The Philosophical Significance of the Concept of Superposition in Quantum Field Theory

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The candidate confirms that the work submitted is his/her/their own and that appropriate credit has been given where reference has been made to the work of others.

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Abstract

I analyse the applicability of the concept of superposition within quantum field theory (QFT) in order to consider the physical interpretation of a simple interacting theory.

First, I consider the significance of 'superposition' in classical physics motivated by the philosophical framework developed by Wilson (2006; 2017), via the analyses of 'superposition' developed by Volkmann (1896; 1910) and Simons (1987) in addition to a historical survey of its application with special reference to Fourier techniques. The concept has a patchwork or façade structure, with application subject to 'prolongation' and 'semantic mimicry'. Proper usage of 'superposition' is associated with identifying partial states and laws that provide a natural description of complicated phenomena supporting physically salient explanations, inductive inferences and counterfactual reasoning.

Secondly, I demonstrate that application of 'superposition' in quantum physics is a prolongation of its classical usage involving new rules of application.

Thirdly, I analyse the historical origins of QFT and the mature theory to indicate where, and how, proper application of 'superposition' is made to free theories, whilst semantic mimicry is involved in the interpretation of interacting theories. Improper and often implicit appeals are made to 'superposition' to incorrectly claim physical interpretations of interacting theories. Two major related failures of 'superposition' are identified, associated with the initial postulation of putative fields and corresponding states associated with different particle types, and with the nonlinearity of the coupled field equations, so that natural descriptions of interacting states using familiar concepts are unavailable. Renormalization is interpreted as a symptom and pragmatic partial remedy for the failure of 'superposition' such that empirically successful calculations are supported using LSZ scattering theory and the Gell-Mann and Low theorem.

Finally, I suggest that the interpretation of QFT is best approached within the context of an engineer's perspective rather than a metaphysician's, having implications for wider philosophical debates.

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Chapter 1

Introduction

1.1 Context and concerns of the thesis

Quantum field theories (QFTs) are some of our most successful scientific theories, purportedly modelling matter and its interactions at the (near) fundamental level. They underpin the 'Standard Model' (SM) of particle physics, and have found application in condensed matter physics. QFTs combine quantum mechanical treatments of phenomena with special relativity. This allows for the creation and annihilation of particles and for antiparticles to be modelled. Jordan (in Born, Heisenberg and Jordan (1926)) and subsequently Dirac (1927) initially but prematurely claimed that QFT resolved the particle-wave paradox of quantum mechanics.

Despite their undoubted success, significant difficulties remain regarding both the physics and the philosophy of QFTs. For instance, the relationship of QFT to General Relativity remains an open question. The conceptual problem of renormalization and its physical interpretation is notorious. It has been taken to suggest that the physics of very short length scales remains opaque even if the difficulties have been circumvented to some extent by renormalization group techniques, so that QFTs have come to be understood as 'effective field theories' (EFTs) rather than 'fundamental theories'. QFTs also inherit the notorious 'measurement problem' from quantum mechanics (QM). Technical difficulties abound such as those arising from Haag's theorem as regards the inequivalence of representations, and from the manipulation of distributions for example. The role of 'virtual particles' in QFT brings yet another interpretative difficulty, as does the use of divergent series in the calculation of scattering amplitudes.

Philosophical interest in QFT has grown in the last 30-40 years and reflects two main responses to these (and other) difficulties. The approaches have been characterized (somewhat inadequately) as 'physicists'' or 'conventional' QFT, and 'philosophers'' or 'axiomatic' algebraic QFT. The former approach regards the philosophical task as the investigation of working QFTs

'warts and all', whilst the latter construes the philosophical task in developing a 'pure' QFT. The difficulty with the former approach is that problems occur that are dealt with in ad hoc and non-rigorous ways so that the value of a philosophical analysis in this context is questioned. The difficulty with the latter is that success to date has only been achieved for simplistic and unrealistic 'toy models', and so the value of a philosophical treatment of such a theory may be questioned. One may pursue either approach, for there is, at least potentially, philosophical value in both.[1]

Here I shall consider how one describes interacting states in QFT and what kind of explanation of interaction processes QFT supports. This places the project within 'physicists' QFT'. We shall see however that, perhaps surprisingly, there is no explicit 'natural' description of interacting states supported by QFT, and thus no explanation of interaction processes available. This surprising failure is explained or diagnosed in terms of the failure of 'superposition' in interacting QFTs, which is manifested in and pragmatically partially accommodated by renormalization.

On the one hand my project stands in continuity with various debates in the philosophy of QFT, such as those which concern the role of particles and / or fields in an 'ontology' of QFT, the use of asymptotic series, and the significance of virtual particles and Feynman diagrams. On the other hand the approach that I take and the framing concerns will be different. For instance renormalization group approaches will not be discussed, and Haag's theorem will play only a minor role. Instead, the significance and applicability of 'superposition' is central. In particular, its relation to the initial selection of fields, to Fourier techniques and to the character of the relevant differential equations as linear or nonlinear will be pivotal. These are largely uncharted areas in philosophy, both in classical and quantum physics.[2] Conceptual analysis will be a driving concern of the thesis, paying attention to the sometimes unexpected and unnoticed behaviour

[1] This is often set up as the 'Fraser-Wallace debate'. For a defence of conventional QFT see Wallace (2006; 2011), and for axiomatic QFT see Fraser (2008; 2009; 2011). For discussion see Baker (2016); Egg, Lamm and Oldofredi (2017); Kronz and Lupher (2019), and for discussion of the interpretation of algebraic and Hilbert space approaches see Ruetsche (2011).

[2] Some of these concerns (e.g. the significance of non-linearity in interacting states and the role that Fourier analysis plays) were anticipated by Redhead (1988), but have not been pursued in any significant way.

of concepts and their failure to apply how and where they are expected, which leads to confusion.

My key claim or result is that 'superposition', in various putative or implicit uses of the concept, fails to be applicable when it has been implicitly assumed to apply, and that as a result of this wrong assumption a number of conceptual or ontological confusions and dilemmas arise. These difficulties are diagnosed by the failure of 'superposition' even at the beginning of laying the foundation of interacting QFTs.

Finally, QFT is a large and diverse area of physics with different approaches, tools and techniques adopted in different domains. I shall focus on the application of QFT to the physics of what has been understood to be 'fundamental particles' from the perspective of canonical QFT. I shall not deal with issues surrounding the 'measurement problem' or alternative interpretations common to philosophical treatments of QM such as Bohmian, GRW or Everettian interpretations. However, the conclusions of the thesis might indicate that such approaches are not well-motivated, and that (following Cartwright (1983), although for slightly different reasons) the measurement problem is not well posed, and thus further research is needed.

The philosophical approach advocated by Mark Wilson is well suited to the kind of philosophical analysis of working QFTs that I wish to pursue. I shall summarize what a 'Wilsonian' approach involves in §1.2 before presenting an overview of the thesis in §1.3.

1.2 A 'Wilsonian' approach outlined

Critical issues in the interpretation of QFT involve the use and interpretation of the superposition concept or principle, Fourier techniques and iterative techniques, all of which are used in a number of areas of both classical and quantum physics. However, the physical significance and use of such concepts and techniques differs from application to application, and have not received the philosophical attention deserved. We obtain a better grasp on the issues involved in QFT if we pay attention to the local application of such concepts and techniques, clarifying what similarities and differences there are when moving

between contexts so as to refine the interpretation of 'superposition' and its physical significance in QFT.

The concerns that I explore dovetail with the philosophical approach to applied mathematics, physical theories and concepts that Mark Wilson has developed in *Wandering Significance* (2006) and *Physics Avoidance* (2017). I shall approach QFT through the lens offered by Wilson's philosophical concerns. These concerns might be characterized as local conceptual clarification that is necessitated by a contrast that he develops between the classical picture of concepts as 'glued' to their reference in a global fashion, and a patchwork picture of concepts that form an atlas (or façade) of 'patches' of local application so that such concepts have 'wandering significance'. This is associated with a careful analysis of the nature of the supportive role that the relevant mathematics and physics plays on each patch in the case of physical theories.

There are various interrelated themes and threads running through his work that characterize his stance. These recur throughout this thesis and so we briefly consider them now to set the philosophical framework. I shall not seek to justify Wilson's framework further, but as the analysis proceeds the philosophical value of the framework should become clear.

1.2.1 'Theory T syndrome'

One way of characterizing Wilson's approach is as an analysis of the role and behaviour of concepts in relation to scientific theories arising from a rejection of what he terms 'Theory T syndrome' or 'Theory T thinking'. He does not define this precisely, although through regular references we acquire understanding of the phrase via its usage. In a discussion of multi-scale modelling he suggests that the result of Theory T thinking is the attempted 'logification' of non-logical relationships between models and theories (2017, 220) which leads to confusion, driven by the assumption that 'in principle' one can calculate behaviours at different scales 'in a bottom-up manner' (2017, 231), reflecting a demand for grounding in fundamental laws (2017, 292). He also characterizes Theory T syndrome as the assumption that '"fundamental science" must strive to capture the fully autonomous behaviours of nature within their mathematical netting' (2017, 185, cf. 420-421), and that there is the 'presumption that suitably

articulated theories "implicitly fix the meanings" of their specialized vocabularies' (2017, 268). Hence, working from the Theory T stance, 'analytic metaphysicians … warmly assure us that someday science will supply us with a perfected Theory T from which all concerns of wobbly reference will be entirely expunged' (2017, 417), whose contours we can anticipate sufficiently well to support metaphysical reflection and analysis of contemporary scientific theories.

Wilson's use of Theory T thinking, which is a foil for the façade approach that he develops, is possibly best captured in terms of the general attitude that any theory can be characterised or 'rationally reconstructed' in terms of some unitary, axiomatized framework that is ready for metaphysical appropriation. Wilson claims that such thinking is to be rejected since it cannot adequately capture the heterogeneity and complexity of working 'scientific theories', modelling practices and their use of applied mathematics. Moreover the Theory T attitude leads to misplaced attempts to prematurely understand theories and associated concepts in a global, axiomatic or metaphysical sense rather than in terms of understanding the actual local supporting 'semantic architecture' and associated inferential pathways in particular applications of 'theories'.

He suggests that it is a good bet that owing to the limitations of human reasoning, understanding and our conceptual abilities the Theory T goal will never be achieved (cf. 2017, 392, 417). Instead of adopting such a Theory T stance and trying to fit existing theories into this mould, Wilson urges us to pay attention to the detailed workings and analyses of current applied mathematics and scientific theories as the locus of philosophical investigation.[3]

The result of 'Theory T thinking', and Wilson's complaint against it, is that it 'dull[s] our appreciation of the strategic subtleties within working science, through advancing inadequate discriminations that fail to distinguish between explanatory structures that are architecturally distinct' (2017, 136). It also leads to 'grand schemes for analysing notions such as "law," "cause," and "counterfactual conditional" in a logic-focused manner' (2017, 152) that pays insufficient attention to the observation that these, and other physical concepts such as 'force' and 'hardness' are 'mutable semantic creatures, able to acclimate themselves to virtually any explanatory landscape into which they happen to be

[3] For discussion of Wilson's Theory T thinking in one approach to QFT see Li (2015).

cast' (2017, 54). This view that such concepts have wandering significance, playing different roles in different scientific theories or modelling practices and are not always amenable to logical analysis contrasts with Theory T thinking in which the reference of such concepts is fixed globally.

Wilson suggests that much science can be characterized in terms of the exploitation of what he terms *descriptive opportunities* that nature offers: 'I talk much of the descriptive opportunities that nature makes available to us – the patterns and strategies of integrated linguistic employment that allow us to reach practical conclusions effectively and swiftly within a suitable environment.' (2017, 318) He defines a descriptive opportunity as 'physical circumstances whose dominant ranges of variation can be adequately captured in a smallish number of descriptive parameters and where significant questions of interest can be addressed through feasible calculation' (2017, 17).

He notes that the *reasoning advantages* that follow from the exploitation of descriptive opportunities can vanish if crude descriptions are replaced with detailed descriptions which attempt to model the actual physical situation more carefully (2017 36), such as modelling the behaviour of a steel beam at different length scales. For many purposes at macroscopic length scales a continuum model of the beam is more reliable and better suited than a more accurate model of the beam paying attention to its atomic structure (cf. 2017, 9-40).

Wilson's recurring concern is that we should pay close attention to the local application of scientific theorizing, concepts and models in different situations:

> [A]cademic philosophers should recognize that their diagnostic duties are more substantial – and less a priori determined – than many current writers presume. [I] … focus centrally upon situations where misidentified explanatory landscapes have created substantive philosophical confusion. [I have] … aimed at steering the reader's attention to the applied mathematics resources that can assist us in diagnosing underlying architecture better (2017, 88-89).

This encapsulates the philosophical approach to QFT that I adopt.

By paying attention to contemporary science and applied mathematics we often find that a scientific theory is a *façade* or a patchwork structure of local applications of theories, models and concepts (cf. 2006, 209, 324). Confusions arise when a concept or modelling technique, along with its mathematical architecture, is 'dragged' from one 'patch' of application to another without realizing that semantic shifts in supporting architecture have occurred. That is,

> doctrines of quite different supportive natures can mimic for one another quite nicely within classical physics, giving rise to a collected bundle of useful assertions that can seem – if we don't look too closely – as if they constitute a unified theory. In truth, however, we confront … a *theory façade* … an uneven pile of pasteboard cutouts that ably masquerade … for an integral metropolis. As such, its atlas structuring is secretly subject to substantial degrees of property dragging, but these semantic displacements occur in quiet ways. (2006, 356)

1.2.2 Concepts, patches and façades

Two interrelated themes in Wilson's work are the nature or behaviour of concepts, and (theory) façades, both of which are associated with 'patches' of local application. He suggests that confusions have arisen from what he terms the 'classical picture' of concepts and predicates in which they are taken to have global reference rather than local or patchwork semantic support and application.[4] Wilson claims that our conceptual grasp is weaker and thinner than the classical picture leads us to believe (2006, 41), and so our 'language employment often fail[s] to submit happily to axiomatic organization at the macroscopic [global] level …[so that] macroscopic doctrine is better arranged as a set of linked, but nonetheless disjoint, patches that shall be called a *façade* [an] odd organization [that] proves natural from a descriptive point of view' (2006, 179). Wilson argues via examples that this façade picture of concepts and theories offers a better description of science than does the classical picture of concepts coupled with generalized accounts of scientific theories interpreted in logical or metaphysical terms. We briefly consider two examples he discusses:

First, the concept of 'weight' and the associated concept of 'weightlessness'. This example is motivated by considering the 'weight' of astronauts in an orbiting spaceship. 'Weight' is commonly defined via impressed gravitational force, distinguished from mass. But if the weight of an astronaut in the space station is so defined, rather than being weightless as often suggested, only a small reduction in weight occurs relative to the astronaut's weight on the Earth's surface. Some authors recognize this problem, so speak instead of only 'apparent weightlessness', but whilst also speaking simultaneously of muscle deterioration in astronauts arising from weightlessness in a manner in which the weightlessness is not construed as merely apparent. For Wilson this illustrates

[4] See 2006, 139-146 for what Wilson terms the 44 'chief theses' of the 'classical view' of properties, predicates and concepts.

how a façade structure is evidenced for the concept of weight (or weightlessness), being understood and applied differently on different patches (2006, 328-335).

Secondly, the concept of hardness. The definition of hardness is given in relation to a context of application or use, with different hardness measures (e.g. Rockwell / Brinell, Knoops / Vickers or Mohr / Bierbaum) arising from different testing methods that relate to the application of the concept for different physical applications. He suggests that it is natural to search for some more fundamental evaluative trait ('essence') upon which each local application depends. But this is not how the concept operates, being distributed in a multi-valued fashion across different local patches of application (2006, 335-345). So 'our usage of the predicate "is hard" displays a fine-grained structure that we are unlikely to have noticed, for our everyday usage is built from local patches of evaluation subtly strung together by natural links of prolongation' (2006, 336). And, 'our employment of "hardness" silently distributes itself into a patchwork of sheets, locally distinguished by a certain vein of probing … that sit over various varieties of material stuffs and continue smoothly into one another' (2006, 338). It is a mythological picture of hardness to suppose that it must represent a single underlying characteristic (2006, 351-352).

These examples illustrate the contrast between the classical 'global' picture of concepts and the patchwork picture of concepts as forming an atlas (or façade) of patches of local application.

More formally, consider a domain D of physical fact that we wish to cover in a linguistically profitable fashion. Over D we erect basic patches or sheets A_i corresponding to localized flat maps on which some physical vocabulary or predicates, etc. will be made available (e.g. truth functions and quantifiers over subdomains covered in the patch). Over each A each predicate "P" will correspond to one or more attributes in D below, under the condition that if attributes φ and ψ are both assigned to "P" on A they will act coextensively within the region of D that it covers. There is a set of local recipes attached to each patch A. Each patch possesses a natural boundary ∂A marked by the fact that if one moves beyond ∂A into B then some of the predicates in A will either shift to new property alignments (which Wilson calls *prolongation*) within B or

the reasoning tools R native to A will no longer lead to sound expectations (which he associates with *semantic mimicry*). Boundary crossings indicate that one should not follow beguiling directives beyond the boundary. This sets up a local grammar on A, but we are often interested in how language behaves over the joins or continuations that connect the patches, which can be smooth, abrupt or overlapping (2006, 377-379). He concludes:

> Quite commonly there will be a small group of recipes or rules of inference R that prove central to creating this bridging between A and B (the role of power series in analytic continuation represents our prototype here). Patches can also sit partially astraddle of one another through fibered connections established by common names. Connected to these will be translation principles τ that regulate how data shall *lift* from one sheet to another … We demand no specific topology in how our atlas of covering patches fits together, so it may be possible to move through the patches in a multi-valued manner.
> An atlas of essentially one patch, that covers its whole domain adequately will be called a *flat structure*; it is essentially the linguistic platform that the classical thinking expects to see, once language has been cleansed of its undesirable ambiguities. …
> [There is a] need for a *preface* or *picture* of our atlas's workings: viz., a schematic overview of how the patch-to-world relationships unfold in the façade. … [A]n agent might be able to employ an atlas quite capably from a practical point of view, yet entertain an erroneous picture of its descriptive workings. … [W]e shall be much interested in *semantic mimicry* where some façade-like construction passes, amongst its employers, for a flat structure: it looks very much like the "first order theory" of the logicians if we don't scrutinize its oddities too closely … A façade assembly should be regarded, in analogy to the two-sheeted Riemann surface for $\sqrt{z}$, as a *strategically informed platform* upon which a stable linguistic usage can be settled … As long as a speaker respects the boundary divides marked by ∂A, she can employ an unevenly founded language to freely express what she wishes locally, while exploiting the boundary restrictions between regions to create an overall employment that may prove more effective and efficient overall. (2006, 379)

Whilst it is not a point that Wilson develops, perhaps façades arise in two different, although related ways that we might label 'theory façade' and 'concept façade'. Examples of 'concept facades' are reflected in Wilson's analysis of 'weight' and 'hardness' whilst 'theory façades' are illustrated in the modelling of billiard ball collisions or the elastic behaviour of steel beams. That is, in a 'concept façade' one discovers the structure and application of a concept – how its semantic support differs in different contexts of application and how these might relate. In a 'theory façade' different models or theories are used on different contextual patches to model the same physical domain according to different interests, perhaps as relates to behaviour at different length scales.

So for instance the behaviour of a steel beam (the same physical phenomenon) is modelled very differently on a macroscopic patch of application, at the microscopic level, and at the atomic level (cf. 2017, 9-40). The formal description of a façade above seems more applicable to a 'concept façade',

whereas a theory façade, and its relation to Theory T syndrome is perhaps more clearly summarized as:

> A descriptive complex of this quilt-like pattern [here, different models of a billiard ball collision] supplies a good example of what I intend by a *façade*: a set of patches or plateaus that are formally inconsistent with one another but are stitched together by "for more details see …" linkages or other bridgework. Often the whole is fabricated in such a manner that, if we don't play close attention to its discontinuous boundary joints and shifts in mathematical setting, we might suppose that we are looking at a theory ready to be axiomatized (2006, 191-192).

In many cases however a façade may manifest both aspects, where concepts are dragged from patch to patch upon which different theoretical models are used to describe the same physical phenomenon according to different questions of interest.

For Wilson, what is important is not the logic or axiomatization of a theory or the attempt to identify a unifying underlying theory, or the identification of a putative 'true' meaning of a concept, predicate or property, but the reliability and explanatory value of the descriptive opportunities appropriated (in the sense that physically salient explanations may be offered, inductive inferences may be drawn, and counterfactual reasoning supported, on a given patch of application), and the clarification of the architecture of the concepts and theories or models employed so as to avoid conceptual confusions (cf. 2017, 282). The alternative to Theory T thinking is paying attention to the developmental histories that have produced overburdened concepts (2017, 152), for such developmental histories shape the use and application of central terms (2006, 36-37),[5] according to a local façade structure rather than a global 'flat' structure'.

1.2.3. Prolongation, property dragging and semantic mimicry

The notions of *prolongation* or *property dragging* and *semantic mimicry* are important for Wilson's analysis of concepts. Brandom helpfully articulates Wilson's notion of 'property dragging' in terms of

> cases where the range of proper application and the inferential consequences of application of some predicate drifts over time, pulled now one way, now another by features of the actual properties of the system of which the users of the predicate are at most only vaguely aware. This is the ''wandering significance'' of the book's title. In place of the classical picture of a sense determining a reference, a conceptual content stably ''gluing'' a predicate to a property, Wilson offers

[5] Wilson cites Duhem, 'The real introduction to the expression of a principle of physics is a historical introduction', suggesting that 'philosophers of language have much to gain from Duhem's investigations' (2017, 153).

> not an account of an alternative adhesive, but simply a more detailed accounting of the machinery of cooperation (and lack of it) between Nature and man that often leads descriptive language along the improving, but often mysterious, developmental paths we frequently witness. [2006, 235-6]
>
> … [P]roperty dragging ought not, Wilson argues, to be thought of as a shortcoming or blemish that one might hope a more perspicuous or detailed account or idiom could eliminate, nor as a merely epistemic difficulty resulting from imperfections of our understanding. It is rather an essential aspect of the procedures that make it possible for us to do as good a job as we simple-minded folks do in describing a messy, complicated world. (Brandom 2011, 189-190)

Maddy also provides a helpful analysis of Wilson's project (2007, 175-196). She suggests that Wilson takes the 'serious work to be the investigation of word-world connections … reject[ing] the call for a single "mechanism of reference"'. However, there is room to be 'more skeptical than Wilson's correlationalist on a prediction: that further study will uncover unified supports for word-world connections in local patches.' (2007, 196). This may help elucidate a difference between Wilson's stance and Wittgenstein's, which have been noted to be similar in many regards (Wilson 2006, xx; Brandom 2011, 200). Both Wittgenstein and Wilson advocate the rejection of the classical, global picture of the reference of concepts in favour of a picture that concepts are associated with local contextual application, relative to 'patches' (Wilson) or 'language games' (Wittgenstein).

However, Wittgenstein frequently discusses 'borderline cases' in which the application of a concept is unclear. A paradigmatic example for Wittgenstein is the analogy between ethical or aesthetic concepts and a watercolour painting of coloured shapes in which the colours and shapes blur into each other and merge so that neither the shapes nor the colours are clearly defined near the boundaries (Wittgenstein 2001 [1945] §§76-77). Wilson's paradigmatic example is that of a Riemann surface, where concepts are defined locally on each sheet of the surface with clear rules regarding the behaviour of the concept when moving between sheets (i.e. local contexts of application). This might indicate that Wilson's picture of concepts is 'locally classical' whereas Wittgenstein's is not. We shall discover that 'superposition', whilst patchwork, perhaps does not always neatly resolve, with there being borderline cases (see chapter 5), being a novel feature of the prolongation of concepts that does not occur in Wilson's analysis, although such borderline cases do not affect my central thesis.

Associated with property dragging, 'semantic mimicry' is of central importance in clarifying the significance of concepts in physical theories and the

conceptual confusions that can arise (Wilson 2006, 567-598; 2017 326-327). Semantic mimicry occurs when a concept is applied on more than one patch where it appears that it has the same semantic support in each case whereas in fact the support differs. This may not be noticed, and it may not be noticed that multiple patches have been adopted. More precisely, Wilson suggests semantic mimicry occurs when

> a local patch A looks as if it is semantically supported in manner *M*, although it is actually propped up in fashion *N*. Grammatical sentences that would be meaningful if *M* represented their proper support do not gather any reading under *N* (… where the *working grammar* of "P" differs from its *apparent grammar*). … Many celebrated puzzles with respect to causation can be aligned with these patterns (2006, 568, cf. 2017, 326-327).

He discusses Euler's approach to the buckling of a thin strut under an applied load to illustrate semantic mimicry. From the full partial differential equation (the wave equation) modelling the strut's behaviour a 'reduced' time-independent differential equation is extracted giving the strut's equilibrium (2017, 65). He contrasts the equations and their different solution techniques to indicate how and where confusions arise. The significance of a numerical method via successive approximation for the reduced equation is contrasted with solution of the full equation, noting that it is possible to mistake the successive approximations of the strut's shape given by the reduced equation for actual shapes of the strut as they unfold in a causal process as modelled by the full time-dependent equation (2006, 579-580).

This is an instance of semantic (and causal) mimicry. 'If we picture this perfectly valid inferential technique as providing a story of how "causal processes" unfold in our strut, we have fallen victim to *semantic mimicry.*' (2006, 580) Causal mimicry is a specific case of semantic mimicry in which the mathematical architecture of one patch whose support might rightly be understood in terms of physical causal processes is transferred to another patch in of similar mathematical architecture but with different support, yet where the physical semantic support (i.e. of physical causal process) is illegitimately transferred to the new patch. Here, it might be supposed that the mathematical technique associated with solving the reduced equation gives rise to the identification of a physical causal process when it does not (2006, 587-588).

I shall use the term 'semantic mimicry' in a slightly narrower sense than Wilson. By semantic mimicry I mean an unnoticed instance of the dragging or

prolongation of a concept (whether explicit or implicit) such that its supporting architecture has changed so much that the concept should no longer be applied in such an instance as the concept would not share sufficient resemblance to its application on other patches. We shall see examples of such mimicry in classical physics in §4.5, which will be important in the analysis of QFT.

1.2.4. Realism, metaphysics and Wilson's approach

I shall now situate Wilson's approach in relation to realism so as to clarify what sort of philosophical claims might follow from a 'Wilsonian analysis' of QFTs, and whether any particular stance is demanded.

Realism *per se* is not a significant theme in Wilson's work, although one should distinguish responses to two approaches to 'realism' that can be discerned, namely a modest 'scientific realism' (or perhaps 'shallow realism') and 'metaphysical realism' (or 'deep realism').[6] Broadly speaking scientific realism may be understood in the sense that scientific theories and terms 'latch on' to the world even if we are not in a position to specify what this 'latching' consists in, where it might be understood in a local, case-by-case manner.[7]

Wilson acknowledges that he operates within a modest scientific realist context:

> I am a scientific realist at heart and have no doubt that all of these varied patterns "fit together somehow." But, at the present moment in scientific time, we don't really know how this "fitting together" formally operates, and nature offers many surprises on this score. (2017, 79)

and:

> I am as confirmed a "scientific realist" as walks this planet and believe that ongoing science gradually accumulates a large set of reliable "truths". However, I don't believe that the truth-rules for these truths can be captured by … simple isomorphisms … Such views … suffer from insufficiently flexible options with respect to the semantics of language. (2017, 361. Cf. 2006, 10.)

However, against the context of Theory T thinking Wilson suggests that his goal is 'not to supply contravening dogmas of my own, but to widen our methodological appreciation for the variety of explanatory landscapes arising within science.' (2017, 79) As we have seen, this entails a rejection of what he terms the 'classical' picture of concepts and descriptive terms – that they are 'glued' or refer rigidly to their referents in the world, a confusion that arises in

[6] For discussion of the distinction see French (2017; 2018).
[7] For analysis of contemporary approaches to scientific realism see Saatsi (2017).

part from the assumption – that terminologies firmly attach to the external world in uniform ways in science. (2017, 382)

Is this an anti-realist stance? Concluding a discussion of Duhem (whom he uses to motivate his development of conceptual wandering), Wilson suggests that 'the proper moral … from Duhem's example is not anti-realism, but patience. Methodological puzzles often take a long time before their underpinnings can be fully rationalized.' (2017, 200)[8] Scientific theories and applied mathematics exploit descriptive opportunities that nature presents in particular patches of application via concepts and descriptive terms that greatly simplify the ways in which one captures the *dominant behaviours* of nature on those patches.

Wilson does not identify as an anti-realist or instrumentalist. But neither does he identify as a traditional pragmatist, for, he suggests, traditional pragmatists are deflationists about meaning and reference, which, he claims, he is not. He 'merely think[s] that referential ties to the natural world ultimately stem from language's practical entanglements with it, in manners that which often employ rather complex forms of data registration' (2017, 282). In this sense he sees himself as a 'semantic pragmatist' (282). One should adopt a stance of epistemic humility and remain silent regarding the truth conditions of concepts or descriptive terms on particular patches, and not overestimate human conceptual capacity (2017, 286, 392).

Whilst Wilson is a modest scientific realist, his stance towards metaphysics is more difficult to articulate. He is clearly critical of much contemporary analytic metaphysics, and in that sense anti-metaphysical. However, he is perhaps best read as metaphysically quietist or neutral rather than anti-metaphysical per se. He regards Theory T thinking as mistakenly developing scientific theories of local and limited applicability into metaphysical doctrines using a 'flat' or global understanding of concepts and descriptive terms coupled with a logic-centred interpretation of the relevant scientific theories and an undisciplined use of the concept of 'law' (cf. 2017, 339, 375). This leads to confusion and mistaken metaphysics, and as such he sees much contemporary metaphysics as having gone astray. For instance, he notes that the application of mereology to fracture mechanics is wholly misleading (2017, 37-38).

[8] Cf. 2017, 415 - he does not advocate anti-realism.

However, Wilson refrains from making predictions regarding future science and philosophy, and so taking these comments in conjunction with those on scientific realism his stance is best characterized as metaphysically quietest although verging on anti-metaphysical: it is difficult to see under what circumstances quietism would give way to metaphysical assertion. But what is important for us is that through conceptual clarification one may be able to identify misplaced metaphysical claims, or misplaced claims that a realist interpretation should be adopted to particular theories or theoretical terms.

1.2.5. Concepts and metaphysics

Wilson argues that both physical and metaphysical concepts such as 'force', 'hardness', 'temperature', 'weight', 'cause', 'law', 'part', 'whole', 'composition', 'natural kind', etc. should be subjected to local 'patchwork' analysis to reveal the differing semantic architectures upon which they are based in the context of local patches of application so as to reveal and diagnose confusions that have arisen in their use.[9] Such analysis may reveal where metaphysical weight cannot be placed, and where it is possible that metaphysical commitments might be sought even if they are deferred. For example, he suggests that 'force' and 'cause' are 'mutable semantic creatures … able to accommodate to any explanatory landscape' (2017, 54, cf. 244).

Two concerns recur throughout Wilson's analyses of the description of physical systems. First, that as highlighted by Hadamard (1923), in the context of PDEs the boundary or accessory conditions are as important in determining the form of a solution as the form of the equation itself (2017, 210), which problematizes the idea of a law as associated with a differential equation (DE) as modelling causal processes as in 'ODE thinking'. Indeed, Wilson suggests that further confusions have arisen owing to different applications of 'cause' in ordinary differential equations (ODE) as distinct from partial differential equations (PDE), noting (again drawing upon Hadamard (1923)) that a change in explanatory landscape occurs in moving from the ODE context to the PDE context (2017, 413-415). Secondly, that the formation of DEs in physics is often

[9] See e.g. force (2017, 27, 28, 33, 86, 320-323); temperature (2017, 28-29); cause (2017, 42, 67, 76-77, 244-247, 252-253, 260, 262, 286); part / whole / mereology / component (2017, 37-39, 42, 241-244, 418); natural kind (2017, 146, 185); law (2017, 152, 339, 375).

not motivated by the ambition of describing real life behaviour at the microscopic scale. Rather, DEs serve as inferential paths taken en route to important conclusions pertaining to higher scale level behaviour as for example in modelling the bending of a steel beam. Different DEs play different explanatory roles in science (2017, 68) and one should pay attention to their local application.

Wilson's preference regarding some philosophical concepts would often appear to be to replace them. He suggests that the notion of projectibility is better construed in terms of being 'suitable for effective reasoning' (2017, 275-276), and 'simplicity' in terms of capturing dominant behaviour (2017, 214). Again, although his analysis of these notions brings clarity regarding their significance, it is unclear whether the rejection of these notions in favour of Wilson's is always philosophically advantageous. At the very least, noting that the behaviour of some system can be characterized by a model that has, at a minimum, the appearance of simplicity, strength and projectibility (even if these are to be understood locally and modestly) might be said to offer an explanation of why some properties seem inherently suitable for describing the system or 'natural', and why this model of behaviour might be characterized as at least 'law-like' or 'reliable'.

In summary, Wilson advocates careful local conceptual and semantic analyses to clarify the use of concepts and descriptive opportunities so as to differentiate and elucidate their applications on different patches. Sometimes this manifests in rejecting the possibility of a realist interpretation of certain terms in a scientific theory or model – and sometimes in a quietest stance in the sense that it is now best to remain silent leaving the possibility that more might be said in the future.

1.2.6. An example: particle concepts

I shall analyse the concepts of superposition, component and particle, and the different semantic architecture of Fourier techniques in different applications, and their significance, as motivated by Wilson's framework in the following chapters. In particular we shall see that attention to local application, property dragging and semantic mimicry illuminate our understanding of QFT.

As an initial example, I briefly consider Falkenburg's 'metamorphoses of the particle concept' (2007, 209-263) from a Wilsonian perspective. Falkenburg notes that the particle concept is used in various senses. She claims that particles are 'experimental phenomena rather than fundamental entities' (2007, 209), and in a manner that has resonances with Wilson, she suggests that

> after the quantum revolution only an *informal* particle concept remains. This concept is more than a mere *façon de parler*. But it is metaphysically more modest than the mereological and causal particle concept associated with classical physics. It has an operational basis which stands in precise relations to the current quantum theories, and it has some typical features, amongst them *statistical independence.* (2007, 210)

She proceeds to analyse eight different particle concepts: classical particles, quantum mechanical particles, light quanta, operational particles, field quanta, group theoretical particles, virtual particles and quasiparticles before going on to consider particles in relation to 'matter constituents' in which bound systems are discussed (210-256). Up to this point her analysis shares some similarities with Wilson's in that she identifies differing local applications of 'particle', which may be related to patches of application.

However, rather than considering these applications as patches of application of 'particle' and studying the semantic architecture of each patch and the descriptive opportunity that it provides, she goes on to consider a general particle concept (258-263). It is not clear what philosophical work is achieved in this consideration of generality – which is perhaps Wilson's point. I shall not discuss these different 'patches' of application of the particle concept here – rather, I am simply anticipating how a Wilsonian analysis might proceed before appropriating some of these particle concepts in my discussion when needed in the following chapters. To analyse these particle concepts first requires a 'Wilsonian analysis' of concepts such as 'superposition' and 'component' as used in the context of applied mathematics in terms of prolongation, dragging and semantic mimicry.

1.3 Thesis overview

A Wilsonian framework to the analysis of QFT is more naturally applied to 'physicists' QFT' rather than 'axiomatic QFT', and this is the route that I am taking. Within the context of 'physicists' QFT' a locus of study of 'effective field theory' using renormalization group analysis has emerged (cf. J. Fraser 2016;

Hancox-Li 2015), which may sit well with Wilson's approach, but this is not the route that I shall take as there are other issues that I wish to pursue. QFT is not a unified theory, but is a façade of local applications of theories. My concern will be primarily with canonical QFT as applied to particle physics.

Thus I shall focus on the concepts of particle, superposition, interaction, and component and the role that they play – especially in the applied mathematical context of (generalized) Fourier techniques – as analysed through the philosophical framework that Wilson provides. Since little philosophical work has been done on the nature and significance of both the superposition concept and Fourier techniques I shall need to spend some time laying a philosophical foundation for these in classical physics before turning to their application in QFT. As we shall see the Wilsonian framework, with a few developments motivated by both Volkmann's (1896; 1900) and Simons' (1987) analyses of the concept of superposition, is well suited to their analysis as a façade structure to the concepts may be identified.

The point is that, as we shall see in chapters 6-7, key aspects of the various architectures of 'superposition' in classical physics which are central to its significance and thus the interpretation of theories and models that rely on it, whether explicitly or implicitly, are retained when the concept is prolonged to quantum physics even if there are significant additional novel aspects to the architecture of 'superposition' in quantum physics. That is, there is both continuity and discontinuity of the semantic architecture of 'superposition' in the concept's prolongation from classical to quantum physics, and it is the aspects of continuity that are central to my analysis. The aspects that are discontinuous are often associated with the 'measurement problem', and are tangential to my argument and not discussed – that is, the philosophical difficulties that I identify via 'superposition' are 'upstream' of or prior to the measurement problem.

Whilst the analysis I present does not supply a metaphysical (deep) or even a scientific (shallow) realist account of QFT, it clarifies where and why the conceptual difficulties occur by considering the misapplication of 'superposition'. My analysis will however indicate where realist interpretations cannot be placed owing to failure of 'superposition' whilst showing where one might seek local and limited shallow realist interpretations of the theory that clarify where, for

instance, a (suitably modest) particle interpretation might be appropriate, with clarity brought as regards what might be meant by a particle interpretation.

However, 'semantic mimicry' often occurs when the particle concept or description, along with the concept of superposition, is 'dragged' from local contexts of descriptive opportunity to try to describe general interacting states via implicit appeal to 'superposition'. This suggests that, according to QFT at least, it is wrong to suppose that matter is in a metaphysical sense composed of particles like electrons, quarks and gluons. However, depending on the strength of the field couplings, a particle description can still provide a reasoning advantage in an 'engineer's sense' (as clarified in chapter 11) if the coupling is weak (such as in QED at low energy), but not if the coupling is strong (such as QCD at low energy).

More fundamentally though, as the conceptual difficulties involved with renormalization highlight, the superposition principle for both fields and particles fails to hold for anything but free QFTs, so that we discover that QFT does not support an interpretation of a (near) 'fundamental level' in terms of entities (be they particles or fields) and their properties. It is the failure to be able to apply the superposition principle that reflects some of the fundamental conceptual difficulties with QFT, rather than our ignorance of short length-scale physics or our inability to (as yet) develop a working quantum theory of gravity. As well as highlighting the failure of 'superposition', renormalization is the means by which its failure is partially accommodated so that empirically adequate calculations may be performed. One can claim genuine knowledge of the world through QFT, but in a restricted sense – such as that of the probabilities of outcomes of scattering experiments for example.

So, in chapter 2 I survey the history of the application of 'superposition' and the sparse philosophical literature on the concept in relation to Paul Volkmann's analysis of the concept (1896; 1900; 1910). Then in chapter 3 I analyse the superposition principle and allied notions of components and vector decomposition for systems with a finite number of degrees of freedom in classical physics. In chapter 4 I consider the façade structure of 'superposition' and its semantic mimics for systems with infinitely many degrees of freedom with reference to Fourier techniques and Sturm-Liouville theory. I discuss linear

systems in relation to 'superposition' in chapter 5 and contrast the application of 'superposition' in linear systems to its failure, and hence semantic mimics in nonlinear systems. In chapter 6 I build upon chapters 2-5 to survey the application of 'superposition' on the new patch of quantum physics, noting the historical origins of the quantum usage of the concept and its continuities and discontinuities with classical usage, and then I consider the historical foundations of QFT in chapter 7.

In chapter 8 I analyse in detail the application of 'superposition' and 'particle' in free QFTs. The importance of the linearity of the field equations in conjunction with the superposition principle is highlighted for exploiting the descriptive opportunities available in free QFTs, being associated with the ability to construct physically meaningful Fock space structures. In chapter 9 I consider how interacting QFTs are constructed perturbatively from free QFTs and introduce scattering theory and the use of Dyson's series in the interaction picture. We'll see that Feynman diagrams and virtual particles are often misinterpreted in a realistic sense owing to semantic mimicry via misapplication of 'superposition'. I develop a conceptual analysis of interacting QFTs in chapter 10, concluding that no particle description of interacting states is supported before considering what kind of 'natural description' is available – which turns out to be very thin even if it does have applications. The conceptual difficulties are diagnosed in terms of the nonlinearity of the coupled field equations, and the initial failure of 'superposition' understood in Volkmann's general sense to the selection of the fields and associated states. Renormalization is the means by which the initial failure of the 'generalized superposition' is partially compensated for, supporting empirically adequate results.

In chapter 11 I indicate how unstable particles are modelled in QFT before completing the discussion of scattering theory. The analysis of bound states is addressed briefly, and strongly coupled theories are compared with weakly coupled theories in the context of an 'engineering approach' to understanding the nature of QFT and the kind of approximate descriptions that might be available in cases where 'superposition' is approximately applicable.

My conclusions are summarized in chapter 12 along with some suggestions for further research.

Chapter 2

Isolation and Superposition in Mathematical Physics: Overview

2.1 Introduction

I start to lay the groundwork for my analysis of QFT in relation to 'superposition' through a Wilsonian analysis of the concept and associated Fourier techniques. The concepts of 'superposition' and 'component' will be seen to have façade structures whose semantic support or physical significance differs between contexts or patches of application (cf. chapter 1). Semantic mimicry can occur leading to conceptual confusion. I'll analyse the differing supporting architectures whilst refraining from metaphysical judgements, considering instead how the superposition principle relates to establishing a 'natural' descriptive opportunity in Wilson's sense. What we shall ultimately discover is that applications of 'superposition' on a 'QFT patch' often fail, being subject to semantic mimicry leading to misplaced metaphysics, such as that of a particle ontology which QFT does not support in general. By clarifying the role of 'superposition' we shall see that the natural descriptions available in QFT are very thin, being available only 'in principle' and, in general, unrelated to familiar particle concepts.

The concept of superposition together with Fourier techniques and related Sturm-Liouville theory are central to mathematical physics, engineering, and in particular, quantum physics. However, there has been rather little philosophical engagement with either the concept of superposition, or with Fourier techniques and associated notions of 'component' and 'composition' and so their roles in the interpretation of physical theories are poorly understood. The fullest philosophical treatment of 'superposition' in mathematical physics of which I am aware is that of Paul Volkmann's (1896, 1910). He studied the processes of isolation and superposition in the context of his project of the development of an epistemology of science. More recently 'superposition' has been discussed by Peter Simons in relation to metaphysical application to

material objects or substances, but not to physics (1987). Isolated discussions of Fourier analysis, sometimes tangential or passing, are found in Redhead (1988), Healey (2013), Wilson (1993; 2006; 2013; 2017) and Liston (1993), but there is no sustained discussion of the technique and its association with superposition, and indeed much of the discussion in the contemporary literature was anticipated by Fourier (1878 [1822]).

There is, however, a growing body of philosophical literature on vector composition, and the composition of forces in particular, even if 'superposition' is not explicitly in view.[1] This literature stems from Cartwright (1980) and Creary (1981), although it has its roots in Mill (1843-1882) and was anticipated by Volkmann (1896).

I'll adopt and adapt examples from this literature to analyse the concepts of superposition and component in relation to vectors in particular in chapter 3, and, in chapters 4-5, Fourier techniques. Much of the recent literature has a metaphysical focus or is concerned with realism. For instance much discussion in the vector composition literature revolves around the issue of causal overdetermination. In my Wilsonian approach I remain metaphysically quietest having noted in chapter 1 difficulties regarding the architecture of concepts such as 'cause', 'force' and 'law'. So, I shall remain silent on a number of questions that arise, and use these terms in a metaphysically neutral Wilsonian fashion.

I shall, however, consider some of the inferential pathways that have been adopted with a view to clarifying what Wilson terms the 'semantic support' or 'supporting architecture' for 'superposition' (and Fourier techniques in the following chapters) as adopted on various patches of application, which may feed in to realism debates. My analysis will focus on the clarification of the semantic architecture of 'superposition' which will help us to see where realist commitments *cannot* be placed in QFT as well as enabling us to diagnose some of the conceptual difficulties with QFT.

In this chapter I survey the application of 'superposition' and the sparse philosophical literature on the concept.

[1] See e.g. the special edition of *Dialectica* (2009, 63.4) dedicated to vectors.

2.2 An overview of the façade structure of 'superposition' and its history

The superposition principle is often introduced in elementary physics in terms of graphical construction, inviting intuitive visualization, for example the interference of waves, or vector addition of forces (Whelan and Hodgson 1978 35, 103-104), or the motion of a vibrating string as 'composed' of its harmonics (Brillouin 1946, 2). Alternatively it has been defined as a property of linear differential equations. In this case the 'superposition principle' states that a linear combinations of solutions of a linear differential equation is also a solution (Courant and Hilbert 1924, 221). This definition encompasses wave interference and string vibrations, since these phenomena are modelled by linear differential equations, but not the vector addition of forces in statics. The superposition principle may also be defined via integral equations (Volterra 1913, 219). The use of 'superposition' in quantum physics – notoriously in thought experiments about the state of a cat – reflects a different understanding of 'superposition' again (Schrödinger 1926a-e, 1935). Already this motivates consideration of the 'wandering significance' and façade structure of 'superposition', 'component' and 'composition'.

However, despite both historical and contemporary 'patchwork' definitions of 'superposition', attempts have been made to identify a common ground for these patches of application. For instance the Penguin Dictionary of Physics defines superposition as, 'A principle that holds generally in physics whenever linear phenomena occur.' (Illingworth 1990, 469). Elasticity, vibrations and waves are cited as exemplifying the superposition principle. This definition is developed in the Wikipedia entry:

> The superposition principle, also known as superposition property, states that, for all linear systems, the net response caused by two or more stimuli is the sum of the responses that would have been caused by each stimulus individually. So that if input *A* produces response *X* and input *B* produces response *Y* then input (*A* + *B*) produces response (*X* + *Y*). (Anon, 'Superposition Principle')

Such a definition might appear to abstract the essential features of various uses of 'superposition', which would suggest that it is a 'flat' concept after all for which the 'classical picture' of concepts suffices. But as we shall see this definition fails to capture usage of 'superposition' adequately.

The history of the concept of 'superposition' in mathematical physics is difficult to trace since the concept has often been used implicitly (e.g. Bernoulli

1753a&b) yet with its explicit usage often implied by later authors (e.g. Brillouin 1946, 2 on Bernoulli). The application of the concept or principle in physics is probably to be traced to Galileo's analysis of projectile motion (Mach 1942 [1933], 181-186; Volkmann 1896, 74-76; cf. Galileo 1914 [1638], 244-294, esp. 262-263). However, we find only passing reference to 'superposing' per se in Galileo, drawing upon the geometrical concept of superposing figures, although the concept appears to be in use implicitly.[2] Galileo's understanding of the physical significance of 'superposing' is illustrated by the interlocutor Sagredo:

> One cannot deny that the argument is new, subtle and conclusive, resting as it does upon this hypothesis, namely, that the horizontal motion remains uniform, that the vertical motion continues to be accelerated downwards in proportion to the square of the time, and that such motions and velocities as these combine without altering, disturbing, or hindering each other, so that as the motion proceeds the path of the projectile does not change into a different curve (1914, 250 [1638, 273])

Bernoulli analysed the motion of a vibrating string, and subsequently Chladni the vibrations of plates, in terms of an implicit principle of superposition, that is, in terms of a mixture of coexistent simple vibrations that exist independently of each other (Bernoulli 1753a, 160; Chladni 2015 [1809]). These examples were later interpreted and developed in terms of 'superposition' explicitly (Fourier 1878 [1822]; Herschel 1830), although Fourier applied 'superposition' primarily in relation to his development of the analysis of heat flow (see chapter 4).

However, in the early-mid 19th century several 'superposition principles' emerged: Kipnis traces a principle of superposition of small motions, a principle of superposition of vibrations, and a principle of superposition of waves (1991, 17-24). Whilst we must be cautious in attempting to read 19th century authors in terms of later debates, as we shall see in chapter 4 in this era the components of superpositions appear to be understood in what we would understand to be a realist sense – the components are taken to exist (meta)physically and are not mathematical artefacts (Fourier, Herschel). For example Herschel argued:

> We may here notice a very remarkable experiment … which shows to what an extent the principle of the *superposition* of vibrating motions and the simultaneous coincidence of different modes of vibration in the same vibrating body, must be admitted in Acoustics. If, instead of one, two … tuning-forks be held over the mouth of a pipe [e.g. an organ pipe] side by side, both nearly in unison with the pipe … The same column of air, then, at the same time, is vibrating as a part of two distinct systems, and each series of vibrations,

[2] See Damerow et al (1992, 266-267); Naylor (1980, 560-561) and Prudovsky (1989, 455) for consideration of Galileo's treatment of projectile motion in relation to 'superposition'.

> however near coincidence they may be brought, continues perfectly distinct and absolutely free from mutual influence. (1830, Art. 205)[3]

In the mid-19th century the superposition principle(s) were often related to small motions, or infinitesimal quantities, and perhaps it was this aspect of 'superposition' that Hilbert (Courant and Hilbert 1924, 221) and perhaps Volterra (1913, 219) focused upon.[4] However, 'superposition' was not applied to force composition, i.e. to the 'parallelogram of forces' even if this would become a paradigmatic example of superposition (Volkmann 1896; Mach 1942 [1933]), and was perhaps closer to Galileo's original application of the concept. For instance, writing before the development of the concept of vectors, [5] Pratt did not interpret the resultant composition of forces in terms of superposition (1841, Art. 11-67), but did explicitly use the 'principle of superposition of small motions' to analyse planetary motion (Art. 288), which he also associates with the 'Principle of the Coexistence of Small Vibrations' (Art. 490-491). Indeed, the application of 'superposition' was sporadic in the 19th century with some authors using the concept rather sparingly (e.g. Thomson and Tait 1888). That is, usage was not consistent.

As with Bernoulli and Chladni, who used 'superposition' without naming it as such, so we may see the use without naming of the concept with regard to the composition of forces or causes. So for instance Mill considers the way in which several agents or causes compose to act jointly, such as in the composition of forces to deduce the joint action from the individual actions:

> To enable us to do this, it is only necessary that the same law which expresses the effect of each cause acting by itself, shall also correctly express the part due to that cause, of the effect which follows from the two together. … This law of nature is called, in dynamics, the principle of the Composition of Forces: and in imitation of that well-chosen expression, *I shall give the name the Composition of Causes to the principle which is exemplified in all*

[3] This is a standard 'realist' move, to adopt a realist stance to or existence of an entity (here, an individual mode of vibration of a column of air) owing to its causal relevance – the 'Eleatic Principle'. However, this example in fact differs in important ways from the resonance of a pipe set up by some arbitrary acoustic source.

[4] The principle of superposition of small motions is stated by Thomson and Tait:

> From similar considerations follows also the general principle of *Superposition of small motions.* It asserts that if several causes act *simultaneously* on the same particle or rigid body, and if the effect produced by each is of the first order of small quantities, the joint effect will be obtained if we consider the causes to act *successively,* each taking the point or system in the position in which the preceding one left it. It is evident at once that this is an immediate deduction from the fact that the second order of infinitely small quantities may be with rigorous accuracy neglected. This principle is of very great use … its applications are of constant occurrence. (1888 Art. 89; cf. Volkmann 1896, 84; Poincaré 1905, 148).

I have been unable to trace the origins of this view.

[5] See Crowe (1967) for the development of the vector concept in the late 19th-century.

> *cases in which the joint effect of several causes is identical with the sum of their separate effects*. (1851, I. 373-374, emphasis added)

Volkmann, and subsequently Mach, would recognize the 'composition of causes' exemplified in the 'Parallelogram Law' of forces, or vector composition more generally, as a paradigmatic example of 'superposition' (e.g. Volkmann 1896, 177), although Volkmann is careful to suggest that we should think not in terms of the composition of forces or causes, but in terms of the composition of their effects (Volkmann 1896, 81).

By the late 19th – early 20th century then the 'superposition principle' was applied in increasingly broad and diverse although sporadic ways, arguably faithfully to Galileo, without distinguishing between local applications (e.g. Volkmann (1896-1910); Mach (1897-1942 [1933])), having acquired a façade structure in the sense that the concept is applied with reference to differing supporting architectures. It came to be interpreted as an 'epistemological principle' rather than a 'natural' or 'metaphysical' principle however.[6] For example Boltzmann argued that understanding

> the superposition principle as a general natural principle, seemed to me very premature; moreover, the requirement of its separate applicability to the three different directions of coordinates makes it pure abstraction, since the coordinate axes only exist in our imagination [Phantasie]. Moreover, when there are several material points, the separation of the various activities [Arbeiten] of each point in its motions is arbitrary. (1896, 45 [my translation])

Here, it is the decompositions of vectors using arbitrary coordinate systems, or coordinate systems chosen for convenience, that is in view.

Entering the 20th century the superposition principle came to be understood in relation to integral (so Volterra) and differential equations (so Hilbert): Volterra, developing Boltzmann's work on the 'heredity principle' in elasticity,[7] articulated some ideas foundational to modelling linear systems in

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[6] This might account for its increasingly sporadic use in the latter 19th century as metaphysics came to be seen as problematic. The two poles of this debate replay geometric pre-Galilean 16th century debates. For instance, Clavius comments on Peletier

> he does not seem to have understood in a satisfactory manner how Geometers use that superposition. For they do not want that superposition to be carried out in reality (for that would be something mechanical), but only in thought and in the mind, which is the task of reason and of intellect. (Clavius 1589, in Mancosu 1996, 121).

Mancosu discusses the widespread philosophical significance of superposition in the 17th century (1996, 32), noting that proofs by superposition 'became central to debates over foundations of the geometry of indivisibles' (33).

[7] Markovitz (1975, 431) discusses Boltzmann's contribution to superposition in the context of elasticity or rheology, 'that the influence of the deformations which are imposed at various times can be superposed', and Ianniello and Israel (1993) link Boltzmann's work to Volterra's.

relation to 'superposition' that we shall consider in chapter 5.[8] The idea is that if a linear system is subjected to a series of 'impulses' at different times, then its response at some later time is constructed from the 'superposition' of the effects of the impulse responses taken individually in isolation. Volterra analyses a thread which twists by a displacement $\omega(t)$ in response to a time-varying torque $M(t)$. He introduces a 'coefficient of heredity' Φ that relates the displacement at t to torques applied at earlier times τ considered as a continuous sum of impulses so that

$$\omega(t) = KM(t) + \int_0^t \Phi(t-\tau)M(\tau)d\tau$$

Volterra comments:

> We can now, as a first approximation, admit that Φ depends on M by a linear relation ... From a physical point of view, this amounts to supposing that the effects of the superposition of the moments of torsion, in the past, are increasing; we said then that heredity is linear. (1913, 219 [my translation])[9]

For convenience I call this patch of application 'Volterra superposition', where superposition is applied in integral form, often associated with idealized 'impulses' (see §5.2.2). This approach to 'superposition' reflects a trajectory through the 20th and into the 21st century (e.g. Jeffreys and Jeffreys 1956, 239; Simmons 2017, 145).

Hilbert understood the superposition principle in terms of linear combinations of solutions to a linear differential equation also being a solution (Courant and Hilbert 1924, 221). Arguably it reflects an attempt to axiomatize 'superposition' understood as the principle of superposition of small motions exemplified by Fourier and Pratt. Courant and Hilbert state:

> The solutions of the homogeneous [differential] equation [L[u] = 0] have the fundamental superposition property: *If u_1, u_2 are two solutions, then for arbitrary values of the constants c_1, c_2, $c_1 u_1 + c_2 u_2$ is also a solution*. (1924, 221 [my translation])

For convenience I call this 'Hilbert superposition'. This statement is set in the context of a chapter on 'Vibration and Eigenvalue Problems in Mathematical Physics', and is thus to be understood in the trajectory of Fourier and Sturm and Liouville, and as with 'Volterra superposition' it establishes a trajectory through

[8] The trajectory of this approach may be traced to Green, through Heaviside, eventually being made rigorous by Schwartz (1950-1951) on distributions. See Lützen (1982) for detailed history. Wilson engages with this trajectory in terms of his own approach, interpreting Heaviside through Schwartz (2006, 476-566) as well as Green through Schwartz (2017, 324-361).

[9] The linear approximation is required for 'superposition' here (cf. §5.3.1), as in the examples we discuss relating to Fourier techniques such as the 1-dimensional wave equation modeling vibrating strings and pulses travelling on ropes.

the 20th-21st century (e.g. Simmons 2017, 133).[10] However, as in the abstract application of 'superposition' to vectors, so abstract application of 'Hilbert superposition' and 'Volterra superposition' removed from a physical context leads to difficulties in the application of the concept.

Finally, Schrödinger applied Hilbert's superposition principle in the quantum context (Schrödinger 1926a-e, directly citing Courant and Hilbert 1924), which Dirac subsequently cast as foundational to quantum physics (1930). As interpreted in conjunction with Born's rule a new patch of application of 'superposition' is formed that I call 'quantum superposition' (see chapter 6) that leads to notorious and surprising dilemmas in the interpretation of quantum theory as famously noted by Schrödinger (1935).

2.3 Philosophical analyses of classical superposition

2.3.1 Paul Volkmann on isolation and superposition

Paul Volkmann regarded 'superposition' together with associated 'isolation' processes as central to science (1896; 1900; 1910). It is worth quoting Volkmann at length:

> The scientific method, which is characterized by the forms of induction and deduction, actually - correctly understood - exhausts everything that can be said about the scientific method. It can therefore only be a matter of explaining the forms of induction and deduction in their application in common cases.
>
> The modes of thought of analysis and synthesis are of particular importance: analysis represents more the purely subjective; inductive - the synthesis, more an objective, deductive moment.
>
> These forms of thinking are based on the fact of experience that the world of phenomena before us, the scientific representation of which is concerned, is not something uniform, indivisible – on the contrary it is composite. It is the task of science to decompose this composite, which is present in experience, into its natural constituent components. Induction seeks out these simple constituents of the world of appearances - we call this thought process analysis - and deduction tests the correctness and tries to reconstruct the world of appearances from these simple elements - we call this thought process synthesis.
>
> The natural sciences offer many examples of how analyses and syntheses are carried out. Chemistry applies names to their specific scientific methods.
>
> As it has developed, physics has increasingly pointed to a certain form of analytical and synthetic method of study, which seems so fundamental today that a discussion seems unnecessary here; I mean the forms of isolation and superposition.
>
> I understand isolation as the inductive attempt within a compound 'region of effects' to track down the elements that retain their effect independently of other simultaneously existing 'effect elements', and superposition as the deductive attempt to construct in reverse the 'region of effects' from the elements found in this way, i.e. to reconstruct the real phenomenon.

[10] In later editions Courant and Hilbert expand the definition to include reference to 'any linear homogenous functional equation' (1953, 276), reflecting 'Volterra superposition'.

> The emphasis here is essentially on the "independent existence of the effects". In the general case of analysis and synthesis, as in chemistry, there is the phenomenon that individual effects, through their coexistence, are modified and become different from what they are in themselves.
>
> The principle of parallelograms of forces, the composition of simple motions, such as in acoustics, optics and heat conduction are typical examples of the method of isolation and superposition. These methodological principles go very far, being independent of particular applications, e.g. whether there is a long-range effect or a short-range effect, whether treating forces, or whether an application of linear differential equations.
>
> At first glance, it may seem that this isolation and superposition process is extremely simple - and it is also fundamental - but the history of physics teaches that the isolation and superposition elements are not quickly or easily discovered. The fact that light consists of a wave motion was discovered relatively early (Huygens), and yet this idea was completely suppressed by the theory of emanation (Newton) for a century. This battle of theories was essentially about whether the intensity of light is the natural isolation or superposition element of a theory of light or not.
>
> Reality is so complicated because the 'effect elements' from which the individual phenomena are formed appear in very different proportions, thereby creating and making possible a diversity that in many cases creates the impression that it is not quantitative ratios to be distinguished, as if they were more about qualities than quantities. (1900, 28-30, my translation)[11]

Elsewhere Volkmann suggests that the abstracted 'isolation elements' are associated with natural laws,[12] which are then combined via superposition to describe the 'concrete phenomena' before us:

> The law is the true scientific term, the abstractum; it understands an aggregate of phenomena from one point of view, that of isolation, of abstraction. The establishment of a law is the conclusion of an isolation process, the law remains a resting place, an 'isolation center' of research. The natural scientist who establishes a law of nature creates a concept; the extraordinary business of naming this term, of creating a word for that concept, is merely a matter of convenience. (1896, 88)

He concluded in 1896 that the superposition principle is more an 'epistemological principle' than a 'natural principle' ('Ich möchte das Superpositionsprincip weniger als Naturprincip wie als erkenntnistheoretisches Princip hinstellen.' (1896, 177-178))

Volkmann's analysis of isolation and superposition, setting to one side his specific philosophical inclinations,[13] might be fruitfully re-appropriated within

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[11] See Howard (1988, 179-180) for an analysis of these themes in Volkmann 1896.

[12] There is a difference here regarding the ability to 'isolate' or to 'abstract' the partial laws in that isolation might suppose the ability to physically isolate a system associated with a law, whilst abstracting a system associated with a law might not suppose the ability to physically isolate that system. For instance in the example discussed in chapter 3 of conjoined gravitational and electrostatic forces acting on a body it is possible to physically isolate a system subject to gravitational force only, that is, a body with mass but not charge. The electrostatic abstraction is different, since charged bodies have mass, that is we cannot physically isolate a system that has charge but not mass. In the context of the model however such isolation is possible, and this abstraction makes sense in that we conceive a body with charge but without mass in establishing what we shall call the partial states and partial laws. For Volkmann's account of superposition it is sufficient that we can abstract even if we cannot isolate in this sense.

[13] Howard suggests that Volkmann is located in the tradition of neo-Kantian critical realism (1988, 176). This is especially clear in Volkmann's 1910 edition, which exhibits frequent engagement with Kant that is absent in the 1896 first edition. Volkmann acknowledges a shift in

the Wilsonian framework, for example by clarifying how descriptive opportunities that relate to physically salient explanations and reliable counterfactual reasoning arise (to use Wilson's terms) in relation to application of isolation and superposition (to use Volkmann's terms). Taking their work together indicates the potential philosophical value of considering the application of 'superposition' without necessarily adopting any particular epistemological or metaphysical stance to the components isolated or abstracted.

The concept of 'law' is perhaps a useful (even if problematic) concept to use in relation to analysing the semantic architecture of 'superposition'. It will be helpful in many situations to appeal to 'partial laws' as the laws obtained through 'isolation' that together compose an 'overall law' characterizing the behaviour of the system modelled. Appeal to such partial laws will not be made quite in Volkmann's sense in which 'law' is used in a rather restricted sense (as 'law of nature') which, if enforced, is unable to capture the architecture of 'superposition', in Fourier techniques in particular (cf. chapter 4).

Volkmann associates the isolation/superposition process with the establishment of 'laws'. He characterizes a law as

> the shortest (kürzeste), most comprehensive (allumfassendste) expression (Ausdruck) for something that happens within a larger domain of appearances, which must happen under all circumstances. (1896, 59)

However Volkmann understands 'law' in his own philosophical context, as Wilson points out, usage of 'law' in physics is varied and the concept has its own façade structure (2017, 65, 152, 339-342). 'Law' is sometimes used in the sense of 'law of nature', and sometimes in relation to a differential equation modelling the behaviour of some system derived from 'laws of nature' as a 'system law'. After Hadamard, we should think in terms of laws as associated with a differential equation *and* the associated boundary conditions in the context of a particular model since both equation and boundary conditions together

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perspective between the editions (1910, vi), although his discussion of isolation and superposition is substantially the same in both. The point is that, for my purposes, one may (re)appropriate the concepts of isolation and superposition as used by Volkmann and others in the 19th-20th centuries without necessarily adopting their philosophical frameworks, so that the concepts may be fruitfully used philosophically whilst acknowledging that their semantic architecture is understood differently in another (here, Wilsonian) framework than their own.

determine the forms the solutions take (2017, 410-416). This patchwork understanding of 'law' will be beneficial in the analysis of 'superposition'.

Volkmann's characterization of laws ('the shortest (kürzeste), most comprehensive (allumfassendste) expression (Ausdruck)') has affinities with the Mill-Ramsey-Lewis 'best system' account, taken in a metaphysically neutral sense. Laws in the Mill-Ramsey-Lewis 'best system' account (Lewis 1973) can be characterized in terms of

> those generalizations which figure as axioms or theorems in the deductive systemization of our empirical knowledge that achieves the best combination of simplicity and strength (where strength has to do with the range of empirical truths that are deducible). (Woodward 2014)

The 'best system' account is much discussed, in particular with regard to questions around the subjectivity of standards for 'simplicity' and for 'balance' or 'best combination' especially with regard to comparisons of rival systems.[14]

What is conceptually important for us is the local analysis of models of particular systems in which we elucidate the differing semantic supports of 'superposition', or note its mimics, in relation to a local understanding of 'law' in the context of that system. I shall, in continuity with the contemporary literature on force composition (cf. §3.1) use the term 'partial law' to denote the 'laws' that are identified by isolation/superposition as outlined by Volkmann and may be considered to 'compose' to form an overall law characterizing the behaviour of the phenomenon or system analysed. In continuity with Fourier (§4.2) I shall adopt the term 'partial system', or 'partial state', to be those states of the system or phenomenon associated with the partial laws identified.[15]

The (metaphysically neutral here) appeal to 'partial laws' is associated with clarifying the role of isolation/superposition in establishing a *robust and*

[14] See e.g. Cohen and Callender (2009); Massimi (2017; 2018) for detailed discussion especially with regard to Lewis' 'best system' account and possible improvements, which we cannot consider in detail.

[15] I have chosen this characterization of isolation/superposition in terms of 'partial states' and corresponding 'partial laws' with a view to its application in quantum physics. In the classical cases considered in chapters 3-5 it will often be the case that either the partial laws or the partial states are in view rather than both. The force composition literature primarily refers to laws rather than states whilst Fourier refers primarily to states, although the most precise characterization of superposition according to the 'Volkmann device' (see below) involves both. However, this leads to some complications especially when PDEs are involved for which there are multiple identifications of states and laws (cf. §§4.2; 4.4.2). The complication appears to reflect Wilson's concern regarding the usage of 'law' and 'cause' in the context of PDEs. It might be considered to arise from multiple applications of superposition, in particular to both spatial and temporal coordinates, an issue that does not arise in relation to ODEs.

reliable descriptive opportunity supporting a reasoning advantage with regard to the partial laws and corresponding states as associated with physically salient explanations of the system's behaviour while also supporting inductive inferences and the ability to manipulate the system by counterfactual reasoning.[16] We know that we have deduced the correct 'isolation centres' (decomposition into partial systems or states) when we can show that we have identified a set of partial laws *as* laws, which we can identify as such on the Mill-Ramsey-Lewis account, where these set of partial laws (and associated states) completely characterize the behaviour of the system and take the same form individually and in combination.

Going a little beyond Wilson then, we might understand the description of a complicated phenomenon obtained through the 'Volkmann device' of isolation/superposition as *natural*, relating to 'natural properties' of the system, although acknowledging the possibility of locally variable subjective aspects to standards of simplicity, balance and the notion of naturalness. For instance simplicity might be understood in terms of syntactic simplicity of the mathematical relationships cited as partial laws within the tradition of mathematical physics in which we stand (cf. Volkmann's 'shortest'), where it may be unclear if simplicity in this context is in fact simplicity simpliciter or simply in relation to our bedrock tradition.

That is, one can characterize the 'Volkmann device' of isolation/superposition in terms of establishing an optimally balanced simple and strong description, for some class of phenomena, in terms of abstracted partial states and their corresponding laws that take the same form individually and in combination. The 'Volkmann device' is understood as forming a natural description, supporting physically salient explanations, inductive inferences and counterfactual reasoning in terms of properties identified as 'natural

[16] I cannot explore here the various accounts of 'scientific explanation' and its relation to laws. The approach developed here would sit well with a classical deductive-nomological account, but as Woodward points out 'explanation' could be understood in what follows in unificationist terms as well. In some cases that we consider a causal-mechanical account might be relevant, but not in all our examples, even in the classical domain. See Woodward (2014) for an overview of the different accounts of 'scientific explanation' and discussion of the Mill-Ramsey-Lewis account of laws in this context.

properties'.[17] One know that one has established isolation centres proper to the model when one has a set of partial laws associated with corresponding partial states that describe the system with an optimal balance of simplicity and strength, perhaps relative to a particular interest (we shall see the need for this caveat in chapter 5 especially) such that in the domain of application one is confident that the isolation centres are correctly identified.

Volkmann's device may be characterized as follows then for a system completely characterizable by two partial laws and two partial states. This may be generalized in an obvious way to the case of *N* laws. We identify via abstraction or isolation two partial laws law_A and law_B associated with partial states/systems $state_A$ and $state_B$ where those abstractable states/systems are given such that law_A is relevant to (in abstraction or counterfactually) $state_A$ only and law_B to $state_B$ only, with each $*_A$ independent of each $*_B$. The *'Volkmann (superposition) device'* is that the overall law law_O and state $state_O$ completely characterizing the overall system are given by

$law_O = (law_A) \odot_{law} (law_B)$

$state_O = (state_A) \odot_{state} (state_B)$

where the $\odot_i$ are appropriate composition relations, traditionally simple addition or vector addition for the law, although Volkmann appears not to require this. Such representation leads to an explanatory and calculational advantage and supports counterfactual reasoning in physically salient terms, with the partial laws and states able to be considered completely independently of each other.[18]

The parallelogram law for the vector addition of forces is a paradigmatic example of isolation and superposition for Volkmann. However, he also considers the composite character of (scalar) temperature laws in terms of the superposition of three partial heat transfer phenomena: radiation, conduction and convection (1896, 76-79). No vector space structure is in view. Rather, it is

[17] The standard realist move is then to claim that if such a description in terms of partial laws and associated states/systems is explanatorily indispensible then an inferential pathway is established supporting a realist interpretation of the partial laws and associated states and properties.

[18] As noted above, in particular examples it may sometimes be either the partial laws (and associated composition) or partial states (and associated composition), rather than both, that are in view with the other being implicit.

the application of three 'partial laws' associated with partial states or systems which we identify as a radiative partial state or system (corresponding to a radiative (partial) law), a conductive system and a convective system (with corresponding partial laws likewise). These may be associated with abstractable physical causal processes and properties considered to produce a composite effect, mathematically modelled via scalar rather than vector addition. It is this composition of partial laws and states rather than any vector space structure that provides the semantic support of 'superposition'. What is important is that the processes are independent, taking the same form individually and in combination. Description of the overall system in terms of these three processes offers physically salient explanations of its behaviour, leads to a reasoning advantage, supports counterfactual reasoning and hence the design of thermal systems for example.

Volkmann's application of 'superposition' is broad. For example as well as general vector decomposition (1896, 82-83) he considers the abstractable properties and laws associated with the weight, hardness and colour of an object to be examples of 'isolation centres' that compose according to superposition to describe the object (1896, 71). He also considers the possibility of understanding chemical compounds in relation to the superposition of chemical elements (1896, 179). Moreover, nothing in Volkmann's discussion requires superposition to be a linear composition relation,[19] even though this is traditionally the case, and we shall consider nonlinear examples, although difficulties then arise with regard to conflicting usage for the same phenomenon. A good example of this is the modulation of radio waves considered in §3.2.

However, this provisional account of 'Volkmann superposition' might be seen to offer a global or 'flat' definition of 'superposition' so that the concept would seem not to have a façade structure after all. As we shall see however, this is not the case once we look more closely at the architecture and application of the concept in particular examples. Some of Volkmann's own examples, as well as further examples will help shape our analysis of 'superposition' and reveal its façade structure. We shall see that adopting different forms of the composition

[19] Since the 1960s there has been interest in the 'nonlinear superposition principle' – see e.g. Jones and Ames (1967); Menini and Tornambè (2011), briefly discussed in §§3.2, 10.2.4.

relation for the same phenomena can lead to 'superposition' being characterized in different ways for the same phenomenon, which is one reason that I am adopting the term 'Volkmann device'. Moreover, there are important (rather than contrived) borderline cases that will need to be clarified as either prolongations or semantic mimics of 'superposition'.

One such example is demonstrated in his analysis of vector decomposition in relation to isolation and superposition. He notes that with regard to the decomposition of vectors representing physical quantities there is an 'innumerable set' of possible decompositions, but that a particular decomposition may be preferred when the decomposition according to some components reflects a physical isolation process, such as in Galileo's analysis of projectile motion. However, even if the decomposition has no special physical meaning, it is, as he claims, mathematically justified and may be preferred for the simplicity of the calculation that results (1896, 82-83). Volkmann recognises but apparently ignores, or at least does not develop, the distinctions between physically significant and purely mathematical decompositions, but in doing so he may well lose some of the philosophical subtleties of the concept of superposition and its patchwork character.

Ignoring the distinction seems undesirable as it leads one to overlook important differences in the architecture of 'component' and 'superposition' where both mathematical and physical supporting architecture is important. Paying attention to the distinction highlights the façade structure of 'component' and helps to highlight cases of mimicry. In some cases the components of a vector may have independent physical origins, and other cases not, although the components may play an important, physically salient explanatory. In other cases the components may be mathematically supported only.

We consider these distinctions at length in chapters 3-5. Indeed, the traditional intuition regarding the significance of 'superposition' is that the components have independent physical or causal origins. Should we stipulate that this is a requirement for application of the concept? I discuss this in more detail in chapter 3 after considering now a different philosophical approach to superposition that would seem to require this to be the case, although the distinction will turn out to reflect the façade character of the concept.

2.3.2 Peter Simons on 'superposition'

Peter Simons considers the possibility of applying 'superposition', understood metaphysically, to material objects or substances in a dynamical sense (1987). In one sense his analysis is tangential to our concerns. Moreover, he also assumes but does not elucidate the non-trivial applicability of 'superposition' to wave interference for example, which is the point at issue for us. However, we may adapt his approach to discuss a common introductory example that motivates the understanding of superposition in mathematical physics, namely to 'pulses' crossing on a rope. This example might evoke the early-mid 19th-century intuition that superposition may be understood to have a metaphysical foundation.

The example that Simons discusses, of the superposition of projected clouds, is problematic in some ways. However, we may appropriate the central conceptual aspects of his analysis in terms of 'trace principles' to help consider the concept of superposition in mathematical physics. Simons considers application of superposition to pivot on the question of whether or not we have determinate means for tracing the kinds said to be superposed through time, where the kinds are understood as specified by a sortal term F that gives necessary and sufficient conditions for the identity of the Fs. The worry is that 'if we allow distinct continuants of a kind to be superposed, then we do not have determinate means for tracing things of the kind in question through time, since they become temporarily indiscernible from one another upon their superposition.' (1987, 221-222) What is required to support the superposition concept for Simons is that of the ability to establish principles to trace continuing or persisting identity (i.e., a trans-temporal identity) within the period of coincidence of the kinds, when the ability to discern such identity is in question. For convenience I call this 'Simons superposition'.

