlation functionals by a special choice of basis junction vectors. This has two complementary ramifications. First, the formula (2.21) for the three-point coefficients of local primary operators now only involves fusion categorical quantities, and the holomorphic defect four-point crossing equation (A.3) reads simply

$$
\mathcal{F}\begin{bmatrix} h_{\mathcal{D}_1} & h_{\mathcal{D}_4} \\ h_{\mathcal{D}_2} & h_{\mathcal{D}_3} \end{bmatrix}^c_{h_{\mathcal{D}_5}}(z) = \sum_{\mathcal{L}_6} \mathcal{F}\begin{bmatrix} h_{\mathcal{D}_2} & h_{\mathcal{D}_1} \\ h_{\mathcal{D}_3} & h_{\mathcal{D}_4} \end{bmatrix}^c_{h_{\mathcal{D}_6}}(1-z) \times (F^{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3}_{\overline{\mathcal{L}}_4})_{\mathcal{L}_5,\mathcal{L}_6} \,. \tag{A.4}
$$

Hence the nontrivial dynamical data is solved if one could determine the explicit values of the F-symbols in this special basis that trivializes the defect three-point correlation functionals. However, actually finding such a basis requires knowing the explicit blocks, for which one must resort to solving the null state decoupling equation [19] or the Wronskian method [20]. Moreover, as demonstrated in the example of Ising in Section 5.2, the F-symbols in such a basis are rather complicated.

---

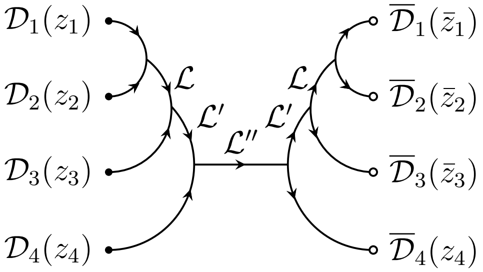
[11] A standard normalization for a block is to require unit coefficient for the leading coefficient in the cross ratio expansion

$$
\mathcal{F}\begin{bmatrix} h_{\mathcal{D}_1} & h_{\mathcal{D}_4} \\ h_{\mathcal{D}_2} & h_{\mathcal{D}_3} \end{bmatrix}^c_{h_{\mathcal{D}_5}}(z) = z^{h_{\mathcal{D}_5} - h_{\mathcal{D}_1} - h_{\mathcal{D}_2}} \left(1 + \mathcal{O}(z)\right) \,. \tag{A.2}
$$

The four-point function of holomorphically-defect-factorized local operators can be evaluated as follows. In the s-channel,

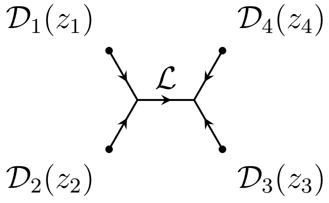
$$\langle \mathcal{O}_1(z_1,\bar{z}_1)\,\mathcal{O}_2(z_2,\bar{z}_2)\,\mathcal{O}_3(z_3,\bar{z}_3)\,\mathcal{O}_4(z_4,\bar{z}_4)\rangle$$

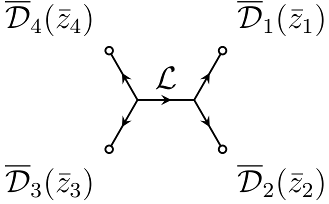
$$= \sqrt{\textstyle\prod_{i=1}^4 \langle \mathcal{L}_i\rangle_{\mathbb{R}^2}} \sum_{\mathcal{L},\mathcal{L}',\mathcal{L}''} \Big\langle \qquad \Big\rangle$$



$$\circ \Big[(F^{\mathcal{L}',\overline{\mathcal{L}}',\overline{\mathcal{L}}_4}_{\overline{\mathcal{L}}_4})_{\mathcal{I},\overline{\mathcal{L}}''}\,(1_{\mathcal{L}',\overline{\mathcal{L}}',\mathcal{I}},1_{\mathcal{I},\overline{\mathcal{L}}_4,\mathcal{L}_4}) \otimes (F^{\mathcal{L},\overline{\mathcal{L}},\overline{\mathcal{L}}_3}_{\overline{\mathcal{L}}_3})_{\mathcal{I},\overline{\mathcal{L}}'}\,(1_{\mathcal{L},\overline{\mathcal{L}},\mathcal{I}},1_{\mathcal{I},\overline{\mathcal{L}}_3,\mathcal{L}_3}) \\ \otimes (F^{\mathcal{L}_1,\overline{\mathcal{L}}_1,\overline{\mathcal{L}}_2}_{\overline{\mathcal{L}}_2})_{\mathcal{I},\overline{\mathcal{L}}}\,(1_{\mathcal{L}_1,\overline{\mathcal{L}}_1,\mathcal{I}},1_{\mathcal{I},\overline{\mathcal{L}}_2,\mathcal{L}_2})\Big] \tag{A.5}$$

$$= \frac{\sqrt{\prod_{i=1}^4 \langle \mathcal{L}_i\rangle_{\mathbb{R}^2}}}{\langle \mathcal{L}_4\rangle_{\mathbb{R}^2}} \sum_{\mathcal{L}} \Big\langle \qquad \Big\rangle \Big\langle \qquad \Big\rangle$$





$$\circ \Big[(F^{\mathcal{L},\overline{\mathcal{L}},\overline{\mathcal{L}}_3}_{\overline{\mathcal{L}}_3})_{\mathcal{I},\mathcal{L}_4}\,(1_{\mathcal{L},\overline{\mathcal{L}},\mathcal{I}},1_{\mathcal{I},\overline{\mathcal{L}}_3,\mathcal{L}_3}) \otimes (F^{\mathcal{L}_1,\overline{\mathcal{L}}_1,\overline{\mathcal{L}}_2}_{\overline{\mathcal{L}}_2})_{\mathcal{I},\overline{\mathcal{L}}}\,(1_{\mathcal{L}_1,\overline{\mathcal{L}}_1,\mathcal{I}},1_{\mathcal{I},\overline{\mathcal{L}}_2,\mathcal{L}_2})\Big]\,.$$

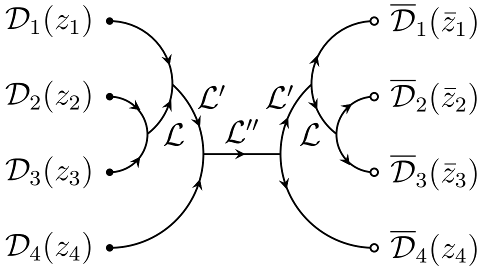
By performing block expansions on the defect four-point functions, and using (2.21) and (2.22), we recover with the usual s-channel conformal block expansion for local operators,

