WARWICK
THE UNIVERSITY OF WARWICK

Picard group of K3 surfaces over finite fields

by

Francesco Broggi

Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Department of Mathematics

June 2020

Contents

Acknowledgments

I want to thank all the people who supported me throughout the years and by whom help I could get this thesis done.
First of all, I want to thank my supervisor Damiano Testa who introduced me to the world of arithmetic geometry and guided me thought those years. I also want to thank Samuele Anni for all the conversations we had while he was at Warwick.
I want to thank Samirk Siksek and Daniel Loughran for agreeing to read my thesis and be my examiners.
I then want to thank the number theory group and algebraic geometry group at Warwick. The seminars and study groups that were organized were a great chance to deepen my mathematical knowledge and gave me the possibility to study a great varieties of very interesting topics.
A special mention for John Cremona who game me access to the computational resources I needed for my thesis.
I want to thank also all the friends I met in Warwick. The italian crew: Cassa, Zampa, Chicco, Serena, Lorenzo, Bruno, Ilaria, Mattia, Alice, Livia, Pizzo, Ferdi, Alessandro, Paolo, Jay and Beniamino. We had a great time together, I am sure our friendship will last for long.
Then I cannot forget about the other PhD students I have met: Chris B., Florian, Ian, Alex W., Alex T., Matthew, Alejandro and Ros.
I can't forget about the friends I met while living on campus: Nicola, Markus, Rachele, Ben, Konstantin and all the others guys from WBS. It was really fun hanging out with you and get distracted from mathematics when I was getting too stressed.
A special mention to Gianlorenzo, Giovanni and Massimo who shared the house for me for almost 3 years. I enjoyed all the time we spent together, even playing complicated board games!
I also want to thank my old time friends: Mirko, Maurizio, Ellena, Carolina, Beppe. I

wouldn't be here if it wasn't for you.
My interest in algebraic geometry goes back to my studies at the University of Milan. Hence, I want to thank my former professors, especially Antonio Lanteri (who with his course in algebraic surfaces gave me the first glimpse of algebraic geometry), Bert Van Geemen and Fabrizio Andreatta who supervised my master and bachelor thesis respectively.
I also want to thank my former coursemates: Mattia, Giacomo, Claudia, Luca, Giulia, Rocco and Simone.
A big thank goes also to my mangers at TiQ, Albino and Alessandro, for having supported me to complete the PhD in the last two years while I was working for them.
Finally, I want to thank my family for the support I receive from them during all those years of study.

Declarations

The work presented in this thesis has been carried out under the supervision of Dr. Damiano Testa.
In Chapters 2 and Chapter 3 are collected some standard results in the field, which are not my own. I have throughout quoted the main sources I have used in the exposition.
In Chapter 4 I have exposed the algorithms we used alongside with some classical and new results to give a full literature review.
Chapter 5 starts from a paper by K. Kedlaya and A. Sutherland and uses their outcome to apply the results from Chapter 4.
I declare that the work presented is my own except for what mentioned above and when otherwise stated. This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree.

Abstract

The aim of this work is to provide a method to find explicitly generators for the Picard group of a K3 surface of degree four defined over a finite field.
This work has been motivated primarily by the difficulty related to this problem, and by the lack of examples in the literature.
Another question related to this problem is if given a K3 surface of degree 4 defined over a finite field, is it possible to determine the minimal degree of a non complete intersection curve lying over it.
We could not answer completely to this question, but we were still able to find algorithms to determine whether or not it contains a conic defined over an extension of the base field.
We were also able to determine an algorithm to find twisted cubics defined over the base field in case it is $\mathbb{F}_2$.

Chapter 1

Introduction

The study of K3 surfaces has seen many developments in the recent years.
While many problems have been solved and our understanding grown, there are still many open question on the arithmetic of those surfaces.
In particular, one big question regards divisors on K3s. In this thesis we will focus on this problem looking at quartic K3 surfaces defined over a finite field.
In this thesis we will try to dig into this problem focusing on K3 surfaces defined over $\mathbb{F}_2$.
Thanks to the work of Degtyarev, Rams, Schütt and Veniani we now have a complete picture of how many lines can be contained in a K3 surface and we also know which configurations of lines are admissible.
However, not much is know if we turn towards higher degree curves.
For example, even though there are examples in the literature of K3 surfaces in characteristic 0 with a big number of conics no reasonable upper bound is known.
Even less is know regarding non plane curves (such as twisted cubics).
Knowing divisors on a surface is a key ingredient when trying to compute the Picard rank. Even a small set of curves could potentially determine a finite index sublattice of the former.
Computing the Picard rank for a K3 surface X over $\mathbb{Q}$ is in most cases a very difficult question to answer. One has to find an upper and lower bound and hope to get the coinciding. The lower bound can be found by finding divisors and then use the Gram matrix.
On the other hand, reducing modulo a prime p we get another K3 surface X_p over a finite field whose Picard rank is greater or equal than $\rho(X)$. Hence, we can get an upper bound for the rank over $\mathbb{Q}$.
Computing the Picard rank over finite fields is normally easier than characteristic 0 since we can use Tate conjecture.
Namely, the factorization of the zeta function tells immediately not only the rank, but also the minimal field extension needed to achieve the geometric rank using a base field extension.
Starting from this, Van Luijk presented the first example of a K3 surface with Picard rank 1 also providing an algorithm to determine if the latter holds.

His method relies on finding two different primes p, q for which the reductions X_p, X_q have Picard rank 2 and different discriminants up to a square. Hence, he has to find a curve on each surface, then the computation of the discriminant is trivial.
Once the discriminants are known, if they lay in two different classes modulo squares, then the K3 has rank 1 over $\mathbb{Q}$.
The starting point for our calculations has been the Tate Conjecture. For K3 surfaces it is now a theorem, thanks to recent works by M. Lieblich, D. Maulik, F. Charles, W. Kim and W. Madapusi Pera. Using this result it is possible to determine the Picard number of a K3 surface defined over a finite field.
However, the problem of finding generators for such group still remains open.
We have first been able to write algorithms that make it possible to determine whether or not a smooth quartic in $\mathbb{P}^3(\mathbb{F}_p)$ contains lines and conics.
With regards to cubic curves, if we have a plane cubic lying in a quartic this would imply that we have a plane section made by the cubic and a line. Hence, when dealing with cubics we should only take into account twisted cubics.
For such curves we had to work in a slightly different way then before. We started fixing a model γ for such a curve and then we determined all possible smooth quartics containing it.
Then, we divided this set into PGL_4 orbits. The cardinality of each of them will be a multiple of the order of $\mathrm{Stab}(\gamma)$, and the latter is isomorphic to PGL_2 . For example, over $\mathbb{F}_2$ it has order 6.
So, we ended up determining the number of quartics in every orbit up to multiplication by an element of the stabilizer of γ.
All twisted cubics are projectively equivalent, hence the former is equivalent to the number of twisted cubics lying over a specific K3.
Once we have these informations it is possible to find the sublattice of the Picard group generated by curves of degree less or equal than 3.
We applied such algorithms to K3 surfaces of degree four defined over $\mathbb{F}_2$. Thanks to Kedlaya and Sutherland, we have the complete list of such surfaces up to PGL_4 action.
Morever, they also provide the list of associated zeta functions, enabling us to compute the geometric Picard number of all these surfaces.
Having the explicit zeta function we know how the Picard number increases over finite extensions of $\mathbb{F}_2$. In particular, we know on which extension the Picard group would be completely defined, making our computations substantially faster.
Using the previously described algorithms, we could determine curves of low degree lying on such surfaces and for many of these surfaces we could find a sublattice of finite index of the Picard group.
In order to achieve this for all the surfaces we would have to look for curves of higher degree, but up to now we have not found a suitable algorithm apart from brute force, which would be computationally unfeasible.

Chapter 2

K3 surfaces

1 Introduction to K3 surfaces

Suppose X is a smooth irreducible projective surface defined over a field K.

Definition 1.1.

- *A prime divisor on X is a curve Y on X. A Weil divisor is a finite formal linear combination $\sum_{i\in I} n_i D_i$ of prime divisors over $\mathbb{Z}$. The divisors form a group $Div(X)$.*
- *A Weil divisor is effective if $\forall i \in I, n_i \geq 0$.*
- *Let f be a rational function on X and v_Z be the discrete valuation defined by a prime divisor Z. We can then define $Div(f) = (f) := \sum_Z v_Z(f)Z$. Such divisors form a subgroup $Princ(X) \leq Div(X)$.*

We now introduce three different equivalence relations for divisors: linear, numerical and algebraic.

Definition 1.2.

- *Two divisors C, D are linearly equivalent if there exists a rational function f such that $C - D = div(f)$.*
- *The Picard group of X is defined as Pic(X):= $\frac{Div(X)}{Princ(X)}$, meaning that it is the group of divisors modulo linear equivalence.*
- *A divisor D is numerically equivalent to zero if $D.E = 0$ for all divisors E. D, E are numerically equivalent if their difference $D - E$ is numerically equivalent to zero.*
- *$Num(X)$ is the group of divisors modulo numerical equivalence.*

For *algebraic equivalence* we follow notation and results from [23].
Let X be a surface and T a non singular irreducible curve. An algebraic family of effective divisors on X is a Cartier divisor D on $X \times T$, flat over T. For any choice of two closed points in T, the two corresponding divisors on X are said to be prealgebraically equivalent. Two divisors C, D are algebraically equivalent if there exists a finite sequence of divisors $\{D_i\}, i \in \{1, \ldots, n\}$ where $D_0 = C$, $D_n = D$ and D_i prealgebraically equivalent to D_{i+1}.

The group of divisors modulo algebraic equivalence is called Néron-Severi group. Its rank, denoted $\rho(X)$ is called Picard number of X. We can then define $Div^0(X)$ to be the subgroup of divisors algebraically equivalent to 0, meaning that we may write $NS(X) = Pic(X)/Pic^0(X)$ where $Pic^0(X)$ is the image of $Div^0(X)$ inside $Pic(X)$.

Definition 1.3. *A K3 surface is a smooth, projective, surface X with trivial canonical sheaf $\omega_X = \mathcal{O}_X$ and irregularity $q = dim\ H^1(X, \mathcal{O}_X) = 0$.*

Definition 1.4. *The rank $\rho := rank(NS(X))$ is called the Picard number of X.*

Proposition 1.5. *[26] Prop 2.1*
The Neron Severi group of X is a finitely generated abelian group.

Proposition 1.6. *[26] Prop 2.4*
For a K3 surface X the following holds:

$$Pic(X) \cong NS(X) \cong Num(X).$$

Moreover, the intersection pairing on $Pic(X)$ is even, non-degenerate and of signature $(1, \rho(X) - 1)$.

For a K3 surface $\rho(X) \leq 22$. If char$K = 0$, the maximum value for the Picard number is 20. The K3 surfaces for which $\rho = 20$ are called *singular*.

Example 1.7. *Kummer surfaces.*
We shall consider an abelian variety A. Since A has a group law, we have an automorphism $i \in Aut(A) : i(x) = -x$ called natural involution.
Since $A = \mathbb{R}^4/\Lambda$, where Λ is a lattice, the set of the fixed point under i

$$Fix_i(A) = \{a : i(a) = a\} = \{a \in A \quad s.t. \quad 2a = 0\} = A[2].$$

is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$.
We can contruct the following commutative diagram (where $\widetilde{A}$ denotes the blow-up on the fixed points):

$$\begin{tikzcd}
A \arrow[d] & & \widetilde{A} \arrow[ll] \arrow[d] \\
A/i & \widetilde{A/i} \cong \widetilde{A}/\widetilde{i} =: Km(A) \arrow[l]
\end{tikzcd}$$

It can be proven that $Km(A)$ is a K3 surface.

A second set of examples comes from complete intersection of hypersurfaces.

Definition 1.8. *Let $X \subset \mathbb{P}^n$ be the m-dimensional variety given by complete intersection of $n - m$ hypersurfaces of degree d_j. Then X is a complete intersection of multidegree $(d_1, \ldots, d_{n-m})$.*

Using adjunction formula (i.e. $K_X = (K_Y \otimes X)|_X$ where X is a codimension 1 subvariety of Y) and the following classical result, we can find more examples of K3 surfaces.

Proposition 1.9. *If we are given $X \subset \mathbb{P}^n$ as in the previous definition, then for $0 < i <$ dim X we have $h^{i,0}(X) = 0$ and $\pi_1(X) = 0$.*

Proposition 1.10. *The complete intersections in $\mathbb{P}^n$ which are K3 surfaces are the following:*

1) $X_4 \subset \mathbb{P}^3$, i.e. a quartic in $\mathbb{P}^3$;

2) $X_{2,3} \subset \mathbb{P}^4$, i.e. the complete intersection of a quadric and a cubic;

3) $X_{2,2,2} \subset \mathbb{P}^5$, i.e. the complete intersection of three quadrics.

2 Singular K3 surfaces

As it was said before, a singular K3 surface has maximal Picard rank. The main example is probably the Fermat quartic.

Example 2.1. *The Fermat quartic*

$$S = \{x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\} \subset \mathbb{P}^3$$

is a singular K3 surface.
One way to prove it is by picking the 48 *lines on S and verify that the Gram matrix has rank* 20 *as shown in [58].*

The main invariant of a singular K3 surface is the *trascendental lattice*: consider the Neron-Severi group NS(X) as a lattice in $H^2(X, \mathbb{Z})$ with cup-product. Then we define the trascendental lattice $T(X)$ as the orthogonal complement

$$T(X) = NS(X)^{\perp} \subset H^2(X, \mathbb{Z}).$$

The Torelli theorem in the case of K3 surfaces becomes particulary interesting if the two surfaces are singular. More precisely, given two K3 surfaces X and Y, the Torelli theorem states that any Hodge isometry $H^2(X, \mathbb{Z}) \cong H^2(Y, \mathbb{Z})$ is induced by a unique isomorphism $X \cong Y$.
Using the fact that for K3 surfaces $H^2(X, \mathbb{Z})$ has rank 22 and signature $(3, 19)$, we can deduce that the trascendental lattice has rank $22 - \rho(X)$ and signature $(2, 20 - \rho(X))$. Hence, if X has maximal Picard rank $T(X)$ is a rank 2, even, positive-definite lattice,

which will be identified with a 2×2 matrix

$$T(X) \longleftrightarrow \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$$

where $a, b, c \in \mathbb{Z}$ and $d = b^2 - 4ac < 0$.
Hence, we can restate the Torelli theorem as follows:

Theorem 2.2. *Two singular K3 surfaces X, Y are isomorphic if and only if there is an isometry $T(X) \cong T(Y)$. Equivalentely the quadratic forms induced by the 2×2 matrices are conjugate under $SL(2, \mathbb{Z})$.*

We also have the following classification:

Theorem 2.3. *The map $X \mapsto T(X)$ gives a bijection*

$$\{\textit{Singular K3 surfaces}\}/\cong \longleftrightarrow \{\textit{positive-definite oriented even lattices of rank two}\}/\cong .$$

The injectivity comes from the restated Torelli theorem which says that two K3 surfaces X, Y are isomorphic if and only if exists and isometry $T(X) \cong T(Y)$.
The proof of the surjectivity is due to Shioda and Inose, who in [54] showed how to produce a singular K3 surface with given trascendental lattice. We will give just a sketch of their work. They started with an abelian variety A with $\rho(A) = 4$ and given quadratic form Q on the trascendental lattice $T(A)$. It was already known that in this case A is the product of two isogenous elliptic curves E, E' with CM in $\mathbb{Q}(\sqrt{d})$. Writing E, E' as complex tori we have $E = E_\tau$, $E' = E_{\tau'}$ where $\tau = \frac{-b+\sqrt{d}}{2a}$ and $\tau' = \frac{b+\sqrt{d}}{2}$. It follows that the Kummer surface of A is a singular K3 surface with intersection form $2Q$. In order to get a singular K3 with intersection form exactly Q, Shioda and Inose chose a particular elliptic fibration on $Km(A)$ that, with a suitable quadratic base change, proves the result.

3 Supersingular K3 surfaces

We have seen that over a field of characteristic zero the Picard number of a K3 surface is at most 20. This is no longer true over fields with positive characteristic.
In this case the Picard number can be equal to the second Betti number, i.e. $\rho(X) = b_2 = 22$. Such K3 surfaces are called *Shioda - supersingular*.
Here are some examples of Shioda-supersingular K3 surfaces X defined over K, $\mathrm{char}(K) = p$:

i) the Kummer surface $Km(E \times E)$ of a self-product of an elliptic curve is a supersingular K3 surface if and only if E is a supersingular K3 surface in characteristic $p > 2$;

ii) the Fermat quartic surface $\sum_{i=0}^{3} x_i^4 = 0$ is supersingular if and only if $p \equiv -1 (\text{mod } 4)$.