If we have two individual separated entities specified by F_1 and F_2 what account do we give of their coincidence and separation, such as if F_1 and F_2 are travelling pulses that cross on a stretched rope? Simons suggests that there are three possibilities: (1) The F_1 and F_2 survive during coincidence even though simultaneously the state of affairs during coincidence falls under the description of another entity F_3; (2) The F_1 and F_2 fuse during coincidence, ceasing to exist,

being replaced by F_3 which is then subject to fission so that the existence of F_1 and F_2 is re-established; (3) The F_1 and F_2 fuse during coincidence, ceasing to exist, being replaced by F_3 which is then subject to fission to create two new entities F_4 and F_5 distinct from F_1 and F_2. Simons claims that in the case where a 'trace principle' can be established, that is the continued identity and persistence of F_1 and F_2 can be established and traced along a causal path, then (1) offers the best account, that is, we understand the F_1 and F_2 to continue to exist in superposition during coincidence (cf. 222-228).

The architecture of 'superposition' then is that *apparently* the identities of F_1 and F_2 are obliterated so that they cease to exist, so that the state of affairs during coincidence is described by some F_3 only, but that in fact their identity can be traced during coincidence so that the F_1 and F_2 coexist with F_3. The description of the system is simultaneously given by both identities.

This account can be filled out using a mathematical model of the rope. Consider a long stretched rope where at each end a short duration pulse is applied so as to cause a wave of finite duration (or pulse) to travel along the rope towards the opposite end. So we have two separated pulses that converge, coalesce (putatively, as superposition) and then diverge again as they traverse the rope. Mathematically, the rope's behaviour is modelled (using suitable approximations and idealizations) by the one-dimensional wave equation

$$\frac{\partial^2}{\partial x^2}\varphi(x,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi(x,t) = 0$$

subject to initial / boundary conditions, where x is the distance along the rope, t the time, φ displacement of the rope and c the velocity of wave propagation. D'Alembert found the most general solutions as functions of the form $f(x - ct)$ and $g(x + ct)$, where f and g represent the displacement of the rope at x at time t. In the context of our model we interpret f as a right-moving wave or pulse and g as a left-moving wave or pulse, where f and g graphically represent their shapes or forms. By 'Hilbert superposition', since f and g are solutions to a linear differential equation their 'superposition' $h = f + g$ is also a solution. But this is just the situation that we have using Simons' analysis; $h = f + g$ represents the state of the rope with the two travelling pulses that converge, coincide and then diverge, with exactly the same form after coinciding as before. Geometrically or

graphically this corresponds to the construction of the superposed wave by adding *f* and *g*'s graphs pointwise to obtain the 'resultant' displacement.

That is, take Simons' sortal term F to be a pulse on the rope and identify F_1 as the pulse $f(x - ct)$ and F_2 as $g(x + ct)$, and F_3 as $(f(x - ct) + g(x + ct))$. The pulses have independent causal origins and their identities may be traced as the terms $f(x - ct)$ and $g(x + ct)$ irrespective of whether they are coincident or not, or of the existence of the other pulse. The pulses coexist and behave independently of each other, composing by simple pointwise summation, so that their identity may be traced though coincidence.

The identity of the pulses and their persistence depends on the nature of propagation that the rope and its boundary conditions supports, which is given by the wave equation subject to boundary conditions. That is, the mode of propagation must support a trace principle. In this case the trace principle demonstrating persistence of the pulses is illustrated mathematically via the superposition of D'Alembert's solutions f and g to the wave equation using Hilbert superposition when taken together with the causal conditions for the production of the pulses. That is, that the transmission of the rope supports the trace principle is characterized by the equation modelling it being linear. If the behaviour of the rope was nonlinear the identity of f and g would not survive their coincidence.[20] Mathematically, in the linearized model of the rope, we can trace the pulses through the period of temporary indiscernibility.

The ability to supply 'trace principles' does not imply the ability to isolate or recover by physical means either of the Fs in superposition even if their identities persist. This might be seen more clearly in a related but more complicated example that we cannot discuss in detail – that of directionally transmitted interfering radio waves of the same carrier frequency. In the region of interference the original signals cannot be recovered, but after interference, i.e. after they have crossed they can be physically recovered. But if the signals are transmitted on different carrier frequencies, even when they 'interfere' or are superposed they can be recovered by physical means, as is crucial to the ability to communicate by radio using tuning circuits.

[20] Real ropes behave nonlinearly, but with reasonable assumptions in many cases the behaviour is approximately linear as modeled by the wave equation. That is, for real world ropes superposition is 'approximately true'.

A Wilsonian analysis is metaphysically neutral however. So although Simons' analysis is suggestive of a metaphysical construal of 'superposition', at least in some applications or on some patches, I do not develop such an account here. Moreover, as Simons sets it up, it is not clear that his account can accommodate application of superposition to Fourier techniques, although his notion of 'trace principle' is helpful in clarifying the application of 'superposition' in this context (chapter 4). However, as we shall see in the following chapters, the availability of 'trace principles' – however understood metaphysically – is one way in which different kinds of vector decomposition can be characterized and distinguished, so that superposition is not to be conflated with vector composition as might be suggested by Volkmann. But in Wilsonian terms what the discussion of Simons' 'trace principle' approach to superposition highlights is the semantic support of the superposition principle and of the pulses *f* and *g* as components in the example considered. It is owing to the ability to establish a physically salient trace principle that supports a descriptive opportunity leading to a reasoning advantage regarding the behaviour of the rope that characterizes this example as an instance of 'superposition'.

The reasoning advantage of construing wave propagation and interference via 'superposition' is nicely illustrated in the example just alluded to – radio signal transmission and the associated technology. I cannot develop this further, but briefly, the superposition principle is exploited to enable the ability to isolate and receive a desired signal sufficiently well from the simultaneous transmission of radio signals on multiple frequencies in the same region in space and time.[21] If the identity of the signals was 'lost' in the region of coincidence one could not tune a radio receiver to receive a faithful signal transmission.

2.4 Summary

We have considered the history and usage of 'superposition' and two different philosophical approaches to the concept. I shall consider various examples in chapters 3-5 to clarify the façade structure of 'superposition', its mimics and their significance.

[21] That is, the isolation is not perfect but idealized, being adequate for practical purposes of communication.

Chapter 3

The façade structure of 'superposition' (1): Classical systems with a finite number of degrees of freedom

I now explore the patchwork structure of the concepts of 'superposition' and 'component' primarily with reference to Volkmann's characterization of the concepts as re-appropriated within a Wilsonian framework. In this chapter I consider application of the concepts to systems with a finite number of degrees of freedom using three types of example: First, the decomposition of vectors representing a force; secondly, the modulation of radio waves; thirdly, principal axis transformations, which will lead into the Fourier techniques discussed in chapter 4. These examples will enable us to consider in detail the differing semantic support of 'superposition', its prolongations and its mimics.

3.1 Force vector composition and superposition

The application of 'superposition' to the composition of forces is motivated by J.S. Mill's discussion of the composition of causes as developed by Volkmann, who reinterprets the composition of forces or causes in terms of the composition of their effects as an instance of the superposition principle. In the more recent literature on force composition, Cartwright (1980; 1983) also cites Mill's analysis to develop her thesis that the laws of physics do not state the facts by considering component and resultant forces. She makes no explicit use of 'superposition', although her discussion, along with the debate initiated with Creary (1981) and the subsequent literature will contribute to my analysis of 'superposition'. [1] The focus of the recent literature concerns logic and metaphysics, often driven by the question of causal overdetermination. However, as Wilson has observed, the concepts of 'force', 'cause' and 'law' have something of a façade structure as noted in chapter 1, and the force composition literature may be seen to reflect several different notions or applications of the concepts of

[1] See Spurrett (2001), the *Dialectica* special edition on vectors (63.4, 2009), and in particular Massin (2009; 2017).

'cause' and 'law' that are often not clearly articulated (cf. Spurrett 2001, 265).[2] As with my appropriation of Simons' analysis in a metaphysically quietest manner within an overall 'Wilsonian' framework, I shall use this literature in a metaphysically neutral sense whilst allowing the concerns raised in the discussion to clarify the architecture of 'superposition' on this patch of application.[3]

Mill discusses an example concerning northeasterly motion caused by two independent forces:

> If a physical body is propelled in two directions by two forces, one tending to drive it to the north, and the other to the east, it is caused to move in a given time exactly as far in *both* directions as the two forces would have separately carried it; and is left precisely where it would have arrived if it had been acted upon first by one of the two forces, and afterwards by the other. (1851, 1.374)

In the contemporary literature Mill's example gives rise to consideration of two related types of example that should be considered separately as it will be important to clarify the distinctions between them. The first type concerns a body acted on by what one would prima facie take as different forces of different physical, causal origins that are understood to compose to produce a resultant overall force, such as a body acted upon by gravitational and electrostatic forces. The second type of example concerns a body propelled northeast without any reference to independent physical components of the resultant force. So, to clarify the distinctions between the two cases, in the first case we could consider a body moving northeast by one rocket propelling it in a northerly direction and a second in an easterly direction (rocket_2); in the second case a single rocket

[2] Frequent appeals are made to the concept of 'force' in the force composition literature that require the concept, and especially the concept of a 'Newtonian force', to bear considerable metaphysical burden, yet the metaphysics of forces remains unclear. For instance Jessica Wilson has argued for a realist stance to Newtonian forces in the context of scientific realism in which Newtonian physics is regarded as a 'special science' (2007). Alternatively Massin seeks a more metaphysical account of Newtonian forces, but considers the possibility that an instrumentalist stance is appropriate (2009, 579). The point is that one should be cautious in seeking to make metaphysical judgements regarding force composition when we do not have a robust metaphysical account of 'force'.

[3] In this vein some of the literature has been concerned with the sense in which vector components are 'parts' of the vector (e.g. Cartwright 1983, 60-61). This is likely to be misleading owing to the shifting architecture of the concept of 'part' that Cartwright appears not to notice (cf. Spurrett 2001, 257). Although Teller develops the notion of 'analytic part' in contrast with 'mereological part' (1995, 140-141) it is unclear what conceptual advantage the concept of 'part' offers over 'component', and so my analysis is conducted simply with regard to 'component' and its semantic architecture, as consistent with usage in mathematical physics and engineering.

propels the body northeast ($rocket_1$).[4] What then are we to say about the northerly and easterly component forces, and the 'resultant' in each case?

These different examples highlight Volkmann's acknowledgement that different physical situations are reflected in vector decomposition in different cases as distinguished in terms of the physical significance of the components. That is, the two cases exemplify two different patches of application of 'superposition', 'component', and 'composition', or possibly a prolongation or mimic of 'superposition' in the second case. In the first case there is a 'story to tell' regarding the physical origins of the individual components, so that either (or both) a 'partial law' or a 'trace principle' exists for particular components in the first but not the second case, which might question whether the second case reflects an application of 'superposition' or of a semantic mimic. However, as indicated in the literature, what complicates the situation is whether we should accept (with Cartwright) or deny (with Creary (1981 151-152); possibly Volkmann (1896, 81-82)) the existence of the resultant force in the first case. The first case is now often discussed in the literature regarding a body acted upon by electrostatic and gravitational forces, which we now consider.

3.1.1 Conjoined gravitational and electrostatic forces and $rocket_2$

In a manner reminiscent of but departing from Volkmann, Cartwright suggests:

> This picture of how nature operates to produce the subtle and complicated effects we see around us is reflected in the explanations that we give: we explain complex phenomena by reducing them to their more simple components. This is not the only kind of explanation we give, but it is an important and central kind. I shall use the language of John Stuart Mill, and call this *explanation by composition of causes.*
>
> It is characteristic of explanations by composition of causes that the laws they employ fail to satisfy the requirement of facticity. The force of these explanations comes from the presumption that the explanatory laws 'act' in combination just as they would 'act' separately. It is critical, then, that the laws cited have the same form, in or out of combination. But this is impossible if the laws are to describe the actual behaviour of objects. The actual behaviour is the resultant of simple laws in combination. The effect that occurs is not an effect dictated by any one of the laws separately. In order to be true in the composite case, the law must describe one effect (the effect that actually happens); but to be explanatory, it must describe another. (1983, 58-59)

There are two important issues. First, what does it mean for 'explanatory laws' (or the 'partial' laws as I introduced them in chapter 2) to '"act" in combination just as they would "act" separately', or to 'have the same form, in or out of combination'? Secondly, in what sense is it important that (partial) laws 'satisfy

[4] Ignoring any other forces.

the requirement of facticity', that is, that they 'describe the actual behaviour of objects'? But it might be to miss the point to worry that the partial laws do not satisfy the facticity requirement in the way that Cartwright envisages. Perhaps it is precisely the observation that we *require* 'partial laws' to take the same form individually in isolation (or abstraction) and together in combination but *do not state the facts when in combination* as providing the semantic architecture of 'superposition' in this sort of situation. This might also enable clarification of the distinction between the 'isolation centres' of a superposition as *components* rather than as *parts*, although this is not a point that I shall pursue.

The question often posed in the literature is that of whether the gravitational and electrostatic forces exist (understood as associated with partial laws that characterize the behaviour of the body), or whether it is the resultant force that exists. Spurrett (2001) and Jessica Wilson (2009) have helpfully summarised the contours of the debate. Jessica Wilson observes that the driving issue is that of causal overdetermination, even if Cartwright's original focus was the facticity of laws. Briefly, it is supposed that one must either deny existence of the component electrostatic and gravitational forces, or deny the existence of the resultant in order to avoid causal overdetermination.[5] This difficulty, although perhaps in a different guise, was anticipated and avoided by Volkmann, who suggested in connection with the 'Theorem of the Parallelogram of Forces' that, 'It is not a question of composing forces or causes, but rather of putting together the expressions and effects of the forces or causes. ... Forces and causes always work separately.' (1896, 82) Set in the terms of the contemporary debate then, Volkmann would appear to deny the existence of the resultant force.

In order to assert the existence of the component forces Creary introduces a distinction between *causal influences* (corresponding to the influences described by 'partial laws') and *causal actions* (corresponding to the fact of the overall effect on the body) (1981, 150-151). Creary argues, contrary to Cartwright, that the component forces are real, being influences of real causes, although the causal influences then become a 'third kind of entity in the causal ontology … [which ground] the facticity of the laws of influence [partial laws]

[5] Massin considers the possibility that one could be a primitivist about vector composition (in the context of the composition of forces) and avoid the overdetermination problem (2017, 825-826). He rejects this position however.

that figure so prominently in … explanations by composition of causes' (152). The point is that there are various ways in which the metaphysics of the situation may be understood, and the façade structure of 'cause' indicated.

However, in adopting a (Mark) 'Wilsonian' approach it is not my goal to provide a metaphysical account of the situation, but rather to clarify the architecture of 'superposition', i.e. whether and if so in what sense we should understand the force acting on the body (or the overall effect of the component forces on the body) as the superposition of electrostatic and gravitational forces (or their effects).[6] Within such an approach, although the façade structure of notions such as 'cause' is noted as a likely source of conceptual confusion (cf. Wilson 2017, 247-267), for my purposes an analysis of that façade structure will not be necessary, since the analysis of 'superposition' can be conducted with reference to explanation rather than to causation.

Indeed, Jessica Wilson indicates how one may circumvent consideration of the metaphysical issues associated with causation in such examples as this by shifting to a focus on explanation (2009, 549-551). Specifically, she is concerned with the question of how to make sense of counterfactually instanced laws in reductive explanations of phenomena, such as in appeal to conjoined electrostatic and gravitational forces. She suggests that

> appeals to partial laws that are only counterfactually instanced in conjoined circumstances may nonetheless be explanatory of phenomena occurring in those circumstances, when the partial laws serve as a determinative basis for the goings-on (and associated laws) that are actually instanced in the circumstances. …
> Actually instantiated composition laws serve, then, as the ultimate reason why appeals to partial laws that are only counterfactually instanced can be explanatory of goings-on in conjoined circumstances. (550-551)

With regard to 'superposition', this would mean that the 'actually instanced composition laws' reflects the superposition principle. I.e., the superposition principle is an assertion that the counterfactually instanced partial laws (that, as Cartwright would put it, 'do not state the facts') are explanatory of the 'goings on in conjoined circumstances'. In this case the 'actually instanced composition law' is that the partial laws (as force laws) take the same form individually and in combination *through vector addition* to explain the facts of the resultant behaviour of the complicated phenomenon. Mathematically speaking, 'taking the

[6] Alternatively, closer to Mill's original example, as the resultant force (or effect) on a body acted upon by two rockets as the superposition of their individual thrusts.

same form' is expressed in terms of the linearity of the isolation-superposition relation or principle. We see then that the superposition principle applies precisely when the explanatory laws do not state the facts regarding the actual behaviour of the body acted upon, but are nonetheless explanatory of the body's behaviour since they take the same form individually and in combination.

Application of 'superposition' is supported in the case of the conjoined electrostatic and gravitational forces because:

First, we can counterfactually identify by isolation or abstraction partial laws (that we can construe *as* laws via the Mill-Ramsey-Lewis account of laws applied in a metaphysically neutral way) that take the same form in and out of combination. That is, the overall law expressing the force acting on the body is given simply as the vector sum of the forces given by the two partial laws. The point is that the form of the partial laws is not modified in combination, and support inductive inferences, physically salient explanations and counterfactual reasoning;

Secondly, the partial laws do not individually state the facts regarding the resultant force on the body (or the effects of the partial laws do not individually state the facts of the overall effect on the body);

Thirdly, in conjunction with the first two observations, the observation that the system is completely characterized (according to this model) by these partial laws according to the Mill-Ramsey-Lewis account indicates that the correct 'isolation centres' have been identified. This means that characterization of the system using these partial laws and the properties that they invoke can be said to offer a natural description of the system.[7] In Wilsonian terms these observations indicate that a descriptive opportunity has been established that leads to a reliable and robust reasoning advantage regarding the behaviour of the system in terms of physically salient features that we use to explain,

[7] As per Volkmann's characterization of isolation-superposition on the one hand and Mill-Ramsey-Lewis' characterization of laws on the other, we know that we have selected the correct 'isolation centers' as we have identified a complete set of partial laws, namely the electrostatic and gravitational force laws. These partial laws may be associated with corresponding partial systems or states, namely the electrostatic and gravitational states. We identify the partial 'laws' obtained via the 'Volkmann device' *as* laws on the Mill-Ramsey-Lewis characterization since such syntactic representation offers an optimal balance of simplicity and strength for the expression of the behavior of the system. We then say that the description of the system in terms of the partial systems or states associated with the partial laws is natural, as inherited from the characterization of the partial laws as laws, which are also associated with natural properties.

counterfactually reason and make predictions about the behaviour of such systems.

'Superposition' is applicable to $rocket_2$ for similar reasons even though we understand the notion of 'law' to differ in the two examples.[8] In both these cases however we can identify independent physical (perhaps 'causal') origins for the component forces and hence the partial laws associated with them. On Simons' account of superposition (cf. §2.3.2) this means that the application of 'superposition' is to assert a 'trace principle' in which the identities of forces that are regarded to have independent physical origins persist in the conjoined situation. Perhaps one might say that this expresses the 19th century intuition of the existence of components of a superposition (cf. §2.2). But is this necessary for the application of 'superposition'? I now compare these examples with $rocket_1$ and arbitrary vector decomposition to consider this question.

3.1.2 Motion north-east, $rocket_1$ and arbitrary vector decomposition

I now consider the second kind of example that Volkmann cites with reference to the application of 'superposition'. That is, the decomposition of a vector representing some physical quantity into components, such that the vector is considered to be the 'superposition' of the isolated components, but where there is no story to tell regarding the independent physical origins of the components. This kind of example is also discussed in the force composition literature with regard to 'motion northeast', no doubt with reference to Mill's original example. However, this case is characterized by the absence of any physical account for the origins of the components of the decomposition of the force vector *as components,* or in other words in the absence of a 'trace principle' associating a component with an independent physical (and perhaps causal) origin. This class of examples, as exemplified by 'motion north-east', corresponds to the north-easterly motion of a body propelled in that direction by a single rocket propelling it north-east ($rocket_1$) rather than by two rockets thrusting in different directions ($rocket_2$).

[8] That is, one might regard the gravitational and electrostatic force laws as 'laws of nature' whilst the force law characterizing the thrust of a rocket is derivative from laws of nature, if one were to model the production process of the rocket's thrust.

In rocket$_1$ decomposition of the force vector into northerly and easterly components is purely conventional – the force could be decomposed according to any coordinate system since it is modelled as a vector. Indeed, the same is true of rocket$_2$ in that the 'resultant force' can be decomposed according to any coordinate system when it is mathematically modelled as a vector. However, in the case of rocket$_2$ one decomposition is preferred or privileged owing to physical rather than mathematical considerations. This privileging is understood in relation to the independent physical origins of the force components.

Although both kinds of example (rocket$_1$ and rocket$_2$) share the mathematical support of a vector space structure as regards the decomposition of the overall force into components, the physical support or semantic architecture of the decomposition is different in each case, and this already introduces a distinction into the possible application of 'superposition'. That is, in the motion or force northeast of rocket$_1$ there are no 'trace principles' to trace the existence of components to physical origins. Moreover, it is harder to see how to associate the components with 'partial laws' *as* laws. For instance, it is not clear that a putative 'partial law' associated with a component would appear in a best system account of the behaviour of the body. This is unlike rocket$_2$ or the conjoined gravitational and electrostatic forces.

The question then is whether or not the decomposition into components of 'rocket$_1$' or 'motion north-east' is an instance of 'superposition'. Whether or not one regards rocket$_1$ as an instance of superposition, it reveals a patchwork structure for the concepts of 'component' and 'composition', indicating that 'superposition' is not to be conflated with 'vector composition'. The semantic support of the force composition and decomposition into components differs between rocket$_1$ and rocket$_2$.

The rocket$_1$ type case can however be 'dragged' in one of two directions, namely, to either: (1) an arbitrary decomposition of the force vector into components without physical context, or, (2) decomposition into components where there is a privileging strategy that arises from a physical context even if the components do not have independent physical origins. Case (1) appears to be a mimic of 'superposition' that is best understood simply as 'vector composition' while case (2), although apparently borderline, is best understood as an instance

of 'superposition' although with a differing semantic architecture than that deduced in §3.1.1. One reason for applying 'superposition' in this case is that, as we shall see in chapter 4, in order to understand the established use of Fourier techniques in terms of 'superposition' will require case (2) to be interpreted as a 'superposition', or at least as supporting the application of the concept.

There is both similarity and difference between these examples of vector decomposition and some of Wilson's examples illustrating semantic mimicry. There is similarity in the sense that as in Wilson's general definitions of semantic mimicry (2006, 379, 568), mimicry occurs because the supporting architecture for the application of a concept (here, 'superposition') has changed, where this shift in support goes unnoticed. This leads to conceptual confusion owing to the misunderstandings that then arise regarding the physical significance of the (mis)application of the concept. There are differences between the application of semantic mimicry in relation to vector composition and Wilson's examples however (2006, 567-598; 2017, 324-361). In his examples, for instance such as when he considers the use of numerical methods in Euler's approach to modelling the buckling of a strut for which 'calculating the next iteration' in the approximation algorithm is mistakenly associated with a causal process (see §1.2.3), the unnoticed shift in supporting architecture is mathematical. [9] However, in these various examples of vector decomposition that I am analysing the mathematical architecture is the same, or at least is shared, with the shift in architecture occurring in relation to how the physical significance of the components of some vector decomposition is understood in relation to application of 'superposition'.[10] Moreover, the question of the applicability of 'superposition' to vector decomposition raises the possibility of there being borderline cases of the application of the concept requiring philosophical judgements to be made, a feature that does not occur in Wilson's examples.

So, with this in mind, I now consider the possibility of physically privileged decompositions of vectors so as to support application of 'superposition'.

[9] Cf. 2017, 88-89 for Wilson's focus on mathematical diagnoses of shifts in supporting architecture.

[10] When we consider semantic mimicry in relation to Fourier techniques in §4.5, the shifts in architecture may be understood mathematically, thus being more similar to Wilson's examples.

3.1.3 Vector decomposition into physically significant components: A body sliding down an inclined plane

Consider a simple model of an object sliding down an inclined plane under gravity. Resolving the gravitational force perpendicular to the plane is explanatorily relevant to the frictional force acting on the object while the component of the gravitational force parallel to the plane is explanatorily (or causally) relevant to the force pulling the object down the plane. Resolving the gravitational force into these (and not some other) components appears 'natural' to the situation in which the physical context supplies a privileging strategy for the decomposition of the force into these components, even though there are no independent physical (causal) origins to the components individually.

The components are well determined in this situation. But altering the angle of inclination of the plane alters the components despite the same gravitational force acting, so that the same gravitational force has innumerably many physically salient or privileged decompositions depending on the physical situation, i.e. the angle of the plane. The preference for the component choice is a question of explanatory relevance in a context, where the choice is determined by the simplicity or efficiency of the ability to explain or to calculate using the components. We might say then that the privileging of one decomposition, relative to a given plane inclination, is supported by the (syntactic) simplicity and strength that that decomposition into a particular set of components offers for supporting physically salient explanations and calculations. Other decompositions are possible, but explanations and calculations with respect to other components would be contorted and 'unnatural', with the ultimate explanation or calculation grounded upon the privileged components (cf. Volkmann 1896, 82-83).

In a Mill-Ramsey-Lewis-esque appeal we might say that the privileged components are privileged because they are associated with partial laws for the system considered, such that explanations and calculations for the system's behaviour are given most simply, and strongly (in the sense that any sliding behaviour can, on this model, be given most simply by these and not some other

partial laws).[11] Then in Volkmann's sense this indicates that we have found the correct 'isolation centres' for the phenomenon so that 'superposition' is applicable to the way the force components as associated with partial laws recombine in characterizing the overall behaviour of the system, in that they take the same form individually (in abstraction) and in linear combination in the 'concrete' phenomenon.

We might say then that the resolution of the gravitational force in this example reflects a different patch of application of 'superposition' from §3.1.1 in which the components are now 'traced forward' to support explanations of the system's behaviour but cannot be 'traced back' to independent physical origins.

3.1.4 Summary of vector decomposition

In the different examples considered the vector representing a physical quantity may, in each case, be considered to be 'composed' of 'components'. However, the metaphysical nature of such is unclear, and the semantic architecture differs in each case even though the mathematical support is the same in each case (vector composition). The question is what one should say about the application of 'superposition' as associated with how we are to understand the physical significance of the components:

$Rocket_2$ and the conjoined gravitational and electrostatic forces share the similar feature that the force components in question have independent physical, causal origins which may be traced, counterfactually perhaps, when acting in combination, and are associated with partial laws that combine linearly to take the same form in and out of combination, but do not state the facts in combination. This is an undisputed application of 'superposition'.

$Rocket_1$ and 'motion north-east' present vectors representing physical quantities that may be decomposed according to the same mathematical architecture as $rocket_2$ and the conjoined electrostatic and gravitational forces, but have a different physical semantic support. In general, such decomposition is underdetermined and of no particular physical significance, undermining the

[11] Note that this implies that appeal to the concept of a partial law in this case using the Mill-Ramsey-Lewis account is contextual, that is, limited in application to a specific type of system or model. The architecture of 'partial law' is therefore different here from that of the examples in §3.1.1.

application of 'superposition'. What one should say about superposition is clarified by dragging the example in one of two directions:

First, the decomposition of a vector representing a physical quantity into arbitrary components without any further physical context appears to reflect a mimic of 'superposition', with such composition / decomposition supported mathematically by not physically. This reflects vector composition / decomposition and not superposition.

Secondly, the case of a body sliding down an inclined plane sits between $rocket_1$ and $rocket_2$ now that there is a way of physically privileging one set of components over others, and with the components supporting inductive inferences, counterfactual reasoning and physically salient explanations. But there is no independent physical or causal origin for each component individually. However, the privileged decomposition is privileged because it may be associated with partial laws characterizing the behaviour of the system that take the same form individually and in linear combination without stating the facts. This suggests that it represents an instance of 'superposition', but on a different patch from the conjoined force examples, that is, it has different supporting architecture from the conjoined force examples. One might dispute that this is a case of 'superposition', but we shall discover in §3.3.2 and chapter 4 in particular that it is in fact necessary to identify this as 'superposition' in order to support the ubiquitous application of 'superposition' in the use of Fourier techniques, especially as understood in the 19th century. However, consideration of this example also indicates that the architecture for the application of 'superposition' to Fourier techniques in the 19th century probably differs from that which was assumed.

The key issue is to note the differing semantic architectures for the decompositions in each case and what physical inferences may be drawn. By considering force vector decomposition we have identified two patches of application of 'superposition' and a mimic.[12]

[12] Although questions of realism are not directly in view in my analysis a few provisional comments may be beneficial. The notion of the actual existence of the partial states and corresponding partial laws identified in terms of superposition appears to reflect an important intuition in the understanding and application of 'superposition' in the early-mid 19th century as demonstrated by Fourier and Herschel (§2.2; cf. chapter 4). But in terms of contemporary

3.2 Modulation of radio signals: conflicting superpositions

I now consider 'modulation' in radio signal transmission as a form of composition related to 'superposition' or the use of the 'Volkmann device' (§2.3.1). The modulated wave was introduced into the Cartwright-Creary debate (§3.1.1) by Sheldon (1985) using a simple but illuminative example. The example will further demonstrate the façade structure of 'superposition', and it has some affinities with the 'separation of variables' considered in Fourier techniques in chapter 4.

Sheldon considers the simple case of a carrier wave c of fixed frequency ϕ and fixed amplitude a represented by

$$c(t) = a\cos(2\pi\phi t)$$

that is 'modulated' using amplitude modulation with a simple signal x of fixed frequency f and amplitude b, that is

$$x(t) = b\cos(2\pi f t)$$

(where normally $b < a, f \ll \phi$) so that the modulated wave y is

$$y(t) = (a + b\cos(2\pi f t))\cos(2\pi\phi t)$$

Sheldon suggests that this is a single wave of fixed frequency ϕ and variable amplitude $(a + b\cos(2\pi f t))$.

However, he observes, using simple trigonometry that y can also be represented as

$$y(t) = \frac{b}{2}\cos(2\pi(\phi - f)t) + a\cos(2\pi\phi t) + \frac{b}{2}\cos(2\pi(\phi + f)t)$$

which he identifies as three waves of fixed amplitudes and fixed frequencies. The question arises of whether the modulated wave is a single wave or three waves. He goes on to note that there are scientific and technological reasons for

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debates it is not clear what the realist ought to say regarding the components of superpositions on various patches of application, at least in terms of common inferential pathways. On the one hand, the indispensable explanatory (or causal) role or relevance of the force components in the examples studied might be taken as a valid inferential pathway that supports a realist commitment. On the other hand, the (in general) underdetermination of the components of the gravitational force acting on an object on an inclined plane would bring such a commitment into doubt, as might the lack of independent physical origins. As we shall see below, and in chapters 4&5 the situation only gets more difficult as more examples are considered, and it becomes increasingly unclear what is at stake in advocating or denying a realist commitment to components of a superposition. So, I refrain from considering 'superposition' in relation to 'realism'. However, it is worth noting that for the kind of examples considered, it would appear that proper application of 'superposition' is a necessary condition for the adoption of a realist commitment to components, whatever that may mean.

adopting both answers. This, he argues, is problematic for both Cartwright and Creary's analysis of component and resultant forces, and concludes that the problem is then to 'give a satisfactory account of criteria for reality' (436).

The realism question is not the question that I wish to pursue (cf. note 12 above). Rather, the question is how 'superposition' is applied in this example. In engineering practice the answer is clear – the modulated wave is the superposition of the three component waves. This representation supports reasoning advantages for the behaviour of the modulated wave that leads to technological advantages for manipulating the modulated wave. For instance, one electronically filters the modulated wave to suppress the carrier and one of the 'side bands' so that only the 'upper' or 'lower' side band is transmitted. This reduces the bandwidth required for transmission, and wastes less power transmitting redundant information. That is, technological explanations and manipulations are usually conducted in terms of viewing the modulated wave as the 'superposition' of the three 'component' waves (using Fourier techniques in general where the signal is a more complicated waveform). But none of the three waves have physical, causal origins, even if as components they have causal or explanatory relevance.

However, the signal and the carrier do have independent physical origins (analogous to the gravitational and electrostatic forces) and their identities may be traced mathematically in the modulated wave, with their identities persisting, taking the same form individually and in combination even if it is not simple linear combination. Moreover, physically, the signal may be recovered by a demodulation process. This is central to radio technology, the whole point of signal transmission using modulation – that one regards the modulated wave as 'composed of' signal and carrier in such a way that the signal can be recovered. It appears then that we may also regard the modulated wave as the 'superposition' of the signal and carrier. However, this should probably not be regarded as a superposition, for the composition is not given as a linear relation, by a simple summation. But in Volkmann's sense, where no reference is made to linearity, one may consider the modulated wave to be the 'superposition' of the carrier and signal, which would mean that 'superposition' is not well-defined since

'superposition' would normally be applied to the three components identified above as composing the modulated wave.[13]

This is a good example then of Wilson's 'prolongation' or 'dragging' of a concept ('superposition') that leads to ambiguous application in some cases for which a physical example (the modulated wave) is located on both patches simultaneously. The ambiguity does not occur in engineering practice since there is a different name already for each composition – 'modulation' and 'superposition' (understood in the traditional 'superposition of waves' sense).[14] Modulation is perhaps a 'generalized' superposition as we may see from Volkmann's analysis, understood in relation to the 'Volkmann device', and this may be the best way to understand it, as a generalization of classical superposition associated with the use of the Volkmann device.[15]

This example further indicates the patchwork nature of 'superposition' that also demonstrates that there can be ambiguity in its application, even if it can be resolved in this case. Moreover, here we see that the trace principles, understood in relation to the independent physical origins of components apply only to modulation and not to superposition as usually understood in this context. Yet it is the three components identified with reference to the usual understanding of 'superposition' that are often used to support inductive inferences, counterfactual reasoning and physically salient explanations.

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[13] Modulation was not mathematically understood in this way until after the last of Volkmann's work on superposition. See Colpitts and Blackwell (1921) for history of early developments; Heising (1921) for an early mathematical treatment; Black (1953) for detailed analysis of modulation techniques.

[14] In signal engineering parlance the wave is the modulated wave considered in the time-domain and the superposition of its frequency-domain components considered in the frequency-domain representation, so there is a question of the domain of interest (perspective?) here. The analysis is usually performed in terms of Fourier transforms since the signal is usually a complicated waveform, such as a voice broadcast. The simple example here illustrates the points I wish to consider without the further complications of appeals to Fourier transforms. For discussion of modulation see e.g. Oppenheim, Willsky and Young (1983, 447-512).
Moreover, there are different kinds of modulation. We consider amplitude modulation, but frequency modulation is another common form of modulation.

[15] One form of 'generalized superposition that is recognized as such is the 'nonlinear superposition principle'. It was stated concisely by Jones and Ames (1967) as an extension of Hilbert superposition. Consider a partial differential equation $L(u)$=0 and let $\{u_i(\boldsymbol{x})\}_{i=1\ldots k}$ be solutions. If $F = F(u_1(\boldsymbol{x}), u_2(\boldsymbol{x}), \ldots, u_k(\boldsymbol{x}), \boldsymbol{x})$ is also a solution of $L(u)$=0 then F, which is not necessarily unique, is called a connecting function, and this constitutes the nonlinear superposition principle (484). The nonlinear superposition principle has been developed in a slightly different way, tracing to Lie, in more recent work. See for instance Menini and Tornambè (2011) for discussion of 'physically motivated examples', although they do not discuss the physical significance of the components.

Decompositions of the modulated wave into signal and carrier on the one hand, or into three components as above, are, in Volkmann's terminology, different decompositions with reference to two different sets of 'isolation centres'. But both decompositions might be considered as exemplifying 'simplicity and strength' with regard to describing modulated signals, both supporting physically salient explanations, inductive inferences and lead to reasoning advantage in engineering technology – this was Sheldon's point.

We should regard Volkmann's analysis of isolation and superposition in relation to the 'Volkmann device' in terms of various prolongations of classical applications of superposition. It is for this reason, to avoid further confusion, that I am calling Volkmann's analysis of isolation and superposition, with all its façade structure in view, as the 'Volkmann device'. What will be most important for us is to be clear what the semantic architecture of the 'Volkmann device' is in any particular setting and what may be inferred from it.

I now consider a further type of example that has often been treated in terms of 'superposition' that evidences another patch of application with different supporting architecture again, namely the principal axis transformation. [16] This case is directly relevant to the Fourier techniques discussed in chapter 4, and lays the groundwork for it.

3.3 Principal axis transformations

3.3.1 Overview

Principal axis transformations, as understood in the context of Fourier techniques, will play an important role in distinguishing physical 'superpositions' of 'simple solutions' of linear differential equations from abstract arbitrary linear combinations of solutions to such equations according to Hilbert superposition. I illustrate the important conceptual points in examples that concern finite-dimensional vector spaces here before considering examples requiring a Hilbert space as analysed with Fourier techniques in chapter 4.

Principal axis transformations may be understood using the theory of spectral decompositions of linear operators. Such transformations arise when

[16] See Goldstein (1980, 198-263) for detailed discussion of principal axis transformations and use of 'superposition' in this context.

physical quantities associated with a physical system are represented by vectors **u** and **v** of finite-dimensional vector spaces related by a (Hermitian) linear transformation **A**. So for instance the evolution of an initial state **u** to a later state **v** is given by **v**=**Au**, so that **A** expresses a 'law' for the behaviour of the system (in some broad sense). In the principal axis transformation a coordinate transformation is performed via a matrix **P** from whatever coordinate system is initially used to represent **u**, **v**, **A** to a coordinate system whose basis consists of the eigenvectors or eigenfunctions (the 'principal axes') of the linear transformation **A.**[17] In the transformed coordinates **A** is represented in eigenvector (principal axis) coordinates by a diagonal matrix **D** of the eigenvalues of **A**, i.e. $\mathbf{D}=\mathbf{P}^{T}\mathbf{AP}$, so that the system's evolution is given simply by scaling the principal axis coordinates individually and summing as vectors. So, a principal axis transformation is a transformation to eigenvector co-ordinates of a system whose behaviour is characterized by a Hermitian linear map.

This means that mathematically the system's behaviour is given in as simple a form as possible when represented in the principal axis coordinates. The overall law characterizing the system's behaviour is expressed as the linear (vector) sum of principal axis components scaled by their eigenvalues. Each scaling of an eigenvector by an eigenvalue can be interpreted as a 'partial law' on the Mill-Ramsey-Lewis account, being associated with the 'partial system' or 'partial state' corresponding to each eigenvector. Representation in the principal axis coordinates achieves the syntactically simplest representation possible for the system's evolution, whilst also being strong in the sense that any evolution can be so expressed. This means that there is no coupling between the behaviours with respect to different axes, so that the behaviours are simple.

The complicated overall behaviour of the system is then interpreted as the 'superposition' of the simple behaviours of the partial states, as the behaviours of the partial states (eigenvectors or eigenstates) take the same form individually (given by scaling by the appropriate eigenvalue) in isolation and in combination (given by simple summing of the scaled eigenstates, because they are uncoupled), whilst not stating the facts in combination. Mathematically, this

[17] The existence of such a basis is ensured by the spectral theorem if **A** is Hermitian, as is the case for many physical systems.

is because the law-like behaviour of the complicated system is represented by a diagonal matrix acting on the eigenvector (principal axis) components. Because the eigenvector behaviours are uncoupled, they may be associated with conserved quantities and so the principal axes may be associated with 'natural' properties of the system.

In Wilson's terms the principal axis transformation presents a robust descriptive opportunity yielding a reasoning advantage to provide physically salient explanations, as predictions and explanations of the system's behaviour are given in the simplest and strongest way in terms of the principal axes. That is, as Wilson will claim in the context of Sturm-Liouville theory (cf. chapter 4), the principal axes are associated with 'hidden' physical properties of the system that are salient to providing physical explanations of its behaviour, supporting counterfactual reasoning.

A paradigmatic example of a principal axis transformation concerns a rotating rigid body for which the matrix representing the inertia tensor I is diagonalized, where the principal axes are the three axes with respect to which the rotary motions are uncoupled, and associated with the property that rotational kinetic energy is conserved for motions about these axes so that there is no energy transferred between the motions about the principal axes (Goldstein 1980, 198-263). However, we turn immediately to a model of a system of masses connected by springs in oscillatory motion as this leads most directly to the Fourier techniques of chapter 4 that are central to QFT.

3.3.2 Example: Spring-mass system

Consider a system S modelled by two masses and three springs in the configuration indicated, in 1-dimensional motion:

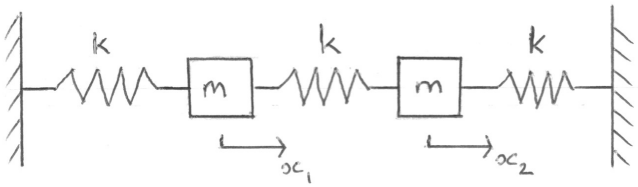

Fig. 1.1 Coupled spring-mass system

The configuration of S is represented by a two-dimensional vector space where the initial coordinate choice is given by the displacements of the masses from

equilibrium as shown. The equation representing the time-evolution of S is obtained from Newton's second law (i.e., 'law' applied in the fundamental sense). The specific equation expressing the behaviour of the system may be considered as the 'overall' or 'system' law characterizing the evolution of the system. A 2x2 Hermitian matrix **A** is introduced whose elements are a function of the topology of S so that the 'overall' or 'system' law is: [18]

$$\ddot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x}$$

with

$$\ddot{\boldsymbol{x}} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -2k/m & k/m \\ k/m & -2k/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The eigenvalues $\lambda_1 = -k/m$, $\lambda_2 = -3k/m$ of **A** corresponding to eigenvectors $\boldsymbol{v}_1 = \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \alpha_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are easily calculated, and the principal axis transformation is given by $\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, where in this context (and in that of Fourier techniques) the principal axes are known as the 'normal modes'. So substituting $\boldsymbol{x} = \mathbf{P}\boldsymbol{\eta}$ (where $\boldsymbol{\eta}$ is the configuration of S in the principal axis coordinates) gives $\ddot{\boldsymbol{\eta}} = \mathbf{D}\boldsymbol{\eta}$ for a diagonal matrix **D** of eigenvalues of **A:**

$$\ddot{\boldsymbol{\eta}} = \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} = \begin{bmatrix} -k/m & 0 \\ 0 & -3k/m \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

The differential equations in the $\boldsymbol{\eta}$-coordinates (normal mode or principal axis coordinates) are now uncoupled, i.e.

$$\ddot{\eta}_1 = -k/m\,\eta_1$$

with general solution $\eta_1 = a_1 \cos\sqrt{k/m}\,t + b_1 \sin\sqrt{k/m}\,t$

and

$$\ddot{\eta}_2 = -3k/m\,\eta_2$$

with general solution $\eta_2 = a_2 \cos\sqrt{3k/m}\,t + b_2 \sin\sqrt{3k/m}\,t$

The general solution in the original $\boldsymbol{x}$-coordinates is then given by $\boldsymbol{x} = \mathbf{P}\boldsymbol{\eta}$ so that:

[18] In general a system is represented by an *N*-dimensional vector space with an *N*x*N* matrix representing the evolution of the system whose entries depend on the topology of the system. 'Law' instantiates here Newton's second law in the particular context of this system and its topology.

$$x_1 = a_1 \cos\sqrt{k/m}\,t + b_1 \sin\sqrt{k/m}\,t + a_2 \cos\sqrt{3k/m}\,t + b_2 \sin\sqrt{3k/m}\,t$$

$$x_2 = a_1 \cos\sqrt{k/m}\,t + b_1 \sin\sqrt{k/m}\,t - a_2 \cos\sqrt{3k/m}\,t - b_2 \sin\sqrt{3k/m}\,t$$

with the coefficients a_i, b_i calculated from the initial conditions.

The point is that we have expressed any complicated motion of the system as a superposition of the 'simple' motions of the normal mode coordinates.

The overall behaviour of S is complicated because considered in terms of the motions of the masses, the motions of the masses couple. It is difficult to reason, predict or explain the system's behaviour using the initial configuration space representation (***x***-coordinates). However, a principal axis transformation **P** can be performed so that the motions are expressed relative to uncoupled principal axis $\boldsymbol{\eta}$-coordinates ('normal mode' coordinates) with respect to which the matrix **D** representing the system's evolution is diagonal.

We can interpret **D** as abstracting or isolating 'partial laws' for S given as simple scaling of the eigenvector (normal mode) coordinates by their corresponding eigenvalues. The partial laws are uncoupled, taking the same form in and out of combination where combination is represented by vector summation of the scaled partial states. The overall behaviour of the system is given by expressing its state as the 'superposition' of its normal mode states which each evolve individually and independently according to the partial law associated with each partial state, so that the overall evolution of S is the vector sum of the partial states evolved by their corresponding partial laws. Representation using normal mode coordinates can be characterized as the representation that optimally balances (minimally, in a syntactic sense) simplicity and strength – any state can be represented by the normal modes and calculations and physically salient explanations are given most simply using the normal mode coordinates as they are uncoupled. So in this kind of example a Mill-Ramsey-Lewis characterization of the partial laws in the context of Volkmann's understanding of isolation and superposition is successful, with the principal axis representation a natural representation. In Wilson's terms we exploit a descriptive opportunity to obtain a reasoning advantage supporting

physically salient explanations and predictions, and the ability to manipulate the system via counterfactual reasoning.

Moreover, the principal axis transformation simultaneously diagonalizes the matrices representing both the system's kinetic and potential energy so that the total energy of each mode constant (Goldstein 1980, 243-263). This further characterizes the normal modes physically. They are constant energy states, so that no energy is transferred between the normal modes. As Wilson will argue in the context of Sturm-Liouville theory, the normal modes are states that 'pick out' macroscopic properties of the system associated with conserved energy (2006, 244-251). So this offers another sense in which the principal axis representation is natural in that the axes or modes are associated with 'natural properties' of S that are otherwise hidden.

This example reflects another architecture of 'superposition'. The principal axes are determined by the system and the normal modes represent 'simple' (where the sense of simplicity is as developed above) vibratory states of the system that are associated with 'partial laws' for the system. If the initial conditions are expressed in the principal axis coordinates, the identity of the normal modes may be traced in the evolution of the system. I shall develop in detail in chapter 4 the idea that there are *two* aspects to the application of 'superposition' in relation to normal modes: first, the normal mode decomposition as an eigenfunction decomposition for the evolution of S; secondly the decomposition of some initial condition in terms of the eigenfunctions. Both decompositions must be available for the procedure to be successful.

However, unless the system is initialized and hence remains in a state corresponding exactly to a normal mode, there need not be any independent physical origin for the normal modes as they appear in the decomposition of the initial conditions of S. Clearly however the normal mode decomposition of the initial conditions has explanatory relevance for the behaviour of S in virtue of their eigenvector property for S.

That is, the normal mode decomposition supports the identification of partial laws that are foundational to supporting physically salient explanations of the system's behaviour and counterfactual reasoning. In this sense the normal

mode decomposition of the initial conditions is similar to the decomposition of the gravitational force of an object sliding down the plane. However, normal mode decomposition of the state of S is dissimilar to the decomposition of the gravitational force in the sense that, additionally, the modes are associated with natural properties of the system through conserved quantities unlike the object on the plane. Moreover the normal modes have independent existence as possible individual states of the system and can be physically *isolated*, unlike the components of the gravitational force acting on the body on the inclined plane for which the components can be *abstracted*, but do not have independent physical existence. So there are two aspects to the application of 'superposition' to normal modes, and this will be important when we consider Fourier techniques in chapter 4, which we explore further there.

3.3.3 Hilbert superposition and principal axes

Contact can be made with 'Hilbert superposition' in the example of §3.3.2. The equation representing S's evolution is a linear differential equation, so 'Hilbert superposition' applies to linear combinations of *any* (and not just normal mode) solutions to the equation,[19] so 'superposition' is semantically supported in mathematical terms according to 'Hilbert superposition', although it is underdetermined or perhaps 'promiscuous' if not further supported on physical grounds, as in the case of the arbitrary decomposition of a force vector (§3.1.2). The normal modes solutions are possible states of the system, being privileged states in that they are 'simple' owing to their eigenvector property in diagonalizing the motion.

That is, there are reasons to privilege the expression of solutions to the differential equation as a sum of normal modes rather than something else, and apply 'superposition' only to such simple solutions. For it is only with respect to the simple solutions that a reasoning advantage for modelling complicated behaviours is obtained (in Wilson's terms); the simple solutions are the proper 'isolation centres' for the phenomenon (in Volkmann's terms), so that it is only with respect to these that we should apply 'superposition'. Beyond that,

[19] Arbitrary solutions will of course be linear combinations of the normal mode solutions as the normal mode solutions span the solution space.

application of 'Hilbert superposition' to arbitrary solutions is an example of 'dragging' that leads to semantic mimicry. That is, 'Hilbert superposition' admits 'too much' as 'superposition' without further qualification, comparable to arbitrary vector decomposition without reference to physically salient simple components.[20]

Finally, contrast the analysis of S with the case of the spring-mass system S' comprising of 2 masses and 2 springs as indicated:

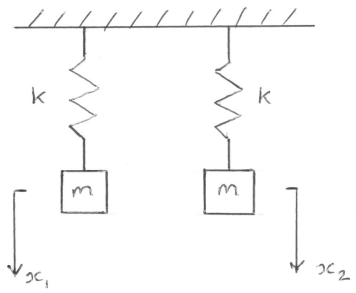

Fig. 1.2 Uncoupled spring-mass system

The configuration space is a two-dimensional vector space as before, with the state represented by ***x***. The initial configuration space representation is already the principal axis or normal mode representation. But intuitively we would not regard the state of S' as given by the 'superposition' of the normal modes as there is no interference or coupling.

This intuitive distinction between the application of 'superposition' to S but not to S' can be captured by Cartwright's worry about 'stating the facts'. In S' the components and associated 'partial laws' state the facts, whereas in S the components do not state the facts. Both cases are instances of 'Hilbert superposition'. However, it seems that one should require as part of the architecture of 'superposition' that the partial states or partial laws do not state the facts when in combination, and so one should not regard S' as exhibiting superposition.

3.4 Summary

By considering various examples we have seen that the superposition concept has 'wandering significance' involving different semantic architectures in

[20] The analogy is very close since the solution space to the differential equation is a vector space, so that a solution is a vector in this space and may be arbitrarily decomposed according to any basis.

different applications. Some of the patches of application might give rise to disputed application of 'superposition' and to semantic mimics. 'Component' also has a differing semantic architecture on different patches of application, and applies to vector composition even when 'superposition' does not. In the next chapter we consider the application of superposition, and its semantic mimics, to Fourier techniques and Sturm-Liouville theory.

Chapter 4

The façade structure of superposition (2): Fourier techniques, superposition and semantic mimics

4.1 Overview

I now continue my analysis of 'superposition' in classical physics, considering systems with infinitely many degrees of freedom for which 'superposition' is associated with Fourier techniques, Hilbert space structures and linear systems analysis. These involve various prolongations as well as mimics of 'superposition', and are foundational to quantum physics as we shall see in chapter 6 onwards. I commence with an overview of Fourier techniques in this chapter and then consider some prolongations to general linear systems analysis in chapter 5. Wilson discusses some of the same examples that I shall analyse, although with a different focus as I indicate in §4.4.1. I shall focus on the application and significance of 'superposition', a concept that Wilson rarely mentions and then only in passing, with a view towards its significance for the analysis of quantum field theory. My analysis is motivated by Wilson in the sense of probing the 'wandering significance' and façade nature of the concept of superposition and Fourier techniques, for which I also draw upon the work of Volkmann and Simons as discussed in chapter 2 in order to extend what might be said about the role and significance of 'superposition'.

The technique introduced by Fourier (1878 [1822]) and developed by Sturm and Liouville can be understood as a principal axis transformation (cf. §3.3) applied to the representation of a physical system modelled by a partial differential equation (PDE) subject to boundary conditions. The linear maps that are 'diagonalized' are differential operators associated with the PDE modelling the system acting on a Hilbert space, and the normal modes, 'partial states' and 'partial laws' are associated with the eigenfunction representations of the differential operators (cf. Goldstein 1980, 200). The PDE with specified boundary conditions represents the overall or system law describing the system's behaviour, being deduced from 'laws of nature' appropriate to the domain of the

model. The normal modes form the 'partial systems/states' associated with the 'partial laws' that compose the 'overall law' according to 'superposition'. This is the originary patch of application of Fourier techniques and associated usage of 'superposition', and has been subject to various prolongations and mimics.

It is crucial to appreciate, as Fourier made clear, that on this originary patch there are *two* aspects to the semantic architecture and application of Fourier decomposition (Fourier 1873 [1822], 133-137) that are often overlooked in subsequent discussions:

The first aspect: a 'separation of variables' is performed on the PDE (a 'generalized superposition', see below) to derive coupled but simpler linear ordinary differential equations (ODEs) that have the form of eigenvalue equations. A set of 'simple solutions' to the eigenvalue equations subject to boundary conditions is constructed. The 'simple solutions' are identified as the 'Fourier modes' and associated partial (evolution) laws.[1] The modes are physically significant, being mutually independent or uncoupled possible 'partial states' of the system (or 'partial systems' in Fourier's terminology) that persist with an invariant form whilst evolving according to their corresponding partial laws. The system's behaviour is given by scaling each mode (partial state) individually and independently by its corresponding partial law, and then summing these 'simple solutions' to obtain the overall state by 'Hilbert superposition'. In this sense the mode states are 'simple', and associated with 'partial laws' as the system's behaviour is expressed most simply in terms of these Fourier modes.

Working backwards, *if* one could express any arbitrary initial state as a linear combination of the simple eigenvalue states (modes) as a 'Fourier series', it is then straightforward to calculate, predict or explain the system's behaviour. One simply scales the modes individually by their corresponding partial laws and then sums them as vectors in the Hilbert space representing the space of possible states of the system (i.e., form their 'Hilbert superposition').[2] This is the physical

[1] 'Mode' is used in different ways. Properly, it refers to eigenfunctions as they arise in this (physical) context, but is often used to refer to individual terms in an abstract Fourier series by 'dragging'. Moreover, complications regarding the identification of modes (partial states) and partial laws arise owing to multiple different uses of superposition (or the Volkmann device), as we shall see below.

[2] The details here depend on the particular example as we shall see in what follows.

motivation for the mathematical *second aspect* of the Fourier technique, an aspect that came to dominate Fourier analysis, namely the Fourier decomposition of an arbitrary function. But this represents a dragging to a new patch if it is applied without reference to the first aspect, similar to examples of vector decomposition in chapter 3.

So, *the second aspect to Fourier's technique* is that having obtained the 'Fourier modes' for the system by separation of variables and eigenfunction decomposition of the PDE modelling the system, in order to establish a reasoning advantage one must be able to 'decompose' the initial conditions (or 'forcing') of the system in terms of these modes (eigenfunctions). For if the initial conditions can be represented as a linear combination of the modes or simple states, then the system's behaviour is given simply by scaling the modes individually by their corresponding partial laws and summing. Establishing the decomposition of the initial conditions into modes is the *second aspect* of Fourier decomposition. It is Fourier's celebrated result, although it involves essentially mathematical rather than physical semantic support even though it may mimic having physical support so that application of 'superposition' may be in dispute. It took over a century to establish all the important mathematical results rigorously regarding abstract Fourier decomposition.[3] The mathematical result is often presented as 'Fourier analysis', but it is only one aspect of Fourier's original technique.

In abstract terms the significance of a function's Fourier decomposition is that the 'modes' (or terms in the Fourier series) form an orthonormal basis for a Hilbert space of functions. This provides the semantic support for the decomposition on this mathematical patch. But it can lead to confusion through semantic mimicry if a function represents a physical quantity that is not explicitly a solution to a linear differential equation modelling the behaviour of some system. Moreover, there is a mathematical structure to Fourier series beyond that of power series expansions of functions, such as Taylor series, for which the individual terms do not form the basis of any vector space. Comparison of Fourier series with power series also leads to confusion through semantic mimicry.

[3] Bollobás (1990, 150) suggests that the final result was obtained by Carleson in 1966.

There are then (at least) four patches of application of Fourier analysis that have different semantic architecture that reflect both 'prolongation' and 'mimicry' in relation to 'superposition':

Patch 1 is Fourier's original 'two aspect' technique as developed in Sturm-Liouville theory modelling the behaviour of physical systems. 'Simple' eigenfunction solutions are found to the PDE modelling the system subject to boundary conditions (1st aspect of Fourier technique) before representing the initial conditions via the simple solutions (2nd aspect) so that the system's behaviour is represented by scaling each mode by its corresponding partial law and summing. It is exemplified by Fourier's heat flow examples (§4.2) and by the vibrating string (§4.4.1), and is foundational for quantum physics. There are several different applications of 'superposition' on this patch as we shall see;

Patch 2 reflects decomposition or representation of functions representing physical quantities by Fourier series without further physical semantic support, so that for instance the Fourier decomposition is not associated with any PDE modelling the system. 'Patch 2' is exemplified by epicyclical astronomy (see §4.5.1). 'Superposition' is mimicked on this patch;

Patch 3 concerns abstract decomposition or representation of an arbitrary function by a Fourier series in which nothing more than a Hilbert space structure is in view (§4.3), exemplified in pure mathematics. Again, 'superposition' is mimicked;

Patch 4 collates further prolongations to linear systems analysis involving ever more intricate subpatch structures, such as Fourier transforms (patch 4a) (ubiquitous in QFT) and ultimately Laplace transforms (patch 4b), where the prolongation of the Fourier technique becomes entangled with 'Volterra superposition' (patch 4c) and its prolongations in the theory of Green's functions and distributions for instance. The architectures of various potential applications of 'superposition' are intricate on this patch (see chapter 5).

Patch 3 is mathematically foundational to both patch 2 and to the second aspect of patch 1 when a PDE is brought into view, by which Fourier's technique stands or falls. However, in moving from (3) to (2) and (3) to (1) additional physical semantic support for the decomposition becomes relevant. Patch 4 may be

considered to draw upon the other patches, with the architecture and application of 'superposition' being intricate and subject to dispute with borderline cases.

The observation that there are different patches of application of Fourier analysis has implications for the application (or not) of 'superposition', and its semantic support, and this has led to confusions in the few recent philosophical mentions of Fourier analysis (e.g. Healey 2013; Vickers 2013). We should understand the significance of Fourier decompositions differently on different patches.

I now consider, through exemplifying examples, these patches and the prolongation or mimicking of 'superposition' and associated physical significance. I analyse patches 1-3 in this chapter and patch 4 in chapter 5.

4.2 Patch 1: Fourier's original example

I now consider the first and original patch of application of Fourier techniques as developed by Fourier (1878 [1822]) by studying his original example. Fourier derives a PDE modelling the heat flow in an ideal two-dimensional slab as

$$\frac{\partial}{\partial t}v(x,y,t) = k\left(\frac{\partial^2}{\partial x^2}v(x,y,t) + \frac{\partial^2}{\partial y^2}v(x,y,t)\right)$$

for some constant k depending on the slab's thermal properties, where v is the temperature at (x, y, t). He deduces the steady-state temperature $\varphi(x, y)$ of a semi-infinite rectangular slab for which the (finite) base, A = [-π/2, π/2] along the y-axis is held at constant temperature 1 and the sides (parallel to the x-axis) held at constant temperature 0. So

$$\frac{\partial^2}{\partial x^2}\varphi(x,y) + \frac{\partial^2}{\partial y^2}\varphi(x,y) = 0 \qquad (\alpha)$$

The boundary conditions impose constraints on $\varphi(x, y)$ that determine the form of the 'macroscopic' temperature distribution. Fourier tackles the problem in two stages, the 'two aspects'. First, he seeks the 'simplest' functions possible satisfying (α) subject to restricted individual boundary conditions. Secondly, he imposes all the boundary conditions simultaneously, and deduces the steady-state temperature distribution after showing that the initial thermal distribution can be 'decomposed' into the simple functions just obtained (1878 [1822], 134).

Fourier deduces simple solutions via the 'separation of variables', that is, solutions of (α) of the form $\varphi(x,y) = X(x)Y(y)$. This marks an implicit application of the 'Volkmann device' in a nonlinear form. That is, we assume that we can abstract away partial states (or partial systems as Fourier terms them) that are associated with partial laws taking the same form individually and in combination. The composition rules are complicated. The decomposition of $\varphi(x,y)$ into the product $X(x)Y(y)$ involves additional structure that relates $X(x)$ to $Y(y)$ via a 'separation constant' as deduced from substituting $X(x)Y(y)$ into (α) to obtain two coupled ODEs. The simple solutions to (α) obtained then are a set of solutions $\varphi_m(x,y) = X_m(x)Y_m(y)$. As the 'initial condition' is considered to be the thermal source on A, the $Y_m(y)$ are identified as the modes or partial states into which the base thermal distribution will be decomposed that evolve according to the 'partial laws' $X_m(x)$.

However, the products $\varphi_m(x,y) = X_m(x)Y_m(y)$ may themselves be considered to form a set of partial states or systems, as indeed Fourier does, that are summed to give the overall state of the slab. That is, the $\varphi_m(x,y)$ are 'simple solutions' to (α) that compose according to (a proper application of, cf. §3.3.3) 'Hilbert superposition' to form the general solution since the $\varphi_m(x,y)$ individually are solutions to (α). This is how 'superposition' is often understood in this context, although the architecture of the concept as arising from the separation of variables as well as 'Hilbert superposition' is rather complicated, involving two applications of the 'Volkmann device', namely the 'separation of variables' leading to eigenfunction decomposition, and 'Hilbert superposition' leading to the construction of the general solution from the 'simple solutions' $\varphi_m(x,y)$. Wilson refers to the Fourier procedure as a 'factoring technique' (2017, 270-278, 377-381). Both 'modulation' (§3.2) and 'separation of variables' may be considered to involve the 'Volkmann device' in its generalized sense, but with differing architectures in each case. I now clarify the procedure explicitly.

By separating variables Fourier obtains:

$$\frac{d^2}{dx^2}X(x) = mX(x)$$

$$\frac{d^2}{dy^2}Y(y) = mY(y)$$

from (α) where m is the separation constant. Note for reference later (the functional analysis framework came after Fourier) that these are two linear ODEs coupled by m that have the form of eigenvalue equations where $D_{\cdot} = \frac{d^2}{d\cdot^2}$ is considered as a self-adjoint linear operator acting on a Hilbert space of functions, and so has a 'diagonal' representation relative to a basis of eigenfunctions of $D_{\cdot}$ by the spectral theorem.

Returning to Fourier's treatment, since the sides of the bar are held at temperature 0, the simplest solution for Y is $Y_m(y) = \cos(my)$, which form the modes. Since the base is held at temperature 1, and the temperature tends to zero far away from the base, the simplest solution for X is $X_m(x) = \exp(-mx)$, which form the partial laws via multiplication of the corresponding modes.[4] To satisfy the boundary conditions separately m is a positive odd integer. Thus a 'simple solution' $\varphi_m(x, y)$ to (α) satisfying two boundary conditions is obtained by recombining the simple solutions:

$$\varphi_m(x, y) = \exp(-mx).\cos(my)$$

This does not satisfy the remaining boundary condition for $Y(y)$ when $x = 0$. However, these simple solutions can be combined to form a general solution

$$\varphi(x, y) = a.\exp(-x).\cos(y) + b.\exp(-3x).\cos(3y) + c.\exp(-5x).\cos(5y) + etc. \quad (\beta)$$

solving (α) which satisfies all the boundary conditions individually (from a later perspective, as an application of 'Hilbert superposition') *if* the 'joint' boundary condition on A, namely $\varphi(0, y) = 1$ so that $Y(y) = 1$ for $x = 0$ *can* be written as such a series, i.e.

$$1 = a.\cos(y) + b.\cos(3y) + c.\cos(5y) + etc. \quad (\gamma)$$

Then the problem is completely solved if the coefficients *a, b, c, …* can be calculated. The steady state temperature distribution $\varphi(x, y)$ is known

[4] This example illustrates Wilson's point regarding the central role that boundary conditions play – we have two identical differential equations (the sign of m is unspecified as yet), but the different boundary conditions lead to very different forms of solution, and are essential to determining it and thus the notion of 'law' in this context. However, see §4.4 for the question of assigning the $X_m(x)$ as either partial laws or partial states in Fourier techniques. This difficulty appears related to Wilson's concern that the notions of law and causation become problematic in the context of PDEs and merits further analysis. The (apparent) difficulty occurs due to multiple applications of superposition.

everywhere with all the boundary conditions satisfied, given by (β) once the coefficients are calculated from (γ).

This Fourier demonstrates (137-154), and he goes on to generalize the result to show that *any* physical base temperature distribution $\varphi(0, y)$ can be written as such a series, that is, 'decomposed into the modes' (168-209)[5] – the famous Fourier series – so that the problem of the steady state heat distribution in the slab is completely solved via the 'simple solutions'.

Fourier considers the physical significance of each term in (β) for $\varphi(x, y)$ by analysing the situation in which the temperature distribution at the source A simply takes the form of a mode. All the coefficients of (β) vanish apart from the one corresponding to the source temperature distribution. Thus if the source temperature distribution is a simple solution of *Y*, the temperature distribution persists in the form of that mode for *Y,* but scaled by *X* as one traverses the slab. Then, supposing that any source temperature distribution $\varphi(0, y)$ can be written in the form (γ), 'in this manner an exact idea might be formed of the movement of heat in the most general case; for it will be seen … that the movement is always compounded of a multitude of elementary movements, each of which is accomplished as if it alone existed.' (137). He concludes

> The equation $v = \frac{4}{\pi} e^{-x} \cos y$ represents also a state of the solid which would be preserved without any change, if it were once formed; the same would be the case with the state represented by the equation $v = \frac{4}{3\pi} e^{-3x} \cos 3y$, and in general each term of the series corresponds to a particular state which enjoys the same property. All these partial systems exist at once in that which equation (α) represents; they are all superposed, and the movement of heat takes place with respect to each of them as if it alone existed. (155)

He suggests

> that the particular values [of the modes] have their origin in the physical problem itself. Each of them expresses a simple mode according to which heat is established and propagated in a rectangular plate, whose infinite sides retain a constant temperature. The general system of temperatures is compounded always of a multitude of simple systems, and the expression for their sum has nothing arbitrary (156)

and that

> This superposition of simple effects is one of the fundamental elements in the theory of heat. It is expressed in the investigation, by the very nature of the general equations, and derives its origin from the principle of the communication of heat. (164)

These references to 'superposition' exemplify the concept for Fourier.

Fourier further claims, 'The fundamental problems of the theory of heat cannot be completely solved, without reducing to this form [i.e., Fourier series

[5] At the intersections of the boundaries there may be a mismatch requiring an infinite heat flow at a point according to the model (159).

for the source *A*] the functions which represent the initial state of the temperatures.' (206) Summing up the treatise after considering other examples he suggests

> The integrals … obtained [that is, the integrals resulting from Fourier's analysis so as to determine the various coefficients required] are not only general expressions which satisfy the differential equations; they represent in the most distinct manner the natural effect which is the object of the problem. … When this condition is fulfilled, the integral is, properly speaking, *the equation of the phenomenon;* it expresses clearly the character and progress of it. … In general, we could not introduce any change in the form of our solutions, without making them lose their essential character, which is the representation of the phenomena. (450-451)

and that

> We have said that each of these solutions gives *the equation proper to the phenomenon,* since it represents it distinctly throughout the whole extent of its course, and serves to determine with facility all its results numerically.
> The functions which are obtained by these solutions are then composed of a multitude of terms, either finite or infinitely small: but the form of these expressions is in no degree arbitrary; it is determined by the physical character of the phenomenon. For this reason, when the value of the function is expressed by a series into which exponentials relative to time enter, it is of necessity that this should be so, since the natural effect whose laws we seek, is really decomposed into distinct parts, corresponding to the different terms of the series. The parts express so many *simple movements* compatible with the special conditions; for each one of these movements, all the temperatures decrease, preserving their primitive ratios. In this composition we ought not to see a result of analysis due to the linear form of the differential equations, but an actual effect which becomes sensible in experiments. (453-454)

I have quoted Fourier at length to clarify the physical significance and motivation of the introduction of the modes and the semantic architecture of 'superposition', which has affinities with both Volkmann's account (1896) (in terms of 'isolation centres' of the phenomenon associated with partial laws) and Simons' account (1987) (in terms of physical 'trace principles') treatments.

Whilst Fourier may read more metaphysically into his analysis than is warranted, so that the architecture supporting 'superposition' is not quite what he envisaged, the physical significance of the modes and the 'partial systems' is clear. He anticipates a number of issues raised in recent philosophical treatments of Fourier decomposition: modes are physical invariants (Fourier 155, cf. Wilson 2006, 2017); modes are not arbitrary or promiscuous (Fourier 156, 453-454, pace Healey 2013b; Vickers 2013); modes are indispensable (Fourier 206, cf. Liston 1993).

I postpone detailed analysis of the multiple applications of 'superposition' until we have considered the later functional analysis perspective further. For now I note that the modes 'pick out' certain physical invariants of the system. Using modes to represent the thermal distribution is to appropriate a descriptive

opportunity offering a reasoning advantage to give physically salient explanations about the thermal distribution of the slab. Moreover the mode representation, as associated with their corresponding partial laws, supplies the syntactic representation that will best balance simplicity and strength so as to support physically salient explanations, inductive inferences and counterfactual reasoning. This gives content to what we mean by a 'natural' representation of the system.[6]

The mode representation is 'strong' in that *any* thermal distribution can be expressed in terms of the modes; it is 'simple' in that the expressions for the behaviour of the modes individually is of the simplest form possible – scaling and then adding. This suggests, on the Mill-Ramsey-Lewis account that we rightly consider the modes as partial states associated with partial laws. Whilst there may be a subjective aspect to the standards employed, it seems inconceivable that a better-balanced representation of the thermal behaviour of the slab could be derived in the context of such a model. Either simplicity or strength will be sacrificed.

According to Wilson's discussion of 'law', in this context the overall or system 'law' L for the system is given by the PDE (α) subject to the boundary conditions. However, to go beyond Wilson, the syntactic form of L that optimally balances simplicity and strength so as to support explanations associated with partial laws and states, calculations, inductive inferences and counterfactual reasoning is given in terms of the mode decomposition. This is what we mean by the representation being 'natural'.[7] This indicates, in Volkmann's terminology, that we have decomposed the system (via two decompositions) into the correct isolation centres to properly represent the overall law for S according to

[6] To recap, as per Volkmann's characterization of isolation-superposition on the one hand and Mill-Ramsey-Lewis characterization of laws on the other, we know that we have selected the correct 'isolation centers' as we have identified a set of partial laws associated with corresponding partial states. We identify the partial 'laws' obtained via the 'Volkmann device' *as* laws on the Mill-Ramsey-Lewis characterization since this syntactic representation offers an optimal balance of simplicity and strength for representing the behavior of the system. We then say that the description in terms of the partial states associated with their corresponding partial laws is natural, as inherited from the characterization of the partial laws as laws.

[7] As noted in §3.1.1, the standard realist move would be to adopt a realist stance to the modes owing to their indispensible explanatory power (cf. Fourier, 450-454 cited above). However, we shall again encounter the underdetermination issues raised in §3.1.2, which suggests that the grammar of realism about components or modes, as well as the application of 'superposition', is subtle.

'superposition'. In Simons' terminology the mode decomposition exhibits the identity of the physically persistent states $Y_m(y)$ and their trace principles $X_m(x)$. The complication however is the question of the origins of the physically invariant states in relation to an arbitrary thermal source, which we consider in more detail below.

I now discuss the mathematical architecture of Fourier decomposition and the functional analysis perspective.

4.3 Patch 3: The mathematical architecture of Fourier representation

Having studied Fourier techniques on their originary patch in which the significance of the Fourier decomposition of the source function was predicated on its physical salience to the system under consideration, I now consider the 'dragging' of the technique to a patch upon which the architecture is solely mathematical. This will help to begin to clarify the architecture of patches 1 and 2.

We begin with Fourier analysis as it is often introduced in textbooks, which reflects the third patch of application of Fourier techniques. The central mathematical idea regarding the representation of a function in terms of 'Fourier modes' is that any (suitably well-behaved) P-periodic function f can be *represented* or *decomposed* as a Fourier series or sum of 'modes':

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2\pi n x}{P} + b_n \sin \frac{2\pi n x}{P} \right)$$

where

$$a_n = \frac{2}{P} \int_0^P f(x) \cos \frac{2\pi n x}{P} \, dx$$

$$b_n = \frac{2}{P} \int_0^P f(x) \sin \frac{2\pi n x}{P} \, dx$$

This may be generalized to further classes of functions with different forms of modes, as in Sturm-Liouville theory considered below. Fourier decomposition is made mathematically rigorous using Hilbert space formalism and functional

analysis.[8] No reference is made to a physical system, and this is a purely mathematical result.

Fourier analysis is performed on a separable Hilbert space H of functions with inner product $(,)$ and orthonormal basis $\{\varphi_k : k = 1,2,\dots\}$. The Fourier decomposition of any $f \in H$ is

$$f = \sum_k c_k \varphi_k$$

where $c_k = (f, \varphi_k)$. So, for the familiar trigonometric modes, formally $H=L^2(\mathbf{T}) \cong L^2(\mathbb{R}/P\mathbb{R})$ where $L^2(\mathbf{X})$ denotes the space of square (Lesbegue) integrable functions on $\mathbf{X}$ with period P with

$$(f, g) = \int_0^P f(t)\overline{g(t)}dt$$

and

$$\varphi_k(t) = \exp(ikt)$$

where $k = 0, \pm 1, \pm 2, \dots$, using the exponential form of the Fourier modes.[9]

Mathematically, the Fourier modes are a basis for H so they form a complete linearly independent set. Crucially though, if the semantic support for the modes is given only mathematically – as a basis for the function space in view – then the modes are not unique, for there are other bases of different form. We might construct a basis for $L^2(\mathbb{R}/P\mathbb{R})$ of square-waves, the 'Haar wavelets' (Haar 1910), rather than trigonometric modes. These, and other more forms of wavelet, are important in digital signal processing. Moreover, van der Pol (1953) demonstrated advantages of both square-wave and 'sawtooth' representations of waveforms in electronics in some situations,[10] observing also the importance of these bases in the foundations of number theory. So in some contexts a square or sawtooth wave basis might be preferred or privileged, so that a non-Fourier basis may sometimes offer the best reasoning advantage.