$$\begin{aligned}
&\langle \mathcal{O}_1(z_1,\bar{z}_1)\,\mathcal{O}_2(z_2,\bar{z}_2)\,\mathcal{O}_3(z_3,\bar{z}_3)\,\mathcal{O}_4(z_4,\bar{z}_4)\rangle \\
&= \frac{\sqrt{\prod_{i=1}^4 \langle \mathcal{L}_i\rangle_{\mathbb{R}^2}}}{\langle \mathcal{L}_4\rangle_{\mathbb{R}^2}} \sum_{\mathcal{L}} \sum_{\mathcal{D}\in\mathcal{H}^{\bar{h}=0}_{\overline{\mathcal{L}}}} \sum_{\overline{\mathcal{D}}\in\mathcal{H}^{h=0}_{\mathcal{L}}} \mathcal{F}\begin{bmatrix} h_{\mathcal{D}_1} & h_{\mathcal{D}_4} \\ h_{\mathcal{D}_2} & h_{\mathcal{D}_3}\end{bmatrix}^{c}_{h_{\mathcal{D}}}(z)\,\mathcal{F}\begin{bmatrix} \bar{h}_{\overline{\mathcal{D}}_1} & \bar{h}_{\overline{\mathcal{D}}_4} \\ \bar{h}_{\overline{\mathcal{D}}_2} & \bar{h}_{\overline{\mathcal{D}}_3}\end{bmatrix}^{\bar{c}}_{\bar{h}_{\overline{\mathcal{D}}}}(\bar{z}) \\
&\qquad \times C_{\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}^\dagger} \otimes C_{\mathcal{D}_3,\mathcal{D}_4,\mathcal{D}} \otimes C_{\overline{\mathcal{D}}_1,\overline{\mathcal{D}}^\dagger,\overline{\mathcal{D}}_2} \otimes C_{\overline{\mathcal{D}}_3,\overline{\mathcal{D}},\overline{\mathcal{D}}_4} \\
&\qquad\qquad \circ \Big[(F^{\mathcal{L},\overline{\mathcal{L}},\overline{\mathcal{L}}_3}_{\overline{\mathcal{L}}_3})_{\mathcal{I},\mathcal{L}_4}\,(1_{\mathcal{L},\overline{\mathcal{L}},\mathcal{I}},1_{\mathcal{I},\overline{\mathcal{L}}_3,\mathcal{L}_3}) \otimes (F^{\mathcal{L}_1,\overline{\mathcal{L}}_1,\overline{\mathcal{L}}_2}_{\overline{\mathcal{L}}_2})_{\mathcal{I},\overline{\mathcal{L}}}\,(1_{\mathcal{L}_1,\overline{\mathcal{L}}_1,\mathcal{I}},1_{\mathcal{I},\overline{\mathcal{L}}_2,\mathcal{L}_2})\Big] \\
&= \sum_{\mathcal{O}} \mathcal{F}\begin{bmatrix} h_{\mathcal{O}_1} & h_{\mathcal{O}_4} \\ h_{\mathcal{O}_2} & h_{\mathcal{O}_3}\end{bmatrix}^{c}_{h_{\mathcal{O}}}(z)\,\mathcal{F}\begin{bmatrix} \bar{h}_{\mathcal{O}_1} & \bar{h}_{\mathcal{O}_4} \\ \bar{h}_{\mathcal{O}_2} & \bar{h}_{\mathcal{O}_3}\end{bmatrix}^{\bar{c}}_{\bar{h}_{\mathcal{O}}}(\bar{z})\,C_{\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}}\,C_{\mathcal{O}_3,\mathcal{O}_4,\mathcal{O}}\,.
\end{aligned} \tag{A.6}$$

Similarly, in the t-channel,

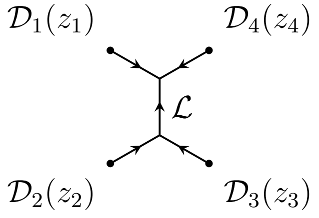
$$\langle \mathcal{O}_1(z_1,\bar z_1)\,\mathcal{O}_2(z_2,\bar z_2)\,\mathcal{O}_3(z_3,\bar z_3)\,\mathcal{O}_4(z_4,\bar z_4)\rangle$$

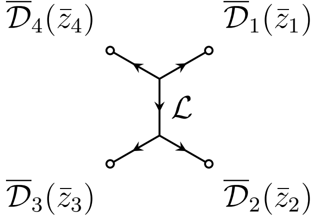
$$= \sqrt{\textstyle\prod_{i=1}^4 \langle \mathcal{L}_i\rangle_{\mathbb{R}^2}} \sum_{\mathcal{L},\mathcal{L}',\mathcal{L}''} \Big\langle \quad \Big\rangle$$



$$\circ \Big[(F^{\mathcal{L}',\overline{\mathcal{L}}',\overline{\mathcal{L}}_4}_{\overline{\mathcal{L}}_4})_{\mathcal{I},\overline{\mathcal{L}}''}\ (1_{\mathcal{L}',\overline{\mathcal{L}}',\mathcal{I}}, 1_{\mathcal{I},\overline{\mathcal{L}}_4,\mathcal{L}_4}) \otimes (F^{\mathcal{L}_1,\overline{\mathcal{L}}_1,\overline{\mathcal{L}}}_{\overline{\mathcal{L}}})_{\mathcal{I},\overline{\mathcal{L}}'}\ (1_{\mathcal{L}_1,\overline{\mathcal{L}}_1,\mathcal{I}}, 1_{\mathcal{I},\overline{\mathcal{L}},\mathcal{L}}) \\ \otimes (F^{\mathcal{L}_2,\overline{\mathcal{L}}_2,\overline{\mathcal{L}}_3}_{\overline{\mathcal{L}}_3})_{\mathcal{I},\overline{\mathcal{L}}}\ (1_{\mathcal{L}_2,\overline{\mathcal{L}}_2,\mathcal{I}}, 1_{\mathcal{I},\overline{\mathcal{L}}_3,\mathcal{L}_3}) \Big] \tag{A.7}$$

$$= \frac{\sqrt{\prod_{i=1}^4 \langle \mathcal{L}_i\rangle_{\mathbb{R}^2}}}{\langle \mathcal{L}_4\rangle_{\mathbb{R}^2}} \sum_{\mathcal{L}} \Big\langle \quad \Big\rangle \Big\langle \quad \Big\rangle$$





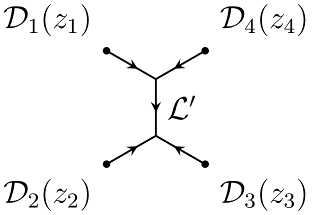
$$\circ \Big[(F^{\mathcal{L}_1,\overline{\mathcal{L}}_1,\overline{\mathcal{L}}}_{\overline{\mathcal{L}}})_{\mathcal{I},\mathcal{L}_4}\ (1_{\mathcal{L}_1,\overline{\mathcal{L}}_1,\mathcal{I}}, 1_{\mathcal{I},\overline{\mathcal{L}},\mathcal{L}}) \otimes (F^{\mathcal{L}_2,\overline{\mathcal{L}}_2,\overline{\mathcal{L}}_3}_{\overline{\mathcal{L}}_3})_{\mathcal{I},\overline{\mathcal{L}}}\ (1_{\mathcal{L}_2,\overline{\mathcal{L}}_2,\mathcal{I}}, 1_{\mathcal{I},\overline{\mathcal{L}}_3,\mathcal{L}_3}) \Big]\,.$$

Hence, by (2.21) and (2.22), we recover the usual t-channel conformal block expansion for local operators