In [1], Artin studied such examples using the formal Brauer group.
We will now give the basic definition of the latter as stated in [26].
Let X be a scheme.

Definition 3.1. *An Azumaya algebra over X is an $\mathcal{O}_X$-algebra $\mathcal{A}$ that is coherent as an $\mathcal{O}_X$-module and étale locally isomorphic to the matrix algebra $M_n(\mathcal{O}_X)$.*
An Azumaya algebra is called trivial if it is isomorphic to $End(E)$ for some locally free sheaf E. Two algebras $\mathcal{A}_1$, $\mathcal{A}_2$ are equivalent if there exists locally free sheaves E_1, E_2 such that

$$\mathcal{A}_1 \otimes End(E_1) \cong \mathcal{A}_2 \otimes End(E_2)$$

as Azumaya algebras.
The set of such equivalence classes is the Brauer group of X under the group structure given by tensor product. The inverse A^{op} of an algebra is the $\mathcal{O}_X - algebra$ obtained by reversing the order of multiplication of A.

Such formal group, is smooth of dimension $h^{2,0} := \dim\,(X, \mathcal{O})$. By the p-rank of the kernel of the multiplication by p, it is possible to define the *height* h of the Brauer Group:

$$p^h = \#ker(p : Br(X) \longrightarrow Br(X)). \tag{2.1}$$

From a joint work ([2]), Artin and Mazure could prove that $\rho(X) \leq 22 - 2h$, hence either $h = \infty$ either $h \leq 10$. In [1], Artin then gave a definition for *Artin-supersingular* K3 surfaces.

Definition 3.2. *A K3 surface defined over a field of positive characteristic is supersingular if and and only if the height of its Brauer group is infinite.*

In this paper, Artin showed that Shioda-supersingular K3s are Artin-supersingular. The converse follows from the proof of Tate's conjecture (as we will show later on).

3.1 The Artin invariant

In his paper [1] from 1974, Artin proved that the moduli space of supersingular K3 surfaces is 9-dimensional and is stratified by the invariant σ_0.
Let X be a supersingular K3 surface over an algebraically closed field k of characteristic p, and let NS(X) be the Neron-Severi group of X. The discriminant of the intersection form on NS(X) is an even power of p:

$$\det \mathrm{NS}(X) = -p^{2\sigma_0}$$

and $\sigma_0 = \sigma_0(X)$ is a positive integer such that $1 \leq \sigma_0 \leq 10$. This integer is called the *Artin invariant* of X.

The stratification of this moduli space is given by

$$\{\sigma_0 = 1\} \subset \{\sigma_0 \leq 2\} \subset \cdots \subset \{\sigma_0 \leq 10\}.$$

Supersingular K3 surfaces with Artin invariant σ_0 form a $(\sigma_0 - 1)$-dimensional family.
We know give some recent results about supersingular K3 surfaces with Artin invariant 1.
Namely, there is a unique (up to isomorphism) supersingular K3 surface with $\sigma_0(X) = 1$ in characteristic p, for every prime.
This particular case have been studied in great detail for fields of small characteristic, but not much is known for supersingular K3 surfaces with big Artin invariant.
Start with the following definition

Definition 3.3. *Let X be a smooth projective surface and C a smooth curve. A genus one fibration is a surjective morphism $S \twoheadrightarrow C$ such that the generic fiber is a curve of arithmetic genus one.*

It is not hard to prove that any genus one fibration on a supersingular K3 surface of Artin invariant 1 admits a section, as shown in [15].
Moreover, for $p = 2$ and $p = 3$ we also know all the possible configurations of the singular fibers. Using this information Elkies and Schütt in [15] classified all the possible genus 1 fibrations of the supersingular K3 surface with $\sigma_0 = 1$ in characteristic 2.
In [49], Sengupta has done the same thing for $p = 3$.

Chapter 3

Tate Conjecture and Picard number for K3 surfaces

1 Tate Conjecture

1.1 Statement

There are many possible ways to express the Tate conjecture, we will follow the formulation by Totaro in [64].
Let X be a smooth projective variety defined over K, a finitely generated field either over $\mathbb{Q}$ or $\mathbb{F}_q$ for some $q = p^n$. Let K_s be the separable closure of K and $G := \mathrm{Gal}(K_s|K)$. We also give the following definition for the *Tate twists*:

Definition 1.1. *Let $\mu_{l^r}(K_s)$ be the group of the l^r roots of unity. We define the Tate twists $\mathbb{Z}_l(1)$ as the inverse limit of $\mu_{l^r}(K_s)$:*

$$\mathbb{Z}_l(1) = \varprojlim_n \mu_{l^n}$$

This means that $\mathbb{Z}_l(1) \cong \mathbb{Z}_l$ but the former has a non trivial action of G.

Consider now a closed subvariety Y of X with codim $(Y) = a$.
Then, Y determines an element of $H^{2a}(X, \mathbb{Z}_l(a))$. Actually, because we are considering subvarieties defined over K, the class of Y lies inside $H^{2a}(X \otimes K_s, \mathbb{Z}_l(a))^G$.
Tate conjecture statement is that the converse holds:

Conjecture 1.2. *Let K, K_s, X, G as before.*
Then, $H^{2a}(X \otimes K_s, \mathbb{Q}_l(a))^G$ is spanned by the classes of codimension-a subvarieties.
In degree two, this gives an isomorphism

$$NS(X) \otimes \mathbb{Q}_l \cong (H^2_{et}(X \times K_s, \mathbb{Q}_l(a)))^G.$$

1.2 History

Characteristic 0

The first setting for which the Tate conjecture for K3 surfaces was proven was in characteristic 0.
The main tool for this proof (and also for proofs for the finite field cases) is the *Kuga-Satake construction.*
We will recall the main definitions and properties as used in [64].
The main idea behind such a construction is to associate the $H^2(X, \mathbb{Z})$ of a K3 surface X with the $H^2(A, \mathbb{Z})$ of an abelian surface A. The reason why it is useful to have such a correspondence while working on the Tate conjecture goes back to the proof by Faltings in 1983 in [21].
He proved Mordell conjecture (which states that every curve of genus at least 2 over a number field has only finitely many rational points) and as part of the proof he proved that the Tate conjecture holds for divisors on an abelian variety defined over a number field.

Finite fields

The proof of the Tate conjecture for K3 surfaces over finite field has seen a lot of people beeing involved. The hardest case is that of supersingular K3 surfaces, a very special class of K3s which has a very different behavior from characteristic 0.
The first result was achieved by Artin and Swinnerton-Dyer in 1973 for K3 surfaces admitting an elliptic fibration.

Definition 1.3. *Let $f : A' \longrightarrow Y$ be a flat proper map of schemes such that every geometric fiber is one of the following:*

a) an elliptic curve;

b) a rational curve with a node;

c) a rational curve with a cusp;

we call such a map f a Weierstrass fibration.
One obtains $f : A' \longrightarrow Y$ from the minimal model A^* by contracting all components of fibers except for the ones containing the identity section. So, A^* is nonsingular and A' normal and we have a morphism $\pi : A^* \longrightarrow A'$ which is an isomorphism outside finitely many points of A (open set of smooth points of A').

This is their main theorem in [3]:

Theorem 1.4. *[3] Let $f : A' \longrightarrow Y$ be a Weierstrass fibration, where Y is the projective line over a finite field K, such that the associated nonsingular model A^* is a K3 surface. Then, if $l \neq p = char\ K$:*

$$\alpha \in H^2(\overline{A'}, \mathbb{Z}_l(1)) \otimes \mathbb{Q}_l$$

is invariant under the action of $G = Gal(\overline{K}|K)$, then it is in the image of $Pic(\overline{A}') \otimes \mathbb{Q}_l$.

Later on, in [46] Rudakov, Shafarevich and Zink proved the case of K3 surfaces of degree 2.
For non-supersingular K3 surfaces in characteristic $p > 5$, the Tate conjecture was proved by Nygaard and Ogus in 1985 in [40].
Their proof of the Tate conjecture relies on the existence of quasicanonical lifting. Roughly speaking, they constructed a lifting Y of a K3 surface X to $\mathbb{C}$ such that the action of the Frobenius map on the crystalline cohomology of X corresponds to an endomorphism of the Hodge structure of Y.
The supersingular case (in characteristic different from 2) has been proved in the last few years by Charles, Kim, Madapusi Pera and Maulik.
Finally, Kim and Madapusi Pera proved Tate conjecture for K3 surfaces defined over a field of characteristic 2 which was the last open case.
It is worth pointing out that for supersingular K3 surfaces, Tate conjectures would imply that the two definitions given in the previous chapter by Artin and Shioda coincide. Here is a sketchy explanation of how such equivalence is established.
The definition of Artin-supersingular makes use of the Brauer group, from which it is normally hard to extract any informations. Instead, it is possible to formulate an equivalent definition using the Frobenius action (see for example page 404 of [26] or section 5.1 in [64]). Namely, this is the definition of supersingular K3 surface given by F. Charles in [10]:

Definition 1.5. *Let X be a K3 surface defined over a finite field. We say that X is supersingular if the Frobenius morphism acts on the second étale cohomology group of X through a finite group.*

As a consequence, after passing to a certain finite extension, we would have that the Galois action on $H^2_{et}(\overline{X}, \mathbb{Q}_l(1))$ is trivial. Hence, Tate conjecture coincides with the claim that $\rho(X) = 22$.
This part of the Tate conjecture was the last the be proved. In fact, the argument of Nygaard and Ogus fails over supersingular (in the sense of Artin) K3 surfaces since the goal would be to show that the Picard number is 22, but any lifting of a K3 surface X to characteristic 0 would have rank 20 at most.
One important step was performed by Lieblich, Maulik, Snowden who gave an equivalent formulation for the Tate conjecture for K3 surfaces defined over finite fields of characteristic $p \geq 5$.

Theorem 1.6. *[32] Let K be a finite field of characteristic $p \geq 5$.*

1) *There are only finitely many isomorphism classes of K3 surfaces over K which satisfies the Tate conjecture over $\overline{K}$;*

2) If there are only finitely many isomorphism classes of K3 surfaces over the quadratic extension K' of K, then every K3 surface over K satisfies the Tate conjecture over K'.

In particular, if $p \geq 5$, the Tate conjecture holds for all K3 surfaces over $\overline{K}$ if and only if there are only finitely many K3 surfaces defined over each finite extension of K.

In 2012 François Charles proved the Tate conjecture for Artin-supersingular K3 surfaces defined over a field of characteristic greater than 3.
His main result is the following:

Theorem 1.7. *[10] Let X be a supersingular K3 surface defined over an algebraically closed field of characteristic $p \geq 5$. Then, the rank of the Néron-Severi has maximal possible value, which is 22 (meaning that Artin-supersingular implies Shioda-supersingular).*

Using results by Nygaard and Ogus this implies the following:

Theorem 1.8. *[10] The Tate conjecture holds for K3 surfaces defined over finite fields of characteristic $p \geq 5$.*
Thanks to [32] this implies that for every such field there exists only finitely many K3 surfaces up to isomorphism.

In 2014 Madapusi Pera proved Tate conjecture for fields K of odd characteristic, proving the case char $K = 3$.

Theorem 1.9. *[34] Let X be a K3 surface over a finitely generated field K of characteristic non equal to 2. Then the Tate conjecture holds for X.*
That means that for any given prime l invertible in K the l-adic Chern class map

$$ch : Pic(X) \otimes \mathbb{Q}_l \longrightarrow H^2_{et}(X_{sep}, \mathbb{Q}_l(1))^G$$

is an isomorphism. Here K^{sep} is a separable closure of K and $G = Gal(K^{sep}|K)$ is the associate absolute Galois group.

Finally, in 2015 Kim and Madapusi Pera completed the proof of Tate conjecture for K3 surfaces in chacteristic 2. They used E. Lau classification of 2-divisible groups to construct canonical models for Shimura varieties of abelian type and as an immediate application they proved

Theorem 1.10. *[30] The Tate conjecture holds for K3 surfaces over finitely generated fields.*

1.3 Zeta function and Weil conjectures

We now introduce some key results known as the *Weil conjectures* (which were proved by Deligne in the case of K3 surfaces) which will make it possible to give equivalent formulations of the Tate conjecture. This will make use of a fundamental tool, the *Hasse-Weil zeta function* of a K3.

Definition 1.11. *Let X be a smooth algebraic variety of dimension d over a finite field $\mathbb{F}_p$, its zeta function is defined by*

$$Z(X,t) := exp\Big(\sum_{n\geq 1} \#X(\mathbb{F}_{p^n})\frac{t^n}{n}\Big).$$

Theorem 1.12. *Suppose X is a smooth projective variety of dimension n over a finite filed $K := \mathbb{F}_q$ for $q = p^k$. Let $Z(X,T)$ be its zeta function. The latter satisfies the following:*

1) *Rationality: $Z(X,T)$ is a rational function of q^{-t};*

2) *Functional equation: $Z(X,T)$ satisfies a functional equation $Z(X, n-T) = \pm q^{E(\frac{n}{2}-T)}Z(X,T)$ for some integer E (which is actually the Euler characteristic);*

3) *Riemann Hypothesis: we can write the zeta function as*

$$Z(X,T) := \frac{P_1(T)P_3(T)\dots P_{2n-1}(T)}{P_0(t)P_2(T)\dots P_{2n}(T)}$$

where the $P_i(T)$ are integral polynomials with roots of absolute value $q^{-n/2}$. Also, if $l \neq p$ we have that

$$P_i(X,T) := det(1 - T Frob|H^i_{et}(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_l))$$

which implies that $P_0(T) = 1 - T$ and $P_{2n}(T) = 1 - q^nT$.

4) *Betti numbers: suppose R is a f.g. $\mathbb{Z}$-algebra with $R \twoheadrightarrow K$ and $R \hookrightarrow \mathbb{C}$ and $\mathcal{X}/Spec\ R$ is a smooth and proper scheme such that $X = \mathcal{X}_K$. Then, the degree of each P_i is the i-th Betti number of the manifold $\mathcal{X}(\mathbb{C})$.*

The rationality of the Zeta function and its functional equation had been proved by Dwork by 1960. The analogue of the Riemann hypothesis was eventually proved by Deligne in 1974, who had verified it for K3 surfaces a few years earlier.

1.4 Artin-Tate formula

Using the zeta function $Z(X,t)$, Tate was able to reformulate the conjecture in [62] as follows.

Theorem 1.13. *Let X be a surface defined over the finite field $\mathbb{F}_q$ and $Z(X,t)$ its zeta function. Thanks to Deligne, from [11], we know that the latter has the form*

$$Z(X,t) := \frac{P_1(t)P_3(t)}{(1-t)P_2(t)(1-q^2t)}.$$

Then, Tate conjecture holds if and only if the Picard rank $\rho(X)$ is equal to the multiplicity *of q ad a reciprocal root of $P_2(t)$. This is equivalent to say that the order of the pole of $Z(X,t)$ at $t = 1$ is equal to $\rho(X)$.*

This statement was then refined by Artin and Tate showing the connection between the Neron-Severi, the frobenius and the Brauer Group.

We can now give a compact form of the Tate conjecture which we will use for the rest of our work.

Theorem 1.14. *[62]*
Let X be a projective smooth surface over a finite field $\mathbb{F}_q$ and $P_2(X,T)$ as above.
Denote by ρ and Δ the rank and the discriminant of the Neron-Severi, respectively, and $Br(X)$ its Brauer group. Also, let $\alpha(X) := \chi(X, \mathcal{O}_X) - 1 + dim(Pic(X))$
Then,

(Tate Conjecture) $\rho(X)$ equals the multiplicity of q as a reciprocal root of $P_2(X,T)$;

$$(3.1)$$

(Artin-Tate) $\#Br(X)$ is finite and $$P_2(X,q^{-s}) \sim \frac{(-1)^{\rho(X)-1}|Br(X)|det(D_i \cdot D_j)}{q^{\alpha(X)}(NS(X):B)^2} \quad \text{as } s \longrightarrow 1. \tag{3.2}$$

where $D_1, \ldots, D_\rho$ are independent elements of NS(X) and $B := \sum \mathbb{Z}D_i$ is subgroup of NS(X) generated by D_i.

In 1975 in [36] Milne proved the equivalence between Tate conjecture and Artin-Tate formula.