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[8] See Bollobás (1990, 141-150) for a mathematical introduction to abstract Fourier analysis, with brief historical notes, and Körner (1988) for detailed introduction.

[9] Equivalence with the familiar trigonometric modes follows from

$$\exp(ikt) = \cos(kt) + i.\sin(kt)$$

and admitting complex coefficients. The exponential form is often more convenient.

[10] See also the exchange of letters on 'Concerning the Frequencies Resulting from Distortion' by various authors in *AJP* (1952-1955) where the question of whether Fourier modes are 'real' is discussed.

However, we have removed the discussion from Fourier's original context. Such 'promiscuity' of decomposition was not in view in Fourier's original application since the trigonometric basis for the decomposition of the base thermal distribution (i.e., the function in $L^2(\mathbb{R}/P\mathbb{R})$ whose decomposition we are considering) was chosen for physical reasons as indicated in §4.2, even if Fourier *also required* the mathematical supporting architecture to justify the 'second aspect'.

This clarifies in one sense how the supporting architecture differs in important ways between Fourier's original application and later abstractions or indeed applications of Fourier techniques in different contexts. Once the physical context of the function decomposed is removed so that the modes are not simultaneously eigenfunctions of a differential operator associated with the PDE modelling a physical system, the decomposition of the function is promiscuous, considered as a function in a suitable Hilbert space that supports various bases.

This is analogous to the examples of vector decomposition discussed in §3.1.2. Mathematically a vector can be decomposed via any basis, but in certain physical situations one decomposition is privileged, associated with the expression of the 'partial laws' of the system under consideration, such as in the decomposition of the gravitational force vector for an object on an inclined plane. Fourier decomposition is always supported in terms of a Hilbert space structure with the modes forming an orthonormal basis for the space. So the Fourier decomposition of the base thermal distribution into modes is comparable with the decomposition of the gravitational force vector, reflecting similar application of 'superposition' as regards how we understand the physical significance of the components.

It is important to note, as I shall develop in §4.5.2, that this Hilbert space structure distinguishes the *representation* of a function by a Fourier series from its *approximation* with a Taylor series where no such structure is in view.[11] This will be important later on when we contrast Fourier decompositions with power series.

[11] For example $\{1, x, x^2, x^3, \dots\}$ summed with suitable coefficients forms a Taylor series approximation to a given function, but does not form a basis for, say $L^2[\mathbf{T}]$ so that this set does not support a Fourier representation on this space.

We briefly consider two examples indicating how Fourier analysis may be 'prolonged' in this abstract approach so that the 'modes' take different forms in different function spaces. These decompositions, when set in a physical context, may be physically significant and support 'superposition' as they may arise in an analogous way (as modes as eigenfunctions of a differential operator in Sturm-Liouville theory, a prolongation of Fourier's technique) to trigonometric modes in Fourier's original application.

First, taking $H=L^2[-1,1]$ (without periodic extension to $\mathbb{R}$ in view) with inner product

$$(f,g)=\int_{-1}^{1} f(t)\overline{g(t)}dt$$

the normalized Legendre polynomials

$$P_n(t)=\sqrt{\frac{2}{2n+1}}.\frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n$$

form an orthonormal basis supporting generalized Fourier decompositions of functions in $L^2[-1,1]$. Physically, these modes will turn out to be important for the study of the two-dimensional wave equation modelling a vibrating circular membrane clamped around its perimeter, such as a drum skin.[12]

Secondly, and more generally still, a 'weighting function' w may be introduced into the inner product so that

$$(f,g)_w=\int f(t)\overline{g(t)}w(t)dt$$

Then, taking $H=L^2(-\infty,\infty)$ with $w(t)=\exp(-t^2)$ and

$$(f,g)_w=\int_{-\infty}^{\infty} e^{-t^2}f(t)\overline{g(t)}dt$$

the Hermite polynomials

$$H_n(t)=(-1)^n e^{\frac{t^2}{2}}\frac{d^n}{dt^n}e^{-t^2}$$

form a basis for generalized Fourier decomposition on $H=L^2(-\infty,\infty)$. These modes are used for the radial solutions of a model of the energy states of a hydrogen

[12] Wilson discusses the vibrations of a system modeled as a two-dimensional membrane clamped around its perimeter in several contexts (e.g. 2006, 113-115, 242-258, 267-268; 2017, 398-404) with a slightly different focus than mine.

atom using Schrödinger's equation with a Coulomb potential, and have physical significance in this context and support 'superposition' (§6.3.4).

The first example of representation in the Legendre basis illustrates how the different patches of application of Fourier decomposition associated with the abstract Hilbert space perspective and Fourier's original physically situated 'two aspect' technique 'come apart' if we consider the architecture of the 'second aspect' to Fourier techniques. That is, any function may be represented in terms of a (generalized) Fourier basis in an appropriate function space, but such representation may lack physical significance, so the construal of the representation in terms of 'superposition' may be inappropriate.

For instance, suppose that if after suitable scaling, etc. we represent the base of the heated slab in §4.2 as the interval [-1,1], we can either perform the familiar trigonometric Fourier decomposition of the function representing the thermal distribution along this boundary as Fourier did: or, mathematically, we can decompose the function via the Legendre polynomial basis.[13] The point is that the *mathematical architecture* for both decompositions of the base thermal distribution function considered abstracted from its physical context as the source distribution of the slab is the same (abstract Fourier decomposition). However, the trigonometric decomposition supports the descriptive opportunity for reasoning about the behaviour of the heated slab, as it picks out states of an invariant form (that may be traced throughout the system) that are associated with 'partial laws' for the physical system (analysable in terms of eigenfunctions of a differential operator modelling the system's behaviour), whilst the Legendre decomposition does not.

In this case then we would consider applying 'superposition' to the trigonometric basis for the base thermal distribution owing to the physical significance of the basis elements in the system, but we would not apply 'superposition' to the Legendre basis representation for the thermal distribution

[13] The function spaces differ here, but this is irrelevant to the physical model. The point is that a thermal distribution given on [-1,1] can either be considered in relation to the space $L^2(\mathbb{R}/P\mathbb{R})$equipped with the trigonometric basis, or in relation to L^2[-1,1] equipped with the Legendre polynomial basis. An example that Fourier might consider for an initial thermal distribution could be decomposed into Legendre polynomials rather than trigonometric modes since the model does not demand either function space. The mathematical supporting architecture is the same in both cases whilst the physical architecture differs, for the Legendre representation no longer supports any reasoning advantage.

function. In a different physical situation, the situation might be reversed for the same mathematical function – that is, for a physical system in which the trace principles or partial laws are associated with the Legendre polynomials and not the trigonometric modes.

4.4 Patch 1 prolonged: Sturm-Liouville theory

Sturm-Liouville theory enables analysis of physical systems whose behaviour is modelled by a linear PDE subject to boundary conditions where the variables can be 'separated'. This means that a single PDE in several variables is reduced to several ODEs coupled by 'separation constant(s)'. It is a development of Fourier's technique, and we can now consider it from the functional analysis perspective.[14]

It is often the case that the PDE involves a time variable that will naturally give rise to partial laws as 'evolution laws'. So, consider a physical system S modelled by a linear PDE in, say, $\psi(x,t)$ subject to 'suitable' boundary conditions. Separating variables,[15] $\psi(x,t) = u(x).T(t)$, and

$$\frac{D_x}{u(x)} = \frac{D_t}{T(t)} = \lambda$$

where D_x, D_t are linear differential operators in x, t respectively, coupled by a separation constant λ. D_t often represents simple harmonic motion as in vibrating systems, or sometimes exponential decay. D_x is often more complicated and specific to the situation, with boundary conditions determining the form of the solutions.

D_x is associated with an eigenvalue equation

$$D_x u_\lambda(x) = \lambda u_\lambda(x)$$

where (under reasonable assumptions) D_x is *self-adjoint*, so the spectral theorem assures (mathematical) existence of real eigenvalues $\{\lambda_1, \lambda_2, \lambda_3, \dots\}$ corresponding to a complete orthonormal set of eigenfunctions $\{u_1(x), u_2(x), u_3(x), \dots\}$, the 'generalized' Fourier modes of D_x.

[14] See Al-Gwaiz (2008) for an introduction to Sturm-Liouville theory. There are technical complications for applications to some systems that I avoid in this discussion (see Titchmarsh 1958-1962).

[15] Where more spatial dimensions are involved we may have, e.g. X(x).Y(y).T(t), etc.

If S's spatial states are represented using the eigenfunctions of D_x (the modes), the representation of D_x has as simple a form as possible,[16] namely a 'diagonal' form (cf. §3.3). That is, intuitively, the diagonal representation of a linear operator is the simplest sufficiently strong representation of that operator, which is clearly the case if one considers simplicity in syntactic terms. The action of D_x is given by scalar multiplication of each eigenfunction $u_n(x)$ by the corresponding eigenvalue λ_n. They are 'simple solutions', the modes, of the 'separated' equation in x in virtue of being eigenfunction solutions. They form a set of abstracted partial states that correspond, via the λ_n, to a set of partial laws given by the $T_n(t)$ via the 'Volkmann device' of the separation of variables. The evolved partial states then form *another* set of (isolatable) partial states, the 'simple solutions' $\psi_n(x,t) = u_n(x).T_n(t)$ of the original PDE, being possible 'simple states' of the system. These $\psi_n(x,t)$ combine via 'Hilbert superposition' so that the full solution subject to the boundary / initial conditions is

$$\psi(x,t) = \sum_n c_n u_n(x).T_n(t)$$

for some constants c_n determined by the initial conditions.[17]

Then if we can express the initial condition, such as the thermal distribution of the base of the heated slab as a 'Fourier series' in the $u_n(x)$ (the second aspect of Fourier's original technique)

$$\psi(x,0) = \sum_n c_n u_n(x)$$

we can exploit a descriptive opportunity by expressing S's behaviour in terms of the modes $u_n(x)$ of D_x, as they evolve individually and independently as scaled by the $T_n(t)$.

The solutions $\psi_n(x,t)$ are simple and strong because any solution can be written as a linear combination of these (strong), and simple because this is, minimally, the simplest syntactic form of an arbitrary solution. The modes are uncoupled, so the behaviour of the system is given by simple scaling of each

[16] Which is the whole point of the spectral theorem.

[17] And for example in two spatial dimensions $\psi_n(x,y,t) = u_n(x).v_n(y).T_n(t)$ and $\psi(x,y,t) = \sum_n c_n u_n(x).v_n(y).T_n(t)$ where the u_n and v_n will in general take different forms, depending on the differential operators and boundary conditions for each separated variable.

mode $\psi_n(x,t)$ by the constant c_n followed by their addition ('Hilbert superposition').

Since there are two applications of the 'Volkmann device', there are two aspects to characterizing S via partial states and associated partial laws. First, from the separation of variables, the eigenvalue equation in x supports the identification of the eigenfunctions $\{u_1(x), u_2(x), u_3(x), \dots\}$ as partial states$_1$ that are associated with the partial laws$_1$ $\{T_1(t), T_2(t), T_3(t), \dots\}$. That is, $T_n(t)$ is the partial law$_1$ associated with the partial state$_1$ (the mode shape) $u_n(x)$, so that the mode shape $u_n(x)$ (as scaled by c_n) evolves according to $T_n(t)$ by simple multiplication. This gives, secondly, a partial state$_2$ $\psi_n(x,t)$, the nth normal mode,[18] which is associated with the partial law$_2$ which is simply the identity, with the partial states$_2$ composing according to 'Hilbert superposition' to give the overall state and its evolution.

The complicated overall behaviour of S is expressed with optimal balance of simplicity and strength in terms of the (Hilbert) superposition of the {partial state$_2$}, the normal modes, which are associated with the trivial {partial law$_2$}, but with the {partial state$_2$} having internal structure of {partial state$_1$}⊗{partial law$_1$}.[19] However we formally 'carve up' the states and laws, we call the representation of S via the $\{u_1(x), u_2(x), u_3(x), \dots\}$ and $\{T_1(t), T_2(t), T_3(t), \dots\}$ and their products a 'natural representation' of the behaviour of S. It is the representation that offers the optimal explanatory power for S's behaviour in terms of the partial laws and states of S, supporting inductive inferences and counterfactual reasoning.

From Simons' perspective on superposition, the partial states (whichever way understood) support a trace principle according to the appropriate form of the partial laws. However, the partial states need not have physical causal origins as such, when in combination. That is, if the initial condition for $\psi(x,0)$ had

[18] Usage of 'mode' and 'normal mode' is often ambiguous, referring both to the $u_n(x)$ and the $\psi_n(x,t)$. One might disambiguate the usage by referring to the former as 'modes' and the latter as 'normal modes'.

[19] The architecture of the partial states and laws, and application of 'superposition' is subtle owing to multiple application of the Volkmann device. One could alternatively suggest that the partial states for S are simply the {partial state$_1$} with corresponding partial laws {partial law$_1$}⊗{partial law$_2$}. However it is often helpful to think of the system in a simple state $\psi_n(x,t)$, with these forming the 'partial systems', as Fourier does.

$\psi(x,0) = u_n(x)$ for some n, then the mode has a physical causal origin, and persists. However, in general

$$\psi(x,0) = \sum_n c_n u_n(x)$$

which is supported mathematically on patch 3. But there is no reason to postulate physical causal origins for the modes individually in this decomposition in the usual circumstance that no causal history is given for the initial state. Rather, the physical significance of the decomposition of the initial state $\psi(x,0)$ is that the modes have physical causal or explanatory relevance in supporting a trace principle for S in virtue of being eigenfunctions of D_x.

However, the trace principle account can be supported counterfactually. That is, if the initial state $\psi(x,0)$ had arisen as the superposition

$$\psi(x,0) = \sum_n c_n u_n(x)$$

in which each $u_n(x)$ did have a physical causal origin individually, then a trace principle is established in virtue of the $u_n(x)$ being eigenfunctions of D_x. As far as the response of S to $\psi(x,0)$ is concerned, it is irrelevant whether its decomposition into modes is supported only in mathematical terms or in terms of being the superposition of terms having independent causal origins.

The Fourier decomposition of $\psi(x,0)$ may be compared with the decomposition of the gravitational force vector for an object on an inclined plane (§3.1.3). Application of 'superposition' to the initial or source condition (Fourier aspect 2) is now clarified as supported in a similar manner to this case even though the eigenfunction decomposition of S's response supports 'superposition' differently (Fourier aspect 1), with superposition according to aspect 1 supporting superposition according to aspect 2 in the Fourier case but not for the object on the plane.[20]

Whatever stance one takes towards the decomposition of $\psi(x,0)$, the modes obtained support a trace principle in virtue of their eigenfunction attribute so that the expression of S's behaviour is properly to be thought of in terms of a superposition of the mode partial states. The support for application

[20] Again, a standard realist move would be to claim that the causal relevance of the modes of $\psi(x,0)$ supports a realist stance towards them. This would then justify application of 'superposition'.

of 'superposition' in relation to the $u_n(x)$ both to the evolution of S, i.e. $\psi(x,t)$, and to the initial condition or source $\psi(x,0)$ applied to S is the same, namely that the $u_n(x)$ are eigenfunctions of D_x. If we write $\psi(x,0) = f(x)$, then application of 'superposition' to the Fourier decomposition of $f(x)$ as a mathematical function is dependent upon the context in which $f(x)$ is considered.

As Wilson has observed (2006, 384-386), there is more to be said regarding the energy properties of normal modes, which is of crucial importance in QFT as it is related to the identification of particles as we shall see. Further discussion in the classical domain will be beneficial prior to turning to quantum physics however, and is best conducted with reference to specific examples. To that end I now turn to the wave equation. Solution of the wave equation subject to boundary conditions is an important application of Sturm-Liouville theory and the superposition principle, and was treated by Fourier (1878 [1822]) who developed and vindicated Daniel Bernoulli's famous controversial analysis (1753a&b).[21] The wave equation is of particular relevance to us as wave equations are central to QFT. We now study two simple paradigmatic examples in classical physics.

4.4.1 The vibrating string (1-dimensional wave equation)

Consider the one-dimensional wave equation modelling the vibrations of a stretched string with fixed ends. It will be illuminative to consider briefly the history of its analysis. The linearized or idealized one-dimensional wave equation modelling the behaviour of waves or pulses on a stretched string

$$\frac{\partial^2}{\partial x^2}\varphi(x,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi(x,t) = 0$$

was first derived by d'Alembert (cf. §2.3.2). Daniel Bernoulli (1753a&b) was first to analyse the vibrating string in a way anticipating Fourier's techniques. He did not use the language of superposition, but of 'mixture of coexistent vibrations' that are 'absolutely independent of each other' (1753a, 160).

Bernoulli's analysis was developed by Herschel (1830) using Fourier's techniques, interpreting the coexistence of simple vibrations (the normal modes

[21] See e.g. Struik (1969, 351-368) for the interesting and controversial history of the analysis of the vibrating string.

or string harmonics in this case) in terms of 'superposition'. As with Fourier, Bernoulli and Herschel appear to give a physical or perhaps metaphysical account of the composition of the phenomenon according to superposition. For instance Herschel appeals to microphysical causal processes to claim that

> any number of the different modes of vibration, of which a cord is thus susceptible, may be going on *simultaneously*, or be, as it were, superposed on each other. This is a consequence of the principle in mechanics of "the superposition of small motions", which, when the excursions of the parts of the system from their places of rest are infinitely small, admits of any or all the motions which, from any causes, they are susceptible, to go on at once without interfering with or disturbing each other. (1830, Art. 164, p.782)

He associates this supporting architecture of 'superposition' here in mechanics with the more geometrical semantic support in which the graph of the general motion of the string is calculated by the pointwise addition of the graphs of the simple motions.

He goes on to justify what we might term a 'realist' interpretation of the harmonics via inference from the causal relevance of the harmonics. A musician's 'trained ear' can detected the harmonic (i.e. normal mode) sounds in the sound produced from a stringed instrument (1830, Art. 166; cf. Art. 205 cited in §2.2).[22] This argument depends on the causal or explanatory role the modes play, that is, with reference to their effects rather than their origins. However, we have seen reasons to be cautious regarding such inferences once the mathematical supporting architecture is clarified via vector or Hilbert space structure. We saw in §3.1.2 that arbitrary components of the decomposition of a force vector have a causal or explanatory role, but they are also underdetermined in general. Thus we should be cautious in making any metaphysical inferences here as the 18th-19th century authors appear to, so that in fact the architecture supporting

[22] This argument could be made more precise in the 19th century by appealing to Rudolf König's 'manometric flame analyser for the analysis of sounds'. This apparatus is essentially a spectral analyser that relates the heights of a series of flames to the harmonics of a musical instrument (see Pantalony 2009). The harmonics may be then said to be causally relevant to the physical distribution of flame heights. Moreover, the explanation of the timbre of a musical instrument is given in terms of its harmonic structure, and the note played identified by the lowest frequency harmonic present. For the realist such observations are suggestive of a realist interpretation of the modes. Related to such an argument to 'realism' about the modes is the appeal to Gibbs phenomenon, in which the Fourier decomposition of some physical quantity is taken to explain the unexpected behavior of physical systems in which some input rapidly changes, which may be modeled as a discontinuity. See Körner (1988) and Hewitt and Hewitt (1979) for discussion of two important examples in the history of technology. I cannot discuss this here, but the inference to realism might be inferred through novel predictive success of the Fourier decomposition in these cases.

application of 'superposition' differs from what they supposed, as I have sought to outline.

That is, we have seen, however, how we can recover application of 'superposition' in Sturm-Liouville theory. As in the decomposition of the gravitational force vector acting on the object of an inclined plane, the decomposition into modes of the string's initial condition is not arbitrary but natural to the system or phenomenon. So whilst on patch 3 a purely mathematical decomposition may be arbitrary, on patch 1, given a physical context, the decomposition is determined as Fourier observed via separation of variables and eigenfunction decomposition.

I now consider the mathematical architecture for the analysis of a vibrating string with fixed ends, of length *L*. The relevant linear PDE is

$$\frac{\partial^2}{\partial x^2}\varphi(x,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi(x,t) = 0$$

subject to the boundary conditions $\varphi(0,t) = \varphi(L,t) = 0\ \forall t$. Separating variables with $\varphi(x,t) = u(x)T(t)$:

$$\frac{d^2T}{dt^2} + \omega^2 T = 0$$

$$\frac{d^2u}{dx^2} + k^2 u = 0$$

where $-k^2$ is the separation constant, ω=kc and $u(0) = u(L) = 0$. The T-equation is a simple harmonic oscillator equation, whilst for the u-equation $\{u_n(x)\} = \{\sin(\frac{n\pi x}{L})\}$ are the familiar trigonometric Fourier modes. $\{u_n(x)\}$ forms an orthonormal basis for the relevant Hilbert space, being the set of eigenfunctions of $D_x = d^2/dx^2$, with the u-equation the eigenvalue equation for D_x.

Recombining the 'simple solutions' the general solution is

$$\varphi(x,t) = u_n(x)T_n(t) = \sum_n \sin\frac{n\pi x}{L}\left(C'_n \sin\frac{n\pi ct}{L} + D'_n \cos\frac{n\pi ct}{L}\right)$$

where the constants are determined by decomposing the initial displacement $\varphi(x,0)$ in terms of the eigenfunctions of D_x as before, as well as the initial velocity. This is a natural representation of the string's behaviour interpreted in terms of 'superposition', optimally balancing simplicity and strength, for the reasons noted above.

It is worth clarifying some issues arising in the context of this specific example. First, the modes are determined as the trigonometric modes of *this* form and not some other in virtue of the $\{u_n(x)\} = \{\sin(\frac{n\pi x}{L})\}$ being eigenfunctions of D_x. That is, this basis offers a natural representation for the behaviour of the vibrating string and not a square-wave or saw-tooth wave basis, or a trigonometric base of some other form. As Fourier puts it, they are 'natural to the phenomenon' and not arbitrary.

Secondly, we may develop the significance of the invariant properties associated with the normal modes. As noted in the analysis of the spring-mass system, the transformation to principal axes (eigenfunctions) diagonalizes the kinetic and potential energy operators, where, moreover, the total energy of each normal mode $\varphi_n(x,t) = u_n(x)T_n(t)$ is constant. This allows us to associate a physical property of the vibrating string with the modes. Namely, that each normal mode represents a partial (and possible) state of constant energy. No energy is transferred between the normal modes in the time-evolution of the system. Wilson pictures the modes as 'energy traps', picking out important macroscopic properties of the system that are related to the boundary conditions (2017, 249, 271, 400). His interest here is in the study of how the Fourier technique offers a procedure of 'semantic lifting' to a new vocabulary or descriptive area for the string (frequency or energy domain representation) that offers a superior reasoning advantage over the original representation of the system in terms of the evolution of the displacement of points on the string (2006, 490-491, 531-532). Wilson does not analyse the modes or the success of the Fourier technique in relation to the concept of superposition and the physical significance and supporting architecture of the concept. The role of the concept of superposition coupled with the observation that the modes are states of constant energy is crucial in QFT where, roughly speaking, the modes in the decomposition of the relevant PDEs are related to states of invariant energy, and relativistically associating energy with mass, modes are then associated with particles.[23]

[23] To return to realism issues, what this will mean is that first, where, and only where, we have a patch 1 type Fourier decomposition we have a particle concept available, so secondly, to deny the existence of Fourier modes on patch 1 will be to deny the existence of particles.

4.4.2 The vibrating circular membrane (2-dimensional wave equation)

Consider the linearized model of a vibrating circular membrane of radius R clamped around its circumference, leading to the two-dimensional wave equation for the membrane's perpendicular deflection $\varphi(r,\theta,t)$ in polar coordinates, subject to appropriate boundary conditions, namely $\varphi(R,\theta,t)=0$. Owing to the boundary conditions polar coordinates are adopted so that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi(r,\theta,t)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\varphi(r,\theta,t)}{\partial\theta^2}-\frac{1}{c^2}\frac{\partial^2\varphi(r,\theta,t)}{\partial t^2}=0$$

Separating variables via $\varphi(r,\theta,t)=U(r,\theta)T(t)$ gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U(r,\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 U(r,\theta)}{\partial\theta^2}+k^2U(r,\theta)=0$$

$$\frac{d^2T}{dt^2}+\omega^2T=0 \qquad (\omega=kc)$$

Separating variables again via $U(r,\theta)=R(r)\Theta(\theta)$ gives

$$\frac{d^2\Theta(\theta)}{d\theta^2}+n^2\,\Theta(\theta)=0$$

$$\frac{d^2\mathrm{R}(r)}{dr^2}+\frac{1}{r}\frac{dR(r)}{dr}+\left(k^2-\frac{n^2}{r^2}\right)R(r)=0$$

With the appropriate boundary conditions we have

$$D_\theta=\frac{d^2}{d\theta^2}$$

$$D_r=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}$$

and eigenvalue equations

$$(D_\theta+n^2)\Theta(\theta)=0$$

$$(D_r+k^2)\mathrm{R}(r)=0$$

The differential operator D_θ is familiar from the vibrating string with the trigonometric Fourier modes $\{\sin(n\theta),\cos(n\theta)\}$ as eigenfunctions with eigenvalues $\{n^2\}$. The eigenfunctions for D_r are the Bessel functions $\{J_n\}$, a set of generalized Fourier modes. The orthonormality conditions for $\{J_n\}$ are rather complicated,[24] but the Bessel functions form an orthonormal set $\{J_n(k_{j,n}r)\}$ on the relevant Hilbert space.

[24] The Bessel functions $J_n(k_{j,n}r)$ are orthogonal (and may be normalized) with respect to the $k_{j,n}$ in the sense that the inner-product $\left(J_n(k_{j,n}r),J_n(k_{l,n}r)\right)=\delta_{jl}$ on the space of functions with

This normal modes are obtained after recombining variables as

$$\varphi_{j,n}(r,\theta,t) = A_{j,n}J_n(k_{j,n}r)(\sin n\theta + B_n \cos n\theta)\left(\sin ck_{j,n}t + C_{j,n}\cos ck_{j,n}t\right)$$

from which, using 'Hilbert superposition' the general solution of the vibrating drum-membrane is (Mathews and Walker, 1970, 266-7):

$$\varphi(r,\theta,t) = \sum_{j,n=1}^{\infty} A_{j,n}J_n(k_{j,n}r)(\sin n\theta + B_n \cos n\theta)\left(\sin ck_{j,n}t + C_{j,n}\cos ck_{j,n}t\right)$$

The modes of vibrating membranes or plates were identified experimentally in the late 18th-century by E.F.F. Chladni and are referred to as 'Chladni modes' (Chladni 2015 [1809]). We now understand the Chladni modes as generalized Fourier modes as just indicated. Wilson comments that in these modes Chladni discovered hidden properties of plates, being the 'normal modes' of their vibration (1993; 2006; 2017).

The semantic architecture of the Fourier technique, the modes and the application of 'superposition' here is conceptually similar to that of the vibrating string with an added layer of structure owing to the move from one to two spatial dimensions. The individual normal modes $\varphi_{j,n}(r,\theta,t)$ are possible independent vibratory states, where the energy of each normal mode is conserved. Any vibratory state can be decomposed as a (Hilbert) superposition of these modes, with the architecture of the application of 'superposition' as before, associating the modes with partial systems and laws that take the same form in and out of (linear) combination without stating the facts. Again, it is crucial that any initial state can be represented in terms of the (product of) modes

$$\psi_{j,n}(r,\theta) = J_n(k_{j,n}r)(\sin n\theta + B_n \cos n\theta)$$

(second aspect to Fourier's technique). The modes represent a natural description of the membrane's vibratory behaviour.

However, it is important to observe several new features that become clear in this example. First, the initial choice of coordinate system (polar) was crucial in establishing the 'descriptive opportunity'. Polar coordinates are in a sense 'natural' here since they support the eigenfunction decompositions in a way that rectangular coordinates would not. The naturalness of the coordinate

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suitable boundary conditions for the problem under consideration (Mathews and Walker 1970, 181-2)

choice in this sense appears to owe to symmetry considerations, with choice of coordinates associated with the separation of variables as an application of the 'Volkmann device'. That is, these coordinates are the correct 'isolation centres' for the system for they support a natural description of the system optimally balancing simplicity and strength.

Secondly, since there are now three applications of the Volkmann device (two separations of variables and Hilbert superposition to the simple 'normal mode' solutions) the architecture of 'superposition' has become more complicated, especially with regard to the different ways in which we can identify partial states and partial laws as discussed above (notes 18-19), and there are now three candidates for 'mode'.

Thirdly, the role of the boundary conditions is crucial. A membrane of a different shape (e.g. a rectangle), whilst satisfying the same wave equation, will have very different solutions owing to the different boundary conditions.[25] The difficulty that this highlights is that once we depart from the analysis of certain 'ideal' shapes to more irregular shapes, such as a guitar soundboard, whilst from Sturm-Liouville theory we may know *in principle* of the existence of modes of the membrane, it may be impossible to represent them explicitly. That is, we might know of their existence whilst not being able to say about them (cf. Wilson 2006, 240-258; 2017, 398-404). This can be taken to illustrate the limitations of considering only idealized 'text book' examples. We can only fully exploit the descriptive opportunity that Sturm-Liouville offers in very limited situations.

4.5 Patch 2 and semantic mimicry

I now consider two examples of 'semantic mimics' for 'superposition' relating to patch 2, namely decompositions of a function representing a physical quantity outside the context of Sturm-Liouville theory: First, the Fourier decomposition of a function representing a physical quantity without reference to any linear equation representing a physical 'law' for the system; Secondly, the construction of a solution to a linear differential equation modelling a physical system by

[25] This highlights Wilson's repeated insistence, drawing upon Hadamard (1915), that we consider not only a differential equation, but the boundary conditions also (e.g. 2017, 413).

iterative techniques or approximation methods. Failing to recognize such mimicry leads to conceptual confusions, as I shall develop in QFT.

4.5.1 Epicyclical astronomy and Fourier analysis

Consider the decomposition of a function f representing an observed planetary trajectory into epicycles. This epicyclical decomposition may be interpreted as the trigonometric Fourier decomposition of the trajectory considered as a parameterized curve $f(t)$ without reference to physical laws, but to an observed trajectory. This example has been used to motivate anti-realism regarding both scientific theories and Fourier modes (cf. Healey 2013).

Hanson (1960) showed how to understand epicycles as terms in a Fourier series: Consider the representation of arbitrary planar periodic motion by the 'superposition' of epicycles (understood geometrically). The trajectory may be represented on $\mathbb{C} = \{z = x + iy = r.\exp(i\theta): x, y, r \in \mathbb{R}; \theta \in [0,2\pi)\}$. An arbitrary (though realistic) 2π-periodic motion of a body expressed as $f(t)$ has Fourier decomposition

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp(nit)$$

The geometric interpretation of this decomposition is a 'geometric superposition' of epicycles. For suppose the motion of a point A is given by $g(t)$ and that B is moving relative to A in a circle of radius ρ, period T and phase α. Then B moves on an epicycle carried by A with

$$z = g(t) + \rho.\exp\left(\frac{2\pi it}{T} + i\alpha\right) = g(t) + a.\exp(ikt)$$

and so on, for C moving relative to B, etc. Arbitrary periodic planar motion can be approximated to any precision via the 'geometric superposition' of N epicycles, equivalent to the corresponding N terms in the Fourier decomposition of $f(t)$.

Epicyclical astronomy is a paradigmatic counter-example to realist stances toward scientific theories, and has been used to claim that Fourier decompositions do not support realist interpretation. For example Richard Healey suggests that the

> mathematical promiscuity of [Fourier] decomposition may prompt one to question its physical significance when one notes that a Ptolemaic analysis of geocentric motion by a system of epicycles and deferents can reproduce any observed planetary motion with

> arbitrary accuracy. But … the enormous utility of the technique in diverse applications throughout physics should at least prompt a more critical examination of the very notion of physical significance. (2013, 51)

But Healey does not distinguish between different applications of Fourier techniques, treating the decomposition of *f* here as 'promiscuous' and apparently imports such promiscuity into the application of Fourier techniques generally.

Two observations: First, the architecture supporting the Fourier decomposition of *f* here on patch 2 is the Hilbert space structure of the function space of which *f* is an element on patch 3. So its decomposition *is* 'promiscuous' as according to this architecture *f* could be decomposed relative to any suitable basis. So the decomposition of *f* here may be compared with the arbitrary decomposition of a vector in which there is no 'physical significance' to the components individually in the sense of the examples in which we have applied 'superposition'. The components here do not support a physical trace principle.

Secondly, as we have seen in the application of Fourier techniques prolonged to Sturm-Liouville theory on patch 1 the Fourier decomposition is not promiscuous or arbitrary on patch 1, as Fourier argued. In such cases on patch 1 there is a physical story to tell regarding the significance of the Fourier basis involving a 'trace principle' as developed above. Indeed, the modes of patch 1 are physically salient partial states of the system, unlike the epicycles on patch 2, which are not as they are not derived from a PDE modelling the behaviour of the system. There are no partial laws available with which to associate epicycles that take the same form in and out of combination, and so 'superposition' (in other than a non-physical geometric sense) is inapplicable to the epicyclical decomposition of *f*.

The epicyclical representation is not a natural representation as it doesn't support inductive inferences or counterfactual reasoning. That is, in the case of the vibrating string we can ask, in terms of the Fourier components, what would happen if the string density and length had been different, and readily supply a physically salient answer. But we cannot do so with the epicycles – if the planets had started in different positions, had different masses, etc. we could not reason about how things would be different from the epicycles.

However, we might ask if there is any reason to privilege the trigonometric basis for the decomposition of *f* into epicycles. Volkmann suggests

that the trigonometric modes as epicycles were preferred as it was supposed that circular motion is the most perfect form of motion (1896, 73). Whether this was an aesthetic or a metaphysical claim, a reason is given to prefer a particular decomposition. But such argument carries little or no weight now. We saw that van der Pol showed that a saw-tooth or square-wave basis might be chosen, and might be preferable in certain instances.

The epicyclical decomposition of planetary trajectories exhibits 'semantic mimicry' in relation to the application of Fourier techniques and 'superposition' from patch 1. If the conceptual 'dragging' associated with epicyclical astronomy is not noticed confusion results, as in Healey's discussion. 'Superposition' does not apply on the patch of Fourier decomposition in which a function is decomposed without reference to a linear differential equation modelling a system's behaviour.

4.5.2 Iterative series techniques contrasted with Fourier series

The second example of semantic mimicry concerns the contrast between a Fourier series solution to a linear differential equation $L[v(x)] = 0$ representing some physical system's behaviour, and a power series solution of the equation. The Fourier series solution is, in exponential form

$$v(x) = \sum_{n=-\infty}^{\infty} c_n \exp(nix)$$

To form a power series solution, suppose that a solution exists of the form

$$v(x) = \sum_{m=0}^{\infty} d_m x^m$$

then solve for the coefficients by iterative methods.[26]

Superficially, in syntactic terms the structure of the solutions appears similar – infinite series of terms that converge to the solution. However, their semantic architecture is different. In the Fourier series individual terms are solutions individually to the differential equation by Hilbert superposition, representing possible physical partial states associated with partial laws and support trace principles, physically salient explanations, etc. so that

[26] See Simmons (2017, chapter 5).

'superposition' applies. This is not so for the power series, for no individual term is a solution to the differential equation, and so Hilbert superposition does not apply, and so individual terms (or their finite linear combinations) do not represent possible partial states. The individual terms are not associated with partial laws that take the same form in and out of combination associated with persisting states. So whilst a reasoning advantage is established through the use of the Fourier modes, no such advantage is obtained with the power series as it is not a natural description since it does not support inductive inferences or counterfactual reasoning, or at least only in a limited sense.[27] The power series overall simply gives an approximate solution in a specific case without physical insight. So 'superposition' does not apply to the power series.

Furthermore, the mathematical architecture of the Fourier series and the power series is different, for the set of individual terms of the Fourier series solution form a basis for the relevant function space, whereas there is no such Hilbert space structure in view or available with regard to the set of individual terms of the power series. So attempts to interpret iterative series as analogous to a Fourier series on patch 1 (or vice versa) is an instance of semantic mimicry.[28] This observation will be important in QFT where the individual terms in an iterative power series (Dyson's series) are mimics for patch 1 Fourier techniques which leads to confusion in the interpretation of Feynman diagrams for the shift in semantic architecture often goes unnoticed (§9.7).

4.5.3 Semantic mimicry: Summary

The two cases of semantic mimicry studied in relation to application of 'superposition' occur when either the behaviour of the system is given in terms of an observed behaviour rather than one modelled through a linear differential equation, and / or the decomposition of the function representing the behaviour

[27] To say that there is no reasoning advantage or physical significance of individual terms in a power series needs nuancing. That is, there are cases in which the coefficient of a particular term in a power series solution can give insight into physically significant trends of dependence on some variable or parameter (cf. Holmes 2013, 1-46). This is, however, a much weaker claim than that required for 'superposition' for at best it enables the identification of a trend rather than actual behaviour.

[28] Cf. Vickers (2013) who interprets Fourier series via assumed analogy with arbitrary infinite series used to represent a physical quantity. This is misleading and another instance of semantic mimicry.

of the system is not constructed using eigenfunction techniques. In each case the representation of the system's behaviour via a series is not a 'natural' description and 'superposition' is inapplicable, unlike examples in Sturm-Liouville theory.

4.6 Summary

We have analysed the differing semantic architectures of Fourier techniques to consider the applicability and significance of 'superposition', considering the physical significance of the concept and contrasting it with mimics. I continue the analysis of prolongations of Fourier techniques in the context of linear systems analysis in chapter 5.

Chapter 5

The façade structure of superposition (3): Linear and nonlinear systems

In this chapter I shall continue to articulate the façade structure of 'superposition'. In §§5.1-2 I consider the application of the concept in the general analysis of linear systems before contrasting such systems with nonlinear systems in §5.3. Superposition is inapplicable to nonlinear systems, with semantic mimics of superposition likely to occur, which we study in §5.3.2. This will form a foundation for considering semantic mimics of superposition, and its inapplicability, with regard to the use of nonlinear equations associated with interacting QFTs.

5.1 Linear systems analysis overview

Linear systems analysis concerns modelling physical systems in which a response or output is related linearly to an input. Often the inputs and outputs are functions of time, and in many cases the systems are invariant to time translations: so-called linear time-invariant (LTI) systems. So for instance in electronics one studies the relationship between the input and output of an 'LCR' AC circuit functioning as a filter in which the input and output are AC voltages. If $f_1(t)$ and $f_2(t)$ are inputs (e.g. voltage signals) to an LTI system S that are (perhaps scaled and) summed and applied as the input $a_1 f_1(t) + a_2 f_2(t)$ then the output is

$$L(a_1 f_1(t) + a_2 f_2(t)) = a_1 L\big(f_1(t)\big) + a_2 L\big(f_2(t)\big)$$

where L is the linear (differential) operator modelling the behaviour of S. If the $f_i(t)$ have physical causal origins we may consider this an application of 'Simons superposition' with the response to the composite input being the linear combination of the (physical) 'partial' inputs taken individually that may be 'traced' in the system response. We consider the composite input as the superposition of the $f_i(t)$, as decomposition of the input in this sense supports physically salient explanations and counterfactual reasoning regarding the

response of S. In this sense describing the composite input as the superposition of the $f_i(t)$ can be considered to support a natural description of the situation.

It remains to express the action of L however, and this may call for a different representation of the composite input, or indeed the $f_i(t)$. We can decompose f in numerous ways owing to S's linearity via 'Hilbert superposition', but most of these would not offer a reasoning advantage or physically insightful or salient explanations of S's behaviour. Put another way, we can ask the question of whether there is a decomposition that is 'natural' or 'privileged' so that it optimizes the simplicity and strength of the representation of the input f so that the components have physical significance and 'superposition' applies in a physically meaningful sense, similar to the Sturm-Liouville examples considered in chapter 4.

Consideration of this question leads to a rich area of mathematical physics. There are two important and related decompositions of an arbitrary input to an LTI system S that I shall consider,[1] the frequency response and the impulse response:

First, one can consider decomposition of f as a 'superposition' of eigenfunctions associated with L (a 'frequency domain' decomposition). This is associated with 'Hilbert superposition' and reflects Fourier's original approach in its 'two aspects' (patch 1), although new semantic architecture is introduced here, establishing a patch 4a where a Fourier transform rather than a Fourier series is employed;

Secondly, f can be decomposed as a 'superposition' of impulses (a 'time domain' decomposition). This is 'Volterra superposition' (patch 4c).

For the same phenomenon (i.e., the physical input represented by f considered as arising from two distinct causal origins) potentially three different applications of 'superposition' may be made: first, the composite input considered as the superposition of the physically separate inputs (Simons superposition); secondly, 'Hilbert superposition' via the Fourier transform, reflecting an eigenfunction decomposition of L; thirdly, 'Volterra superposition' via impulse responses of S. If each of these is genuinely an instance of

[1] Another important decomposition relates to the use of wavelets, and is particularly important in digital signal processing. I do not consider such decomposition as it is not relevant to my analysis of QFT.

superposition, it indicates the façade structure of both the concepts of superposition and component in a way comparable with Wilson's analysis of 'hardness' (cf. §1.2.2), since their applications depend on the context of physical interest. Perhaps then 'superposition' is promiscuous in application, to use Healey's term (§4.5.1), although not arbitrary, and it is natural in its application.

5.2 Linear systems analysis

5.2.1 Frequency domain (eigenfunction) decomposition (patch 4a)

For an LTI system S, $\{\exp(s + i\omega t) : s, \omega, t \in \mathbb{R}\}$ forms a set of eigenfunctions for the linear operator L as above acting on the relevant Hilbert space (Lathi 2010, 195). If we take s=0, then the eigenfunction decomposition of the input $f(t)$ as above gives the *Fourier transform* of f:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)e^{i\omega t}d\omega$$

irrespective of the physical origins of f, although by linearity one may regard this decomposition as the sum of the Fourier transforms of the $f_i(t)$ composing $f(t)$ as above. However, it will be the Fourier transform of $f(t)$ simpliciter that plays the indispensible explanatory role regarding S's response even if counterfactual reasoning in terms of the $f_i(t)$ also remains supported.

One may construe the Fourier transform as the limit of a Fourier series of a P-periodic function as $[-P/2, P/2] \to (-\infty, \infty)$, and so it may be considered a decomposition of a signal into a continuous set of trigonometric modes. It is convenient to use the complex exponential form, but the crucial difference between this and the Fourier series is the necessary introduction of the integral over a continuum of modes, and with infinite limits. This significantly alters the semantic architecture of 'superposition' and the Fourier technique, invoking additional mathematical structure – that is, the Fourier transform is not only the limit of a Fourier series.

Bracketing the difficulties this introduces for a moment, this means that if we represent an arbitrary input to S by the Fourier integral, the output is given simply by scaling the modes $\{\exp(i\omega t)\}$ in virtue of their being eigenfunctions of L. The Fourier transform supplies a 'diagonal' representation of L, namely $H(\omega)$,

which is identified as a property of S known as its 'frequency response'. That is, the output $u(t)$ to an arbitrary input $f(t)$ is given by (Lathi 2010, 719):

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)H(\omega)e^{i\omega t}d\omega$$

As per the 'two aspects' of Fourier techniques, decomposition of f into the eigenfunctions of L with coefficients $\tilde{f}(\omega)$ via the Fourier transform exploits a descriptive opportunity that affords a natural decomposition, optimally balancing simplicity and strength to support physically salient explanations and counterfactual reasoning. $H(\omega)$ supplies the 'partial laws' corresponding to the 'partial states' $e^{i\omega t}$ (as indexed continuously by ω) that describe the behaviour of the system, i.e. by pointwise multiplication of the $e^{i\omega t}$ by $H(\omega)$. This superficially looks like the Fourier technique on patch 1 in chapter 4, and so we would privilege decomposition of f via $\{\exp(i\omega t)\}$ and call this its 'superposition' in the context of S.

However, there are two obstacles: First, for many functions that one would wish to consider, the Fourier transform is undefined, if the transform is defined as above. That is, even 'nice' functions such as $\sin(at)$ do not have a Fourier transform in the sense of the limit of a Fourier series as it does not decrease sufficiently rapidly as $t \to \pm\infty$. This can be dealt with mathematically using the theory of distributions (see below), and indeed Laurent Schwartz claimed that setting the definition of the Fourier transform in the distributional context is 'inevitable, in a direct or camouflaged form' (Schwartz 1950, 7).[2] But this is to prolong the Fourier technique to a new patch of application with differing supporting architecture in which extra mathematical structure is invoked (distributions, and the dual space of a Hilbert space of functions).

Secondly, setting aside the issue that we are dealing with complex-valued functions,[3] the modes $\{\exp(i\omega t)\}$ are unphysical in the sense that one must consider them as 'everlasting' inputs, i.e. starting at $t = -\infty$. The modes must be considered as idealized inputs or states rather than possible physical states,

[2] See Lützen (1982, chapter 3) for historical development; Stein and Shakarchi (2003, 129-145) for mathematical treatment of the Fourier transform in its distributional setting.
[3] This difficulty may be overcome by considering the representation in terms of amplitude and phase of real-valued trigonometric functions, but the exponential form is often more convenient.

unlike the Fourier series case. This idealization introduces another 'dragging' of 'superposition' and the Fourier technique.[4]

We should pause to consider the implications. For the vibrating string the possibility remains open for considering the string's motion as composed of the physical harmonics (the Fourier modes) in a (metaphysically) serious sense, where moreover it is physically conceivable that the initial condition had the form of a mode. But with the Fourier transform we can decompose functions that are not periodic, unlike the Fourier series case, but into physically inconceivable 'everlasting modes'. We could consider a function f representing a physical quantity, say the output of a voltage generator which is initially switched off, switched on for a short time, then switched off again, with the signal *f* fed into an LCR circuit as our LTI system. The function f representing the voltage as a function of time is zero everywhere apart from a short interval. But the Fourier transform of the voltage signal gives non-zero modes extending from $t = -\infty$ to $t = \infty$. Intuitively it appears wrong to regard the voltage signal as composed of the modes in any physical (or indeed metaphysical) sense whilst the voltage generator is switched off. This would suggest that whilst the modes in the Fourier transform have a form of physical significance in supporting explanations, inductive inferences and counterfactual reasoning, and pick out what we could call a physical property of the system known as its 'frequency response', they appear to be 'components' in a different sense from the Fourier modes on patch 1, not having the same physical significance of Fourier modes in Sturm-Liouville systems.

However, perhaps the difference is only apparent, or at least perhaps the distinction is too caught up in realist intuitions, which is something that I have been careful to avoid. In the Fourier series case we could consider a periodic function that is mostly zero on some finite interval. Its Fourier series decomposition will be non-zero everywhere on the interval however, like the Fourier transform. This suggests that we should be careful regarding the sense in

[4] Cf. Wilson (2006, 518-566, esp. 540), and the further discussion below in §5.2.3. The point is that unlike the cases of Fourier techniques considered in chapter 4 involving discrete modes for which the modes represented possible (even if unlikely) physical states or initial conditions of the system as modeled, the continuum modes of the Fourier transform represent idealized inputs or states, so that 'superposition' is no longer supported in terms of the modes being physically possible states or inputs for finite times.

which we view the modes of a Fourier series as 'physically significant', and avoid seeking to infer a realist understanding of the modes prematurely, despite their role in supporting (indispensible) physically salient explanations and counterfactual reasoning.[5] The modes in both cases are associated with properties of a system and not some input to a system, for which the modes become significant only in virtue of the role that they play in the system considered.

We gain a reasoning advantage supporting inductive inferences and counterfactual reasoning by considering the representation of the input to an LTI system via the continuous set of eigenfunctions of L, namely $\{\exp(i\omega t) : \omega, t \in \mathbb{R}\}$, for which the response of L is given by the frequency response $H(\omega)$, which is the 'diagonal' representation of L and may be regarded as a property of the system. The modes support a 'trace principle' in the sense discussed in chapter 4. This is of particular significance in systems exhibiting 'resonance', where the presence of components $\exp(i\omega t)$ near resonance are explanatorily relevant to the resonant behaviour. It is especially this feature that engineers exploit in the design of mechanical and electrical systems.

In QFT Fourier transforms are ubiquitous. The Fourier transform converts between position and momentum representations, it is used in the solution of the wave equations and it is used in explicit representations of propagators. Moreover, the Heisenberg uncertainty principle can be understood as a property of Fourier transforms (Stein and Shakarchi 2003, 158-161), apart from any physical semantic support or context.

5.2.2 Time-domain (impulse) decomposition (patch 4c)

Volterra applied 'superposition' in the calculation of the twist ω of a thread in response to a time-varying torque M considered as an integral of impulses $M(t)$, so that the overall response (twist) of the thread at time t is the 'Volterra superposition' of the 'impulse responses' of the thread over earlier times (Volterra 1913, 219-221):

[5] There is a further complication in that if we consider two identical but 180° out of phase sine waves as AC voltage inputs that are summed, their composite is always zero, but it seems correct to regard the zero voltage input as composed of two sinusoidal signals.

$$\omega(t) = KM(t) + \int_0^t \Phi(t-\tau)M(\tau)d\tau$$

where Φ is the 'coefficient of heredity' that relates the twist of the thread at time t to an instantaneous torque M applied at an earlier time τ.

This introduces a further idealization in that we study the response of a linear system S to an 'instantaneously' applied torque of infinitesimally short duration, that is, as an idealized impulse, and it is this idealization that complicates the architecture of 'superposition'. We decompose an arbitrary input $f(t)$ to S as a 'superposition' of impulses in the time-domain, rather than as a 'superposition' of eigenfunctions in the frequency domain, and calculate the response or output of S as the superposition of the 'impulse responses'.

In a modern perspective, this involves the introduction of 'generalized functions' or distributions and Green's functions. The impulse is modelled, in idealized form, by the Dirac-δ functional or, distribution. Distributions are mathematical entities that invoke new structure so as to enable sense to be made of, for example, the Dirac-δ considered as a limit of functions, or to enable one to define Fourier transforms of functions such as $\sin(2\pi at)$.

Very briefly, mathematically speaking distributions are understood as linear functionals acting on a 'suitable space' of functions, so that a 'distribution' is an element of the dual space of the appropriate function space. That is, suppose f is a locally integrable function and φ is any smooth function of 'sufficiently rapid decrease', then a distribution T_f may be associated with f via[6]

$$\langle T_f, \varphi\rangle = \int f(x)\varphi(x)dx \quad \forall\varphi \in \mathcal{S}(\mathbb{R})$$

$\langle \delta, \varphi\rangle = \varphi(0)$ defines the Dirac-δ, and one may interpret $\sin(2\pi at)$ as a distribution (substituting $\sin(2\pi at)$ for $f(x)$) enabling its Fourier transform $\mathcal{F}$ to be defined distributionally as

$$\mathcal{F}\{\sin 2\pi at\}(\omega) = \frac{1}{2}i[\delta(\omega + a) - \delta(\omega - a)]$$

However, the physical significance of the Dirac-δ as an impulse might be more readily understood in terms of the idealized limit of short-duration inputs

[6] $\mathcal{S}(\mathbb{R})$ is the Schwarz space of test-functions. Other spaces of test-function may be used, but use of the Schwarz space is natural with regard to the Fourier transform. The theory of distributions is a large and important area of mathematics that I cannot develop here.

than as a mathematical distribution. That is, an impulse is not a physically possible input, but rather the abstraction or idealization of a form of input that will be valuable in modelling the behaviour of the system, i.e. establishing a reasoning advantage. [7]

A Green's function $G(t,s)$ for S is defined as the 'impulse response' of S, that is, the response at t given the application of a unit impulse at s. So if the homogenous linear system is modelled as

$$L[v(t)] = 0$$

then

$$L[G(t,s)] = \delta(t-s)$$

where $G(t,s)$ is the relevant Green's function.[8] Then an arbitrary input $f(t)$ can be represented as a 'superposition' of suitably scaled impulses:

$$f(t) = \int \delta(t-s)f(s)ds$$

From this, and knowledge of G, the response $u(t)$ to f can be calculated to be

$$u(t) = \int [G(t,s)]f(s)ds$$

So the response to f may be calculated if the impulse response $G(t,s)$ is known. $G(t,s)$ establishes a 'trace principle' or 'partial law' for the impulse $\delta(t-s)$. It is common to write the impulse response as h so that

$$u(t) = \int h(t-s)f(s)ds$$

This is 'Volterra superposition', which clearly resembles 'superposition' as we have analysed it so far, the $h(t-s)$ for individual t,s taking the same form in and out of linear combination, whilst not stating the facts when in combination. However, as was the case with the 'frequency response' just considered, the semantic architecture of 'superposition' is becoming ever more intricate and difficult to unravel as appeal is made to idealized inputs and distributions. As with the 'frequency response' H we consider the 'impulse

[7] See Bueno and French (2018) for philosophical discussion of the Dirac-δ.
[8] The Green's function need not be unique, although for many important systems it is. I do not consider the details here.

response' h as a property of S, with 'superposition' applied to the input f in each case supported by the properties of H and h respectively.[9]

Green's functions will play a central role in QFT, being associated with particle propagators, with the Dirac-δ often associated with the introduction or removal of a particle from the system as a source term, and with the choice of test function associated with the distributional definition of the Dirac-δ interpreted as the wave function of a particle.

5.2.3 Integral transforms: Generalizations on patch 4

Integration of Green's functions considered as impulse responses is an example of an *integral transform* by a *kernel*, that is (Arfken 1985, 794)

$$g(\alpha) = \int_a^b f(t)K(\alpha, t)dt$$

The Fourier transform is another example of such a transform with $K(\alpha, t) = e^{i\alpha t}$.

Another important transform in linear system design and analysis is the *Laplace transform.* The Laplace transform of f is defined as

$$F(s) = \int_0^\infty f(t)e^{-st}dt \qquad s = \sigma + i\omega : \sigma, \omega \in \mathbb{R},$$

It is an integral transform with $K(\alpha, t) = e^{-\alpha t}$ where $\alpha \in \mathbb{C}$. The Laplace transform introduces the possibility of yet another patch of application for 'superposition' involving further abstractions, complications and differing semantic architecture.

---

[9] There are however important relationships between impulse and frequency response. The *convolution* of two functions is defined

$$f * g = \int g(y)f(x - y)dy$$

The Fourier transform of a convolution has a simple form:

$$\widetilde{f * g} = \tilde{f}.\tilde{g}$$

That is, if f and g are time-domain functions, their convolution in the time domain corresponds to simple multiplication in the frequency domain. The response of an LTI system is the convolution of its impulse response with the input:

$$u(t) = h(t) * f(t)$$

Taking Fourier transforms

$$\tilde{u}(\omega) = \tilde{h}(\omega).\tilde{f}(\omega)$$

But $\tilde{h}(\omega) = H(\omega)$, the frequency response, and $\tilde{f}(\omega)$ is the Fourier transform of the arbitrary input $f(t)$, that is, the coefficients of its eigenfunction decomposition. So the two descriptions of the system's response, and the allied notions of superposition, are closely related. One involves a description or choice of simple components in the time domain, the other in the frequency domain. Both descriptions involve 'idealized' and in that sense 'unphysical components' mathematically described via distributions.

Wilson discusses the Laplace transform alongside the use of impulse and frequency responses in the context of Heaviside's operational methods (2006, 518-566). He understands concepts such as impulse response, frequency response, etc. as 'design parameters' that code basic properties of response latent in the system: 'In short, the physical significance of "solution" shifts from *representing a possible history of the circuit* to covering more abstract inclinations to reshape signals fed into the wire.' (2006, 540) In this sense, as reflects engineering practice in the design of control systems for instance, these integral transform techniques form a 'toolkit' that establishes reasoning advantages in relation to the design and analysis of such systems.

The architecture of 'superposition' is intricate and complex here, as are decisions on the applicability of terms such as 'partial law' qua 'law'. For in the Mill-Ramsey-Lewis sense, it would appear that there are several complementary rather than competing ways of construing a 'best system' in the context of the design and analysis of such systems.[10] For my purposes however, it is sufficient to note that it is beneficial to discuss impulse and frequency responses in terms of 'partial laws' so as to forge a link with 'superposition' in the context of LTI systems analysis, heeding Wilson's analysis on the façade structure of 'law', whether or not it is desirable to invoke the Mill-Ramsey-Lewis characterization.

These abstract integral transforms provide further 'prolongation' of 'superposition' to ever more abstract settings, although useful in system design, and indeed Jefferys and Jefferys term this 'generalized superposition' (1956, 404). In this generalized approach the semantic architecture of 'superposition' differs on a case-by-case basis. However, as it is the Fourier rather than Laplace transform that is ubiquitous in QFT I shall not consider the Laplace transform further.

5.2.4 Linear systems summary

There are various ways in which we can consider the input to an LTI system as 'composed of' a 'superposition' of components. In each case we can identify the physical significance of the components (partial states) and associate them with

[10] This class of example might lead to a 'perspectival' account of the best system account (cf. Massimi 2018).

partial laws, even if appeal to Mill-Ramsey-Lewis is problematic here. Each appears to reflect a genuine although different application of 'superposition' with differing supporting architecture, with different decompositions preferred in different reasoning or design contexts.

5.3 Nonlinear systems and semantic mimics

5.3.1 The contrast between linear and nonlinear systems

The behaviours of the various linear systems considered here and in chapter 4 arise in the context of models that approximate or idealize the behaviour of physical systems so that a linear model is obtained that supports 'superposition'. That is, if $f = a_1 f_1 + a_2 f_2$ is the input or initial / source condition to a system modelled by a linear operator L_{lin}, then the response is

$$L_{lin}[f] = a_1 L_{lin}[f_1] + a_2 L_{lin}[f_2]$$

by either 'Hilbert superposition' or 'Volterra superposition'. The ways in which we understand the semantic architecture of 'superposition' in the various cases and thus the sense in which we can speak of the 'composition' of the input or response varies from application to application.

We can decompose f in various ways, but decompositions into (generalized) Fourier modes are usually available, at least in principle, for systems modelled with linear differential equations. Whilst the physical significance of such decompositions must be analysed locally, minimally the observation that such decompositions are 'superpositions' is associated with the physical salience for the system of the simple components identified.

In practice most systems modelled with linear differential or integral equations exhibit a weak degree of nonlinearity when the linearizing idealizations or approximations are removed. However, in many cases the nonlinearity is sufficiently weak that it may be ignored for practical purposes, and 'superposition' is approximately true, so that

$$L_{realworld}[f] \approx a_1 L_{realworld}[f_1] + a_2 L_{realworld}[f_2]$$

This contrasts with the behaviour of nonlinear systems for which the behaviour is modelled by a nonlinear operator J_{nonlin} in which the nonlinearity is significant and cannot be ignored, as will be the case in interacting QFTs:

$$J_{nonlin}[f(\boldsymbol{x}, t)] \neq a_1 J_{nonlin}[f_1(\boldsymbol{x}, t)] + a_2 J_{nonlin}[f_2(\boldsymbol{x}, t)].$$

That is, solutions or responses to different inputs or initial conditions do not combine in any simple way according to 'superposition', however broadly understood. This means that one cannot construct solutions or responses to nonlinear differential or integral systems from simple solutions or inputs, and one must adopt iterative or approximation techniques to solve for the response of a nonlinear system that likely introduces semantic mimicry. That is, we cannot establish a descriptive opportunity to form a reasoning advantage in modelling nonlinear systems from 'simple' solutions or idealized inputs and responses.[11] Whilst the input may be decomposed by familiar means, the components obtained will lack physical salience.

For example, we can usually form a Fourier decomposition or Fourier transform of an input f to a nonlinear system or, indeed of the response of a nonlinear system modelled as a function u. But the semantic architecture or the physical significance of the decompositions of f or u differ here from that of the linear system as the individual terms in the decomposition are not, or are not related to, eigenfunctions or 'simple solutions' to the equation modelling the nonlinear system. So the significance of such Fourier decomposition is comparable with that of the epicycles of epicyclical astronomy (§4.5.1). That is, the Fourier decomposition of, or application of 'superposition' to an input to or response of a nonlinear system mimics that of its significance in a linear system. Confusion arises when the shifting semantic support is not noticed.

That is, for the nonlinear system the semantic support of the Fourier decomposition is solely that of the mathematical decomposition relative to a basis of a function in a suitable Hilbert space, but where the decomposition, and Hilbert space structure, has no physical salience with respect to the system, or mathematical salience to the nonlinear equation modelling it, unlike in the Sturm-Liouville systems (§4.4). So, knowledge of the response of a nonlinear system to any individual term in a Fourier decomposition of the input tells us nothing about the system's response to the input, as the responses do not take the same form individually and in combination.

[11] Leaving aside application of the nonlinear superposition principle in special cases, where the physical significance of the 'components' is not at all clear. See Menini and Tornambé (2011) for examples. Such cases do not occur in the QFT that I consider, so I shall not consider this further.

This observation will prove important in QFTs with interactions. Fourier decompositions can be performed on solutions of the nonlinear coupled field equation, but such decompositions lack physical significance since the terms of the Fourier decomposition are not solutions of the field equation. Rather, such Fourier decompositions have only the mathematical support as in §4.3, and no physical salience, which can lead to confusion via semantic mimicry.

5.3.2 Volterra series and semantic mimicry

Semantic mimicry can occur in relation to the Volterra series technique for the solution of nonlinear integral equations. Here, Volterra extended his 'heredity principle' to allow for better modelling of physical systems to allow for nonlinearity once the linear idealization is removed, although there is an important shift in the semantic architecture in moving beyond the linear approximation. This method is relevant to us given formal similarities between the Volterra series and Dyson's series in QFT.[12]

Consider a nonlinear system with input $u(t)$ and output $y(t)$ where their relationship is given by a time-invariant functional operator $\mathcal{H}[\cdot]$.[13] One expresses $\mathcal{H}$ as a series of operators of different orders so that

$$y(t) = \sum_{j=0}^{\infty} y_j(t) = \sum_{j=0}^{\infty} \mathcal{H}_j[u(t)]$$

where

$$\mathcal{H}_j[u(t)] = \int_{\tau_j=-\infty}^{\infty} \dots \int_{\tau_1=-\infty}^{\infty} h_j(\tau_1, \dots, \tau_j) \prod_{r=1}^{j} u(t-\tau_r) d\tau_1 \dots d\tau_j$$

$\mathcal{H}_j$ is the j-th order Volterra operator and h_j the j-th order Volterra kernel. If j=1 is sufficient to characterize the system's behaviour (i.e. the kernels and operators are all zero for j>1) then the system is linear and

$$\mathcal{H}[u(t)] = \mathcal{H}_1[u(t)] = \int_{\tau=-\infty}^{\infty} h(\tau) u(t-\tau) d\tau$$

where h is the Green's function, the impulse response as in §5.2.2 (cf. Dunn 2013, 40).

---

[12] Greiner and Reinhardt note that in QFT the Green's functions $G^{(n)}(x_1,\dots x_n)$ obtained via the generating functional W[J] or Z[J] are the coefficients of the Volterra series of the generating functional W[J] (1996, 380).

[13] The following draws upon Dunn (2013); Rugh (1981); Carassale and Kareem (2010); Schetzen (1980).

In the nonlinear case the j=1 term is no longer sufficient to characterize the system's behaviour. Heuristically, the higher-order h_j terms can be considered as 'higher-order' impulse responses (Dunn 2013, 54-57). Whilst it may be tempting to consider such 'higher-order responses' as a 'superposition' of kernels or responses of various orders, this is a misapplication (cf. Dunn 2013, 40). No individual term h_j is a solution to the equation for some impulse, and so does not represent a possible response of the system, whether idealized or not. The h_j are not associated with any partial laws or trace principles.[14] The individual terms are *correction terms* that function together as a whole to approximate the system's behaviour.[15] For instance h_2 does not individually characterize a response that could actually be obtained from an input, idealized or not. Rather, it is a second-order correction term to the linear approximation from h_1. h_3 is then a correction to h_2, and so on. The Volterra series solution is a perturbation about a linear solution (Schetzen 1980, 150), and may be thought of as a power series with 'memory' (Schetzen 1980, 8).[16]

Individual terms in a Volterra series solution to a nonlinear differential equation can 'mimic' the role of individual terms in a Fourier series (or transform) solution. The individual terms in the Volterra series do not compose the solution by superposition, but are a series of corrections to a lower order estimate of the solution, as they do not represent partial laws that take the same form in and out of combination, or support trace principles.

5.4 Summary

To summarize our study of superposition in classical physics, we note that within classical physics the concepts of isolation, component and superposition, and Fourier techniques, have façade structures. There are a number of local

[14] Perhaps there is a difficulty again in application of the Mill-Ramsey-Lewis account of laws in denying that the h_j are associated with laws. That is, if the Volterra series offers our simplest and strongest account of the behaviour of some nonlinear system then there is nothing to deny the h_j the status of (partial) laws. But this is intuitively wrong, and we should ensure that the h_j do not count as laws. The h_j do not support inductive inferences or counterfactual reasoning, other than identifying trends (cf. Holmes 2013, 1-46), and are not abstractable.

[15] Worse still, in a number of important cases (such as Dyson's series in QFT), the series obtained is an asymptotic series in the sense that the series diverges even though the first few terms provide a good approximation to the solution. See e.g. Holmes (2013, 1-46).

[16] Also see Maia et al (1997, 404-405).

related patches of application upon which the semantic architecture differs, and semantic mimicry can occur if the underlying architecture is not scrutinized.

'Superposition' can in one sense be 'promiscuous' in its application, but it supports multiple 'natural' decompositions so that it is not arbitrary where there is more than one genuine application. Decomposition according to Volkmann's 'isolation / superposition' device may be understood in Wilsonian terms as the establishment of a descriptive opportunity that supports physically salient explanations and counterfactual reasoning. Moreover, 'superposition' is associated with the ability to abstract partial states or responses of a system that correspond to partial laws that take the same form individually and in combination whilst not stating the facts individually, supporting some form of 'trace principle'.

Important contrasts are to be drawn between linear and nonlinear systems, and Fourier decompositions performed either within or outside the context of eigenfunction decompositions of operators modelling the behaviour of the system.

In the next chapter we finally turn to 'quantum superposition'.

Chapter 6

Quantum superposition

6.1 Overview

I shall now consider the 'prolongation' of 'superposition' and Fourier techniques to new patches in quantum physics. As before, 'superposition' and the Fourier techniques have subpatch structures, so that the physical significance of components differs from context to context, and semantic mimicry occurs in QFT especially. What we shall see over the following chapters is that in QFT the particle concept (on the 'field quanta' patch, cf. §1.2.6) both in terms of type and number stands or falls with the applicability of 'superposition'.

Here, and in chapter 7, I briefly consider the origins of the application of 'superposition' and Fourier techniques in quantum physics that are foundational to their on-going use. I consider non-relativistic quantum mechanics (NRQM) first here in chapter 6, and QFT in chapter 7. Ehrenfest (1925) applied the concept of superposition to quantized normal modes of the electromagnetic field in fledgling QFT, and Schrödinger introduced 'superposition' to NRQM (1926a-e), again in relation to the normal modes of quantum systems. Fourier and eigenfunction techniques were developed simultaneously in, and were foundational to, NRQM and the emerging QFT, although 'quantum superposition' is perhaps primarily associated with Schrödinger. Since QFT inherits many aspects of 'quantum superposition' from NRQM, and because there are fewer conceptual difficulties in NRQM than in QFT, I briefly consider superposition in NRQM first even though my ultimate focus is on QFT. These brief and selective historical surveys will help to clarify the significance and limitations of 'superposition', especially in QFT, which are perhaps more clearly seen by considering their historical origins than in later developments of the theory.[1] The discussion will be somewhat broad-brush to orient the reader for the more detailed analyses in subsequent chapters.

[1] Cf. Wilson, citing Duhem (with qualified approval), 'The real introduction to the expression of a principle of physics is a historical introduction.' (*Analysis of Mach's Science of Mechanics*, p.117 in Wilson 2017, 153).

There are two important issues regarding the semantic architecture of 'quantum superposition' that I shall not consider beyond a few remarks. First, there is the issue of what characterizes a system as 'quantum'. I adopt a characterization set out by Ruetsche as a standard account in NRQM:

> In the Hamiltonian quantization scheme, a classical theory cast in Hamiltonian form is quantized by promoting its canonical observables to symmetric operators $\hat{q}_i, \hat{p}_i$ acting on some Hilbert space $\mathcal{H}$ and obeying commutation relations corresponding to the fundamental Poisson brackets of the classical theory. The familiar Heisenberg form of the canonical commutation relations (CCRs), representing a quantization of [a] classical theory with phase space $\mathbb{R}^{2n}$ and canonical observables q_i and p_i is …
>
> $$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{p}_i, \hat{q}_j] = -i\hat{I}\delta_{ij}$$
>
> where $\hat{I}$ is the identity operator [and h is suppressed via natural units]. (2011, 36, but cf. 117-122).[2]

The CCRs form the 'physical core of a quantum theory' as Ruetsche puts it. This treatment extends to QFT, with $\hat{q}_i$, $\hat{p}_i$ replaced by fields $\hat{\varphi}_i$, $\hat{\pi}_i$ with the commutator adopted for bosonic fields, and the anticommutator for fermionic fields. The Hilbert space upon which the (field) operators (as distributions) act is the space of states of the system, and the states evolve linearly, or unitarily, according to Schrödinger's equation. This is often taken as 'axiomatic' of what constitutes a quantum system, with the interpretation of the Hilbert space of states as the locus of the interpretation of the theory according to 'physicists' QFT', although complications arise.[3]

Secondly, there is the notorious 'measurement problem' which has given rise to various (perhaps) speculative interpretations of or additions to quantum theory (e.g. Bohmian, GRW, Everettian).[4] The 'measurement problem' has been bound up with 'quantum superposition' often with respect to consideration of macroscopic systems in order to problematize the superposition concept, following Schrödinger (1935). However, the measurement problem, as often characterized, conflates different issues so it is not clear that the problem is properly posed (Cartwright 1983, 163-216). From a Wilsonian perspective the 'measurement problem' might be understood as reflecting 'Theory T thinking'

[2] As Ruetsche goes on to note, there are mathematical difficulties with the familiar Heisenberg CCRs, so that the Weyl form of the CCRs is preferred, although I shall not consider this as it is tangential to the analysis that I wish to present.

[3] In particular, as regards unitarily inequivalent representations. However, the operators form a C* algebra, and there are advantages of interpreting the theory in terms of such algebras rather than their Hilbert space representations. This raises the question of where the proper locus is for the interpretation of QFT, as discussed by Ruetsche (119-147), and forms the basis of the 'Fraser-Wallace debate' (see chapter 1). 'Algebraic' interpretations bring their own difficulties, and cannot (as yet) accommodate interactions, which is the focus of this study.

[4] See Lewis (2016) for a recent philosophical overview and discussion.

and results from an attempt to apply a single theory or theoretical framework to what might be considered to be the theory façade of quantum physics, especially in the context of interactions of quantum systems with macroscopic systems. That is, 'linear Schrödinger evolution' and 'nonlinear Born rule / von Neumann collapse' might be regarded as two patches of the theory façade of quantum physics, and one should not necessarily expect to discover a unifying theoretical framework within the context of NRQM, especially when NRQM is pressed into the description of macroscopic systems.[5] 'Quantum superposition' may be discussed fruitfully to philosophical benefit without addressing the measurement problem.

6.2 The origins of quantum superposition in NRQM

I now review the origins of 'superposition' in quantum physics. Schrödinger explicitly introduced it in a collection of six papers on wave mechanics in 1926. This collection comprised a series of four papers, 'Quantization as Problem of Proper Values' (QPPV) I-IV, and two other papers. He repeatedly draws upon Courant and Hilbert (1924) and applies their definition of superposition ('Hilbert superposition', 1924, 221-320), situated in a discussion of mechanical vibrations modelled by Sturm-Liouville theory, to the quantum context.[6]

In QPPV I Schrödinger introduces the 'wave-function' into modelling the atom, which he associates with a vibration process (as per Courant and Hilbert), but does not elaborate (1926a, 9). There are no references to superposition in QPPV I-II, despite introducing analogies of the wave-function with optics in QPPV

[5] Recall Wilson's theory façade: 'A descriptive complex [of various models applied in different contexts of some phenomenon] of this quilt-like pattern supplies a good example of … a façade: a set of patches or plateaus that are formally inconsistent with one another but are stitched together by "for more details see …" linkages or other bridgework. Often the whole is fabricated in such a manner that, if we don't pay close attention to its discontinuous boundary joints and shifts in mathematical setting, we might suppose that we are looking at a theory ready to be axiomatized' (2006, 191-192). In other words, philosophical dilemmas stemming from the measurement problem might be diagnosed by observing that it has been incorrectly supposed that quantum physics has a 'Theory T' structure rather than a façade structure. Such façade structures are ubiquitous in physics and engineering but do not lead to the same levels of metaphysical angst as in quantum physics, possibly because the latter is taken to be a more fundamental theory. As we shall see however, our conceptual ignorance at the level of quantum theories is much greater than we often think.

[6] For the centrality of Courant and Hilbert (1924) to the development of quantum physics see Coleman (2019, 1103).

II, and the introduction of 'linear aggregates' of 'proper vibrations', i.e. the 'normal modes', again in QPPV II:

> it is not just the *single proper vibration* that furnishes a *possible state of vibration,* but an arbitrary, finite or infinite, *linear aggregate* of such vibrations. (1926b, 34).