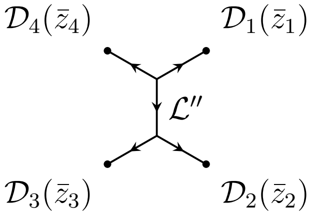
$$\begin{aligned} &\langle \mathcal{O}_1(z_1,\bar z_1)\,\mathcal{O}_2(z_2,\bar z_2)\,\mathcal{O}_3(z_3,\bar z_3)\,\mathcal{O}_4(z_4,\bar z_4)\rangle \\ &= \frac{\sqrt{\prod_{i=1}^4 \langle \mathcal{L}_i\rangle_{\mathbb{R}^2}}}{\langle \mathcal{L}_4\rangle_{\mathbb{R}^2}} \sum_{\mathcal{L}} \sum_{\mathcal{D}\in\mathcal{H}^{\bar h=0}_{\overline{\mathcal{L}}}} \sum_{\overline{\mathcal{D}}\in\mathcal{H}^{h=0}_{\mathcal{L}}} \mathcal{F}\begin{bmatrix} h_{\mathcal{D}_2} & h_{\mathcal{D}_1} \\ h_{\mathcal{D}_3} & h_{\mathcal{D}_4}\end{bmatrix}^c_{h_{\mathcal{D}}}(1-z)\,\mathcal{F}\begin{bmatrix} \bar h_{\overline{\mathcal{D}}_2} & \bar h_{\overline{\mathcal{D}}_1} \\ \bar h_{\overline{\mathcal{D}}_3} & \bar h_{\overline{\mathcal{D}}_4}\end{bmatrix}^{\bar c}_{\bar h_{\overline{\mathcal{D}}}}(1-\bar z) \\ &\qquad \times C_{\mathcal{D}_2,\mathcal{D}_3,\mathcal{D}^\dagger} \otimes C_{\mathcal{D}_1,\mathcal{D},\mathcal{D}_4} \otimes C_{\overline{\mathcal{D}}_2,\overline{\mathcal{D}}^\dagger,\overline{\mathcal{D}}_3} \otimes C_{\overline{\mathcal{D}}_1,\overline{\mathcal{D}}_4,\overline{\mathcal{D}}} \\ &\qquad\quad \circ \Big[(F^{\mathcal{L}_1,\overline{\mathcal{L}}_1,\overline{\mathcal{L}}}_{\overline{\mathcal{L}}})_{\mathcal{I},\mathcal{L}_4}\ (1_{\mathcal{L}_1,\overline{\mathcal{L}}_1,\mathcal{I}}, 1_{\mathcal{I},\overline{\mathcal{L}},\mathcal{L}}) \otimes (F^{\mathcal{L}_2,\overline{\mathcal{L}}_2,\overline{\mathcal{L}}_3}_{\overline{\mathcal{L}}_3})_{\mathcal{I},\overline{\mathcal{L}}}\ (1_{\mathcal{L}_2,\overline{\mathcal{L}}_2,\mathcal{I}}, 1_{\mathcal{I},\overline{\mathcal{L}}_3,\mathcal{L}_3}) \Big] \\ &= \sum_{\mathcal{O}} \mathcal{F}\begin{bmatrix} h_{\mathcal{O}_2} & h_{\mathcal{O}_1} \\ h_{\mathcal{O}_3} & h_{\mathcal{O}_4}\end{bmatrix}^c_{h_{\mathcal{O}}}(1-z)\,\mathcal{F}\begin{bmatrix} \bar h_{\mathcal{O}_2} & \bar h_{\mathcal{O}_1} \\ \bar h_{\mathcal{O}_3} & \bar h_{\mathcal{O}_4}\end{bmatrix}^{\bar c}_{\bar h_{\mathcal{O}}}(1-\bar z)\, C_{\mathcal{O}_2,\mathcal{O}_3,\mathcal{O}}\, C_{\mathcal{O}_1,\mathcal{O},\mathcal{O}_4}\,. \end{aligned} \tag{A.8}$$

Let us perform two more F-moves on the last line of the s-channel expression (A.5) to arrive at

$$\cdots = \frac{\sqrt{\prod_{i=1}^{4} \langle \mathcal{L}_i \rangle_{\mathbb{R}^2}}}{\langle \mathcal{L}_4 \rangle_{\mathbb{R}^2}} \sum_{\mathcal{L},\mathcal{L}',\mathcal{L}''} \langle$$



$$\rangle\langle$$



$$\rangle \qquad \text{(A.9)}$$

$$\circ \left[(F^{\overline{\mathcal{L}}_1,\overline{\mathcal{L}}_2,\overline{\mathcal{L}}_3}_{\mathcal{L}_4})_{\overline{\mathcal{L}},\mathcal{L}'} \otimes (F^{\mathcal{L}_4,\mathcal{L}_3,\mathcal{L}_2}_{\overline{\mathcal{L}}_1})_{\overline{\mathcal{L}},\mathcal{L}''} \right]$$
$$\circ \left[(F^{\mathcal{L},\overline{\mathcal{L}},\overline{\mathcal{L}}_3}_{\overline{\mathcal{L}}_3})_{\mathcal{I},\mathcal{L}_4}\ (1_{\mathcal{L},\overline{\mathcal{L}},\mathcal{I}}, 1_{\mathcal{I},\overline{\mathcal{L}}_3,\mathcal{L}_3}) \otimes (F^{\mathcal{L}_1,\overline{\mathcal{L}}_1,\overline{\mathcal{L}}_2}_{\overline{\mathcal{L}}_2})_{\mathcal{I},\overline{\mathcal{L}}}\ (1_{\mathcal{L}_1,\overline{\mathcal{L}}_1,\mathcal{I}}, 1_{\mathcal{I},\overline{\mathcal{L}}_2,\mathcal{L}_2}) \right].$$

Compared to the last line of the t-channel expression (A.7), we see that crossing symmetry of holomorphically-defect-factorized local operators is a consequence of

1. Crossing symmetry (A.3) of holomorphic defect operators, and
2. The fusion categorical identity

$$\begin{aligned}
&\sum_{\mathcal{L}} \left[(F^{\overline{\mathcal{L}}_1,\overline{\mathcal{L}}_2,\overline{\mathcal{L}}_3}_{\mathcal{L}_4})_{\overline{\mathcal{L}},\mathcal{L}'} \otimes (F^{\mathcal{L}_4,\mathcal{L}_3,\mathcal{L}_2}_{\overline{\mathcal{L}}_1})_{\overline{\mathcal{L}},\mathcal{L}''} \right] \\
&\qquad \circ \left[(F^{\mathcal{L},\overline{\mathcal{L}},\overline{\mathcal{L}}_3}_{\overline{\mathcal{L}}_3})_{\mathcal{I},\mathcal{L}_4}\ (1_{\mathcal{L},\overline{\mathcal{L}},\mathcal{I}}, 1_{\mathcal{I},\overline{\mathcal{L}}_3,\mathcal{L}_3}) \otimes (F^{\mathcal{L}_1,\overline{\mathcal{L}}_1,\overline{\mathcal{L}}_2}_{\overline{\mathcal{L}}_2})_{\mathcal{I},\overline{\mathcal{L}}}\ (1_{\mathcal{L}_1,\overline{\mathcal{L}}_1,\mathcal{I}}, 1_{\mathcal{I},\overline{\mathcal{L}}_2,\mathcal{L}_2}) \right] \\
&= \delta_{\mathcal{L}',\overline{\mathcal{L}}''}\, (F^{\mathcal{L}_1,\overline{\mathcal{L}}_1,\overline{\mathcal{L}}'}_{\overline{\mathcal{L}}'})_{\mathcal{I},\mathcal{L}_4}\ (1_{\mathcal{L}_1,\overline{\mathcal{L}}_1,\mathcal{I}}, 1_{\mathcal{I},\overline{\mathcal{L}}',\mathcal{L}'}) \otimes (F^{\mathcal{L}_2,\overline{\mathcal{L}}_2,\overline{\mathcal{L}}_3}_{\overline{\mathcal{L}}_3})_{\mathcal{I},\mathcal{L}'}\ (1_{\mathcal{L}_2,\overline{\mathcal{L}}_2,\mathcal{I}}, 1_{\mathcal{I},\overline{\mathcal{L}}_3,\mathcal{L}_3})
\end{aligned} \qquad \text{(A.10)}$$

for fusion categories that admit a gauge in which the cyclic permutation map is trivial. If not, the identity involves extra cyclic permutation maps/F-symbols.

B Unitarity bound on the conformal Regge limit at infinite boost

Consider a quantum field theory that hosts a finite set of simple topological defect lines (TDLs) $\{\mathcal{L}_i \mid i = 1, \dots, n\}$ obeying a fusion ring R under fusion.[12] Let the fusion coefficients be N_{ij}^k, and let $\mathbf{N}_i$ denote the matrix whose (j,k) component is given by N_{ij}^k. Associativity implies that $\mathbf{N}_*$ furnish a representation of the fusion ring, called the regular representation **reg**, which is the direct sum of irreducible representations, $\mathbf{reg} = \bigoplus_{a=1}^{n_r} \mathbf{r}_a$. We write $\mathbf{r} < \mathbf{reg}$ if $\mathbf{r} \in \{r_1, \dots, r_{n_r}\}$.

---

[12] The full set of simple TDLs may be infinite. Here we consider a finite subset closed under fusion. The generalization to the countably infinite case utilizes the corresponding generalization of the Perron-Frobenius theorem to the Krein–Rutman theorem.