Theorem 1.15. *Let X be a smooth projective surface over a finite field k of characteristic not equal to 2. If Tate conjecture holds, then also does Artin-Tate formula.*

Since our aim is to use Tate conjecture to determine the Picard rank of certain K3 surfaces, we need to compute efficiently get the coefficients of $P_2(X,T)$.
Van Luijk showed how to approach this problem in [65] using a result from [6], page 5.

Lemma 1.16. *Let V be a vector space of dimension n and T a linear operator on V. Let t_i denote the trace of T^i.*
Then, the characteristic polynomial of T is equal to

$$f_T(x) = det(x \cdot Id - T) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n, \tag{3.3}$$

where c_i are defined recursively by

$$c_1 := -t_1 \qquad \text{and} \qquad -kc_k = t_k + \sum_{i=1}^{k-1} c_i t_{k-i}. \tag{3.4}$$

For us, the polynomial f in the above Lemma will be $P_2(X,t) = \sum_{i=0}^{2} 2c_i x^i$.
To apply this lemma and get the coefficients of $P_2(X,t)$, we need to calculate the traces of the Frobenius operator. This can be achieved using the Lefschetz Trace formula:

Theorem 1.17. *Lefschetz Trace formula*
Let V be a n-dimensional projective variety with good reduction at a prime p, then for $l \neq p$ and $q = p^k$

$$\#V(\mathbb{F}_q) = \sum_{i=0}^{2n} (-1)^i \, Tr(Frob \ , H^i_{et}(V, \mathbb{Q}_\ell)).$$

For a K3 surface X the latter reads as follows:

$$Tr(Frob^k) = \#X(\mathbb{F}_{q^k}) - q^{2k} - 1. \tag{3.5}$$

Hence, we need to compute the coefficients t_k from 1.16, we need the number of points over extensions of the ground field.
In conclusion, via 1.16, we can calculate recursively the coefficients c_k.
Since point counting is normally very computationally expensive to achieve, one should avoid to compute all such coefficients directly.
Thanks to the second Weil conjecture, we can apply the following functional equation to relate the coefficient c_i to c_{22-i}:

$$q^{22} P_2(X,T) = \pm T^{22} P_2(X, q^2/T). \tag{3.6}$$

Suppose we have calculated the traces t_k for $k = 1, \ldots, 11$.We will use c_{11} to get rid of the sign ambiguity.
This is resolved as follows: if $c_{11} \neq 0$, it is possible to conclude that the sign of the functional equation is positive since if that happens, for both sides of 3.6 the coefficient of x^{11} is $c_{11}p^{22}$. Obviously, if the sing was negative such coefficient would be 0.
In contrast, if $c_{11} = 0$ we need to compute also c_{12} to get rid of the sign ambiguity by comparing the coefficients of x^{12}, i.e. solve $p^{22}c_{10} = \pm c_{12}p^{20}$.

2 Computing the Picard number

Thanks to the results we described in the previous section, we can now determine the Picard rank of a K3 surface X over $\mathbb{F}_p$.
Since we want to use Tate conjecture, we need to determine the zeta function of X.
For K3 surfaces we will apply a normalization of the zeta function, as given in [63], using the following isomorphism of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ modules, where μ_{l^n} is the group of l^n roots of unity:

$$H^2_{et}(\overline{X}, \mathbb{Q}_l(1)) \cong H^2_{et}(\overline{X}, \mathbb{Q}_l) \otimes_{\mathbb{Z}_l} (\mathbb{Q}_l \otimes_{\mathbb{Z}_l} \varprojlim \mu_{l^n}) \tag{3.7}$$

Let X be a K3 surface over a finite field $\mathbb{F}_q$. From its definition, we deduce that $P_0(X,T) =$

$P_3(X,T) = 1$. Hence, its zeta function is:

$$Z(X,T) = \frac{1}{(1-T)L(X,qT)(1-q^2T)} \tag{3.8}$$

where

$$L(X,T) := \det(1 - t\text{Frob}|H^i_{et}(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_l(1))) = \prod_{i=1}^{22}(1-\gamma_i T), \tag{3.9}$$

and γ_i have absolute value 1. Hence, in this case Tate conjecture says that the Picard number of X is determined by the number of γ_i that are roots of unity.
The first step will be to use a point counting algorithm to determine $\#X(\mathbb{F}_p)\dots\#X(\mathbb{F}_{p^{11}})$. There are different methods to get those numbers, here is a survey of some of those result based on [16].
The first approach one could think of to determine the number of rational points over a variety $X \subset \mathbb{P}^N(\mathbb{F}_{p^k})$ could be to determine all the $\mathbb{F}_q$ rational points, namely check for all the N-tuples $(x_0,\dots,x_{N-1})$ which ones satisfies the equation that define X. Clearly, for the purpose of computing the zeta function, the explicit calculation of the rational points is redundant. Moreover, the complexity of such method is $\mathcal{O}(p^{(N-1)k})$.
One could slightly improve the method when X is given by a polynomial $f := f_1(x_1,\dots,x_M) + f_2(x_{M+1},\dots,x_N)$. Using convolution it is possible to improve the above method to $\mathcal{O}(q^{max(M,N-M)k})$.
However, this method is still far from being computationally efficient.
To achieve better results one should use algorithms which do not require to find explicitly the rational points.
For example, Harvey's p-adic method as shown in [24].
He presented two algorithms: the first takes as input an arithmetic scheme X and computes $Z(X_{\mathbb{F}_P}, t)$ in time $p^{1/2+\mathcal{O}(1)}$. The second one computes the zeta function for all $p < N$ in time $Nlog^{3+\mathcal{O}(1)}(N)$.
The beauty of his method is that it is completely elementary and it is very general in theory. However, it probably can be implemented only for very specific varieties, such as K3 surfaces of degree 2, as shown in [16].
For specific types of K3s it is possible to determine an *ad hoc* algorithm. In chapter 4 we will have a close look at the algorithm implemented by K. Kedlaya and A. Sutherland for quartic surfaces which uses Zinoviev's formulas.
From point counting, we can then apply Lefschetz Trace formula to get the trace of $Frob^i$ for $0 < i \leq 11$. Then, applying Newton's identities we can obtain the first 11 coefficients for the Hasse-Zeta function of X.
Finally, the full equation for $Z(X,t)$ is given by its functional equation.
As we have seen at the beginning of the chapter, Tate's conjecture has been finally proven for all K3 surfaces defined over finite fields.

Hence, the following holds:

$$rank(NS(X \times_K \overline{K})) = \sum_{\chi} ord_{T=\chi/q}(P_2(X,t)) \tag{3.10}$$

where the sum is over the roots of unity χ.
However, using 3.9, this is equivalent to find the complex roots of unity of the normalization of $P_2(X,T)$ that we have called $L(X,T)$.
The same approach can be used to compute the Picard rank of X over $\mathbb{F}_{q^k}$ by simply counting the number of roots of $L(X,T)$ which belong to $\mathbb{F}_{q^k}$.
Namely, suppose we have $L(X,T) = f_{i_1}(T)^{e_1} \dots f_{i_n}(T)^{e_n} G(T)$ where f_j is the jth-cyclotomic polynomial and G the non cyclotomic factor of $P_2(X,T)$. Such decomposition is sometimes written (for example in [63]) as $L = L_{alg} L_{trc}$, with

$$\begin{aligned} L_{alg} &= \textstyle\prod_{\gamma_i \in \mu_\infty} (1 - T\gamma_i) & (3.11)\\ L_{trc} &= \textstyle\prod_{\gamma_i \notin \mu_\infty} (1 - T\gamma_i) & (3.12) \end{aligned}$$

where μ_∞ is the group of complex roots of unity.
Then we have that $\rho(\overline{X}) = \sum_j \deg f_j \cdot e_j = 22 - \deg(L_{trc})$ and the minimal degree of $\mathbb{F}_q$ to achieve the geometric rank is $N = l.c.m(i_1, \dots, i_N)$.
We will show some explicit examples in the next chapter.

Chapter 4

Divisors on K3 surfaces

We now show our methods to detect all lines and conics over smooth quartic surfaces. For curves of degree 3, we could find a method to find twisted cubics if they are define over the base field.

For every family of curves we show the method and the code used. The computer program on which we implemented our algorithms is Magma.

We also add some examples of these computations. These examples come from the database of quartics over $\mathbb{P}^3(\mathbb{F}_2)$ provided by Kedlaya and Sutherland. What made their list of quartic K3s very useful for us, was that they also provided the computation for zeta function of every quartic.

However, our methods can be applied for any characteristic $p > 0$.

1 Lines

1.1 Literature review

Probably, the most famous examples concerning lines contained in an algebraic surface is the case of smooth cubics in $\mathbb{P}^3$.

Proposition 1.1. *Let S be a smooth cubic surface defined over $\mathbb{P}^3(K)$ where K an algebraically closed field. Then, S contains 27 lines.*

Moving towards K3 smooth quartics, the first attempt was made by Segre in 1945 [48] where he claimed that a line on a smooth complex quartic intersects at most 18 other lines. This result was used to prove that the maximum number of lines would be 64. However, as pointed out by Rams and Schütt in [43], a mistake in his arguments actually yields a bound of 72 lines on such a surface.

The erroneous claim was that a line on a complex quartic surface does **not** intersect at most 18 other lines.

This problem was then studied by S. Rams and M. Schütt. In their first joint paper [43] they dealt with smooth quartic surfaces defined over a field of characteristic $p \geq 0, p \neq 2, 3$. In particular, they proved the following:

Proposition 1.2. *[43]*

1) *A line l on a geometrically smooth quartic surface S in $\mathbb{P}^3(K)$ intersect at most 20 other lines provided that char $(K) \neq 2, 3$.*

2) *If l meets more than 18 lines on S, then S can be given by a quartic polynomial*

$$x_3x_1^3 + x_4x_2^3 + x_1x_2q(x_3, x_4) + x_3g(x_3, x_4)$$

where $q, g \in \overline{K}[x_3, x_4]$ are homogeneous of degree 2 resp. 3.

3) *The line $l = \{x_3 = x_4 = 0\}$ meets 20 lines on S if and only if $g_4|g$.*

This result leads to their main theorem:

Theorem 1.3. *Let K be a field of characteristic $p \geq 0$ with $p \neq 2, 3$. Then any geometrically smooth quartic surface over K contains at most 64 lines.*
Also, this bound is sharp since in any characteristic not 2 and 3 such bound can be attained by the Schur's quartic

$$\{x^4 - xy^3 = z^4 - zw^3\} \subset \mathbb{P}^3.$$

In characteristic 3 the Schur's surface contains 112 lines, motivating Rams and Schütt towards a study of the two remaining characteristics:

Theorem 1.4. *[42] Let K be a field of characteristic 3. Then, any smooth quartic surfaces over K contains at most 112 lines.*

Proposition 1.5. *[42] Let K be a field of characteristic 2. Then, any smooth quartic surface over K contains at most 84 lines.*

As they pointed out in the paper, they could not determine any example to show that bound could have been sharp.
Infact, thanks to a subsequent paper, they improved such bounds as follows:

Theorem 1.6. *[44] Let K be a field of characteristic 2. Then, any smooth quartic surface over K contains at most 64 lines.*

Again, the bound they found was not sharp (we will see how it has been improved to 60). In section 8 of the paper they gave an explicit example of a smooth quartic containing 60 lines over $\mathbb{F}_{16}$.
Namely, they considered geometrically irreducible quartic surfaces with an action by the symmetric group S_5.
Such surfaces form a one-dimensional pencil that can be written as $S_\mu = \{s_1 = s_4 + \mu_2^2\}$ in $\mathbb{P}^4$.
Choosing $\mu_0 = 1 + \alpha^2 + \alpha^3$ (where α is the fifth rooth of unity) it can be showed that S_{μ_0} contains the line given by $\{s_1 = x_3 + x_2 + (\alpha^3 + \alpha + 1)x_1 = x_4 + (\alpha^3 + \alpha^2 + \alpha + 1)x_2 + \alpha x_1 = 0\}$ and its entire S_5-orbit (which consists of 60 lines).

While Rams and Schutt focused on smooth surfaces, D.C. Veniani worked with quartic surfaces admitting isolated rational double points over an algebraically closed field. He started considering the field of definition to have characteristic different from $2, 3$.
His main result was

Theorem 1.7. *[67] Let K be an algebraically closed field of characteristic $p \geq 0, p \neq 2, 3$ and let $X \subset \mathbb{P}^3(K)$ be a surface of degree 4 over K admitting only isolated rational double points and singularities. Then, X contains at most 64 lines.*

He then worked on characteristic 2 and 3. In the first case, he was able to prove that:

Theorem 1.8. *[66] Let X is a K3 quartic surface (hence, a quartic surface in $\mathbb{P}^3$ with at most isolated rational double points as singularities) defined over a field of characteristic 2. Then, X contains at most 68 lines.*
Such bound is attained by a one dimensional family $\mathcal{X}_{68}$ with one type A_3 singularity. If X contains 68 lines, it is projectively equivalent to a member of such family.

$$X_{68} := \lambda x_0 x_1^2 x_2 + x_1^4 + x_1 x_2^3 + x_0^3 x_3 + x_0 x_2 x_3^2 = 0$$

For char $K = 3$ he managed to improve the results of [42]:

Theorem 1.9. *[68] If X is a K3 quartic surface defined over a field of characteristic 3, then X contains 112 lines or at most 67. Moreover, if X contains 112 lines it means that X is projectively equivalent to the Fermat quartic surface.*

A similar improvement for char $K = 2$ was found by Degtyarev in [12]. He managed to get sharp bounds for the number of lines contained in a K3 surface defined over a field of characteristic 2 or 3. Moreover, he was able to determine accurate bounds in the case of supersingular surfaces.
Let X be a non singular quartic surface defined over $\mathbb{P}^3(K)$ where $K = \overline{K}$. Also, let Fn X be the set of lines contained in X.

Theorem 1.10. *[12] Assume that char $K = 2$ and X is supersingular. Then either $|Fn\ X| = 40$, and there are five configurations or $|Fn\ X| \leq 32$, and this bound is sharp.*

Theorem 1.11. *[12] Assume that char $K = 3$ and X is supersingular. Then either $|Fn\ X| = 112$ and X is the Fermat quartic or $|Fn\ X| = 58$ and there are three configurations or $|Fn\ X| \leq 52$, and this bound is sharp.*

Theorem 1.12. *[12] Assume that char $K = 2, 3$ and X is not supersingular. X contains at most 60 lines.*

1.2 Determining lines on a K3 surface

Below we give a description of the method we followed to explicitly determine the lines contained in a K3 surface defined over a finite field.

Suppose now we are looking at a quartic $\mathcal{X} : f(W, X, Y, Z) = 0$ in $\mathbb{P}^3(\mathbb{F}_q)$. Using the Tate conjecture we know its Picard rank and the degree N of the minimal extension for which it is attained.
Every line is given by a parametrization of the form

$$\begin{aligned} \mathcal{L} : \mathbb{A}^1 &\longrightarrow \mathbb{P}^3 \\ [1, t] &\longmapsto [a_0 + b_0 t : a_1 + b_1 t : a_2 + b_2 t : a_3 + b_3 t]. \end{aligned}$$

We will divide all possible lines into 6 families :

$A_1 := \{\mathcal{L}$ given by $[1 : t : a_0 t + a_1 : a_2 t + a_3]$ for some $a_i \in \mathbb{F}_q, i = 0, 1, 2, 3\}$;

$A_2 := \{\mathcal{L}$ given by $[1 : a_0 : t : a_1 t + a_2]$ for some $a_i \in \mathbb{F}_q, i = 0, 1, 2\}$;

$A_3 := \{\mathcal{L}$ given by $[1 : a_0 : a_1 : t]]$ for some $a_i \in \mathbb{F}_q, i = 0, 1\}$;

$A_4 := \{\mathcal{L}$ given by $[0 : 1 : t : a_0 t + a_1]$ for some $a_i \in \mathbb{F}_q, i = 0, 1\}$;

$A_5 := \{\mathcal{L}$ given by $[0 : 1 : a_0 : t]$ for $a_0 \in \mathbb{F}_q\}$;

$A_6 := \{\mathcal{L}$ given by $[0 : 0 : 1 : t]\}$.