It is in the third paper in the collection ('The Continuous Transition from Micro- to Macro-Mechanics') that the first appeal to superposition is made, by analogy with classical vibrating mechanical systems analysed with Sturm-Liouville theory, as per Courant and Hilbert. This is in the context of a discussion of the 'proper vibrations' (normal modes) of the quantized simple harmonic oscillator (QSHO):

> As in the differential equation of a vibrating string or of any other vibrating system, ψ [the "wave-function"] is given as a superposition of pure time harmonic (*i.e.* "sinusoidal") vibrations, the frequencies of which agree exactly with the spectroscopic "term frequencies" of the micro-mechanical system. For example, in the case of the linear Planck oscillator … we get ψ as the superposition of the following proper vibrations:
>
> $$\psi_n = e^{-\frac{x^2}{2}} H_n(x) e^{2\pi i \nu_n t}$$
> $$\nu_n = \frac{2n+1}{2}\nu_0; n = 0,1,2,3, \ldots$$
>
> The H_n's are the polynomials [Courant and Hilbert, 76] named after Hermite. … At first sight it appears very strange to try to describe a process, which we previously regarded as belonging to particle mechanics, by a system of such proper vibrations. (1926c, 41-42)

He goes on to relate macro-systems to micro-systems, but without reference to superposition. In QPV III he again discusses Courant and Hilbert's presentation of Sturm-Liouville theory, although without reference to superposition, referring instead to 'the *simultaneous existence* of … two proper vibrations' (1926d, 83) in language reminiscent of Bernoulli or Herschel (cf. chapter 4). In QPPV IV Schrödinger addresses the physical significance of the wave-function and superposition in NRQM:

> the following conception … allows the true meaning of ψ to stand out more clearly. $\psi\bar{\psi}$ is a kind of *weight-function* in the system's configuration space. The *wave-mechanical* configuration is a *superposition* of many, strictly speaking of *all,* point-mechanical configurations kinematically possible. Thus, each point-mechanical configuration contributes to the true wave-mechanical configuration with a certain *weight,* which is given precisely by $\psi\bar{\psi}$. If we like paradoxes, we may say that the system exists, as it were, simultaneously in all the positions kinematically imaginable, but not "equally strongly" in all. … This new interpretation may shock us at first glance, since we have often previously spoken in such an intuitive concrete way of the "ψ-vibrations" as though of something quite real. But there is something tangibly real behind the present conception also, namely, the very real electrodynamically effective fluctuations of the electric space density. The ψ-function is to do no more and no less than permit of the totality of these fluctuations being mastered and surveyed mathematically by a single partial differential equation. (1926e, 120)

He anticipates the difficulties associated with the interpretation of the wave-function that have pervaded quantum physics, concluding:

> Our inability to give more accurate information … is intimately connected with the fact that, in [the appropriate Schrödinger's equation], we have before us only the substitute – extraordinarily convenient for the calculation, to be sure – for a real wave equation of probably the fourth order (1926e, 123).

In the remaining papers, and in four lectures delivered in 1928 later published together with the papers (1982), he does not make any reference to superposition or co-existent vibrations.

It was later, in response to Einstein, Podolsky and Rosen's famous paper (1935), that Schrödinger introduced his notorious thought experiment with a cat. He supposed that 'quantum superposition' could be prolonged to macroscopic objects with the same semantic architecture regarding system evolution and measurement to argue that the then orthodox interpretation of quantum mechanics was untenable (Schrödinger 1935). So whilst initially (1926) Schrödinger introduced 'superposition' to quantum physics by simple 'prolongation' of the concept via Hilbert's treatment of Sturm-Liouville theory in classical physics, as further layers of interpretation develop, and 'superposition' becomes entangled with other issues involving measurement that may arise from a failure to recognize the nature of NRQM as a theory façade, the concept is dragged to the point where conceptual confusions arise, as in the cat thought experiment. This indicates that care is required in clarifying the proper application of 'superposition' in the quantum context.

However, whilst Schrödinger may have lost confidence in 'superposition' in quantum physics, Dirac made the concept foundational, as reflected in the first chapter of *The Principles of Quantum Mechanics* (1930). In the first edition Dirac defines the superposition principle in quantum mechanics without reference to Sturm-Liouville theory, but with reference to measurement:

> We may say that a state A may be formed by a superposition of states B and C when, if any observation is made on the system in state A leading to any result, there is a finite probability for the same result being obtained when the same observation on the system on one (at least) of the two states B and C. The Principle of Superposition says that any two states B and C may be superimposed in accordance with this definition to form a state A and indeed an infinite number of different states A may be formed by superposing B and C in different ways. This principle forms the foundation of quantum mechanics. It is completely opposed to classical ideas, according to which the result of any observation is certain and for any two states there exists an observation that will certainly lead to two different results. (1930, 15)

Although Dirac significantly rewrote *Principles* in subsequent editions, his approach to 'superposition' changes little, suggesting later that

> *the superposition that occurs in quantum mechanics is of an essentially different nature from any occurring in the classical theory,* as is shown by the fact that the quantum

> superposition principle demands indeterminacy in the results of observations in order to be capable of a sensible physical interpretation (1958, 14).

However, whilst there are important differences between the semantic architecture of 'superposition' in classical and quantum contexts (or patches), Dirac overstates the discontinuity, as might be indicated by Schrödinger's original prolongation of the concept via 'quantized Sturm-Liouville theory'. That is, unlike Schrödinger's early comments on 'superposition', Dirac's characterization or definition of 'superposition' pays insufficient attention to the supporting architecture of the prolongation of the concept from its classical home to its quantum application. But confusion arises when 'superposition' is *defined* via measurement, rather than measurement outcomes being *explained* via components of a 'superposition' with quantum architecture.

Moreover, as in classical physics, application of 'superposition' in quantum physics may be disputed. Reflecting Dirac, Zeh describes the superposition principle as 'the main axiom of quantum theory' (1970, 69-76). However, quantum mechanics may be developed without reference to it. In von Neumann's *Mathematical Foundations of Quantum Mechanics* it is the uncertainty principle that is central rather than the superposition principle (1955 [1932] 92, 108), and there do not appear to be any references to 'superposition', even though von Neumann regularly deals with linear combinations of vectors (or rays) in Hilbert spaces.[7] Moreover, Feynman (2006a [1965]) rarely refers to 'superposition'. More recently, in a philosophical context, 'superposition' is absent in Wallace's statement and summary of 'orthodox' quantum mechanics, only mentioned briefly in the context of Everettian interpretation (2019), while Teller (1995) repeatedly applies the concept in quantum physics, in the context of QFT in particular.

6.3 'Superposition' in 'orthodox' or 'textbook' NRQM

6.3.1 Overview of 'quantum superposition' in NRQM

As in classical physics, there are different patches of 'superposition' in quantum physics. 'Superposition' is initially established on a quantum patch because a quantum system is characterized by a state that evolves according to a linear partial differential equation (Schrödinger's equation). The superposition

[7] Cf. Hughes (1989) for discussion of the use of 'superposition' by Dirac but not von Neumann.

principle is properly applied to (simple) solutions of Schrödinger's equation via 'Hilbert superposition', i.e. to possible states that the system can occupy that can be regarded as 'simple'.

Moreover, observables in NRQM are associated with self-adjoint operators acting on the Hilbert space of states and thus have eigenstate representations by the spectral theorem. Such eigenstates are 'simple' states associated with that observable, being the possible states that the system may be measured in, with the value of that observable being the eigenvalue corresponding to the eigenstate. That is, the eigenstates of the observable of interest form a complete set of simple states for that observable, which are thus simple solutions to Schrödinger's equation *for that observable*. These eigenstates are then partial states that evolve according to partial laws (from Schrödinger's equation), taking the same form individually and in linear combination, whilst not stating the facts in the sense discussed in chapter 3.

This is 'quantum superposition', where a new rule emerges to interpret the coefficients of a superposition of eigenstates of an observable as the amplitude for measuring the system in that state. However, as we saw in chapter 5 decomposition of the quantum state according to 'superposition' is 'natural' but promiscuous although not arbitrary since in general different observables have different eigenstate decompositions for the same state. Moreover, the architecture of 'superposition' as applied to the energy observable differs from that of others owing to the role of the Hamiltonian in Schrödinger's equation, and hence significance of its eigenstates as considered in the Sturm-Liouville context. I now clarify these comments, but we may already see that 'quantum superposition' is in continuity with, rather than essentially different from, classical superposition.

6.3.2 Schrödinger picture

First I consider 'superposition' in NRQM in the Schrödinger picture by developing Wallace's (2019) concise summary of 'orthodox' quantum mechanics. Although Wallace does not refer to 'superposition', he presents a succinct

summary suitable for my purposes.[8] The 'structural core' of quantum mechanics consists of:

> 1. States: The possible states of a quantum system are represented by normalised vectors in some complex Hilbert space.
> 2. Observables: To any physical quantity used to describe the system (often called an 'observable') is associated a self-adjoint operator on that same Hilbert space.
> 3. Dynamics: The state of a quantum system evolves over time according to the *Schrödinger equation*:
>
> $$\frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar}\widehat{H}|\psi(t)\rangle$$
>
> where $\widehat{H}$ is the self-adjoint operator corresponding to the system's energy.

Wallace adds three further components taken to comprise orthodox NRQM: The Born rule; the projection postulate (the collapse law), and the eigenvector-eigenvalue link:

> The Born (probability) rule: Suppose some quantity O has associated operator $\hat{O}$, which can be written
>
> $$\hat{O} = \sum_i o_i \widehat{\Pi}(i)$$
>
> where the o_i are the distinct eigenvalues of the operator and $\widehat{\Pi}(i)$ projects onto the subspace of states with eigenvalue o_i. … Then if O is measured on a quantum system with state $|\psi\rangle$, then:
>
> 1. The only possible outcomes of the measurement are the eigenvalues o_i of the operator;
> 2. The probability of the measurement giving result o_i is
>
> $\Pr(O = o_i) = \langle\psi|\hat{P}(i)|\psi\rangle$

Notably, there is no reference to superposition, but in similar treatments elsewhere there are. For instance, after setting out a similar set of postulates for NRQM, Shankar introduces the principle of superposition:

> When we say that $|\psi\rangle$ is an element of a vector space we mean that if $|\psi\rangle$ and $|\psi'\rangle$ represent possible states of a particle so does $\alpha|\psi\rangle + \beta|\psi'\rangle$. This is called the *principle of superposition.* (1994, 117)

Although it is not explicitly mentioned, such 'superposition' is grounded upon 'Hilbert superposition' applied to Schrödinger's equation: since $|\psi\rangle$ and $|\psi'\rangle$ are possible states they evolve linearly according to Schrödinger's equation. In one sense Shankar's postulate is redundant as it may be deduced from the linearity of Schrödinger's equation.

The question of philosophical or interpretative importance is then whether or not, and if so how, 'superposition' does useful work beyond the notion of Hilbert space vector (or ray) addition. That is, does paying attention to 'superposition' help clarify the physical significance or interpretation of the theory? When we study QFT we shall see that it does in the sense that

[8] For simplicity I only consider pure states here. For philosophical discussion of mixed states and the density matrix formalism see e.g. Friedrich (2015).

consideration of the applicability of the concept indicates how the mathematical structures and terms do or do not support a physical interpretation.

'Quantum superposition' stands in continuity with classical 'superposition' as a 'prolongation' of the concept onto a new patch via 'Hilbert superposition' applied to the (linear) Schrödinger equation. Its new or additional semantic architecture is indicated by the Born rule, projection postulate and eigenvalue-eigenvector link, which locate the discontinuities between the classical and quantum concept which might be understood as arising from the promotion of canonical observables to operators and the imposition of the CCRs, marking the patch as specifically 'quantum' (cf. Ruetsche above).

6.3.3 Energy eigenfunction decomposition

The simplest case to consider is when we are interested in the observable energy, and thus energy eigenstates. When we are interested in energy eigenfunction decompositions in quantum physics we can carry over our analysis of Sturm-Liouville systems from chapter 4 directly, and supplement 'superposition' with the Born rule to establish its quantum application.

The point is that if we express a quantum Sturm-Liouville type system's state in terms of eigenfunctions of the Hamiltonian, we establish a descriptive opportunity leading to a reasoning advantage via a natural description of the system's state that supports physically salient explanations and counterfactual reasoning according to the Fourier technique. The state evolves simply by the independent scaling of the energy eigenfunctions individually and summing. The eigenfunctions may be interpreted as a complete set of independent partial states that evolve according to independent 'partial laws' that take the same form individually and in linear combination whilst not stating the facts. The eigenfunctions persist in and identify the state, with a 'trace principle' established in the evolution of the state via Schrödinger's equation. The eigenstates are states of constant energy, so may be considered to 'pick out' important properties of the system.

The Fourier technique of representing the quantum system in this way using energy eigenfunctions is analogous to the 'two aspects' of Fourier's analysis (cf. chapter 4). That is, we deduce the eigenfunctions as 'simple states'

that persist in form as they are eigenfunctions of the relevant differential operator(s) of the eigenvalue equations obtained after separating variables of Schrödinger's equation (first aspect). We then express the initial state as a 'superposition' of these eigenstates (second aspect), which may be interpreted as 'partial states' that evolve as a 'superposition' according to 'partial laws' that take the same form individually and in linear combination. Just as in the classical case there are two different usages of 'superposition' here.

This 'two aspect' technique provides a natural description in the sense that it supplies the representation that optimally balances simplicity and strength of the system's behaviour and supports physically salient predictions, explanations and counterfactual reasoning, if one is interested in features of the system related to energy, such as the emission spectra of atoms. This interest will be especially important in QFT, where energy eigenstates will be associated with particles and their states. However, on the quantum patch the coefficients of the Fourier decomposition have a different semantic architecture from the classical patch, being interpreted as the amplitude for observing the system in that eigenstate upon taking a measurement of energy.[9]

6.3.4 Example: The hydrogen atom

I illustrate these ideas via a common model of the hydrogen atom in which the electron is bound to the nucleus (proton) by a Coulomb potential, neglecting spin.[10] This was an important example historically, being used to explain the emission spectrum of the hydrogen atom, since the energy states obtained are used to explain the possible electromagnetic absorptions / emissions.

As in classical applications of Sturm-Liouville theory, we begin with a second-order linear partial differential equation (Schrödinger's equation) subject to boundary conditions. We separate variables as before to obtain coupled differential equations as eigenvalue equations. As in the vibrating membrane in §4.4.2 this is done repeatedly and requires appropriate

[9] In our extended Wilsonian perspective our analysis is complete. A further question arises in realism debates however as regards what should be said about the quantum system prior to measurement. Within our Wilsonian framework we may remain silent on this point, although it is interesting to note that one aspect of 'superposition' is that the components of a superposition do not state the facts individually, an observation that might feed in to the realism debate.

[10] See e.g. Bolton and Freake (2009) for the model and analysis. Their treatment is followed here.

coordinates. Having found the modes (energy eigenfunctions), the initial state is decomposed according to this basis, thus supporting a natural description of the atom's state according to 'superposition'.

Begin with Schrödinger's equation:

$$i\hbar\frac{\partial}{\partial t}\,\Psi(\mathbf{x},t)=\widehat{H}\Psi(\mathbf{x},t)$$

and separate variables to isolate the time-dependency to obtain, after recombining variables,

$$\Psi(\mathbf{x},t)=\psi(\mathbf{x})e^{-iEt/\hbar}$$

The time-independent Schrödinger equation obtained,

$$\widehat{H}\psi(\mathbf{x})=E\psi(\mathbf{x})$$

is an eigenvalue equation solved using Sturm-Liouville theory as before. Substituting explicitly for $\widehat{H}$:

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(\mathbf{x})-\frac{e^2}{4\pi\varepsilon_0 r}\psi(\mathbf{x})=E\psi(\mathbf{x})$$

where

$$\mu=\frac{m_e m_p}{m_e+m_p}$$

and m_e=electron mass and m_p=proton mass. In spherical polar coordinates the time-independent equation is

$$\left(-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{2\mu r^2}\hat{L}^2-\frac{e^2}{4\pi\varepsilon_0 r}\right)\psi(\boldsymbol{r})=E\psi(\boldsymbol{r})$$

where $\hat{L}$ is the orbital angular momentum operator. Separating variables again via $\psi(\boldsymbol{r})=R(r)Y(\theta,\varphi)$ we obtain separated coupled eigenvalue equations as ODEs as per chapter 4 with

$$\left(-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+\frac{1}{2\mu r^2}K-\frac{e^2}{4\pi\varepsilon_0 r}\right)R(r)=ER(r)$$

$$\hat{L}^2Y(\theta,\varphi)=KY(\theta,\varphi)$$

where K is the separation constant. We find 'simple solutions' as eigenfunction solutions $R_{n,l}(r)$ (radial eigenfunctions) and $Y_{l,m}(\theta,\varphi)$ ('spherical harmonics'), and build the general solution via 'Hilbert superposition' after recombining variables, just as before. We have:

$$\hat{L}^2Y_{l,m}(\theta,\varphi)\;=l(l+1)\hbar^2Y_{l,m}(\theta,\varphi)$$

$$\left(-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+\frac{l(l+1)\hbar^2}{2\mu r^2}-\frac{e^2}{4\pi\varepsilon_0 r}\right)R_{n,l}(r)=E_nR_{n,l}(r)$$

and from the spherical harmonics:

$$\widehat{L_z}=m\hbar Y_{l,m}(\theta,\varphi)$$

where $\widehat{L_z}$ is the z-component of the orbital angular momentum, n= principal quantum number, l= azimuthal quantum number and m= magnetic quantum number.

The $R_{n,l}$ are the radial components of the wave-function, being functions of the associated Laguerre polynomials, and $Y_{l,m}$ are the spherical harmonics, being functions of the associated Legendre polynomials. These form the generalized Fourier modes associated with the relevant differential operators.

The point is that the solutions $\psi_{n,l,m}(r,\theta,\varphi)=R_{n,l}(r)Y_{l,m}(\theta,\varphi)$ form a complete orthonormal set of eigenfunction solutions to the time-independent Schrödinger equation, simultaneously being eigenfunctions of energy, orbital angular momentum and the z-component of angular momentum. Mathematically this is because the operators corresponding to these observables commute, which ensures that we can choose simultaneous eigenfunctions of these observables. This means that we can completely specify or identify the state $\psi_{n,l,m}$ by the eigenvalues of these eigenstates, labelling them with *n,l,m* which are properties of the eigenstate since upon measurement of the atom in this state these values are obtained with certainty. That is, the state $\psi_{n,l,m}(r,\theta,\varphi)$ has properties of energy *n,* orbital angular momentum *l* and z-component of angular momentum *m*. These eigenstates 'persist' or may be 'traced' in the evolution of the system according to independent evolution via the partial law of multiplication by $e^{-iE_nt/\hbar}$, so that the eigenvalues are invariants of the evolution identifiable, upon measurement, as properties of the hydrogen atom.

As in the classical theory, the general (initial) state of the hydrogen atom is then expressed as a superposition (second aspect of the Fourier technique) of these eigenstates or simple states:

$$\psi(r,\theta,\varphi)=\sum_{n,l,m}c_{n,l,m}\psi_{n,l,m}(r,\theta,\varphi)$$

The eigenstates $\psi_{n,l,m}$ are then natural states with which to express the identity of a general state of the hydrogen atom as they are uncoupled and evolve

independently with time, after recombining variables, with the evolution taking the simplest form possible, namely scaling according to

$$\Psi_{n,l,m}(r,\theta,\varphi,t) = \psi_{n,l,m}(r,\theta,\varphi)e^{-iE_n t/\hbar}$$

being analogous to the 'normal modes' identified in classical systems in chapter 4. The general state is then the superposition of these 'normal modes' according to 'Hilbert superposition':

$$\Psi(r,\theta,\varphi,t) = \sum_{n,l,m} c_{n,l,m} e^{-iE_n t/\hbar}\psi_{n,l,m}(r,\theta,\varphi)$$

That is, the energy eigenstates $\psi_{n,l,m}(r,\theta,\varphi) = R_{n,l}(r)Y_{l,m}(\theta,\varphi)$ are persisting 'partial states' that evolve according to 'partial laws' of multiplication by $e^{-iE_n t/\hbar}$, where the partial states and laws take the same form in and out of linear combination whilst not stating the facts,[11] so this is 'superposition'. Interpretation is conducted on the quantum patch however, so Born's rule interprets the coefficients of the superposition

$$\Psi(r,\theta,\varphi,t) = \sum_{n,l,m} c_{n,l,m} e^{-iE_n t/\hbar}\psi_{n,l,m}(r,\theta,\varphi)$$

as the amplitude of obtaining that state and its corresponding eigenvalues upon measurement. This is the difference from the classical application of 'superposition', which ultimately owes to the imposition of the CCRs.

In Wilsonian terms, expressing the state of the hydrogen atom by the generalized Fourier modes $\psi_{n,l,m}(r,\theta,\varphi) = R_{n,l}(r)Y_{l,m}(\theta,\varphi)$ is to appropriate a descriptive opportunity supporting a reasoning advantage, enabling certain properties of the hydrogen atom to be identified, and physically salient explanations of the behaviour of the hydrogen atom to be offered, such as its emission spectrum, as well as supporting counterfactual reasoning, such as 'what if' the charge or mass of the electron were different. In addition we should regard the description via the $\psi_{n,l,m}(r,\theta,\varphi) = R_{n,l}(r)Y_{l,m}(\theta,\varphi)$ as a natural description, optimally balancing simplicity and strength of our representation of the behaviour of the hydrogen atom with respect to energy considerations via independent partial states that may be associated with natural properties of the hydrogen atom. Application of 'superposition' is proper as each eigenstate has physical significance as an abstractable 'partial state' associated with a 'partial

[11] None of the partial states describe the actual state of the atom when in combination.

law' that takes the same form individually and in of linear combination, without stating the facts.[12]

6.3.5 Other observables

A complication arises in 'quantum superposition' not found in the classical analysis of Sturm-Liouville systems in that the eigenstate decomposition that supports physically salient explanations associated with a given observable may not arise from an eigenvalue equation derived from the separation of variables of Schrödinger's equation. As we saw, tangentially, in the analysis of the hydrogen atom, it is a result of linear algebra that self-adjoint operators associated with commuting observables possess a basis of simultaneous eigenfunctions (with real eigenvalues). This means that as in the analysis of the hydrogen atom, for any operator commuting with the Hamiltonian simultaneous eigenstates may be chosen, being eigenstates of energy and the observable in question. But for operators that do not commute with the Hamiltonian, no such simultaneous basis of eigenfunctions exists.

However, since observables are represented by self-adjoint operators and so possess a 'diagonal' eigenfunction representation, if the state is initially represented via this eigenfunction basis of states, by the linearity of Schrödinger's equation and 'Hilbert superposition' these eigenfunctions are partial states that may be associated with corresponding 'partial laws' via Schrödinger's equation, and have the same form individually and in combination, being a genuine application of 'superposition'.

This may be compared with the dual aspects of the Fourier technique in which the initial state is expressed in terms of the eigenfunctions of the system of interest, except now additional sets of eigenfunctions are relevant. This means that 'quantum superposition' is promiscuous, but it is not arbitrary and supports natural descriptions *relative to* observables of interest. Although such promiscuity may arise in a novel way on the quantum patch, promiscuity is not novel in general in that as we saw in chapter 5 various appeals to different

[12] Given this, and the explanatory role of the eigenstates, the scientific realist is likely to understand the generalized Fourier modes here in 'realist' terms. However, anyone who is, in principle, antirealist concerning Fourier modes ought then to be antirealist about these states of the hydrogen atom.

'superpositions' are supported in classical systems.[13] This complication will not arise in QFT since the observables of interest commute and so we may always choose a basis of simultaneous eigenstates to represent the state.

However, it may sometimes be more natural to consider eigenstates of observables in relation to the Heisenberg picture, as we shall see in a moment. But to glimpse ahead to QFT, the Fourier decomposition of both the field (wave) and state equations are foundational. The field (wave) equation is interpreted in terms of the Heisenberg picture, with the wave equation being the Heisenberg equation of motion for the field. The state equation is interpreted in the Schrödinger picture, with the state evolving according to Schrödinger's equation. So, I now consider the Heisenberg picture.

6.4 Time evolution and the Heisenberg picture

One issue that is central to QFT that I now consider first in NRQM is that one can associate time dependency either with the operator associated with the observable of interest (Heisenberg picture), or with the state (Schrödinger picture, as just discussed). In QFT the field (wave) equations are interpreted in the Heisenberg picture whilst the state equation (Schrödinger's equation) is interpreted in the Schrödinger picture.

A unitary operator $\widehat{U}$, the 'time-evolution' operator, can be introduced so that (using Dirac notation)

$$|\psi(t)\rangle = \widehat{U}(t)|\psi(0)\rangle$$

where $\widehat{U}$ satisfies Schrödinger's equation:[14]

$$\frac{d\widehat{U}(t)}{dt} = -\frac{i}{\hbar}\widehat{H}\widehat{U}(t)$$

with formal solution

$$\widehat{U}(t) = e^{-\frac{i}{\hbar}\widehat{H}t}$$

[13] However, there is perhaps less novelty than is apparent. The quantum state encapsulates all the information about the system under consideration, whereas for instance in our model of the heated slab we consider only thermal information and not vibratory information for instance. So if we wanted to model all the characteristics of a vibrating heated slab, we would need to express its 'state' in terms of superpositions of eigenfunctions of different physical characteristics. That is, in both classical and quantum applications decompositions according to different sets of eigenfunctions associated with different physical properties are required.

[14] Some authors start with unitary time evolution as a postulate or axiom, and deduce Schrödinger's equation, e.g. Greiner (1994, 402).

If one takes an initial state decomposed into a 'superposition' of eigenstates of a self-adjoint operator $\hat{A}$ corresponding to some observable (as above), by the linearity of $\widehat{U}$ the state evolves as a superposition of eigenstates, i.e.

$$|\Psi(\mathrm{t})\rangle = \widehat{U}(t)\sum_n c_n|\psi_n(0)\rangle$$

$$= \sum_n c_n\widehat{U}(t)|\psi_n(0)\rangle$$

Then

$$\hat{A}|\Psi(\mathrm{t})\rangle = \textstyle\sum_n c_n\hat{A}\widehat{U}(t)|\psi_n(0)\rangle$$

However, we may associate the time-evolution operator $\widehat{U}$ with the observable rather than with the state, assuming the state is 'fixed' (at its t=0 value). This is the Heisenberg picture, and the Heisenberg picture operator is

$$\hat{A}_H(t) = \widehat{U}^{\dagger}(t,0)\hat{A}_s\widehat{U}(t,0)$$

where in the Schrödinger picture $\hat{A}_s = \hat{A}$ is time-independent and

$$|\psi_s(t)\rangle = \widehat{U}(t)|\psi(0)\rangle$$

In the Heisenberg picture $|\psi\rangle_H = |\psi(0)\rangle$, and if the Hamiltonian has no explicit time dependency we have the *Heisenberg equation of motion*

$$\frac{d\hat{A}_H(t)}{dt} = -\frac{i}{\hbar}\left[\hat{A}_H(t),\widehat{H}\right]$$

So for an arbitrary observable A represented by a self-adjoint operator $\hat{A}$ in the Schrödinger picture, we may form an orthonormal basis of the Hilbert space of states associated with the system consisting of the eigenstates of $\hat{A}_H(0) = \hat{A}$, and express the (fixed) initial state at t=0 (for which the Heisenberg and Schrödinger pictures coincide) as a superposition of these eigenstates. We then allow $\hat{A}(t)$ to act on this eigenstate decomposition of the (fixed) state $|\psi\rangle_H = |\psi(0)\rangle$. The eigenstates of $\hat{A}(t)$ coincide with those of $\hat{A}(0)$ as may be seen from the above, so measurement of A at time t yields an eigenstate of $\hat{A}(0)$. In this picture, as regards the application of 'superposition' the partial states are the states obtained from the decomposition of $|\psi\rangle_H = |\psi(0)\rangle$ as the eigenfunctions of $\hat{A}_H(0) = \hat{A}$, and the partial laws are given via the independent action of $\hat{A}(t)$ on each partial state.

In both Schrödinger and Heisenberg pictures analysis is dependent on the quantum state's decomposition at some reference time into the eigenstates of the observable of interest, and the evolution of state or operator being linear. This is foundational for establishing a natural description of the system in both pictures so as to support physically salient explanations, since the independent eigenstates persist as such owing to linearity.

6.5 Dirac / interaction picture

The Dirac or 'interaction' picture combines the Heisenberg and Schrödinger pictures, with operators and state both carrying time dependency. It is used for systems that (one assumes) can be modelled by well-understood and computable 'free' evolutions coupled via a complicated interaction typically introduced as a 'small' perturbation to the free system. The Hamiltonian is split into a solvable 'free' part $\widehat{H}_0$, and an interaction term $\widehat{H}'$, i.e.

$$\widehat{H} = \widehat{H}_0 + \widehat{H}'$$

In the interaction picture the states carry the time dependency from $\widehat{H}'$, and the operators the time dependency from $\widehat{H}_0$. The operators satisfy the Heisenberg equation of motion for $\widehat{H}_0$, and the state satisfies Schrödinger's equation for $\widehat{H}'$. This procedure or picture is especially important for calculations in QFT, as we shall see in detail in chapter 9. The physical interpretation of superposition is more obscure in this picture, and I do not seek to develop it here, for as we shall see semantic mimicry occurs in the perturbative application of the picture in QFT.

6.6 The quantized simple harmonic oscillator

I complete the discussion of NRQM where I began, with Schrödinger's first application of 'superposition' to the quantized simple harmonic oscillator (QSHO), but now using a different method to deduce the energy eigenstates that is developed in QFT. Indeed, the QSHO historically was, and remains, foundational to QFT, forming a bridge from NRQM to QFT. It shows in a simpler context how to establish an eigenfunction representation or 'diagonalization' of the Hamiltonian using 'raising' and 'lowering' operators (a-operators) to change coordinates to achieve a 'natural description' of the QSHO in terms of the

superposition of independent partial states and corresponding partial laws, since they take the same form individually and in combination.

Commence with the SHO Hamiltonian[15]

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2$$

and quantize by promoting p and x to operators

$$\widehat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2 \hat{x}^2$$

and imposing the CCR

$$[\hat{x}, \hat{p}] = i.$$

Change variables in order to 'factorize' the Hamiltonian:

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\omega}\hat{x} + \frac{i}{\sqrt{\omega}}\hat{p}\right)$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\omega}\hat{x} - \frac{i}{\sqrt{\omega}}\hat{p}\right)$$

Then

$$[\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0$$

$$[\hat{a}, \hat{a}^\dagger] = 1$$

by substitution. Substituting these in the Hamiltonian, and using the CCRs, gives[16]

$$\widehat{H} = \omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

The 'number operator' is defined:

$$\widehat{N} = \hat{a}^\dagger\hat{a}$$

and since any state $|\psi\rangle$ that is an eigenstate of $\widehat{H}$ is also an eigenstate of $\widehat{N}$ as the operators commute, labelling these eigenstates of $\widehat{N}$ by their eigenvalues, i.e.

$$\widehat{N}|n\rangle = n|n\rangle$$

then

$$\widehat{H}|n\rangle = \omega\left(n + \frac{1}{2}\right)|n\rangle$$

Here n is a non-negative integer, so there is a 'ground state' of lowest energy $|0\rangle$ corresponding to n=0, and the spectrum of $\widehat{H}$ is

[15] The analysis here loosely follows Hatfield (1992, 16-19), working now in natural units.
[16] Note for use in QFT that by the CCRs

$$\widehat{H} = \tfrac{1}{2}\omega(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) \qquad (*)$$

$$E_n = \omega\left(n + \frac{1}{2}\right)$$

This is the energy of the QSHO if measured in state $|n\rangle$. The possible energy states or energy levels are equally spaced, and we normalize the states so that $\langle n|n\rangle = 1$ and then $\hat{a}^\dagger$ is interpreted as a *raising operator* owing to its action on the state $|n\rangle$,

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

and $\hat{a}$ is interpreted as a *lowering operator* since

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

That is, the a-operators raise or lower the state of the system by one energy level. The interpretation of this observation will be important in QFT where such raising / lowering of energy state is interpreted as raising / lowering the particle content of the state by one quantum (or particle), via relativistic considerations.

Any excited eigenstate of the system may be built from the ground state, and these are the eigenstates of the Hamiltonian, the energy eigenstates of the quantized SHO:

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle$$

As these eigenstates form a basis for the Hilbert space of states any general state of the QSHO may be expressed as a superposition of these eigenstates:

$$|\psi\rangle = \sum_n c_n|n\rangle$$

The eigenstates, and hence any such linear combination of eigenstates, are solutions to Schrödinger's equation by Hilbert superposition. This is properly understood as a superposition since the eigenstates as 'partial states' associated with 'partial laws' have physical significance individually (being states having the property of a determinate energy) and take the same form in and out of (linear) combination, persisting in the evolution of the system according to Schrödinger's equation.

Calculations may be performed in either Schrödinger or Heisenberg picture using this eigenstate basis representation of the Hilbert space of states. These energy eigenstates can be expressed in the coordinate representation as

wave-functions $\psi_n(x,t)$ evolving according to Schrödinger's equation. In these coordinates the ground state eigenstate is

$$\psi_0(x,0) = \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} e^{-\omega x^2/2} \equiv |0\rangle$$

and the excited states

$$\psi_n(x,0) = \frac{1}{\sqrt{2^n n!}}\left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} H_n(\sqrt{\omega}x) e^{-\omega x^2/2} \equiv |n\rangle$$

where the H_n are the Hermite polynomials. Since the partial states $\psi_n(x,0)$ are eigenstates of the Hamiltonian, their time evolution is calculated simply by scaling each state according to its associated partial law by recombining variables:

$$\psi_n(x,t) = e^{-i\hat{H}t}\psi_n(x,0) = e^{-iE_n t}\psi_n(x,0)$$

where $E_n = \omega\left(n + \frac{1}{2}\right)$ as before.

The general initial state is the superposition (according to second aspect of the Fourier technique)

$$\psi(x,0) = \sum_n c_n \psi_n(x,0)$$

where c_n is the amplitude of measuring the system in the nth energy eigenstate. This initial state evolves by scaling the coefficients individually as a superposition, after recombining variables as

$$\psi(x,t) = \sum_n c_n e^{-iE_n t}\psi_n(x,0)$$

owing to the linearity of Schrödinger's equation, and the observation that the ψ_n are a complete set of eigenstates of the Hamiltonian (Hilbert superposition).

The energy eigenfunction representation offers a natural representation of the state of the QSHO for the reasons set out above. Such representation exploits a descriptive opportunity as the eigenfunctions are simple partial states associated with simple 'partial laws' optimally balancing simplicity and strength that take the same form in and out of linear combination, support physically salient explanations, inductive inferences and counterfactual reasoning, as with the hydrogen atom model. The energy eigenstates correspond to states that the system can occupy upon measurement of well-defined and constant energy.

6.7 Summary

We have seen how ‘superposition’ and the Fourier techniques of the classical domain are naturally prolonged via ‘Hilbert superposition’ in particular, together with an extension of eigenfunction techniques, to a new quantum patch owing to the promotion of observables to operators and the imposition of the CCRs. This leads to additional architecture to ‘quantum superposition’ as might be characterized by Born’s rule, and application of ‘superposition’ relative to eigenstates of an observable. We have considered the QSHO in some detail as preparatory for QFT, upon which we finally embark in the next chapter.

Chapter 7

Superposition and the early foundations of QFT

7.1 Overview

I briefly consider some aspects of the original development of QFT. This will help to bring in to focus some of the conceptual difficulties that have subsequently become obscured. My aim is not to provide a detailed historical analysis of QFT,[1] but rather to highlight the origins and emergence of certain foundational concepts, techniques and ideas that have shaped QFT. I shall develop a detailed analysis of 'superposition' as applied to QFT in the following chapters, but here we see how Fourier techniques are foundational to QFT, as well as locating the origins of cases of semantic mimicry that have brought conceptual confusion.

7.2 The origins of QFT and particle descriptions via Fourier techniques

The origins of QFT might be traced to three papers treating electromagnetic energy fluctuations in a cavity: Ehrenfest, who explicitly adopts the concept of superposition in his analysis (1925); Born and Jordan (1925); and Born, Heisenberg and Jordan (1926). According to Duncan, 'a truly quantum-field-theoretic calculation is employed [in Born, Heisenberg and Jordan (1926)] to resolve the conundrum of the wave-particle duality of light first raised by Einstein's results of 1909' (2012, 19).

The model of the system, and associated theory, is set out by Born and Jordan (1925). The analysis of the system is performed in continuity with the Sturm-Liouville techniques discussed in chapter 4:

> A cavity with electromagnetic oscillations constitutes a system of infinitely many degrees of freedom. Nevertheless, the basic principles developed in the preceding sections … are sufficient to handle this case as well, given that it goes over to a system of *uncoupled* oscillators once analyzed in terms of eigenmodes. *There is hardly any possible doubt, how such a system is to be treated*. In particular, the circumstance that the basic equations of electromagnetism are linear is of importance, for it then follows that the virtual oscillators (eigenmodes) are *harmonic*, and it is precisely for harmonic oscillators, in contradistinction to other systems, that the validity of energy conservation is independent of the quantum condition. (In Duncan (2012, 18), Duncan's emphasis)

[1] For historical studies from various perspectives that overlap in places with my concerns see e.g. Carson (1996a&b); Brown and Rechenberg (1996); Duncan (2012); Schweber (1994).

Whilst 'superposition' is not mentioned explicitly here, the importance of the linearity of the relevant differential equations is. This gives rise to the eigenfunction decomposition of the electromagnetic field into Fourier modes in a similar way to the examples considered in chapter 4.

Such association with the Fourier techniques of classical physics is more explicit in Ehrenfest, who introduces a one-dimensional model of cavity radiation to develop an analogy of the electromagnetic field with the simple model of a vibrating string in which he explicitly applies 'superposition' to quantized normal modes of the electromagnetic field (1925). This analogy allowed Jordan, in Born, Heisenberg and Jordan (1926) (the '3M-paper') to pursue Ehrenfest's analysis through use of the Fourier mode (eigenmode) solutions as in the vibrating string when the quantum condition of the commutator between the canonically conjugate coordinates is introduced. Although Jordan did not introduce the 'raising' and 'lowering' operators (the *a*-operators) that would become central to QFT from the quantized simple harmonic oscillator (QSHO), Duncan comments that

> the calculations of the 3M paper involve only the p_j and q_j matrices, and their commutation relation, [but] are mathematically perfectly equivalent to results obtained [with the raising and lowering operators a_j and $a_j^\dagger$]. The introduction of operators which raise or lower the excitation level of the individual eigenmodes will become central in our later development of the modern formalism of quantum field theory. To the extent that the excitation levels $\{n_j\}$ are identified (as they clearly are in the 3M paper) with the number of light quanta (i.e., photons in modern terminology) with frequency ω_j, operators raising and lowering these levels are clearly identifiable as the particle creation and destruction operators of modern field theory. Later … they will turn out to be the technical tool ideally suited to the introduction of physically sensible local interactions as well as dealing effortlessly with the statistics of properly symmetrized multi-particle states. (2012, 23)

We see the beginnings of the development of the semantic architecture of the eigenfunction decompositions obtained in QFT and, implicitly, 'superposition'. There is an association between the eigenfunction description of the state and its particle description in QFT.

Provisionally, and anticipating what is to come, we may say that when the evolution of the state is modelled via a suitable linear wave equation, Fourier techniques support a natural descriptive opportunity for which the 'partial states' may be interpreted as 'particles', with the partial states evolving

according to corresponding partial laws via the wave equations. This becomes a way of 'cashing out' the particle-wave duality of quantum physics.[2]

Duncan suggests that two vital foundational tasks remained following these papers: first, modelling the interaction of light with matter, begun by Dirac ca. 1927; secondly, 'the extension of the notion of field quantization to the treatment of matter fields, in particular fields with elementary excitations of fermionic character, which in consequence could never possess a classical counterpart analogous to the electromagnetic field', a task taken up by Jordan beginning ca. 1927 (2012, 28-29). However, confusion arose from attempts to use 'first-quantization' ideas for the electron (i.e., treating it with a relativistic wave equation to describe a single, or a fixed number of particles) whilst applying the 'second-quantized' formalism (i.e., quantum field theoretic) to the electromagnetic field. I now consider 'second quantization'.

7.3 'Second quantization'

Jordan was an early champion 'of the notion that wave-particle duality extended to a coherence in the mathematical formalisms used to describe radiation (specifically, the electromagnetic field) and matter (which seems to denote for Jordan and co-workers the aggregate behaviour of massive particles of either bosonic or fermionic type).' (Duncan 2012, 40) Jordan claimed that treatment of an N-particle system in terms of Schrödinger wave-functions in abstract $3N$-dimensional configuration space should be replaced by a treatment involving a single quantum field $\varphi(\mathbf{x},t)$ defined on space-time. Jordan and Klein (1927) adopted this procedure, subsequently known as 'second quantization', of replacing wave-functions with quantized fields promoted to operators. They did not carry it through relativistically however, and they did not apply it to fermions. Jordan and Wigner (1928) applied second quantization to fermions, but again not relativistically, an omission remedied by Heisenberg and Pauli (1929) in a paper 'which put in place the formalism of Lagrangian field theory, still … at the core of modern field theory.' (Duncan, 43)

[2] In the scientific realist's terms, one might expect that one's stance toward a particle description in QFT ought then to be similar to one's stance towards the harmonics of a vibrating string (cf. Redhead 1988, 19-21). Redhead compares interacting states with the vibrating string, but this introduces semantic mimicry, as we shall see in chapters 9-10.

Briefly, and I shall fill in the details in chapter 8, the (free) field $\varphi(\mathbf{x},t)$ satisfying a suitable (linear) wave equation, such as Dirac's equation for fermions, is quantized by 'promoting' it to an operator acting on the quantum state of the system and introducing the CCRs between $\varphi(\mathbf{x},t)$ and its conjugate momentum field $\pi(\mathbf{x},t)$, from which a Hamiltonian is obtained. Using the Fourier mode (eigenfunction) solution of the wave equation, 'raising' and 'lowering' operators (the *a*-operators) are obtained as the (operator-valued) coefficients of the modes of the wave equation indexed by their momenta k. These *a*-operators are then used to transform co-ordinates for the Hamiltonian from representation in $\varphi(\mathbf{x},t)$ and $\pi(\mathbf{x},t)$ coordinates to *a*-coordinates, in order to 'diagonalize' the Hamiltonian via its eigenfunction representation. The *a*-operators perform a similar role here to that of their role in the QSHO, except that that the second quantized system may be thought of as a collection of QSHOs, one for each k. The action of the Hamiltonian on the state is then simply characterized since the Hamiltonian is diagonalized. This enables one to introduce a particle interpretation, with the *a*-operators raising or lowering the state by one quantum, from which a particle description is obtained once wave-functions are incorporated. Such particles are interpreted as particles of a type corresponding to the field. The ability to form this 'natural description', or exploit this 'descriptive opportunity' in terms of a physical particle description is due to the linearity of *both* field and state equations such that Fourier techniques are supported.

It is important to note the double application of 'Hilbert superposition' (in the quantum context) here, once to the field equation for $\varphi(\mathbf{x},t)$ to obtain the *a*-operators through Fourier decomposition, and then to Schrödinger's equation for the state $|\Phi\rangle$ using the eigenfunction decomposition for the Hamiltonian obtained via the *a*-operators. This double application of the superposition principle using eigenfunction coordinates establishes a descriptive opportunity in which a natural description of the quantum system is given as a particle description. That is, the identity and persistence conditions of the quantum system from which physically insightful explanations are made are given via a particle description. The ability to do this depends on the linearity of *both* PDEs (i.e. for field and state). We shall see that this fails for the field equations when

interactions are introduced. This will imply that no metaphysical or even natural particle description is available in general (chapter 10), although for weakly coupled theories a particle description may offer a good 'engineering approximation' to the state however (chapter 11).

The formalism and mathematical structure required to carry this procedure through, with a sufficiently fine-grained structure for the eigenstates of the Hamiltonian, was developed by Fock (1932): the ubiquitous Fock space structure of QFT. However, it would not be until 1940, when the spin-statistics theorem was proved by Pauli, that the relationships between the requirements of microcausality, the form of the wave equation, the form of the commutators, spin and the form of the Fock space structure as symmetric or antisymmetric would emerge.

'Second quantization' established a quantized description of both 'matter' and 'radiation' from both particle and wave perspectives, when matter and radiation are considered independently without interacting with each other, i.e. as 'free', so that the field equations for both are linear. The belief that one can meaningfully abstract independent, free matter and radiation fields and states is an innocent looking assumption, no doubt made according to metaphysical presumptions that still linger, but in fact it reflects an implicit application of the 'Volkmann device' already that leads to various confusions, as manifested in the need for renormalization as we shall see.

A general picture of particle interactions according to QFT emerged in the 1930s through a series of (variously co-authored) papers by Yukawa in which the interaction (but not matter) field is subjected to second quantization. Yukawa's work extended QFT from fledgling QED to nuclear forces, leading to an understanding of particle interactions in general as exchange processes of 'virtual quanta', using perturbation theory. The picture depends on the innocent looking but, as we shall see, conceptually problematic implicit application of the 'Volkmann device' to suppose that independent free matter and radiation states can be abstracted and meaningfully used to represent interacting states.

Problems with divergences occurred in the 1930s that would not be solved until the late 1940s by renormalization techniques and a fully second-quantized treatment of interaction. Indeed, one aspect that Yukawa's treatment

lacks is the notion of associated fields and states for both matter and 'radiation', that is, a fully 'second quantized' treatment of both matter and radiation. This point is clarified in Feynman's discussion of 'second quantization' where he distinguishes between the *electron field* Ψ that satisfies Dirac's equation, and the *electron wave-function* χ that satisfies Schrödinger's equation, where Ψ acts on χ with the Hamiltonian expressed in terms of Ψ (1949a). The developments of the 1930s led to the establishment of the first 'working' QFT by Dyson, Feynman and Tomonaga, and subsequently to further developments culminating in the Standard Model by the 1970s.

It will be beneficial to study Yukawa's papers, which, whilst a notable triumph on the one hand, also exacerbate the emerging conceptual confusions of QFT initiated by the implicit use of the 'Volkmann device' to identify putative matter and radiation fields that take the same form individually (as free) and in combination (in interaction).

7.4 Yukawa: Interaction as virtual quanta exchange

7.4.1 Interaction as exchange

Yukawa, in a series of co-authored papers relating to nuclear forces, 'On the Interaction of Elementary Particles', offers an account of elementary interactions using quantum mechanical perturbation methods to postulate a new type of particle. He does not develop a fully second-quantized account of matter and radiation, but rather a second-quantized account of the interaction fields. This led him to picture interactions in terms of the exchange of 'virtual quanta', a picture that remains pervasive in accounting for how fundamental particles interact.[3] Historically, the concept of particle interaction via exchange or migration of a shared particle traces to (at least) Heisenberg (1932-1933), although the formulation of interaction as the exchange of quanta in QFT is generally credited to Yukawa (1935).

[3] E.g. Lancaster and Blundell (2014, 159-161), and also the essays in Brown and Harré (1988). Also see Brink for a commentary on Yukawa's work (1965, 83-114). Brink describes the early calculations of the meson exchange interaction as utilizing 'quantum mechanical perturbation theory' (87).

In the first paper (1935), Yukawa proposes that interaction between nucleons is mediated by a potential, considered as a field, the *U*-field, and just as the electromagnetic field has quanta associated with it, so should the *U*-field:

> interaction between the elementary particles [nucleons] can be described by means of a field of force, just as the interaction between the charged particles is described by the electromagnetic field. …
> In the quantum theory this field should be accompanied by a new sort of quantum, just as the electromagnetic field is accompanied by the photon. (1935, 49)

He concludes, in a manner that might sit comfortably within a causal-mechanical framework that:

> The interaction of elementary particles are described by considering a hypothetical quantum which has the elementary charge and the proper mass and which obeys Bose's statistics. …
> Such quanta, if they ever exist and approach the matter close enough to be absorbed, will deliver their charge and energy to the latter. (1935, 57)

Elementary 'matter particles' are modelled by Yukawa by wave-functions Ψ which satisfy Dirac's equation. The state of the matter particles is considered to be acted upon by a Hamiltonian, and although the matter particles are in some sense 'fixed', transitions between protons and neutrons are allowed, being regarded as transitions of nucleon state owing to isospin.[4] The mediation of interactions between such 'matter particles' is developed via a potential field *U.* This field has quanta associated with it, where *U* satisfies a linear wave equation, and is used to construct the Hamiltonian, by analogy with the electromagnetic field. The total Hamiltonian is taken to comprise of a Hamiltonian of the free matter particles H_M, the free interaction field H_U, and an interaction term H' introduced as a perturbation to the independent free Hamiltonians, so that $H = H_M + H_U + H'$ (Yukawa and Sakata 1937, 1084-1091).

In this sense Yukawa founds his treatment on quantum mechanical perturbation theory, but utilizes 'second quantization' to model the interaction potential in field theoretic terms. If one is to read a (meta)physical interpretation into the interacting system so modelled (rather than regarding this as 'merely' a perturbative approximation to crunch the numbers apart from offering a natural description), as we find in these papers, it requires implicit application of the 'Volkmann device' to isolate (free) matter and interaction components, supposing these to retain their identities when interacting.

[4] That is, protons and neutrons are considered to be different isospin states of the same particle.

That is, the matter and interaction states and associated fields are construed in terms of partial states and partial laws that take the same form individually and in combination according to some complicated composition relations owing to the interaction. But this application of the 'Volkmann device', and its significance, goes largely unnoticed so that semantic mimicry is likely to occur when H' is expressed as a perturbative expansion, and mathematical correction terms in a perturbative approximation scheme applied to a set of nonlinear differential equations are interpreted as physical processes (cf. §5.3).

The U-field, its Hamiltonian H_U and the interaction term coupling the U-field to the matter particles via the Hamiltonian component H' are expressed in terms of the 'normal (mode) coordinates' of the free U-field, since the equation for the isolated free U-field is linear. That is, the U-field, the Hamiltonian H_U and the interaction term coupling the U-field to the matter particles via the Hamiltonian component H' are each then expressed in terms of the Fourier decomposition of the U-field. Associated with this Fourier decomposition are the raising and lowering a-operators, where the a-operators are interpreted as raising or lowering the state by one U-quantum. This is set out in the third paper:

> [W]e begin with the construction of the linear equation for the new field [mediating the interaction], which can be considered as a generalization of Maxwell's equations for the electromagnetic field. … The new field equations can be derived from the Lagrangian, so that the canonical variables and the Hamiltonian can be determined in the usual way. We can then go over into the quantum theory by constructing the commutation relations and the equations of motions for these variables. … We can decompose the field variables into Fourier components ... (Yukawa, Sakata and Taketani 1938, 319-320)

H' is treated as a perturbation to the total free Hamiltonian and is then written as a series expansion with respect to an interaction parameter (i.e. charge). With H' expressed in terms of the raising and lowering operators of the free U-field, the order of each term in the perturbative expansion is then interpreted to correspond to that number of exchange processes. Each term is interpreted as modelling such exchange via a sum over the possible intermediate states occurring at that order, giving rise to the 'matrix elements' of the theory:

> The interaction between the neutron and the proton, which is caused by virtual absorption and emission of the heavy [U] quanta, can be calculated from the Hamiltonian … as second order effect, by straightforward application of perturbation theory. Namely, we first transform the variables for the unperturbed system with the Hamiltonian $\overline{H}_0 = \overline{H}_v + \overline{H}_M$ into the normal coordinates [i.e., via the Fourier decomposition of the interaction field], then express the perturbation energy $\overline{H}'$ in terms of these coordinates and perform the calculation to the second order. (Yukawa, Sakata and Taketani 1938, 334)

This procedure is recognizable in the foundations of contemporary QFT. In the literature of the era it is usual to consider only the second or fourth order terms, with these interpreted as corresponding to a single or double exchange of interaction quanta, with the number of exchanges required accounted for by considering charge conservation.

For instance, returning to the second paper, Yukawa and Sakata note that the matrix element corresponding to the second-order perturbation energy 'shows that the interaction between the proton and neutron can be described by the exchange force of Heisenberg type' (Yukawa and Sakata 1937, 1088). They also consider the force between like particles via the *U*-field, noting that three intermediate stages are required, so that 'the interaction between like particles can be deduced as a fourth-order process due to the interaction of the heavy particles with the *U*-field' (Yukawa and Sakata 1937, 1089).

The authors understand different order terms of the 'perturbation energy' expansion to correspond to different physical (although, as we shall see, virtual and intermediate) exchange processes that mediate the interaction via quanta associated with the (force-)field mediating the interaction, an idea that would be appropriated in the context of Feynman diagrams and lingers through QFT until now. The problem of divergences at the higher orders, and 'self-energy' is not developed in these papers.[5]

There is a lack of clarity regarding the semantic support of the perturbative procedure and its interpretation.[6] That is, there is little or no consideration given as to whether the treatment developed is simply a mathematical approximation method, or a model of the actual physical processes involved, and these two aspects seem to be confused.

It does not appear that the various 'perturbation energy terms' were interpreted as successive correction terms to generate an approximation of the actual physical 'force' or cross-section for example. Rather, individual 'correction terms' are interpreted as corresponding to different physical processes that can occur as might be construed in a causal-mechanical framework. One may, on this account, legitimately focus on one particular term in order to understand a

[5] See Heitler (1936, 97, 177-185) for development of these points in QED.
[6] See Wentzel (1947) for some discussion of the perturbative method in this era, although he does not get to the heart of the interpretative issues.

particular physical process that contributes to the interaction, thus initiating a trajectory that would take in Feynman diagrams in the years to follow in which interactions in fundamental physics are understood in terms of (virtual) particle exchanges. The procedure is an exemplar of semantic mimicry, as we shall see in more detail in chapter 9.

Two problems arose in this approach in the 1930s, both of which have led to continued conceptual confusion: first, energy is not conserved during the intermediate processes so modelled; secondly, divergences are encountered in the fourth-order calculations. The first problem was quickly circumvented by postulating 'virtual quanta', which we shall see is the result of semantic mimicry in chapter 9, whilst the resolution of the second problem would have to wait until the development of renormalization techniques sometime later, which are a symptom of and a partial compensation for the failure of the initial application of the 'Volkmann device'.

7.4.2 Virtual processes / quanta

In Yukawa's third paper the 'virtual presence' of quanta is mentioned, but the meaning of this term is not explained, although it appears to be understood with reference to failure to conserve energy during the interaction process (Yukawa, Sakata and Taketani 1938, 329). The emission of virtual quanta in intermediate states is explicitly distinguished here from the creation of quanta in terms of their 'proper energy' (337).

Some other references to virtual particles / processes or intermediate states in this era help illuminate the concept.[7] Condon suggests that the language of 'virtual level' is a 'mode of speech' used to refer to unstable energy levels (1939, 808-809). Alternatively, Heitler comments:

> The probability [of a given final state] is appreciable only when the *energy of the final state is equal to the energy of the initial state* … The *energy* is therefore *conserved for all transitions from or into the continuous spectrum.* For transitions from or to the intermediate states … the energy is, of course, in general not conserved. (1936, 90)

He discusses the Heisenberg energy-time uncertainty relation (ETUR) in a slightly different context (113), but does not apply it to virtual quanta as Wick

[7] It is unclear where the concepts originated. In 1924 Slater spoke of a 'virtual radiation field' (Slater 1924; Bohr, Kramers and Slater 1924), and we saw that Born and Jordan discussed the electromagnetic field in a cavity in terms of 'virtual oscillators' (1925), but it is not clear whether or how these concepts are associated with virtual quanta.

does when he deduces the range of forces. Wick comments on virtual processes that

> in Yukawa's theory the interaction between heavy particles is carried by the semi-heavy particles, by means of simple emission and absorption processes … these are not, of course, actual emission and absorption processes, which would be contrary to the energy principle; they are called, therefore, virtual transitions. (1938, 994)

It is not entirely clear how he construes the contrast between 'virtual' and 'actual' given the wider context of the discussion. The ETUR, in conjunction with virtual quanta, is adopted as a new physical principle appealed to in order to allow energy to be conserved as regards the measureable outcome of an interaction whilst also allowing the interaction to be interpreted in terms of virtual quanta exchange where energy is not conserved.

The ETUR allows 'property dragging' in Wilson's sense with respect to energy and energy conservation. Virtual quanta and intermediate states appear to be understood as physical, but not satisfying energy conservation. Owing to the status of energy conservation as a law, some way of adapting the description of the exchange process was required. Some new process or principle must be postulated so that energy conservation is not violated *at least in terms of what can in principle be measured* (cf. Wilson 2006, 372-373).

The empirical success of Wick's calculation of the range of forces is taken to lend support to the ETUR and the postulated mechanism of interaction.[8] The concept of virtual quanta would find its ultimate expression in Feynman diagrams, although it is important to note the 'prehistory' of the concept just outlined: In a study of the origins of the concept of exchange forces Carson concludes that 'it would be wrong to say that Feynman's picture owed nothing to what had gone before. The language of exchange, and all the complex of ideas associated with it, did not originate with him.' (1996b, 130) Feynman himself was rather measured about the interpretation of the diagrams. As Brink observes,

> In Feynman's original paper the diagrams have a definite significance: Each one represents a particular term in the perturbation expansion of the interaction between the electromagnetic field and the charges. Subsequent diagrams have been used in many

[8] However, the ETUR has been considered problematic, as it is not clear what was or is meant by uncertainty in time, as time is not an operator but a parameter, both in NRQM and QFT. See Fox (2008), Hilgevoord (1996; 1998; 2005), Busch (1990a&b); 2008 and Bunge (1970).

> contexts, often in a rather loose way to give a physical picture of some interaction processes. (1965, 84)[9]

However, Feynman would often describe the charge on an electron in terms of the probability that it will emit a virtual photon (2006a [1985], 91).[10]

In these various treatments however, the semantic mimicry associated with seeking a physical interpretation to perturbative correction terms went unnoticed (cf. §5.3).

7.5 Divergences, self-energy and the roots of renormalization

Yukawa's approach emerged as the 'canonical' approach to interactions, as may be seen for instance in the paper by Fröhlich, Heitler and Kemmer (1938). Yukawa's basic conceptual framework is adopted, but with a vector rather than scalar field, and a more realistic model adopted. We see again the physical significance attributed to terms in the perturbative expansion of the interaction Hamiltonian. Here however the problem of divergences in the calculations at short distances is raised although the problem is incorrectly diagnosed. We see a prevailing object-based particle concept with regard to the nucleons, that is, they are treated as spatio-temporally located objects with extension, and it is this that (as they see it) may mitigate the divergence problem since noting this feature establishes a natural scale limit. For example,

> The fact that the fourth order of approximation is greater than the second order for small distances means, of course, that the *whole theory diverges for small distances* and our results can therefore only have a very qualitative significance. As a main result of our theory we can only say that the nuclear particles will have a finite radius of the order of equation [$d = 1/2\lambda$]. In this connexion we want to emphasize that the fourth order of the neutron-proton forces does not lead to any exchange force. The only way of exchanging two heavy electrons between a proton and a neutron is the successive emission of a positive and negative heavy electron by the proton or neutron and re-absorption by the other particle. It does obviously not lead to any exchange of charge. (1938, 169-170)

However, this problem of divergence is one of the key difficulties encountered in the development of QFT, and remains one of the main conceptual problems of QFT, even if renormalization group techniques have been taken to alleviate such concerns to some extent at least.

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[9] It is important to note that the concept of a Feynman diagram and the associated theories has a façade structure, although we cannot argue or develop this point in detail. That is, the physical significance of a diagram, and their role in relation to cloud chamber photographs on the one hand and modeling interactions as we have just set it out is supported by different semantic architectures, and this leads to confusion (cf. Kaiser 2005; Passon 2019).

[10] Some treatments of virtual quanta (e.g. Fox 2008; Harré 1988) discuss the concept solely in terms of Feynman diagrams.

Condon, discussing interactions in nuclear theories, recognizes that self-interaction is associated with the problem of divergences:

> But these calculations are divergent just as in the case of the calculation of the self-energy of a charged particle which arises from its being coupled to the electromagnetic field. The proper handling of those divergent calculations in quantum electrodynamics and in the theory of the electron neutrino fields is a complete mystery which has baffled all attempts at resolution up to now. The solution of the mystery will probably call for an entirely new approach to these questions – at present, theory is up against a stone wall. (1939, 814)

The 'complete mystery' owes to the (unexpected) failure of the (implicit) 'Volkmann device' as we shall see.

Although the problem of divergences would ultimately be 'solved' at one level in terms of renormalization techniques, there is the more fundamental difficulty that remains unaddressed – the failure of the Volkmann device, which is also associated with the nonlinearity of the coupled field equations – a difficulty that was anticipated by Heitler as we are about to see, but has essentially been ignored.

7.6 Perturbation theory, nonlinearity and Heitler's worry

The use and interpretation of the perturbation method, especially in terms of virtual intermediate states or quanta, is of central importance to being able to perform calculations of interactions in QFT. Moreover, once the need to accommodate self-interaction is recognized, the interpretation of the method is seen to be central to understanding any state in QFT, including 'free' particles, as they are always self-interacting.

It is worth considering some passages from the first edition of Heitler's textbook to clarify the emerging conceptual difficulties. This will highlight, albeit in inchoate form, at an early stage in the development of QFT a recognition of the difficulty of the conceptual separation of the quantum system in terms of 'pure' isolated uncoupled fields or particles that are taken to have the same form individually and in combination, that is, the implicit use of the 'Volkmann device' in the perturbative treatment of interactions.

Under the heading of 'perturbation theory' Heitler remarks:

> The equations describing the behaviour of an electron interacting with a radiation field are far too complicated to be solved exactly. In all applications of the theory, therefore, the interaction energy is treated as small, and approximate solutions are obtained which are correct only to the first order in this energy. Apart, however, from the mathematical difficulties of proceeding to a higher degree of approximation it appears, as we shall see, that only the first order approximation has physical significance; the higher orders do not

> correspond to reality. This corresponds to a deep-seated limitation of the present theory. (1936, 87)

He goes on to clarify that, having written the Hamiltonian as $H = H_0 + H'$, where H_0 is the Hamiltonian for the system considered as a non-interacting system, and *H'* is the 'perturbing term', 'perturbation theory [here] means an expansion of the transition probability in powers of the perturbation function *H*" (91).

However, despite noting that the method is an approximation technique, he associates terms in the perturbative expansion with physical emission / absorption processes (97). But he notes that the higher-order terms in the power series obtained for *H'* are divergent, and so physically meaningless, even though an apparently physical interpretation can be accorded to them in terms of exchange processes (93, 102, 177-185). This exemplifies the confusion, or at least lack of clarity regarding the use of a perturbative method coupled with a determination to read physical significance into the expansion obtained.

Despite this difficulty, Heitler offers some important reflections on the limitations of QFT (as QED) following consideration of 'positive electrons' and pair creation, features of QFT that I have not discussed so far.[11] In the terms of my analysis, Heitler recognizes in an inchoate fashion that the 'Volkmann device' does not straightforwardly apply to enable the isolation of 'matter particles' from 'radiation quanta' (or fields) that take the same form individually and in combination:

> the idea of an *electromagnetic field in vacuo* has to be *abandoned*. Even if no particles are present an electromagnetic field can give rise to the creation of pairs. Since, however, for this purpose a minimum energy of $2mc^2$ is required, pairs can only be created if, in the Fourier expansion of the field, frequencies higher than $2mc^2/\hbar$ or wave-lengths smaller than $\lambda_0/2$ occur. If this is the case, a 'pure field' *in vacuo* no longer exists. **In the future theory the electromagnetic field and the 'field' representing the positive and negative electrons will be intimately connected, neither of them having a physical meaning independently from the other**. (192, bold emphasis added)

This is noteworthy as essentially Heitler questions the implicit application of the 'Volkmann device' foundational to QFT.

He continues:

> If, however, the field is weak enough to give only a small probability for the creation of a pair, the 'field' representing the pairs and the electromagnetic field can be separated. [Volkmann device] The creation and annihilation of pairs can then be considered as the result of a *perturbation*. … From the above consideration it follows that it has only a limited meaning to speak of a *wave packet* of light or of a single electron with an *extension smaller than* $\lambda_0 \equiv \hbar/mc$. In the Fourier expansion of such a wave packet essentially waves

[11] Positive electrons, or positrons emerge naturally from Dirac's equation, with pair creation emerging from a second-quantized treatment of the matter field coupled to the radiation field.

> of a quantum energy mc^2 occur. Therefore, in the wave packet an indefinite number of pairs is present. [Moreover, we must consider that the] *creation of pairs in intermediate states* gives rise to some processes which are impossible in principle according to ordinary electrodynamics. … Processes of this sort [various scattering examples are given] can never be described by the present electrodynamics. Formally **they can be obtained only from a *non-linear electro-dynamics for the vacuum,* i.e. from a theory in which the *principle of superposition of two fields is not valid*.** (193-194, bold emphasis added)

This final remark is of key importance, one of the few recognitions of the failure of the 'Volkmann device' as associated with the nonlinearity of the coupled field equations, so that Hilbert superposition fails for their decomposition, yet it would appear to have been overlooked in subsequent developments of QFT. The observation will be central to my analysis in which I draw attention to the conceptual difficulties in the interpretation of QFT owing to nonlinearities associated with various failures of 'superposition'.[12]

7.7 Summary and anticipation in relation to 'superposition'

To summarize this chapter and anticipate what will follow in detail in chapters 8 and 10 in particular, there are three major applications of 'superposition' in QFT: (1) The initial use of the 'Volkmann device' to isolate from the quantum state partial states associated with different quanta types with the partial laws given in terms of the wave equation corresponding to each quanta type, such that the partial laws and states take the same form individually and in combination; (2) The Fourier mode decomposition of the field (wave) equations according to 'Hilbert superposition' to introduce raising and lowering operators which supports, (3) The eigenfunction decomposition of the Hamiltonian relative to a basis of eigenstates which has the finer-grained structure of a Fock space through the decomposition of the wave equations, where 'Hilbert superposition' is supported by the linearity of Schrödinger's equation. The establishment of the Fock space is then taken to supply a particle description of the quantum state.

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[12] We may gain some insight into the significance of Heitler's last remark via a comment in Schweber, Bethe, de Hoffmann. They hint at the importance or assumption of linearity as physically central in developing quantum theories owing to the perceived centrality of the superposition principle. In discussing derivation of Dirac's equation, they suggest that the Dirac wave function 'will have to satisfy a first-order linear differential equation', with footnote: 'The linearity is required in order that the superposition principle of quantum mechanics hold.' (1956, 12). The confusion here is that the quantum state always evolves linearly according to Schrödinger's equation so that superposition holds, whilst the field equations are nonlinear when interactions are introduced. It is this that problematizes the way that the state is represented as components (e.g. as electrons and photons in QED), and thus the initial application of the 'Volkmann device'.

All three applications are unproblematic in free theories, but (1) and (2) fail in interacting (coupled, nonlinear) theories so that (3) is not supported according to (1) and (2). To anticipate the remainder of the study, this means that there is no physically significant partitioning of the overall state into partial states associated with particle types, and that there is no physically significant Fock basis via which to represent the Hamiltonian, so that there is no particle description available for interacting theories. The need for renormalization is a symptom of these difficulties, and the application of renormalization is a partial compensation for the failure of 'superposition' in the sense that it allows empirically adequate results to be obtained. However, by the linearity of Schrödinger's equation and the self-adjointness of the Hamiltonian there is always a basis of eigenstates with which the overall state can be represented in a natural way (as an application of Sturm-Liouville theory), but such a representation does not make contact with familiar concepts of particle type and number, except in limited circumstances. When the coupling is weak however 'superposition' might be regarded as approximately true so that 'to an engineer's approximation' the state might be considered to be composed of the different particle types as assumed in the use of the Volkmann device.

This conceptual framework was (and, in general, remains) largely unrecognized in QFT, with Heitler representing a lone voice anticipating these difficulties in the 1930s. The difficulties likely went unrecognized owing to conceptual confusions with the use of perturbation methods when coupled with misplaced metaphysical assumptions, subsequently ignored perhaps owing to the astonishing empirical successes of QFT from the late 1940s onwards.

In the remainder of the thesis I clarify and analyse these claims in detail in relation to the applicability of 'superposition', exploring the consequences for how we should understand QFT.

Chapter 8

Superposition, field quanta and particles in free QFTs

In this chapter I'll show how superposition and particle concepts are established, along with their semantic architecture, for linear, free QFTs. For my purposes consideration of spinless, massive scalar fields will suffice,[1] with a view to analysing the 'toy model' of 'scalar Yukawa theory' in subsequent chapters. This is the simplest interacting model that will support the conceptual analysis that I wish to develop.[2] Once the 'field quanta' particle characterization is established, I consider the 'group theoretic' particle characterization to note that like 'field quanta' it is only supported on free rather than interacting QFTs. I'll consider virtual particles in chapter 9.

8.1 Orientation and overview

There are various patches of application of 'particle' within QFT as set out by Falkenburg (2007). She considers the particle concept in terms of 'field quanta', 'virtual particles', 'quasiparticles' and 'group theoretic particles'. Of these 'field quanta' is perhaps the notion most used in elementary QFT. As we shall see, the applicability of a particle concept in QFT depends on application of the superposition principle, which depends on the linearity (and independence) of the relevant PDEs. Particle descriptions are associated with coordinate representations of the state and fields offering natural 'descriptive opportunities' that support physically salient explanations and inductive inferences concerning the quantum state modelled. That is, when applicable, a description of the state in terms of particle types, numbers and states is a natural description obtained by Fourier techniques. The availability of such natural descriptions is limited to rather specific contexts in which the system can be considered to be free, such as

[1] Development of these concepts for free vector and spinor fields is similar but involves technical issues and further structure that would obscure the main conceptual points, so are not discussed.
[2] Scalar Yukawa theory is often adopted as a toy model for just these purposes in introductory texts even if it may not be pressed as a physical theory. Note that although φ^4 theory is often discussed, it does not have sufficient structure to illuminate the conceptual issues that I wish to explore, which require a theory characterized by more than one field type.

that of asymptotic states, although it may be 'approximately available' in a way akin to an 'engineer's model' in further – but still limited – contexts as we shall see in chapters 10-11.

In QFT in canonical 'second quantized' form prior to application of the particle concept, there are two types of entity to consider – the state, and the fields that act upon the state. One seeks a particle description of the state via the fields, with the fields associated with different particle types. The fields act on the state to evolve the state in terms of their corresponding particle numbers and states. The fields satisfy relativistic field or wave equations, and are considered in the Heisenberg picture. These PDEs are linear when the fields are free (so that 'quantum Hilbert superposition' applies), and nonlinear when the fields interact, so that 'superposition' does not apply. Moreover, the fields are operator-valued distributions requiring integration against suitable 'test-functions' to model realistic particle wave-packets. The state satisfies Schrödinger's equation, and may be considered in the Schrödinger picture, although often the interaction picture is used in calculations as we shall see in chapter 9. Schrödinger's equation is always linear, so that 'quantum Hilbert superposition' always applies to the *overall* state whether or not interactions occur. The difficulty that arises for describing interactions relates to constructing natural representations for the state and Hamiltonian.

Free QFTs may be developed in either the Heisenberg or Schrödinger pictures using Fourier techniques. As in application of superposition in NRQM (cf. chapter 6), application of Fourier techniques in either case depends on a suitable decomposition of the state at some reference time into simultaneous eigenfunctions/states of the Hamiltonian and momentum operators to ensure relativistic invariance, i.e. one seeks eigenstates of the 4-momentum operator. Moreover, one wishes to construct eigenstates with a suitably fine-grained structure to obtain a natural representation that optimally balances simplicity and strength. A particle representation of the system requires a Fock (or equivalent) structure to be established.[3]

[3] A difficulty arises even for free QFTs in that free QFTs admit unitarily inequivalent representations, leading to 'incommensurable' particle concepts, so that a privileging strategy is required (Ruetsche 2011, 117-126, 190-220). Such a strategy is available when a QFT is defined via a linear field equation on Minkowski spacetime, for this generates *the* Fock space

There are two different representations that have been adopted to construct the fine-grained structure of the eigenstates of 4-momentum. One is the Schrödinger wave-functional representation, which has sporadically been developed, but is rarely used. It is more usual to construct the canonical Fock space representation. Both approaches give rise to a particle description as 'field quanta'. These two representations are equivalent in the case of free fields,[4] but difficulties arise when interactions are introduced for both representations that require the use of perturbative techniques. I shall work primarily with the canonical Fock representation. The Dirac (interaction) picture is used to perform calculations on interactions perturbatively as we shall see in chapter 9. The increasingly popular 'path integral' approach will not be considered as it introduces further conceptual difficulties, but can be shown to be equivalent to the Fock and wave-functional representations for free fields, whilst again requiring perturbative methods to deal with interactions,[5] so that it does not circumvent the conceptual difficulties involved in the representation of an interacting state.

In each representation or approach there are conceptual difficulties in the application of 'superposition' and 'particle' in QFT, for instance in considering what it means to speak of a 'superposition of particles'. Complications arise from:

1. The assumption that the state can be characterized by particle types that can be isolated or abstracted and considered independently of each other. In addition it is assumed that associated with each particle type is a field that can be considered independently of each other field, and that the Hamiltonian can be represented via such fields as a concrete 'coordinate

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representation via the identification of a unique vacuum element (cf. Ruetsche, 190-204). All our QFTs will be associated with field equations on Minkowski spacetime so this problem does not arise. The problem of unitarily inequivalent representations is unavoidable when interactions are introduced however (Haag's theorem, see Ruetsche, 250-253), which I discuss in chapters 9-10. Moreover, it has been demonstrated that an observer uniformly accelerating through a Fock vacuum state observes particles – the Unruh effect (Ruetsche, 190-191). Perhaps this is unsurprising since consideration of non-inertial frames is beyond QFT's domain of applicability, so that it is not our 'Theory T'.

[4] See Hatfield (1992) for discussion and comparison of the two approaches; Jackiw (1988; 1990) for detailed development of the Schrödinger wave-functional representation and Kuhlmann (2020) for brief philosophical discussion. As may be inferred from Hatfield, the wave-functional representation does not circumvent the difficulties encountered in attempting to apply the Fock representation to interacting theories, although space prevents discussion in detail here.

[5] See Hatfield (1992).

representation' of the Hamiltonian operator. These assumptions reflect an initial implicit application of the 'Volkmann device';

2. Consideration of 'superposition' and 'particle' in relation to both the (operator-valued distributional) field satisfying a relativistic wave equation (e.g. Klein-Gordon or Dirac) and the quantum state, satisfying Schrödinger's equation. That is, there are two PDEs to which 'superposition' may be applied;
3. Nonlinearity is introduced to the field equations but not to the evolution of the *overall* state when interactions are considered so, that 'quantum Hilbert superposition' fails for the field but not state equation, provided that the overall state is considered and not the 'partial states' associated with putative particle types obtained from initial use of the Volkmann device in (1);
4. Use of the Dirac picture and perturbative techniques to deal with interactions, leading to series expansions requiring interpretation in terms of 'virtual quanta' with the likelihood of semantic mimicry.

Free (i.e., non-interacting) particles or fields may be dealt with fairly easily, as we consider in this chapter. The introduction of interactions greatly complicates the situation via nonlinear behaviour for which difficulties associated with renormalization are symptomatic.[6]

8.2 Initial application of the 'Volkmann device'

The 'Volkmann device' is implicitly applied at the very beginning of the development of a QFT. One *already supposes* that it is possible to abstract or to isolate fields associated with different particle types that identify 'partial states' that are assumed to partition and compose the quantum state such that associated with each partial state are partial laws (e.g. Schrödinger evolution for each partial state individually, or the relativistic wave equation for the corresponding independent field individually) that take the same form

[6] I shall not consider two conceptual difficulties that arise in QFT as they do not alter the conclusions that will be reached in chapters 10-11. Namely that: first, the fields introduced are distributions, but the product of distributions is in general mathematically undefined but appears not to inhibit the development of QFT; secondly, the implications of various 'no-go' theorems for the ontology of QFT, the significance of which remain the subject of debate. See Bigaj (2018); Oldofredi (2018); Halvorson and Clifton (2002).

individually and in combination. That is, one supposes that one can analyse the quantum state via distinct and independent particle and field types.