On a cylinder, a TDL wrapped on the spatial circle acts as an operator on the Hilbert space. If the theory is unitary and if there is a unique vacuum, then every TDL acts on the vacuum with a positive eigenvalue. In other words, the cylinder loop expectation value is positive, $\langle\mathcal{L}_i\rangle_{\mathrm{S}^1\times\mathbb{R}} > 0$ for all $i = 1,\ldots,n$. This set of numbers solve the abelianized fusion ring,

$$\langle\mathcal{L}_i\rangle_{\mathrm{S}^1\times\mathbb{R}}\,\langle\mathcal{L}_j\rangle_{\mathrm{S}^1\times\mathbb{R}} = \sum_k N_{ij}^k\,\langle\mathcal{L}_k\rangle_{\mathrm{S}^1\times\mathbb{R}}\,, \tag{B.1}$$

and furnishes a one-dimensional representation of R. The relation between the $\langle\mathcal{L}_i\rangle_{\mathrm{S}^1\times\mathbb{R}}$ and $\langle\mathcal{L}_i\rangle_{\mathbb{R}^2}$ was discussed in Footnote 4, and in particular,

$$|\langle\mathcal{L}_i\rangle_{\mathrm{S}^1\times\mathbb{R}}| = |\langle\mathcal{L}_i\rangle_{\mathbb{R}^2}|\,. \tag{B.2}$$

The abelianized fusion ring relation (B.1) can be interpreted as saying that $\langle\mathcal{L}_*\rangle_{\mathrm{S}^1\times\mathbb{R}}$ is a simultaneous eigenvector of $\mathbf{N}_i$ with eigenvalue $\langle\mathcal{L}_i\rangle_{\mathrm{S}^1\times\mathbb{R}}$. Due to the non-negativity of N_{ij}^k, Perron-Frobenius (PF) theory applies: the positive vector $\langle\mathcal{L}_*\rangle_{\mathrm{S}^1\times\mathbb{R}}$ is the PF eigenvector of every $\mathbf{N}^i$, and the PF eigenvalue $\langle\mathcal{L}_i\rangle_{\mathrm{S}^1\times\mathbb{R}}$ is the spectral radius of $\mathbf{N}^i$, *i.e.*

$$\left|\frac{v^\dagger\mathbf{N}_i v}{v^\dagger v}\right| \leq \langle\mathcal{L}_i\rangle_{\mathrm{S}^1\times\mathbb{R}} \quad \forall i = 1,\ldots,n, \quad \forall v\in\mathbb{C}^n\,. \tag{B.3}$$

Proposition 1 *In a unitary (1+1)d quantum field theory on a cylinder with a unique vacuum, let R be the fusion ring of a finite set of topological defect lines $\{\mathcal{L}_i \mid i = 1,\ldots,n\}$. Denote by $\widehat{\mathcal{L}}_i$ the operator corresponding to wrapping $\mathcal{L}_i$ on the spatial circle. For any state $|V\rangle$ transforming in an irreducible representation $\mathbf{r} < \mathbf{reg}$ with respect to the ring R, the following inequality holds*

$$\left|\frac{\left\langle V^\dagger\widehat{\mathcal{L}}_i V\right\rangle}{\langle V^\dagger V\rangle}\right| \leq \langle\mathcal{L}_i\rangle_{\mathrm{S}^1\times\mathbb{R}}\,, \quad \forall i = 1,\ldots,n\,. \tag{B.4}$$

*In particular, if R is a group, then because every irreducible representation $<$ **reg**, the above inequality always holds.*

Corollary 1 *In conformal field theory, if a local operator W is holomorphically-defect-factorized through a topological defect line $\mathcal{L}$ that generates under fusion a finite sum of simple objects, and if V (not necessarily holomorphically-defect-factorized) transforms in an irreducible representation $\mathbf{r} < \mathbf{reg}$, then in the infinite boost limit* (4.11), $|r[W,V]| \leq 1$. *If $\mathcal{L}$ is invertible, then there is no restriction on V.*

For example, for the Verlinde lines in rational conformal field theory, the matrices $\mathbf{N}_i$ commute with each other. By the Verlinde formula [50], the simultaneous eigenvectors are related to the modular S-matrix by

$$\mathbf{N}_i \cdot \mathbf{v}_m = \frac{S_{im}}{S_{0m}} \mathbf{v}_m \,, \quad (\mathbf{v}_m)_j = \frac{S_{jm}}{S_{0m}} \,. \tag{B.5}$$

The PF eigenvector is the zeroth eigenvector $\mathbf{v}_0 = S_{*m}/S_{0m} = \langle \mathcal{L}_* \rangle_{\mathrm{S}^1 \times \mathbb{R}}$. The expression in (5.11) is the ratio between the k-th eigenvalue and the PF eigenvalue. Hence, its absolute value should be less than one.

C Free boson orbifold theory

This appendix concerns the holomorphic-defect-factorization of twist fields on the orbifold branch of the $c = 1$ free boson theory. We first examine the dual descriptions at special rational points, and cast the topological defect lines as Verlinde lines. We then review the basic definition and properties of Riemann theta functions that are used to express general correlators, and collect details of the computation of the torus Regge limit.

C.1 Holomorphic-defect-factorization at rational points

Holomorphic-defect-factorization can be explicitly examined at the following rational points via dual descriptions:

$$\begin{aligned}
&\text{(a)} \quad \frac{S^1_{r=1}}{\mathbb{Z}_2} = S^1_{r=2} \,, \\
&\text{(b)} \quad \frac{S^1_{r=\sqrt{3}/\sqrt{2}}}{\mathbb{Z}_2} = \frac{\mathcal{SM}(4,6)}{(-1)^F} \\
&\text{(c)} \quad \frac{S^1_{r=\sqrt{2}}}{\mathbb{Z}_2} = \text{Ising}^2 \,, \\
&\text{(d)} \quad \frac{S^1_{r=\sqrt{3}}}{\mathbb{Z}_2} = \frac{SU(2)_4}{\mathrm{U}(1)} \,, \\
&\text{(e)} \quad \frac{S^1_{r=\sqrt{6}}}{\mathbb{Z}_2} = \frac{\mathcal{SM}(4,6) \otimes (-1)^{\text{Arf}}}{(-1)^F} \\
&\text{(f)} \quad \frac{S^1_{r=2\sqrt{2}}}{\mathbb{Z}_2} = \text{Sym}^2 \, \text{Ising} \,, \quad \ldots \,,
\end{aligned} \tag{C.1}$$

where $\mathcal{SM}$ denotes an $\mathcal{N} = 1$ super-Virasoro minimal model.

(a) In the $S^1_{r=1}/\mathbb{Z}_2$ theory, all local operators including the twist fields are exponential operators in the $S^1_{r=2}$ description, holomorphically-defect-factorized through U(1) symmetry lines with $\left\langle \mathcal{L}_{(\theta_m,\theta_w)} \right\rangle_{\mathbb{R}^2} = 1$. To be specific, the twist fields are the exponential operators with $m = \pm 1$, $w = 0$ in $S^1_{r=2}$. The topological defect lines are the symmetry lines $\mathcal{L}_{\pm(\frac{\pi}{4},\pi)}$, which act on an exponential operator $\mathcal{O}_{m,w}$ by a phase $e^{\pm i\pi(\frac{m}{4}+w)}$.

(b) The twist fields of $S^1_{r=\sqrt{3}/\sqrt{2}}/\mathbb{Z}_2$ are the two weight $(\frac{1}{16},\frac{1}{16})$ operators (one in NS and one in R) in the bosonized $\mathcal{SM}(4,6)$ description. They are holomorphically-defect-factorized through Verlinde lines with $\langle \mathcal{L} \rangle_{\mathbb{R}^2} = \sqrt{6}$.

(c) The twist fields of $S^1_{r=\sqrt{2}}/\mathbb{Z}_2$ are the σ_1 and σ_2 operators in the Ising^2 description, which are holomorphically-defect-factorized through the Kramers-Wannier duality lines $\mathcal{N}_1$ and $\mathcal{N}_2$ with $\langle \mathcal{N}_1 \rangle_{\mathbb{R}^2} = \langle \mathcal{N}_2 \rangle_{\mathbb{R}^2} = \sqrt{2}$.