Indeed we will not only be interested in lines defined over the ground field, but over any finite extensions of $\mathbb{F}_q$.
Suppose we want to determine if $\mathcal{X}$ contains lines belonging to the first family.
Then, for the general line $\mathcal{L}$ in A_1, we want to determine whether or not there is a choice of coefficients such that $g(t) := f(\mathcal{L}(1, t))$ is identically zero.
Since the scheme obtained from the coefficients of g must have dimension 0 (or -1 meaning that it is empty), its degree would give us the number of points. Such points will be defined over an extension of the ground field which still needs to be determined. We show in a moment how we managed to find it.
The following code gives us the number of lines in each family A_i.

```
LINES:=function(f,N)
                    K:=GF(q);
                    R<W,X,Y,Z>:=PolynomialRing(K,4);
                    CountLines:=[];


//first family
          AR0<[a0]>:=PolynomialRing(K,4);
          A0:=Spec(AR0);
          T0<t0>:=PolynomialRing(AR0);
          eva0 := hom< R->T0 | [1,t0,a0[1]*t0+a0[2],a0[3]*t0+a0[4]] >;
          coef:=Coefficients(eva0(equ));
          Fa0:=Scheme(A0,coef);
```

```
        DimDeg:=[Dimension(Fa0),Degree(Fa0)];
        Append(~CountLines,DimDeg);
end function;
```

And then we have to repeat this procedure for the other 5 families.
Once we have computed the lines, we need to find their defining equations, since our aim is to compute the intersection matrix of the sublattice of the Picard group.
Here is our general idea.
First of all, we need to find on which extension of $\mathbb{F}_q$ the lines are defined. We will work on each family separately.
Secondly, given a rational point on the corresponding affine scheme, we determine the three points of $\mathbb{P}^3$ that will generate the subscheme.
Finally, we will write the equations of the two plane whose intersection is the line we are looking for.
We now explain in more details those three steps.
First of all we want to determine the smallest $\mathbb{F}_q$ finite extension on which all the lines are defined. This is required for the computation of the intersection numbers between divisors to obtain the Gram matrix of the K3 surface X.
Thanks to the Tate conjecture, we know the minimal field extension degree N for the Picard lattice to be fully defined. However, the lines could be defined over a smaller field whose degree will be denoted Min_Exp.
We definitely could avoid this discussion and use directly N to define the lines (and later conics and twisted cubics), but this would be computationally inefficient.
Hence, to determine the minimal field of definition of the set of lines for X, we simply compared the number of rational points of the base change of the scheme $Fa0$ with its degree.
We already know that if there is any rational point, it would be defined over $\mathbb{F}_{2^N}$. To get the minimal extension, we looped over its subfields, meaning all the $\mathbb{F}_{q^k}$ where k divides N, to find the smallest one on which the number of rational points is equal to the degree of the scheme.

```
lines:=[];
Div:=Divisors(N);
n:=1;
while #RationalPoints(BaseExtend(Fa0,GF(2^Div[n]))) ne Degree(Fa0)
        do n+:=1;
end while;
Min_Exp:=Div[n];
```

We now construct the rational point of the base change over the extension previously determined.

```
BC0:=BaseExtend(Fa0, GF(2^Min_Exp));
points0:=RationalPoints(BC0);
```

```
for P in points0 do
Space<W,X,Y,Z>:=ProjectiveSpace(GF(2^(Min_Exp)),3);
```

Once we have a rational point P we are able to determine the two points that generate the affine line $\mathcal{L}$. However, since we want to work with projective schemes, we need to find a third point that combined with the other two generates the line in $\mathbb{P}^3$.

```
//we have now to determine the third point
M01:=Transpose(Matrix([[1,0,P[1],P[4]],[1,1,P[1]+P[2],P[3]+P[4]],[0,0,1,0]]));
m01:=Determinant(Matrix([M01[2],M01[3],M01[4]]));
m02:=Determinant(Matrix([M01[1],M01[3],M01[4]]));
m03:=Determinant(Matrix([M01[1],M01[2],M01[4]]));
m04:=Determinant(Matrix([M01[1],M01[2],M01[3]]));
if m01 ne 0 then
        base:=[1,0,0,0];
elif m02 ne 0 then
        base:=[0,1,0,0];
elif m03 ne 0 then
        base:=[0,0,1,0];
else
        base:=[0,0,0,1];
end if;
```

Once this is done, using the minors of the matrix of coefficients of the three points we obtain equations for two planes whose intersection is $\mathcal{L}$.

```
M02:=Transpose(Matrix([[1,0,P[2],P[4]],[1,1,P[1]+P[2],P[3]+P[4]],base]));
n01:=Determinant(Matrix([M02[2],M02[3],M02[4]]));
n02:=Determinant(Matrix([M02[1],M02[3],M02[4]]));
n03:=Determinant(Matrix([M02[1],M02[2],M02[4]]));
n04:=Determinant(Matrix([M02[1],M02[2],M02[3]]));
M:=Scheme(Space,[m01*W+m02*X+m03*Y+m04*Z,n01*W+n02*X+n03*Y+n04*Z]);
Append(~lines,M);
end for;
```

then do the same for the other families.

1.3 Examples

We now give some examples of lines contained in smooth quartic K3 surfaces defined over $\mathbb{F}_2$.

Remark 1.13. *Let X be such a K3 surface defined over $\mathbb{F}_p$, where p is a prime, and l a line contained in X such that it is defined over $K := \mathbb{F}_{p^N}$.*
Then, the action of Frobenius (and its powers) gives another line still contained in X.

Namely, suppose that $K^* :=< \zeta >$ as a cyclic group, we have that $Frob^i : \zeta \mapsto \zeta^{p^i}$ for $1 \le i \le N-1$. Then, $Frob^i(l)$ is still a line and it is contained in X.
This result can also be applied to divisors of higher degree.

We shall start considering the following example.
Take $\mathcal{X}_1$ defined by:

$$f_1 = W^3Z + WXYZ + WZ^3 + X^4 + X^2YZ + XY^3 + XYZ^2 + XZ^3 + Y^4 + Y^3Z + Z^4.$$

This a quartic K3 surface the characteristic polynomial of the Frobenius is

$$\phi_{\mathcal{X}_1}(x) = (x-1)^3(x+1)^5(x^2+1)^5(2x^4 - x^3 - x + 2)$$

and hence it has geometric Picard number $1 \cdot 3 + 1 \cdot 5 + 5 \cdot 2 = 18$ and this rank is achieved lifting the surface over $K := \mathbb{F}_{2^4}$ since $l.c.m.(1,2,4) = 4$.
From the first part of the code we can see that the sum of the degrees of the six schemes $Fa1, Fa2, \dots, Fa6$ is 40, giving us the number of lines over K. If we look for lines defined over subfields of K we end up not finding any. Hence, we are forced to work over K. With z we denote a generator of $K^* \simeq \mathbb{Z}/15$ as a cyclic group.
As it was said in remark 1.13, we don't need to give the entire list, only one generator per Frobenius orbit.
For sake of clarity, we start giving one explicit example of such orbit.
Let L_0 be the scheme over $\mathbb{F}_{2^4}$ given by

$$W + z^7X + Z;$$
$$z^{14}X + Y + z^7Z$$

The Frobenius morphism acts by $Frob : z \mapsto z^2$, hence we get the line L_1 given by:

$$W + z^{14}X + Z;$$
$$z^{13}X + Y + z^{14}Z$$

Similarly, via $Frob^2, Frob^3$ we get

$$L_2 := V(W + z^{13}X + Z, z^{11}X + Y + z^{13}Z);$$
$$L_3 := V(W + z^{11}X + Z, z^7X + Y + z^{11}Z)$$

Here is the list of the orbits (for each of them we get 4 lines, hence 40 in total):

Scheme over $GF(2^4)$ defined by $W + z^7X + Z; z^{14}X + Y + z^7Z$;
Scheme over $GF(2^4)$ defined by $zW + z^7X + Z; X + zY + z^6Z$;
Scheme over $GF(2^4)$ defined by $z^3W + z^7X + Z; z^{10}X + z^3Y + z^9Z$;

Scheme over $GF(2^4)$ defined by $Z; z^7X + Y$;

Scheme over $GF(2^4)$ defined by $W + Z; z^7X + Y + z^{11}Z$;

Scheme over $GF(2^4)$ defined by $W + zY + Z; X + z2^Y + zZ$;

Scheme over $GF(2^4)$ defined by $zW + z^2Y + Z; zX + z^2Y + Z$;

Scheme over $GF(2^4)$ defined by $zW + z^9X + Z; z^{12}X + zY + z^6Z$;

Scheme over $GF(2^4)$ defined by $z^5W + z^2X + Z; zX + z^5Y + z^{10}Z$;

Scheme over $GF(2^4)$ defined by $z^7W + zX + Z; z^6X + z^7Y + Z$.

Second example:

Consider now the quartic $\mathcal{X}_0$ over $\mathbb{F}_2$ given by

$$f_0 := W^3Z + WX^2Z + WXY^2 + WXYZ + WYZ^2 + X^3Y + X^2YZ + XY^3 + Y^2Z^2 + Z^4.$$

The characteristic polynomial of Frobenius is $\phi_{\mathcal{X}_0}(x) := (x+1)(x-1)^5(x^2+1)(2x^2+3x+2)(x^6+x^3+1)^2$.

Hence, we can deduce that its geometric Picard number is $1+5+2+6\cdot 2 = 20$ and such group would be defined over and extension of degree $l.c.m.(1, 2, 4, 9) = 36$. Here are the number of lines over all the subextensions:

k	Number of Lines
1	2
2	4
3	2
4	8
6	4
9	11
12	8
18	13
36	18

We write below the lines defined over extensions of $\mathbb{F}_2$ up to degree 12. For any of these fields K_i we denote by z the generator of K_i^* as a cyclic group.

Scheme over $\mathbb{F}_2$ defined by Z, Y;

Scheme over $\mathbb{F}_2$ defined by Z, X.

Scheme over $\mathbb{F}_{2^2}$ defined by Z, Y;

Scheme over $\mathbb{F}_{2^2}$ defined by $zW + z^2X + Z, zY + Z$;

Scheme over $\mathbb{F}_{2^2}$ defined by $z^2W + zX + Z, z^2Y + Z$;

Scheme over $\mathbb{F}_{2^2}$ defined by Z, X.

Scheme over $\mathbb{F}_{2^3}$ defined by Z, Y;

Scheme over $\mathbb{F}_{2^3}$ defined by Z, X.

Scheme over $\mathbb{F}_{2^4}$ defined by Z, Y;

Scheme over $\mathbb{F}_{2^4}$ defined by $z^5W + z^{10}X + Z, z^5Y + Z$;

Scheme over $\mathbb{F}_{2^4}$ defined by $z^6W + z^2X + Z, z^6Y + z^{11}Z$.

Scheme over $\mathbb{F}_{2^4}$ defined by $z^{10}W + z^5X + Z, z^{10}Y + Z$;

Scheme over $\mathbb{F}_{2^4}$ defined by $z^3W + zX + Z, z^3Y + z^{13}Z$;

Scheme over $\mathbb{F}_{2^4}$ defined by $z^9W + z^8X + Z, z^9Y + z^{14}Z$;

Scheme over $\mathbb{F}_{2^4}$ defined by $z^{12}W + z^4X + Z, z^{12}Y + z^7Z$;

Scheme over $\mathbb{F}_{2^4}$ defined by Z, X.

Scheme over $\mathbb{F}_{2^6}$ defined by Z, Y;

Scheme over $\mathbb{F}_{2^6}$ defined by $z^{21}W + z^{42}X + Z, z^{21}Y + Z$;

Scheme over $\mathbb{F}_{2^6}$ defined by $z^{42}W + z^{21}X + Z, z^{42}Y + Z$;

Scheme over $\mathbb{F}_{2^6}$ defined by Z, X.

Scheme over $\mathbb{F}_{2^9}$ defined by Z, Y;

Scheme over $\mathbb{F}_{2^9}$ defined by $z^{151}W + z^{330}X + Z, z^{151}Y + z^{443}Z$;

Scheme over $\mathbb{F}_{2^9}$ defined by $z^{302}W + z^{149}X + Z, z^{302}Y + z^{375}Z$;

Scheme over $\mathbb{F}_{2^9}$ defined by $z^{372}W + z^{170}X + Z, z^{372}Y + z^{445}Z$;

Scheme over $\mathbb{F}_{2^9}$ defined by $z^{331}W + z^{165}X + Z, z^{331}Y + z^{477}Z$;

Scheme over $\mathbb{F}_{2^9}$ defined by $z^{186}W + z^{85}X + Z, z^{186}Y + z^{478}Z$;

Scheme over $\mathbb{F}_{2^9}$ defined by $z^{421}W + z^{338}X + Z, z^{421}Y + z^{494}Z$;

Scheme over $\mathbb{F}_{2^9}$ defined by $z^{93}W + z^{298}X + Z, z^{93}Y + z^{239}Z$;

Scheme over $\mathbb{F}_{2^9}$ defined by $z^{466}W + z^{169}X + Z, z^{466}Y + z^{247}Z$;

Scheme over $\mathbb{F}_{2^9}$ defined by $z^{233}W + z^{340}X + Z, z^{233}Y + z^{379}Z$;

Scheme over $\mathbb{F}_{2^9}$ defined by Z, X.

Scheme over $\mathbb{F}_{2^{12}}$ defined by Z, Y;

Scheme over $\mathbb{F}_{2^{12}}$ defined by $z^{1365}W + z^{2730}X + Z, z^{1365}Y + Z$;

Scheme over $\mathbb{F}_{2^{12}}$ defined by $z^{1638}W + z^{546}X + Z, z^{1638}Y + z^{3003}Z$;

Scheme over $\mathbb{F}_{2^{12}}$ defined by $z^{2730}W + z^{1365}X + Z, z^{2730}Y + Z$;

Scheme over $\mathbb{F}_{2^{12}}$ defined by $z^{819}W + z^{273}X + Z, z^{819}Y + z^{3549}Z$;

Scheme over $\mathbb{F}_{2^{12}}$ defined by $z^{2457}W + z^{2184}X + Z, z^{2457}Y + z^{3822}Z$;

Scheme over $\mathbb{F}_{2^{12}}$ defined by $z^{3276}W + z^{1092}X + Z, z^{3276}Y + z^{1911}Z$;

Scheme over $\mathbb{F}_{2^{12}}$ defined by Z, X

2 Conics

2.1 Literature review

As we have seen in the previous section, thanks to the recent works we do have a sharp bound for the maximum number of lines which a smooth quartic surface may contain.
For conics such a bound is still unknown.
Here are some examples of quartic surfaces containing a big number of conics.
In 1994, W.Barth and T.Bauer in [4] determined quartics with 352 conics.
The following year, T.Bauer in [5] found a two-dimensional family of quartics in $\mathbb{P}^3$ containing 16 mutually disjoint conics and altogether exactly 432 conics.
He also states in his paper that a Chern class computation, by the methods of [27], would show that the maximal number of conics contained in a smooth quartic in $\mathbb{P}^3$ should be 5016. However, this bound is of course far from being sharp.
Define $G \leq \mathrm{Aut}(\mathbb{P}^3(\overline{\mathbb{Q}}))$ the subgroup generated by the following transformations:

$$[x : y : z : w] \mapsto [y : x : w : z], [z : w : x : y], [x : y : -z : -w], [x : -y : z : -w].$$

The family of quartic surfaces invariant under the action of G can be parametrized by $\mathbb{P}^4$ and it was shown by Eklund [13] that the general element of such family contains at least 320 conics.
The same family was studied by F. Bouyer to answer the question of what is the smallest extension for the conics on it to be defined in case such a surface is defined over a number field [7].
He was also able to show in another paper [8] that for some families of quartics invariant under G the general element has Picard group generated by lines and conics.
We also have the recent paper by J.A.D. Maia, A. R. Silva, I. Vainsencher and F. Xavier [35] where they proved the following:

Theorem 2.1. *[35] For all $d \geq 5$ the locus of surfaces in $\mathbb{P}^3$ of degree d containing a conic* *is a variety of codimension* $2d-7$ *and degree*

$$\binom{d}{4}(d^2-d+8)\cdot(d^2-d+6)\cdot$$
$$(207d^8-288d^7+498d^6+5068d^5-15693d^4+31732d^3-37332d^2+9280d-47040)/967680. \tag{4.1}$$

For $d=4$ *the correct degree is 5016/2 due to Bezout.*

In order to determine all conics contained in a quartic K3 surface, we adapted what we have done to find lines to this case.
Suppose we have $\mathcal{X} : \{f(w,x,y,z)=0\}$ a smooth quartic in $\mathbb{P}^3(\mathbb{F}_q)$. Take $k \in \mathbb{N}$ and π a plane in $\mathbb{P}^3_{\mathbb{F}_{q^k}}$.
We want to see if exists π such that $f|_\pi = q_1(w,x,y,z)\cdot q_2(w,x,y,z)$ where at least one of the $q_i \in \mathbb{F}_{q^k}$ is an irreducible homogeneous polynomial of degree 2.
Indeed, we can either have a section given by two conics or by one conic and two lines.
We will divide all the possibilities into 16 families. This number comes from the total number of possible plane sections and the number of potentially irreducible conics lying over the section.
Namely, we can cut the quartic with four families of planes in $\mathbb{P}^3(\mathbb{F}_q)$:

plane: $w = a_0\cdot x + a_1\cdot y + a_2\cdot z$;

plane: $x = a_0\cdot y + a_1\cdot z$;

plane: $y = a_0\cdot z$;

plane: $z = 0$;