For a free, non-interacting system this is straightforward, so that isolated partial states, fields and laws taking the same form in and out of combination can be identified to characterize the system completely in terms of different particle types and states, so that this is a proper application of the Volkmann device, as I shall set out more precisely below (§8.5). The unquestioned and unnoticed application of the Volkmann device probably has a phenomenological explanation in the sense that it is apparently possible to detect or track free electrons, photons, neutrons, etc. so that such application appears so trivial as to not merit comment, with, prima facie, the composition of a state understood simply as the state being composed of these particles. As we shall see this leads to various dilemmas and confusions in the interpretation of QFT, as manifested by renormalization.

I shall return to consider the initial application of the Volkmann device in more detail after analysing individual free QFTs.

8.3 Fock space construction via the Klein-Gordon equation for a neutral scalar field

8.3.1 Overview: From second quantization to the Fock construction

I consider a QFT associated with the Klein-Gordon equation as the relativistic wave equation.[7] So suppose that the (partial) quantum system is characterized completely and in isolation via the action on the state of a single Hermitian, free or uncoupled relativistic scalar field $\varphi_0(x) = \varphi_0(\boldsymbol{x}, t)$ of mass m satisfying the (linear) Klein-Gordon equation.[8] It is non-interacting (even with itself) and neutral (uncharged) so that there is no distinction between its associated particles and antiparticles, which are spinless and satisfy bosonic statistics.

The Klein-Gordon equation is

$$\frac{\partial^2}{\partial t^2}\varphi_0(\boldsymbol{x}, t) - \nabla^2\varphi_0(\boldsymbol{x}, t) + m^2\varphi_0(\boldsymbol{x}, t) = 0$$

[7] What follows is loosely based around Hatfield (1992, 42-48), and cf. Fraser (2008, 843-847) for the quanta interpretation of Fock space.

[8] The suffix '0' indicates free (and 'bare', i.e., not renormalized, which will be denoted as φ_{phys}) fields that satisfy the free field equations rather than the interacting, coupled field equations. I denote the field satisfying the coupled field equations as φ.

or more compactly

$$(\partial^2 + m^2)\varphi_0(x) = 0.$$

$\varphi_0(x)$ is quantized canonically by promoting it to an operator $\hat{\varphi}_0(x)$, obtaining the conjugate momentum field and Hamiltonian from the associated Lagrangian, and imposing the canonical commutation relations (CCRs). The field operates in the Heisenberg picture on the quantum state $|\Sigma\rangle$, which satisfies Schrödinger's equation in the Schrödinger picture. The Klein-Gordon equation is the Heisenberg equation of motion for the field. Eigenfunction (Fourier) decomposition of both field and state may be performed as an application of 'quantum Hilbert superposition', the ability to do so being mathematically assured by the linearity of the PDEs and the self-adjointness of the operators.

Central to the analysis is the decomposition of the state at some reference time into simultaneous eigenfunctions of energy and 3-momentum (i.e., eigenfunctions of the 4-momentum operator) that are also +1 eigenvalue eigenstates of the permutation operator (owing to Bosonic statistics), reflecting the second aspect of the Fourier technique in the relativistic quantum context. These eigenfunctions persist in form, evolving linearly and independently according to Schrödinger's equation via 'Hilbert superposition', reflecting the first aspect of the Fourier technique. This procedure enables a 'field quanta' interpretation as a natural description of the system reflecting the 'two aspects' of Fourier techniques (cf. chapter 4) as applied on the quantum patch to the state and Schrödinger's equation. What is novel here is that the Fourier mode solution of a *second* PDE is required (the wave equation, here the Klein-Gordon equation) to 'diagonalize' the Hamiltonian (and 4-momentum) so as to construct the eigenstates of the Hamiltonian and momentum operators.

This construction enables the Hilbert space of states to be endowed with a finer-grained Fock structure. The eigenstates form a basis of this Fock space supporting a 'field quanta' description of the state, in turn allowing one to establish a particle description of the system as a natural description once wave-functions are introduced. That is, there is a further complication in that the fields are distributions and give rise to field quanta that are completely delocalised. To obtain realistic particles the fields must be integrated against suitable 'test-functions' (wave-functions) to obtain realistic particle wave-packets with

appropriate spreads of position / 3-momentum subject to the Heisenberg uncertainty relation.

A particle interpretation is understood as a natural description of the quantum system in the sense that it is a description that best balances simplicity and strength so as to support inductive inferences and physically insightful explanations.[9] It is analogous to use of eigenfunction or 'normal mode' coordinates in chapter 4 transposed to the quantum patch, where for instance the state of a vibrating string is described in terms of its harmonics (§4.4.1; cf. Ehrenfest 1925). This particle concept reflects Falkenburg's 'Field Quanta' characterization (2007, 224-229), and is that commonly adopted in the physics literature (e.g. Bjorken and Drell 1965, §12.5).

8.3.2 Single particle state construction[10]

Whilst it is commonplace to associate a QFT with a relativistic wave equation, it is preferable to take the Lagrangian associated with the wave equation as the foundation of a QFT. The Lagrangian is relativistically invariant, and from it the conjugate momenta and Hamiltonian are readily obtained mathematically, as in classical physics, and the field equation deduced. Moreover, conserved quantities and their physical significance are deduced from Noether's theorem via the Lagrangian, and interactions may be introduced through imposing local gauge invariance on the Lagrangian, as is foundational to the Standard Model.

The relevant classical action S and Lagrangian (density) $\mathcal{L}$ leading to the Klein-Gordon equation is

$$S[\varphi_0] = \int d^4x \mathcal{L}(x) = \frac{1}{2}\int d^4x\left(\partial^\mu \varphi_0(x)\partial_\mu \varphi_0(x) - m^2\varphi_0^2(x)\right)$$

giving the conjugate momentum

[9] To recap, as per Volkmann's characterization of isolation-superposition on the one hand and Mill-Ramsey-Lewis' characterization of laws on the other, we know that we have selected the correct 'isolation centers' as we have identified a set of partial laws associated with corresponding partial states. We identify the partial 'laws' obtained via the 'Volkmann device' *as* laws on the Mill-Ramsey-Lewis characterization since such syntactic representation offers an optimal balance of simplicity and strength. We then say that the description in terms of the partial states associated with the partial laws is natural, as inherited from the characterization of the partial laws as laws.

[10] The mathematics that follows is 'standard bookwork' (e.g. Hatfield 1992, 42-48) glossed with my own conceptual analysis.

$$\Pi_0^\mu(x) = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \varphi_0(x)\right)}$$

so

$$\Pi_0^\mu(x) = \pi_0(x) = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \varphi_0(x)\right)} = \partial_0 \varphi_0(x) \equiv \dot{\varphi}_0(x) \qquad (*)$$

The Hamiltonian density is

$$\mathcal{H}_0 = \partial^0 \varphi_0(x) \partial_0 \varphi_0(x) - \mathcal{L}_0$$

and the Hamiltonian

$$H_0 = \frac{1}{2}\int d^3x(\pi_0^2(x) + |\nabla\varphi_0(x)|^2 + m^2\varphi_0^2(x))$$

To 'second quantize' promote the fields to operators and impose the equal-time CCRs:

$$[\hat{\varphi}_0(\boldsymbol{x}, t), \hat{\pi}_0(\boldsymbol{y}, t)] = i\delta(\boldsymbol{x} - \boldsymbol{y})$$
$$[\hat{\varphi}_0(\boldsymbol{x}, t), \hat{\varphi}_0(\boldsymbol{y}, t)] = [\hat{\pi}_0(\boldsymbol{x}, t), \hat{\pi}_0(\boldsymbol{y}, t)] = 0$$

These commutators are adopted for the Klein-Gordon equation as it models a field with bosonic statistics.[11]

The Heisenberg equations of motion for the fields are

$$\dot{\hat{\varphi}}_0 = i\left[\hat{H}_0, \hat{\varphi}_0\right]$$

reproducing (*) as an operator equation, and

$$\dot{\hat{\pi}}_0 = i\left[\hat{H}_0, \hat{\pi}_0\right]$$

which is the Klein-Gordon equation

$$(\partial^2 + m^2)\hat{\varphi}_0(x) = 0$$

In the Heisenberg picture $\hat{\varphi}_0(x) = \hat{\varphi}_0(\mathbf{x}, t)$ operates on a fixed state $|\Sigma_0\rangle$ describing the system (by convention, at t=0, $|\Sigma_0\rangle_{t=0}$). The system's evolution can be considered in either the Heisenberg or Schrödinger picture, and I shall consider both.

The first step in constructing the Fock representation is to transform $\hat{\varphi}_0(x) = \hat{\varphi}_0(\boldsymbol{x}, t)$ to eigenmode coordinates or 'simple solutions' of the Klein-Gordon equation. If the field is considered over a finite region, such as a cubical box of side L with periodic boundary conditions we have the Fourier series solution, the 'mode expansion':

[11] Fermionic systems as associated for example with Dirac's equation require anticommutators. The form of the commutator is chosen to preserve causality. See Peskin and Schroeder (1995, 54-56).

$$\hat{\varphi}_0(x) = \hat{\varphi}_0(\boldsymbol{x}, t) = \frac{1}{L^{3/2}} \sum_{\boldsymbol{k}} \frac{1}{2\omega_k} \left(\hat{a}_{0,\boldsymbol{k}} e^{-ik\cdot x} + \hat{a}^{\dagger}_{0,\boldsymbol{k}} e^{ik\cdot x} \right)$$

where $\boldsymbol{k}_i = \frac{2\pi n_i}{L}$ for integer n_i, and $\omega_{\boldsymbol{k}} = (\boldsymbol{k}^2 + m^2)^{1/2}$, the relativistic energy-mass relation. However, the field is generally considered over all space-time, so the Fourier transform is obtained in the limit

$$\hat{\varphi}_0(x) = \hat{\varphi}_0(\boldsymbol{x}, t) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{1}{2\omega_k} \left(\hat{a}_0(\boldsymbol{k}) e^{-ik\cdot x} + \hat{a}^{\dagger}_0(\boldsymbol{k}) e^{ik\cdot x} \right)$$

where $\boldsymbol{k}$ is any real number.[12] We should note that as per §5.2, introduction of the Fourier transform involves a more subtle semantic architecture than in the case of the Fourier series. However, I shall not need to develop the differences in architecture, and I shall be able to contrast free and interacting QFTs without clarifying the differences. All I shall need to note is that the asymptotic representations (i.e., in the infinite limits) obtained are idealized.

The volume element for the integral is the Lorentz invariant measure d^4k where $k = (k^0, \boldsymbol{k})$.[13] But owing to the 'mass-shell' condition $\omega_{\boldsymbol{k}} = (\boldsymbol{k}^2 + m^2)^{1/2}$, and since $\hat{\varphi}_0$ satisfies the Klein-Gordon equation, integration over d^4k may be simplified via:

$$\int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{1}{2\omega_{\boldsymbol{k}}} = \int \frac{d^4k}{(2\pi)^4} 2\pi\delta^4(k^2 - m^2)\theta(k^0)$$

where it is understood that $k^0 = \omega_{\boldsymbol{k}}$ on the LHS to satisfy the mass-shell condition.[14]

The solutions $\hat{\varphi}_0(x)$ of the Klein-Gordon equation are constructed as a 'Hilbert superposition' of the 'simple', i.e. Fourier solutions of the PDE since it is a linear PDE. However, as the fields and the coefficients obtained are operator-

[12] Note that in the field decomposition one can associate the time dependency with either the waves as $e^{\pm ik\cdot x}$ (e.g. Hatfield 1992, Duncan 2012, Haag 1996) or the a-operators (e.g. Schwartz 2014, Fraser 2008, Teller 1995), depending on the placement of the $e^{\pm ik^0}$ factor. For my analysis it is preferable to place the time dependency in the 'wave-functions'. The time dependency of the a-operators will neatly distinguish between free and interacting fields since the a-operators *must* carry time dependency in the interacting case (see §10.2.1). As regards the diagonalization of the Hamiltonian in what follows in the free field case, it is unimportant where the time dependency is placed as both placements yield identical representations of the Hamiltonian.

[13] It turns out that the ability to conduct the 'frequency splitting' process involving the establishment of a complex structure is of central importance to the establishment of the Fock space structure in what follows, although we shall not consider this aspect of the theory, as per standard treatments. See Baker (2009).

[14] Different normalization conventions are used giving rise to different factors in the CCRs and integrals in the definition of the operators that follow. Here I follow Hatfield's convention.

valued distributions, and since it may be shown that the operators $\hat{\varphi}_0(x)$ are not observables, their physical significance is unclear for this 'second-quantized' interpretation of the wave equation. However, owing to the role that the Fourier solution of the Klein-Gordon equation will play in the Fock construction, the 'simple solutions' have physical salience in this sense owing to their role in establishing the identity of partial states and corresponding partial laws. So such representation might be understood as a 'superposition', even if it involves another subtle prolongation of the concept.

I now articulate the physical significance of the Fourier solutions of the Klein-Gordon equation, demonstrating that 'superposition' does apply by elucidating the physical salience of the modes, indicating their role in the construction of a Fock structure and ultimately a particle description of the state.

We express $\hat{a}_0(\boldsymbol{k})$ and $\hat{a}_0^\dagger(\boldsymbol{k})$ in the mode expansion in terms of $\hat{\varphi}_0(x)$ and $\hat{\pi}_0(\boldsymbol{x}, t)$

$$\hat{a}_0(\boldsymbol{k}) = \int d^3\boldsymbol{x} e^{ik\cdot x}(\,\omega_{\boldsymbol{k}}\hat{\varphi}_0(\boldsymbol{x}, t) + i\hat{\pi}_0(\boldsymbol{x}, t))$$

$$\hat{a}_0^\dagger(\boldsymbol{k}) = \int d^3\boldsymbol{x} e^{-ik\cdot x}(\,\omega_{\boldsymbol{k}}\hat{\varphi}_0(\boldsymbol{x}, t) - i\hat{\pi}_0(\boldsymbol{x}, t))$$

and calculate the commutators to give the algebra of the a-operators:

$$\left[\hat{a}_0(\boldsymbol{k}), \hat{a}_0^\dagger(\boldsymbol{k}')\right] = (2\pi)^3 2\omega_{\boldsymbol{k}}\delta^3(\boldsymbol{k} - \boldsymbol{k}')$$

$$[\hat{a}_0(\boldsymbol{k}), \hat{a}_0(\boldsymbol{k}')] = \left[\,\hat{a}_0^\dagger(\boldsymbol{k}), \hat{a}_0^\dagger(\boldsymbol{k}')\right] = 0$$

What is of particular importance is that it may be shown that, first, the a-operators for the free field are time-independent (cf. Duncan 2012, 250), and secondly, a coordinate transformation on the Hilbert space of states may be performed to a coordinate system of eigenstates of the 4-momemtum operator (i.e., simultaneous eigenstates of the Hamiltonian and 3-momentum operators satisfying the correct relativistic relationship) via the a-operators since the a-operators establish a 'diagonal' representation of the Hamiltonian (and 4-momentum) in terms of (uncoupled and independent over $\boldsymbol{k}$) operators:[15]

[15] If we use the CCRs on the a-operator representation of the Hamiltonian then

$$\hat{H}_0 = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\omega_k\,\hat{a}_0^\dagger(k)\hat{a}_0(k) + \frac{1}{2}\int d^3k\omega_k\delta^3(0)$$

so that the energy diverges owing to the contribution from the δ. This is not unexpected, since each 'oscillator' has non-zero ground state energy. The difficulty is avoided by 'normal ordering',

$$\widehat{H}_0 = \frac{1}{2}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{k}}}\omega_{\boldsymbol{k}}\left(\hat{a}_0^\dagger(\boldsymbol{k})\hat{a}_0(\boldsymbol{k}) + \hat{a}_0(\boldsymbol{k})\hat{a}_0^\dagger(\boldsymbol{k})\right)$$

or, after 'normal ordering':

$$:\widehat{H}_0: = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{k}}}\omega_{\boldsymbol{k}}\hat{a}_0^\dagger(\boldsymbol{k})\hat{a}_0(\boldsymbol{k})$$

If we compare this with the QSHO in NRQM (§6.6), we may see that (modulo relativistic normalization factor), as Hatfield puts it,

> the Hamiltonian is the continuous sum of harmonic oscillator Hamiltonians, one for each $\boldsymbol{k}$. By comparing the commutators … we see that $\hat{a}_0^\dagger(\boldsymbol{k})$ is a creation (raising) operator, while $\hat{a}_0(\boldsymbol{k})$ is a destruction (lowering) operator. The particle interpretation results from considering $\hat{a}_0^\dagger(\boldsymbol{k})$ as an operator that creates a particle of energy $\omega_{\boldsymbol{k}}$ and momentum $\boldsymbol{k}$, while $\hat{a}_0(\boldsymbol{k})$ destroys such a particle.
> The ground state or bare vacuum is the state in Fock space that satisfies
> $$\hat{a}_0(\boldsymbol{k})|0\rangle_{\boldsymbol{k}} = 0$$
> and is normalised so that $\langle 0|0\rangle_{\boldsymbol{k}} = 1$. The state $\hat{a}_0^\dagger(\boldsymbol{k})|0\rangle_{\boldsymbol{k}}$ is a state containing one particle of energy $\omega_{\boldsymbol{k}}$ and momentum $\boldsymbol{k}$, $\left(\hat{a}_0^\dagger(\boldsymbol{k})\right)^2|0\rangle_{\boldsymbol{k}}$ contains two such particles, and so on. (1992, 44, notation adapted)

This requires careful analysis. The simultaneous diagonal representations of the Hamiltonian and 3-momentum operators give their actions on the state in a form best balancing simplicity and strength, thus being a representation in terms of 'partial laws' as I have defined the concept.

Consider first the 'simplest' eigenstates of the form $\hat{a}_0^\dagger(\boldsymbol{k})|0\rangle_{\boldsymbol{k}} = |\boldsymbol{k}\rangle$ which are seen to be eigenstates since

$$\widehat{H}_0|\boldsymbol{k}\rangle = \widehat{H}_0\hat{a}_0^\dagger(\boldsymbol{k})|0\rangle_{\boldsymbol{k}} = \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{k}'}}\omega_{\boldsymbol{k}'}\hat{a}_0^\dagger(\boldsymbol{k}')\hat{a}_0(\boldsymbol{k}')\hat{a}_0^\dagger(\boldsymbol{k})|0\rangle_{\boldsymbol{k}} = \omega_{\boldsymbol{k}}\,|\boldsymbol{k}\rangle$$

and

$$\widehat{\boldsymbol{P}}_0 = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{k}}}\boldsymbol{k}\,\hat{a}_0^\dagger(\boldsymbol{k})\hat{a}_0(\boldsymbol{k})$$

so that similarly

$$\widehat{\boldsymbol{P}}_0|\boldsymbol{k}\rangle = \boldsymbol{k}|\boldsymbol{k}\rangle$$

The 4-momentum operator is

$$\hat{P}_0^\mu = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{k}}}k^\mu\,\hat{a}_0^\dagger(\boldsymbol{k})\hat{a}_0(\boldsymbol{k})\Big|_{k^0=\omega_k}$$

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which eliminates the vacuum energy; only energy differences (i.e. above the ground state) are usually considered physically important. See Hatfield (45-46). These solutions are somewhat ad hoc and unsatisfying, but we shall not dwell on the problem here, noting that the 'workarounds' do in fact remove the problem as far as the formal theory is concerned. We take operators to be normal-ordered as per usual practice.

Owing to the 'mass-shell' condition $\omega_{\boldsymbol{k}} = (\boldsymbol{k}^2 + m^2)^{1/2}$ the eigenstates of $\hat{P}_0^{\mu}$ of the form $\hat{a}_0^{\dagger}(\boldsymbol{k})|0\rangle_{\boldsymbol{k}} = |\boldsymbol{k}\rangle$ are interpreted as single 'phion' quanta states, as phions of (rest) mass *m* and 3-momentum $\boldsymbol{k}$.

By analogy with the QSHO the action of $\hat{a}_0^{\dagger}(\boldsymbol{k})$ on an arbitrary state $|\Sigma\rangle$ raises the 3-momentum of the state by $\boldsymbol{k}$ and energy by $\omega_{\boldsymbol{k}} = (\boldsymbol{k}^2 + m^2)^{1/2}$. This is interpreted as raising the state by a quanta of energy-momentum given by the Lorentz invariant scalar $k \cdot k = m^2$. That is, the action of $\hat{a}_0^{\dagger}(\boldsymbol{k})$ is interpreted as adding a *quantum* of mass *m* and momentum $\boldsymbol{k}$ to the state *of a type corresponding to the field* φ_0, i.e. adding a 'phion'. There are two complications here, however, arising first from the fact that eigenstates of the form $\hat{a}_0^{\dagger}(\boldsymbol{k})|0\rangle_{\boldsymbol{k}} = |\boldsymbol{k}\rangle$ are only *some* of the eigenstates of the 4-momentum operator, and secondly that particle statistics and the permutation operator come in to play for these other eigenstates as we shall consider in a moment.

Before doing so, note that we can form linear combinations of the states $\hat{a}_0^{\dagger}(\boldsymbol{k})|0\rangle_{\boldsymbol{k}} = |\boldsymbol{k}\rangle$. The action of linear combinations of the $\hat{a}_0^{\dagger}(\boldsymbol{k})$ on the ground state(s) $|0\rangle_{\boldsymbol{k}}$ is to create a state of one quanta of mass *m* in a 'Hilbert superposition' of momentum states, i.e.

$$\sum_k c_{\boldsymbol{k}} \hat{a}_0^{\dagger}(\boldsymbol{k})|0\rangle$$

where we write the ground state of the system

$$|0\rangle = \prod_{\boldsymbol{k}} |0\rangle_{\boldsymbol{k}}$$

That is,

$$\sum_k c_{\boldsymbol{k}} \hat{a}_0^{\dagger}(\boldsymbol{k})|0\rangle$$

represents the general state of a single quantum of mass *m,* whose 3-momentum will have the value $\boldsymbol{k}$ on measurement with probability $|c_k|^2$. The set of such states forms a Hilbert space $\mathbb{H}$, which will be identified as the one-quanta sector of the Fock space for the system. That this sum is properly a 'Hilbert superposition' is supported by Schrödinger's equation as we now see, which also assists us to clarify the physical significance of the interpretation of the eigenstates as 'field quanta'.

The success of the quanta interpretation as a 'natural description' follows from the 'two aspects' of the Fourier technique that we have studied repeatedly, when applied to the quantum domain in which the eigenfunctions of more than one operator are in view as in chapter 6. The point is that the eigenfunctions of the Hamiltonian just constructed (i.e. the $\hat{a}_0^\dagger(\boldsymbol{k})|0\rangle_{\boldsymbol{k}}$) satisfy the eigenvalue equation

$$\widehat{H}_0|\Sigma\rangle = \omega_{\boldsymbol{k}}|\Sigma\rangle$$

which is the time-independent Schrödinger equation. If one now considers the evolution of the state in the Schrödinger picture, after recombining variables in relation to the full Schrödinger equation

$$i\frac{\partial}{\partial t}|\Sigma\rangle = \widehat{H}_0|\Sigma\rangle$$

the eigenstates constructed (i.e. the $\hat{a}_0^\dagger(\boldsymbol{k})|0\rangle_{\boldsymbol{k}}$) form the partial states that evolve according to the partial law of multiplication by $e^{-i\omega_{\boldsymbol{k}}t}$, i.e. the simple or 'normal mode' solutions (of this restricted set of eigenstates) of Schrödinger's equation take the form

$$e^{-i\omega_{\boldsymbol{k}}t}\hat{a}_0^\dagger(\boldsymbol{k})|0\rangle_{\boldsymbol{k}}$$

This means that if the initial state may be expressed as above as

$$|\Sigma\rangle_0 = \sum_k c_{\boldsymbol{k}}\hat{a}_0^\dagger(\boldsymbol{k})|0\rangle$$

then it evolves according to Schrödinger's equation as

$$|\Sigma\rangle_t = \sum_k c_{\boldsymbol{k}}e^{-i\omega_{\boldsymbol{k}}t}\hat{a}_0^\dagger(\boldsymbol{k})|0\rangle$$

The 3-momentum of $|\Sigma\rangle_t$ will have the value $\boldsymbol{k}$ on measurement with probability $|c_k|^2$ since the exponential term contributes only a phase factor.[16]

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[16] To make contact with subsequent chapters, for a general (not necessarily free) system which evolves according to

$$i\frac{\partial}{\partial t}|\Sigma\rangle_t = \widehat{H}|\Sigma\rangle_t$$

where $|\Sigma\rangle_t$ denotes the state of the system at time *t,* then the system's time evolution may be expressed by introducing a unitary operator $U(t)$ or $U(t_0, t)$ so that

$$|\Sigma\rangle_t = U(t_0, t)\ |\Sigma\rangle_{t_0}$$

where, formally

$$U(t_0, t) = e^{-i\widehat{H}(t-t_0)}$$

Much QFT concerns the calculation and interpretation of this exponential. The point here is that if $|\Sigma\rangle_{t_0}$ can be expressed as a 'superposition' (Fourier aspect 2) of eigenstates of the Hamiltonian (and 4-momentum) then (Fourier aspect 1) the system evolves according to a (Hilbert) superposition of these eigenstates individually and independently according to the

That is, these set of eigenfunctions of the Hamiltonian constructed via the Fourier decomposition of the Klein-Gordon equation are eigenfunction solutions to Schrödinger's equation. They are simple, independent 'partial states' that persist in form in the evolution of the system, evolving according to the independent partial laws as just stated, with the partial states and laws taking the same form individually and in combination. This is the first aspect of the Fourier technique. The second aspect was to express the initial state in terms of these eigenstates as indicated, which is mathematically supported by the self-adjointness of the Hamiltonian.

However, for these eigenstates to be relativistically invariant, which is what we require in a relativistic theory and to ultimately support a 'particle' interpretation, they must also be eigenstates of the momentum operator, and it is this further feature that allows identification of the eigenstates as 'field quanta', which support a natural description of the system owing to the two aspects of the Fourier technique just considered when set in the quantum context with additional operators involved.

So, the construction of the natural description in terms of quanta is subtle, involving two PDEs and the eigenfunctions of two different operators. There are three further complications that I have alluded to that I now consider in detail: the construction of realistic particles from field quanta (§8.3.3); the provisional construction of 'multi-particle' eigenstates of the 4-momentum operator (§8.3.4); the imposition of the CCRs via the permutation operator leading to the full Fock space construction (§8.3.5).

8.3.3 Construction of realistic particles

'Realistic' *particles* are obtained from the quanta $\hat{a}_0^\dagger(\boldsymbol{k})|0\rangle$ formed from the distributions $\hat{a}_0^\dagger(\boldsymbol{k})$ via their integration against suitable 'test functions' to form single particle states as superpositions of momentum states:

$$|\boldsymbol{k}\rangle_f = \int d^3\boldsymbol{k}' f_k(\boldsymbol{k}')\hat{a}_0^\dagger(\boldsymbol{k}')|0\rangle$$

corresponding partial law obtained from $U(t_0, t)$. For free systems $|\Sigma\rangle_{t_0}$ has a Fock structure that persists in the evolution of the system, whereas for interacting systems it does not, as we shall see.

where the 'test function' $f_k(\boldsymbol{k}')$ is interpreted as a normalized momentum space wave-packet of mean momentum $\boldsymbol{k}$.

$|\boldsymbol{k}\rangle_f$ models a realistic particle localized in position and 3-momentum according to the Heisenberg uncertainty relation. In physical terms the form of f is determined by the circumstances of the particle production, so that it is not purely put in 'by hand', although as we'll see in subsequent chapters it will often be convenient to choose f of a particular (though realistic) form to ensure that particles are and remain well-separated when considering multi-particle states, as is physically the case in scattering experiments for example.

8.3.4 Provisional construction of multi-quanta / particle eigenstates of 4-momentum

The states $\hat{a}_0^\dagger(\mathbf{k})|0\rangle$ are only some of the eigenstates of the Hamiltonian (and 4-momentum), so we have not yet shown how to decompose an arbitrary state into eigenstates so as to complete both aspects of Fourier's technique for all states. The additional eigenstates of the 4-momentum operator can be constructed by induction (or 'aggregation') via repeated application of $\hat{a}_0^\dagger(\boldsymbol{k}')$ to such states, i.e. as $\hat{a}_0^\dagger(\boldsymbol{k_1}) \ldots \hat{a}_0^\dagger(\boldsymbol{k_N})|0\rangle$. I now consider the construction.

To recap, the eigenstates $\hat{a}_0^\dagger(\boldsymbol{k})|0\rangle$ form a basis for a Hilbert space $\mathbb{H}$, interpreted as the 1-quanta, and hence 1-particle sector of Fock space. But these states do not span the whole state space, that is, there are further solutions to Schrödinger's equation that are not expressible as linear combinations of these eigenstates. To span the whole solution space we introduce what are provisionally interpreted as multi-quanta (and hence particle) eigenstates by repeated application of $\hat{a}_0^\dagger(\boldsymbol{k})$, where $\boldsymbol{k}$ may differ on each application. Linear combinations of these 'multi-quanta' states $\hat{a}_0^\dagger(\boldsymbol{k_1}) \ldots \hat{a}_0^\dagger(\boldsymbol{k_N})|0\rangle$ are solutions to Schrödinger's equation by Hilbert superposition, giving additional Hilbert sub-spaces of the full space of states.

For each $\boldsymbol{k}$, the action of $\hat{a}_0^\dagger(\boldsymbol{k})$ on an arbitrary state $|\Sigma\rangle$ is interpreted as adding one phion of 3-momentum $\boldsymbol{k}$, and removing one phion for $\hat{a}_0(\boldsymbol{k})$. This interpretation follows from the change in energy and momentum eigenvalues of the state formed, and the relativistic relationship. By induction we provisionally

interpret a state formed by the N-fold application of $\hat{a}_0^\dagger(\boldsymbol{k_i})$ on $|0\rangle$ as an N-phion state where the phions have 3-momenta $\boldsymbol{k_1}, \ldots, \boldsymbol{k_N}$. So we can construct multi-quanta and hence particle states by integrating states of the form $\hat{a}_0^\dagger(\boldsymbol{k_1}) \ldots \hat{a}_0^\dagger(\boldsymbol{k_N})|0\rangle$ against suitable test functions $f_{\boldsymbol{k_1}\ldots\boldsymbol{k_N}}(\boldsymbol{k'_1}, \ldots, \boldsymbol{k'_N})$. It may be verified that these are eigenstates of $\widehat{H}_0$, $\widehat{\boldsymbol{P}}_0$ and $\hat{P}_0^\mu$. Naively then, for fixed N, the set of linear combinations of these n-phion states forms a Hilbert space, the N-quanta sector of (pre-)Fock space, which is the N-fold tensor product of the individual single particle Hilbert spaces, $\bigotimes_N \mathbb{H}$.[17] The direct sum of all these N-quanta Hilbert spaces, when symmetrized,[18] taken over all N itself forms a Hilbert space, the Fock space for the system. It is according to this Hilbert space that the decomposition of the initial state is conducted as the 'second aspect' of the Fourier technique to support a natural description of the state in terms of quanta and hence particles.

The interpretation of the states $\hat{a}_0^\dagger(\boldsymbol{k_1}) \ldots \hat{a}_0^\dagger(\boldsymbol{k_N})|0\rangle$ as N-quanta (particle) states is consistent with the 'number operator':

$$\widehat{N}_0 = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{1}{2\omega_{\boldsymbol{k}}} \hat{a}_0^\dagger(\boldsymbol{k}) \hat{a}_0(\boldsymbol{k})$$

which 'counts' the number of quanta/particles in a state, with the number given as the eigenvalue for a determinate particle number state. This enables us to verify (or at least check the consistency of) the interpretation of

$$\hat{\varphi}_0(x)|0\rangle = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{1}{2\omega_k} \left(\hat{a}_0(\boldsymbol{k}) e^{-ik\cdot x} + \hat{a}_0^\dagger(\boldsymbol{k}) e^{ik\cdot x}\right)|0\rangle$$

as a single phion at definite space-time location x, but totally delocalised in 3-momentum $\boldsymbol{k}$, being in a superposition of momentum states (cf. $\hat{a}_0^\dagger(\boldsymbol{k})$ as introducing a phion of definite momentum but totally delocalised in space). As with the a-operators, the field operators are distributions that require integration against test functions to model realistic particle wave-packets:

$$\hat{\varphi}_{0,f}(x) = \int d^3x' f_x(x') \hat{\varphi}_0(x)$$

models a single-particle wave-packet $|x\rangle_f = \hat{\varphi}_{0,f}(x)|0\rangle$ at mean location x.[19]

[17] Naively and 'pre-Fock' since we have yet to impose the statistics.
[18] Antisymmetrized for fermions.
[19] The 'smearing functions' are often implicitly understood and omitted to de-clutter the equations.

A realistic *N*-particle state (without consideration of particle statistics) is constructed as, again for 'suitable' *f:*

$$|\boldsymbol{k_1} \dots \boldsymbol{k_N}\rangle_{f,unsymmetrized} = \int d^3\,\boldsymbol{k'_1} \dots \boldsymbol{k'_N} f_{\boldsymbol{k_1}\dots\boldsymbol{k_N}}(\boldsymbol{k'_1}, \dots, \boldsymbol{k'_N}) \hat{a}_0^\dagger(\boldsymbol{k'_1}) \dots \hat{a}_0^\dagger(\boldsymbol{k'_N})|0\rangle$$

However, these states are unphysical in that we have not applied the CCRs from the appropriate particle statistics. So, to implement the bosonic statistics we must construct from such eigenstates a set of eigenstates that are simultaneously eigenstates of the permutation operator with eigenvalue +1.

8.3.5 Imposing the CCRs via the permutation operator and the Fock space construction

We have not yet considered the imposition of the CCRs on the 'multi-particle' states.[20] This requires the eigenstates already constructed to be, in addition, the +1 eigenvalue eigenstates of the permutation operator $\mathfrak{P}$. That is, the imposition of the CCRs requires that the structure of an *N*-quanta sector of the state space as spanned by the eigenstates of the 4-momentum operator is not simply that of the tensor product of Hilbert spaces as it may appear thus far, but the symmetrized tensor product in the case of bosons (as here) or anti-symmetrized tensor product for fermions.

Consider an (unsymmetrized) eigenstate of the 4-momentum operator in an *N*-quanta sector of pre-Fock space $\bigotimes_N \mathbb{H}$ provisionally interpreted as an *N*-phion state whose momenta are $\boldsymbol{k_1}, \dots, \boldsymbol{k_N}$, namely the eigenstate $\hat{a}_0^\dagger(\boldsymbol{k_1}) \dots \hat{a}_0^\dagger(\boldsymbol{k_N})|0\rangle$. Then then the permutation operator $\mathfrak{P}_{ij}: \bigotimes_N \mathbb{H} \to \bigotimes_N \mathbb{H}$ defined by (cf. Szekeres 2004, 404-409)

$$\mathfrak{P}_{ij}\hat{a}_0^\dagger(\boldsymbol{k_1}) \dots \hat{a}_0^\dagger(\boldsymbol{k_i}) \dots \hat{a}_0^\dagger(\boldsymbol{k_j}) \dots \hat{a}_0^\dagger(\boldsymbol{k_N})|0\rangle$$
$$= \hat{a}_0^\dagger(\boldsymbol{k_1}) \dots \hat{a}_0^\dagger(\boldsymbol{k_j}) \dots \hat{a}_0^\dagger(\boldsymbol{k_i}) \dots \hat{a}_0^\dagger(\boldsymbol{k_N})|0\rangle$$

has, neglecting parastatistics, eigenvalue +1 for Bosonic statistics or -1 for Fermionic statistics. Then, any permutation $\mathfrak{P}$ can be expressed as a product of such permutations, and the normalized 4-momentum eigenstates that are simultaneously +1 eigenvalue eigenstates of the permutation operator are

[20] The need to impose the CCRs of the relevant form is due to the requirement to establish relations on the fields to ensure that there is no propagation of the fields outside the light-cone.

$$|\boldsymbol{k}_{n_1}, \ldots, \boldsymbol{k}_{n_M}\rangle = \frac{1}{\sqrt{N!}\sqrt{n_1!}\ldots\sqrt{n_M!}} \sum_{\mathfrak{P}} \mathfrak{P}\left(\left(\hat{a}_0^\dagger(\boldsymbol{k_1})\right)^{n_1} \ldots \left(\hat{a}_0^\dagger(\boldsymbol{k_M})\right)^{n_M} |0\rangle\right) \quad (**)$$

for spin-0 bosons as here, where n_i denotes the number of quanta in state $\boldsymbol{k}_i$ and $\sum_{i=1}^{M} n_i = N$.[21] In our phion example, these eigenstates $|\boldsymbol{k}_{n_1}, \ldots, \boldsymbol{k}_{n_M}\rangle$ form a basis for the appropriate *N* quanta state-space.

The summation (**) is not to be interpreted as a superposition for no individual term in the sum has physical significance as a 'partial state'. Rather, the expression is a mathematical artefact arising from the difficulty of representing the symmetrized (or antisymmetrized) subspaces in terms of the tensor product space that arises in our construction of the eigenstates via the Fourier solution to the wave equation.

The overall, appropriately symmetrized space of states, the Fock space (Fock 1932), is then the direct sum of these *N* quanta sectors. The symmetric Fock space (for bosons) over the (single particle) Hilbert space $\mathbb{H}$ is

$$\mathfrak{F}_s(\mathbb{H}) = \oplus_{n=0}^{\infty} [\otimes_n \mathbb{H}]_s$$

where $[\otimes_n \mathbb{H}]_s$ is the symmetrized *n*-fold tensor product of $\mathbb{H}$, and the antisymmetric Fock space (for fermions):

$$\mathfrak{F}_a(\mathbb{H}) = \oplus_{n=0}^{\infty} [\otimes_n \mathbb{H}]_a$$

where $[\otimes_n \mathbb{H}]_a$ is the antisymmetrized *n*-fold tensor product of $\mathbb{H}$, with $\otimes_0 \mathbb{H} = |0\rangle$, the ground state with no particles.[22]

There are two different compositional issues associated with each '*n*-quanta' Hilbert space $\mathbb{H}_{n,\sigma} \equiv [\otimes_n \mathbb{H}]_\sigma$, namely that of the interpretation of an appropriately symmetrized eigenstate, and the interpretation of linear combinations of such states for fixed *n*. A further issue arises regarding the composition of Fock states given by linear combinations of states from different quanta-number sectors of Fock space. Thus in total there are three issues regarding the interpretation of Fock states in relation to 'superposition' that I address in turn:

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[21] For fermions $|\boldsymbol{k}_{s,1}, \ldots, \boldsymbol{k}_{s,N}\rangle = \frac{1}{\sqrt{N!}} \sum_{\mathfrak{P}} (-1)^{\mathfrak{P}} \mathfrak{P}\left(\hat{a}_{0,s}^\dagger(\boldsymbol{k_1}) \ldots \hat{a}_{0,s}^\dagger(\boldsymbol{k_N})|0\rangle\right)$
in the same spin state *s* and the sum is taken over all permutations $\mathfrak{P}$. I cannot discuss spin here as this introduces further technical issues.

[22] In this analysis 'superposition' applies to both bosonic and fermionic systems even though the details of the supporting architecture differs according to the statistics which shows up in phenomenological differences between bosonic and fermionic systems. It is not 'superposition' that distinguishes such systems, but the local rules of application of 'superposition' on each patch.

First, $|\boldsymbol{k_1} \ldots \boldsymbol{k_N}\rangle$ is interpreted as an *N*-quanta, or specifically here an *N*-phion, state. That is, intuitively perhaps, one wishes to consider the eigenstate to be *'composed'* of *N* phions. The desire to do so may be further motivated by integrating this eigenstate against suitable test-functions to obtain a realistic *N*-particle state where each particle is associated with its own wave-packet appropriately localized so that each particle is separated in position and momentum from the others. If the wave-packets are chosen to be well-separated and remain so, the state corresponds intuitively to *N* well separated particles, so that it seems natural to interpret this state as composed of *N* particles. Moreover, one might consider the physical formation of such a state as the 'bringing together' of *N* single particle states of independent and spatially well-separated physical origins, i.e. as associated with *N* individual systems for which the *N* states are combined in a single state to form a single system, as in a particle scattering experiment. Intuitively we want to say that the single state so formed is composed of *N* quanta or particles.

Complications to this picture arise when the wave-packets overlap. But, leaving this to one side, the situation is more complicated when the particle statistics are considered. The nature of the composition of the states $|\boldsymbol{k}_{n_1}, \ldots, \boldsymbol{k}_{n_M}\rangle$ or $|\boldsymbol{k}_{s,1}, \ldots, \boldsymbol{k}_{s,N}\rangle$ might be addressed with reference either to the concepts of 'individuality' (French and Krause 2006), 'separability' (Howard 2011), or 'superposability' perhaps.[23] Leaving the metaphysical issues to one side, the question for us is that of whether we should regard the putative *N*-particle or quanta state *as* a 'superposition' of *N* particles or quanta. Or, should we regard the state as a simple 'isolation centre' without further structure according to the 'Volkmann device' (so that describing it as an *N*-quanta state is misleading) in our account of a natural description of the state?

The key question is perhaps that of how we identify the 'simple elements' as the partial states associated with partial laws that take the same form in and

[23] The question of whether of *n*-quanta states may be considered as mereological compositions of objects for example, or as 'aggregations', is difficult. Such questions are discussed in the literature often in relation to particle statistics, e.g. Auyang (1995); da Costa and Holik (2015); Dieks (1990); Saunders (2006a&b); French and Krause (2006). In particular, da Costa and Holik note that the formalism of the *n*-particle Schrödinger equation in NRQM is not equivalent to the Fock space formalism (512), where perhaps the 'ontology' is built in to the modelling assumptions.

out of combination according to isolation/superposition and the 'Volkmann device'. Can we identify the simple partial states as the $\hat{a}_0^\dagger(\boldsymbol{k})|0\rangle$ as composing (according to the 'Volkmann device') to form *N*-quanta states understood as the *N*-fold composition (generalized superposition) of the $\hat{a}_0^\dagger(\boldsymbol{k})|0\rangle$, or are the '*N*-quanta' states themselves the simple partial states?

It is construction and structure via the *a*-operators applied in aggregation in conjunction with the energy and momentum eigenvalues of the eigenstates that motivates the interpretation of these eigenstates *as* *N*-quanta states of *N* quanta with the momenta indicated, as already suggested. Moreover, the number operator interprets such states as *N*-quanta states. Such an interpretation is, from our perspective, to (attempt to) appropriate a descriptive opportunity that leads to reasoning advantages and physically salient explanations, as we shall see when we consider scattering of like particles. That is, more can be said about the states if we can interpret putative multi-quanta states *as* multi-quanta states than by simply identifying the states by their total mass and momentum for instance, as we shall see in scattering theory.

The composition of the putative *N*-quanta states from lower quanta number states does not arise through simple combination as would be the case if we had simply a tensor product structure as suggested by our naïve construction. A simple tensor product structure arises with distinguishable particles (cf. §8.5 below) for which the single quanta states and the 'partial laws' associated with them take the same form individually and in combination so that 'superposition' in a generalized sense applies to states composed of unlike particle types, and perhaps this is one feature that adds to the confusion here.

Here however, for identical quanta with bosonic or fermionic statistics, owing to the symmetrized or antisymmetrized tensor product structure of the Fock space the partial laws associated with single quanta states take a *different* form individually and in combination in the *N*-quanta states. This is perhaps most clearly seen with fermionic statistics. Two isolated single quantum fermionic systems, considered as 'partial systems' with 'partial laws' each with the identical state $\hat{a}_{0,s}^\dagger(\boldsymbol{k})|0\rangle$ *cannot* be combined to form a new overall system for which the partial laws are the same individually and in combination,

whatever the structure of the combination, for $\hat{a}^{\dagger}_{0,s}(\boldsymbol{k})\hat{a}^{\dagger}_{0,s}(\boldsymbol{k})|0\rangle = 0$ owing to the fermionic statistics. As Auyang puts it, 'The α in $|\alpha\alpha'\alpha''\rangle_{\pm}$ [where the α are states of three putative particles composing the state and ± denotes the symmetrized or antisymmetrized state] no longer means the state of an individual particle. … Individual particle states are no longer viable. The multi-particle system takes over.' (1995, 164-165) This can be taken to indicate the failure of the Volkmann device and the inapplicability of 'superposition' in even a generalized sense with regard to the decomposition of '*N*-quanta' states in general.

So we should not identify the *N*-quanta states as 'superpositions' of *N*-quanta, even in Volkmann's extended sense. The point is that the '*N*-quanta' states are themselves simple states *according to the Volkmann device*. The trouble is that it is often helpful to regard or to refer to such states as composed of *N* quanta, even though this might also be a root of conceptual confusion in quantum physics.[24]

Ultimately however, as we shall see in subsequent chapters when we consider interactions, perhaps confusion stems here from the observation that we overstate the significance of particle descriptions. As Wald puts it, 'it always should be borne in mind that the notion of "particles" – while quite useful in certain contexts – plays no fundamental role in the formulation of quantum field theory' (1994, 51). To this end I shall not dwell on this issue and simply note that although it will often be convenient, and indeed support physically salient explanations in scattering theory, to speak of *N*-quanta states as though they are composed of *N* quanta that the supporting semantic architecture is unclear.

Turning now to the second issue that I raised regarding the application of superposition to the Fock construction, there is the question of the composition of linear combinations of the simultaneous eigenstates of 4-momentum and

[24] Although this is not a point that I wish to argue or develop, perhaps the difficulty may be partially resolved by noting the difference between 'particles' and 'quanta', and differences in applications of the Volkmann device *in general* and *in restricted contexts*. That is, suitable and restricted choice of the wave-functions $f_{\boldsymbol{k_1}\ldots\boldsymbol{k_N}}(\boldsymbol{k'}_1, \ldots, \boldsymbol{k'}_N)$ may support application of the Volkmann device to *particles* subject to stipulations about the relationships between the wave-functions so as to support the Volkmann device, as in particle scattering experiments in which at the start of the experiment we can consider a natural description of the state to be given in terms of *N* particles and their states.

permutation operators forming the basis of an N-quanta Fock space sector. An element of $\mathbb{H}_{N,S}$ for our phion system is of the form, suitably normalized,

$$|\Sigma\rangle_N = \sum_{\substack{\boldsymbol{k}_1 \ldots \boldsymbol{k}_M \\ \sum n_i = N}} c_{1\ldots M} |\boldsymbol{k}_{n_1}, \ldots, \boldsymbol{k}_{n_M}\rangle$$

This is straightforwardly interpreted, by 'Hilbert superposition', as the states $|\boldsymbol{k}_{n_1}, \ldots, \boldsymbol{k}_{n_M}\rangle$ (putatively, of N-quanta) in a superposition of momentum states, so that the amplitude for measuring the N phions in momentum states $\boldsymbol{k}_{n_1}, \ldots, \boldsymbol{k}_{n_M}$ is $c_{1\ldots M}$. This is properly a 'superposition' for analogous reasons as given above for the single phion states, namely that the $|\boldsymbol{k}_{n_1}, \ldots, \boldsymbol{k}_{n_M}\rangle$ are simple solutions of Schrödinger's equation that are also eigenstates of 4-momentum and now the permutation operator. This also extends to systems of higher spin, taking account of the spin components also as a superposition.

Finally, turning to the third issue raised regarding the Fock construction, an arbitrary Fock space state $|\Sigma\rangle$ for our phion system may be represented as a 'Hilbert superposition' taken over all the N-quanta eigenstates over all momenta

$$|\Sigma\rangle = \lim_{N\to\infty} \sum_{N} \sum_{\boldsymbol{k}_{N_{n_1}}, \ldots, \boldsymbol{k}_{N_{n_M}}} c_{N_{n_1}\ldots N_{n_M}} |\boldsymbol{k}_{N_{n_1}}, \ldots, \boldsymbol{k}_{N_{n_M}}\rangle$$

(and similarly again for higher spin systems). That is, for a given N, the state is interpreted as a superposition of all phion number states of up to N phions each in a superposition of 3-momemtum states, so that the probability of measuring the state as an N-quanta (particle) state with the quanta having momenta $\boldsymbol{k}_{N_{n_1}}, \ldots, \boldsymbol{k}_{N_{n_M}}$ is $\left|c_{N_{n_1}\ldots N_{n_M}}\right|^2$. This is properly 'Hilbert superposition' applied both to momenta and the number of quanta (particles) in the state. That is, the state may be a superposition of, say, a 2-phion and 5-phion state with the probability of obtaining either given by Born's rule.

I have now elucidated the interpretation of the Fock space structure for the neutral scalar field in relation to superposition, and turn now to the charged scalar field.

8.4 The Fock construction for the charged scalar field

I now consider how a conserved quantity known as 'charge' is introduced to a QFT, which also introduces antiparticles, and will lead to the possibility of

interactions. The introduction of a charged massive scalar field marks the introduction of the other field type required for the scalar Yukawa theory. However, I consider here an isolated free system completely specified in terms of a single field / particle type with no interactions.[25]

Much of the Fourier techniques, application of superposition and the interpretation developed for the neutral scalar field carries over, but with additional physical structure arising from the incorporation of charge so that the state can be considered to be composed of particles and antiparticles as an application of the Volkmann device in the free theory, even though there is a subtlety regarding charge conservation.

A charged scalar field of mass M is modelled by a complex scalar field ψ and its Hermitian conjugate $\psi^\dagger$ with Lagrangian:

$$\mathcal{L} = \partial^\mu \psi_0^\dagger(x) \partial_\mu \psi_0(x) - M^2 \psi_0^\dagger(x) \psi_0(x)$$

The momenta are:

$$\Pi^\mu_{\psi_0}(x) = \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \psi_0(x) \right)}$$

$$\Pi^\mu_{\psi_0^\dagger}(x) = \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \psi_0^\dagger(x) \right)}$$

and

$$\Pi^0_{\psi_0}(x) = \pi_{\psi_0}(x) = \pi_0(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_0)} = \partial_0 \psi_0^\dagger \equiv \dot{\psi}_0^\dagger$$

$$\Pi^0_{\psi_0^\dagger}(x) = \pi_{\psi_0^\dagger}(x) = \pi_0^\dagger(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_0^\dagger)} = \partial_0 \psi_0 \equiv \dot{\psi}_0$$

with Hamiltonian density:

$$\mathcal{H}_0 = \pi_0^\dagger \pi_0 + \nabla \psi_0^\dagger \cdot \nabla \psi_0 + M^2 \psi_0^\dagger \psi_0$$

The fields are quantized by promotion to operators with CCRs imposed as before:

$$\left[\hat{\psi}_0(\boldsymbol{x}, t), \hat{\pi}_0(\boldsymbol{y}, t) \right] = i\delta(\boldsymbol{x} - \boldsymbol{y})$$

[25] If we understand the property of charge in QFT via the coupling between fields corresponding to different particle types, then perhaps it is curious that one can introduce the property of charge through a Lagrangian global internal symmetry without reference to other fields. The gauge principle that became central to interacting QFTs requires the imposition of a local symmetry that introduces another field. I shall not pursue this however beyond a few comments in §9.2.

$$\left[\hat{\psi}_0(\boldsymbol{x},t),\hat{\psi}_0(\boldsymbol{y},t)\right]=[\hat{\pi}_0(\boldsymbol{x},t),\hat{\pi}_0(\boldsymbol{y},t)]=0$$

and

$$\left[\hat{\psi}_0^\dagger(\boldsymbol{x},t),\hat{\pi}_0^\dagger(\boldsymbol{y},t)\right]=i\delta(\boldsymbol{x}-\boldsymbol{y})$$
$$\left[\hat{\psi}_0^\dagger(\boldsymbol{x},t),\hat{\psi}_0^\dagger(\boldsymbol{y},t)\right]=\left[\hat{\pi}_0^\dagger(\boldsymbol{x},t),\hat{\pi}_0^\dagger(\boldsymbol{y},t)\right]=0$$

$\hat{\psi}_0$ and $\hat{\psi}_0^\dagger$ are hermitian conjugates but each satisfy the Klein-Gordon equation individually. The Klein-Gordon equations

$$(\partial^2+M^2)\hat{\psi}_0(x)=0$$
$$(\partial^2+M^2)\hat{\psi}_0^\dagger(x)=0$$

are derived from the Lagrangian via

$$\dot{\hat{\pi}}_{\psi_0^\dagger}=i\left[\hat{H}_0,\hat{\pi}_{\psi_0^\dagger}\right]$$

and

$$\dot{\hat{\pi}}_{\psi_0}=i\left[\hat{H}_0,\hat{\pi}_{\psi_0}\right]$$

as before. The Fourier mode representations are obtained, although careful labelling is required to support identification of particle ('psion') and anti-particle ('anti-psion') states,

$$\hat{\psi}_0(x)=\hat{\psi}_0(\boldsymbol{x},t)=\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{k}}}\left(\hat{a}_0(\boldsymbol{k})e^{-ik\cdot x}+\hat{b}_0^\dagger(\boldsymbol{k})e^{ik\cdot x}\right)$$

$$\hat{\psi}_0^\dagger(x)=\hat{\psi}_0^\dagger(\boldsymbol{x},t)=\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{k}}}\left(\hat{b}_0(\boldsymbol{k})e^{-ik\cdot x}+\hat{a}_0^\dagger(\boldsymbol{k})e^{ik\cdot x}\right)$$

where as before $\omega_{\boldsymbol{k}}=(\boldsymbol{k}^2+M^2)^{1/2}$. The a-operators are associated with and construct the Fock space for the particles of the theory and the b-operators with the construction of the Fock space of the antiparticles. The Fock spaces are constructed as before from the simultaneous eigenstates of the 4-momentum and +1 eigenstates of the permutation operators, being relativistically invariant 'simple' states that persist in the evolution of the system and satisfy the correct statistics.[26]

The tensor product of the particle and anti-particle Fock spaces is then constructed to form the overall Fock space for the theory. This tensor product

[26] The CCRs for the a- and b-operators are as before, being time-independent, with the commutators mixing a- and b-operators vanishing.

does not require any symmetrization, as the particles are distinguishable from the anti-particles. That is,

$$|\Sigma_{overall}\rangle = |\Sigma_{particle\ sector}\rangle \otimes |\Sigma_{antiparticle\ sector}\rangle$$

The action of the operators on their individual spaces is naturally extended to this space, so for instance the a-operators operate on the overall Fock space as $\hat{a}_0^\dagger(\boldsymbol{k}) \otimes \iota_{ap}$ and the b-operators as $\iota_p \otimes \hat{b}_0(\boldsymbol{k})$, etc., using the obvious notation.

The number operators are defined for each field as before, and the total (normal ordered) Hamiltonian is

$$:\widehat{H}_0: = \frac{1}{2}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{k}}}\omega_{\boldsymbol{k}}\left(\hat{a}_0^\dagger(\boldsymbol{k})\hat{a}_0(\boldsymbol{k}) + \hat{b}_0^\dagger(\boldsymbol{k})\hat{b}_0(\boldsymbol{k})\right)$$

or, written out fully,

$$:\widehat{H}_0: = \frac{1}{2}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{k}}}\omega_{\boldsymbol{k}}\left(\hat{a}_0^\dagger(\boldsymbol{k}) \otimes \iota_{ap}\hat{a}_0(\boldsymbol{k}) \otimes \iota_{ap} + \iota_p \otimes \hat{b}_0^\dagger(\boldsymbol{k})\iota_p \otimes \hat{b}_0(\boldsymbol{k})\right)$$

I follow usual practice and omit the identity operations, but the actual structure should be kept in mind.

What this structure together with the decomposed form of the Hamiltonian implies is that the overall state may be understood in relation to application of the Volkmann device. Separation or isolation of particle and antiparticle states is achieved, so that individual particle and antiparticle states persist (following the decomposition of the initial state), being associated with corresponding partial laws that take the same form individually and in combination, but subject to charge conservation.[27] The partial laws for each are given by the relevant component of the Hamiltonian via Schrödinger's equation as before, so that the partial laws take the same form individually and in linear combination.

The Lagrangian has additional structure via an internal U(1) symmetry. That is, if we perform a global transformation:

$$\psi_0 \to e^{i\alpha}\psi_0$$

[27] This means that given a particle or antiparticle state as a partial state, in the free theory the partial states persist and are associated with partial laws that take the same form individually and in combination. The relevance of charge conservation is that any change to the particle number will require a corresponding change to the antiparticle number. But this situation does not arise in the free theory, and so the tensor product of the particle and antiparticle states can be regarded as an instance of 'superposition' even if the conservation of charge constraint lurks in the background but does no work in describing evolution in the free theory.

$$\psi_0^\dagger \to e^{-i\alpha}\psi_0^\dagger$$

the Lagrangian is unchanged. By Noether's theorem, associated with this symmetry is a conserved Noether current, ${J_0}^\mu$:

$${J_0}^\mu = \Pi^\mu_{\psi_0} D\psi_0 + \Pi^\mu_{{\psi_0}^\dagger} D\psi_0^\dagger$$

where $D = \frac{\partial}{\partial\alpha}\Big|_{\alpha\to 0}$

so $${J_0}^\mu = i\left[\left(\partial^\mu\psi_0^\dagger\right)\psi_0 - \left(\partial^\mu\psi_0\right)\psi_0^\dagger\right]$$

This current ${J_0}^\mu$ may be promoted to an operator on quantization, although normal ordering is required to remove ordering ambiguities. From the current operator a charge operator is deduced:

$$\hat{Q}_0 = \int d^3x \hat{J_0}^0 = i\left[\left(\partial^0\hat{\psi}_0^\dagger\right)\hat{\psi}_0 - \left(\partial^0\hat{\psi}_0\right)\hat{\psi}_0^\dagger\right]$$

$$= \frac{1}{2}\int d^3k \left(-\hat{a}_0^\dagger(\boldsymbol{k})\hat{a}_0(\boldsymbol{k}) + \hat{b}_0(\boldsymbol{k})\hat{b}_0^\dagger(\boldsymbol{k}) - \hat{a}_0(\boldsymbol{k})\hat{a}_0^\dagger(\boldsymbol{k}) + \hat{b}_0^\dagger(\boldsymbol{k})\hat{b}_0(\boldsymbol{k})\right)$$

so

$$:\hat{Q}_0 \coloneqq \int d^3k \left(\hat{b}_0^\dagger(\boldsymbol{k})\hat{b}_0(\boldsymbol{k}) - \hat{a}_0^\dagger(\boldsymbol{k})\hat{a}_0(\boldsymbol{k})\right)$$

The conserved charge is given by the difference between the number of anti-particles and particles. Conventionally, the Noether current is defined to be positive for particles, so, changing sign:

$$:\hat{Q}_0 \coloneqq \int d^3k \left(\hat{N}^{(a)}_{0,k} - \hat{N}^{(b)}_{0,k}\right)$$

8.5 Quantum systems with different particle types: The initial use of the 'Volkmann device' revisited

I now consider a quantum system characterized completely by different particle and field types, without coupling or interaction, as supported by successful application of the Volkmann device. For concreteness, and anticipating the scalar Yukawa theory, suppose that the overall state $|\Sigma\rangle_{overall}$ is completely described by two spinless quanta (particle) types, phions, being neutral spin 0 bosons of mass m, and psions / anti-psions, being spin 0 bosons of mass M and charge $|g|$. The associated fields $\hat{\phi}_0$, $\hat{\psi}_0$ and $\hat{\psi}_0^\dagger$ each satisfy a Klein-Gordon equation individually, and each act individually on the state to raise or lower the phion or

(anti)psion content. That is, the field equations completely characterizing the system are linear and uncoupled.

The overall state may be decomposed into isolated (free) phion and psion/anti-psion component states since

$$|\Sigma\rangle_{overall} = |\Phi_0\rangle \otimes |\Psi_0\rangle \quad (A)$$

where further

$$|\Sigma\rangle_{overall} = |\Psi_{particle\ sector}\rangle \otimes |\Psi_{antiparticle\ sector}\rangle$$

There is no symmetrization of either tensor product as the particles are distinguishable, and each component $|\Phi_0\rangle$, $|\Psi_{particle\ sector}\rangle$ and $|\Psi_{antiparticle\ sector}\rangle$ is represented by an element of its own symmetrized Fock space, i.e. by linear combinations of *N*-quanta states as given above. The action of the fields is given by

$$\hat{\phi}_0|\Sigma\rangle_{overall} = \hat{\phi}_0 \otimes \iota_\psi|\Phi_0\rangle \otimes |\Psi_0\rangle = \left(\hat{\phi}_0|\Phi_0\rangle\right) \otimes |\Psi_0\rangle \quad (B)$$

and likewise for the Ψ_0-fields $\hat{\psi}_0$ and $\hat{\psi}_0^\dagger$.

The overall Hamiltonian $\widehat{H}_{0,overall}$ is the sum of the two independent Hamiltonians $\widehat{H}_{0,\Phi}$ and $\widehat{H}_{0,\Psi}$, each acting on their component of the tensor product space (and as the identity on the other):

$$\widehat{H}_{0,overall}|\Sigma\rangle_{overall} = \widehat{H}_{0,\Phi} \otimes \iota_\psi|\Phi_0\rangle \otimes |\Psi_0\rangle + \iota_\varphi \otimes \widehat{H}_{0,\Psi}|\Phi_0\rangle \otimes |\Psi_0\rangle \quad (C)$$

where normally the identity maps are suppressed so that

$$\widehat{H}_{0,overall} = \widehat{H}_{0,\Phi} + \widehat{H}_{0,\Psi}$$

This construction of the combined Fock space structure, and associated Hamiltonian, enables us to view a general state as a composite of psion, anti-psion and phion 'partial states' that each have the same Fock structure individually here and in combination (given by the tensor product). Moreover, the partial laws for each sector or partial state are the same individually and in combination owing to the additive structure of the Hamiltonian coupled with the observation that each field acts simply as the identity on the other component of the overall state.

That is, the decomposition of the state into phion and psion components and associated fields is an application of the Volkmann device owing to this 'superposition' structure. This means that the particle interpretation established on the two component spaces individually as above carries directly over to the

combined, overall space by the Volkmann device. A general state in this tensor product space evolves as a Hilbert superposition for which Born's rule is applicable, with the coefficient of each term giving the amplitude of measuring the system in a given state as before in terms of phion and psion content and states. The QFTs constructed in §§8.3-4 take the same form individually and in combination, whilst not stating the facts, in a system completely characterized by phions and psions, provided that there are no interactions.

The crucial point is that a free system supports application of the Volkmann device in the first step of establishing a QFT for a 'composite' system owing to (*A*-*C*) holding (cf. §2.3.1). That is, when the fields taken to characterize a quantum system completely are free and not coupled application of the Volkmann device is supported so that a QFT can be developed individually and in isolation for each field (as we did above). Moreover, each field individually and in isolation supports a particle description of the (partial) state associated with each field individually and in isolation. The QFTs associated with each field and the corresponding particle description of the (partial) state then take the same form in combination as they do individually in the sense just established.

Moreover, it may be seen from (*A*-*C*) that the converse holds – a quantum system which supports the Volkmann device in order to identify a set of fields (with associated Lagrangians) completely characterizing the system is a free (non-interacting) system.

This is the initial application of the Volkmann device that was mentioned in chapter 7 in relation to the historical development of QFTs, the applicability of which apparently seemed so obviously true as to be unnoticed. However, the crunch will come when we consider interactions as in our 'toy model' of scalar Yukawa theory in which the different field types are coupled. This means that (*A*-*C*) no longer hold so that the initial application of the Volkmann device fails. The field equations are coupled and nonlinear, with the wave equation for one field now acting on the putative partial state associated with the other field. Moreover, Hilbert superposition is no longer supported for the wave equations or the evolution of the putative partial states owing to the nonlinearity of the wave equations.

The failure of the Volkmann device has far-reaching consequences that I explore in the subsequent chapters. Renormalization, and the need to introduce 'virtual particles', are indicators of the failure of the Volkmann device. They offer partial compensation, generally only as regards supporting calculations, in limited circumstances. As QFT developed during the mid-20th century, Heitler's lone voice of concern (§7.6) was ignored as it appeared inconceivable that one could not identify independent fields and corresponding partial states, with some form of application of the Volkmann device implicitly assumed. For instance Schwinger suggested that QED should have

> the following essential features – explicit covariance with respect to Lorentz transformations, *and a natural division between the properties of independent fields and the effects of field interactions*. As the simplest example of the latter, we consider … the phenomena of vacuum polarization and the self energies of photon and electron, which arise from the coupling between the matter and electromagnetic fields and their vacuum fluctuations. (Schwinger 1949, 651-652, emphasis added)

This appeal to 'independent fields' is an application of the Volkmann device in the context of examples that Heitler had raised as problematic, and leads to the need for renormalization as a partial remedy. More recently it has perhaps become preferable to work with the renormalized rather than 'bare' fields, which might be taken as an attempt to re-establish the Volkmann device with reference to different 'isolation centres'. But as we shall, this fails to re-establish the applicability of the Volkmann device in the context of general interactions.

From the Wilsonian perspective the initial application of the Volkmann device for free systems establishes a descriptive opportunity leading to a reasoning advantage, supporting physically salient explanations by enabling one to describe a complicated quantum system in terms of different types of particles in different states. This is a natural description of the system, which is supported in a rather complicated way by application of the 'two aspects' of Fourier's technique in addition to the Volkmann device. As we shall study in detail in subsequent chapters, semantic mimicry occurs with regard to application of the Volkmann device and the interpretation of virtual particles when interactions are introduced.

8.6 The 'group theoretic' characterization of particles

My discussion of the particle concept in QFT has centred on Falkenburg's 'Field Quanta' patch. I now discuss another important particle concept in QFT, namely

Falkenburg's 'Group Theoretic' characterization which is associated with Wigner's classification of particles (Wigner 1939). This is in one sense a more powerful approach than that of 'field quanta' and might be considered as an alternative approach when the 'field quanta' characterization encounters difficulties. However, as with 'field quanta' it is only applicable to free fields satisfying linear field equations, so it cannot be used to support particle descriptions of interacting theories. Wigner's approach shows us how to classify particle types and derive the associated field equations. But it is in another sense more restricted in its application than 'field quanta', being more a 'classificatory' approach that identifies what is necessarily required to identify an entity as a particle in a relativistic setting, and it does not show us how to characterize a state as composed of particles.

Wigner's group-theoretic approach might be considered as supplying either the classification, identification or definition of particles in QFT. Particles are identified as states that transform under irreducible unitary representations of the proper orthochronous Poincaré group $ISO(1,3)$ (Schwartz 2014, 110). Wigner (1939) first classified such representations as representations that may be embedded in fields, where the fields are the free field wave equations for single relativistic particles. He showed that the irreducible unitary representations are uniquely classified by two parameters: mass m and spin J, where m is a non-negative real number and J is a non-negative half integer.

This means that all particles can be classified by two parameters identified as mass and spin, so the pair (m, J) can be taken to 'define' a particle in QFT (cf. Streater 1988, 144; Schwartz 2014, 110). It is, however, a mathematical classification of group representations. It does not say which are physically instantiated. It gives what is mathematically necessary but not physically sufficient for what we intuitively identify as a particle (cf. Kuhlmann 2010, 93). Falkenburg suggests that Wigner's classification, if taken as a definition, leads to a '*very general* meaning of the term "particle"' (2007, 231), with particles no longer local, and are considered to be non-interacting (232), as the definition requires *linear free-field equations*.

In order to construct a unitary QFT one must embed the irreducible representations into objects with space-time indices, that is, scalar fields, vector

fields, tensor fields and spinor fields from which Lagrangians can be deduced (Schwartz 2014, 111) and the dynamics of the theory developed. This is straightforward for spin 0, i.e. J=0, where one puts one degree of freedom, i.e., the mass m into a scalar field (Schwartz 2014, 114), leading to the Klein-Gordon equation. The cases for J>0 are more complicated as extra degrees of freedom must be embedded in the fields. I shall not consider the details but simply note that, for example, the Dirac equation can be deduced in this way.[28] Indeed, Kuhlmann suggests that, 'One success of Wigner's approach is that relativistic wave equations for all possible types of free particles, such as the Klein-Gordon or the Dirac equation, can be derived in a systematic fashion without heuristic ad hoc moves.' (2010, 93) Perhaps this is the real significance and power of Wigner's approach – it provides a way of identifying all the possible free, linear wave equations to be considered as candidates for canonical QFTs.

In subsequent chapters I develop a contrast between linear and nonlinear field theories. It is worth stressing that Wigner emphasised that his approach was applicable only to linear (free) theories, and not to nonlinear theories: 'It should be emphasized … that … [o]ur analysis is necessarily restricted to free particles and does not lead to any assertions about possible interactions.' (Bargmann and Wigner 1948, 213; cf. Wigner 1939, 151).

Falkenburg briefly considers the significance of interactions in relation to Wigner's particle concept, suggesting that 'particles are considered to be *primarily non-interacting.* They are considered to be independent of the rest of the world, like the substances of traditional metaphysics. … Particles are subject to *uncoupled* field equations.' (2007, 232) She does not develop this point, although it is worth analysing in more detail. Wigner's project is based on the observation that, 'If the wave functions in question refer to a free particle and satisfy relativistic wave equations, there exists a correspondence between the wave functions describing the same state in different Lorentz frames.' (Bargmann and Wigner 1948, 211) This crucial point – that linear or free field equations are required for this approach – has perhaps become obscured. That is, the 'group theoretic' particle characterization does not provide a way of circumventing the difficulties that occur with the 'field quanta' characterization

[28] See Schwartz (2014, 114-138).

when moving from free (linear) to interacting (nonlinear) theories, so that it does not offer a way of 'prolonging' the particle concept into interacting theories.

Finally, Wigner's approach classifies 'elementary systems' that are to be regarded as 'particles' but not necessarily 'elementary particles'. Newton and Wigner note the difficulty of defining and contrasting 'elementary particle' and 'elementary system', suggesting that the defining characteristic of an 'elementary particle' is not always clear cut: 'it is that it should not be useful to consider the particle as a union of other particles'. So they suggest that whilst a hydrogen atom is an elementary system but not an elementary particle, the case of the π-meson is more ambiguous. They conclude that the π-meson should be regarded as an elementary particle owing to the difference between its properties and those expected 'from a compound consisting of a μ-meson and a neutrino' (Newton and Wigner 1949, 400). They also consider protons and neutrons as elementary particles, writing prior to QCD, but perhaps the question of the composition of strongly interacting 'bound states' such as those of quarks and gluons nicely illustrates the interpretative difficulties of 'elementarity'. I shall return to this in chapter 11.

8.7 Summary and anticipation of interacting QFTs

We have seen how a particle description in free (non-interacting) QFT via particle types and their states arises as a natural description of the state according to 'field quanta' via several applications of 'superposition':

First, an initial implicit appeal to the Volkmann device to allow the state to be considered to be composed of isolatable partial systems comprising of phion and psion component states (for example) that take the same form individually and in combination, associated with corresponding independent partial laws taking the same form individually and in combination;

Secondly, Hilbert superposition is applied to Schrödinger's equation, reflecting classical Fourier techniques, with the caveat that the interpretation of the coefficients is given by Born's rule on this quantum patch of application. The task is to construct explicitly the eigenstates of the Hamiltonian that are simultaneously eigenstates of the momentum and permutation operators to support a particle interpretation, and associated laws, as a natural description;

Thirdly, to construct such eigenstates, Hilbert superposition is applied to the wave equations, again using Fourier techniques. This enables the construction of the Fock basis to support a particle description of the system as a natural description.

Of these three applications of 'superposition', only the second – Hilbert superposition applied to the overall (NB not partial) state in Schrödinger's equation – survives in interacting theories, as we shall see. This problematizes the particle concept and the way that we understand and describe interacting states, with renormalization a symptom of the failure of the Volkmann device and a partial cure as a pragmatic response to the problem.

Finally, we considered the 'group theoretic' approach to particle characterization, noting that it will not offer a way around the failure of 'field quanta' in interacting theories, even if it supports the systematic construction of relativistic wave equations associated with different types of free fields and particles.

Chapter 9

Introducing interactions, Dyson's expansion and Feynman diagrams

9.1 Introduction and overview

In chapter 8 we saw how 'free' QFTs were constructed so as to establish particle descriptions of quantum systems as natural descriptions by appeal to 'superposition' via Fourier techniques. I now begin my analysis of interacting theories where, as per the historical foundations of QFT (chapter 7), an interaction is introduced as a 'perturbation' to the established free theories via a parameter 'charge' that couples the fields. Iterative series techniques are then applied to solve for the behaviour of the interactions of the system provided that the coupling is sufficiently small to support the use of such techniques, since the iterative series is a power series in the parameter 'charge'. It has been commonplace to endow individual terms in the series expansion obtained with a physical interpretation even though difficulties are encountered, not least from the need for renormalization. Haag outlines this approach to QFT thus:

> To study [QED's] physical consequences one started from the free field theory, resulting if $\mathcal{L}_I$ [the perturbative interaction term coupling the fields in the Lagrangian] is neglected, as a zero order approximation. There we have the Fock space of noninteracting photons, electrons and positrons. $\mathcal{L}_I$ is considered as a perturbation producing transitions between particle configurations. The elementary process is the emission or annihilation of a photon combined with the corresponding change of momentum of an electron or positron or the creation or annihilation of an electron-positron pair. (1996, 51)

We shall see that this approach incorporates a form of semantic mimicry, and that there is no particle description for interacting states in QFT, whether via virtual particles, or otherwise. This conclusion is not novel (cf. Fraser 2008; Redhead 1988) but analysis of interacting QFT in terms of the failure of the applicability of superposition,[1] and of the Volkmann device in the initial attempt to isolate fields and corresponding states in particular, is novel, which enables development of the conceptual analysis of interacting QFTs, offering a new perspective on renormalization for example.

[1] Fox (2008) considers what he terms the 'superposition argument' with regard to virtual particles, although without thorough analysis of the concept of superposition and its application (or failure).

I shall demonstrate in this and the following chapter that the application of the Volkmann device is deeply problematic and leads to conceptual confusion. Renormalization is a symptom of the failure of the Volkmann device and a pragmatic partial remedy employed to support calculations. Failure of the Volkmann device means that the concept of 'interaction' is also faulty at the (near) fundamental level in that the concept is prolonged from 'everyday' usage in relation to familiar objects to QFT via semantic mimicry. This has far-reaching consequences for our conceptual grasp of fundamental physics.[2] I shall not undertake a philosophical analysis of 'interaction' here; rather, I shall highlight the difficulties as they arise in relation to the Wilsonian framework adopted.