(d) The twist fields of $S^1_{r=\sqrt{3}}/\mathbb{Z}_2$ are the two weight $(\frac{1}{16},\frac{1}{16})$ operators in the $\mathbb{Z}_4$ parafermion theory, which is a diagonal theory. The twist fields are each holomorphically-defect-factorized through a Verlinde line with $\langle \mathcal{L} \rangle_{\mathbb{R}^2} = \sqrt{3}$.

(e) The twist fields of $S^1_{r=\sqrt{6}}/\mathbb{Z}_2$ are the two weight $(\frac{1}{16},\frac{1}{16})$ operators (one in NS and one in R) in the bosonization of the tensor product of $\mathcal{SM}(4,6)$ with the $(-1)^{\mathrm{Arf}}$ topological field theory. They are holomorphically-defect-factorized through Verlinde lines with $\langle \mathcal{L} \rangle_{\mathbb{R}^2} = \sqrt{6}$.

(f) The twist fields of $S^1_{r=2\sqrt{2}}/\mathbb{Z}_2$ in the language of Sym^2 Ising include the weight $(\frac{1}{16},\frac{1}{16})$ operators, $\frac{1}{\sqrt{2}}(\sigma_1+\sigma_2)$ in the untwisted sector and the replica twist field ground state. It is holomorphically-defect-factorized through the Verlinde line $\mathcal{L}$ ($\equiv \mathcal{N}_1 + \mathcal{N}_2$ before the symmetric product orbifold) with $\langle \mathcal{L} \rangle_{\mathbb{R}^2} = 2\sqrt{2}$.

We observe a pattern: If $r^2 = u/v$ with u, v coprime, then the twist fields in the $S^1_r/\mathbb{Z}_2$ theory are holomorphically-defect-factorized through a topological defect line $\mathcal{L}$ with $\langle \mathcal{L} \rangle_{\mathbb{R}^2} = \sqrt{uv}$. That this is true for all rational r^2 is proven in Appendix C.3.

C.2 Riemann and Jacobi theta functions

The Riemann theta function is defined as

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z|\tau) = \sum_{n\in\mathbb{Z}^g} \exp(i\pi(n+\alpha)\cdot\tau\cdot(n+\alpha) + 2i\pi(n+\alpha)\cdot(z+\beta))\,. \tag{C.2}$$

By definition, it changes characteristic under shifts in z:

$$\begin{aligned}\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z+k|\tau) &= \theta\begin{bmatrix}\alpha\\ \beta+k\end{bmatrix}(z|\tau)\,,\\ \theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z+k\tau|\tau) &= \exp(-i\pi k^2\tau)\,\theta\begin{bmatrix}\alpha+k\\ \beta\end{bmatrix}(z|\tau)\,.\end{aligned} \tag{C.3}$$

When $g = 1$ and α, β take values in $\frac{1}{2}\mathbb{Z}$, they are the Jacobi theta functions

$$\begin{aligned} &\theta_1(z|\tau) = -\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|\tau)\,, \quad \theta_2(z|\tau) = \theta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(z|\tau)\,, \\ &\theta_3(z|\tau) = \theta\begin{bmatrix}0\\ 0\end{bmatrix}(z|\tau)\,, \quad \theta_4(z|\tau) = \theta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(z|\tau)\,. \end{aligned} \tag{C.4}$$

Thus

$$\begin{aligned} &\theta_1(z+\frac{1}{2}|\tau) = -\theta_2(z|\tau)\,, \quad \theta_2(z+\frac{1}{2}|\tau) = \theta_1(z|\tau)\,, \\ &\theta_3(z+\frac{1}{2}|\tau) = \theta_4(z|\tau)\,, \quad \theta_4(z+\frac{1}{2}|\tau) = \theta_3(z|\tau)\,, \\ &\theta_1(z+\frac{\tau}{2}|\tau) = e^{-\frac{i\pi}{4}\tau}\theta_4(z|\tau)\,, \quad \theta_2(z+\frac{\tau}{2}|\tau) = e^{-\frac{i\pi}{4}\tau}\theta_3(z|\tau)\,, \\ &\theta_3(z+\frac{\tau}{2}|\tau) = e^{-\frac{i\pi}{4}\tau}\theta_2(z|\tau)\,, \quad \theta_4(z+\frac{\tau}{2}|\tau) = -e^{-\frac{i\pi}{4}\tau}\theta_1(z|\tau)\,. \end{aligned} \tag{C.5}$$

Next consider modular transformations, and restrict to $z = 0$ for simplicity. For Riemann theta functions,

$$\begin{aligned} &\theta\begin{bmatrix}\alpha\\ \beta-\alpha+\frac{1}{2}\end{bmatrix}(0|\tau+1) = \varepsilon_T\, e^{i\pi\alpha(1-\alpha)}\,\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0|\tau)\,, \\ &\theta\begin{bmatrix}-\beta\\ \alpha\end{bmatrix}(0|-\frac{1}{\tau}) = \varepsilon_S\, e^{-2i\pi\alpha\beta}\sqrt{\tau}\,\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0|\tau)\,. \end{aligned} \tag{C.6}$$

For Jacobi theta functions,

$$\begin{aligned} &\theta_4(0|\tau+1) = \varepsilon_T\,\theta_3(0|\tau)\,, \quad \theta_3(0|-\frac{1}{\tau}) = \varepsilon_S\,\sqrt{\tau}\,\theta_3(0|\tau)\,, \\ &\theta_3(0|\tau+1) = \varepsilon_T\,\theta_4(0|\tau)\,, \quad \theta_2(0|-\frac{1}{\tau}) = \varepsilon_S\,\sqrt{\tau}\,\theta_4(0|\tau)\,, \\ &\theta_2(0|\tau+1) = \varepsilon_T\,e^{\frac{i\pi}{4}}\,\theta_2(0|\tau)\,, \quad \theta_4(0|-\frac{1}{\tau}) = \varepsilon_S\,\sqrt{\tau}\,\theta_2(0|\tau)\,, \end{aligned} \tag{C.7}$$

with $\varepsilon_T = \varepsilon_S = 1$.

C.3 Torus Regge limit of twist field two-point function

Recall that in the torus Regge limit, $z \to 1$ and $\bar{z} \to 0$, after stripping off the leading OPE divergence and the overall $e^{2i\pi h}$ phase, one computes the vacuum-normalized defect partition function $\widehat{Z}_{\mathcal{L}}(\tau,\bar{\tau})$. If $\widehat{Z}_{\mathcal{L}}(\tau,\bar{\tau})$ has a discrete expansion in q, $\bar{q}$, and if the coefficients are integers up to a common real factor, then one can strip it off and define a Cardy-normalized defect partition function $Z_{\mathcal{L}}(\tau,\bar{\tau})$ with positive integer degeneracies. This common factor is inverse $\langle\mathcal{L}\rangle_{\mathbb{R}^2}$ of the defect, so

$$\widehat{Z}_{\mathcal{L}}(\tau,\bar{\tau}) = \frac{Z_{\mathcal{L}}(\tau,\bar{\tau})}{\langle\mathcal{L}\rangle_{\mathbb{R}^2}}\,. \tag{C.8}$$

The covering space formalism for computing general correlators of orbifolds [66, 67] was developed in [82, 83], and applied to the $c = 1$ free boson theory in great detail in [84, 85]. In particular, our notation and formulae follow [85] closely.