Once we have a plane quartic $Plane\_Qua := Plane\_Qua(x_1,x_2,x_3)$, all possible conics that can occur as a factor of this plane quartic have to belong to one of the following families:

$C_0 : x_1^2 + a_1\cdot x_1\cdot x_2 + a_2\cdot x_1\cdot x_3 + a_3\cdot x_2^2 + a_4\cdot x_2\cdot x_3 + a_5\cdot x_3^2$

$C_1 : x_2^2 + a_1\cdot x_1\cdot x_2 + a_2\cdot x_1\cdot x_3 + a_3\cdot x_2\cdot x_3 + a_4\cdot x_3^2$

$C_2 : x_3^2 + a_1\cdot x_1\cdot x_2 + a_2\cdot x_1\cdot x_3 + a_3\cdot x_2\cdot x_3$

$C_3 : x_1\cdot x_2 + a_1\cdot x_1\cdot x_3 + a_2\cdot x_2\cdot x_3$

So, we start by defining the basic elements:

```
conics:=function(qua,N)
K:=GF(q);
ConicsList:=[];
Div:=Divisors(N);
```

Once we have N we can start construction the first family taking a general plane section and look for smooth conics of type C_0:

```
//first family
A<[a]>:=PolynomialRing(K,8);
AF:=FieldOfFractions(A);
R<[y]>:=PolynomialRing(AF,4);
P:=Proj(R);
plane:=a[1]*y[2]+a[2]*y[3]+a[3]*y[4];
Plane_Qua:=Evaluate(qua,[plane] cat y[2..4]);
conic:=y[2]^2 +a[4]*y[2]*y[3]+a[5]*y[2]*y[4]+
+a[6]*y[3]^2+a[7]*y[3]*y[4]+a[8]*y[4]^2;
```

Once we have the plane section and the general conic, we divide the former by the latter and build the scheme defined by the coefficients of the remainder.
Using Magma the easiest way is by using the command *Quotrem*. Again, the degree of the scheme will tell us the number of conics.

```
Q, rem:=Quotrem(Plane_Qua,conic);
equ:=[Numerator(cc) : cc in Coefficients(rem)];
conic1:=Scheme(Spec(A),equ);
dim1:=Dimension(conic1);
deg1:=Degree(conic1);
```

As before, we look for the smallest extension of the ground field where all the conics lying in this family are defined.

```
n:=1;
while # RationalPoints(BaseExtend(conic1,GF(2^Div[n]))) ne deg1
        do n+:=1;
end while;
Min_Exp:=Div[n];
```

We can now construct explicitly the conics, taking into account that we do not want to work with the singular ones, since they already came out while looking for lines.

```
C1:=BaseExtend(conic1,GF(2^Min_Exp));
Base<W,X,Y,Z>:=ProjectiveSpace(GF(2^Min_Exp),3);
P1:=RationalPoints(C1);
ConicsList:=[];
for p in P1 do
        c_base:=X^2 +p[4]*X*Y+p[5]*X*Z+p[6]*Y^2+p[7]*Y*Z+p[8]*Z^2;
        plane:=W+p[1]*X+p[2]*Y+p[3]*Z;
                cc:=Scheme(Base,[c_base,plane]);
                if IsNonSingular(cc) then
                        Append(~ConicsList,cc);
```

```
                end if;
end for;
```

We have to repeat this procedure for the other families.

2.2 Examples

The first example we will show a quartic which contains many conics defined over a big extension of the ground field, which we could have not been able to determine using the first approach showed. Consider the K3 surface $\mathcal{X}_2$ defined by

$$\begin{aligned} f_2 :=& W^3Z + WX^2Z + WY^3 + X^4 + X^3Y + X^2Y^2 + \\ &+ XY^3 + XY^2Z + XYZ^2 + XZ^3 + Y^4 + Y^3Z + Y^2Z^2 + YZ^3 + Z^4 \end{aligned}$$

over $\mathbb{F}_2$.
As we have done before we need the characteristic polynomial of Frobenius to compute the Picard rank:

$$\phi_{\mathcal{X}_2}(x) = (-2)(x-1)^2(x+1)^2(x^2-x+1)(x^2+1)(x^2+x+1)(x^4+1)(x^8+1).$$

Hence the geometric Picard number is $2+2+2+2+2+4+8 = 22$ and the Picard group will be completely defined over $K := \mathbb{F}_{2^{48}}$ since $l.c.m(1,2,6,4,3,8,16) = 48$.
We now determine the number of conics over the extensions $\mathbb{F}_{2^k}$ for $k \in Divisors(48)$:

k	Number of Conics
1	0
2	0
3	0
4	0
6	0
8	16
12	0
16	64
24	64
48	112

So, to obtain them all we have to work over K.
Here are two examples of conics defined over $\mathbb{F}_{2^{48}}$. With z we denote its generator as an extension of $\mathbb{F}_2$.

Scheme over $GF(2^{48})$ defined by
X^2+

$+ (z^{45} + z^{44} + z^{42} + z^{39} + z^{38} + z^{37} + z^{35} + z^{34} + z^{33} + z^{32} + z^{31} + z^{30} + z^{29} +$

$+ z^{27} + z^{26} + z^{25} + z^{24} + z^{22} + z^{18} + z^7 + z^5 + z^4 + z^2 + 1)XY+$

$(z^{47} + z^{45} + z^{43} + z^{42} + z^{40} + z^{39} + z^{38} + z^{37} + z^{36} + z^{35} + z^{34} + z^{33} + z^{32} +$

$+ z^{31} + z^{29} + z^{27} + z^{24} + z^{22} + z^{19} + z^{17} + z^{16} + z^{15} + z^{11} + z^9 + z^8 + z^7 + z^5 + z^4 + z^3 + z + 1)Y^2+$

$+ (z^{46} + z^{43} + z^{42} + z^{41} + z^{40} + z^{38} + z^{37} + z^{36} + z^{34} + z^{33} + z^{29} + z^{27} + z^{26} + z^{25} + z^{22} + z^{21} + z^{19} +$

$+ z^{18} + z^{16} + z^{14} + z^{11} + z^{10} + z^9 + z^8 + z^7 + z^3 + z + 1)XZ+$

$+ (z^{42} + z^{39} + z^{35} + z^{34} + z^{31} + z^{29} + z^{28} + z^{27} + z^{26} + z^{25} +$

$+ z^{21} + z^{17} + z^{16} + z^{15} + z^{14} + z^{13} + z^{10} + z^9 + z^7 + z^2 + z)YZ+$

$+ (z^{47} + z^{46} + z^{45} + z^{44} + z^{42} + z^{41} + z^{39} + z^{37} + z^{36} +$

$+ z^{35} + z^{34} + z^{33} + z^{32} + z^{31} + z^{24} + z^{21} + z^{20} + z^{19} +$

$+ z^{17} + z^{14} + z^{12} + z^{11} + z^8 + z^7 + z^4 + z^2 + z + 1)Z^2,$

$W + (z^{47} + z^{46} + z^{45} + z^{44} + z^{43} + z^{40} + z^{37} + z^{36} + z^{35} + z^{33} + z^{32} + z^{30} + z^{26} + z^{25} + z^{24} + z^{23}$

$+ z^{20} + z^{18} + z^{16} + z^{13} + z^{12} + z^{10} + z^5 + z^3 + z^2 + z)X+$

$+ (z^{47} + z^{42} + z^{41} + z^{38} + z^{37} + z^{33} + z^{31} + z^{28} + z^{26}$

$+ z^{24} + z^{22} + z^{21} + z^{19} + z^{18} + z^{16} + z^{15} + z^{14} + z^{12} + z^9 + z^7 + z^5 + z^4 + z + 1)Y$

$+ (z^{47} + z^{46} + z^{45} + z^{44} + z^{42} + z^{41} + z^{40} + z^{38} + z^{37} + z^{35} + z^{34} + z^{32} + z^{30} + z^{28} + z^{27} + z^{24} + z^{23} +$

$z^{21} + z^{17} + z^{15} + z^{14} + z^{10} + z^9 + z^8 + z^7 + z^5 + z^3 + z^2 + z + 1)Z,$

Scheme over $GF(2^{48})$ defined by

X^2+

$+ (z^{45} + z^{41} + z^{40} + z^{31} + z^{30} + z^{29} + z^{28} + z^{26} + z^{24} + z^{23} + z^{22} + z^{21} + z^{16} + z^{15} +$

$+ z^{13} + z^{12} + z^9 + z^8 + z^6 + z^3 + z)XY+$

$+ (z^{41} + z^{34} + z^{32} + z^{31} + z^{27} + z^{25} + z^{24} + z^{23} + z^{19} + z^{18} +$

$+ z^{17} + z^{16} + z^{15} + z^{14} + z^{13} + z^{12} + z^8 + z^3 + z^2 + z + 1)Y^2$

$+ (z^{47} + z^{45} + z^{43} + z^{42} + z^{41} + z^{39} + z^{35} +$

$+ z^{34} + z^{29} + z^{26} + z^{25} + z^{24} + z^{20} + z^{19} + z^{18} + z^{16} + z^{14} + z^{13} + z^{12} + z^{10} + z^5 + z^4)XZ+$

$+ (z^{45} + z^{44} + z^{43} + z^{42} + z^{40} + z^{39} + z^{36} + z^{34} + z^{31} + z^{30} + z^{29} + z^{28} + z^{26} + z^{24} +$

$+ z^{23} + z^{20} + z^{19} + z^{18} + z^{16} + z^{12} + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z)YZ+$

$+ (z^{47} + z^{37} + z^{36} + z^{34} + z^{33} + z^{32} + z^{31} + z^{30} + z^{28} + z^{27} + z^{25} + z^{24} + z^{23} + z^{22} + z^{21} + z^{20} +$

$+ z^{18} + z^{14} + z^{13} + z^{12} + z^{11} + z^{10} + z^8 + z^7 + z^4 + z)Z^2,$

$W + (z^{47} + z^{46} + z^{45} + z^{44} + z^{43} + z^{42} + z^{41} + z^{34} + z^{33} + z^{32} + z^{31} + z^{30} + z^{26} + z^{23} + z^{22} +$

$+ z^{21} + z^{20} + z^{17} + z^{15} + z^{12} + z^9 + z^7 + z^6 + z^5 + z^3)X+$

$+ (z^{47} + z^{45} + z^{41} + z^{39} + z^{38} +$

$$+z^{36}+z^{32}+z^{28}+z^{27}+z^{26}+z^{25}+z^{24}+z^{23}+z^{22}+z^{21}+z^{20}+z^{19}+z^{16}+z^{15}+z^{12}+$$
$$+z^{11}+z^{10}+z^{5}+z^{4}+z+1)Y+$$
$$+(z^{45}+z^{44}+z^{43}+z^{42}+z^{40}+z^{38}+z^{36}+z^{34}+z^{33}+z^{32}+$$
$$+z^{31}+z^{29}+z^{26}+z^{22}+z^{21}+z^{20}+z^{19}+z^{18}+z^{16}+z^{10}+z^{9}+z^{8}+z^{7}+z^{6}+z^{4}+z^{2}+z)Z$$

We now go back to the previous examples, starting with

$$\mathcal{X}_0 : f_0 = W^3Z+WX^2Z+WXY^2+WXYZ+WYZ^2+X^3Y+X^2YZ+XY^3+Y^2Z^2+Z^4.$$

This example is interesting since it shows how sometimes we can work over smaller extensions then the one determined by the Tate conjecture. In particular, in this case if we had directly looked for conics over the extension of degree 36, which a priori should be the one to look at, we would have only find conics which are base change of conics defined over smaller fields. So, instead of looking for rational points over $\mathbb{F}_{2^{36}}$, we can work over $\mathbb{F}_{2^9}$ which is much smaller. On a single example the time difference can be hardly noticed, but later on we will explain how we want to implement this method for a large number of quartic surfaces. Hence, we are trying to speed up the computations as much as we can.

k	Number of Conics
1	1
2	1
3	1
4	1
6	1
9	19
12	12
18	19
36	19

Thanks to remark 1.13, we can give the list of conics defined over $\mathbb{F}_{2^9}$ up to Frobenius action.
Hence, we get the following:

Scheme over $\mathbb{F}_{2^9}$ defined by
$X^2+z^{245}XY+z^{456}XZ+z^{182}YZ+z^{143}Z^2$,
$W+z^{175}X+z^{273}Y+z^{252}Z$,
Scheme over $\mathbb{F}_{2^9}$ defined by
$z^{14}XY+Y^2+z^{279}XZ+z^{480}YZ+z^{10}Z^2$,
$W+z^{378}X+z^{140}Y+z^{483}Z$,
Scheme over $\mathbb{F}_2$ defined by
X^2+WY+Y^2,

Z;

where the first two conics give rise to other 8 conics each and the last one is obviously invariant under the action of the Frobenius. So, this is enough to determine the list of 19 conics contained in $\mathcal{X}_0$.

Similarly, we consider $\mathcal{X}_1$ defined by
$W^3Z + WXYZ + WZ^3 + X^4 + X^2YZ + XY^3 + XYZ^2 + XZ^3 + Y^4 + Y^3Z + Z^4$
and we look for conics over $\mathbb{F}_2$, $\mathbb{F}_{2^2}$ and $\mathbb{F}_{2^4}$ we can see that in the first two cases we have no conics, whereas over the latter we can find 84 of them.

3 Twisted cubics

3.1 Definition and basic properties

The main reference for this section is *Algebraic Geometry* by Hartshorne. Assume that we are working over an algebraically closed field K.

Definition 3.1. *The twisted cubic is the curve $C \subset \mathbb{P}^3$ defined by $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^3$ where $f(s:t) := (s^3 : s^2t : st^2 : t^3)$.*

Proposition 3.2 (Hartshorne, page 159)**.** *Let C be a non singular curve of degree 3 in $\mathbb{P}^3(K)$, for K be an algebraically closed field.*
Then, one of the following has to happen:

- *C is a planar curve;*
- *C can be obtained from the twisted cubic γ by an automorphism of $\mathbb{P}^3(K)$.*

Remark 3.3. *The result above can be generalized to any field k as follows.*
Suppose $C \subset \mathbb{P}^3(k)$ be a cubic not contained into any plane Π.
Let $\mathcal{O}_C(1)$ be the restriction of the line bundle associated to the plane sections of the curve and Ω_C be the canonical line bundle of C.
From the above result by Hartshorne, we know that over $\bar{k}$ we have that $C \cong \mathbb{P}^1$.
Now, $\Omega_C \otimes \mathcal{O}_C(1)$ is a line bundle defined over k which has two sections (via cohomology and base change).
Hence, the associated morphism $C \longrightarrow \mathbb{P}^1(k)$, defined over k, is an isomorphism over $\bar{k}$. The condition for the associated morphism to be an isomorphism does not depend on the field. Hence, we can deduce that C is isomorphic to $\mathbb{P}^1(k)$. In conclusion, the morphism $i : C \hookrightarrow \mathbb{P}^3$ is the morphism associated to the linear system $\mathcal{O}_{\mathbb{P}^1(k)}(3)$, hence $i(C)$ is the twisted cubic up to an automorphism of $\mathbb{P}^3$.

We now see another way to define the twisted cubic (thanks to the previous result we can refer to *the* twisted cubic, rather than *a* twisted cubic).

Definition 3.4. *A variety X of dimension r in $\mathbb{P}^n$ is a strict complete intersection if $I(X)$ can be generated by $n-r$ elements.*
If X can be written as the intersection of $n-r$ hypersurfaces we say that X is a set theoretic complete intersection.

Proposition 3.5.