In order to model interactions and perform calculations in QFT, corresponding to each field *type* (e.g. $\hat{\psi}^{\dagger}(x)$, $\hat{\psi}(x)$, $\hat{\varphi}(x)$) there are three field *species* $\hat{X}(x), \hat{X}_0(x), \hat{X}_{in/out}(x)$ required:

The $\hat{X}_0$ are the bare, free fields introduced in chapter 8 that give rise to a Fock space structure of states constructed from the 'bare' vacuum $|0\rangle$. As I shall outline below, the $\hat{X}_0$ species will be associated with the Feynman diagrams and virtual particles that arise in conjunction with Dyson's expansion in the iterative solution method in the interaction picture. Realistic interpretation of Feynman diagrams arises through semantic mimicry as associated with semantic mimics of 'superposition'. The states associated with the $\hat{X}_0$ do not support a natural description or physical interpretation in interacting systems (§9.7).

The $\hat{X}$ are the coupled fields that I am about to introduce that satisfy the coupled field equations. Ultimately they will be interpreted as *interpolating fields* in scattering theory as they interpolate between 'physical' initial and final states in scattering experiments associated with the $\hat{X}_{in/out}$ (see below). The $\hat{X}$ act on the 'physical' vacuum $|\Omega\rangle$, but this field species does not support a physically meaningful Fock space structure and hence particle interpretation, or indeed any natural description of the system, owing to the nonlinearity of the field equations that they satisfy since 'superposition' fails (§10.2). The $\hat{X}$ will be seen to arise from an implicit application of the Volkmann device in the initial selection of

[2] There is surprisingly little philosophical discussion of the concept of 'interaction', other than in relation to the mind-body problem.

fields as associated with putative different particle types. Many interpretative difficulties arise because this is an improper application of the device.

Finally, the $\hat{X}_{in/out}$ are associated with the idealized 'well-separated' (thus non-interacting) physical or renormalized in-/out-states of scattering experiments. These free fields also act on the physical vacuum $|\Omega\rangle$ and support a Fock space structure with a physical particle interpretation, but only in the asymptotic region. However, the $\hat{X}_{in/out}$ and associated states cannot be prolonged to describe general interacting states or offer a physically salient explanation of interaction, which is shown to be a problematic concept at this (near) fundamental level (§10.2).

So, I shall show here and in chapter 10 that there is no particle, or even natural description of general interacting states associated with any of the field species $\hat{X}(x), \hat{X}_0(x), \hat{X}_{in/out}(x)$. In chapter 11 I indicate how, despite this, reliable empirically adequate results can be obtained using these fields. Such success will depend in part on the success of renormalization techniques to compensate for the failure of the Volkmann device. That is, one can reliably know the *outcome* of, for example, scattering experiments (up to probabilities) without having any description or explanation of the *process* of interaction. This means that one cannot endow a bound state with internal structure via QFT. However, for weakly coupled theories an 'approximate' natural description of interactions and bound states is often available that might be compared with an 'engineer's model' rather than a metaphysical description.

9.2 Introducing an interaction

I analyze interacting QFTs using scalar Yukawa theory. Assume that a system is modelled by charged scalar psion fields of mass *M* associated with 'matter' and 'anti-matter' particles that interact via a neutral scalar phion field of mass *m*. These fields are coupled by an interaction term $g\hat{\psi}^\dagger(x)\hat{\psi}(x)\hat{\varphi}(x)$, where g is the coupling or charge, so the overall Hamiltonian is assumed to be

$$\hat{H} = \hat{H}_0 + \hat{H}' \qquad \text{where } \hat{H}' = g\hat{\psi}^\dagger(x)\hat{\psi}(x)\hat{\varphi}(x)$$

and $\hat{H}_0$ is the sum of the free field Hamiltonians derived in chapter 8.

It is important to note, as per Haag above, that this standard approach to introducing an interaction is *already a perturbative approach* before the use of any series expansion techniques. The interaction term was introduced by assuming that it is meaningful to write the overall Hamiltonian as the sum of the free Hamiltonians with the addition of an interaction term introduced as a perturbation. That is, crucially, one assumes that it is still physically meaningful to apply the Volkmann device that supports the identification of isolated or abstracted different field/state *types* such that the abstracted components $\hat{\psi}^{\dagger}(x), \hat{\psi}(x), \hat{\varphi}(x)$ and associated partial states take the same form individually and in combination. Or, minimally, one supposes that there is some trace principle by which a general interacting state can be identified in terms of such physically meaningful abstracted entities.

It is important to emphasize that *all* interacting QFTs share these problematic assumptions regarding the applicability of the Volkmann device, so that they might be considered to be 'effectively' perturbative theories,[3] and so lead to problematic conceptualizations of 'interaction'. That is, it might appear that the gauge argument, and gauge field theory, especially when considered via its mathematical supporting architecture given in terms of differential geometry,[4] which led to the development of the Standard Model, offers a way of introducing interactions without recourse to the introduction of a perturbation.[5] That is, a coupling can be introduced or deduced by imposing local gauge symmetry on the Lagrangian associated with a 'matter field' according to a chosen symmetry group.

For example, consider the Lagrangian for a complex scalar field theory

$$\mathcal{L}_0 = (\partial^{\mu}\chi_0)^{\dagger}\big(\partial_{\mu}\chi_0\big) - \mu^2\chi_0^{\dagger}\chi_0$$

[3] It is common to apply the term 'perturbative QFT' to weakly coupled QFTs that are solved using iterative series techniques. But this is a *further* application of perturbative techniques in the series expansion approximation. See J. D. Fraser (2020) for recent philosophical discussion of the significance of a perturbative approach to QFT. However, Fraser does not clearly distinguish the *two* applications of a perturbation method.

[4] See Auyang (1995) and Healey (2007) for the setting of gauge theory in differential geometry.

[5] Early explorations of gauge theory were conducted by Pauli (1941) and Yang and Mills (1954), although it did not flourish until the 1960s.

with global U(1) symmetry according to the transformation $\chi_0(x) \rightarrow \chi_0(x)e^{i\alpha}$.[6] Now impose local invariance on the Lagrangian according to a local transformation with a local U(1) symmetry via (dropping the '0' suffix)

$$\chi(x) \rightarrow \chi(x)e^{i\alpha(x)}$$

The Lagrangian is invariant under this transformation if the derivative ∂ is replaced by the covariant derivative D via the 'minimal substitution'

$$D_\mu = \partial_\mu + iqA_\mu(x)$$

whilst requiring that the 'gauge field' $A_\mu(x)$ introduced transforms according to

$$A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{q}\partial_\mu\alpha(x)$$

where q is identified as the charge. This procedure introduces interactions via a 'gauge field' as the field 'mediating interactions' according to the specified symmetry group.

Gauge field theory is too big a topic to treat here, meriting further research as regards its relationship with perturbative methods.[7] The important question for us is that of whether or not the 'gauge argument' provides a means of introducing interactions that bypasses the perturbative foundation of interacting QFT in such a way that the Volkmann device applies.

Clearly the mathematical architecture of the gauge approach differs from that of a perturbative technique, having a much richer mathematical structure and context in differential geometry. However, in terms of its physical interpretation in the context of QFT it would appear that the gauge approach does not support application of the Volkmann device. In this regard the introduction of an interaction in the gauge approach is comparable in its consequences with the perturbative introduction of the coupling. Both approaches entail the implicit assertion that it is meaningful to identify abstracted fields as components associated with partial laws and states that take the same form individually and in combination. But this is not the case when the

[6] See Lancaster and Blundell (2014, 127-128) for this example.

[7] The standard philosophical treatment is Healey (2007). He does not address the significance of the technique in relation to the introduction of an interaction as a perturbation however but notes that common justifications for the 'gauge argument' fail (159-167). Guttmann and Lyre (2000) discuss the possibility of distinguishing 'physically significant' from 'merely mathematical' parts of gauge theories, but likewise do not tackle the question of the relationship between gauge and perturbative methods.

fields are interpreted in their physical rather than mathematical setting, as demonstrated by the need to renormalize the fields still in gauge theory to compensate for the failure of the Volkmann device.[8]

So it appears the gauge argument indicates the correct way to introduce a 'perturbation' corresponding to a particular kind of interaction, even if it is more than a perturbative method *per se*, given that one has *already decided* to isolate a particular 'matter field' by implicit application of the Volkmann device. In other words, one is *already committed* to what is essentially a perturbative approach in that one has already decided that it is meaningful to identify isolated or abstracted 'matter' and 'interaction' fields before coupling them with what might be thought of as a 'perturbation'.

I now show how calculations may be performed on interacting theories in the limited context of scattering theory within the (problematic) 'doubly' perturbative framework: that is, introducing an interaction as a perturbation and then proceeding to solve the coupled equations obtained by iterative methods in the interaction picture, leading to Dyson's expansion and its interpretation in terms of Feynman diagrams when applied to scattering states. For this 'calculational' procedure to be successful requires that the theory is sufficiently weakly coupled. The approach developed in this chapter is preliminary, for we shall see that it requires renormalization as well as setting in the more rigorous context of the LSZ scattering theory via the Gell-Mann and Low theorem, as we shall see in chapter 11.

9.3 Interactions in Scalar Yukawa Theory

9.3.1 Introduction of an interaction as a perturbation

From chapter 8 the Hamiltonians for free psion and phion systems are:

$$H_{\varphi_0} = \frac{1}{2}\int d^3x\left(\pi^2_{\varphi_0}(x)(x) + |\nabla\varphi_0(x)|^2 + m^2\varphi_0^2(x)\right)$$

$$H_{\psi_0} = \frac{1}{2}\int d^3x\left(\pi^\dagger_{\psi_0}(x)\pi_0(x) + \nabla\psi_0^\dagger(x)\cdot\nabla\psi_0(x) + M^2\psi_0^\dagger(x)\psi_0(x)\right)$$

associated with field equations

$$(\partial^2 + m^2)\hat{\varphi}_0(x) = 0$$

[8] I shall characterize the failure of the Volkmann device more precisely in §10.5 once the analysis of interacting QFT is completed, but see §8.5 for its application to free theories.

$$(\partial^2 + M^2)\hat{\psi}_0(x) = 0$$

$$(\partial^2 + M^2)\hat{\psi}_0^\dagger(x) = 0$$

After quantizing, Fourier techniques give normal mode solutions:

$$\hat{\varphi}_0(x) = \hat{\varphi}_0(\boldsymbol{x}, t) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2\omega_k}\left(\hat{a}_0(\boldsymbol{k})e^{-ik\cdot x} + \hat{a}_0^\dagger(\boldsymbol{k})e^{ik\cdot x}\right)$$

$$\hat{\psi}_0(x) = \hat{\psi}_0(\boldsymbol{x}, t) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{k}}}\left(\hat{a}_0(\boldsymbol{k})e^{-ik\cdot x} + \hat{b}_0^\dagger(\boldsymbol{k})e^{ik\cdot x}\right)$$

$$\psi_0^\dagger(x) = \psi_0^\dagger(\boldsymbol{x}, t) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{k}}}\left(\hat{b}_0(\boldsymbol{k})e^{-ik\cdot x} + \hat{a}_0^\dagger(\boldsymbol{k})e^{ik\cdot x}\right)$$

These Fourier solutions enable construction of diagonal representations of the Hamiltonians, and also the 4-momentum operators. In particular

$$:\widehat{H}_{\varphi_0}: = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{k}}}\omega_{\boldsymbol{k}}\hat{a}_0^\dagger(\boldsymbol{k})\hat{a}_0(\boldsymbol{k})$$

$$:\widehat{H}_{\psi_0}: = \frac{1}{2}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{k}}}\omega_{\boldsymbol{k}}\left(\hat{a}_0^\dagger(\boldsymbol{k})\hat{a}_0(\boldsymbol{k}) + \hat{b}_0^\dagger(\boldsymbol{k})\hat{b}_0(\boldsymbol{k})\right)$$

Symmetrized Fock space representations for the fields and CCRs, with vacuum elements $|0\rangle_{\varphi_0}$ and $|0\rangle_{\psi_0}$, giving rise to particle interpretations of the states may be obtained as in chapter 8.

As per §8.5, if we consider a free system composed of and completely specified by non-interacting psions and phions with overall state $|\Sigma\rangle_{overall}$, then the overall or total (free) Hamiltonian is the sum of the individual (free) Hamiltonians:

$$\widehat{H}_{0,overall} = \widehat{H}_{\varphi_0} + \widehat{H}_{\psi_0}$$

and similarly for the Lagrangians, $\mathcal{L}_{0,overall} = \mathcal{L}_{\varphi_0} + \mathcal{L}_{\psi_0}$. Operators act independently on their corresponding sector of the overall Fock space, and as the identity on the other, with

$$|\Sigma\rangle_{overall} = |\Phi_0\rangle\otimes|\Psi_0\rangle \text{ and } |0\rangle_{overall} = |0\rangle_{\varphi_0}\otimes|0\rangle_{\psi_0}$$

This reflects proper application of the Volkmann device to establish a natural description of the system in terms of isolated phion and psion partial states and fields that take the same form individually and in combination, with the partial states in each sector evolving independently and linearly according to Schrödinger's equation.

Difficulties arise when an interaction is introduced via[9]

$$\mathcal{L}' = -g\hat{\psi}^{\dagger}(x)\hat{\psi}(x)\hat{\varphi}(x)$$

or

$$\widehat{H}' = g\hat{\psi}^{\dagger}(x)\hat{\psi}(x)\hat{\varphi}(x)$$

where g is the coupling constant or charge. Since the CCRs are independent of g the CCRs take the same form here as with the free fields (Haag 1996, 54). The total Hamiltonian is:

$$\widehat{H} = \widehat{H}_0 + \widehat{H}'$$

where, on the perturbative assumption, it is supposed that $\widehat{H}_0$ has the same form as $\widehat{H}_{0,overall}$ except with each field species $\hat{X}_0$ being replaced by $\hat{X}$.

The overall state $|\Sigma\rangle$ evolves linearly according to Schrödinger's equation with total Hamiltonian $\widehat{H}$:

$$i\frac{\partial}{\partial t}|\Sigma\rangle = \widehat{H}|\Sigma\rangle$$

From the overall Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}'$$

deduce the coupled field equations:

$$(\partial^2 + m^2)\hat{\varphi}(x) + g\hat{\psi}^{\dagger}(x)\hat{\psi}(x) = 0$$

$$(\partial^2 + g\hat{\varphi}(x) + M^2)\hat{\psi}(x) = 0$$

$$(\partial^2 + g\hat{\varphi}(x) + M^2)\hat{\psi}^{\dagger}(x) = 0$$

We would like to solve these equations to enable us to form a natural description of the interacting state by appeal to 'superposition' and Fourier / eigenfunction techniques in relation to these fields. We shall see that this cannot be done however, even though successful calculations can be supported.

9.3.2 Consequences of the introduction of an interaction as a perturbation: Overview

Before showing how calculations proceed in the interaction picture, leading to Dyson's series and its Feynman diagram interpretation, I pause to remark on several conceptual difficulties that are now introduced and that I shall analyze in

[9] The absence of the '0' suffix indicates that the fields are now solutions to the coupled rather than free equations. As we shall see, they act on a different Hilbert space.

detail in chapters 10-11 to indicate that no natural description of the interacting state is available via the fields of any species:

First, introduction of an interaction in this fashion is already to proceed via a perturbation method whilst assuming the applicability of the Volkmann device as already noted.

Secondly, the coupled field equations are nonlinear so 'Hilbert superposition' no longer applies. The solution space does not have a Hilbert space structure and we cannot apply the superposition principle to support the Fourier technique *in both aspects* in relation to the decomposition of $\hat{\varphi}(x)$, $\hat{\psi}^{\dagger}(x)$ and $\hat{\psi}(x)$ to obtain Fourier series solutions as in the free field case so as to construct a Fock space structure for the system's state-space (cf. Reed and Simon 1979, 318).[10] So we cannot use the 'normal modes' of the field equations to diagonalize the Hamiltonian (and 4-momentum) to construct a natural description of the state understood as a particle description as in the free theory.

Thirdly, the overall state $|\Sigma\rangle$ still evolves linearly (unitarily)[11] according to Schrödinger's equation with total Hamiltonian $\widehat{H}$:

$$i\frac{\partial}{\partial t}|\Sigma\rangle = \widehat{H}|\Sigma\rangle$$

where we do not make any assumptions regarding the representation of $\widehat{H}$. That is, we do not suppose that it is represented in terms of $\hat{\varphi}(x)$, $\hat{\psi}^{\dagger}(x)$ and $\hat{\psi}(x)$ 'coordinates', or that it may be decomposed into free and interacting components. It is *no longer the case* that, in general, the overall state space has the form $|\Sigma\rangle = |\Phi\rangle \otimes |\Psi\rangle$ where $\hat{\varphi}(x)$ acts non-trivially only on $|\Phi\rangle$, and *no longer the case* that $|\Phi\rangle$ evolves linearly as a partial state independently of $\hat{\psi}^{\dagger}(x)$ and $\hat{\psi}(x)$ (and similarly for $\hat{\psi}^{\dagger}(x)$ and $\hat{\psi}(x)$, and associated states). In other words, it is no longer the case that putative partial states associated with their corresponding fields take the same form individually and in combination such that they evolve according to partial laws that take the same form individually and in combination. This marks the failure of the Volkmann device.

[10] Note we can Fourier decompose the field at a given time (second aspect) without relativistic considerations – but the decomposition has a different semantic architecture from a Fourier series solution (cf. epicycles contrasted with normal modes in §4.5.1). In particular, it does not support the construction of a Fock space. I discuss this in detail in chapter 10.

[11] This is often taken as an axiom of QFT, as is the nature of the Hamiltonian to ensure this.

However, Hilbert superposition *is* still applicable to the overall state-space, so the solution space of Schrödinger's equation has a Hilbert space structure, and there exists, in principle, a diagonal representation of the Hamiltonian in terms of energy eigenstates since $\hat{H}$ is unitary and Schrödinger's equation is linear. Moreover, the 'in principle' eigenstates of the Hamiltonian can still be chosen to be eigenstates of the 4-momentum operator as they commute.[12] But the eigenstates and the diagonal representation are not given in relation to the fields $\hat{\varphi}(x)$, $\hat{\psi}^{\dagger}(x)$ and $\hat{\psi}(x)$ and associated states $|\Phi\rangle$ and $|\Psi\rangle$ and their Fock structures. So, we can say little about the eigenstates in general as we do not know how to represent the Hamiltonian explicitly in a way that supports application of the Volkmann device, apart from in the idealized asymptotic context in which the Volkmann device *is* applicable to the $\hat{\varphi}_{in/out}(x)$, $\hat{\psi}^{\dagger}_{in/out}(x)$ and $\hat{\psi}_{in/out}(x)$ and associated states. In general however, the best we can do is to characterize the eigenstates by their total invariant mass and momentum (and any other quantum numbers needed to specify the state) (Schweber 1961, 652 cf. §10.3.1).

So in the light of our discussion on the eigenfunction representation of systems in Sturm-Liouville theory in chapter 4, the best, in principle, 'natural description' of the interacting system available at this stage is in terms of these 'in principle' eigenstates. This description is, however, of limited value as we cannot form explicit expressions for these states, and it is unclear how, in general, to relate them conceptually to physical 'renormalized' particles associated with the $\hat{X}_{in/out}$, or indeed to those obtained from the free $\hat{X}_0$ fields.[13] The description does however have value, such as in the derivation of the spectral resolution (§11.2).

Fourthly, although we do not as yet (and will see in §10.2, cannot) have a Fock space structure associated with the $\hat{X}$ field species, we may suppose on

[12] Whilst we defined the momentum operator in terms of fields and the a-operators in chapter 8, it is taken as axiomatic that there exists an operator $\hat{P}^{\mu}$ that is an infinitesimal generator of translations, interpreted as the momentum operator. That is, we can define a momentum operator on the interacting system without reference to fields or a-operators. See §10.3.

[13] Note that we cannot define a number operator for interacting states as decomposition according to the a-operators is not physically meaningful, so that we cannot deduce a particle content this way, as discussed in chapter 10. Moreover, it is not clear what (if anything) is meant by particle statistics or indeed the permutation operator here.

physical grounds that there is a lowest energy (unique)[14] 'vacuum' state corresponding to a relativistically invariant state of no particles, or the ground state for the interacting system. The interacting system ground state $|\Omega\rangle$ differs from that of the non-interacting system, $|0\rangle$, since the Hamiltonian (written in terms of $\hat{\varphi}(x)$, $\hat{\psi}^{\dagger}(x)$ and $\hat{\psi}(x)$ 'coordinates') is a function of g, and so one expects the lowest energy eigenstate (i.e., the vacuum or ground state) to be a function of g also (Haag 1996, 55). The relationship between them is important, and we note for now that $|\Omega\rangle \neq |0\rangle$, which will lead to inequivalent Hilbert space structures for the states associated with the different field species.

Fifthly, Haag's theorem implies that representations of the CCRs of the free bare, free physical and interacting fields $\hat{X}_0$, $\hat{X}_{in}$ and $\hat{X}$ are unitarily inequivalent (Haag 1996, 54-55; cf. Ruetsche 2011; Earman and Fraser 2006; Miller 2018), even though the CCRs have the same form for the free and interacting theories as noted above (Haag, 54). A consequence of Haag's theorem (cf. Streater and Wightman 1964, 161-162) is that there is no unitary map V satisfying

$$V(t)\hat{X}(\boldsymbol{x},t)V^{-1}(t) = \hat{X}_0(\boldsymbol{x},t)$$

This means that since the particle concept for a QFT is defined from the field, an implication of Haag's theorem is that particle concepts (if available) in free and interacting theories are 'incommensurable' (to use Ruetsche's term), since they are unitarily inequivalent, which is taken as a requirement for the equivalence of particle concepts (cf. Wigner 1939, 152; Ruetsche 2011, 14-15, 24-30).

Having oriented ourselves to the conceptual difficulties resulting from the introduction of an interaction, we now proceed by considering how calculations are nonetheless performed.

9.4 The interaction picture, Dyson's series and asymptotic states

9.4.1 The interaction picture

Calculations in QFT are often performed using the interaction picture.[15] The Hamiltonian is split into putative free and interacting components as above, $\hat{H} =$

[14] This excludes theories with vacua subject to spontaneous symmetry breaking.
[15] What follows is 'standard bookwork', e.g. Lancaster and Blundell (2014, 167-168).

$\hat{H}_0 + \hat{H}'$. The interaction picture operators are then constructed to carry time dependency according to the free component of the Hamiltonian according to

$$\hat{O}_I(t) = e^{i\hat{H}_0(t-t_0)}\hat{O}e^{-i\hat{H}_0(t-t_0)}$$

whilst the interaction picture states are constructed via

$$|\sigma_I(t)\rangle = e^{i\hat{H}_0 t}|\sigma(t)\rangle$$

This enables use of the well-understood free fields in iterative calculations since it may be shown that

$$\hat{X}_I(x) = \hat{X}_0(x)$$

Moreover,

$$\hat{H} = \hat{H}_S = \hat{H}_H$$

$$\hat{H}_0 = \hat{H}_{0,S} = \hat{H}_{0,H}$$

$$\hat{H}_I'(t) = e^{i\hat{H}_0(t-t_0)}\hat{H}_S' e^{-i\hat{H}_0(t-t_0)} \neq \hat{H}_S'$$

In the scalar Yukawa theory, in the interaction picture

$$\hat{\mathcal{H}}_I'(t,\mathbf{x}) = \hat{\mathcal{H}}_I'(x) = g\hat{\psi}_I^\dagger(x)\hat{\psi}_I(x)\hat{\varphi}_I(x)$$

but the interaction picture field operators are just the free Heisenberg field operators, so

$$\hat{\mathcal{H}}_I'(x) = g\hat{\psi}_0^\dagger(x)\hat{\psi}_0(x)\hat{\varphi}_0(x)$$

This is a crucial result. We have expressed the interaction term in the full Hamiltonian using the free fields, which are the solutions to the free-field equations, for which we have obtained Fourier series solutions.

The interaction picture state $|\sigma_I(t)\rangle$ evolves according to Schrödinger's equation

$$i\frac{\partial}{\partial t}|\sigma_I(t)\rangle = \hat{H}_I'(t)|\sigma_I(t)\rangle$$

where $\hat{H}_I'(t)$ is expressible in terms of the free fields via the above, allowing us to deduce the interaction picture time evolution operator $\hat{U}_I(t,t_0)$:

$$i\frac{\partial}{\partial t}\hat{U}_I(t,t_0) = \hat{H}_I'(t)\hat{U}_I(t,t_0)$$

such that

$$|\sigma_I(t)\rangle = \hat{U}_I(t,t_0)|\sigma_I(t_0)\rangle$$

and

$$\hat{U}_I(t,t_0) = e^{i\hat{H}_0(t-t_0)}\hat{U}(t,t_0)e^{-i\hat{H}_0(t-t_0)}$$

with

$$\widehat{U}_I(t,t) = 1$$

This means that *if* we can choose suitable physically meaningful coordinates to represent $|\sigma_I(t_0)\rangle$ as a 'superposition' of partial states that evolve individually and independently according to corresponding partial laws given by an eigenfunction representation of $\widehat{U}_I$, then we can form a natural description of the interacting system, according to the 'two aspects' of the Fourier technique.

However, solving for $\widehat{U}_I(t,t_0)$ and decomposing $|\sigma_I(t)\rangle$ according to Fourier techniques are both problematic as we shall see. To construct an explicit (but approximate rather than analytic) solution one must turn to iterative methods rather than Fourier techniques, which turn out to be inapplicable even in principle. The use of approximation methods often leads to semantic mimicry in relation to the physical significance of the results obtained (e.g. in terms of the interpretation of Feynman diagrams) by failing to recognize that the semantic architecture of the results has shifted from that of the two aspects of the Fourier technique. The mimicry arises in a manner comparable with the examples that we studied in which 'superposition' is mimicked such as in epicyclical astronomy (§4.5.1); power series solutions (§4.5.2); and iterative solution techniques applied to nonlinear systems, such as the Volterra series in particular (§5.3.2). I now consider the iterative solution method adopted.

9.4.2 Solving for $\widehat{U}_I(t,t_0)$: Dyson's series as a sum of correction terms rather than a 'superposition'

Although $\widehat{U}_I$ satisfies a linear differential equation, so that Hilbert superposition applies, in practice solving for $\widehat{U}_I$ requires iterative techniques.[16] Such techniques supply an asymptotic series solution in powers of the coupling constant g known as Dyson's expansion (cf. Dyson 1949a&b) expressed in terms of time-ordered products $T\left[\widehat{H}_I'(t_1)\ldots\widehat{H}_I'(t_n)\right]$of the $\widehat{H}_I'(t_i)$, where time-ordering is necessary since the $\widehat{H}_I'(t)$ do not commute at unequal times (Peskin and Schroeder 1995, 84-85).

[16] Although $\widehat{U}_I$ is a linear operator the 'coordinates' $\widehat{\psi}_0^\dagger(x), \widehat{\psi}_0(x), \widehat{\varphi}_0(x)$ with respect to which it is represented are nonlinearly related as they are coupled by the interaction term and relevant wave equations.

This supposes that g is 'sufficiently small', that is, the theory is sufficiently 'weakly coupled', for the iterative procedure to be successful.[17] The result is:

$$\widehat{U}_I(t,t_0) = 1 + \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \ldots \int_{t_0}^{t_{n-1}} dt_n T\left[\widehat{H}_I'(t_1) \ldots \widehat{H}_I'(t_n)\right]$$

or, 'formally':

$$\widehat{U}_I(t,t_0) = T\left[e^{-i\int_{t_0}^{t} dt' \widehat{H}_I'(t')}\right]$$

Inserting the interaction picture form for the scalar Yukawa interaction gives $\widehat{U}_I(t,t_0)$ as an iterative series in g:

$$\widehat{U}_I(t,t_0) = 1 + \sum_{n=0}^{\infty} \frac{(-i)^n g^n}{n!} \int_{t_0}^{t} d^4x_1 \ldots \int_{t_0}^{t_{n-1}} d^4x_n T\left[\hat{\psi}_0^\dagger(x_1)\hat{\psi}_0(x_1)\hat{\varphi}_0(x_1) \ldots \hat{\psi}_0^\dagger(x_n)\hat{\psi}_0(x_n)\hat{\varphi}_0(x_n)\right]$$

However, as an iterative series Hilbert superposition does not apply to individual terms, as the individual terms in the series are correction terms and not solutions of the DE that $\widehat{U}_I$ satisfies (cf. §5.3.2). Interpretation of this result is likely to involve semantic mimicry as Dyson's series is not an eigenfunction decomposition as would be required in the application of the Fourier technique.

9.4.3 Choice of a coordinate representation for $|\sigma_I(t)\rangle$

Ideally one would seek to establish a physically meaningful Fock (or similar) representation for $|\sigma_I(t)\rangle$ as per the free theory to offer a natural description of the interacting system. But owing to the nonlinear coupling of the fields we cannot do this in general for the $\hat{X}$, as we shall see in detail in §10.2 (cf. Reed and Simon 1979, 318). However, even if we could form a physically meaningful Fock representation for the space of states $|\sigma_I(t)\rangle$, since we only have an iterative expansion for $\widehat{U}_I$ it is not clear what physical significance we should attach to the action of individual terms or sums of terms in the action of iterative expansion for $\widehat{U}_I$ on any given representation of $|\sigma_I(t)\rangle$, since the iterative expansion for $\widehat{U}_I$ is not a superposition of eigenfunction solutions but a series of correction terms to the solution of the evolution equation. This is not an application of Fourier

[17] Success need not mean convergence. Success is achieved if a good approximation is obtained by summing the first few terms even though the series may diverge. Such series are (perhaps misleadingly) termed 'asymptotic series'. The Dyson series often diverges, as Dyson noted for QED (1952). See Holmes (2013, 7-19) for asymptotic approximation techniques.

techniques, and so the procedure does not support a natural description of the evolving interacting state in relation to the fields $\hat{\psi}_0^\dagger(x), \hat{\psi}_0(x), \hat{\varphi}_0(x)$.

There is however an idealized and restricted context in which we can construct a Fock structure for the $|\sigma_I(t)\rangle$, namely in the infinite time limits via the $\hat{X}_{in/out}$. As we shall see, although this will *not* help us to form a natural description of general interacting states it will help us to perform calculations in LSZ scattering theory (§11.3). This will require a further idealization to relate calculations in the interaction picture fields $\hat{X}_0$ associated with the vacuum $|0\rangle$ with the $\hat{X}_{in/out}$ associated with the vacuum $|\Omega\rangle$ in terms of the Gell-Mann and Low theorem (§11.4).

In the remainder of this chapter I survey the difficulties that will be encountered before indicating how calculations proceed in the interaction picture fields $\hat{X}_0$ below, and analyze the semantic mimicry that often arises with regard to the physical significance of the procedure.

9.4.4 Asymptotic states

If we restrict attention to scattering systems where, in a sense that requires clarification, we suppose that the idealized initial and final states are free (i.e., in the asymptotic time limits), being modelled by well-separated non-interacting particles as supported by observation, a Fock space structure may be constructed for these idealized spaces of initial and final states. This means that we can describe the asymptotic, idealized physical initial 'in' and final 'out' states by well-understood free-field theories using Fock representations constructed from the $\hat{X}_{in}$ (and $\hat{X}_{out}$) satisfying free, linear Klein-Gordon equations.

This construction, and the interpretation of the relationships between the $\hat{X}_{in}$, the $\hat{X}$, and the $\hat{X}_0$ – all of which are required in scattering theory – introduces a number of difficulties. We now outline these difficulties, developing some of the observations in §9.3.2, to provide orientation for the discussion that follows in the remainder of this and the subsequent chapters.

First, as just outlined, calculation of a general state is given by the action of a (probably) divergent iterative series on an initial state so that at best an

approximate description of the state during interaction is obtained in actual calculations.

Secondly, it is assumed that the Hilbert space of the idealized asymptotic in/out states $\mathcal{H}_{in} = \mathcal{H}_{out}$ may be identified with the Hilbert space $\mathcal{H}$ of the full theory, a hypothesis known as asymptotic completeness (AC) (Duncan 2012, 267-268). AC is almost universally assumed, yet its validity remains a fundamental outstanding problem (§10.3.3). It is important however to distinguish between the equivalence of the Hilbert spaces and the ability to translate between or even construct physically meaningful Fock spaces on those Hilbert spaces.

Thirdly, conceptually the description of the asymptotic yet 'physical' 'in' and 'out' states (and associated fields) as free requires clarification. Motivated by physical observation, well-separated particles may be identified as non-interacting and hence free. However, such 'physical particles' in the asymptotic regions are understood to be always 'self-interacting'. Such self-interaction is 'absorbed' into the definition of the asymptotic states and fields, so they are modelled as free, without interaction or a property of charge.[18] Such ever-present self-interaction that is absorbed by the asymptotic fields is associated with renormalization which 'absorbs' the failure of the initial application of the Volkmann device in the limited context of idealized asymptotic states.

To enable asymptotic states to be modelled as free particles one introduces non-interacting asymptotic in/out free fields $\hat{X}_{in/out}$ that (heuristically, and naively) 'correspond to' the interacting fields $\hat{X}$ in the asymptotic limit. The $\hat{X}_{in/out}$ are free fields for which the self-interaction is absorbed by a shift in mass,[19] a procedure known as *mass renormalization*. The asymptotic correspondence between the $\hat{X}$ and the $\hat{X}_{in/out}$ is established as a weak convergence relationship using a *field strength renormalization* parameter (see the fourth point below and §11.3.1).

The renormalized asymptotic fields $\hat{\psi}^{\dagger}_{in}(x) = \hat{\psi}^{\dagger}_{out}(x)$, $\hat{\psi}_{in}(x) = \hat{\psi}_{out}(x)$ and $\hat{\varphi}_{in}(x) = \hat{\varphi}_{out}(x)$ satisfy (linear, free) Klein-Gordon equations with mass

[18] This suggests qualification to the sense in which one should regard the asymptotic states as 'physical'.
[19] See Coleman (2019, 205-210).

parameters shifted to the physical or measured (renormalized) value for free particles associated with the fields,[20] i.e.

$$\left(\partial^2 + m_{phys}^2\right)\hat{\varphi}_{in}(x) = 0$$
$$\left(\partial^2 + M_{phys}^2\right)\hat{\psi}_{in}(x) = 0$$
$$\left(\partial^2 + M_{phys}^2\right)\hat{\psi}_{in}^{\dagger}(x) = 0$$

Since the fields are considered free *ex hypothesi*, it is meaningless to seek to introduce a coupling between the fields. These fields simply model the phenomenological notion of freely propagating well-separated particles. However, these in/out fields satisfy the same Hamiltonian as the interacting fields and are defined relative to the same vacuum $|\Omega\rangle$, assuming AC. The *representation* of the Hamiltonian differs according to the fields used (i.e. $\hat{X}_{in}$ or $\hat{X}$), with the representation in terms of the asymptotic fields only valid in the asymptotic regions under the stipulation that the particle states modelled remain well separated.

Use of the $\hat{X}_{in}$ fields enables the construction of a Fock space for the asymptotic states using Fourier techniques via a_{in}-operators (etc.). So for $\hat{\varphi}_{in}$ for example:

$$\hat{\varphi}_{in}(x) = \hat{\varphi}_{in}(\boldsymbol{x}, t) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2\omega_k}\left(\hat{a}_{in}(\boldsymbol{k})e^{-ik\cdot x} + \hat{a}_{in}{}^{\dagger}(\boldsymbol{k})e^{ik\cdot x}\right)$$

where now $\omega_{\boldsymbol{k}} = \left(\boldsymbol{k}^2 + m_{phys}^2\right)^{1/2}$. So a particle interpretation can be developed for the asymptotic states as per chapter 8.

Fourthly, we need to relate $\hat{X}_{in}$ to $\hat{X}$ satisfying the coupled field equations to establish a scattering theory. But this is not straightforward, and the relationship is established in Haag-Ruelle and LSZ scattering theory as discussed in §11.3. One might be tempted to suggest (as is sometimes done 'heuristically') that

$\hat{X}(\boldsymbol{x}, t) \to \hat{X}_{in}(\boldsymbol{x}, t)$ as $t \to -\infty$

However, this is not the case. Difficulties arise since the action of $\hat{X}_{in}(\boldsymbol{x}, t)$ on an idealized asymptotic state is (when appropriately 'smeared') to introduce a single idealized, asymptotic physical particle localized around $\boldsymbol{x}$, whereas the

[20] What mass renormalization and consideration of the asymptotic states shows is that the mass parameters *m, M* in the free and interacting field equations in $\hat{X}_0$ or $\hat{X}$ do not correspond to the measured masses of the particles m_{phys}, M_{phys} associated with the corresponding fields. The mass shift is often infinite, which is one of the traditional worries regarding renormalization.

action of $\hat{X}(\boldsymbol{x}, t)$ is more complicated (cf. §11.2). In particular, the action of $\hat{X}$ on $|\Omega\rangle$ differs from the action of $\hat{X}_{in}$ on $|\Omega\rangle$. Ultimately, for calculational purposes, we shall require only that $\hat{X}|\Omega\rangle$ has non-zero overlap with single particle states $|p\rangle$ associated with $\hat{X}_{in}$, that is $\langle p|\hat{X}|\Omega\rangle \neq 0$, in which case $\hat{X}$ is known as an *interpolating field*. We shall see (§11.3) that the relationship between the $\hat{X}$ and the $\hat{X}_{in}$ is one of weak convergence according to

$$_{in}\langle\beta\big|\hat{X}_f(t)\big|\alpha\rangle_{in} \rightarrow Z_X^{\frac{1}{2}}{}_{in}\langle\beta\big|\hat{X}_{in,f}(t)\big|\alpha\rangle_{in},\ t \rightarrow -\infty$$

where $Z_X^{\frac{1}{2}}$ is the 'wave function' or 'field strength' renormalization factor, and for a suitable 'test function' f to generate a single particle wave-packet.

It is the interpolating fields $\hat{X}$ that are foundational for calculations in LSZ scattering theory, although they are underdetermined (cf. §11.3), and we must be cautious in how we understand the relationship between the $\hat{X}$ and the $\hat{X}_{in}$ as we shall see that, for example, the $\hat{X}_{in}$ support a particle concept on the physical idealized asymptotic states that cannot be prolonged to general interacting states while the $\hat{X}$ do not support any particle concept (§10.2).

Fifthly, to perform calculations in the interaction picture we need to relate the $\hat{X}_{in}$ to $\hat{X}_0$ as well as $\hat{X}$ and their associated Hilbert spaces since our expression for $\widehat{U}_I$ was constructed to enable calculations given in terms of the free, bare $\hat{X}_0$ that act on $\mathcal{H}_0$. That is, the actual evolution of the system is given by an operator acting on the full Hilbert space of interacting states $\mathcal{H}$ while the expression for $\widehat{U}_I$ is given in terms of operators acting on $\mathcal{H}_0$. The relationship is established using the Gell-Mann and Low theorem (§11.4) via the 'adiabatic hypothesis'.

The idea is that an 'adiabatic switching' term is introduced into $\widehat{H}_I'(t)$ so that the interaction is slowly 'turned off' in the infinite asymptotic limits. The adiabatically 'switched off' Hamiltonian is the original free Hamiltonian represented in the free $\hat{X}_0$ fields of the original masses acting independently on the bare vacuum $|0\rangle$ for large $|t|$. To do this an even function $s(t)$ is introduced such that $s(t)$=1 for $|t|$<T, for some large but finite T, and $s(t)$ = 0 for $|t|$>>T, with s slowly and smoothly varying monotonically between 0 and 1 in between. The adiabatically switched Hamiltonian is then:

$$\widehat{H} = \widehat{H}_0 + s(t)\widehat{H}'$$

The free asymptotic *bare* states are described by the free QFT for the relevant field types $\hat{X}_0$ with mass *m* or *M* and the free Hamiltonian $\hat{H}_0$, for which a natural (quanta/particle) description in terms of a Fock space has been constructed using Fourier techniques as in chapter 8. These 'bare' free particles, fields and states are 'abstracted' by the adiabatic switching, introduced for calculational convenience,[21] and do not represent the physical asymptotic particles associated with the $\hat{X}_{in}$ of measured mass.

It is supposed that the adiabatic switching term evolves the Hamiltonian sufficiently slowly such that the bare states associated with $\hat{X}_0$ may be 'traced' into the idealized physical (renormalized) asymptotic states associated with $\hat{X}_{in}$ and ultimately to the interacting states associated with $\hat{X}$ to meaningfully relate to the $\hat{X}_0$ states to the $\hat{X}_{in}$ and $\hat{X}$ states. It is supposed that the interaction term is fully 'switched on' when the particles are still very well separated such that they are considered not to interact with each other, i.e. as modelled by the renormalized asymptotic free fields and states associated with $\hat{X}_{in}$. Thus we need to distinguish 'bare' asymptotic states and 'physical', 'dressed' or 'renormalized' asymptotic states, which are both idealized.

However this 'tracing' is problematic since the three field species are, according to Haag's theorem, unitarily inequivalent. This means that, for instance, the particle description constructed from the $\hat{X}_0$ is 'incommensurable' with the particle description constructed from the $\hat{X}_{in}$, and we shall see on other grounds (nonlinearity of the coupled field equations) that in fact no particle description can be constructed from the $\hat{X}$ (§10.2). The purpose of application of the Gell-Mann and Low theorem will then not be to *translate between descriptions* of the state according to different field species, but to *support calculations* using the $\hat{X}_0$.

Finally, a scattering based account cannot be used to analyse bound states or unstable particles, for by definition bound states exist *as* bound states in the asymptotic limits and not as their constituents, whilst unstable particles do not have asymptotic states. A scattering based account is thus limited in its application, only applicable to specific physical situations involving scattering,

[21] That is, one assumes that the interaction can be 'abstracted away' to leave the bare field.

which are often more relevant in the laboratory (specifically, the particle accelerator) than to general quantum states in the world outside the laboratory (cf. Cartwright 1999). However, much of QFT and its application to understanding fundamental physics is concerned with scattering theory owing to its association with particle physics and the desire to discover fundamental particles and their interactions. Moreover, as we shall see in chapter 11 some of the tools developed in scattering theory are applied to the analysis of bound states.

The task is now to consider the interpretation, in relation to 'superposition' and semantic mimicry in particular, of the relationships between the field species $\hat{X}_0$, $\hat{X}_{in}$ and $\hat{X}$ as all of these are required in scattering theory. In the remainder of this chapter I consider the role of the $\hat{X}_0$ in iterative calculations in the interaction picture, indicating that they do not support a natural description of the interacting state. In chapter 10 I show that the $\hat{X}_{in}$ and $\hat{X}$ do not support natural descriptions of the interacting state either, before showing in chapter 11 that although none of the $\hat{X}_0$, $\hat{X}_{in}$ and $\hat{X}$ support descriptions of interactions, they do however support empirically successful calculations.

9.5 The S-matrix and Dyson's series

With these difficulties noted and placed to one side for now I consider the role of the $\hat{X}_0$ in performing scattering calculations via their use in Dyson's expansion as applied in the *S*-matrix. Bearing in mind the comments above, we should be careful, however, to distinguish between the *S*-matrix relating the physical asymptotic states in $\mathcal{H}_{in} = \mathcal{H}_{out}$, which we denote as *S,* and the *S*-matrix relating the bare asymptotic states in $\mathcal{H}_0$, which we denote S^0. What we want, and will develop in LSZ scattering theory (§11.3) is *S,* but it will ultimately be calculated in terms of S^0.

The *S*-matrix is defined as a map between the physical asymptotic in- and out-states,[22] so that

$$S_{\beta\alpha} = {}^{out}\langle\beta|S|\alpha\rangle^{in} = \langle\beta|\alpha\rangle$$

[22] The approach originated with Wheeler (1937) and Heisenberg (1943). See Coleman (2019, 138-140). The example that follows loosely follows Lancaster and Blundell, so we now adopt their normalization conventions.

where $|\alpha\rangle^{in}$ has a Fock space representation via the action of the $\hat{X}_{in}$ on $|\Omega\rangle$ interpreted as a state comprising of physical well-separated (free) particles, and with the S-operator defined as

$$S = \lim_{\substack{t_i \to -\infty \\ t_f \to \infty}} \widehat{U}(t_i, t_f)$$

without adiabatic switching.

If the Hamiltonian is now adiabatically switched then we define the S^0operator as above,[23] with the adiabatic switching included in the definition of $\widehat{U}$.[24] Then S^0 is a map between the bare rather than physical asymptotic states so that

$$S^0_{\beta\alpha} = {}^0\langle\beta|S^0|\alpha\rangle^0$$

where $|\alpha\rangle^0$ has a Fock space representation via the action of the $\hat{X}_0$ on $|0\rangle$ interpreted as a state comprising of bare well-separated (free) particles, and similarly for ${}^0\langle\beta|$. The point is then that since we have an iterative solution for $\widehat{U}_I$, i.e.

$$\widehat{U}_I(t, t_0) = 1 + \sum_{n=0}^{\infty} \frac{(-i)^n g^n}{n!} \int_{t_0}^{t} d^4x_1 \dots \int_{t_0}^{t_{n-1}} d^4x_n T\left[\hat{\psi}_0^\dagger(x_1)\hat{\psi}_0(x_1)\hat{\varphi}_0(x_1) \dots \hat{\psi}_0^\dagger(x_n)\hat{\psi}_0(x_n)\hat{\varphi}_0(x_n)\right]$$

and since

$$S^0 = \lim_{\substack{t_i \to -\infty \\ t_f \to \infty}} \widehat{U}_I(t_i, t_f)$$

then we may calculate S^0 iteratively via

$$S^0 = \lim_{\substack{t_0 \to -\infty \\ t \to \infty}} \left(1 + \sum_{n=0}^{\infty} \frac{(-i)^n g^n}{n!} \int_{t_0}^{t} d^4x_1 \dots \int_{t_0}^{t_{n-1}} d^4x_n T\left[\hat{\psi}_0^\dagger(x_1)\hat{\psi}_0(x_1)\hat{\varphi}_0(x_1) \dots \hat{\psi}_0^\dagger(x_n)\hat{\psi}_0(x_n)\hat{\varphi}_0(x_n)\right]\right)$$

The actual evaluation is performed for matrix elements $S^0_{\beta\alpha} = {}^0\langle\beta|\alpha\rangle^0$, interpreted as the amplitude $\mathcal{A}^0_{\beta\alpha}$ for an initial idealized bare asymptotic state $|\alpha\rangle_0$ to evolve, by scattering, to a final state $|\beta\rangle_0$, states that have natural (although idealized) descriptions in terms particle contents.

[23] This distinction is sometimes obscured in the literature.

[24] The adiabatic switching term is implicitly understood here without being explicitly stated, as is usual practice in the literature.

For example, consider two psions scattering off each other. The asymptotic initial bare state comprises two well-separated bare psions with momenta p_1 and p_2 :[25]

$$|\alpha\rangle_0 = |p_1 p_2\rangle = (2\pi)^3\left(4E_{p_1}E_{p_2}\right)^{\frac{1}{2}}\hat{a}^\dagger(p_1)\hat{a}^\dagger(p_2)|0\rangle$$

Similarly for the bare asymptotic final two psion state:

$$|\beta\rangle_0 = |q_1 q_2\rangle = (2\pi)^3\left(4E_{q_1}E_{q_2}\right)^{\frac{1}{2}}\hat{a}^\dagger(q_1)\hat{a}^\dagger(q_2)|0\rangle$$

These bare asymptotic states are 2-quanta sectors of Fock space for the free psion fields $\hat{\psi}_0^\dagger(x)$ and $\hat{\psi}_0(x)$ as per §8.3.5. The scattering amplitude is:

$$\mathcal{A}^0_{\beta\alpha} = S^0_{\beta\alpha} = {}_0\langle\beta|S^0|\alpha\rangle_0 = (2\pi)^6\left(16E_{p_1}E_{p_2}E_{q_1}E_{q_2}\right)^{\frac{1}{2}}\langle 0|\hat{a}(q_1)\hat{a}(q_2)S^0\hat{a}^\dagger(p_1)\hat{a}^\dagger(p_2)|0\rangle$$

Inserting the iterative solution for S^0, Dyson's expansion,[26]

$$\mathcal{A}^0_{\beta\alpha}$$

$$= (2\pi)^6\left(16E_{p_1}E_{p_2}E_{q_1}E_{q_2}\right)^{\frac{1}{2}}\Bigg[1$$

$$+\sum_{n=0}^{\infty}\frac{(-i)^n g^n}{n!}\int d^4x_1 \ldots \int d^4x_n T\left[\langle 0|\hat{a}(q_1)\hat{a}(q_2)\hat{\psi}_0^\dagger(x_1)\hat{\psi}_0(x_1)\hat{\varphi}_0(x_1)\ldots\hat{\psi}_0^\dagger(x_n)\hat{\psi}_0(x_n)\hat{\varphi}_0(x_n)\hat{a}^\dagger(p_1)\hat{a}^\dagger(p_2)|0\rangle\right]\Bigg]$$

This complicated expression is simplified by a result for *free fields* due to Wick, namely that for free fields $\hat{A}, \hat{B}, \hat{C}, \ldots \hat{Y}, \hat{Z}$:

$$\langle 0|T\left[\hat{A}\hat{B}\hat{C}\ldots\hat{Y}\hat{Z}\right]|0\rangle = \langle 0|T\left[\hat{A}\hat{B}\right]|0\rangle\langle 0|T\left[\hat{C}\hat{D}\right]|0\rangle\ldots\langle 0|T\left[\hat{Y}\hat{Z}\right]|0\rangle$$

$$+\langle 0|T\left[\hat{A}\hat{C}\right]|0\rangle\langle 0|T\left[\hat{B}\hat{D}\right]|0\rangle\ldots\langle 0|T\left[\hat{Y}\hat{Z}\right]|0\rangle$$

$$+\ldots$$

which may be stated that 'the VEV [vacuum expectation value] of a time-ordered string of operators is given by the sum of products of all possible combinations of VEVs of time ordered pairs' (Lancaster and Blundell, 173).

Many terms are obtained in application of Wick's result to Dyson's expansion, but a number of these do not contribute to the scattering amplitude. The non-zero terms arise from three kinds of expression, which are derived from

[25] Symmetrization is usually considered in the calculation of the Feynman diagrams later, and is not introduced at this point. We should keep in mind the cautions in §8.3.5 regarding the identification of *N*-quanta states *as N* quanta states. In scattering experiments the incoming particles may be identified as separate particles since they have distinct physical origins and so may be separated according to their wave-functions. It is usually practice to omit the test or 'smearing' functions at this stage.

[26] As is customary adiabatic switching terms are omitted.

and evaluated via the Fourier mode decomposition of the free fields as solutions to the (free) wave equations.

The first kind of expression arises from terms of the form $\langle 0|T[\hat{a}(q_1)\hat{a}^\dagger(p_1)]|0\rangle$ and is evaluated as simply

$$\langle 0|T[\hat{a}(q_1)\hat{a}^\dagger(p_1)]|0\rangle = \delta(q_1 - p_1)$$

The second kind of expression are terms of the form $\langle 0|T[\hat{\psi}_0(x)\,\hat{a}^\dagger(p_1)]|0\rangle$ which are evaluated as:

$$\langle 0|T[\hat{\psi}_0(x)\,\hat{a}^\dagger(p_1)]|0\rangle = \frac{1}{(2\pi)^{3/2}}\frac{1}{\left(2E_{p_1}\right)^{1/2}}e^{-ip_1\cdot x}$$

and similarly for other terms, with the appropriate sign in the exponential. These terms are interpreted as linking the bare asymptotic states with the interacting states.

The third kind of expression is given by terms of the form $\langle 0|T[\hat{\varphi}_0(x_1)\,\hat{\varphi}_0^\dagger(x_2)]|0\rangle$. This is the free Feynman propagator for the field, $\Delta^0_{F,\varphi}(x_1 - x_2)$ which is evaluated as

$$\langle 0|T[\hat{\varphi}_0(x_1)\,\hat{\varphi}_0^\dagger(x_2)]|0\rangle = \Delta^0_{F,\varphi}(x_1 - x_2) = \int \frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x_1-x_2)}\frac{i}{p^2 - m^2 + i\varepsilon}$$

commonly interpreted as associated with the exchange of a virtual phion-(anti)phion pair. Although this is a valid mathematical result, there are a number of difficulties with its physical interpretation, to which I now turn.

9.6 The Feynman propagator and 'virtual particles'

9.6.1 Two-point functions: Introduction

I'll first set the Feynman propagator in the context of some important conceptual tools that are related to particle concepts, namely Green's functions (cf. §5.2.2) and vacuum expectation values (VEVs). These are associated with propagators and correlation functions. I illustrate the concepts with the spin-0, charged, massive scalar field $\hat{\psi}_0(x)$ (§8.4) before considering the Feynman propagator in detail, which will indicate how virtual particles are introduced as the result of semantic mimicry.[27]

[27] Since a neutral scalar field is Hermitian, equivalent results follow, noting that $\hat{\varphi}_0^\dagger(x) = \hat{\varphi}_0(x)$ with the interpretation that each particle is its own antiparticle.

The action of $\hat{\psi}_0^\dagger(x)$ on the state is interpreted as introducing a quantum / removing an anti-quantum at *x*, whilst $\hat{\psi}_0(y)$ removes a quantum / introduces an anti-quantum at *y* (cf. §8.3.4).[28] So the action on the vacuum, $\hat{\psi}_0(y)\hat{\psi}_0^\dagger(x)|0\rangle$, for $x^0 < y^0$, represents a state where a particle is introduced at *x* and removed at *y*. The VEV for $\hat{\psi}_0(y)\hat{\psi}_0^\dagger(x)$, namely $\langle 0|\hat{\psi}_0(y)\hat{\psi}_0^\dagger(x)|0\rangle$, is interpreted as the amplitude for introducing (or emitting) a particle at *x* to the vacuum or ground state, then removing (or absorbing) it at *y*, restoring the vacuum. So this VEV is understood to represent the amplitude for a particle to propagate from *x* to *y* for $x^0 < y^0$. Alternatively, $\langle 0|\hat{\psi}_0^\dagger(y)\hat{\psi}_0(x)|0\rangle$ ($x^0 < y^0$) is the amplitude to introduce an antiparticle at *x* to the vacuum and remove it at *y*, restoring the vacuum: the amplitude for an antiparticle to propagate from *x* to *y* (cf. Lancaster and Blundell 2014, 155).

These VEVs are examples of two-point field correlation functions, or Wightman functions $W_0(y, x)$. Two related functions are the (free-field) Pauli-Jordan function:

$$\Delta^0(x - y) = \left[\hat{\psi}_0(x), \hat{\psi}_0^\dagger(y)\right]$$

which is a solution of the homogenous Klein-Gordon equation, and the (free) Feynman propagator $\Delta_F^0(x - y)$, which we now consider in detail. Note for future reference that these '2-point' functions can be extended to multi-point functions so that rather than modelling the propagation of a single particle, multi-point functions are introduced in interacting theories to model scattering, which brings considerable complications (cf. §11.3).

9.6.2 The free Feynman propagator

The Feynman propagator for the psion field is

$$\Delta_F^0(x - y) = \langle 0|T\{\hat{\psi}_0(x)\hat{\psi}_0^\dagger(y)\}|0\rangle$$

where *T* denotes the time-ordered product of the fields:

$$\langle 0|T\{\hat{\psi}_0(x)\hat{\psi}_0^\dagger(y)\}|0\rangle$$
$$= \Theta(x^0 - y^0)\langle 0|\hat{\psi}_0(x)\hat{\psi}_0^\dagger(y)|0\rangle + \Theta(y^0 - x^0)\langle 0|\hat{\psi}_0(y)\hat{\psi}_0^\dagger(x)|0\rangle$$

[28] From hereon we follow standard practice and refer to 'particles' rather than 'quanta', recognizing that this is shorthand for quanta integrated against wave-functions to obtain a localized particle wave-packet.

using the Heaviside function $\Theta(z)$ where

$$\Theta(z) = \begin{cases} 1 & z \geq 0 \\ 0 & z < 0 \end{cases}$$

It is sometimes denoted $G_F^0(x-y)$ or $G_F^0(x,y)$.

The (free) Feynman propagator is a Green's function for the Klein-Gordon equation:

$$(\partial^2 + m^2)\Delta_F^0(x-y) = -\delta^4(x-y)$$

One can show that:

$$\Delta_F^0(x-y) = \langle 0|T\{\hat{\psi}_0(x)\hat{\psi}_0^\dagger(y)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{p^2 - m^2 + i\varepsilon}$$

This derivation requires the construction and calculation of an integral which is achieved through representation of the Heaviside function on the complex p^0-plane, so that the integral can be calculated from a contour on the complex plane. This requires the introduction of 'small contour displacements' around poles implemented by the addition of the $i\varepsilon$ terms, which are, it is understood, ultimately taken to zero even though this is not usually stated explicitly.[29] The contour displacements are chosen to implement the Feynman boundary conditions, i.e. to give the Feynman propagator.[30] This procedure does however introduce semantic mimicry regarding the interpretation of p as we now see.

We have[31]

$$\Delta_F^0(x-y) = \Theta(x^0 - y^0)\langle 0\big|\hat{\psi}_0(x)\hat{\psi}_0^\dagger(y)\big|0\rangle + \Theta(y^0 - x^0)\langle 0\big|\hat{\psi}_0(y)\hat{\psi}_0^\dagger(x)\big|0\rangle$$

$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_p} e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \left(e^{iE_p(x_0-y_0)}\Theta(y^0 - x^0) + e^{-iE_p(x_0-y_0)}\Theta(x^0 - y^0) \right)$$

So far so good, since the variables represent physical quantities. However, the Heaviside functions appear non-covariant, and are tricky to work with, so appeal is made to the purely mathematical result that, introducing a dummy variable z that has no physical significance, on the complex z-plane,

$$\Theta(x^0 - y^0) = \lim_{\varepsilon \to 0} i \int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{e^{-iz(x_0-y_0)}}{z + i\varepsilon}$$

[29] This is a standard application of Cauchy's theorem for contour integration in the complex plane.
[30] Other displacements may be used to give different Green's functions.
[31] Following Lancaster and Blundell (2014, 158-159); Schwartz (2014, 75-77).

It is important to stress that z does not represent a physical quantity. The next move in the derivation of the propagator is to make the substitution $z' = z + E_p$, which again is a purely mathematical device, and write $z' = p^0$. Again this is for mathematical convenience but crucially p_0 *is not a physical quantity anymore* but a 'dummy variable' once we substitute this representation of the Heaviside function in the expression for the Feynman propagator with the variables so defined. Now define $p = (z', \mathbf{p}) = (p^0, \mathbf{p})$. Although $\mathbf{p}$ is still physical 3-momentum, neither p_0 nor p are the physical quantities as previously defined. This means that $(p^0)^2 \neq \mathbf{p}^2 + m^2$, although one can write $E_p^2 = \mathbf{p}^2 + m^2$ with E_p a function of the physical quantities $\mathbf{p}$ and m rather than the new p.

The substitutions allow us to rewrite:

$$\Delta_F^0(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{(p^0)^2 - E_p^2 + i\varepsilon}$$

but where p^0 and p are now 'dummy variables' in the sense just indicated. They are not used in their usual sense of representing physical quantities. Since $E_p^2 = \mathbf{p}^2 + m^2$ (which still bears a physical interpretation as just indicated) the denominator may be written mathematically in terms of the dummy variables p^0 and p as $(p^0)^2 - E_p^2 + i\varepsilon = p^2 - m^2 + i\varepsilon$ to give the final result,

$$\Delta_F^0(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{p^2 - m^2 + i\varepsilon}$$

where p is no longer the physical 4-momentum. However, p often appears to retain its interpretation as the 4-momentum by semantic mimicry, which leads to conceptual confusion, especially with regard to 'virtual particles' as we consider in a moment. Interpretative difficulties continue, for the Fourier transform of $\Delta_F^0(x-y)$, namely:

$$\tilde{\Delta}_F^0(p) = \frac{i}{p^2 - m^2 + i\varepsilon}$$

is interpreted as the free Feynman propagator in p-space or 'momentum space'.[32] But this is misleading, for the Fourier transform pair (x, p) no longer corresponds to physical space-time and physical 4-momentum coordinates in which $p^0 = (\boldsymbol{p}^2 + m^2)^{1/2}$ supports a physical interpretation. Instead the coordinates are

[32] E.g. Greiner and Reinhardt (1996, 100-115).

physical space-time x and 'p-space' where now p has a physical 3-momentum component ***p,*** and the unphysical dummy variable p^0.

So, owing to this semantic mimicry one should not interpret

$$\Delta_F^0(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{p^2 - m^2 + i\varepsilon}$$

as the superposition of momentum space propagators. It is a convenient mathematical expression, with mathematical rather than physical semantic architecture.

9.6.3 Virtual particles

However, and especially in the context of Feynman diagrams that I shall consider below, when the mass-shell condition is not satisfied $\tilde{\Delta}_F^0(p)$ is interpreted as the momentum space propagator for a 'virtual particle' of 4-momentum p. This leads to further confusion regarding the existence of 'off mass-shell' particles and the failure to conserve energy during interaction, interpreted as exchange of virtual particles (Lancaster and Blundell 2014, 159-161; cf. §7.4). That is, 'ordinary' particles are associated with the propagator

$$\tilde{\Delta}_F^0(p) = \frac{i}{p^2 - m^2 + i\varepsilon}$$

in the special case when *p does* represent physical 4-momentum, and so satisfies the mass-shell condition.[33] Virtual particles are, by prolongation, associated with the same propagator when p does not represent physical 4-momentum, as is necessitated in the evaluation of Feynman diagrams.

The 'virtual particle' concept is one of the particle concepts that Falkenburg considers (2007, 233-238). It reflects an attempt to appropriate a particle concept for interactions, but does so via semantic mimicry by falsely attributing a physical interpretation to the p-space propagator

[33] An important deduction is made from the form of the propagator. That is, one notes that the mass of the particle associated with the field is given by the location of the pole of the propagator, considered as a function on the complex plane. This means that if one is able to deduce by some means an expression for a propagator of a field, its mass may be inferred from the location of the pole. This is important in regard to the identification of quasiparticles in some contexts, and in regard to the identification of renormalized particles, or renormalization conditions in interacting theories (cf. §11.2). See Lancaster and Blundell (2014, 276-279) for an overview of the importance of this observation; Schwartz (2014, 330-333) for further discussion noting complications in relation to renormalized mass.

$$\tilde{\Delta}_F^0(p) = \frac{i}{p^2 - m^2 + i\varepsilon}$$

The shift in semantic architecture of the Fourier transform pair has not been noticed. The adjective 'virtual' is applied in recognition that $p^0 \neq (\boldsymbol{p}^2 + m^2)^{1/2}$, which should alert one to the possibility of semantic mimicry in that one may have a mathematical artefact without direct physical significance.

However, rather than recognizing that this *p*-space propagator is now a mathematical artefact arising from the introduction of a dummy variable in order to enable an integral to be evaluated, often the *p*-space propagator is taken as indicative of a somewhat mysterious new type of physical particle concept that exists only in the intermediate stages of interactions (cf. §7.4) used to 'explain' interactions in a causal-mechanical sense.

Indeed, a whole explanatory architecture arises to account for virtual particles, in particular appeal to the so-called 'energy-time uncertainty relationship' (ETUR) which suggests that energy (given via p^0, the dummy variable) is 'borrowed' for the short time required for particles to interact via the exchange of virtual particles.[34]

There is nothing mysterious about terms such as $\frac{i}{p^2-m^2+i\varepsilon}$ representing exchanges of 'off mass-shell' virtual particles, for the integral over such terms is a mathematical device that does not support the physical interpretation of individual terms in general. Endowing individual terms with a physical interpretation arises from semantic mimicry unless there is some physical justification for some particular value of *p*.

[34] See Wick (1938) and e.g. Lancaster and Blundell (2014, 159-160) for a recent overview of the common interpretation. Difficulties with the interpretation have been raised, e.g. Bunge (1970) and Fox (2008). For detailed discussion of ETUR see Busch (1990a&b; 2008) and Hilgevoord (1996; 1998; 2005). Appeal to the ETUR seems problematic since the Heisenberg uncertainty relations are construed in terms of operators, and time is not an operator but a parameter. However, the Heisenberg uncertainty relations are 'localization' relationships between Fourier transform pairs of variables (Stein and Shakarchi 2003, 158-161). So as *t* and p^0 are corresponding variables of a Fourier transform pair a p^0 / *t* localization relationship may be deduced mathematically as satisfying the Heisenberg relationship in its (mathematical) Fourier transform context. But this does not support physical interpretation in terms of 'energy borrowing' as p^0 is a dummy variable.

9.7 Feynman diagrams, interactions and renormalization

9.7.1 Feynman diagrams in the context of Dyson's series

We have a series expansion for the 'matrix elements' $\mathcal{A}^0_{\beta\alpha}$ of the scattering amplitude $\mathcal{A}^0$, that is, expressions for amplitudes for all possible scattering scenarios in the theory for bare, idealized asymptotic states using the adiabatic hypothesis. The amplitudes are each given by the Dyson series expansion of the S^0-matrix folded in with the given in/out states. Each term of the resulting expression is simplified using Wick's result, for which the resulting terms can be 'visualized' as a Feynman diagram via the Feynman propagators represented as lines which join at nodes.[35]

In this context Feynman diagrams are understood to model interactions via the exchange of virtual particles. It is then often said that the interaction that results in scattering can be understood as, or explained as, a 'superposition' of the putative processes of virtual particle interactions represented by the Feynman diagrams (cf. Falkenburg 2007, 236-238; Teller 1995, 140-142; Weingard 1988, 43-58; cf. 1982).[36] Such interpretation is an example of semantic mimicry, as we have begun to see with regard to appeal to the 'virtual particle' concept.

Let $\mathcal{A}^{(n)}$ denote the *n*th-order term in the scattering amplitude for the case under consideration. The second-order term contains only two non-zero terms:

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[35] Introduced in Feynman (1949a&b).

[36] Discussion of virtual particles and Feynman diagrams in the philosophical literature has generally been conducted with reference to realism debates, with a realist stance usually rejected, although some authors adopt thin forms. See Fox (2008) for a summary of arguments against realist interpretation, and Valente (2011) for a response to Fox advocating a 'thin realism' perhaps comparable with Falkenburg (2007). Valente argues that we can see 'virtual quanta as an explanatory nexus – through an extension of the concept of quanta – of the quantized exchange of energy and momentum between real particles even if it turns out to be an *intricate one*' (49), concluding that virtual quanta have epistemic relevance which goes beyond the 'formal tools' reading (50-51). Falkenburg argues that '*infinitely many* virtual particles *together* may be considered to cause a *real collective effect. …* The transition probability stems from *all* virtual field quanta involved in the superpositions of the relevant lowest and higher order Feynman diagrams.' (237) Alternatively Kuhlmann concludes that 'a realistic interpretation of Feynman diagrams is excluded in a substance ontology while it seems possible in a process ontology.' (2010, 131) I simply note that much discussion involves a faulty understanding or application of 'superposition' (cf. Falkenburg and Fox). It is the question of the applicability of 'superposition' that I develop in detail in the Wilsonian perspective, which ultimately suggests that a realist interpretation of virtual particles and Feynman diagrams is untenable, resulting from semantic mimicry.

$$\mathcal{A}^{(2)} = \frac{-(2\pi)^6}{2}.g^2.\left(16E_{p_1}E_{p_2}E_{q_1}E_{q_2}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}d^4x_1\int_{-\infty}^{\infty}d^4x_2$$

$$\left(\langle 0|T[\hat{a}_{q_1}\hat{\psi}^\dagger(x_1)]|0\rangle\langle 0|T[\hat{a}_{q_2}\hat{\psi}^\dagger(x_2)]|0\rangle\langle 0|T[\hat{\psi}(x_1)\hat{a}^\dagger_{p_1}]|0\rangle\langle 0|T[\hat{\varphi}(x_1)\hat{\varphi}(x_2)]|0\rangle\langle 0|T[\hat{\psi}(x_2)\hat{a}^\dagger_{p_2}]|0\rangle\right.$$

$$\left.+\langle 0|T[\hat{a}_{q_1}\hat{\psi}^\dagger(x_2)]|0\rangle\langle 0|T[\hat{a}_{q_2}\hat{\psi}^\dagger(x_1)]|0\rangle\langle 0|T[\hat{\psi}(x_1)\hat{a}^\dagger_{p_1}]|0\rangle\langle 0|T[\hat{\varphi}(x_1)\hat{\varphi}(x_2)]|0\rangle\langle 0|T[\hat{\psi}(x_2)\hat{a}^\dagger_{p_2}]|0\rangle\right)$$

The *formal* interpretation of $\mathcal{A}^{(2)}$ is straightforward in terms of the propagators discussed above, which have 'inherited' physical interpretation from the 1930s treatments of virtual particle exchange (cf. chapter 7).

The key interaction term in $\mathcal{A}^{(2)}$ is $\langle 0|T[\hat{\varphi}(x_1)\hat{\varphi}(x_2)]|0\rangle$. It is interpreted as the *free* Feynman propagator representing the exchange of a virtual phion-(anti)phion pair between x_1 and x_2 as we saw above. Such terms represent 'internal lines' of the Feynman diagrams as in the diagrams below. The other two-point VEVs are interpreted as introducing or removing asymptotic, idealized bare particles of the free theory to or from the scattering process (cf. §9.5). These are the 'external legs' of the Feynman diagrams. Within each of the products each VEV can be visualized as a line, joined at nodes (representing a point-like coupling of the fields) where the arguments of the field functions agree.

The two nonzero terms in $\mathcal{A}^{(2)}$ can be visualised as Feynman diagrams:

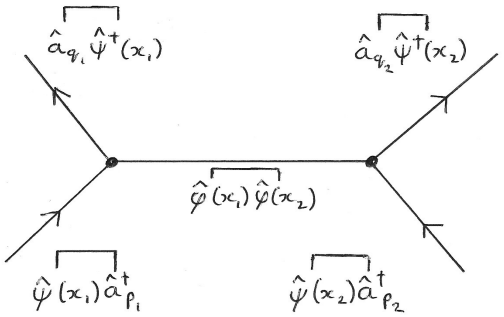

Fig. 9.1 The first term – the t-channel process

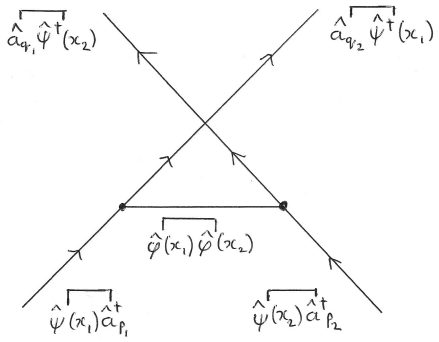

Fig. 9.2 The second term – the u-channel process

The story told is that the diagram represents two incoming psions that interact by the exchange of a virtual phion-(anti)phion pair between x_1 and x_2, and then separate. The psion coupling is restricted to these two points only, the propagation being free otherwise. However, this 'process' is integrated over all x_1 and x_2, sometimes interpreted as a 'superposition' of exchange processes that contribute to the scattering. The amplitude for the process can then be calculated from the expressions above.

However, for the simple internal line depicted in the two diagrams the (dummy) p value in the explicit form of the propagator $\langle 0 \big| T\big[\hat{\varphi}_0(x_1)\, \hat{\varphi}_0^\dagger(x_2)\big] \big| 0\rangle = \Delta^0_{F,\varphi}(x_1 - x_2)$ is fixed by the incoming and outgoing momenta at some off mass-shell value q applying momentum conservation at the nodes.[37] We do not require integration over all (dummy) p, so that here

$$\langle 0 \big| T\big[\hat{\varphi}_0(x_1)\, \hat{\varphi}_0^\dagger(x_2)\big] \big| 0\rangle = \Delta^0_{F,\varphi}(x_1 - x_2) = \frac{i}{q^2 - m^2 + i\varepsilon}$$

The usual interpretation is that this line then represents the exchange of a virtual phion pair of off mass-shell momentum q. But such interpretation of the propagator in p-space involves semantic mimicry (see above), so this expression in q is a mathematical artefact and does not represent a physical process.

The diagrams depicted only represent the second-order 'processes' in g, indicated diagrammatically in that there are two nodes to the Feynman diagrams. Higher-order 'processes', that (recall) are simply correction terms in an iterative series expansion, must be added. I shall not set out the lengthy mathematical expressions of these, but instead note the character of the Feynman diagrams that arise from the mathematics and the implications.

9.7.2 Renormalization

The order of the 'process' (correction term) corresponds to the number of nodes in its associated Feynman diagram. The scattering amplitude calculation formally requires summation of the $\mathcal{A}^{(n)}$ to all orders. In practice, the calculation is truncated after a few terms on the assumption that g is small, so that the

[37] q=q_1-p_1 in the t-channel process and q=q_1-p_2 in the u-channel.

contribution from the higher terms rapidly diminishes.[38] However, a serious difficulty emerges in that the Feynman diagrams with 'loops' that arise at higher orders lead to divergent integrals, at any given order, since the *p*-values in the loops can take arbitrarily high values, and the integrals diverge with *p*. So for example one of the $\mathcal{A}^{(4)}$ diagrams will consist in one of the above diagrams but with the simple internal phion line replace with the loop diagram:

Fig. 9.3 Simple loop diagram on the internal phion propagator

In this case the loop momenta can take any value in *p*-space, with the loop interpreted as a virtual psion-antipsion process. But this gives a divergent integral to be evaluated in the calculation of the scattering amplitude.

This is an indication for the need for, and is corrected by, renormalization, a mathematical procedure applied to cancel the divergences and arrive at empirically correct results. Renormalization compensates for the failure of the Volkmann device as far as the ability to perform calculations is concerned.[39] The

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[38] Dyson's series is an 'asymptotic series' in the sense that often the summation of the first few terms provides a good approximation despite the series diverging.