In the $S^1/\mathbb{Z}_2$ orbifold theory, the bosonic field X is double valued. When computing the partition function on a Riemann surface, there are distinct topological sectors distinguished by whether X flips sign around each nontrivial cycle. On a closed Riemann surface of genus g, those sectors are labeled by $\varepsilon_i \in \frac{1}{2}\mathbb{Z}_2$ around a-cycles and $\delta_i \in \frac{1}{2}\mathbb{Z}_2$ around b-cycles, for $i = 1, \ldots, g$. In a given sector described by ε_i, δ_i, the double-valued field X on Σ_g can be lifted to a single-valued field X on a double-sheeted cover $\widetilde{\Sigma}_g$. The cover $\widetilde{\Sigma}_g$ is a replica-symmetric genus $2g$ Riemann surface, and its modulus is described by the period matrix $\Pi_{\varepsilon_i,\delta_i}$ of Prym differentials (replica-symmetric holomorphic one-forms on $\widetilde{\Sigma}_g$). The modulus $\Pi_{\epsilon_i,\delta_i}$ is fixed by the period matrix τ of Σ_g, the sector ε_i, δ_i, and the positions of twist fields; this relation will be explicitly given for $g = 1$ later.

The twist-field two-point function on a genus-g Riemann surface Σ_g is given in Dijkgraaf-Verilinde-Verlinde (5.13) to be

$$\langle \mathcal{E}(z,\bar{z})\mathcal{E}(0)\rangle_{\Sigma_g(\tau,\bar{\tau})} = 2^{-g} \sum_{\varepsilon_i,\delta_i \in (\frac{1}{2}\mathbb{Z}_2)^g} Z^{\mathrm{cl}}(r, \Pi_{\varepsilon_i,\delta_i}, \overline{\Pi}_{\varepsilon_i,\delta_i})\, Z^{\mathrm{qu}}_{\varepsilon_i,\delta_i}(\tau,\bar{\tau})\,, \tag{C.9}$$

where

$$Z^{\mathrm{qu}}_{\varepsilon,\delta}(\tau,\bar{\tau}) = Z^{\mathrm{qu}}_0(\tau,\bar{\tau}) \left| c\begin{bmatrix}\varepsilon_i\\ \delta_i\end{bmatrix}(\tau)\right|^{-2} . \tag{C.10}$$

Let us explain the pieces comprising this formula.

1. $Z^{\mathrm{cl}}(r, \Pi_{\varepsilon_i,\delta_i}, \overline{\Pi}_{\varepsilon_i,\delta_i})$ is the classical contribution to the partition function. It is a solitonic sum over momentum and winding on the two-sheeted cover $\widetilde{\Sigma}_g$ of Σ_g,

$$Z^{\mathrm{cl}}(r, \Pi_{\varepsilon_i,\delta_i}, \overline{\Pi}_{\varepsilon_i,\delta_i}) = \sum_{p,\bar{p} \in \Gamma_r} \exp\left[\frac{i\pi}{2}(p \cdot \Pi_{\varepsilon_i,\delta_i} \cdot p - \bar{p} \cdot \overline{\Pi}_{\varepsilon_i,\delta_i} \cdot \bar{p})\right] , \tag{C.11}$$

   where

$$\Gamma_r = \left\{ \Big(\frac{m_i}{r} + w_i r, \frac{m_i}{r} - w_i r \Big) \mid m_i, w_i \in \mathbb{Z} \right\} , \quad (i = 1, \ldots, g)\,. \tag{C.12}$$

2. $Z^{\mathrm{qu}}_{\varepsilon_i,\delta_i}(\tau,\bar{\tau})$ is the quantum contribution to the partition function. And $Z^{\mathrm{qu}}_0(\tau,\bar{\tau})$ is a common factor shared by all distinct topological sectors, that only depends on the period matrix τ of Σ_g.

3. Finally,

$$c\begin{bmatrix}\varepsilon_i\\ \delta_i\end{bmatrix}(\tau)^{-1} = E(z,0)^{-\frac{1}{8}} \frac{\theta\begin{bmatrix}\gamma_i + \frac{\varepsilon_i}{2}\\ \delta_i\end{bmatrix}(\frac{1}{2}\int_0^z \omega \mid 2\tau)}{\theta\begin{bmatrix}\gamma_i\\ 0\end{bmatrix}(0 \mid 2\Pi_{\varepsilon_i,\delta_i})} , \tag{C.13}$$

where $\gamma_i \in (\frac{1}{2}\mathbb{Z}_2)^g$ is arbitrary, ω is the holomorphic one-form on Σ_g, and $E(z,0)$ is the prime form, the closest thing to z that respects the global structure of the Riemann surface. At short distances, $E(z,0) \sim z$.

We now specialize to $g = 1$. The classical solitonic sum $Z^{\mathrm{cl}}(r, \Pi_{\varepsilon,\delta}, \overline{\Pi}_{\varepsilon,\delta})$ is just the free compact boson partition function with τ set to $\Pi_{\varepsilon,\delta}$. The common factor in the quantum contributions to the partition function is

$$Z_0^{\mathrm{qu}}(\tau, \bar{\tau}) = \frac{1}{|\eta(\tau)|^2} \,. \tag{C.14}$$

The prime form on a torus is

$$E(z, q) = \frac{\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z \mid \tau)}{\partial_z \theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z \mid \tau)\big|_{z=0}} \tag{C.15}$$

The Abel map is $\mathfrak{z} = \frac{1}{2}\int_0^z \omega$, where ω is the holomorphic one-form on $\Sigma_{g=1}$. The Schottky relation (the arbitrariness of γ_i mentioned before)

$$\frac{\theta\begin{bmatrix}0\\ 0\end{bmatrix}(\mathfrak{z} \mid 2\tau)}{\theta\begin{bmatrix}0\\ 0\end{bmatrix}(0 \mid 2\Pi_{0,0})} = \frac{\theta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(\mathfrak{z} \mid 2\tau)}{\theta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(0 \mid 2\Pi_{0,0})} \tag{C.16}$$

implicitly defines $\Pi_{0,0}$ as a function of $\mathfrak{z}$ and τ. Let $\Pi(\mathfrak{z}, \tau) \equiv \Pi_{0,0}(\mathfrak{z}, \tau)$, then the rest of $\Pi_{\varepsilon,\delta}(\mathfrak{z}, \tau)$ are related via half-integer shifts of $\mathfrak{z}$

$$\Pi_{\varepsilon,\delta}(\mathfrak{z}, \tau) = \Pi(\mathfrak{z} + \delta + \varepsilon\tau, \tau) \,. \tag{C.17}$$

Using the Schottky relation (C.16) together with the identities

$$\begin{aligned}
\theta\begin{bmatrix}\gamma\\ 0\end{bmatrix}(\mathfrak{z} + \frac{1}{2} \mid 2\tau) &= \theta\begin{bmatrix}\gamma\\ \frac{1}{2}\end{bmatrix}(\mathfrak{z} \mid 2\tau) \,, \\
\theta\begin{bmatrix}\gamma\\ 0\end{bmatrix}(\mathfrak{z} + \frac{\tau}{2} \mid 2\tau) &= \exp(-\frac{i\pi}{16})\, \theta\begin{bmatrix}\gamma + \frac{1}{4}\\ 0\end{bmatrix}(\mathfrak{z} \mid 2\tau) \,, \\
\theta\begin{bmatrix}\gamma\\ 0\end{bmatrix}(\mathfrak{z} + \frac{1+\tau}{2} \mid 2\tau) &= \exp(-\frac{i\pi}{16})\, \theta\begin{bmatrix}\gamma + \frac{1}{4}\\ \frac{1}{2}\end{bmatrix}(\mathfrak{z} \mid 2\tau) \,,
\end{aligned} \tag{C.18}$$

we find that in the limit of the two twist fields colliding $z \to 0$, the period matrix Π behaves as

$$\Pi_{0,0}(0, \tau) = \tau \,, \quad \Pi_{0,\frac{1}{2}}(0, \tau) = i\infty \,, \quad \Pi_{\frac{1}{2},0}(0, \tau) = i0^+ \,, \quad \Pi_{\frac{1}{2},\frac{1}{2}}(0, \tau) = -1 + i0^+ \,. \tag{C.19}$$