- *Let X be a variety in $\mathbb{P}^n$ such that $X = V(\mathfrak{I})$ where $\mathfrak{I}$ is an ideal generated by r elements. Then, dim $X \geq n-r$.*
- *A strict complete intersection is a set-theoretic complete intersection.*
- *The converse is false.*

We will now show that the twisted cubic is a set theoretic intersection of a quadric and a cubic, but not a strict complete intersection.
Let us consider the following quadrics contained in $\mathbb{P}^3(K)_{[x,y,z,t]}$:

$$Q_0 := \{yz - xt = 0\}; \quad Q_1 := \{y^2 - zx = 0\}. \tag{4.2}$$

They both contain the line $y = x = 0$, and they residually intersect in points (x, y, z, t) s.t. $\frac{x}{y} = \frac{y}{z} = \frac{z}{t} =: \theta$.
Thus, the residual intersection is the twisted cubic γ. Note that the quadric $Q_2 := \{z^2 - yt = 0\}$ contains the twisted cubic, but not the line $y = x = 0$, hence γ is the intersection of the three quadrics Q_0, Q_1, Q_2.
However, the twisted cubic is not a set-theoretic complete intersection of any choice of $\{Q_i, Q_j\}$, $i, j \in \{0, 1, 2\}$ since the intersection would also contain a line:

$$Q_0 \cap Q_1 = \gamma \cup \{y = x = 0\} \tag{4.3}$$
$$Q_0 \cap Q_2 = \gamma \cup \{y = t = 0\} \tag{4.4}$$
$$Q_1 \cap Q_2 = \gamma \cup \{y = z = 0\} \tag{4.5}$$

Now, $I :=< q_0, q_1, q_2 >$ where $Q_i := \{q_i = 0\}$.
Hence, such generators are

$$q_0 = yz - xt, \tag{4.6}$$
$$q_1 = y^2 - zx, \tag{4.7}$$
$$q_2 = z^2 - yt. \tag{4.8}$$

No linear form can vanish on γ and the above 3 generators are linearly independent, hence I cannot be generated by less then 3 polynomials. This implies that γ cannot be a strict complete intersection.
To conclude that $I(\gamma) = I$, we need to show that I is a radical ideal. Here is the proof.
$I \subset \sqrt{I}$ is trivial.
For the converse, let us pick $P := P(x, y, z, t)$ that vanishes on γ. We will prove that $P \in I$.
We claim that we can write $P = R(x, t) + S(x, t)y + T(x, t)z + i(x, y, z, t)$, where $R, S, T \in K[x, t]$ and $i \in I$.
The proof of the latter is by induction on the degree of P.
Hence, we have that

$$0 = P(s^3, s^2t, st^2, t^3) = R(s^3, t^3) + S(s^3, t^3)s^2t + T(s^3, t^3)st^2 + 0 \tag{4.9}$$

Hence, comparing the monomials of degree modulo 3 we can deduce that no cancellation can occur, hence $R = T = S = 0$.
We can now use this fact to show that γ is a set theoretic complete intersection. Namely,

we will show that $\gamma = V(y^2 - xz) \cap V(z^3 - 2yzt + xt^2)$.
First of all, $z^3 - 2yzt + xt^2 = z(z^2 - yt) + t(xt - yz)$, hence lies in I.
This implies that $\gamma \subseteq V(y^2 - xz) \cap V(z^3 - 2yzt + xt^2)$. To prove the converse, we have that

$$(z^2 - yt)^2 = z(z^3 - 2yzt + xt^2) + t^2(y^2 - xz) \tag{4.10}$$
$$(xt - yz)^2 = x(z^3 - 2yzt + xt^2) + z^2(y^2 - xz). \tag{4.11}$$

This means that $< q_2, q_0 > \in \sqrt{q_1, z^3 - 2yzt + xt^2}$, hence $I \subseteq \sqrt{q_1, z^3 - 2yzt + xt^2}$ proving that $\gamma = V(y^2 - xz) \cap V(z^3 - 2yzt + xt^2) \subseteq \gamma$.
This proves that the twisted cubic is a set theoretic complete intersection.

3.2 Code

As before, suppose we are given a K3 surface X defined over a finite field $\mathbb{F}_q$.
The idea to find twisted cubic will somehow resemble what we have previously done for lines and conics.
We were not able to proceed using the exact same method, since we would be dealing with schemes of dimension 40, thus computing the degree of those schemes is not feasible.
Hence, we used a method similar to the first approach used for conics.
Start considering the *fundamental twisted cubic* γ, namely the one given by the minors of the matrix
$$\begin{bmatrix} W & X & Y \\ X & Y & Z \end{bmatrix}$$
Hence, it is given by the equations

$$\begin{cases} f_1 := WY - X^2 = 0 \\ f_2 := XZ - Y^2 = 0 \\ f_3 := WZ - XY = 0 \end{cases}$$

Our approach aims to find all possible K3 surfaces of degree four which contain γ and then compare X with this set.
If a quartic contains γ, it means that we should have three polynomials g_1, g_2, g_3 over $\mathbb{F}_q$ homogeneous of degree 2 such that $X := g_1 \cdot f_1 + g_2 \cdot f_2 + g_3 \cdot f_3 = 0$.
Each g_i has 10 coefficients, so running the coefficients over $\mathbb{F}_q$ we get a first set $\mathfrak{C}_0$ of quartics.
However, not each one of those quartic surfaces is actually a K3, we need to get rid of the singular ones. Call this new set $\mathfrak{C}_1$.
Once this is done, we think of this set as a disjoint union of quartics depending on the PGL_4 orbit they lay in, meaning that we write $\mathfrak{C}_1 = \sqcup G_i$.
As we have reminded in the previous section, all twisted cubics are projectively equivalent. However, this action is not fully transitive: every twisted cubic γ_i is stable under the

action of a subgroup of PGL_4 that we denote by $Stab(\gamma)$, independent on the choice of the conjugacy class of γ_i, which contains the image of the isomorphisms of $\mathbb{P}^1$.
To make things clear, the twisted cubic can be written as

$$\begin{aligned}\gamma :&\mathbb{P}^1 \longrightarrow \mathbb{P}^3\\ &(s:t) \longmapsto (s^3 : s^2t : st^2 : t^3)\end{aligned}$$

Hence every linear transformation on $\mathbb{P}^1$ can be mapped into an element of PGL_4 which fixes γ.
Let's now consider a PGL_4-class of quartics $G_k := \{X_0, \ldots, X_n\}$.
From what we have said before, we can write the class G_k as a distinct union of classes of the Stabilizer. In general, we will have that the size of every such class is a multiple of $\#Stab(\gamma)$.
Identifying the quartics modulo the action of $Stab(\gamma)$, we finally have the set $\mathfrak{C}$ we were looking for.
To determine whether X contains a twisted cubic we have to determine if there is any element of its PGL_4 orbit which lays in $\mathfrak{C}$.
Here is the code we are using for this first part of the computations for smooth quartics over $\mathbb{F}_2$.

```
K:=GF(2);
Ring<W,X,Y,Z>:=PolynomialRing(K,4);
f1:=X^2+W*Y;
f2:=Y^2+X*Z;
f3:=W*Z+X*Y;
D:=[W^2,W*X,W*Y,W*Z,X^2,X*Y,X*Z,Y^2,Y*Z,Z^2];
V<[a]>:=VectorSpace(K,30);

TwistedCubics:={};
for P in V do
        g1:=&+[P[i]*D[i] : i in [1..10]];
        g2:=&+[P[i+10]*D[i] : i in [1..10]];
        g3:=&+[P[i+20]*D[i] : i in [1..10]];
        //we are working over GF(2)
        cub:=g1*f1+g2*f2+g3*f3;
        Include(~TwistedCubics,cub);
end for;
Space<W,X,Y,Z>:=ProjectiveSpace(K,3);
for Q in TwistedCubics do
S:=Scheme(Space,Q);
if IsSingular(S) then
Exclude(~TwistedCubics, Q);
```

```
end if;
end for;
```

Once this is done, we have to divide all quartics lying in the set *TwistedCubics* into PGL_4 orbits.
An equivalent approach to this problem would be to enumerate all the twisted cubics defined over $\mathbb{F}_2$. This can be achieved starting from the fundamental twisted cubic and via PGL_4 find the complete list of such curves over $\mathbb{F}_2$.
Once this is done, it is sufficient to check if they are contained in a K3 surface X.

3.3 Examples

Here is an example.
Consider the following K3 surface defined over $\mathbb{F}_2$:

$$W^3Z + W^2Y^2 + WY^3 + X^4 + X^2Y^2 + X^2YZ + XY^2Z + XYZ^2 + XZ^3 + Y^3Z + Z^4.$$

This quartic contains 8 twisted cubic defined over its base field:

$$\begin{aligned}
&\gamma_0 := [X^2 + XZ + YZ + Z^2 + XW + YW + W^2,\\
&XY + XZ + YZ + Z^2 + XW + YW + ZW,\\
&Y^2 + Z^2 + XW + W^2];
\end{aligned}$$

$$\begin{aligned}
&\gamma_1 := [X^2 + XZ + Z^2 + ZW,\\
&XY + XZ + XW + YW,\\
&YZ + Z^2 + XW + YW + ZW];
\end{aligned}$$

$$\begin{aligned}
&\gamma_2 := [X^2 + YZ + ZW,\\
&XZ + YW + W^2,\\
&Z^2 + XW];
\end{aligned}$$

$$\begin{aligned}
&\gamma_3 := [X^2 + XZ + Z^2 + ZW,\\
&XY + XZ + ZW,\\
&YZ + Z^2 + XW + YW;
\end{aligned}$$

$$\begin{aligned}
&\gamma_4 := [X^2 + Y^2 + XW + ZW + W^2,\\
&XZ + Z^2 + XW + YW + W^2,\\
&YZ + Z^2 + XW + YW + ZW];
\end{aligned}$$

$$\gamma_5 := [X^2 + XZ + YZ + Z^2 + XW + YW + ZW + W^2,$$
$$XY + XZ + XW + YW,$$
$$Y^2 + Z^2 + XW + ZW + W^2];$$

$$\gamma_6 := [X^2 + YZ + XW,$$
$$XZ + XW + YW + ZW + W^2,$$
$$Z^2 + XW + ZW];$$

$$\gamma_7 := [X^2 + Y^2 + Z^2 + XW + W^2,$$
$$XZ + XW + YW + ZW + W^2,$$
$$YZ + Z^2 + XW + YW].$$

4 Computing the sublattice

Once we have found divisors over a quartic K3 surface X, we need to compute the sublattice generated by such divisors.
First of all, since every time we tried to work over the smallest possible extension of $\mathbb{F}_2$, we need to have them defined over the same field. The easiest way to do it is to go back to zeta function of X to determine the degree of the minimal extensions on which all the generators of the Picard group are defined. Once we have this degree n_X, we can lift all the divisors we have found over $\mathbb{F}_{2^{n_X}}$.
We can now build their intersection matrix M.
We can construct it as follows: suppose we have a list of divisors $\{H, L, C, TC\}$ where H is the hyperplane section, L is the set of lines, C the set of irreducible conics and TC the set of twisted cubics.
Every entry of M denotes the intersection number between two divisors.
Namely, in the first line we have $M[0,0] = 4$, $M[0,i] = 1$ for $i \leq \#L$, $M[0,i] = 2$ for $\#L \leq i \leq \#C + \#L$ and $M[0,i] = 3$ for $\#L + \#C \leq i \leq \#C + \#L + \#TC$.
On the diagonal we have $M[i,i] = 2 \cdot g(D) - 2$ for any divisor D. Since we are considering rational curves the value would be -2 on the whole diagonal apart from $M[0,0]$.
For the other entries, we have to compute the degree of the intersection between the two divisors. Since the intersection scheme is either empty or zero-dimensional, this would give us the number of points of the intersection.

```
GramMatrix:=function(gen,gen1,gen2)
        Mat:=[];
        Line:=[4];
        for i in [1..#gen1] do
                Line:=Append(Line,1);
        end for;
```

```
    for i in [1..#gen2] do
        Line:=Append(Line,2);
    end for;
    Mat:=Append(Mat,Line);
    for S in gen do
        Line:=[];
        if S in gen1 then
            Line:=Append(Line,1);
        else
            Line:=Append(Line,2);
        end if;
        for D in gen do
            if S eq D then
                Line:= Append(Line,2*ArithmeticGenus(S)-2);
            else
                Line:=Append(Line, Degree(S meet D));
            end if;
        end for;
        Mat:=Append(Mat,Line);
    end for;
    return Mat;
end function;
```

4.1 Examples

Let's pick the examples considered in the previous sections. We will recall the defining equation, the geometric Picard $\rho(\overline{\mathcal{X}})$ and the minimal degree N for achieving such rank. Also, all those three examples contain no twisted cubic defined over the base field.
$\mathcal{X}_0 : f_0 = W^3Z+WX^2Z+WXY^2+WXYZ+WYZ^2+X^3Y+X^2YZ+XY^3+Y^2Z^2+Z^4$.
N=36; $\rho(\overline{\mathcal{X}_0}) := 20$.
It has 17 lines and 19 conics giving a sublattice of whose intersection matrix has rank 20, hence it is of finite index.
Let's now consider the surface $\mathcal{X}_1$ defined by

$$f_1 := W^3Z + WXYZ + WZ^3 + X^4 + X^2YZ + XY^3 + XYZ^2 + XZ^3 + Y^4 + Y^3Z + Z^4,$$

N:=4, $\rho(\overline{\mathcal{X}_1}) := 18$.
We have 40 lines and 84 conics and they give a sublattice of finite index.
Finally, we consider $\mathcal{X}_2$ defined by

$$f_2 := W^3Z + WX^2Z + WY^3 + X^4 + X^3Y + X^2Y^2 +$$

$$+ XY^3 + XY^2Z + XYZ^2 + XZ^3 + Y^4 + Y^3Z + Y^2Z^2 + YZ^3 + Z^4.$$

N:=48 ; $\rho(\overline{\mathcal{X}_2}) := 22$;

We have 16 lines, 112 conics and we get a sublattice of finite index.

This will not of course happen every time, but in general for a good number of quartics such a sublattice is achieved.

We will show statistics about this fact when considering smooth quartics over $\mathbb{F}_2$.

Chapter 5

K3 surfaces defined over $\mathbb{F}_2$

1 A census of quartics over $\mathbb{F}_2$

We now give an overview of the paper *A census of zeta functions of quartic K3 surfaces over* $\mathbb{F}_2$ by K. Kedlaya and A. V. Sutherland.
This paper aims to detect all the smooth quartic surfaces over $\mathbb{F}_2$ and their zeta function. To start this investigation, recall that the zeta function of an algebraic variety X defined over $\mathbb{F}_q$ with $q = p^e$ is

$$Z(X,t) := \exp\Big(\sum_{n=1}^{\infty} \#X(\mathbb{F}_{q^n})\frac{t^n}{n!}\Big).$$

For K3 surfaces via the Weil Conjecture we have the following:

Theorem 1.1. *Let X be a K3 surface over $\mathbb{F}_q$. Then*

$$Z(X,t) = \frac{1}{(1-t)(1-qt)(1-q^2t)L(qt)} \tag{5.1}$$

where $L(t)$ is a polynomial of degree 21 and all of its roots with absolute value 1.

They computed the set $\mathcal{Q}$ of PGL_4 equivalence classes for smooth quartics and it turns out to contain 528,257 elements. Via point counting they could also compute all their zeta functions, and this set turns out to have 52,755 elements.