[39] Cf. §10.5. Renormalization involves the addition of 'counter-terms' to the Lagrangian, chosen to cancel exactly the divergences encountered in the integrals and force the calculated outcomes to agree with empirically obtained results in simple experiments. In the scalar Yukawa theory there are six renormalization conditions required to cancel divergences to all orders and force empirically adequate results: The Lagrangian of the non-renormalized theory is:

$$\mathcal{L} = \partial^\mu \hat{\varphi} \partial_\mu \hat{\varphi} - m^2 \hat{\varphi}^2 + \partial^\mu \hat{\psi}^\dagger \partial_\mu \hat{\psi} - M^2 \hat{\psi}^\dagger \hat{\psi} - g \hat{\psi}^\dagger \hat{\psi} \hat{\varphi}$$

Define the renormalized Lagrangian $\mathcal{L}_{ren}$ using renormalized fields $\hat{\varphi}'$, $\hat{\psi}'^\dagger$, $\hat{\psi}'$ and counterterms

$$\mathcal{L}_{ren} = \mathcal{L} + \mathcal{L}_{ct}$$

where

$$\mathcal{L}_{ct} = A\hat{\varphi}' + B\partial^\mu \hat{\varphi}' \partial_\mu \hat{\varphi}' - C\hat{\varphi}'^2 + D\partial^\mu \hat{\psi}'^\dagger \partial_\mu \hat{\psi}' - E\hat{\psi}'^\dagger \hat{\psi} - F\hat{\psi}'^\dagger \hat{\psi}' \hat{\varphi}'$$

and {*A,B,C,D,E,F*} are parameters determined by the renormalization conditions:

1. $\langle \Omega | \hat{\varphi}' | \Omega \rangle = 0$ fixes A
2. $\langle q | \hat{\varphi}'(0) | \Omega \rangle = 1$ fixes B, where $|q\rangle$ is a single phion state of momentum q
3. The physical phion mass, m_{phys} fixes C
4. $\langle p | \hat{\psi}'(0) | \Omega \rangle = 1$ fixes D, where $|p\rangle$ is a single (anti)psion state of momentum p
5. The physical psion mass, M_{phys} fixes E
6. The physical definition of charge g fixes F, although this is dependent upon the renormalization point of the total momenta of the scattering states, in older approaches to renormalization at least (the analysis of the role of the parameters is more complicated, but more illuminative, in the newer renormalization group approaches).

The B and D parameters relate to the 'wave function renormalization' parameters Z_φ and Z_ψ of $\hat{\varphi}'$ and $\hat{\psi}'$, the others to mass and charge renormalization. By determining the parameters using experiments calculations can be performed using the new species $\hat{X}_0'$ at any order without divergences – see Coleman (2019, 300-343) for details. The notion of 'renormalizability' of a theory, and its necessity, is not straightforward and there are various senses in which a theory

compensation is partial in the sense that renormalization supports calculations, but does not support the ability to identify fields and states that take the same form individually and in combination, so that a description of interacting states can be given. Renormalization does not construct or identify isolated fields that appear in the Lagrangian and field equation with finite, constant parameters that take the same form individually and in combination.

9.7.3 The failure of Feynman diagrams to describe interacting states

I am developing the argument that QFT does not support a particle description of interacting states, or explanations of interactions in terms of particle processes owing to the failure of 'superposition' in the initial application of the Volkmann device, which is further associated with the failure of 'superposition' to apply to the coupled wave equations so that they cannot support the Fock decomposition of the state.

However, it might be objected that a particle description of interacting states is physically supported by Feynman diagrams in the interaction picture, as inherited from the Fock structure of the $\hat{X}_0$. By Haag's theorem such a particle description would be unitarily inequivalent to (and so perhaps incommensurable with, to use Ruetsche's term) any particle description associated with the $\hat{X}_{in}$ or the $\hat{X}$ (cf. §10.2.2). But appeal to Haag's theorem does not rule out a particle description of interacting states in terms of the $\hat{X}_0$. I am developing the stronger claim here that there is *no* particle description of interacting states via the $\hat{X}_0$ by considering the semantic architecture of the Feynman diagrams as they arise from Dyson's expansion as applied to the asymptotic states. In §10.2 I show that there is no particle description, in general, in terms of the $\hat{X}_{in}$ or the $\hat{X}$ either by considering the inapplicability of 'superposition' to interactions in QFT.

The claim that Feynman diagrams do not represent interacting states via the $\hat{X}_0$ is established, even setting aside renormalization issues, by

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may be renormalizable. See e.g. Weinberg (1995, 516-525) for discussion of whether renormalizability is necessary in the traditional sense. Traditionally, 'physical' QFTs were required to be renormalizable in the sense that the divergences at all orders could be cancelled using a finite number of parameters as 'counter-terms'. Conceptual improvements have been gained via the application of so-called 'renormalization group' techniques, although these still involve the addition of counter-terms to the Lagrangian. See Cao (1993); Butterfield and Bouatta (2014) for the development of renormalization techniques, with philosophical discussion.

demonstrating that they arise in the context of a mimic of the Fourier technique for which 'superposition' is inapplicable. We do not have the proper application of the Fourier technique in the use of Dyson's series, having a mimic instead for which 'superposition' may be (improperly) applied, leading to the false conclusion that Feynman diagrams represent interacting states. The conclusion – that Feynman diagrams do not represent interacting states – is not novel, and reflects the consensus view that has emerged in the philosophical literature, which is usually framed in terms of realism rather than natural description.

The application and interpretation of Feynman diagrams with regard to Dyson's expansion stands in direct continuity with and represents the completion of the program(s) of the 1930s in which particle interactions are understood in terms of virtual particle exchanges, a picture which sits comfortably within a causal-mechanical framework of explanation. Perhaps Feynman diagrams inherited a 'realist' interpretation from these models of the 1930s and the physical interpretations with which they were endowed, even though Feynman (and Dyson) were more measured in their physical interpretation of the diagrams. A realistic interpretation would seem supported, prima facie, by bubble chamber tracks apparently indicating particle interactions that look somewhat like Feynman diagrams. So apparently there is phenomenological support for a realistic interpretation of Feynman diagrams and the explanations that they appear to offer. But Feynman diagrams are established on different patches of theory façades – their semantic support differs in different applications. [40]

To summarize the situation here, according to the common realistic interpretation of terms in Dyson's series as applied to the calculation of scattering amplitudes via Feynman diagrams, an interaction is understood in terms of a (double) 'infinite superposition' of discrete emission / absorption processes that punctuate free propagations of particles. That is, for a single process integration over space-time is required for every vertex location,

[40] See Passon (2019); Kaiser (2005). There are different contexts of use of Feynman diagrams in which their semantic architecture differs – comparison with bubble chamber tracks is often misleading. An interesting case meriting further analysis concerns the role of Feynman diagrams in the context of perturbative expansions in polymer field theory where Sam Edwards claimed 'polymers are their own Feynman diagrams' (Goldenfeld 2016, 10; cf. Edwards 1978, 285).

supposedly giving rise to one 'superposition' of like diagrams, and then every process at every order is summed, giving rise to another putative 'superposition' of processes. In both cases summation is incorrectly interpreted as superposition. This story is misleading as indicated by analysing the semantic support of the expansion for $\mathcal{A}$ via Dyson's series for $\widehat{U}_I$.

We have a Fock space representation of the bare, free asymptotic in-states, which can be considered as Fourier decompositions of the states, exploiting a natural descriptive opportunity via the $\hat{X}_0$. So far so good. The difficulty arises when we consider the evolution of the initial bare asymptotic states, which we calculate via the iterative expansion for $\widehat{U}_I$. That is, we wish to consider the evolution of the state or the representation of

$$|\sigma_I(t)\rangle = \widehat{U}_I(t,-\infty)|\sigma_I(-\infty)\rangle = \widehat{U}_I(t,-\infty)|\alpha\rangle_0$$

where $\widehat{U}_I$ satisfies

$$i\frac{\partial}{\partial t}\widehat{U}_I(t,t_0) = \widehat{H}'_I(t)\widehat{U}_I(t,t_0)$$

If we could solve for $\widehat{U}_I$ explicitly and represent it in diagonal form with respect to a basis of its eigenfunctions, a basis which must also be the Fock basis for the bare asymptotic states in order to apply the two aspects of Fourier's technique, then we could support the same kind of interpretation for the system as repeatedly achieved in chapter 4 in terms of 'superposition'. That is, we would have a natural description of the system which here would be a particle description for which an initial (bare) particle state evolved as a superposition of particle states by 'Hilbert superposition', eventually resolving into bare asymptotic final out-states as a superposition of particle states.

But we do not have this, and any such interpretation is an instance of semantic mimicry for the expansion for $\widehat{U}_I$ is not a Fourier solution for which individual terms have physical significance associated with persisting partial states via Hilbert superposition. We cannot solve explicitly for $\widehat{U}_I$, let alone calculate its eigenfunctions, even if we know in principle that they exist. What we have instead is an iterative series expansion for $\widehat{U}_I$. So, when applied to the state, i.e. as $\widehat{U}_I(t,-\infty)|\alpha\rangle_0$ individual terms of the resulting series are not possible states traced through the system. This is because the individual terms represent correction terms to lower-order approximations, and so they are unphysical and

superposition does not apply as in the classical examples considered that also mimic 'superposition' (cf. §4.5; §5.3). So we do not have an application of the Fourier technique to support 'superposition'.

There is therefore no natural physical description of the system during interaction as might be obtained via individual terms in Dyson's series in terms of superpositions of particles. It is the individual terms of Dyson's series that are interpreted by or represented as Feynman diagrams, and so the Feynman diagrams do not support 'superposition' or a natural description of the interacting system as representing physical processes or supporting explanations of interactions. So whilst the S^0-matrix maps superpositions of Fock basis states to superpositions of Fock basis states to give amplitudes for given processes, there is nothing in the calculation of the matrix elements to support any physical interpretation or explanation of the iterative series used to calculate the amplitudes, and nothing is said about the nature of the intermediate states or processes. The S^0-matrix simply provides the amplitudes for particular outcomes given a particular initial state.

Three final observations are worth making: First, in the context that we have considered Feynman diagrams provide a notational opportunity to visualize a complicated mathematical approximation method that greatly simplifies its application and calculation.

Secondly, Feynman diagrams *might* be said to offer a reasoning advantage in that regard, and in regard to 'explanations' of the success of extremely successful calculation in, for instance, the calculation of the Lamb shift and the magnetic moment of the electron (cf. Peskin and Schroder 196-198; Schweber 1994). It is an interesting question, although not one that I shall pursue, of just *what kind* of explanation they offer, which might be compared with other iterative solutions to differential equations in which the individual terms represent corrections to cruder approximations that identify physical trends with reference to some parameter(s), and so have *indirect* (rather than representational) physical significance and explanatory power in this narrower sense (cf. Holmes 2013, 1-46).[41] That is, Feynman diagrams may assist in

[41] Cf. Meynell (2008; 2018) on the interpretation of Feynman diagrams in relation to understanding. Falkenburg suggests that for all practical purposes it is possible to single out 'the

supporting explanations and offer a reasoning advantage, but not because they offer a natural or true description.

Finally, in general, Dyson's series diverges, being an asymptotic series (Duncan 2012, 376). That is, summation of the first few terms provides a good approximation to the desired solution, before diverging after a finite number of terms, even after renormalization.[42] But if the coupling is too strong, the procedure fails, and alternative methods must be employed, as in QCD.[43]

9.8 Conclusion

I have shown how interactions are introduced perturbatively to the free theories of chapter 8 by implicit but improper use of the Volkmann device as a generalized superposition principle, outlining a number of difficulties.

I have indicated how S^0-matrix elements are calculated, subject to renormalization, whilst highlighting the conceptual difficulties with the interpretation of the solution method. The Feynman diagrams obtained do not support natural descriptions (or realist interpretations) of interacting systems. In particular, the explicit mathematical form of the Feynman propagator arising in the evaluation of the Feynman diagrams is incorrectly given a physical interpretation in terms of virtual particles owing to semantic mimicry.

However, I have not addressed the question of how to relate the S^0 matrix which associates idealized bare, free asymptotic states associated with the $\hat{X}_0$ under the assumption of the adiabatic hypothesis to the S matrix mapping between idealized physical free asymptotic states associated with the $\hat{X}_{in}$. This I shall do in chapter 11, but first, having shown that the $\hat{X}_0$ do not support particle descriptions of interacting states, I show that neither the $\hat{X}_{in}$ nor the $\hat{X}$ do either in chapter 10, owing to the inapplicability of 'superposition'.

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real effect of a single Feynman diagram (or virtual field quantum)' such as in Lamb shift and gyromagnetic ratios (2007, 237-8). But this is misleading – we should say that use of the lowest-order correction term to the linear approximation (i.e. lowest-order correction to a Coulomb potential for instance) refines an already very good approximation in terms of some physical parameters, just like correction terms in a perturbative solution to a classical DE.

[42] Cf. Miller (2018); J.D. Fraser (2020).

[43] A further complication arises in that other iterative solution techniques can be employed to approximate $\hat{U}_I$. In particular the Magnus expansion (Magnus 1954) has been developed as an alternative iterative solution procedure in QFT that has some advantages over Dyson's series. See Blanes, Casas, Oteo and Ros (2009) for discussion.

Chapter 10

Conceptual analysis of perturbative and non-perturbative QFT in relation to the applicability of 'superposition'

10.1 Introduction and overview

In the previous chapter I described the construction of an interacting scalar Yukawa theory via the introduction of a perturbation $\widehat{H}' = g\hat{\psi}^{\dagger}(x)\hat{\psi}(x)\hat{\varphi}(x)$ to the free Hamiltonian $\widehat{H}_0$ represented in terms of the scalar fields $\hat{\psi}^{\dagger}(x), \hat{\psi}(x), \hat{\varphi}(x)$. In this construction it is supposed that such physically meaningful fields, and partial states associated with them, can be identified that take the same form individually (i.e., as if they were free fields and states) and in combination (i.e., in the interacting theory). I began to show how this reflects application of the Volkmann device, which goes unnoticed, perhaps owing to lingering ontological commitments that would support such an assumption. We saw that although the mathematical architecture associated with introducing an interaction via gauge theory differs from the introduction of an interaction term as a 'mere' perturbation, both approaches in fact share the same initial modelling assumption of the Volkmann device, which leads to conceptual confusion owing to semantic mimicry regarding the physical interpretation of the fields and associated (putative) states, as we shall see in more detail in this chapter. The need for, and difficulties associated with, renormalization in both cases reflects the failure of the initial use of this device. I shall call QFTs constructed in these ways 'perturbative QFTs'.[1]

Although Haag's theorem is often considered a central difficulty for the development of interacting QFTs, we shall see that the failure of the Volkmann device is a more fundamental difficulty, related to the conceptually problematic nature of the perturbative assumption and the resulting nonlinearity of the coupled field equations. Renormalization is required to obtain empirically adequate results to compensate for the failure of the Volkmann device in

[1] Usage of 'perturbative QFT' is inconsistent. It is sometimes used in the more restrictive sense of QFTs utilizing the iterative series expansions as distinct from, e.g. lattice field theory (cf. §9.2).

conjunction with the use of three different species $\hat{X}$, $\hat{X}_0$ and $\hat{X}_{in}$ of each field type $\hat{\varphi}, \hat{\psi}, \hat{\psi}^\dagger$. In this chapter then I shall demonstrate that there is no particle description of a general interacting state in terms of any of the species $\hat{X}$, $\hat{X}_0$ and $\hat{X}_{in}$ owing to the failure of 'superposition' before considering the kind of natural description of interacting states that is available in principle. The conclusion that there is no particle description for interacting QFTs is not novel, but the diagnosis of the unavailability of a particle description and consideration of the kind of description that QFTs offer via analysis of 'superposition' and the Volkmann device is.

So, in §10.2, in dialogue with Fraser (2008), I show firstly that there is even in principle no 'field quanta' characterization of interacting states via the $\hat{X}$. That is, no 'field quanta' concept is available for theories with interactions. This owes to the nonlinearity of, and hence failure of 'superposition' applied to, the coupled field equations quite apart from Haag's theorem. Secondly, I clarify the implications of Haag's theorem and situate them in relation to the implications of nonlinearity, and briefly consider why Haag's theorem might not forbid empirically adequate results via the $\hat{X}_0$ and $\hat{X}_{in}$, even if further clarificatory work remains. Thirdly, I indicate that various proposals for different architectures of 'particle' for interacting states fail. This means that there is no particle characterization of interacting states. Moreover, through attention to the applicability, or the failure of the applicability of 'superposition' I clarify that no natural description of the general interacting state is available via any of the species $\hat{X}$, $\hat{X}_0$ or $\hat{X}_{in}$, with any attempt to prolong a particle concept from the $\hat{X}_0$ or $\hat{X}_{in}$ failing, with such attempts reflecting a form of semantic mimicry.

Since the introduction of interactions as perturbations is problematic, in §10.3 I consider the foundations of non-perturbative QFT before showing in §10.4 that, *in principle*, a natural description *does* exist for interacting states in QFT but that it cannot, in general, be understood in relation to familiar particle or field concepts. Moreover, since such a natural description is available only 'in principle' and cannot be explicitly constructed, knowledge of its existence has limited value (cf. Wilson 2017, 21).

The surprising failure of the ability to characterize interacting states in QFT by particles, and even particle types, is diagnosed in §10.5 in terms of the

failure of 'superposition' to apply in its more general guise of the Volkmann device. Use of the device is implicitly assumed as I have repeatedly noted, probably owing to lingering bedrock metaphysical assumptions regarding the nature of 'matter' and 'radiation'. The failure of the device casts doubt on the ability to isolate or abstract physically meaningful fields associated with putative partial states as is supposed in the initial application of such 'generalized superposition' in the perturbative approach. Semantic mimicry is likely to occur in relation to physical interpretation of the fields and associated putative partial states owing to the failure of this general form of superposition where it has been implicitly assumed. The need for renormalization procedures is symptomatic of the failure of the Volkmann device. Its failure to apply is partially compensated for by such renormalization procedures, which support the ability of QFTs to provide empirically adequate results.

10.2 The absence of a particle description of interacting states

In this section I demonstrate the inapplicability of the particle concept to characterize general interacting states.

10.2.1 The nonlinearity of the coupled field equations and the decomposition of $\hat{X}$

To develop a particle description of a quantum state via 'field quanta' requires the establishment of a Fock space structure for the states. Recall from chapter 8 that this is achieved via the Fourier solution of the relevant (free) field equation and application of Hilbert superposition in order to diagonalize the Hamiltonian using a basis of the Hilbert space of states of eigenfunctions of the 4-momentum (and relevant permutation) operator. This endows the Hilbert space of states with a physically meaningful Fock space structure, so supporting a particle interpretation (cf. Reed and Simon 1979, 318).

Now, having constructed the (perturbative) coupled field equations to model the interacting system, the obvious choice of field species for construction of a particle description is the $\hat{X}$, that is, the field species that appear as solutions of the coupled field equations (cf. Fraser 2008, 849). I show that even in principle the $\hat{X}$ fail to support a Fock structure for interacting states owing to the

nonlinearity of the coupled field equations, and I analyse the architecture of the Fourier decomposition of $\hat{X}$.

I first clarify the availability and architecture of the Fourier decomposition of the $\hat{X}$. For we can, taking $\hat{\varphi}$ as an example, for any given t, Fourier decompose $\hat{\varphi}$ to obtain a Fourier series/integral *representation* of the field, although the individual terms in the representation are not *Fourier solutions* to the wave equation and so mimic the physical salience of terms in a Fourier solution (cf. §4.5) as we now consider. Duncan notes that for the *free* $\hat{\varphi}_0(\boldsymbol{x}, t)$ the a-operators may be defined via

$$\hat{a}_0^\dagger(k,t) = -i\int d^3x \left(\frac{e^{-ik\cdot x}}{\sqrt{(2\pi)^3 2E(k)}} \frac{\overleftrightarrow{\partial}}{\partial t} \hat{\varphi}_0(\boldsymbol{x}, t) \right) = \hat{a}_0^\dagger(k) = \hat{a}_{0,k}^\dagger$$

where $k_0 = \sqrt{\boldsymbol{k}^2 + m^2}$ and, crucially, they are time-independent. One may define a-operators for the interacting field $\hat{\varphi}(\boldsymbol{x}, t)$ by prolongation:

$$\hat{a}^\dagger(k,t) = -i\int d^3x \left(\frac{e^{-ik\cdot x}}{\sqrt{(2\pi)^3 2E(k)}} \frac{\overleftrightarrow{\partial}}{\partial t} \hat{\varphi}(\boldsymbol{x}, t) \right)$$

However, the a-operators are now time dependent in the interacting case, unlike for the free fields. The time dependency is also very complicated (Duncan 2012, 250). So, if we attempted to factorize the Hamiltonian in the interacting case with the time-dependent operators, when applied to Schrödinger's equation we would no longer be able to separate variables in the way that we did in the free field case in order to establish persistent states that support a particle description that may be 'traced' through the evolution of the system. But we may go further to indicate why it is important to distinguish between the time independency in the free case and dependency in the interacting case.

The procedure for defining the (interacting) $\hat{a}^\dagger(k,t)$ utilizes only the *formal* (or 'second aspect' of) Fourier decomposition of $\hat{\varphi}(\boldsymbol{x}, t)$ over $\boldsymbol{x}$ at a fixed t, obtaining the $\hat{a}^\dagger(k,t)$ as the Fourier term coefficients. Definition of the $\hat{a}^\dagger(k,t)$ by prolongation of the free field definition introduces a different semantic architecture for the Fourier terms. In the free field case the terms are the Fourier or eigenfunction 'simple solutions' to the relevant wave equation ('first aspect') whereas in the case of the coupled equations they are not solutions of the relevant (coupled) wave equation, as indicated by its nonlinearity, so that

'Hilbert superposition' is inapplicable. The interpretation of the Fourier decompositions differs significantly between the two cases. The difference may be compared with the difference in interpretation between the harmonics of the vibrating string and the epicycles of planetary motion (chapter 4) – the individual terms have physical significance in the former case but not the latter as indicated by the failure of 'superposition' in the latter case, so that the terms of the latter are semantic mimics of the former (cf. §4.5). Returning to the coupled field equation, the putative 'Fock space' structures would have different bases (in some complicated way) if one attempted to construct such spaces at different times via $\hat{a}^{\dagger}(k, t_1)$ and $\hat{a}^{\dagger}(k, t_2)$, so that the 'particle' concept associated with each would differ even, if it a particle description could be supported at any given time. That is, the particle concept, if it could be supported at all, would not support the trans-temporal identity of such particles.

But even if one could translate between the (continuously infinite) putative particle concepts required to describe the interacting state there is a further problem that prevents a physical particle concept being established via the $\hat{a}^{\dagger}(k, t)$. That is, these decompositions are constructed on fixed-time slices, so that the decompositions are not relativistically invariant, and do not define a relativistic particle concept as required by QFT. Fraser observes, using ϕ^4 theory:[2]

> It is possible to carry out this Fourier decomposition; however, plugging ϕ(x) into the interacting field equation does not yield the constraint $k^2=m^2$. The consequence is that, unlike the free-field case, k will, in general, not be timelike: $k^2 \neq m^2$, so there is no guarantee that $k^2>0$. As a result, the decomposition in terms of functions $b^+(k), b^-(k)$ [the Fourier coefficients in interacting ϕ^4 theory] is typically not covariant (Roman, 1969, p.119). Furthermore, if $b^+(k), b^-(k)$ were promoted to field operators, they would also fail to be covariant in general; the field operators $\hat{b}^+(k), \hat{b}^-(k)$ would be inertial reference frame dependent, and therefore not candidates for physical fields.
>
> This is a fatal flaw for the strategy of using Fourier decomposition of an interacting field to obtain a [Fock space] representation for it. *A fortiori,* this procedure does not yield a quanta interpretation for an interacting system. (2008, 850).

The attempted Fock space construction using the coupled field equations fails. This is because $\hat{a}^{\dagger}(k, t)$ lack physical significance owing to the nonlinearity of the field equation. They are not associated with solutions of the field equations, and do not allow diagonalization of the Hamiltonian according to a Fourier technique.

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[2] A self-interacting scalar field theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^{\mu}\phi(x)\partial_{\mu}\phi(x) - \frac{1}{2}m^2\phi^2(x) - \frac{1}{4!}\lambda^4\phi^4(x).$$

The variables of Schrödinger's equation cannot be separated to form solutions in terms of physically meaningful eigenfunctions constructed from the wave equations. So the $\hat{a}^{\dagger}(k,t)$ do not support a physical particle concept that persists in the evolution of the interacting system.[3] Moreover, construction of the $\hat{a}^{\dagger}(k,t)$ involves choice of a time-slice such that they are not covariant operators and so cannot support a (relativistically acceptable) physical particle concept.

Diagnosing the problem of the lack of a description of interacting states in terms of nonlinearity and the failure of 'superposition' establishes a stronger result than that in the literature on the subject (Fraser 2008; Huggett 2000; Huggett and Weingard (1994), and (in one sense) a stronger result than Haag's theorem. Fraser suggests that

> In response to the failure of the method of Fourier decomposing an interacting field to yield a quanta interpretation, one might consider generalizing the construction. Instead of Fourier decomposing the classical interacting field into functions of the form $e^{ik.x}$, one might attempt to decompose it into functions of some other form. A suggestion along these lines is mooted in Huggett and Weingard (1994) and Huggett (2000). Huggett floats—but ultimately rejects—the possibility of extending the oscillator analogy to the interacting case in the following way: "[f]or an interacting field the oscillators do not move independently, but as if they were interconnected: there might be further springs, one between any pair of bobs' (p. 628). Translated into the terms of the present discussion, the suggestion is that instead of decomposing the field into independent oscillators—the plane waves $e^{ik.x}$ —the field should be decomposed into coupled oscillators, which are represented by functions of some other form. Huggett and Weingard suspect that this is not possible (p. 376). This is a reasonable conjecture because the proposal faces significant obstacles from two sources. First, there is no guarantee that a function can be decomposed using an arbitrary set of functions; the set of functions of the form $e^{ik.x}$ is special in this respect. Second, even if a workable alternative to Fourier analysis were identified, this resulting decomposition might very well fail to be Lorentz covariant. Since these challenges are both substantial, it seems safe to conclude that it is not possible to obtain an analogue of the Fock representation suited to an interacting field by applying an analogue of the mathematical construction that produces the Fock representation for a free field. (2008, 852)

Huggett and Weingard, after discussing an extended oscillator analogy, conclude: 'Our intuitions are that there is no such set of modes [for the decomposition of the field with which to represent the Hamiltonian], for such a decomposition would seem to make QFT a simpler problem than it is.' (1994, 376)

However, viewing the situation in terms of the applicability of 'superposition', that is, its failure in the context of the nonlinear coupled field equations indicates immediately that the possibility of any such generalized Fourier construction is ruled out so that it is firmly established that there is no

[3] This does not mean that there is not a basis of eigenfunctions of $\hat{H}$ for $\mathcal{H}$. It means that they cannot be constructed from the wave equation, so that a particle description is not available. See §§10.3-4.

particle or quanta concept available for interacting states. This is because that, for such a decomposition to be successful in establishing a particle description for interacting states we would require that the individual modes of the putative generalized Fourier decomposition would be solutions of the field equations in order to enable the diagonal representation of the Hamiltonian. That is, the failure of 'superposition' to apply to the coupled field equations owing to their nonlinearity immediately rules out the use of the 'dual aspects' of the Fourier techniques, since the 'modes' are not solutions, that would be required to support the decomposition of the interacting state into eigenfunctions of the Hamiltonian that are given in terms of the relevant field types, and associated with quanta or particles via the establishment of Fock structures.

As noted above, a (perhaps generalized) Fourier decomposition can be performed on the coupled fields, but the decomposition obtained is to be compared with the epicycles of epicyclical astronomy rather than the harmonics of the vibrating string, with the individual terms lacking physical significance (cf. §§4.5 & 5.3). This is because 'superposition' is mimicked since we do not have both aspects of the Fourier technique satisfied as in the vibrating string or the free field case.

The door is left open, however, to representations of interacting states based on the nonlinear superposition principle, which I consider but reject in §10.2.4. Even if a nonlinear superposition principle could be supported, the existence of which is speculative and unmotivated, the decomposition obtained would not be at all like our familiar particle or quanta concepts.

The failure to establish a Fock structure owing to nonlinearity adds clarity to the observation that a 'number operator' cannot be established on interacting states via the a-operators (cf. Redhead 1988, 20-21). Moreover, we cannot define the charge operator in terms of the a-operators for the same reason, although we can use Noether's theorem to deduce charge conservation in terms of the fields selected. This suggests that it is preferable to construe charge as a property of the overall quantum state in relation to the selected fields and not as a property of particles that is aggregated. Haag suggests, 'the deeper significance of the fields is to effect a local change of charge, not of particle number' (1996, 48) since, we might add, we cannot characterize interacting systems in terms of

particles or particle numbers, except in the idealized asymptotic regions where the concepts are associated with the $\hat{X}_{in/out}$.

10.2.2 Comments on Haag's theorem

A full treatment of Haag's theorem and its implications is beyond my scope, and tangential to my thesis.[4] It has been discussed in detail by, e.g., Barton (1963); Earman and Fraser (2006) and Duncan (2012). It is frequently alluded to in the philosophical literature but rarely in the physics literature. It appears to be a surprising result, and importantly, it undermines the interaction picture. The immediate question is then why empirically successful results may be obtained using the interaction picture even if Haag's theorem undercuts its use (cf. e.g. Earman and Fraser 2006; Duncan 2012; Miller 2018; Teller 1995, 115). I do not claim to resolve this problem. Rather, I shall clarify what is undermined and point to some proposed resolutions, and relate Haag's theorem to the significance of the nonlinearity of interacting field equations.

The theorem may be stated in various forms, but a brief summary suitable for my purposes is that, given some reasonable assumptions, representations of the CCRs of the free and interacting fields $\hat{X}_0$ and $\hat{X}$ (and, as we shall see, $\hat{X}_{in}$) are unitarily inequivalent (cf. §9.3.2). A consequence of Haag's theorem is that there is no unitary map $V(t)\colon \mathcal{H}_0 \to \mathcal{H}$ such that

$$V(t)\hat{X}(\boldsymbol{x},t)V^{-1}(t) = \hat{X}_0(\boldsymbol{x},t)$$
$$V(t)\hat{\Pi}(\boldsymbol{x},t)V^{-1}(t) = \hat{\Pi}_0(\boldsymbol{x},t)$$

The implications of the theorem reach further than the unitary inequivalence of free and interacting fields. Haag showed that the representations of two free scalar fields with different masses are unitarily inequivalent (see Duncan 2012, 359-363 for discussion). Thus Haag's theorem is not directly related to the nonlinearity of the coupled field equations. It is a separate issue since two free field equations of the same form but with different mass parameters are both linear, but give rise to unitarily inequivalent fields and hence particle concepts according to Haag's theorem.

[4] The initial result, subsequently developed e.g. Hall and Wightman (1957), is credited to Haag (1955).

Discussion of Haag's theorem is often conducted with reference to this example of free scalar fields of differing masses. But this example does not capture all the conceptual difficulties involved with the use of the interaction picture for coupled (and hence nonlinear) field equations. Indeed, the implications of nonlinearity are stronger than those of Haag's theorem in the sense that owing to nonlinearity there is *no* 'field quanta' particle concept available for interacting fields: Haag's theorem implies the weaker result that *if* there is a 'field quanta' particle concept for interacting fields *then* it is different ('incommensurable' as Ruetsche (2011) puts it) from that of any free field, but says nothing about whether *there is* a particle concept available for interacting fields or not. However, Haag's theorem indicates that free fields with different masses are unitarily inequivalent. The issue of nonlinearity does not feature here, so Haag's theorem says more than can be gleaned from consideration of nonlinearity in this sense.

In terms of our three field species $\hat{X}, \hat{X}_{in}$ and $\hat{X}_0$ the upshot is that Haag's theorem demonstrates that the species $\hat{X}_{in}$ and $\hat{X}_0$ are unitarily inequivalent to each other as well as to $\hat{X}$, so these fields support inequivalent, 'incommensurable' particle concepts (cf. Ruetsche 2011), even if they are available. Owing to the fact that $\hat{X}_{in}$ and $\hat{X}_0$ satisfy linear field equations they do both support particle concepts via the Fock construction in their (idealized) domains of applicability. The $\hat{X}$ satisfy nonlinear field equations and so do not support a particle concept quite apart from Haag's theorem.[5]

We now consider the empirical success of the interaction picture in relation to Haag's theorem. Duncan's analysis (2012, 359-370) is illuminative and has been well received (e.g. Miller 2018; Butterfield 2015),[6] although his treatment is limited to discussion of free scalar fields of differing mass. Duncan suggests that one must be careful to note what Haag's theorem *does not* say, commenting:

[5] It seems open to question the sense in which unitary inequivalence implies physical inequivalence, especially in terms of 'incommensurability', although this is not an issue that I pursue.

[6] Miller follows Duncan, but concludes that the 'real problem' here is to account for the success of asymptotic series (2018, 814-818). However, asymptotic series are used elsewhere in classical physics, so this question is not specific to QFT.

> there is no difficulty whatsoever in establishing a well-defined unitary relation between the in- and out- states of an interacting field theory: the overlaps ${}_{out}\langle\beta|\alpha\rangle_{in} = S_{\beta\alpha}$ are taken between states living in spaces spanned by a complete basis of eigenstates of *the same Hamiltonian operator H* [see §9.5.1]. Indeed, the Haag-Ruelle and LSZ scattering theories … lead to a perfectly well-defined, and *unitary* S-matrix, on the basis of exactly the same axiomatic framework which can be used to establish the validity of Haag's theorem. The LSZ formula … gives a rigorous connection between well-defined Green functions (time-ordered products of the full Heisenberg fields) and this unitary S-matrix [see §11.3.2]. Direct non-perturbative evaluation of the Green functions … therefore completely circumvents any difficulty with the non-existence of the interaction picture (363-364).

There are two issues that need to be distinguished:

First, Haag's theorem undermines the ability to *describe* interacting states at finite times using the interaction picture, so it would not seem possible to prolong a free field particle concept to describe general interacting states. Following the discussion above on nonlinearity this is unsurprising, and we simply accept that we cannot describe interacting states via the $\hat{X}_{in}$ or $\hat{X}_0$, although there is a weak convergence relation between the $\hat{X}$ and $\hat{X}_{in}$ in the asymptotic limits (§11.3.1), even if the $\hat{X}$ and $\hat{X}_{in}$ are unitarily inequivalent by Haag's theorem.[7] We may restate Duncan's point to note that Haag's theorem is silent on the question of whether we can *calculate* to a good approximation scattering amplitudes using the interaction picture, and perhaps this is all that we should require, which brings us to the second point.

Secondly, we might ask whether (or what kind of) an explanation is owed regarding the ability to perform empirically adequate calculations iteratively in the interaction picture so as to well-approximate a unitary *S*-matrix as a map between the asymptotic states.

Duncan offers an explanation in terms of the role that renormalization procedures play, at least in the case of two free scalar fields of differing masses. He locates the difficulty that Haag's theorem raises in terms of the description of the time development of an infinite-volume continuum field theory (359). If the fields were quantized in a finite box of volume V on discrete points (i.e., introducing IR and UV cut-offs) the interaction picture is well-defined:

> The resultant theory, at the price of loss of Poincaré invariance, is now a quantum-mechanical system with a finite number of independent degrees of freedom, and the interaction picture makes perfect sense. The problem is now transferred to the issue of regaining sensible (in particular, Poincaré invariant!) results in the limit when these

[7] As Duncan notes, the same Hamiltonian and Hilbert space is used throughout – the issue is the representation of the Hamiltonian in terms of fields and the physical particle concepts associated with those fields and the regions in which they are valid.

> cutoffs are removed, *after the perturbative expansion of the n-point functions needed for evaluation of the S-matrix has been performed.* (369).[8]

He concludes that

> the proper response to Haag's theorem is simply a frank admission that the same regularizations needed to make proper mathematical sense of the dynamics of an interacting field theory at each stage of a perturbative calculation will do double duty in restoring the applicability of the interaction picture at intermediate stages of the calculation (370).

Although Duncan's proposal is plausible there are outstanding issues. For instance, the details need to be worked through for an interacting theory with different field types (e.g. scalar Yukawa theory) in which nonlinearity is involved. Indeed, further investigation of the significance of working with a cut-off theory in relation to the implications of nonlinearity would be illuminative as this introduces a feature not usually discussed in conjunction with Haag's theorem.

Alternatively, Barton proposes a possible 'heuristic' explanation by suggesting that in the context of renormalized perturbation theory it may be enough that $\widehat{U}_I$ is an 'improper transformation' and not a unitary transformation (1963 132, 158-159). Earman and Fraser reject this proposal, stating that unitary equivalence is a 'demonstrable necessity' without further discussion (2006, 308-309). However, Barton's point merits further attention by clarifying what properties of $\widehat{U}_I$ are required to *support calculations* even if $\widehat{U}_I$ *does not support descriptions*.

Duncan's and Barton's proposals merit further research, although I cannot undertake this here. The key question is, once we have abandoned the project of *describing* interacting states, what is the remaining problem to solve? The question may turn out to be more closely related to that of the success of asymptotic series approximation methods in general than to specific issues in quantum physics.[9]

10.2.3 Other attempts to establish a particle patch

We saw in chapter 9 how Dyson's series (the iterative series expansion for $\widehat{U}_I$) failed to support a physical particle description of interacting states via the $\hat{X}_0$, quite apart from Haag's theorem. We saw above that the $\hat{X}$ do not support a

[8] The unitary equivalence of representations involving a finite number of degrees of freedom as associated with the imposition of UV and IR cutoffs is assured by the Stone – von Neumann theorem. See Ruetsche (2011).

[9] Cf. §9.6; J.D. Fraser (2020); Miller (2018).

particle description for interacting states owing to nonlinearity, again quite apart from Haag's theorem. It would seem that the only physically meaningful particle concept for interacting QFTs is then that associated with the $\hat{X}_{in}$, although this is only available in the very restricted context of idealized asymptotic states of non-interacting particles and cannot be used to describe interacting states. Here one might appeal to Haag's theorem to suggest that any attempt to somehow prolong the particle concept associated with the $\hat{X}_{in}$ fails.

However, Bain (2000) seeks to establish a prolongation of the $\hat{X}_{in}$ particle concept to general interacting states by 'sidestepping' the implications of Haag's theorem. He acknowledges that there is no (free field) occupation number for interacting states, but claims this should not prevent a particle interpretation of such states, by reconsidering what we mean by 'particles'. That is, he suggests that 'a "particle" be considered a system that minimally possesses an asymptotic state (i.e., a system that is free for all practical purposes at asymptotic times)'. Furthermore, there are 'two types of system that we might consider to be particles: "asymptotic" particles defined directly [via the $\hat{a}_{in,k}$], and "interacting" particles [defined via $\hat{a}_k(t)$]', the latter having the former as asymptotic states for which there is an occupation number operator. He claims that we should view 'both types of system not as distinct types of particle; but rather, as different states in which a particle can be found', referring to such a system as an "LSZ particle"; that is, a system capable of possessing both states. Then, 'a viable interpretation of interacting QFT can be had, based on the notion of an LSZ particle.' (Bain 2000, 394)

If successful, this would represent a prolongation of the 'field quanta' particle concept associated with the renormalized asymptotic free fields to interacting states. However, as we saw above, owing to the nonlinearity of the coupled field equations there is no such 'interacting particle' concept available via the $\hat{a}_k(t)$, or any physically salient concept supported by the $\hat{a}_k(t)$ during interaction. So, we have only the asymptotic particles, or systems that possess asymptotic states with which to attempt to define an interacting state in terms of particles. The situation is far less favourable than Bain envisages.

It might be clear then that Bain's proposal is to be rejected, as Fraser suggests:

> The underlying problem is a weakness in Bain's approach to ontology. The point at issue is whether entities with certain properties—particlelike properties—exist. … [T]he question is whether there are entities with particlelike properties at finite times … [but Bain] does not point to any evidence for the existence of particlelike entities in the presence of an interaction (e.g., that states of an interacting system possess the expected energies for states in which a definite number of quanta are present). (Fraser 2008, 856-857)

However, in the context of our metaphysically quietest Wilsonian framework arguably there might be more work to do depending on how one understands the nature of a particle concept. For instance, does Bain's particle concept support a reasoning advantage to offer physically salient explanations? We've seen that there is no particle concept, as Fraser suggests, via the $\hat{a}_k(t)$, so all that survives of Bain's claim is that 'a "particle" be considered a system that minimally possesses an asymptotic state'. Is this sufficient to establish a meaningful particle concept for interacting states? In other words, even if we abandon the goal of offering a particle description of an interacting state at a finite time, could the interacting state meaningfully inherit an asymptotic particle description that would characterize the interacting state in terms of particles?

Surely it does not. Although one might be able to *label* an interacting state via its asymptotic particle content(s), such a 'particle' concept is at best an idle wheel, and at worst subject to underdetermination.[10] It is not clear how the concept would support physically salient explanations or a reasoning advantage. Moreover, it is unclear what could be said about bound states or unstable states, or QFTs that do not possess asymptotic states. Bain considers this last point in the context of QCD to suggest that quarks cannot be regarded as particles (400).

In summary, Bain's attempt to prolong the asymptotic particle concept to interacting states fails, and we shall see in §10.3 more reason to be cautious even of the claim that the asymptotic states label interacting states.

10.2.4 The nonlinear superposition principle

The last hope[11] for establishing a particle concept on interacting states might be via the nonlinear superposition principle.[12] This would reflect a natural

[10] That is, is the interacting state characterized via the in- or out-states (or both)?

[11] Fraser considers another alternative approach to defining particles on interacting states, an '"axiomatic" method of specifying a Hilbert space representation by stipulating formal conditions' (2008, 843) which, she argues, fails (852-855). I shall not discuss this as the approach is tangential to my thesis, and accept her conclusion without discussion.

extension of Fourier techniques and Hilbert superposition to nonlinear systems using a form of the Volkmann device.

It would require construction of a set of functions $\{\hat{u}_i(x)\}$ as 'simple functions' and a 'connecting function' $F = F(\hat{u}_1(x), \hat{u}_2(x), \dots, x)$ where F and the $\hat{u}_i(x)$ individually are solutions to the coupled wave equations. One would then, presumably, need to diagonalize the Hamiltonian via the $\hat{u}_i(x)$ and establish a 'nonlinear Fock space' structure on the Hilbert space of interacting states in such a way that the state could inherit a particle description. The existence of such a representation is unknown however and cannot be assumed, and it is possible that its non-existence might be demonstrated mathematically.

The possible success of such a method in QFT is speculative and physically unmotivated. Even if such a construction could be performed, the physical significance and interpretation of the 'nonlinear Fock space' is unclear, especially with regard to how the 'particle' concept that would arise might be related to the familiar asymptotic particle concept.

If the method were to be successful it would mark an application of the Volkmann device in its most general sense and thus support a natural description of a general interacting state. However, the physical meaning of such a description is not clear even if it were to be available, and it might be misleading to think of it in terms of a particle concept.

10.2.5 Summary

We have now seen that there is no particle concept applicable to general interacting states in QFT in terms of any of the candidate field species $\hat{X}, \hat{X}_{in}$ or $\hat{X}_0$, which are, moreover, all unitarily inequivalent according to Haag's theorem. None of these field species support even a natural description of general interacting states, as results from the failure of 'superposition'.

I now consider a different approach to the construction of a natural description of general interacting states whose existence is assured even if its physical meaning is unclear in general.

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[12] See e.g. Jones and Ames (1967); Menini and Tornambé (2011).

10.3 Non-perturbative QFT

I have indicated some conceptual difficulties arising in 'perturbative' QFTs, that is, interacting theories constructed from the implicit application of the Volkmann device. The failure of the device will be considered in more detail in §10.5. Here however I clarify what can be said with more confidence using a non-perturbative approach to QFT founded on a set of axioms known as the 'Wightman axioms' (Streater and Wightman 1964, 96-106). These axioms 'incorporate the essential features of a relativistic quantum field theory' (Duncan 2012, 253).[13] Even in this more rigorous setting difficulties remain: first, as regards the asymptotic completeness hypothesis (AC), which is the assumption that the full Hilbert space of states $\mathcal{H} = \mathcal{H}_{in} = \mathcal{H}_{out}$ (recall $\mathcal{H} \neq \mathcal{H}_0$);[14] secondly, with regard to the identification and interpretation of fields chosen to act on $\mathcal{H}$ or $\mathcal{H}_{in}$. The Wightman axioms incorporate the essential features of a relativistic QFT, but do not address failure of the Volkmann device. However, in Duncan's extension of the Wightman axioms, problems of field selection and their physical interpretation is highlighted by drawing attention to the underdetermination of interacting fields as interpolating fields in scattering theory.

My discussion of the axioms will be informal and concentrate only on the conceptual issues required to illuminate the analysis that I wish to pursue. The discussion is based upon Duncan's treatment and development of the axioms (2012, 253-268) owing to his detailed commentary, although his interpretation goes beyond Streater and Wightman's in sometimes problematic but illuminative ways.

10.3.1 The 'state axioms'

Axiom (Ia) states, 'The state space $\mathcal{H}$ is a separable Hilbert space. It carries a unitary representation $U(\Lambda, a)$ … of the proper inhomogenous Lorentz group (i.e.

[13] Duncan and others have restated and developed the Wightman axioms, e.g. Haag (1996); Strocchi (2013).

[14] The proof of this assumption relates to the 'Yang-Mills Existence and Mass Gap' problem. It is one of the Clay institute's seven Millenium Prize Problems, the brief statement of which is to, 'Prove that for any compact simple gauge group G, a non-trivial quantum Yang-Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$. Existence includes establishing axiomatic properties at least as strong as those cited in Streater & Wightman (1964), Osterwalder & Schrader (1973) and Osterwalder & Schrader (1975).' (Jaffe and Witten, 'Quantum Yang-Mills Theory', http://www.claymath.org/sites/default/files/yangmills.pdf, accessed 12-07-2019).

the Poincaré group) … ' (Duncan 2012, 254). This is unproblematic, although Duncan's comments are:

> Our Hilbert space $\mathcal{H}$ is a countable direct sum of multi-particle spaces corresponding to a definite number of particles. The multi-particle space corresponding to a fixed finite number of particles is a finite tensor product of separable L^2 spaces, each with a countable basis, and is therefore itself separable. The separability of $\mathcal{H}$ follows trivially. The reader is free to visualize $\mathcal{H}$ as the space of in-states $\mathcal{H}_{in}$ (or out-states $\mathcal{H}_{out}$) … with the action of the $U(\Lambda, a)$ given by $e^{iP\cdot a}U_H(\Lambda)$ (254)

First, this assumes asymptotic completeness (AC), which is problematic as we consider below (IIIb). The second problem relates to the possibility of a particle interpretation, or the possibility of establishing a physically meaningful Fock space structure on the space $\mathcal{H}$ of general interacting states. Assuming AC, *mathematically speaking* $\mathcal{H}$ inherits the Fock structure from $\mathcal{H}_{in}$ constructed from the $\hat{a}^{\dagger}_{in}(\mathbf{k})$, as it is the same Hilbert space. But as we have seen, for interacting states the physically salient structure of $\mathcal{H}$ is, according to QFT, constructed from $\hat{\varphi}$ and not $\hat{\varphi}_{in}$, and we have seen that no such *physically salient* Fock structure can be constructed from $\hat{\varphi}$ owing to the nonlinearity of the coupled field equations. Whatever 'interpolating fields' are chosen (IIIa), the equations they satisfy are nonlinear, and so unsuitable for the construction of a *physically meaningful* Fock space structure (§10.2). The fact (assuming AC) that $\mathcal{H}$ *mathematically* shares the same 'Fock space' structure as $\mathcal{H}_{in}$ is irrelevant, for it does not support a physical interpretation on $\mathcal{H}$, but only $\mathcal{H}_{in/out}$.

The situation is analogous to the comparison between epicycles in astronomy and harmonics on the vibrating string (chapter 4). The spatial state of the planet and the spatial state of the vibrating string share the same (up to isomorphism) Hilbert space structure, but the Fourier modes of the string have physical significance in virtue of their being simple solutions to the wave equation whereas the epicycles of the planets do not have physical significance as they are not associated with partial laws and states associated with solutions to a PDE modelling the behaviour of the system. Semantic mimicry occurs in both cases – physical interpretation of a mathematical structure that is physically meaningful in one context is implicitly but illegitimately smuggled across or prolonged to different physical situations (cf. chapters 4-5).

Axiom (Ib): 'The infinitesimal generators P_μ of the translation subgroup $T(a) = U(1, a)$ of the Poincaré group have a spectrum p_μ restricted to the forward light cone, $p_0 \geq 0, p^2 \geq 0$.' (Duncan 2012, 254) Duncan comments,

> In accordance with our intuition of asymptotic completeness – [1] that all Heisenberg states of the system correspond to field disturbances which eventually resolve into a finite number of well-separated stable particles of finite energy, and with individual four-momenta on or within the forward light-cone – [2] the total energy-momentum p_μ of any state of the system must be resolvable into a sum of four-vectors … (254)

The first part, [1], is unproblematic, assuming AC, although the interpretation as extended via [2] is problematic however, for it may be understood as an attempt to implicitly 'smuggle in' a particle interpretation from the asymptotic states to the general state, which we have seen we cannot. One might *label* a general state by its asymptotic content (which may include bound states), but this is different from saying that the composition of its asymptotic content prolongs into the general interacting state. What we should say is that the P_μ are infinitesimal generators of the translation subgroup (with the spectrum as Duncan indicates) so that one may assign a total momentum p_μ to a general state, and that *in the idealized asymptotic region* p_μ may be resolved into a sum of four-vectors corresponding to the momenta of isolated particles (and bound states), whilst remaining silent on the structure of the eigenstates of P_μ in the context of a general interacting state.

It is noteworthy that P_μ is defined without reference to fields, and importantly we shall see in §10.4 that it does support, in principle, a natural eigenstate representation of general interacting states, even if it is unclear how to explicitly construct or relate such a representation to established physical concepts, such as asymptotic particles that correspond to our phenomenological particle notion.

Duncan's Axiom (Ic) stipulates the existence of a unique normalized 'vacuum' state $|\Omega\rangle$ with isolated eigenvalue $p_\mu = 0$, and, (Id) a 'mass gap'. That is,

> the squared-mass operator $P^2 = P_\mu P^\mu$ has an isolated eigenvalue $m^2 > 0$, and the spectrum of P^2 is *empty* between 0 and m^2. The subspace $\mathcal{H}_1$ of $\mathcal{H}$ corresponding to the eigenvalue m^2 carries an irreducible spin-0 representation of the HLG. These are the single particle states of the theory. The remaining spectrum of P^2 is continuous, and begins at $(2m)^2$. (254)

Haag-Ruelle theory (§11.3.1) requires this assumption of an isolated eigenvalue m^2 (Glimm and Jaffe, 274). Duncan comments that (Id) excludes QED with a massless photon, so that in calculations the photon is given a small mass and the

massless limit taken after calculations (Duncan, 255). He interprets the spectrum via the in-states for the two-particle subspace:

> In the two-particle subspace (say, for $|p_1, p_2\rangle_{in}$), the squared-mass operator gives $(p_1 + p_2)^2 = 2m^2 + 2p_1 \cdot p_2$, with $p_1 \cdot p_2 > m^2$, so the spectrum of P^2 in this subspace is $[(2m)^2, \infty)$. Overall, the spectrum of P^2 is therefore $\{0, m^2, [(2m)^2, \infty)\}$. … We assume no bound states, e.g., one-particle mass hyperboloids at $p^2 = 4m^2 - \mathcal{E}$. (255)

Again, we must exercise caution here. Duncan's (Ic)-(Id) goes beyond Wightman's statement (Streater and Wightman 1964, 96-106), and more recent restatements of the axioms (e.g. Haag 1996, 56-58; Strocchi 2013, 69-72). Wightman interprets $P_\mu P^\mu = m^2$ as the squared mass operator and comments that the eigenvalues of P_μ lie in or on the forward light cone (97), while Haag (56) and Strocchi (70) refrain from introducing a mass interpretation, noting simply that the spectrum of the energy-momentum operators P_μ is contained in the closed forward cone. Duncan's gloss on (Id), interpreting the spectrum of P^2 via a particle interpretation of the idealized asymptotic in-states, is only justified as a physical interpretation on the asymptotic states. Streater and Wightman, Haag and Strocchi do not attempt to prolong this valid physical interpretation of the asymptotic states to a general interacting state.

The physical interpretation of the asymptotic states as composed of particles having properties cannot be prolonged to general states as we have seen. The physical interpretation of the Fock decomposition that identifies particle states in the asymptotic regions associated with the eigenvalues of the P-operators does not prolong to a physically salient decomposition of interacting states. What we can say, however, is that if the spectrum of the physical P^2 on the asymptotic states is (neglecting bound states) $\{0, m^2, [(2m)^2, \infty)\}$, then as P_μ supports a conservation law, this is also the spectrum of P^2 on a general interacting state. Crucially however, the general state does not support any further physical interpretation as the theory does not support the identification of any substructure to such states. That is, the $(2m)^2$ state does not support the physical interpretation of being a 'two particle state' in a general interacting state, unlike in the asymptotic region. It is simply a state of invariant mass $(2m)^2$ and we remain silent beyond that. For later reference (§11.2), the lowest mass state of the continuous part of the spectrum is known as the 'threshold mass', $\mu^2_{threshold}$, as it will not always be $(2m)^2$.

10.3.2 The 'field axioms'

The fields appear crucial in characterizing a general state in a natural way since it is via the fields that we obtain a 'natural' coordinate system for $\mathcal{H}_{in}$ that we interpret as a particle description. We would like to obtain a natural coordinate system for $\mathcal{H}$ through the fields too, although this fails owing to the failure of 'superposition' for interacting theories, owing specifically to the nonlinearity of the coupled field equations which results from the failure of the initial application of the Volkmann device. This is to think in terms of 'physicists' QFT', which is my project, in which the physical interpretation of the theory centres on $\mathcal{H}$ rather than on the algebra of the field operators as in algebraic QFT.[15] Either way, the fields play a central role. Indeed, Streater and Wightman note, 'To be a field theory, a relativistic quantum theory must have enough fields so its states can be uniquely characterized using fields and functions of fields.' (1964, 100)

Duncan's Axiom (IIa) states, 'An operator-valued (tempered) distribution $\varphi(x)$ exists such that for any Schwartz test function $f(x)$ [infinitely times continuously differentiable functions of fast decrease], the smeared field

$$\varphi_f \equiv \int f(x)\varphi(x)d^4x$$

is an unbounded operator defined on a dense subset $D \subset \mathcal{H}$. Moreover, $\varphi_f D \subset D$, allowing the definition of arbitrary (finite) products of smeared fields.' (256) This is the limited case for a system characterized by a single field. Indeed, Wightman stipulates (in general) the existence a set of operators $\varphi_1(f) \ldots \varphi_n(f)$ together with their adjoints, and that *D* is a linear set containing $|\Omega\rangle$ (98).

Duncan comments that the axiom is *motivated by* equivalent statements for a free scalar field where a dense subset of $\mathcal{H}$ is obtained by considering all normalizable *n*-particle states obtained via Fourier decomposition of the field (256). Note however that no physical interpretation is claimed at this stage in the general statement (IIa). Physical interpretation is only claimed in the context of its motivation, and is not supported in general.

[15] A key question is then whether failure of the Volkmann device to the identification of the fields renders the algebraic approach problematic also.

Duncan also comments (257) that this enables the definition of vacuum expectation values of products of smeared fields, which can be written as overlaps of Wightman distributions ('functions'):

$$\langle\Omega|\phi_{f_1}\dots\phi_{f_n}|\Omega\rangle = \int f_1(x_1)\dots f_n(x_n)\langle\Omega|\phi(x_1)\dots\phi(x_n)|\Omega\rangle d^4x_1\dots d^4x_n$$

with Wightman function:

$$W(x_1,\dots x_n) = \langle\Omega|\phi(x_1)\dots\phi(x_n)|\Omega\rangle$$

The $W(x_1,\dots x_n)$ may be interpreted as 'Green's functions' or correlation functions of the theory. We have studied the time-ordered two-point function for free fields, interpreted as a particle propagator, and will consider the time-ordered two-point function for interacting fields in §11.2. The time-ordered *n*-point function will be central to LSZ scattering theory (§11.3.2).

Axiom (IIb) states the field transformation law under the unitary representation of the Poincaré group $U(\Lambda, a)$ (cf. (Ia)) (Duncan, 257). (IIc) stipulates that if f_1, f_2 are Schwartz functions of compact support on space-like separated regions then $[\varphi_{f_1}, \varphi_{f_2}] = 0$ for bosonic fields or $\{\varphi_{f_1}, \varphi_{f_2}\} = 0$ for fermionic fields (258) (the 'causality' axiom). The final field axiom (IId) stipulates that the set of states obtained by applying arbitrary polynomials in the smeared fields φ_f (or more generally $\varphi_1(f)\dots\varphi_n(f)$) to the vacuum $|\Omega\rangle$ is dense in $\mathcal{H}$ - the 'cyclicity of the vacuum' (259). This is associated with the (physically interpreted) Fock structure of free fields in the asymptotic limits. The cyclicity of the vacuum prolongs (mathematically) to Hilbert space states associated with interacting fields but lacks a physical interpretation in general.

10.3.3 The 'particle-field duality axioms'

We have now defined a relativistic quantum field theory (Streater and Wightman, 101), but as yet we have not made a connection with scattering theory or indicated if, and if so, what kind of physically salient natural description is supported. In particular, there is no explicit introduction of a Lagrangian or Hamiltonian associated with Schrödinger's equation, and the fields are not explicitly associated with field equations.

To begin the process of interpretation – or to consider how empirical QFTs relate to this abstract treatment – Duncan adds a third group of axioms associated with scattering theory and particle interpretations.

Axiom (IIIa) associates asymptotic states interpreted in terms of a physical particle content with fields 'interpolating' between them:

> For some one-particle state $|\alpha\rangle = \int g(\vec{k})|\vec{k}\rangle\, d^3k$ $(g(\vec{k}) \in L^2)$ with discrete eigenvalue m^2 of the squared-mass operator … the smeared field $\phi_f(x)$ has a non-vanishing matrix element from this single-particle state to the vacuum, $\langle\Omega|\phi_f(x)|\alpha\rangle \neq 0$. … If this situation holds, we call $\phi_f(x)$ an *interpolating Heisenberg field* for the given particle. (267, notation adapted)

Duncan implicitly assumes a field $\hat{\varphi}_{in}$ satisfying a free, linear wave equation via which the asymptotic states are endowed with a Fock space structure, with the structure interpreted in terms of idealized physical (phenomenological, but non-interacting) particles. This implicitly requires the identification of a self-adjoint Hamiltonian operator satisfying Schrödinger's equation.

I consider the interpolating fields further in chapter 11 in the context of LSZ scattering theory, noting here that the interpolating fields (of which the $\hat{X}$ fields when suitably smeared are paradigmatic examples) are underdetermined. I shall suggest that this is associated with the failure of 'superposition'. The role of the interpolating fields is simply to associate asymptotic in-states, interpreted as physical particles, with asymptotic out-states, again interpreted as physical particles, via some suitable algorithm. The direct physical significance, or their relationship to a true description, of these interpolating fields is unclear.

Finally, Axiom (IIIb), is the asymptotic completeness axiom (AC), namely that $\mathcal{H} = \mathcal{H}_{in} = \mathcal{H}_{out}$. Duncan comments that from the cyclicity of the vacuum axiom $\mathcal{H}$

> can be regarded as the space generated by application of the smeared fields to the vacuum … [T]his axiom again connects the particle concepts (the asymptotic in- and out-states) with a space $\mathcal{H}$ defined in terms of the action of the basic field(s) of the theory. … [T]his assumption is almost unavoidable physically, as it incorporates a vast amount of phenomenological experience of particle interactions. … [However, the] assumed unitarity of the S-matrix [only requires] $\mathcal{H}_{in} = \mathcal{H}_{out}$, with both of these asymptotically defined spaces being (perhaps) proper subsets of … $\mathcal{H}$. Indeed, the Haag-Ruelle scattering theory … can only establish the existence of the asymptotic states as such subsets. Moreover, even in the few cases where we have maximum mathematical control … the validity of Axiom IIIb remains … ((Glimm and Jaffe, 1987), p.275), "a very deep (and open) mathematical question." Our attitude … in the absence of conclusive evidence to the contrary, will simply be to assume the validity of asymptotic completeness (Duncan 267-268).

If AC holds then any general interacting state can be *labelled* by its asymptotic particle content (Glimm and Jaffe 1987, 274). We have seen above that a

physically meaningful Fock space structure cannot be established on the space of general interacting states $\mathcal{H}$, so that we cannot say that a general interacting state has a particle description even if it can be labelled by an asymptotic particle description in scattering theory.

Might one expect AC to fail? The axiom has received little attention from philosophers despite its foundational importance.[16] Duncan, and Glimm and Jaffe accept AC on the basis of physical assumptions. These assumptions might be influenced by an underlying metaphysical picture of particles as the basic entities of nature. On such a picture one is unlikely to question the validity of AC. But, especially having undercut the assumption of a particle ontology, one might be more willing to question AC. For instance, in the idealized infinite limit-taking procedure is information about the interacting state 'lost', being 'washed out' in the limit? Are there interacting states not accessible from scattering states? This is not clear.

The position that we worked towards in §10.2 was that mathematically speaking $\mathcal{H}$ may well inherit the Hilbert space structure of $\mathcal{H}_{in/out}$ (i.e. AC), but not its physical significance or interpretation, so that there may be $|\Omega\rangle \in \mathcal{H}$ that is cyclic for a set of smeared fields $\varphi_1(f) \ldots \varphi_n(f)$, but where the states generated do not support a physical interpretation. That is, one may accept AC mathematically but deny any physical interpretation of $\mathcal{H}$ generated by the $\varphi_1(f) \ldots \varphi_n(f)$.

These are important issues that I cannot pursue here, being tangential to my thesis, and so as per usual practice I shall accept the axiom as a *mathematical* axiom without importing any physical interpretation.

10.3.4 Limitations of the non-perturbative approach

The constructive approach provides a useful framework within which analysis of QFTs may be situated, clarifying for example the conceptual foundations of Haag-Ruelle and LSZ scattering theory (§11.3). However, importantly, there is nothing, prima facie, in this axiomatic framework that motivates a conceptual resolution of the failure of the Volkmann device with regard to the initial choice of fields.

[16] Ruetsche mentions AC in passing (2011, 254-256), noting that there is 'room to wonder whether the postulate is correct' (254).

The axioms do not identify any field types that would support a natural description of the general interacting system in the sense that we have identified. That is, such that the general interacting system may be analysed via isolated or abstracted partial states associated with partial laws that take the same form individually and in combination in a way that, moreover, optimally balances simplicity and strength so as to support physically salient explanations and counterfactual reasoning. This is indicative of the problematic nature of the concept of '(near) fundamental interaction', indicating that the general interacting system modelled by QFT remains a 'black box' that does not possess a description in terms of simpler components, according to QFT at least, as we shall consider in more detail in §10.5.

There is a further limitation in the axiomatic framework that is also associated with scattering theory in the perturbative framework in that we cannot analyse or describe bound states or unstable particles. Rather, we can only label them by their asymptotic content. Thus far the $\hat{X}_{in}$ that we have introduced only describe 'elementary' particle states, and these do not span $\mathcal{H}_{in/out}$. To span $\mathcal{H}_{in/out}$ we must add the bound states associated with wave equations

$$\left(\partial^2 + \mu_{phys,i}^2\right)\hat{\chi}_{phys,i}(x) = 0, \qquad i = 1,2,3, \ldots$$

where $\mu_{phys,i}$ is the asymptotic physical mass of bound state i with associated free field $\hat{\chi}_{phys,i}$ which we intuitively consider to be composed of interacting phions and psions, conceived as what the state might be considered to be 'made from'. That is, if we perform a low-energy scattering experiment with an asymptotic in-state described by a number of elementary psions and phions and obtain as the out-state a single 'particle', we might consider the out-state particle as a *bound state* 'composed' of the corresponding psions and phions comprising the in-state. But the theory does not allow us to regard the bound states as *composed of* – we must think of them as *made from*. That is, the identities of the entities that are brought together to form a bound state do not persist in the bound state once formed to support a 'trace principle' (cf. §2.3.2).

This is a limitation of the theory, as the theory does not allow us to analyse or to describe a bound state *as* a bound state of elementary particles

(although perhaps we can label it as such) – the 'elementary system' is the bound state. Neither does the theory allow us to model unstable particles directly, as by definition these cannot form asymptotic states, as they will have decayed before becoming an asymptotic state in virtue of being an unstable particle. We consider approaches to modelling unstable particles in §11.2.3 and bound states in §11.5, indicating the philosophical difficulties in forming a natural description of some of the most ubiquitous entities in nature.

10.4 A natural description of general interacting states

We have seen that we cannot establish a physically meaningful particle description on general interacting states via coupled field equations. This is because we cannot construct an explicit eigenstate representation of P_μ (and hence $\widehat{H}$) via the fields with the finer-grained structure of a Fock space using Fourier techniques. That is, we cannot form a natural description of the system by representing $\widehat{H}$ via the coupled fields.

However, if we identify the Hamiltonian $\widehat{H}$ for the interacting system via P_μ as defined in (Ia) *without reference to fields* as in the non-perturbative treatment, and assuming that the operators are self-adjoint and that the state $|\Sigma\rangle \in \mathcal{H}$ satisfies Schrödinger's (linear) equation

$$i\frac{\partial}{\partial t}|\Sigma\rangle = \widehat{H}|\Sigma\rangle$$

then Fourier techniques apply. $|\Sigma\rangle$ has a spectral decomposition into eigenstates of $\widehat{H}$ that are simultaneously eigenstates of P_μ. By Hilbert superposition this grants, in principle, a natural description of an arbitrary interacting state $|\Sigma\rangle$, supporting inductive inferences and physically salient explanations, for similar reasons as hold in the case of the classical examples in chapter 4. This eigenstate basis has the finer-grained structure of a physically salient Fock basis for $\mathcal{H}_{in/out}$ in the asymptotic limit (but only in the asymptotic limit), recovering the familiar particle concept in such limits, although bound states and associated fields also need to be included. It is not clear, however, how to impose any statistics as we have not made reference to fields.

One difficulty is that this natural eigenstate decomposition on $\mathcal{H}$ exists only 'in principle'. Owing to the failure of the Volkmann device there is no natural

representation for $\widehat{H}$ in general via coupled fields, which is what we require to model 'interactions'. In the perturbative approach, although fields may be chosen it is not a natural choice of representation owing to the failure of the Volkmann device, requiring renormalization to support calculations via Dyson's expansion. What we have, in principle, is a natural (eigenstate) description of the state, using Fourier techniques, broadly understood, applying Hilbert superposition to Schrödinger's equation. But such a description is in general more coarse-grained than we would like, and only in the idealized asymptotic limits may the eigenstates be endowed with a natural finer-grained structure using smeared fields to support a physical particle interpretation. The asymptotic fields and states may be regarded as a 'coordinate system' for the idealized asymptotic states that cannot be prolonged to general states.

The general eigenstate decomposition cannot be stated explicitly as we do not have a suitable coordinate system with which to express the eigenstates in terms of familiar concepts. That is, we cannot relate description of the eigenstates to existing or empirical physical concepts other than in the asymptotic limits, so that we cannot offer physically salient explanations of interactions or descriptions of interaction processes. The best we can do is to note that, in principle, the total relativistic mass and momentum of the state labels each eigenstate, being the associated eigenvalue of P_μ, which is known from the asymptotic states and momentum conservation. Moreover, if the general state is associated with or labelled by an asymptotic state that is associated with fields, then by Noether's theorem the total charge(s) of the state associated with the fields are also known by charge conservation with reference to the chosen field. So, we can label the general eigenstates by their asymptotic total momenta and charge in relation to a field, but we cannot endow them with any finer-grained structure as we can with the asymptotic states.

It appears then that there is rather little that can be said about the eigenstates of general interacting states and that such 'in principle' identification may be an 'idle wheel'.[17] However, the 'in principle' knowledge of such states does in fact allow for development of the Källén-Lehmann spectral resolution,

[17] Cf. Wilson's repeated concerns with the value of 'in principle' results (2006; 2017).

which assists analysis of unstable particles and some types of quasiparticle (§11.2).

10.5 Failure of the Volkmann device and its consequences

I now clarify the sense in which the Volkmann device fails in interacting theories in relation to the initial identification or selection of fields and putative 'partial states' associated with them.

In §8.5 we considered a proper application of the Volkmann device in a system whose state $|\Sigma\rangle_{overall}$ is completely characterized by the action of two isolated free field types $\hat{\phi}_{free}$, $\hat{\psi}_{free}$ (and $\hat{\psi}^{\dagger}_{free}$). The overall state may be decomposed into 'partial states' of phion and psion/antipsion component states

$$|\Sigma\rangle_{overall} = |\Phi_{free}\rangle \otimes |\Psi_{free}\rangle$$

without symmetrization of the tensor product since the particles are distinguishable. [18] Each component $|\Phi_{free}\rangle$, $|\Psi_{free}\rangle$ can be represented independently by an element of its own Fock space by further application of 'superposition', i.e. can be represented by linear combinations of what we call *N*-quanta states constructed via the relevant wave equation using Hilbert superposition applied to this and the state equations, using Fourier and eigenfunction techniques.[19]

We can do this because the action of the fields is given by

$$\hat{\phi}_{free}|\Sigma\rangle_{overall} = \hat{\phi}_{free} \otimes \iota_{\psi}|\Phi_{free}\rangle \otimes |\Psi_{free}\rangle = \left(\hat{\phi}_{free}|\Phi_{free}\rangle\right) \otimes |\Psi_{free}\rangle$$

and similarly for $\hat{\psi}_{free}$. In particular, the overall Hamiltonian and 4-momentum operators can be represented by the $\hat{\phi}_{free}$ and $\hat{\psi}_{free}$ so that each field only acts non-trivially only on its corresponding partial state, which means that the overall Hamiltonian can be decomposed into components $\hat{H}_{free,\varphi}$, $\hat{H}_{free,\psi}$ such that

$$\hat{H}_{free,\varphi}|\Sigma\rangle = \hat{H}_{free,\varphi} \otimes \iota_{\psi}|\Phi_{free}\rangle \otimes |\Psi_{free}\rangle = \left(\hat{H}_{free,\varphi}|\Phi_{free}\rangle\right) \otimes |\Psi_{free}\rangle$$

and similarly for $\hat{H}_{free,\psi}$, with $\hat{H}_{free,overall} = \hat{H}_{free,\varphi} + \hat{H}_{free,\psi}$.

This means that the actions of $\hat{\phi}_{free}$ and $\hat{\psi}_{free}$, and hence the corresponding Hamiltonian and 4-momentum operators, take the same form

[18] Note further $|\Psi_{free}\rangle = |\Psi_{particle\ sector}\rangle \otimes |\Psi_{antiparticle\ sector}\rangle$.

[19] Cf. §8.3.5 for an indication of the difficulties associated with regarding an *N*-quanta state as being composed of *N* quanta.

individually on their corresponding partial states and in combination on the overall state. So partial laws associated with the partial states take the same form individually and in combination owing to the form of the action of the fields. This enables Fock structures to be constructed on $|\Phi_{free}\rangle$ and $|\Psi_{free}\rangle$ which take the same form on these partial states as on $|\Phi_{free}\rangle\otimes|\Psi_{free}\rangle$, thus supporting a particle description of the overall state in terms of particle type, number and state.

This is a proper application of the Volkmann device – we have analysed the complicated overall state and its behaviour by isolating two partial states that take the same form individually in isolation and in combination, such that associated with each partial state are partial laws given in terms of their corresponding fields that take the same form on each individual partial state and on the overall state in combination, whilst not stating the facts.

This leads, after the Fock construction on each partial state, to a natural description of the overall system best balancing simplicity and strength (cf. chapter 4, esp. n.7) via multiple applications of 'superposition':

(1) The initial 'generalized' isolation / superposition discussed here into the fields and states associated with different particle types using the Volkmann device;

(2) The Fourier series solutions to the wave equations derived from the fields obtained from (1) using Hilbert superposition;

(3) The decomposition of the Hilbert space sectors associated with particle types postulated in (1) into eigenstates of the 4-momentum and permutation operators to construct Fock spaces states using (2) and Hilbert superposition and Fourier techniques applied to Schrödinger's equation;

(4) To complete the dual aspect of the Fourier technique, the initial state is expressed in terms of the eigenstates of the 4-momentum and relevant permutation operators in each partial state obtained from (1).

This 'Volkmann-Fourier' procedure supplies a natural description of the quantum system interpreted as a particle description that supports physically salient explanations, inductive inferences and counterfactual reasoning for the complicated overall system in terms of the simple partial states and laws associated with them.

This contrasts with the failure of the procedure for interacting systems in QFT, for which the coupled field equations are nonlinear when considered in the perturbative approach.[20] The problem is that however one seeks to identify abstracted fields and corresponding partial states, the fields will always couple with each other in general (by definition, to introduce an interaction). This means that any such field will act non-trivially not only on the putative partial state that one attempted to isolate and associate with that field, but also on other putative partial states too. This means that the 'partial states', and the corresponding 'partial laws', no longer take the same form individually and in combination so that the Volkmann device fails.[21] What we would require for application of the Volkmann device in the interacting case is:

(I) That we could decompose the overall state via

$$|\Sigma\rangle = |\Phi\rangle \circledast_{state} |\Psi\rangle$$

where the form of composition is to be specified, but must be able to support a 'trace principle' for the persistence of the putative components so that we can meaningfully identify the isolated or abstracted partial states in the same way individually and in combination;

(II) The identification of corresponding fields such that the action of the fields is given by

$$\hat{\varphi}|\Sigma\rangle = \hat{\varphi} \circledast_{field} \iota_{\psi}|\Phi\rangle \circledast_{field} |\Psi\rangle = (\hat{\varphi}|\Phi\rangle) \circledast_{field} |\Psi\rangle$$

with the form of composition to be specified, etc.;

(III) These partial states and fields completely characterize the system, with the partial states and corresponding partial laws constructed via the fields taking the same form individually and in combination, according to (I) and (II).

This is what Schwinger (1949, 651-652; cf. §8.5) presumed, that there exist 'bare' independent fields (i.e. isolated or abstracted fields taking the same form individually and in combination) that support representations of interacting QFTs according to the Volkmann device. Schwinger's conceptual assumption remains prevalent even after the introduction of renormalization group methods for example. For instance, Cheng and Li suggest that

[20] The non-perturbative approach does not show how to identify or to relate interacting fields.
[21] As we have seen, in some restricted contexts, such as the idealized free asymptotic states, fields (the $\hat{X}_{in/out}$) and associated partial states can be chosen to satisfy the Volkmann device, but the applicability of such a representation is restricted to this limited context.

> there is no way to switch off the interaction; hence quantities in the absence of interaction, called the *unrenormalized* or the *bare* quantities, are not measurable. For example, in quantum electrodynamics … the bare mass [of the electron] cannot be measured because the electron interacts with the virtual photon field constantly and there is no way to turn off this interaction (1988, 30)

The assumption that Schwinger and Cheng and Li make appears to be that there exist (in some unspecified, although apparently metaphysical sense) bare fields that can be abstracted by mathematical or theoretical means, even if not physically isolated (i.e. 'measured'), that take the same form (as the bare fields) individually in isolation (even if only in a conceptually abstracted sense) and in combination.[22] If this were the case application of the Volkmann device would be supported, albeit possibly with some complicated composition relation that we are not (yet) able to specify.