We are now ready to examine the torus Regge limit. To recap, the torus two-point function is a sum of four terms

$$\langle \mathcal{E}(z,\bar{z})\mathcal{E}(0)\rangle_{\Sigma_{g=1}(\tau,\bar{\tau})} = \frac{1}{2\,|\eta(\tau)|^2} \sum_{\varepsilon,\delta\in\frac{1}{2}\mathbb{Z}_2} Z^{\mathrm{cl}}(r,\Pi_{\varepsilon,\delta},\overline{\Pi}_{\varepsilon,\delta}) \left| c\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau)\right|^{-2} . \tag{C.20}$$

Under $z \to z+1$,

$$\begin{aligned} &Z^{\mathrm{cl}}(r,\Pi_{\varepsilon,\delta},\overline{\Pi}_{\varepsilon,\delta}) \to Z^{\mathrm{cl}}(r,\Pi_{\varepsilon,\delta+\frac{1}{2}},\overline{\Pi}_{\varepsilon,\delta})\,, \\ &\left| c\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau)\right|^{-2} \to e^{\frac{i\pi}{8}}|E(z,0)|^{-\frac{1}{4}} \frac{\theta\begin{bmatrix}\frac{\varepsilon}{2}\\ \delta+\frac{1}{2}\end{bmatrix}(\frac{1}{2}\int_0^z\omega \mid 2\tau)}{\theta\begin{bmatrix}0\\ 0\end{bmatrix}(0\mid 2\Pi_{\varepsilon,\delta+\frac{1}{2}})} \frac{\overline{\theta\begin{bmatrix}\frac{\varepsilon}{2}\\ \delta\end{bmatrix}}(\frac{1}{2}\int_0^z\omega \mid 2\bar{\tau})}{\overline{\theta\begin{bmatrix}0\\ 0\end{bmatrix}}(0\mid 2\overline{\Pi}_{\varepsilon,\delta})}\,, \end{aligned} \tag{C.21}$$

where we have set $\gamma=0$ without loss of generality. The $e^{2i\pi h}=e^{\frac{i\pi}{8}}$ phase will henceforth be stripped off. In the further $z,\,\bar{z}\to 0$ limit, in each term the limiting Π and $\overline{\Pi}$ each takes one of the four values given in (C.19), and the combined limits of the four terms are summarized in Table 1. It suffices to examine say the first and third limits in Table 1, as the remaining two are related by complex conjugation.

(ε,δ)	$(\Pi,\overline{\Pi})$
$(0,0)$	$(i\infty\,,\ \bar{\tau})$
$(0,\frac{1}{2})$	$(\tau\,,\ -i\infty)$
$(\frac{1}{2},0)$	$(i0^+\,,\ -1+i0^+)$
$(\frac{1}{2},\frac{1}{2})$	$(-1+i0^+\,,\ -i0^+)$

Table 1: Limits of the four terms in the torus two-point function of twist fields.

For the first limit, $\Pi\to i\infty$ projects the solitonic sum to $p=0$. We see a dichotomy between rational and irrational r^2. If irrational, then the only term with $p=0$ is $p=\bar{p}=0$; if $r^2=u/v$ is rational, then $p=0$ corresponds to

$$(m,w)\in\{n\times(u,v)\mid n\in\mathbb{Z}\}\,. \tag{C.22}$$

So

$$\begin{aligned}
\lim_{\Pi\to i\infty} Z^{\rm cl}(r,\Pi,\bar\tau) &= \lim_{\Pi\to i\infty,\,\overline{\Pi}\to\bar\tau} \sum_{p,\bar p\in\Gamma_r} \exp\left[\frac{i\pi}{2}(p^2\Pi-\bar p^2\bar\tau)\right] \\
&= \lim_{\Pi\to i\infty} \sum_{m,w\in\mathbb{Z}} \exp\left[\frac{i\pi}{2}(\Big(\frac{m}{r}+wr)^2\Pi-(\frac{m}{r}-wr)^2\bar\tau\Big)\right] \\
&= \begin{cases} \displaystyle\sum_{n\in\mathbb{Z}} \exp\left[-2i\pi uvn^2\bar\tau\right] = \overline{\theta\begin{bmatrix}0\\0\end{bmatrix}}(0|2uv\bar\tau) & r^2=u/v \text{ rational}\,, \\ 1 & r^2 \text{ irrational}\,. \end{cases}
\end{aligned} \tag{C.23}$$

Next consider the third limit. The $\Pi,\overline{\Pi}$-dependent factors are the classical solitonic sum together with the denominator of (C.21),

$$\lim_{\Pi\to i0^+,\,\overline{\Pi}\to -1-i0^+} \frac{Z^{\rm cl}(r,\Pi,\overline{\Pi})}{\theta\begin{bmatrix}0\\0\end{bmatrix}(0\mid 2\Pi)\,\overline{\theta\begin{bmatrix}0\\0\end{bmatrix}}(0\mid 2\overline{\Pi})}\,. \tag{C.24}$$

The limit can be easily taken by first performing a modular transformation. Writing $\Pi'\equiv -1/\Pi$ and $\overline{\Pi}'\equiv -1/\overline{\Pi}$, and noting

$$Z^{\rm cl}(r,\Pi,\overline{\Pi}) = \sqrt{\Pi'\,\overline{\Pi}'}\times Z^{\rm cl}(r,\Pi',\overline{\Pi}')\,,\quad \theta\begin{bmatrix}0\\0\end{bmatrix}(0\mid 2\Pi) = \sqrt{\frac{\Pi'}{2}}\times\theta\begin{bmatrix}0\\0\end{bmatrix}(0\mid\frac{\Pi'}{2})\,, \tag{C.25}$$

(C.24) becomes

$$\begin{aligned}
&\lim_{\Pi\to i0^+,\,\overline{\Pi}\to -1-i0^+} \frac{Z^{\rm cl}(r,\Pi,\overline{\Pi})}{\theta\begin{bmatrix}0\\0\end{bmatrix}(0\mid 2\Pi)\,\overline{\theta\begin{bmatrix}0\\0\end{bmatrix}}(0\mid 2\overline{\Pi})} \\
&= \lim_{\Pi'\to i\infty,\,\overline{\Pi}'\to 1-i0^+} \frac{2\,Z^{\rm cl}(r,\Pi',\overline{\Pi}')}{\theta\begin{bmatrix}0\\0\end{bmatrix}(0\mid\frac{\Pi'}{2})\,\overline{\theta\begin{bmatrix}0\\0\end{bmatrix}}(0\mid\frac{\overline{\Pi}'}{2})} \\
&= \begin{cases} \displaystyle\lim_{\overline{\Pi}'\to 1-i0^+} \frac{2\,\overline{\theta\begin{bmatrix}0\\0\end{bmatrix}}(0|2uv\overline{\Pi}')}{\overline{\theta\begin{bmatrix}0\\0\end{bmatrix}}(0\mid\frac{\overline{\Pi}'}{2})} = \sqrt{\frac{2}{uv}}\,\exp(-\frac{i\pi}{4}) & r^2=u/v \text{ rational}\,, \\ \displaystyle\lim_{\overline{\Pi}'\to 1-i0^+} \frac{2}{\overline{\theta\begin{bmatrix}0\\0\end{bmatrix}}(0\mid\frac{\overline{\Pi}'}{2})} = 0 & r^2 \text{ irrational}\,. \end{cases}
\end{aligned} \tag{C.26}$$

Collecting everything, the final results are summarized as follows.