1.1 Survey of the work by Kedlaya and Sutherland

The vector space of homogeneous polynomials of degree 4 over $\mathbb{F}_2$ is 35 dimensional. So, every such polynomial f can be identified with a vector $v(f) = (v_0, v_1, \ldots, v_{34})$ where $v_i \in \mathbb{F}_2$ is the coefficient of the $i-th$ term (considering lexicographical ordering of the monomials in $\mathbb{F}_2[W,X,Y,Z]_4$). For a better labeling, this vector is translated into a positive integer which will be less that 2^{35}.
The group $PGL_4(\mathbb{F}_2)$ acts on homegeneous quartics via linear change of coordinate and it can be identified with a subgroup G of $GL_{35}(\mathbb{F}_2)$ of order 20,160. Since they were interested in quartic surfaces up to isomorphisms, it is sufficient to consider the G-orbits

of V. Each one of them will be represented by a unique lexicographically minimal element v.
To determine the number of orbits we will use Burnside Lemma:

Lemma 1.2. *Let G be a finite group that acts on a set X. Let X/G be the set of orbits of X. For every $g \in G$ let $X^g := \{x \in X \text{ s.t. } g \cdot x = x\}$. Then*

$$|X/G| = \frac{1}{|G|} \cdot \sum_{g \in G} |X^g|.$$

Hence,

$$\#(V/G) = \frac{1}{|G|} \sum_{g \in G} \#V^g = \frac{\#C}{\#G} \sum_{C} (\#\mathbb{F}_2)^{\dim\ _1(C)} = 1,732,56.$$

Here the second sum runs over the conjugacy classes and $dim_1(C)$ denotes the 1-eigenspace of the conjugacy class C.
This brute force approach would not be feasible for fields bigger that $\mathbb{F}_2$: over $\mathbb{F}_3$ there are 4,127,971,480 orbits, and over $\mathbb{F}_4$ 100,304,466,278,983. Once the list of orbits is known, it is possible to restrict it to those which define a K3 surface. Namely, those are the orbits represented by a vector $v(f)$ for which the polynomial $f \in \mathbb{F}_2[W, X, Y, Z]$ is irreducible, or for which the singular locus defined by the Jacobian matrix of f is nonempty.
Appling those conditions it turns out that 528,257 satisfy them. It is now time to find the zeta functions for such quartics, hence it is necessary to count the rational points of $f(W, X, Y, Z) = 0$ over extensions of $\mathbb{F}_2$ (from the functional equation of the zeta function we know it suffices to stop at the degree 11 field extension).
The naive approach to point counting would be to simply iterate over points $(x_0, y_0) \in \mathbb{F}_{2^m}^2$ and find the roots of $g(W) = f(W, x_0, y_0, 1)$ that lie over $\mathbb{F}_{2^m}$. Of course one should consider the case $Z = 0$, but in this way we are simplifying the calculations.
Zinoviev's formulas in [70] provide a method to find such roots by giving an explicit $n \times n$ linear systems of equations whose solutions correspond to the roots of g.
Going back to the point counting, it is worth noticing that a general polynomial g defined as $g(W) = f(W, x_0, y_0, 1)$ does not have degree 4.
For all but 34 of the surfaces in $\mathcal{Q}$ the degree of the defining polynomial $f(W, X, Y, Z)$ in W is at most 3. In the typical case, after making g monic and applying a linear change of variables, we may assume that $g(W) = W^3 + g_1 W + g_0$. It is hence possible to precompute a lookup table T indexed by pairs $(g_0, g_1) \in \mathbb{F}_{2^m}^2$ whose entries record the number of roots of the polynomial $W^3 + g_1 W + g_0$. Each entry of T is an integer $0 \leq n \leq 3$, thus $\#T = 2^{2r+1}$.
It is faster to compute T than to instantiate $f(W, x_0, y_0, 1)$ at every pair $(x_0, y_0) \in \mathbb{F}_2^2$ and this process can be accelerated by ordering the pairs (g_0, g_1) in such a way to make it easier to compute the matrices appearing in Zinoviev's formulas. This will make the computation of T worthwhile and we can reuse the same table for every $X \in \mathcal{Q}$. Thus,

the cost of point counting is dominated by the time required to compute $f(W, x_0, y_0, 1)$. In order to perform all the point counting required to get the zeta functions, a better way is to reverse this approach. This means to loop over pairs (x_0, y_0) and for each pair count the solutions to $f(W, X, Y, Z) = 0$ over finite extensions of $\mathbb{F}_2$ (and $f \in \mathcal{Q}$) instead of iterating over surfaces given by $f(W, X, Y, Z) = 0$.
This allows to instantiate 35 homogeneous quartic polynomials at $X = X_0, Y = y_0, Z = 1$ just once for each pair and then for every $f \in \mathcal{Q}$ compute $f(W, x_0, y_0, 1)$ as an $\mathbb{F}_2$-linear combination of those.
This algorithm works fast enough and for every $X \in \mathcal{Q}$ we can write $L(T) = 1 + a_1 T + \cdots + a_{21} T^{21}$ with $a_1, \ldots, a_{12}$ known. Using the zeta functional equation it is possible to determine all the others coefficients.
Using this method, they were able to provide the following outputs:

- the list $\mathcal{Q}$ of all quartic K3s over $\mathbb{F}_2$;
- the list of all possible zeta functions together with the associated K3s.

In order to make the .txt file shorter and more readable, the list $\mathcal{Q}$ is given as of integers $N_X \leq 2^{35}$ together with the number of rational points defined up to $\mathbb{F}_{2^{11}}$.
Every K3 $X \in \mathcal{Q}$ is associated to an integer N_X by writing the list of coefficents (which is a series of 0s and 1s of length 35) which is then converted to base 10.

2 Survey of the results

We now show how we used the algorithms described in previous chapter applied to the list $\mathcal{Q}$ determined by Kedlaya and Sutherland.
As seen in chapter 3, from the existing literature we have already have a complete picture of the number of lines contained in a K3 surface.
Thanks to our computations, we can add conics and (partially) twisted cubics to this picture.
However, we could not get any results for curves of higher degrees.

2.1 Preliminary computations

First of all, we have to translate the list given by Kedlaya and Sutherland into quartic surfaces since they provide the elements of $\mathcal{Q}$ as integer $N \leq 2^{35}$.
The algorithm used to perform this first step was pretty straightforward, we simply had to reverse what they did.
Here is one easy example. The first K3 in their list is identified as 2147491859, which after easy computations turns out to represent the quartic $X^3Y + Y^4 + W^3Z + YZ^3 + Z^4$.
Secondly, we have to use the Tate conjecture to determine the geometric Picard rank for all these surfaces and the minimal extension of the ground field to achieve it.
We used Sage for this first part, whereas for the rest of the computations we found out that MAGMA was performing much better.
Here is the distribution of the geometric Picard rank for our set $\mathcal{Q}$.
Ranks:= [0, 87312, 0, 140397, 0, 74575, 0, 66842, 0, 61929, 0, 36065, 0, 26966, 0, 19271, 0, 9139, 0, 3390, 0, 2371].

2.2 Lines

Since we could not determine a degree independent algorithm, we approached such computation by degree of the divisors. Hence the first curves analyzed were lines.
As stated in the previous chapter, sharp bounds for the number of lines have already been found. Nevertheless, we still needed to have the explicit equations for each $X \in \mathcal{Q}$.
We used the method described in chapter 4, at the same time we computed the rank of the Picard sublattice obtained by such divisors.
The following vector contains the distribution of lines on the quartics belonging to $\mathcal{Q}$.
The first entry identifies the number of K3 surfaces for which we found no lines and so on.
NumberOfLines:=[268406, 98965, 46986, 20173, 47773, 12344, 6796, 8484, 7300, 1565, 3082, 1456, 1152, 954, 731, 135, 858, 198, 101, 138, 221, 58, 102, 33, 59, 28, 25, 9, 33, 14, 7, 3, 22, 3, 3, 0, 13, 3, 3, 0, 21]
It is worth noticing that for such quartics the number of lines is at most 40, which agrees with the bound of the paper by Degtyarev [12].
In his paper he showed that the number of lines on a quartic K3 in characteristic 2 is at

most 60. In case X is supersingular, the maximum is either 40 or at most 32.
Here are the data for supersingular K3. We have 2371 such surfaces and the number of lines are
LinesSS:=[856, 122, 57, 14, 725, 16, 67, 92, 132 ,10, 50, 35, 30, 4 ,4, 5, 68, 18, 0, 0, 37, 0, 8, 0, 6, 0, 0, 0, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 12].

2.3 Conics

Once done with lines, we then moved to irreducible conics. For such divisors not much is known, in particular there are very few examples in the literature of K3s with a big number of conics. Even though we restrict our analysis only to K3s of degree 4 over the field $\mathbb{F}_2$, we could find examples containing up to 672 conics.
To obtain the complete description of the conics contained in the quartic K3s over $\mathbb{F}_2$, we worked as described in the previous chapter.
Proceeding in this way we were able to find the exact number of conics contained in every K3 over the field $\overline{\mathbb{F}_2}$ and their equations.
Here is the vector with the number of conics we found.
NumberOfConics:=[290764, 28357, 62528, 9982, 28793, 5344, 13867, 5646, 15670, 3354, 8946, 1625, 7480, 1219, 4796, 1071, 4625, 595, 3755, 1181, 4155, 594, 1905, 356, 1930, 513, 1055, 302, 1185, 183, 1024, 316, 1597, 235, 617, 148, 859, 129, 325, 105, 644, 97, 545, 144, 355, 122, 241, 43, 312, 74, 231, 79, 295, 70, 401, 336, 1488, 87, 312, 89, 343, 87, 138, 73, 222, 20, 214, 38, 140, 10, 146, 38, 228, 16, 103, 34, 101, 31, 88, 32, 226, 28, 72, 11, 115, 17, 59, 12, 42, 9, 117, 32, 96, 11, 25, 1, 86, 9, 36, 11, 35, 4, 19, 16, 72, 12, 34, 4, 30, 2, 23, 5, 72, 5, 48, 4, 32, 2, 19, 7, 48, 6, 15, 6, 11, 0, 12, 10, 78, 0, 27, 8, 25, 1, 19, 7, 11, 0, 3, 4, 45, 2, 17, 4, 29, 5, 6, 0, 12, 0, 9, 5, 19, 2, 22, 1, 11, 0, 16, 4, 61, 1, 250, 3, 11, 16, 11, 0, 284, 0, 11, 0, 4, 0, 0, 0, 60, 0, 4, 0, 2, 0, 2, 0, 2, 0, 12, 0, 7, 1, 0, 0, 40, 0, 2, 0, 5, 0, 3, 2, 10, 0, 6, 0, 1, 0, 0, 0, 24, 0, 0, 0, 1, 0, 0, 0, 9, 0, 15, 0, 3, 0, 0, 0, 55, 0, 6, 0, 0, 0, 3, 4, 4, 0, 0, 3, 2, 0, 0, 0, 16, 0, 1, 0, 4, 0, 4, 0, 2, 0, 0, 0, 0, 0, 3, 0, 5, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 16, 0, 0, 0, 4, 0, 3, 0, 0, 0, 0, 0, 0, 0, 8, 0, 0, 0, 0, 0, 12, 0, 0, 0, 0, 0, 0, 0, 4, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 8, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 9, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 12, 0, 0, 0, 0, 0, 0, 0, 7, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 7, 0, 16];
The greatest number of conics for such quartics is 672.

Below is the list of such K3 surfaces (all of which are supersingular):

$$
\begin{aligned}
f_0 :=& W^3Z + W^2Y^2 + WY^3 + WZ^3 + X^4 + X^2Y^2 + X^2YZ + X^2Z^2 + XY^2Z+\\
&+ XYZ^2 + Y^3Z + Z^4,\\
f_1 :=& W^3Z + W^2Y^2 + WY^3 + WZ^3 + X^4 + X^2Y^2 + X^2YZ + X^2Z^2 + XY^2Z+\\
&+ XYZ^2 + Y^3Z + Y^2Z^2 + YZ^3,\\
f_2 :=& W^3Z + W^2Y^2 + WY^3 + WZ^3 + X^4 + X^2Y^2 + X^2YZ + X^2Z^2 + XY^2Z + XYZ^2 + Y^4 + YZ^3,\\
f_3 :=& W^3Z + WXY^2 + X^4 + XZ^3 + Y^4 + Y^3Z,\\
f_4 :=& W^3Z + W^2Y^2 + WY^3 + WZ^3 + X^4 + X^2Y^2 + X^2YZ + X^2Z^2 + XY^2Z+\\
&+ XYZ^2 + Y^4 + Y^2Z^2 + Z^4,\\
f_5 :=& W^3Z + WY^3 + WYZ^2 + X^4 + X^2Y^2 + X^2YZ + X^2Z^2 + XY^2Z + XYZ^2 + Z^4,\\
f_6 :=& W^3Z + WY^3 + WYZ^2 + X^4 + X^2Y^2 + X^2YZ + X^2Z^2 + XY^2Z + XYZ^2 + Y^2Z^2 + YZ^3,\\
f_7 :=& W^3Z + WY^3 + WYZ^2 + X^4 + X^2Y^2 + X^2YZ + X^2Z^2 + XY^2Z + XYZ^2+\\
&+ Y^4 + Y^3Z + Y^2Z^2 + Z^4,\\
f_8 :=& W^3Z + WY^3 + X^4 + X^2Z^2 + XZ^3 + YZ^3,\\
f_9 :=& W^3Z + W^2Y^2 + WY^2Z + WZ^3 + X^4 + X^2Y^2 + X^2YZ + X^2Z^2 + XY^3 + XYZ^2 + XZ^3 + YZ^3,\\
f_{10} :=& W^3Z + WXY^2 + X^4 + XZ^3 + Y^4 + Y^3Z + Z^4,\\
f_{11} :=& W^3Z + W^2Y^2 + WXY^2 + WZ^3 + X^4 + XZ^3 + Y^4 + Y^3Z + Y^2Z^2,\\
f_{12} :=& W^3Z + WY^3 + WYZ^2 + X^4 + X^2Y^2 + X^2YZ + X^2Z^2 + XY^2Z + XYZ^2 + Y^4 + Y^3Z + YZ^3,\\
f_{13} :=& W^3Z + WX^2Z + WXZ^2 + WY^3 + WYZ^2 + WZ^3 + X^4 + X^3Y + X^2Y^2 + X^2Z^2+\\
&+ XY^3 + XY^2Z + Y^4 + Y^3Z + YZ^3,\\
f_{14} :=& W^3Z + W^2Y^2 + WXY^2 + WZ^3 + X^4 + XZ^3 + Y^4 + Y^3Z + Y^2Z^2 + Z^4,\\
f_{15} :=& W^3Z + W^2Y^2 + WX^2Y + X^4 + X^3Z + X^2Y^2 + X^2Z^2 + XY^3 + Y^4 + Y^3Z + YZ^3,
\end{aligned}
$$

2.4 Twisted cubics

We want to find the number of twisted cubics defined over $\mathbb{F}_2$ lying on every quartic of $\mathcal{Q}$. First of all, we have computed the image of the vector space given by the coefficients of the g_i (see notation from section 3.2) into the vector space of all the smooth quartics over $\mathbb{F}_2$. This does not give the exact number of smooth quartic surfaces which contain a twisted cubics since we still have to identify quartics lying in the same PGL_4 orbit. At this point we have a set $\mathcal{C}$ of around 590,000 quartics.
For each quartic $Q \in \mathcal{C}$ we have determined the number N_Q of all its conjugates lying in $\mathcal{C}$. Since over $\mathbb{F}_2$ the order of $Stab(\gamma)$ is 6, in general N_Q will be a multiple of 6.
Here is the number of twisted cubics defined over $\mathbb{F}_2$ for the quartic K3s:

$$[453092, 57235, 13736, 2716, 998, 243, 128, 25, 19, 4, 3, 0, 0, 0, 0, 1].$$

We also computed this number considering only K3s for which we found no lines or conics: [141640, 20946, 1369,58].
This means that for 141640 K3s we still have not found any divisors.
We tried to carry out the same computation over $\mathbb{F}_4$, but in this situation the size of the orbit of the twisted cubic gets too big.
It is a well-know result that gives the size of $\mathrm{PGL}(n, \mathbb{F}_q)$:

Proposition 2.1.
$$\#PGL(n, \mathbb{F}_q) = q^{\frac{n(n-1)}{2}} \prod_{i=2}^{n} (q^i - 1).$$

Proof. Notation: $\mathrm{PGL}(n, q) = \mathrm{PGL}(n, \mathbb{F}_q)$. Same will be for $\mathrm{GL}(n, q)$.
Since the order of GL (n, q) is the number of order basis over $\mathbb{P}^n(\mathbb{F}_q)$, we have that

$$\#\mathrm{GL}\ (n, q) = (q^n - 1) \cdot (q^n - q) \cdot \dots \cdot (q^n - q^{n-1}) = q^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} (q^i - 1)$$

Dividing by the order of its center (which is $q - 1$), we get the result. □

Hence, the size of $\mathrm{PGL}(4, \mathbb{F}_4)$ is 987,033,600.
This number makes it impossible to proceed with the same method used for $\mathbb{F}_2$.

3 Subgroups of the Picard

Once we have the list of divisors, we computed the sublattice generated by such divisors and its index.
Here is the computation only using lines:

- Rank of the sublattice:
  [268406, 98965, 46986, 57878, 16249, 11842, 9854, 7053, 2684, 3424, 1480, 1494, 673, 451, 239, 344, 103, 74, 29, 23, 0, 6].
- Index of the sublattice obtained: the first entry denotes the number of K3s for which we found a sublattice of finite index:
  [34816, 23268, 59424, 17332, 21772, 12229, 25588, 10129, 16533, 6983, 10280, 4081, 6615, 2561, 4002, 1291, 1531, 366, 871, 57, 122, 0, 0].