1. If $r^2 = u/v$ is rational, then the vacuum-normalized torus partition function twisted by $\mathcal{L}_{\mathcal{E}}$ is

$$\begin{aligned}\widehat{Z}^{\mathcal{L}}_{S^1_r/\mathbb{Z}_2}(\tau,\bar\tau) = \frac{1}{2\,|\eta(\tau)|^2}\Bigg\{ &\Bigg(\theta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(0\mid 2\tau)\,\overline{\theta\begin{bmatrix}0\\ 0\end{bmatrix}}(0\mid 2uv\bar\tau) + (\tau\leftrightarrow\bar\tau)\Bigg) \\ &+\sqrt{\frac{2}{uv}}\Bigg(\exp(-\frac{i\pi}{4})\,\theta\begin{bmatrix}\frac{1}{4}\\ \frac{1}{2}\end{bmatrix}(0\mid 2\tau)\,\overline{\theta\begin{bmatrix}\frac{1}{4}\\ 0\end{bmatrix}}(0\mid 2\bar\tau) + +(\tau\leftrightarrow\bar\tau)\Bigg)\Bigg\}.\end{aligned} \tag{C.27}$$

The modular S transform gives the vacuum-normalized defect partition function

$$\begin{aligned}\widehat{Z}_{\mathcal{L},\,S^1_r/\mathbb{Z}_2}(\tau,\bar\tau) = \frac{1}{4\sqrt{uv}\,|\eta(\tau)|^2}\Bigg\{ &\Bigg(\theta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(0\mid \frac{\tau}{2})\,\overline{\theta\begin{bmatrix}0\\ 0\end{bmatrix}}(0\mid \frac{\bar\tau}{2uv}) + (\tau\leftrightarrow\bar\tau)\Bigg) \\ &+\sqrt{2}\Bigg(\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{4}\end{bmatrix}(0\mid \frac{\tau}{2})\,\overline{\theta\begin{bmatrix}0\\ \frac{1}{4}\end{bmatrix}}(0\mid \frac{\bar\tau}{2}) + (\tau\leftrightarrow\bar\tau)\Bigg)\Bigg\}.\end{aligned} \tag{C.28}$$

Note that $\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{4}\end{bmatrix}(0\mid\frac{\tau}{2})$ defined in (C.2) as a q-series has coefficients that are $\frac{1}{\sqrt{2}}$ times integers due to the combinations of $e^{\pm\frac{i\pi}{4}}$, $e^{\pm\frac{3i\pi}{4}}$ phases. The $\frac{1}{\sqrt{2}}$ is compensated by the overall $\sqrt{2}$ factor to produce integer coefficients. The planar loop expectation value $\langle\mathcal{L}\rangle_{\mathbb{R}^2} = \sqrt{uv}$ is the smallest number such that

$$Z_{\mathcal{L},\,S^1_r/\mathbb{Z}_2}(\tau,\bar\tau) = \langle\mathcal{L}\rangle_{\mathbb{R}^2}\times\widehat{Z}_{\mathcal{L},\,S^1_r/\mathbb{Z}_2}(\tau,\bar\tau) \tag{C.29}$$

is Cardy-normalized, *i.e.* has a character expansion with positive integer coefficients.[13]

2. If r^2 is irrational, and suppose the twist field factorizes through some $\mathcal{L}$, then the torus Regge limit gives

$$\begin{aligned}\widehat{Z}^{\mathcal{L}}_{S^1_r/\mathbb{Z}_2} &= \frac{1}{2|\eta(\tau)|^2}\left(\theta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(0\mid 2\tau) + \overline{\theta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}}(0\mid 2\bar\tau)\right) \\ &= \frac{1}{2|\eta(\tau)|^2}\left(1+\sum_{n=1}^{\infty}(-)^n(q^{n^2}+\bar q^{n^2})\right).\end{aligned} \tag{C.30}$$

At $c=1$, an irreducible module with primary weight $h=n^2$ has a null state at level $2n+1$, so the degenerate character is

$$\chi_{h=n^2}(\tau) = \frac{q^{n^2}-q^{(n+1)^2}}{\eta(\tau)}\,. \tag{C.31}$$

---

[13] For each of the two pieces in braces in (C.28), there are terms in the q, $\bar q$-expansion with coefficient 2. But when the two pieces are combined, all terms have coefficients that are multiples of 4.

The character expansions of the partition function and twisted partition function are thus

$$
\begin{aligned}
Z_{S^1_r/\mathbb{Z}_2}(\tau,\bar\tau) &= \frac{1+\left|\theta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(0|2\tau)\right|^2}{2|\eta(\tau)|^2} + \cdots = \sum_{\substack{n,\bar n=0\\ n-\bar n\in 2\mathbb{Z}}}^{\infty} \chi_{h=n^2}\bar\chi_{\bar h=\bar n^2} + \cdots\,, \\
\widehat{Z}^{\mathcal{L}}_{S^1_r/\mathbb{Z}_2}(\tau,\bar\tau) &= \sum_{\substack{n,\bar n=0\\ n-\bar n\in 2\mathbb{Z}}}^{\infty} (-)^n \chi_{h=n^2}\bar\chi_{\bar h=\bar n^2}\,.
\end{aligned}
\tag{C.32}
$$

In other words, $\mathcal{L}$ annihilates all non-degenerate modules and acts on the degenerate modules by a sign. The modular transform of (C.30) is

$$
\begin{aligned}
\widehat{Z}_{\mathcal{L},\,S^1_r/\mathbb{Z}_2}(\tau,\bar\tau) &= \frac{1}{2\sqrt{2}|\eta(\tau)|^2}\left(\frac{1}{\sqrt{i\bar\tau}}\theta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(0\mid\frac{\tau}{2}) + \frac{1}{\sqrt{-i\tau}}\overline{\theta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(0\mid\frac{\bar\tau}{2})}\right) \\
&= \frac{1}{\sqrt{2}|\eta(\tau)|^2}\sum_{n=0}^{\infty}\Big(\frac{q^{\frac{(2n+1)^2}{16}}}{\sqrt{i\bar\tau}} + \frac{\bar q^{\frac{(2n+1)^2}{16}}}{\sqrt{-i\tau}}\Big) \\
&= \frac{1}{|\eta(\tau)|^2}\sum_{n=0}^{\infty}\int_0^{\infty} dp\left(q^{\frac{(2n+1)^2}{16}}\bar q^{\frac{p^2}{4}} + q^{\frac{p^2}{4}}\bar q^{\frac{(2n+1)^2}{16}}\right)\,.
\end{aligned}
\tag{C.33}
$$

The defect Hilbert space has a spectrum of primary operators continuous in twist,

$$
(h,\bar h) = (\frac{(2n+1)^2}{16},\,\frac{p^2}{4})\,,\ (\frac{p^2}{4},\,\frac{(2n+1)^2}{16})\,,\quad n\in\mathbb{Z},\ p\in\mathbb{R}\,.
\tag{C.34}
$$

The topological defect line has no well-defined $\langle\mathcal{L}\rangle_{\mathbb{R}^2}$.

Finally, the defect partition function (C.33) at an irrational point coincides with the $uv\to\infty$ limit of that (C.28) at rational points,

$$
\begin{aligned}
\lim_{uv\to\infty}\widehat{Z}_{\mathcal{L},\,S^1_r/\mathbb{Z}_2}(\tau,\bar\tau) &= \frac{1}{4|\eta(\tau)|^2}\lim_{uv\to\infty}\frac{1}{\sqrt{uv}}\left(\theta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(0\mid\frac{\tau}{2})\,\overline{\theta\begin{bmatrix}0\\ 0\end{bmatrix}(0\mid\frac{\bar\tau}{2uv})} + \text{c.c.}\right) \\
&= \frac{1}{4|\eta(\tau)|^2}\lim_{uv\to\infty}\frac{1}{\sqrt{uv}}\sum_{m,n\in\mathbb{Z}}\left(q^{\frac{(2n+1)^2}{16}}\bar q^{\frac{m^2}{4uv}} + q^{\frac{m^2}{4uv}}\bar q^{\frac{(2n+1)^2}{16}}\right) \\
&= \frac{1}{|\eta(\tau)|^2}\sum_{n=0}^{\infty}\int_0^{\infty} dp\left(q^{\frac{(2n+1)^2}{16}}\bar q^{\frac{p^2}{4}} + q^{\frac{p^2}{4}}\bar q^{\frac{(2n+1)^2}{16}}\right)\,.
\end{aligned}
\tag{C.35}
$$

The inverse planar loop expectation value $1/\langle\mathcal{L}\rangle_{\mathbb{R}^2}$, which is related to an F-symbol via (2.11), vanishes in this limit and transits to an integration measure for the p-integral over the continuous spectrum.

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