This means that for 6.5% of the surfaces we have already found a sublattice of finite index. Then, we can add conics and see how much of the Picard lattice is being filled by such curves. We also found the minimal field extension over which the conics are defined, which was an important data to allow us to compute the intersection numbers for every pair of curves (C_1, C_2) lying on X.
Thus, it was possible to determine the Gram matrix $G(X)$ which gave us the following results:

- Ranks of the Picard sublattice using lines and conics:
[164013, 116446, 52233, 67709, 27003, 24510, 17170, 16526, 8258, 10833, 4536, 5690, 2416, 3372, 1448, 2197, 761, 923, 225, 295, 34, 1659];

- Index of the sublattice obtained: the first entry denotes the number of K3s for which we found a sublattice of finite index:
[72458, 52256, 78959, 23439, 30313, 14822, 29057, 11274, 18488, 6104, 9621, 3580, 6590, 1903, 3069, 584, 1203, 102, 402, 20, 0, 0, 0];

Hence, for 13.7% of the surfaces we were able to find a sublattice of finite index.
Twisted cubics:

- Ranks of the Picard sublattice using lines and conics and twisted cubics over $\mathbb{F}_2$:
[141640, 123148, 57894, 72123, 28725, 27097, 17244, 16713, 8512, 11347, 4404, 5863, 2496, 3433, 1420, 2271, 763, 951, 220, 298, 36, 1659];

- Index of the sublattice obtained: the first entry denotes the number of K3s for which we found a sublattice of finite index:
[91621, 55112, 75025, 24365, 30780, 16648, 28700, 11956, 18680, 6647, 9229, 3909, 6460, 2224, 2895, 711, 1139, 122, 374, 20, 0 0 0];

This tells that for 17.3% of the K3s we found a sublattice of finite index. Also, for 73.18% of the K3s we found at least one curve.

4 Application

As we already said, we hoped to find good algorithms to understand more about the distribution of curves contained in the K3s lying in Ω and at the same time derive some datas regarding their Picard lattice.
Even though we could not fully solved this problem, even a partial algorithm can still be of some use.
For example, we could get examples of K3s containing 672 conics, more than any other example found in the literature.
Another possible application of our methods could come from the work of Van Luijk in [65].
Namely, in this paper he produced an example of a K3 of degree 4 with geometric Picard number 1.

4.1 K3 surfaces with $\rho = 1$

We go back to the work by Van Luijk in [65].
Let $R := \mathbb{Z}[x, y, z, w]$ be the homogeneous coordinate ring of $\mathbb{P}^3_{\mathbb{Z}}$.
Consider the following family of quartics $X_h := \{wf_1 + 2zf_2 = 3g_1g_2 + 6h\}$, where $f_i, g_i \in R$

defined by :

$$
\begin{aligned}
f_1 =& x^3 - x^2 + -x^2z + x^2w - xy^2 - xyz + 2xyw + xz^2 + 2xzw + y^3 + \\
&+ y^2z - y^2w + yz^2 + yzw - yw^2 + z^2w + zw^2 + 2w^3, \\
f_2 =& xy^2 + xyz - xz^2 - yz^2 + z^3, \\
g_1 =& z^2 + xy + yz, \\
g_2 =& z^2 + xy;
\end{aligned}
$$

and h is an homogeneous polynomial of degree 4.

Theorem 4.1. *[65] Let $h \in R$ homogeneous polynomial of degree 4. Then, the quartic X_h is smooth over $\mathbb{Q}$ and has geometric Picard number 1. The Picard group Pic$\overline{X_h}$ is generated by the hyperplane section*

The proof works in this way:

a) To have a bound of the Picard number over $\mathbb{Q}$ one can find a prime p of good reduction, then $\mathrm{Pic}(X_h)$ injects into $\mathrm{Pic}(X_h)_{\mathbb{F}_p}$.

b) Thanks to the Tate conjecture computing the Picard number of the reduction is relatively easy: once the zeta function is know it is sufficient to count the number of eigenvalues counted with multiplicity. One should aim to find a prime such that the Picard number of the reduction is 2.

c) If the latter is the case, if we had Picard number 2 over $\overline{\mathbb{Q}}$, then the discriminants of the Picard groups would belong to the same square class. Hence, the aim is to find two primes such that different square classes arise for the discriminant. This would imply that X_h has geometric Picard number 1.

In order to determine the discriminants of two reductions, he found two primes (2 and 3) for which $X_{\mathbb{F}_2}$ and $X_{\mathbb{F}_3}$ have Picard number 2. Then, to get the discriminant he had to find one curve for each reduction.
This is where our algorithms could be useful, enabling to determine more examples of such K3 surfaces.

4.2 New examples

We could potentially use our methods to determine new examples of K3 sufaces of Picard number 1.
In order to apply Van Luijk method one needs to know the discriminant of the Picard group, which can be obtained by knowing its generators.

5 Open questions

We now state some questions which remain open.
First of all, considering what we have done counting conics, we know the maximum number of conics lying over a quartic K3 surface defined over $\mathbb{F}_2$.

Question 5.1. *The following questions regarding conics are still open:*

- *Which is the maximum number of conics contained in a general K3 surface defined over $\mathbb{F}_2$?*
- *Which is the maximum number of conics contained in a quartic K3 surface defined over bigger finite fields of characteristic 2?*

Next, for twisted cubics we could only determine such curves when defined over the base field $\mathbb{F}_2$.

Question 5.2. *What is the maximum number of twisted cubics contained in a quartic K3 defined over $\mathbb{F}_2$? And over bigger fields of characteristic 2?*

Bibliography

[1] Artin, M.: *Supersingular K3 surfaces*, Annales scientifiques de lÉ.N.S. 4 série, tome 7, no 4 (1974), p. 543-567

[2] Artin, M., and Mazur, B., *Formal groups arising from algebraic varieties.* Annales scientifiques de l'cole Normale Suprieure 10.1 (1977): 87-131

[3] M. Artin and H. P. F. Swinnerton-Dyer, *The Shafarevich-Tate conjecture for pencils of elliptic curves on K3 surfaces*, Invent. Math. 20 (1973), 249266

[4] W. Barth, T. Bauer, *Smooth quartic surfaces with 352 conics*, manuscripta math. 85 (1994),409-417

[5] T. Bauer, *Quartic surfaces with 16 skew conics*, J. reine and angew. Math. 464 (1995), 207-217.

[6] P. Borwein and T. Erdlyi, *Polynomials and polynomial inequalities*,Graduate Texts in Mathematics 161, Springer, New York, 1995.

[7] F. Bouyer, *On the Monodromy and Galois Group of Conics Lying on Heisenberg Invariant Quartic K3 Surfaces*, Preprint: arXiv 1511.01781.

[8] F. Bouyer, *The Picard Group of Various Families of -invariant Quartic K3 Surfaces*; Acta Arithmetica, Vol. 186 (2018), issue 01, pp 61-86.

[9] Caselli, M.: *Local Heights and Densities for Curves of Low Genus*, PhD Thesis, University of Warwick, 2018.

[10] Charles, F.: *The Tate conjecture for K3 surfaces over finite fields.* Invent. Math. 194 (2013), no. 1, 119145

[11] P. Deligne, *La conjecture de Weil I*, Publ. Math. IHES 43, 1974, 273-307.

[12] A. Degtyarev, *Lines in supersingular quartics*, 2016,to appear. arXiv:1604.05836.

[13] David Eklund, *Curves on Heisenberg invariant quartic surfaces in projective 3-space*, arXiv e-print 1010.4058, October 2010.

[14] Elkies, N. D., Schütt, M.: *Modular forms and K3 surfaces*, Adv. Math. 240, no. 20 (2013), pp. 106131.

[15] Elikies, M., Schütt, M.: *Genus 1 fibrations on the supersingular K3 surface in characteristic 2 with Artin invariant 1*, , Asian J. Math. 19 (2015), 555582.

[16] A.-S. Elsenhans, J. Jahnel: *Point counting on K3 surfaces and an application concerning real and complex multiplication*: Proceedings of the ANTS XII conference (Kaiserslautern 2016), LMS Journal of Computation and Mathematics 19(2016)12-28

[17] A.-S. Elsenhans, J. Jahnel: *The Picard group of a K3 surface and its reduction modulo p*, Algebra & Number Theory 5(2011)1027-104

[18] A.-S. Elsenhans, J. Jahnel: *On the computation of the Picard group for certain singular quartic surfaces*, Mathematica Slovaca 63(2013)215-228

[19] A.-S. Elsenhans, J. Jahnel: *On the computation of the Picard group for K3 surfaces*, Mathematical Proceedings of the Cambridge Philosophical Society 151(2011)263-270

[20] A.-S. Elsenhans, J. Jahnel: *On Weil polynomials of K3 surfaces*, in: Algorithmic number theory, Lecture Notes in Computer Science 6197, Springer, Berlin 2010, 126-141

[21] G. Faltings, *Finiteness theorems for abelian varieties over number fields*, Translated from the German original, Invent. Math. 73 (1983), no. 3, 349366,

[22] Goto, Y.,*The Artin invariant of supersingular weigthed Delsarte surfaces*, J. Math. Kyoto Univ. (JMKYAZ). 36-2 (1996) 359-363.

[23] Hartshorne, R.: *Algebraic Geometry*. Grad. Texts in Math., vol. 52. Springer, Berlin (1977)

[24] Harvey, D.: *Computing zeta functions of arithmetic schemes*, Proc. Lond. Math. Soc. 111 (2015), 13791401

[25] Hirschfeld, J.W.P.: *Finite Projective Spaces of Three Dimensions*, Oxford University Press, Oxford (1985)

[26] Huybrechts, D., *Lectures on K3 surfaces.* 2014

[27] S. Katz, *On the finiteness of rational curves on quintic threefolds*, Compositio Math.60 (1986), 151-162.

[28] Kedlaya, K.; Sutherland, A.:*A census of zeta functions of quartic K3 surfaces over F2*, LMS J. Comput. Math. 19 (2016), suppl. A, 111.

[29] Keum , J., *A note on elliptic K3 surfaces* Trans. Amer. Math. Soc. 352 (2000), no. 5, 20772086.

[30] W Kim, K. Madapusi Pera, *The Tate conjecture for K3 surfaces in odd characteristic*,Invent. Math.201 (2015)625668

[31] Kondo, S., Shimada, I., *On a certain duality of Nron-Severi lattices of supersingular K3 surfaces* Algebr. Geom. 1 (2014), no. 3, 311333

[32] Lieblich, Max; Maulik, D.; Snowden, A.:*Finiteness of K3 surfaces and the Tate conjecture*,Ann. Sci. c. Norm. Supr. (4) 47 (2014), no. 2, 285308. 14J28 (14F20 14G15)

[33] Livné, R: *Motivic orthogonal two-dimensional representations of Gal*($\bar{\mathbb{Q}}/\mathbb{Q}$), Israel J. of Math., **92**, 1995,103-167.

[34] Madapusi Pera, K.: *The Tate conjecture for K3 surfaces in odd characteristic*, Invent. Math. 201 (2015), no. 2, 625668

[35] Maia, J. A. D.; Silva, A. R.; Vainsencher, I.; Xavier, F.: *Enumeration of surfaces containing a curve of low degree*. J. Pure Appl. Algebra 217 (2013), no. 8, 13791394.

[36] J.S. Milne, *On a Conjecture of Artin and Tate*, Annals of Mathematics,102(1975),517-533

[37] Milne, J.S.: *Tate conjecure over finite field*, talk at the The Tate Conjecture workshop at the American Instituteof Mathematics in Palo Alto, CA, July 23July 27, 2007.

[38] Miranda, R: *The basic theory of elliptic surfaces*, Dottorato di ricerca in matematica, ETS Editrice Pisa, 1989.

[39] Miranda, R., Persson, U.: *On extremal Rational Elliptic Surfaces*, Math. Z., **201**, 1986, 537-558.

[40] N. Nygaard, A. Ogus, *Tate's conjecture for K3 surfaces of finite height*, Ann. of Math. (2) 122 (1985), no. 3, 461507,

[41] Piateski-Shapiro I.I. Shafarevich I.R. *A torelli theorem for surfaces of type K3*, Izv. Akad. Nauk SSSR, Ser. Mat.35, 530572 (1971)

[42] Rams, S.; Schtt, M.: *112 lines on smooth quartic surfaces (characteristic 3).* Q. J. Math. 66 (2015), no. 3, 941951

[43] Rams, S.; Schtt, M.: *64 lines on smooth quartic surfaces*. Math. Ann. 362 (2015), no. 1-2, 679698.

[44] Rams, S., Schutt, M.: *At most 64 lines on smooth quartic surfaces (characteristic 2)*

[45] Ribet, K.:*Galois representations attached to eigenforms with Nebentypus*, in: J.- P. Serre, D. B. Zagier (eds.) *Modular Functions of one Variable V*, Bonn, 1976,Lect. Notes in Math. **601**,Springer, 1977,17-52

[46] A. N. Rudakov, I. R. Shafarevich, and T. Zink, *The influence of height on degenerations of algebraic surfaces of type K3*, Math. USSR Izv. 20 (1983), 119135.

[47] A. N. Rudakov and I. R. Shafarevich, *Surfaces of type K3 over fields of finite characteristic*, Current problems in mathematics, vol. 18, pp. 115207, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1981.

[48] Segre, B.: *The maximum number of lines lying on a quartic surface*, Quart. J. Math., Oxford Ser. 14 (1943), 8696.

[49] T. Sengupta, *Elliptic fibrations on supersingular K3 surface with Artin invariant 1 in characteristic 3*, arXiv:1105.1715 (2011).

[50] Shafarevich, I. R:: *On the arithmetic of singular K3 surfaces*, in: *Algebra and Analysis* (Kazan 1994) , de Gruyter (1996), 103-108.

[51] Shimada, I., *On normal K3 surfaces.* Michigan Math. J. 55 (2007), no. 2, 395416.

[52] Shimada, I., *Transcendental lattices and supersingular reduction lattices of a singular K3 surface* Trans. Amer. Math. Soc. 361 (2009), no. 2, 909949

[53] Shioda, T., *Supersingular K3 surfaces with big Artin invariant* J. Reine Angew. Math. 381 (1987), 205210.

[54] Shioda,T., Inose, H.: *On singular K3 surfaces*, W. L. Baily Jr., T. Shioda (eds.), *Complex analysis and algebraic geometry* Iwonami Shoten, Tokyo 1977, 119-136.

[55] Schütt, M: *Elliptic fibrations of some extremal elliptic K3 surfaces*, Rocky Mountain Journal of Mathmatics **37**, 2007, 609-652.

[56] Schütt, M: *CM newforms with rational coefficients*, Ramanujan J., **19**, 2009, 187-205.

[57] Schütt, M.: *Arithmetic of a singular K3 surface*, Michigan Math. J. **56** 2008, 513-527.

[58] Schütt, M.: *Two lectures on the arithmetic of K3 surfaces*, In Arithmetic and geometry of K3 surfaces and Calabi Yau threefolds, volume 67 of Fields Inst. Commun., pages 7199. Springer, New York, 2013.

[59] Schütt, M., Shioda, T.: *Elliptic surfaces* Algebraic geometry in East Asia - Seoul 2008, Advanced Studies in Pure Mathematics **60**, 2010, 51-160.

[60] Silverman, J.H., *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics 151 (Springer, 1994).

[61] Srinivas, V.: *The Tate conjecture for K3 surfaces: a survey of some recent progress*, Acta Math. Vietnam. 39 (2014), no. 1, 6985.

[62] Tate, J.: *On a conjecture of Birch and Swinnerton-Dyer and a geometric analogue*, Dix Exposes sur la Cohomologie des Schemas, North-Holland, Amsterdam, 1968

[63] Taelman, L.: *K3 surfaces over finite fields with given L-function.* Algebra and Number Theory 10 (2016)

[64] B.Totaro: *Recent progress on the Tate conjecture*, Bull. Amer. Math. Soc. (N.S.) 54 (2017), no. 4, 575590

[65] Van Luijk, R.:*K3 surfaces with Picard number one and infinitely many rational points*,Algebra & NumberTheory1(2007), 115

[66] Veniani, D.: *Lines on K3 quartic surfaces in characteristic 2.* Q. J. Math. 68 (2017), no. 2, 551581

[67] Veniani, D.: *The maximum number of lines lying on a K3 quartic surface.* Math. Z. 285 (2017), no. 3-4, 11411166.

[68] Veniani, D.: *Lines on K3 quartic surfaces in characteristic 3*, 15 August 2017, preprint arXiv:1608.04209

[69] Zarhin, Ju. G., *Endomorphisms of Abelian varieties over fields of finite characteristic*, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), no.2, 272–277, 471.

[70] V. Zinoviev, *On the solutions of equations of degree*, Research Report RR-2829, INRIA, 1996,