We have already seen that the $\hat{X}_0$ and the $\hat{X}_{in}$, and associated states, which can be isolated as such, do not take the same form individually in free theories and in combination in general interacting theories, as they cannot be prolonged to general interacting systems. But, moreover, we do not know how to choose $\hat{X}$ fields so as to satisfy simultaneously the conditions I-III of the Volkmann device and support a natural description of the system and a concept of interaction, at least according to the current conceptual framework of QFT.

It would appear that the only way in which the Volkmann device might be supported would by appeal to a form of compositions $\circledast_i$ involving the non-linear superposition principle in a rather general form (i.e., not simple multiplication or tensor product). But it is entirely speculative as to whether such a principle might be supported. Even if it were, the physical significance of the components identified, and their relation to familiar concepts, is not at all clear (cf. §10.2.4). It would require a new conceptual framework involving some sort of 'generalized non-linear Hilbert space' structure, whatever that might look like. This difficulty is perhaps that which Heitler anticipated (1936; cf. §7.6).

If we remain within the standard, established Hilbert space conceptual framework for the representation of states in QFT, this problem – of the inability

[22] This point goes unrecognized in some attempts to supply analogies using examples in classical physics to illuminate renormalization. For example Coleman uses Green's analysis (1834) of the comparison of the 'bare mass' of a sphere and its 'effective mass' when moving in a fluid as an analogy for mass renormalization in QFT (2019, 207). But this is misleading. In Green's example one can isolate the bare sphere, but one cannot meaningfully abstract bare entities and their properties in QFT.

conceptually to abstract bare fields and states according to the Volkmann device – is marked by the need for renormalization, however the fields and parameters appearing in the coupled field equations are chosen. Indeed, compensating for the absence of such a decomposition of states and fields according to the Volkmann device in order to produce an empirically adequate theory is a, if not the, goal of renormalization, the need for which is highlighted when one attempts to calculate the scattering amplitudes via Dyson's expansion.[23]

It is important here to distinguish between this case of the failure of the Volkmann device which is associated with the non-linearity of the coupled field equations in QFT, and models of physical systems that model interactions in a different fashion whilst using non-linear differential equations in which the Volkmann device, or something similar, enables the abstraction or isolation of entities that are said to interact, even when the 'interaction' is non-linear. I have in mind the Lotka-Volterra predator-prey model in which the numbers of (typically) foxes and rabbits present in a population are modeled by non-linear differential equations including an 'interaction' between the rabbits and foxes.[24] In the Lotka-Volterra case one identifies distinct types of entities, the rabbits and foxes, in which the identity of that type of entity persists in the evolution of the system. But the non-linear coupling relates here to the *numbers* of the entities and not to their *identities* as types. In QFT the non-linear differential equations

[23] Historically the need for renormalization was recognized as it arose in the context of the evaluation of divergent integrals in the evaluation of 'loop diagrams' owing to their unrestrained *p*-values. Renormalization was the procedure used to force finite results that agreed with experimental results via infinite shifts in parameters. However, renormalization is required even without infinite parameter shifts, in a finite theory (Cheng and Li 1988, 30). For instance in the application of QFT to condensed matter physics a scale parameter establishes a natural momentum limit in the evaluation of the relevant integrals, so that renormalization requires a finite parameter shift (cf. Aitchison and Hey 2013, 1.304). Thus divergence is not the real issue that drives renormalization. As Lancaster and Blundell put it, 'Renormalization is not, therefore, an exercise in hiding infinities, it's an exercise in making a theory describe real life.' (2014, 291). The driving issue is the failure of the Volkmann device when it is tacitly assumed in the construction of the interacting theory. The failure of the Volkmann device is initially compensated for by regularization of the theory and the introduction of counter-terms, which ultimately leads to the modification of parameters (cf. §9.7.3) to force the theory to agree with experiment. This is the case whether one considers early approaches to renormalization, or modern renormalization group (RG) techniques. RG techniques may be said to compensate for the failure of the Volkmann device in conceptually more illuminative ways. The comparison of this perspective on RG techniques, that is, in relation to the failure of the Volkmann device, to the usual perspective of the role of scale in RG methods merits further analysis, which is beyond my scope here. See e.g. Butterfield and Bouatta (2014); Cao (1993) for overviews of the development of renormalization techniques with discussion of their philosophical significance.

[24] I.e., the foxes eat the rabbits. See Simmons (2017, 507-512).

concern the identities of the types of entities and not simply their number or properties.

Perhaps the difference can be expressed in terms of the claim that in the Lotka-Volterra case an ontology of the model is already chosen from which the behaviour of the numbers of the two types of entities is then modeled by functions that appear in a non-linear differential equation, whereas in QFT it is the mathematical entities that are nonlinearly related that are used to attempt to represent or to deduce the ontology of the model via the identities or descriptions of putative partial states constructed.

So for instance a rabbit has the same form in a population composed of rabbits without foxes, and in a population composed of rabbits and foxes. What differs is the evolution of the number of rabbits present over time in the different populations. The *number of* rabbits does not take the same form in an individual population of rabbits, and in combination, i.e. in a population of rabbits and foxes. This is where the failure of 'superposition' in the Lotka-Volterra model occurs, that is, in terms of the *numbers* of the two entities and the law characterizing their evolution. Renormalization is not required. In the case of interacting QFT the situation is different since we cannot identify partial states (that are assumed to represent the ontology of the system) whose identity takes the same form individually (i.e., in a free theory) and in combination in the interacting theory. We can conceptually abstract, and of course physically isolate, rabbits in the context of the Lotka-Volterra model, but not the putative partial (bare) states of interacting QFT. Renormalization is the means by which one compensates for this difficulty, that one cannot identify entities (rather than their number or properties) that take the same form individually and in combination. In order to set up a model, one is forced to attempt to do so, but one will always fail, with the failure partially compensated for by renormalization.

Application of the Volkmann device *is* supported in QFT with suitable choice of fields/states in some very restricted contexts and in a conceptually limited way, such as the idealized physical (renormalized) asymptotic fields/states of the scalar Yukawa theory via the $\hat{X}_{in}$. However, the $\hat{X}_{in}$ and their associated Fock spaces are associated with free theories, and so do not support a

concept of interaction or any physical explanations in terms of charge, and cannot be prolonged to describe interactions as per §10.2.3.

Different theoretical contexts in QFT – which we should note exhibit a façade structure – involve different situations in which such a limited 'workaround' to the failure of the Volkmann device is possible, such as associated with asymptotic freedom in QCD (cf. Bain 2000) in which idealized free fields may be obtained at very short length scales / high energies, or in terms of quasiparticles in condensed matter physics (cf. Lancaster and Blundell 2014).

What one can say regarding general interacting states in QFT is that one has a decomposition of an arbitrary state $|\Sigma\rangle$ into a superposition of eigenstates $|p, \alpha\rangle$ upon which one can choose fields $\hat{\varphi}$ and $\hat{\psi}$ – that are underdetermined (cf. §§10.3.3; 11.3) – that act in some coupled, complicated way. It is the failure of the Volkmann device characterized by the nonlinearity of the coupled field equations in conjunction with the putative partial states being required to be associated with the ontology of the theory, whatever fields are chosen, that diagnoses why fields and associated states cannot be abstracted or isolated to describe general interacting states. This failure requires renormalization to achieve empirically adequate results. According to the Volkmann device, successful identification of the 'isolation centres' would be marked by the absence of the need for renormalization, and if this were possible it would reflect successful application of the non-linear superposition principle.

In practice we are forced to introduce the $\hat{X}$ coordinates to decompose the Hamiltonian, as we cannot do otherwise, which act on the Hilbert space of states from which iterative calculations yield empirically adequate results using the interaction picture fields $\hat{X}_0$ after renormalization. But the field types associated with the $\hat{X}$ coordinates form a complicated and counterintuitive coordinate system that does not behave nicely as one would expect good coordinate systems to behave that support (or depend upon) 'superposition', and it is this that forces the need for renormalization. No 'coordinate' choice of fields is 'natural' for interacting states as no choice of coordinates directly supports physically salient explanations or picks out physical features or properties of the system as, say, in Sturm-Liouville theory (cf. chapter 4). Moreover, as we shall consider in more detail in §11.3.2 the interacting fields $\hat{X}$ as they are utilized in scattering theory

are underdetermined for general interacting states. The role of the $\hat{X}$ is clarified as *supporting calculations* (after renormalization) as 'interpolating' between the idealized asymptotic in- and out-states, rather than *supporting descriptions* of general interacting states.

Failure of the Volkmann device is associated with a further difficulty raised earlier: How can we meaningfully talk of 'interactions' if we cannot identify components that are said to be interacting? The concept of interaction has a patchwork structure. One patch is associated with interaction between objects such as in the example of the macroscopic interaction between two charged metal spheres, another with isolatable entities such as linearly propagating waves that interfere. Another patch with different architecture might be interactions between people. In each case there are isolatable or abstractable entities with their own independent identity taking the same form individually and in combination that interact, so that the individual identity of the entities may be traced during the interaction (cf. Simons trace principle in §2.3.2).[25] We have supposed that we can prolong the concept of interaction to the fundamental level without special difficulty but, counter-intuitively, the prolongation fails and we don't know what it means to talk of interacting states in QFT beyond saying that initial idealized state A 'interacts' with initial idealized state B to produce final idealized state C.

In other words, we can talk of asymptotic particle in-states via the $\hat{X}_{in}$ that may be said to interact in the sense that there is an outcome of asymptotic particle out-states that differ from the in-states. This is what we mean by particle interactions. But we must remain silent regarding the description or process of the evolving 'interacting state' that associates the in- and out-states. This implies, moreover, that there is a difficulty regarding how we understand bound states as composed of interacting particles (see §§11.5-6).

The difficulty in applying the concept of interaction at or near the fundamental level, the failure of the concept of superposition at this level of description owing to the nonlinearity of the coupled field equations, and the

[25] I am assuming that this is a necessary condition to speak meaningfully of interaction in the full sense of the concept entailing a description of interaction, although perhaps this could be challenged.

introduction of an interaction as a 'perturbation' (or perhaps 'effective perturbation' via gauge methods) and renormalization are intimately related via the failure of the Volkmann device, presenting a serious obstacle for any attempt to develop a 'fundamental level' metaphysics using QFT.

As we shall see in chapter 11, it is better to view QFT as possessing a façade structure of local patches of applications that involve reliable approximations, idealizations and sometimes ad hoc 'fixes' as associated with an engineer's approach to modelling a system or phenomenon rather than as revealing the fundamental nature of reality.[26]

10.6 Summary

We have seen that there is no particle description available for interacting states in perturbative QFT owing to the failure of 'superposition' to apply at two critical (associated) moments – in the selection of the fields and associated states in relation to the Volkmann device, and in relation to the nonlinearity of the coupled field equations. I related the discussion of the implications of nonlinearity of the coupled field equations to Haag's theorem, also briefly considering some implications of the latter.

The failure of 'superposition' in these two senses is not addressed in the constructive, axiomatic foundations of non-perturbative QFT, and is associated with a problem with the concept of 'interaction' at the (near) fundamental level owing to the inapplicability of the initial attempt to use the Volkmann device to identify partial states and associated fields that take the same form individually and in combination. The failure of the Volkmann device is partially compensated for by renormalization procedures.

However, it is possible in principle to form a natural description of a general interacting state in non-perturbative QFT owing to the linearity of Schrödinger's equation and application of Hilbert superposition, but without the ability to construct the required eigenstates in terms of fields associated with familiar particle concepts.

[26] The concept of an 'effective field theory' as associated with RG approaches would be an interesting avenue to explore in this perspective although it is beyond what can be covered here.

Chapter 11

Unstable particles, scattering theory and bound states: An 'engineer's approach'

11.1 Overview and the 'engineering perspective' toward QFT

In this chapter I consider the application of some of the results of the non-perturbative framework (§§10.3-4) before completing my analysis of scattering theory. I briefly discuss the modelling of bound states before commenting on the significance of the coupling strength of a theory for the 'approximate application' of superposition. The various modelling situations manifest the façade nature of QFT, which I interpret to suggest that QFT is best understood in relation to the perspective of an engineer's rather than a metaphysician's.

So, first, in §11.2 I shall demonstrate a practical application of the 'in principle' knowledge of the eigenstates of the 4-momentum operator in establishing the Källén-Lehmann spectral resolution of the 2-point propagator or correlation function for any field. Study of the propagator enables a better understanding of interacting fields (§11.2.1) and, importantly, it allows unstable particles, and some types of quasiparticle, to be modelled and characterized (§11.2.2). However, the physical interpretation of the characterization of unstable particles especially in terms of 'superposition' is unclear.

Secondly, I consider scattering theory in §11.3. The asymptotic relationship between the idealized asymptotic fields $\hat{X}_{in/out}$ and the interacting fields $\hat{X}$ satisfying the coupled wave equations is seen to be established by Haag-Ruelle theory in §11.3.1. In §11.3.2 I show how to relate the $\hat{X}_{out}$ to the $\hat{X}_{in}$ via the $\hat{X}$-fields using the important result of Lehmann, Symanzik and Zimmermann (1955), the 'LSZ scattering theory', in terms of multi-point correlation functions or Green's functions. The $\hat{X}$-fields are interpreted here as underdetermined 'interpolating fields', as introduced in §10.3, since they 'interpolate' between the in- and out-states. Although we already know (chapter 10) that we do not have any natural description of interacting states, the LSZ result is a vital result that leads towards supporting the ability to calculate scattering amplitudes.

Thirdly, in §11.4 I show how the Gell-Mann and Low theorem demonstrates that Dyson's expansion, and its Feynman diagram interpretation in the interaction picture, can be used to approximate the Green's functions that are central to the LSZ result given in terms of the $\hat{X}$-fields. Together, these results enable iterative approximations of S-matrix elements using the $\hat{X}_0$-fields, subject to successful renormalization. So, we now have a scattering theory that, whilst not offering any description of scattering processes, enables empirically adequate calculations of scattering amplitudes to be performed.

Fourthly, I discuss how calculations of the properties and behaviour of bound states may be performed in some cases using the Gell-Mann and Low theorem even in the absence of a natural description of any internal structure of so-called 'bound states' in §11.5.

Finally, in §11.6 I consider the possibility of conceptualizing bound states as 'approximately composed' of constituent particles as a 'working picture' even if not as a true metaphysical description. The ability to do so depends on the strength of the coupling of the theory, indicating that this is possible for weakly but not strongly coupled theories. This means that our 'working picture' of the nucleus of an atom will differ significantly from that of its electron configuration, since the former is modelled partly in terms of a strongly-coupled theory (QCD) and the latter a weakly-coupled theory (QED), where moreover the electron configuration is adequately modelled by NRQM for most practical purposes.

What these various discussions of particular applications of QFTs indicate is the façade nature of QFT and that an 'engineering perspective' reflects a good stance to take towards the various theories, models and modelling tools in QFT. By an 'engineering perspective' or approach I mean that the form of conceptual models and kind of knowledge that QFT supplies is associated with an engineer's approach to locally applicable models of the world, where such models support physically salient explanations, reliable calculations and counterfactual reasoning in a given domain in the patchwork form that Wilson has suggested, even if the models cannot be said to offer 'natural descriptions' of the phenomena modelled. This means that QFT does not supply knowledge of the world in a form suitable for metaphysical reflection as a 'Theory T', but it offers

more than a set of algorithms that support empirically adequate (and useful) calculations and no more.

Such an approach to QFT was anticipated by Dirac:

> The system of approximations I shall use will be somewhat similar to the approximations that engineers use in their calculations. Engineers have to get results and there are so many factors occurring in their problems that they have to neglect an awful lot of them; they don't have time to study everything seriously and they develop a sort of feeling as to what can be neglected and what can't. I believe that physicists working in quantum field theory will have to develop a similar sort of feeling as to what can be neglected and what can't. The final test is whether the resulting theory is coherent and in reasonable agreement with experiment. (1966, 2-3)

What I am suggesting goes slightly beyond this, and might be summed up in the context of QFT in terms of the ability to apply the Volkmann device 'approximately', and to compensate for its failure in situations in which it does not apply approximately. That is, within the conceptual framework of QFT as we have it, the engineering approach involves developing a feel for how to model the system in such a way as one can apply the Volkmann device as a good approximation (such as in modeling the asymptotic states), and / or find ways of compensating for its failure (such as in renormalization).[1]

This 'engineering approach' does not quite fit with realist, empiricist or pragmatist stances. Whether or not particular conceptual approximations are sufficient to support a realist stance to the entities featuring in an engineering model, such as 'constituent' particles of a bound state, is an open question that I cannot pursue here. Moreover, as we have no idea how to de-idealize the models or remove the approximations one cannot specify 'in virtue of what' that reliable models that might support a modest realist stance are obtained.[2]

11.2 An application of the eigenstates of P_μ: the Källén-Lehmann resolution and unstable particles

11.2.1 The Källén-Lehmann resolution

I demonstrate the value of knowledge of the eigenstates of P_μ 'in principle' by considering the derivation of the Källén-Lehmann spectral resolution of the

[1] Of course physics and engineering are separate disciplines. My point is not that fundamental physics is 'really engineering', but that the stance that one should adopt to models and theories of QFT is comparable to an engineer's rather than a metaphysician's, and that QFT does not fulfill the 'promissory notes' of appeal to fundamental physics in some naturalized metaphysics.

[2] Perhaps what is surprising is that the tacit expectation to be able to dispense with idealizations and approximations at the level of fundamental physics is not fulfilled by QFT, so that the sense in which scientific models can be said to explain needs careful nuancing (cf. Bokulich 2011).

propagator, or correlation function (Källén 1952; Lehmann 1954). I shall make some clarifications in the interpretation of the propagator and indicate how unstable particles are characterized in QFT via their propagators. This is a significant result since most of the so-called fundamental particles in Nature are unstable (Weldon 1976, 2030).

Following §10.3, assume that a self-adjoint 4-momentum operator P_μ with Hamiltonian $\widehat{H} = P_0$ exists for an interacting QFT with Hilbert space $\mathcal{H}$, without reference to fields. Label the (Lorentz invariant) eigenstates $|p^{(k)}, \alpha\rangle$ by their energy $p_0^{(k)}$, momentum $\mathbf{p}^{(k)}$ and any other quantum numbers α required (Schweber 1961, 652). The total energy-momentum defines the mass of each state via $m^{(k)2} = p_\mu^{(k)} p^{(k)\mu}$ where $m^{(k)2}$ is the total energy of $|p^{(k)}, \alpha\rangle$ in its rest frame. These eigenstates enable derivation of the Källén-Lehmann spectral resolution for *any* field $\hat{\chi}$ acting on $\mathcal{H}$.

Take the scalar Yukawa theory as an example and introduce $\hat{\chi}$ as a neutral scalar field acting on $\mathcal{H}$, where $\hat{\chi}$ might or might not appear in the Lagrangian.[3] Consider the two-point function

$$G_\chi^{(+)}(x-y) = \langle\Omega|\hat{\chi}(x)\hat{\chi}(y)|\Omega\rangle$$

and insert the completeness relation or 'resolution of the identity' using the physical eigenstates $|p^{(k)}, \alpha\rangle$:[4]

$$\mathbb{I} = \sum_{|p^{(k)},\alpha\rangle} |p^{(k)}, \alpha\rangle \langle p^{(k)}, \alpha|$$

so that

$$G_\chi^{(+)}(x-y) = \langle\Omega|\hat{\chi}(x)\hat{\chi}(y)|\Omega\rangle = \sum_{|p^{(k)},\alpha\rangle} \langle\Omega\big|\hat{\chi}(x)|p^{(k)}, \alpha\rangle\langle p^{(k)}, \alpha|\hat{\chi}(y)\big|\Omega\rangle$$

$$= \sum_{|p^{(k)},\alpha\rangle} \big|\langle\Omega|\hat{\chi}(0)|p^{(k)}, \alpha\rangle\big|^2 e^{-ip^{(k)}\cdot(x-y)}$$

[3] See Schweber (1961, 660-662) for what follows. See Weinberg (1995, 457-462) for charged scalar field and Peskin and Schroeder (1995, 216) for fermionic fields.

[4] In the standard derivation the completeness relation is written as a summation, although as Weinberg observes it 'includes integrals over continuous labels as well as sums over discrete labels.' (1995, 458). This step warrants closer scrutiny; here we follow the standard derivation.

after simplifying using translation and Lorentz invariance. Introduce the positive definite quantity $\rho_\chi\left(p^{(k)}\right) = (2\pi)^3 \sum_\alpha \left|\langle\Omega|\hat{\chi}(0)|p^{(k)},\alpha\rangle\right|^2$ which we may rewrite as

$$(2\pi)^3 \sum_\alpha |\langle\Omega|\hat{\chi}(0)|p,\alpha\rangle|^2 = \Theta(p^2)\Theta(p_0)\rho_\chi(p^2)$$

so

$$G_\chi^{(+)}(x-y) = \langle\Omega|\hat{\chi}(x)\hat{\chi}(y)|\Omega\rangle = \frac{1}{(2\pi)^3}\int d^4p\,\Theta(p^2)\Theta(p_0)\rho_\chi(p^2)e^{-ip\cdot(x-y)}$$

Moreover, since

$$\Theta(p^2) = \int_0^\infty d\mu^2\,\delta(p^2-\mu^2)$$

where p and μ are understood to represent physical momentum and mass,

$$G_\chi^{(+)}(x-y) = \int_0^\infty d\mu^2\rho_\chi(\mu^2)\frac{1}{(2\pi)^3}\int d^4p\,\Theta(p_0)\delta(p^2-\mu^2)e^{-ip\cdot(x-y)}$$

$$= \int_0^\infty d\mu^2\rho_\chi(\mu^2)\,G_0^{(+)}(x-y;\mu^2)$$

where

$$G_0^{(+)}(x-y;\mu^2) = \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}}{2E(\mathbf{p})}e^{-ip\cdot(x-y)} = \Delta_+(x-y;\mu^2)$$

the two-point function for a free neutral scalar field of mass μ, where $E(\mathbf{p}) = \sqrt{\mathbf{p}^2-\mu^2}$. Duncan explains,

> [$G_0^{(+)}(x-y;\mu^2)$ is] the invariant function arising from the two-point function of a *free,* canonically normalized scalar field of mass μ … This is a remarkable result - that the Wightman two-point function of an arbitrary scalar interacting Heisenberg field can be written as the positively weighted average of the corresponding free field Wightman functions for fields of varying mass, with a positive weight function containing all the non-trivial interaction physics of the theory – this is the *Källen-Lehmann* representation of the two-point function.' (Duncan 2012, 291 (notation adapted))

Or, as Brown puts it: 'the two-field function of a general interacting scalar field can be expressed as a superposition of the corresponding free field functions of variable mass. If the theory were that of a free field of mass μ, one would have $\rho_\chi(p^2) = \delta(p^2-\mu^2)$' (1992, 285, notation adapted).

The Feynman propagator for $\hat{\chi}$ in the interacting theory is (cf. Peskin and Schroeder 214-215):

$$G_\chi^F(x-y) \equiv \Delta_{F,\chi}(x-y) \equiv \langle\Omega|T\{\hat{\chi}(x)\hat{\chi}(y)\}|\Omega\rangle = \int d\mu^2\rho_\chi(\mu^2)\,G_0^F(x-y;\mu^2)$$

That is, the Feynman propagator for $\hat{\chi}$ is the 'superposition' of Feynman propagators of free fields of (continuously) varying mass, weighted by ρ_χ.

In what sense are these expressions 'superpositions'? Do the terms have physical significance individually and take the same form in and out of combination? In what sense do the individual components $G_0^F(x-y;\mu^2)$ have physical significance? We have not yet introduced any non-physical quantities, although we have exploited various mathematical representational devices. In particular, at this point p may be interpreted as physical 4-momentum, and $G_0^F(x-y;\mu^2)$ represents physical propagation of a state associated with a free field of mass μ, so this might be considered to be an application of 'superposition'.[5]

Difficulties arise when the Feynman propagator is introduced as an explicit function of p – the form most useful – since p is now a dummy variable rather than physical momentum (§9.6.2). This often results in semantic mimicry manifesting in interpretations of virtual 'off mass shell' states as we saw. However, although the interpretation of p in what follows is not straightforward for this reason, the interpretation of μ and $\rho_\chi(\mu^2)$ is already established physically as above.

The explicit representation in terms of p is then (cf. §9.6.2):

$$G_\chi^F(x-y) = \langle\Omega|T\{\hat{\chi}(x)\hat{\chi}(y)\}|\Omega\rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \int_0^\infty d\mu^2 \frac{i\rho_\chi(\mu^2)}{p^2-\mu^2+i\varepsilon}$$

and taking the Fourier transform of $G_\chi^F(x-y)$ gives the p-space representation:

$$\tilde{G}_\chi^F(p) \equiv \tilde{\Delta}_{F,\chi}(p) = \int_0^\infty \frac{d\mu^2}{2\pi} \frac{i\rho_\chi(\mu^2)}{p^2-\mu^2+i\varepsilon}$$

$\tilde{G}_\chi^F(p)$ is defined for $p^2 \in \mathbb{R}, p^2 \geq 0$, and may be considered a function of p^2, namely $\tilde{G}_\chi^F(p^2)$. However, it will prove useful to consider the continuation of $\tilde{G}_\chi^F(p^2)$ onto the complex p^2-plane, denoted $\widetilde{D}'(p^2)$ where now $p^2 \in \mathbb{C}$. This is a further shift away from a physical interpretation of p using another mathematical device (analytic continuation on $\mathbb{C}$), even if it will have physical implications as we shall see. The analytic structure of $\widetilde{D}'(p^2)$ will prove

[5] Typically the propagator is interpreted with reference to particles, but it is not clear that the state associated with the propagator is best interpreted as a 'particle'. It might be better to interpret it as an 'elementary system' in the Newton-Wigner sense (cf. §8.6).

important once the physical spectrum of the theory, i.e. the set of eigenvalues and corresponding eigenstates of the 4-momentum operator of the theory are introduced as per §10.3.1. The location of poles and branch cuts of $\widetilde{D}'(p^2)$ in relation to the lowest mass eigenstate coupled to $\hat{\chi}$ leads to different forms of $G^F_{\chi}(x-y)$. $\hat{\chi}$ may be associated with *stable* particle types that persist in asymptotic states, as in scattering theory if $\hat{\chi}$ couples with an isolated eigenstate of mass less than the threshold mass (case 1), or *unstable* particles or indeed certain types of *quasiparticles* if it only couples with eigenstates of mass greater than the threshold mass (case 2).

11.2.2 Case 1: $\hat{\chi}$ couples with an isolated pole of $\widetilde{D}'(p^2)$

The first case reflects the scalar Yukawa theory as it has been implicitly set up thus far. That is with a neutral scalar $\hat{\varphi}$ field of mass m coupled to charged scalar fields $\hat{\psi}$ and $\hat{\psi}^{\dagger}$ of mass M where the physical or renormalized masses satisfy $m^2_{phys} < \mu^2_{threshold}$, i.e. m^2_{phys} is below the threshold mass (§10.3.1). Taking $\hat{\chi}$ as this $\hat{\varphi}$, the spectrum of states coupled to $\hat{\varphi}$ is

$$\left\{0, m^2_{phys}, \left[\min\left\{\left(2m_{phys}\right)^2, \left(2M_{phys}\right)^2\right\} = \mu^2_{threshold}, \infty\right)\right\}.^{6}$$

That is, as in our discussion of (Ib) in §10.3, we know that as regards the asymptotic states, $\hat{\varphi}$ 'excites' renormalized single particle states, the single particle state having invariant mass m^2_{phys} which may be associated with $\hat{\varphi}_{in}$.[7] It excites states of mass $\left(Nm_{phys}\right)^2$ possessing any total momentum $\geq \left(Nm_{phys}\right)^2$, and since $\hat{\varphi}$ couples with $\hat{\psi}$ and $\hat{\psi}^{\dagger}$, it will excite physical psion-anti psion pair states of mass $\left(2M_{phys}\right)^2$, $\left(4M_{phys}\right)^2$ and so on, subject to any selection rules. The spectrum of P^2 consists of a discrete point at $\mu^2 = m^2_{phys}$, a discrete point at bound states with $\mu^2 = m^2_b$ (which we ignore for present purposes) and continua starting at the eigenvalues associated with the eigenstates that correspond to multi-particle states in the asymptotic regions, with the lowest commencing at $\mu^2_{threshold}$. Although the particle interpretation of the eigenstates in the asymptotic regions cannot be prolonged to the general interacting states, their

[6] Neglecting bound states for now, and note that depending on the theory selection rules may require, for instance, the lowest mass multi-particle phion state is, e.g. $3m_{phys}$ (Barton, 1963, 55).
[7] For simplicity I omit the 'smearing' function required to produce single particle states here.

spectrum can due to conservation laws and the unitarity of the evolution of the states, giving the analytic structure of the propagator (cf. §10.4). $\rho_\varphi(\mu^2)$ is the amplitude for coupling $\hat{\varphi}$ with eigenstate of invariant mass μ^2.

Mathematically speaking (i.e. p^2 is only interpreted as the physical 4-momentum when it is in the spectrum of P^2 and thus associated with physical states), $\widetilde{D}'(p^2)$ is the continuation on the complex p^2-plane of

$$\tilde{G}_\varphi^F(p) \equiv \tilde{\Delta}_{F,\varphi}(p) = \int_0^\infty \frac{d\mu^2}{2\pi} \frac{i\rho_\varphi(\mu^2)}{p^2 - \mu^2 + i\varepsilon}$$

$\widetilde{D}'(p^2)$ has an isolated pole at $p^2 = m_{phys}^2$, isolated poles corresponding to bound states, and branch cuts commencing at the invariant mass-squared eigenstates that resolve as multi-particle states in the idealized asymptotic regions:

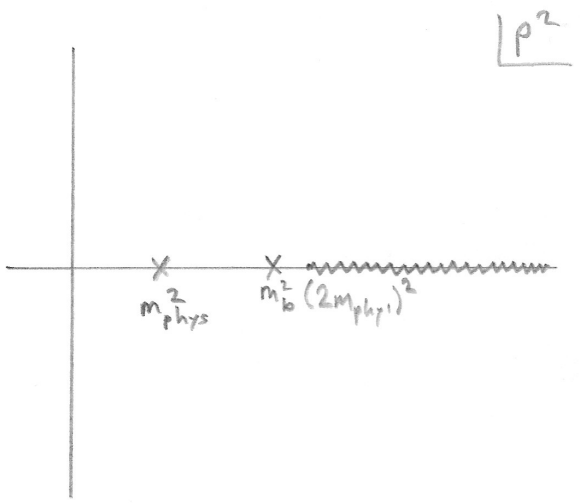

Fig. 11.1 Analytic structure of $\widetilde{D}'(p^2)$ for field of mass less than threshold mass

The propagator has the mathematical form, inserting now a bound state, (cf. Schweber 1964, 663; Schwartz 2014, 467-470)

$$\tilde{G}_\varphi^F(p^2) = \widetilde{D}'(p^2)\Big|_{\lim_{\varepsilon\to 0^+} p^2+i\varepsilon,\ \ p^2\in\mathbb{R}^+}$$

$$= \frac{iZ_\varphi}{p^2 - m_{phys}^2 + i\varepsilon} + \frac{iB}{p^2 - m_b^2 + i\varepsilon} + \int_{\mu^2_{threshold}}^\infty \frac{d\mu^2}{2\pi} \frac{i\rho_\varphi(\mu^2)}{p^2 - \mu^2 + i\varepsilon}$$

The pole at $p^2 = m_{phys}^2$ is associated with the propagation of a free phion. Z_φ is the 'wave function' or 'field strength' renormalization factor as it gives the amplitude for $\hat{\varphi}$ to excite an idealized physical phion of invariant mass m_{phys}^2 as associated with the Fock space structure generated by $\hat{\varphi}_{in}$. The choice of $\hat{\varphi}$ as an 'interpolating field' (cf. §11.3.2) is underdetermined in scattering theory as any

such field coupling with this physical single phion state will suffice. The general form of the spectral weight is sketched:

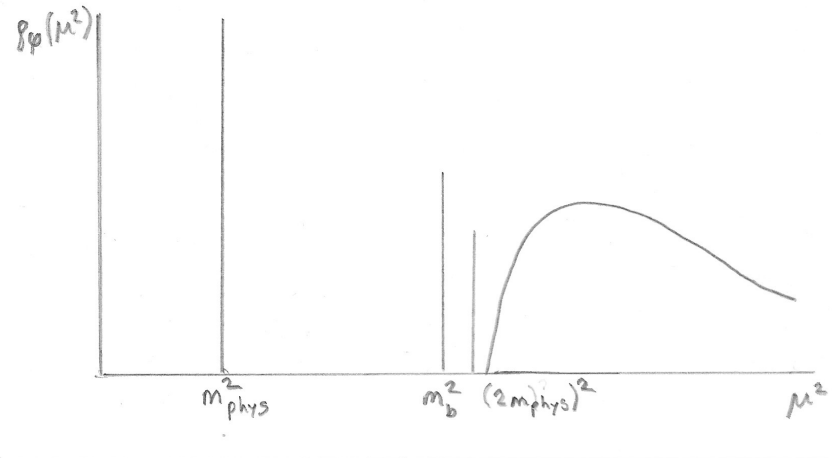

Fig. 11.2 Spectral weight for field of mass less than threshold mass

This is a valuable result as it enables one to identify the asymptotic physical particle associated with the chosen field with its physical mass and field strength renormalization Z_φ from the pole of the momentum space propagator. Moreover, since fields require integration against test functions $f(p)$ to produce realistic particle wave-packets, a 'momentum filter' may be introduced to isolate, for instance, the single physical phion contribution by choosing the support of $f(p)$ such that it only contains p such that p^2 is in a small region containing m^2_{phys}, as used in the Haag-Ruelle theory in §11.3.1.

Although the 'above threshold' states are often interpreted as 'the multi-particle states', they should simply be interpreted as eigenstates of given invariant mass without finer structure, such as a particle interpretation, as this interpretation is unsupported, resulting from semantic mimicry as the 'above threshold' states do not support a physically meaningful Fock structure, except in the idealized asymptotic states (cf. §10.2-3).

11.2.3 Case 2: Lowest mass state $\hat{\chi}$ excites > threshold mass: unstable particle or quasiparticle

$i\tilde{G}^F_\varphi(p^2)$ has an imaginary part for $p^2 > \mu^2_{threshold}$ (Barton 1963, 54).[8] This leads to a complication (that can be exploited) if in the scalar Yukawa theory $\hat{\varphi}$ has

[8] The comment is somewhat awkward since some, such as Barton, relocate the *i* factor so that our *G, D* is their *iG, iD*.

mass greater than the threshold mass. In such a case $\hat{\varphi}$ no longer couples with a lowest mass state corresponding to an isolated pole in $\widetilde{D}'(p^2)$. This complicates the analysis of the action of $\hat{\varphi}$ in the propagator. I omit the mathematical details and state the result with minimal discussion of the mathematical derivation. Very briefly, $i\tilde{G}_{\varphi}^{F}(p^2)$ now always has an imaginary part, with there being a discontinuity in $\widetilde{D}'(p^2)$ in the complex p^2-plane across the branch cut. This necessitates the definition of the propagator $\tilde{G}_{\varphi}^{F}(p^2)$ using the continuation on the Riemann surface associated $\widetilde{D}'(p^2)$, displacing the pole in the propagator associated with $\hat{\varphi}$ below the real axis: [9]

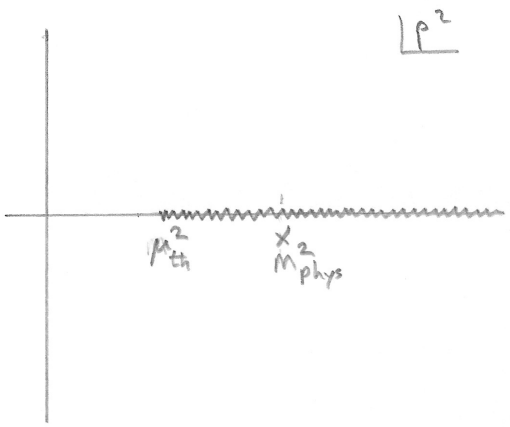

Fig. 11.3 Analytic structure of $\widetilde{D}'(p^2)$ for field with mass greater than threshold mass

The spectral weight is now sketched:

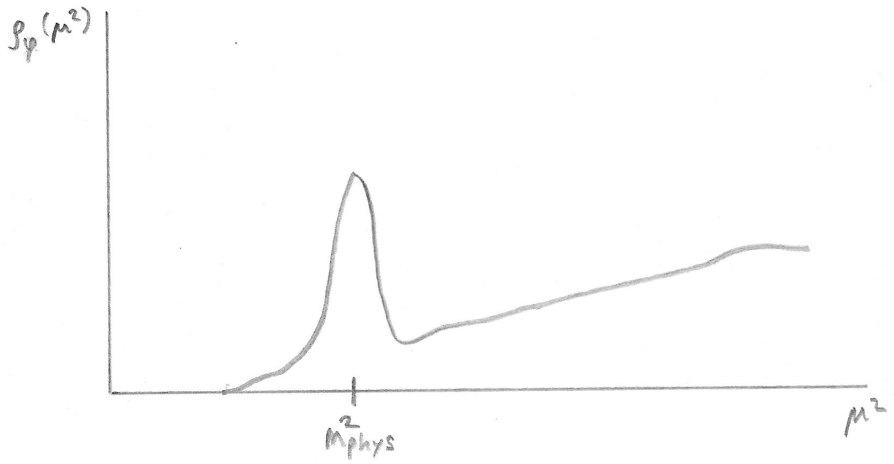

Fig. 11.4 Spectral weight for field associated with unstable particles or quasiparticles

The form of the propagator is now:

[9] See Brown (1992, 298-308); Coleman (2019, 355-362) for what follows.

$$\tilde{G}_{\varphi}^{F}(p) = \frac{iZ_{\varphi}}{p^2 - m_{phys}^2 - i\Gamma} + G_{smooth\ background}$$

where $\Gamma = -Z_{\varphi}\mathrm{Im}\,\widetilde{D}'^{-1}(m_{phys}^2)$. The time dependency of $\tilde{G}_{\varphi}^{F}(p)$ may be expressed, for t sufficiently large, as

$$\tilde{G}_{\varphi}^{F}(\mathbf{p}, t) = \int \frac{dp^0}{2\pi} e^{-ip^0 t} \tilde{G}_{\varphi}^{F}(p) \approx \frac{iZ_{\varphi}}{2\sqrt{\mathbf{p}^2 - m_{phys}^2 - i\Gamma}} e^{-i\sqrt{\mathbf{p}^2 - m_{phys}^2 - i\Gamma}\,t} + \frac{1}{(kt)^N}$$

The pole dominates if $\Gamma \ll m_{phys}$ is not 'too large'. The 'resonance' associated with the pole term may be interpreted as an unstable phion with mean proper lifetime $\frac{m_{phys}}{\Gamma}$. That is, near the pole (Coleman 2019, 360)

$$\tilde{G}_{\varphi}^{F}(p) \approx \frac{iZ_{\varphi}}{p^2 - m_{phys}^2 - i\Gamma}$$

which is interpreted as the Feynman propagator for an unstable particle, and may be understood as a 'dispersion relation'.[10] As in case 1, the spectral resolution enables one to identify a free 'renormalized' particle state associated with the field.

Understood as an elementary theory, the phion associated with this field is unstable, decaying into a psion-antipsion pair, and as such does not have any asymptotic states or fields associated with it. Calculations may be performed with Feynman diagrams using iterative series techniques, but with 'on mass-shell' intermediate states associated with unstable particles.[11] This theoretical approach is also associated with the modelling of certain types of quasiparticle (Lancaster and Blundell 2014, 276-278),[12] for as noted above, the $\hat{\chi}$-field need not be an 'elementary' field.

Whilst we can model or perform calculations of the decay of unstable states into stable states, the physical interpretation of the mathematical representation derived for unstable particles is not at all clear. The form of the propagator or dispersion relation suggests that unstable particles are modelled as a 'resonance' of a continuum of eigenstates of P_{μ} subject to a dispersion

[10] This may also be interpreted in relation to the Breit-Wigner distribution giving the probability for the production of final states of given momentum from an impulsive excitation via the $\hat{\chi}$-field – it peaks at m_{phys}^2 (Coleman 2019, 360-361).

[11] See e.g. Coleman, Brown or Peskin and Schroeder for examples.

[12] See Anderson (1997, 126-132) for detailed application of this theoretical approach to modeling quasiparticles in metals.

relation,[13] unlike a stable idealized asymptotic particle that is associated with a single isolated eigenstate of P_μ. To be clear, the unstable particle is not here considered as a bound state of its decay products.

However, it is not clear if such a 'resonance' is to be interpreted as a superposition of eigenstates of P_μ, whatever the physical interpretation of these eigenstates. Although the eigenstates coupled to $\hat{\chi}$ might exist as independent isolated states, in the context of the characterization of an unstable particle they cannot be abstracted such that they have the same form individually and in combination owing to the dispersion relation: the eigenstates associated with the unstable particle propagator have physical meaning, in the context of an unstable particle, collectively as a continuum of states subject to a dispersion relation. That is, the partial state and partial laws associated with individual eigenstates in the context of an unstable particle depends on their combination in that relation, even though the eigenstates can be 'traced' in combination using the dispersion relation. It is a borderline case of application of 'generalized superposition' at best, exemplifying the conceptual difficulties and perhaps limitations involved in the application and interpretation of the concept.

It is not clear the sense in which we might consider unstable particles as 'composed' of eigenstates. It may simply be a matter of calculational convenience that they can be modelled in this way. What is achieved however is a way of modelling the behaviour of unstable particles in relation to their free propagation even if it is not clear how one should conceptualize an unstable particle. The point is perhaps that our understanding of unstable particle models is best considered in relation to an engineer's approach as indicated above.

11.3 Scattering theory and the relationships between the $\widehat{X}_{out}$, $\widehat{X}_{in}$ and $\widehat{X}$ fields and states

11.3.1 Haag-Ruelle scattering theory: relating $\widehat{X}_{in/out}$ to $\widehat{X}$

I now consider how the axioms from the non-perturbative approach introduced in §10.3 are utilized in Haag-Ruelle scattering theory to establish a rigorous relation between the asymptotic fields and states associated with the $\widehat{X}_{in}$ and

[13] There is a different approach to modeling unstable particles using an extended Hilbert space which I do not consider here – see e.g. Weldon (1976); Kuksa (2015); Levy (1959).

those of the full interacting theory that are associated with the $\hat{X}$. This will be preparatory for discussion of the LSZ scattering theory in §11.3.2. The results here and in LSZ theory depend on the careful construction of suitable 'smeared fields' associated with particle wave-packets in the asymptotic states that are related to the $\hat{X}$ fields. For simplicity we work with the neutral scalar field $\hat{\varphi}(x)$. We construct two types of smeared field – one modelling a single particle state in the asymptotic regions, and a general smeared field, where both types are solutions to the Klein-Gordon equation so that the dynamics of the wave-packet are correct for modelling propagation according to the scalar field.[14]

The first type of smeared field is constructed in two stages such that it produces only time-independent single particle states (in the asymptotic limit) from the vacuum. Form the smeared field $\hat{\varphi}_1(x)$ by integrating $\hat{\varphi}(x)$ against a test-function $f^{(1)}(x)$ whose Fourier transform has support in the region $am_{phys}^2 < p^2 < bm_{phys}^2$, where $0<a<1$ and $1<b<4$. Making the spectral assumption as per §10.3.1, this ensures that $\hat{\varphi}_1(x)$ produces exactly one-particle states from the vacuum. Using covariantly normalized one particle states, $\langle \mathbf{k}'|\mathbf{k}\rangle = 2E(k)\delta^3(k'-k)$,

$$\langle \mathbf{k}|\hat{\varphi}_1(x)|\Omega\rangle = \langle \mathbf{k}|\hat{\varphi}(0)|\Omega\rangle \tilde{f}^{(1)}(\mathbf{k})e^{ik\cdot x}$$

Switching to non-covariantly normalized states for what follows,

$$\langle \mathbf{k}|\hat{\varphi}_1(x)|\Omega\rangle = \frac{1}{(2\pi)^{3/2}\sqrt{2E(k)}}\tilde{f}^{(1)}(\mathbf{k})e^{ik\cdot x}$$

Then for $g(\mathbf{x},t)$ a solution to the Klein-Gordon equation which has a smooth rapidly decreasing momentum wave-function:

$$g(\mathbf{x},t) = \int \frac{d^3p}{2E(p)}\tilde{g}(\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}-E(p)t)}, \qquad E(p) = \sqrt{\mathbf{p}^2 - m^2}$$

the required smeared field is

$$\hat{\varphi}_{1,g}(t) \equiv -i\int d^3x\{\, g(\mathbf{x},t)\frac{\overleftrightarrow{\partial}}{\partial t}\hat{\varphi}_1(\boldsymbol{x},t)\}$$

If multiple applications of $\hat{\varphi}_{1,g}(t)$ are made the state is no longer time-independent, but has a well-defined strong limit for $t \to \pm\infty$, the central result of the Haag-Ruelle scattering theory. For the non-covariantly normalized state $|\mathbf{k}\rangle$, for the single particle wave-function

[14] For what follows see Duncan (2012, 268-276).

$$\Phi_{1,g}(\mathbf{k}) \equiv \langle \mathbf{k} | \hat{\varphi}_{1,g}(t) | \Omega \rangle = (2\pi)^{3/2} \frac{\tilde{g}(\mathbf{k}) \tilde{f}^{(1)}(\mathbf{k})}{\sqrt{2E(k)}}$$

So, 'Any desired (fast-decreasing) momentum-space wave-function of our single-particle state can evidently be obtained as a product of appropriately chosen factors $\tilde{g}(\mathbf{k})$ and $\tilde{f}^{(1)}(\mathbf{k})$.' (Duncan, 271) The scattering theory developed involves the study of the limits as $t \to \pm\infty$ of states obtained from the vacuum by fields $\hat{\varphi}_{1,g}(t)$. Define the time-dependent state:

$$|\Phi, t\rangle \equiv \hat{\varphi}_{1,g_1}(t) \hat{\varphi}_{1,g_2}(t) \dots \hat{\varphi}_{1,g_m}(t) |\Omega\rangle$$

Then *Haag's asymptotic theorem* states that $|\Phi, t\rangle$ converges strongly in the limit $t \to -\infty$ to the m-particle in-state:

$$|\Phi\rangle_{in} = |g_1, g_2, \dots g_m\rangle_{in} \equiv \int d^3k_1 \dots d^3k_m \Phi_{1,g_1}(\mathbf{k}_1) \dots \Phi_{1,g_m}(\mathbf{k}_m) |\mathbf{k}_1, \dots \mathbf{k}_m\rangle_{in}$$

The convergence is a result of wave-packet spreading, although the convergence is faster if the states have disjoint support in momentum space, corresponding to the notion of well-separated particles travelling in different directions.

Haag's asymptotic theorem enables derivation of the 'direct connection between the interpolating Heisenberg field $\phi(x)$ and the free in (resp. out) fields $\phi_{in}(x)$ (resp. $\phi_{out}(x)$)' (Duncan, 278). This is done by defining a smeared field $\phi_g(t)$ similar to the $\phi_{1,g}(t)$ fields but where the initial smearing function $f^{(1)}(x)$ is a general Schwarz function $f(x)$. $\phi_g(t)|\Omega\rangle$ is not a single particle state, nor is it time-independent. We have (for non-covariantly normalized states):

$${}_{in}\langle \mathbf{k} | \hat{\varphi}_f(x) | \Omega \rangle = \frac{Z^{1/2}}{(2\pi)^{3/2}\sqrt{2E(k)}} \tilde{f}\,(k) e^{ik\cdot x}$$

where the 4-momentum k is on mass-shell for the state $|\mathbf{k}\rangle_{in}$, i.e., $k_0 = E(k) = \sqrt{\mathbf{k}^2 + m^2}$. Moreover:

$$\Phi_g(\mathbf{k}) \equiv \langle \mathbf{k} | \hat{\varphi}_g(t) | \Omega \rangle = (2\pi)^{3/2} \frac{\tilde{g}(\mathbf{k}) \tilde{f}\,(\mathbf{k})}{\sqrt{2E(k)}}$$

Now consider a smeared field $\hat{\varphi}_{in,g}(t)$ defined analogously from the free field $\hat{\varphi}_{in}(x)$ associated with the Fock space structure of states of the asymptotic region. We have

$$\left(\partial^2 + m_{phys}^2\right) \hat{\varphi}_{in}(x) = 0$$

and

$$\hat{\varphi}_{in}(x) = \hat{\varphi}_{in}(\boldsymbol{x}, t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E(k)} \left(\hat{a}_{in}(\boldsymbol{k}) e^{-ik\cdot x} + \hat{a}^{\dagger}_{in}(\boldsymbol{k}) e^{ik\cdot x} \right)$$

where $E(k) = \left(\mathbf{k}^2 + m^2_{phys} \right)^{1/2}$ and

$$\hat{\phi}_{in,g}(t) = \int d^3k \, \Phi_g(\mathbf{k}) \hat{a}^{\dagger}_{in}(\boldsymbol{k})$$

Then *provided that asymptotic completeness holds* the weak equivalence relationship between $\hat{\varphi}_g(t)$ and $\hat{\varphi}_{in,g}(t)$ as $t \to \infty$, that is

$$_{in}\langle \beta | \hat{\varphi}_g(t) | \alpha \rangle_{in} \to Z^{\frac{1}{2}} {}_{in}\langle \beta | \hat{\varphi}_{in,g}(t) | \alpha \rangle_{in}, \; t \to -\infty$$

for arbitrary in-states $|\alpha\rangle_{in}, |\beta\rangle_{in}$ where Z is the wave-function or field strength renormalization factor, may be rigorously established. This *asymptotic condition* is the starting point for LSZ scattering theory. (Duncan, 281).

11.3.2 LSZ scattering theory

The Haag-Ruelle theory makes a precise connection between the 'interpolating' or 'Heisenberg' field $\hat{\phi}(x)$ of the coupled field equations and the asymptotic in- and out-states (assuming AC), provided that $\hat{\phi}(x)$ has a non-vanishing matrix element from the vacuum to the single particle state (Duncan 281-282), i.e.

$$_{in}\langle \mathbf{k} | \hat{\phi}(x) \, | \Omega \rangle = \frac{Z^{1/2}}{(2\pi)^{3/2} \sqrt{2E(k)}} e^{ik\cdot x} \quad \text{with } Z \neq 0$$

By considering the *S*-matrix element for the scattering of *n* incoming scalar particles with momentum space wave-functions $\widehat{\Phi}_{g_1}(\mathbf{k}), \ldots, \widehat{\Phi}_{g_n}(\mathbf{k})$ into *m* outgoing particles with momentum space wave-functions $\widehat{\Phi}_{g'_1}(\mathbf{k}), \ldots, \widehat{\Phi}_{g'_m}(\mathbf{k})$ defined as above with disjoint support in momentum space, that is:

$$S_{g'_1 \ldots g'_m, g_1 \ldots g_n} = {}_{out}\langle g'_1, \ldots g'_m | g_1, \ldots g_n \rangle_{in}$$

it may be shown that:[15]

$$S_{g'_1 \ldots g'_m, g_1 \ldots g_n} = \left(iZ^{\frac{1}{2}} \right)^{m+n} \int \prod_{i=1}^{n} \prod_{j=1}^{m} d^4x_i \, d^4x'_j g_i(x_i) g'^{*}_j(x'_j) (\partial^2_{x_i} + m^2) (\partial^2_{x'_j} + m^2) \cdot {}_{out}\langle \Omega | T\{ \hat{\varphi}(x'_1) \ldots \hat{\varphi}(x'_m) \hat{\varphi}(x_1) \ldots \hat{\varphi}(x_n) \} | \Omega \rangle_{in}$$

This is the LSZ reduction formula (Lehmann, Symanzik and Zimmerman 1955), 'giving the multi-particle S-matrix element in terms of an integral involving the

[15] See Duncan (281-289). For simplicity I quote the result for phions, although analogous results may be obtained with charged scalar and fermionic fields.

vacuum-expectation-value of the time-ordered product of the *n+m* Heisenberg interpolating fields (the *n+m* point *Feynman amplitude*) for the particle undergoing scattering' (Duncan, 286). Conventionally the limit is taken in which the wave packets approach plane wave solutions with sharp momenta, i.e. we take

$$g_k(x) = \frac{1}{\sqrt{(2\pi)^3 2E(k)}} e^{-ik\cdot x}$$

so that 'the LSZ formula gives the S-matrix element as a Fourier transform of the distribution obtained by applying the Klein-Gordon operators … to the Feynman amplitude for *n+m* fields' (Duncan, 286):

$$S_{k'_1 \dots k'_m, k_1 \dots k_n}$$

$$= \left(iZ^{\frac{1}{2}}\right)^{m+n} \int \prod_{i=1}^{n} \prod_{j=1}^{m} d^4x_i \, d^4x'_j \frac{1}{\sqrt{(2\pi)^3 2E(k_i)}} \frac{1}{\sqrt{(2\pi)^3 2E(k'_j)}} e^{ik'_j\cdot x_j - ik_i\cdot x_i} (\partial^2_{x_i}$$

$$+ m^2)(\partial^2_{x'_j} + m^2) \cdot \, {}_{out}\langle\Omega|T\{\hat{\varphi}(x'_1) \dots \hat{\varphi}(x'_m)\hat{\varphi}(x_1) \dots \hat{\varphi}(x_n)\}|\Omega\rangle_{in}$$

The 'Feynman Green functions' for the full interacting theory are defined:

$$G(x'_1, \dots x'_m, x_1, \dots x_n) = {}_{out}\langle\Omega|T\{\hat{\varphi}(x'_1) \dots \hat{\varphi}(x'_m)\hat{\varphi}(x_1) \dots \hat{\varphi}(x_n)\}|\Omega\rangle_{in}$$

$$\tilde{G}(k'_1, \dots k'_m, k_1, \dots k_n)$$

$$= \int d^4x'_1 \dots d^4x'_m d^4x_1 \dots d^4x_n \, e^{i\sum k'_j\cdot x_j - i\sum k_i\cdot x_i} \, {}_{out}\langle\Omega|T\{\hat{\varphi}(x'_1) \dots \hat{\varphi}(x'_m)\hat{\varphi}(x_1) \dots \hat{\varphi}(x_n)\}|\Omega\rangle_{in}$$

where the momenta are arbitrary and do not satisfy the mass-shell condition. Integrating by parts gives (Duncan, 286-287):

$$S_{k'_1 \dots k'_m, k_1 \dots k_n}$$

$$= \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{-iZ^{\frac{1}{2}}(k_i^2 - m_{phys}^2)}{\sqrt{(2\pi)^3 2E(k_i)}} \frac{-iZ^{\frac{1}{2}}(k'^2_j - m_{phys}^2)}{\sqrt{(2\pi)^3 2E(k'_j)}} \tilde{G}(k'_1, \dots k'_m, k_1, \dots k_n)$$

So for the S-matrix amplitudes to be finite and non-zero, $\tilde{G}(k'_1, \dots k'_m, k_1, \dots k_n)$ has simple poles in the $k^2 - m^2_{phys}$ for each of the incoming and outgoing states identified as idealized asymptotic, free physical (dressed or renormalized) particles associated with the free $\hat{\varphi}_{in}$ of mass m_{phys}. The 'Green functions' however involve the 'bare, interacting' $\hat{\varphi}$-fields of mass m of the interacting theory, the 'interpolating fields', satisfying the coupled field equations.

Coleman summarizes the significance of the LSZ result: 'if you know the Green's functions exactly, then you know the S-matrix elements exactly, and *a fortiori* if you have an approximation for Green's functions, you have an approximation for the S-matrix elements.' (2019, 355) The Gell-Mann and Low theorem (§11.4) enables us to relate the Green's functions of the interacting theory with the correlation functions obtained using the free fields in the interaction picture calculated with Dyson's expansion (chapter 9). That is, using the Gell-Mann and Low theorem and Dyson's expansion we can, after renormalization, generate an approximation for the Green's functions and thus an approximation for the *S*-matrix elements.

Before considering the Gell-Mann and Low theorem I make two observations on the LSZ result. First, asymptotic completeness (AC) is assumed, and the implications of the failure of AC for the LSZ result are unclear. In this sense the LSZ result is not rigorously justified, although AC is not usually doubted.

Secondly, as Duncan observes, *any* almost local field 'with a non-vanishing vacuum to single particle matrix element' can be used, so that '[e]ven if the particle corresponds to an elementary local field in the theory, *there is no unique interpolating field giving the correct S-matrix for scattering!*' (Duncan, 287). The correlation (Green's) function and normalization constant will be different for different interpolating fields; 'only the multiple pole residue of the on-mass-shell limit of its Fourier transform is guaranteed to be independent of the choice of field, as it gives the presumably unique physical S-matrix amplitude for the scattering of a specific stable particle.' (Duncan, 287). Or, as Coleman puts it,

> The only thing that was required in driving the LSZ reduction is that somehow we could get our hands on a local [interpolating] field with a non-zero vacuum to one-particle matrix element, that makes *some* kind of particle out of the vacuum. It can make any other kind of junk it wants, as long as it has a non-zero matrix element. We don't demand that the field satisfy the canonical commutation relations. (Coleman, 298)

This means that the (interpolating) $\hat{X}$ fields are underdetermined in the LSZ theory, for any fields that have non-zero overlaps with the asymptotic single particle states suffice. This may be considered to be a consequence of the failure of the Volkmann device to apply to interacting fields/states (§10.5), as manifested in the need for renormalization, and compensated for by it. We do not have a natural description of general interacting states in relation to fields,

with the fields being chosen for calculational convenience apart from the application of 'superposition'.

One might try to compare choice of interpolating fields with the choice of components of a vector decomposition as in §3.1.2 for calculational convenience, as Volkmann discusses (1896, 82-83), which is a borderline case for application of 'superposition'. Arbitrary components of a decomposed vector may have explanatory relevance, even if the components are underdetermined. The situation here is, however, worse since the interpolating fields require renormalization, as they do not behave as abstractable components that take the same form individually and in combination as in the arbitrary vector decomposition case. That is, even though a vector decomposition may be arbitrary but support calculations, the purpose of the choice is to isolate components that take the same form individually and in combination in order to simplify analysis. But the failure of 'superposition', associated with the requirement for renormalization, which relates to the nonlinearity of the coupled field equations, shows this not to be the case here for the choice of the interpolating fields and associated states. On the one hand, the interpolating fields are chosen to enable calculation, even if they are underdetermined, but they do not take the same form individually and in combination owing to the non-linearity of the coupled field equations, and so are not to be compared with the components of a decomposed vector.

These observations clarify that the situation is worse than the observation in §10.2 that we lack a particle description as a natural description of interacting states as associated with the $\hat{X}$ fields, and that the Volkmann device fails in the selection of the $\hat{X}$ fields. The only natural description that we have of interacting states is given 'in principle' in terms of the evolution of the system described in terms of the eigenstates of the 4-momentum operator, apart from any reference to fields. But now it is not clear that we have a 'field theory' at all, or rather, field theory is introduced in order to enable calculations and not because it supports a description of interacting states.

11.4 The Gell-Mann and Low theorem

The Gell-Mann and Low theorem supports iterative approximations in QFT by connecting 'bare states $|\alpha\rangle$ (eigenstates of H_0) to the corresponding in- and out-states ... $|\alpha\rangle_{in} = U(0,-\infty)|\alpha\rangle,\ |\alpha\rangle_{out} = U(0,+\infty)|\alpha\rangle$' (Duncan 2012, 245). It was derived to treat bound states in QFT (Gell-Mann and Low 1951) and relates ground states of the free Hamiltonian H_0 to those of the full Hamiltonian where an interaction is introduced as a perturbation i.e. H=H_0+gV. The result continues to be employed in the analysis of bound states in QFT (see §11.5), but it is also used to relate the 'Green's functions' for the coupled $\hat{X}$-fields acting on $|\Omega\rangle$ of the LSZ result as above to the 'Green's functions' of the interaction picture fields $\hat{X}_0$ acting on $|0\rangle$ which can be calculated by Dyson's iterative expansion to obtain approximations for the S-matrix elements, after renormalization. The proof of the result was originally given by perturbative methods, although more recently Molinari (2007) has obtained a proof without perturbation techniques.

The result is established using adiabatic switching, defining the time-dependent operator:

$$\hat{H}_\varepsilon(t) = \hat{H}_0 + e^{-\varepsilon|t|} g\hat{V}$$

so that $\hat{H}_\varepsilon$ interpolates between the free Hamiltonian in the asymptotic infinite time limits and the full Hamiltonian at t=0. Let $\hat{U}_\varepsilon(t,s)$ be the evolution operator for $\hat{H}_\varepsilon$, and introduce the interaction picture evolution operator

$$\hat{U}_{\varepsilon I}(t,s) = e^{it\hat{H}_0}\hat{U}_\varepsilon(t,s)e^{-is\hat{H}_0}$$

The original statement of the theorem is that if $|\Psi_0\rangle$ is an eigenstate of H_0 with eigenvalue E_0, then if the

$$|\Psi^{(\pm)}\rangle = \lim_{\varepsilon\to 0^+} |\Psi_\varepsilon^{(\pm)}\rangle = \lim_{\varepsilon\to 0^+} \frac{\hat{U}_{\varepsilon I}(0,\pm\infty)|\Psi_0\rangle}{\langle\Psi_0|\hat{U}_{\varepsilon I}(0,\pm\infty)|\Psi_0\rangle}$$

exist, they are eigenstates of $\hat{H}$.

This result can be applied in scattering theory to show that

$$G(x'_1,\dots x'_m,x_1,\dots x_n) = {}_{out}\langle\Omega|T\{\hat{\varphi}(x'_1)\dots\hat{\varphi}(x'_m)\hat{\varphi}(x_1)\dots\hat{\varphi}(x_n)\}|\Omega\rangle_{in}$$
$$= \frac{\langle 0|T\{\hat{\varphi}_I(x'_1)\dots\hat{\varphi}_I(x'_m)\hat{\varphi}_I(x_1)\dots\hat{\varphi}_I(x_n)\}S_0|0\rangle}{\langle 0|S_0|0\rangle}$$
$$= \frac{\langle 0|T\{\hat{\varphi}_0(x'_1)\dots\hat{\varphi}_0(x'_m)\hat{\varphi}_0(x_1)\dots\hat{\varphi}_0(x_n)\}S_0|0\rangle}{\langle 0|S_0|0\rangle}$$

The final expression is evaluated using Dyson's series and Wick's theorem as in chapter 9, although without using the a-operators, and it can be shown that (Lancaster and Blundell 2014, 204-205):

$$\frac{\langle 0|T\{\hat{\varphi}_0(x'_1)\ldots\hat{\varphi}_0(x'_m)\hat{\varphi}_0(x_1)\ldots\hat{\varphi}_0(x_n)\}S_0|0\rangle}{\langle 0|S_0|0\rangle} = \sum\begin{pmatrix}\text{Connected diagrams with}\\ n+m\text{ external lines}\end{pmatrix}$$

So, after renormalization, an algorithm is available to approximate the S-matrix elements in the LSZ theory so that scattering amplitudes can be calculated.[16] The failure of 'superposition' has been compensated for in order to support calculations even if not descriptions or explanations of interaction processes.

I now consider application of the Gell-Mann and Low theorem to calculations involving bound states.

11.5 Analysis of bound states using the Gell-Mann and Low theorem

'Bound states' are ubiquitous; matter is constituted from 'bound states' in the form of nucleons, atoms, molecules and ionic structures, so it is important to clarify our conceptualization and analysis of bound states in QFT. Intuitively we regard a bound state as composed of particles whose mutual interaction binds them, as a particular type of interacting state. For instance, a hydrogen atom is considered to be composed of a proton and an electron bound by electromagnetic interaction (or the exchange of 'virtual photons' on common accounts, that we have seen to be problematic) modelled by QED. Alternatively, nucleons are often construed as bound states composed of three quarks interacting via gluon fields according to QCD, or nuclei are considered to be bound states of protons and neutrons modelled by QCD and electroweak theory.

However, my analysis indicates that such descriptions or 'pictures' are misleading, for we have seen that there is no particle description of general interacting states, of which bound states are a kind. The decomposition of bound states into partial states associated with different field types is not supported in QFT owing to the failure of 'superposition' (§10.5). We cannot prolong the identity of the idealized asymptotic free physical 'fundamental' particles into interacting states as associated with the $\hat{X}_{in}$, so we cannot consider bound states

[16] There are a number of technical details required to implement the algorithm that I do not consider here. See the standard texts for details. The point is that such an algorithm exists so that reliable calculations may be supported.

to be 'composed of' rather than 'made from' fundamental particles as discussed in §10.3.4. We saw there that the concept of 'interaction' between putative components of a bound state is meaningless in QFT, because interacting QFT does not support the identification of such components owing to the failure of the Volkmann device.

I noted in §10.4 that 'in principle' natural descriptions of general interacting states are available as superpositions of eigenstates of the 4-momemtum operator, but in general we can say very little about such eigenstates. We cannot relate them to familiar particle/field notions outside the asymptotic context. Moreover, we saw that bound states are eigenstates corresponding to an isolated invariant mass eigenvalue in the spectrum of the 4-momentum operator, without any finer-grained (e.g. Fock) structure. So, we describe a bound state in QFT simply as an eigenstate of the 4-momentum operator with an (isolated) invariant mass just below the threshold mass. As regards a natural description of bound states, this is all there is to say in QFT.

However, as I shall indicate, in iterative perturbation theory an 'approximate structure' may be modelled in some circumstances according to an 'engineer's model'. In terms of the Newton-Wigner approach, free bound states are 'elementary systems' owing to their transformation properties (cf. §8.6). Are they 'elementary particles'? This depends on Newton and Wigner's somewhat loose distinction between elementary systems and elementary particles, namely whether or not it is 'useful to consider the particle as a union of other particles' (Newton and Wigner 1949, 400). That is, it is often *useful to consider* a hydrogen atom as a bound state of an electron and a proton, even if it is not strictly the case according to QFT. I'll consider Newton and Wigner's distinction in more detail with reference to an 'engineer's approach' in relation to the strength of the coupling of the theory below. First however I consider 'usefulness' in relation to the ability to perform successful calculations based on the assumption that a bound state is a composite state, composed of its asymptotic particle content.

There is no universal approach for analysing bound states in QFT, and Ligterink and Weber suggest that it is a truly hard problem (2009, 115). They survey various approaches, developing their own analysis in Yukawa theory in two papers (2001; 2009) using the 'generalized Gell-Mann and Low theorem'. I

now discuss their approach, being an indicative approach to the modelling of bound states. They summarize their approach:

> it is in principle necessary to determine the complete state vectors in Fock space [for a bound state]. However, such a state vector is a linear combination of states with definite numbers of particles, from the lowest component with two or three particles to states with an arbitrarily large number of particles and antiparticles. In many cases, the latter higher Fock states contribute only little to the characteristics of the bound state … and one can hope to sensibly truncate the number of particles in the Fock state. This will not be true, however, in intrinsically nonperturbative problems such as bound states in QCD.
>
> Simple-minded truncations in the particle number lead to many problems ... In … this paper, the Fock states are reconstructed from a projection to their lowest components with the least number of particles (which we will sometimes call their constituents). From these components, the complete states can be generated in a perturbative expansion similar to covariant perturbation theory, but with the advantage that bound states can be described in addition to scattering states. A truncation in the order of the perturbative expansion corresponds to a truncation in the number of particles in the full state. However, for a given finite order of the expansion, covariance is presumably broken to higher orders.
>
> It is assumed that the complete Fock state is determined uniquely by its lowest Fock component .... This is indeed true to every order in the perturbative expansion. .... In practice, the lowest Fock components of the full eigenstates are determined as eigenstates of a certain effective Hamiltonian acting in the subspace of lowest particle number. For the construction of the effective Hamiltonian that reflects the presence of the higher Fock states, the framework of the Gell-Mann–Low theorem … is extended. In the original Gell-Mann–Low approach the asymptotic, or free, scattering states are being evolved to interacting states and back to free states, to be able to describe the scattering process as a unitary matrix between free states. Clearly, for the description of bound states one has to go beyond the description of (physical) scattering processes, and the proper interacting states need to be defined in terms of the "free states", or lowest Fock states, in this case. The Gell-Mann–Low evolution from free states to interacting states precisely yields this transformation. (2009, 118)

That is, they suppose that the full Hilbert space representing the bound state can be well-approximated via what we intuitively consider its free particle content, where the free particles are adiabatically evolved into the interacting system of the bound state. Calculations are performed using an iterative series expansion truncated at some order as in scattering theory using a simplified 'effective Hamiltonian'.

Their approach may prove 'empirically adequate', providing good approximations to the properties and behaviour of bound states. But it does not provide a natural description or analysis of bound states as would be obtained by proper application of Fourier techniques and 'superposition'. Ligterink and Weber appear unaware of the conceptual problems, for instance that a particle description of a general interacting state via a Fock structure is unavailable, even in principle. They appear to regard the difficulties as essentially calculational, unaware of the semantic mimicry reflected in the prolongation of the asymptotic particle concept to the bound state. However, they only anticipate success in

weakly coupled theories. This is likely to be because 'superposition' is 'approximately true' even if it is strictly false in weakly coupled theories, with higher-order correction terms being small after renormalization, so that it is a good modelling assumption to suppose that a bound state is composed of component particles. This is not the case in strongly coupled theories for which superposition is not even approximately true, and perturbative techniques fail.

In conclusion, the concept of a 'bound state' is a misnomer in QFT, resulting from semantic mimicry. We should regard a 'bound state' as, for instance, the state formed by bringing together certain asymptotic fundamental particles in a certain way, but remain silent on its composition otherwise. As we shall see in §11.6, in some circumstances – in weakly coupled theories – the model, and the picture of a bound state as 'composed of' the particles it is 'made from' or 'scatters into' offers a reasonable conceptual or perhaps 'metaphysical' approximation to the situation, as it does for the composition of a general interacting state, such as two electrons scattering at low energy. Such models in weakly coupled theories support reliable calculations and explanations in the sense of an 'engineer's model', even if it does not offer a true or even natural description of a bound state.

I now address the distinction that the coupling strength indicates in relation to an 'engineering approach'.

11.6 The distinction between strongly and weakly coupled theories – an example of the application of the 'engineering approach'

I noted in §9.2 that although calculations in strongly coupled theories, whether gauge theories or not, are not performed using perturbative series expansion techniques, such theories are nonetheless perturbative in the sense that an interaction term is introduced via tacit but improper appeal to the Volkmann device. There is a difference from a *calculational* perspective between weakly and strongly coupled theories, but is there a *conceptual* difference? In one sense there is not – 'superposition', and in particular the initial application of the Volkmann device, fail in interacting theories in general and so there is no description of a general state in terms of isolatable or abstractable, simple partial states associated with partial laws via different fields that take the same form

individually and in combination. However, in another sense there is a conceptual difference. There is a difference of degree according to the strength of the coupling,[17] relating to how far one might regard application of 'superposition' and the Volkmann device to apply *approximately* or 'well enough' to support explanations and calculations within the context of an 'engineer's model' in a given domain.

If the coupling is sufficiently weak, whilst 'superposition' fails, it may hold 'well enough' to support an approximate description of the system that supports reliable inductive inferences, calculations, explanations and counterfactual reasoning for many purposes in a given domain. That is, for a weakly coupled theory use of 'superposition' *almost* yields a natural description of a general interacting state in terms of different particle numbers and types, which might be 'good enough' for many purposes. For many purposes it may be a good model to consider the general interacting state as composed of particles as identified from the asymptotic states. That is, the particle identity of the asymptotic states can be meaningfully traced or prolonged *approximately* to the interacting region, as in Ligterink and Weber's analysis of bound states above.

In interacting QFTs, as the coupling increases application of 'superposition' becomes more problematic in the analysis of general interacting states, both conceptually and calculationally, both in the initial application of the Volkmann device and in the decomposition of partial states associated with the chosen fields into a description of particle number and type. In weakly coupled theories at low energies the approximate applicability of 'superposition' enables the identification of an approximate finer-grained description of the eigenstates of the 4-momentum operator (which we know exist in principle apart from perturbative schemes) in terms of the familiar asymptotic particle types and numbers sufficient for calculations and explanations from an 'engineer's perspective', but this is not possible in strongly coupled theories. There is no natural criterion for the distinction – instead we might state that superposition is approximately true if the calculations based on it give results within 5% of the measured data – but we could specify 1% or 10%. However, it is difficult to

[17] Note however that as renormalization techniques highlight, the coupling of a theory is not constant but a parameter that varies with energy or length scale of interest. This means that a theory may be regarded as strongly coupled in one context but weakly coupled in another.

specify the kind of stance in advance appropriate to the model entities and explanations provided. For instance, is an explanation of the property of charge in terms of virtual particle exchanges a good explanation after all? We know this to be false, resulting from semantic mimicry, but it might provide a good model leading to a reasoning advantage for some purposes.

From an engineering perspective we can to a good conceptual and calculational approximation regard a hydrogen atom as a bound state of an electron and a proton using QED, which ultimately gives rise to the conceptual scheme (or science) of chemistry, supporting explanations and predictions. In this domain one generally need not model the electronic structure with QFT, and NRQM supports reliable explanations and calculations in many cases, where here the composite character of the hydrogen atom as a bound state of a proton and an electron is essentially put in 'by hand' as a modelling assumption in NRQM (§6.3.4). But such a model has the form of an 'engineer's model' rather than a 'true description' of the hydrogen atom, and so does not support analysis in familiar compositional terms.[18] Improved predictions, for example the energy spectrum of the hydrogen atom, are obtained by modelling the system to low orders in QED, but it is not clear what kind of explanation is really offered here, say from the scientific realist's perspective, for instance, with regard to explanations given in terms of 'vacuum polarization' regarding the hyperfine structure of the hydrogen atom. The point is that it is meaningful to think of such explanations, in a weakly coupled theory, in terms of small corrections to a linear model in which superposition holds. This is no longer the case for strongly coupled theories such as QCD.

Even from a modest 'engineering perspective' it is questionable whether a model of the composition of nucleons in terms of quarks bound by gluons according to QCD is supported at least insofar as 'composition' is usually understood. This is often discussed in relation to 'colour confinement', and owes

[18] Cf. McKenzie and Muller (2017) who develop a model of the hydrogen atom as a bound state in NRQM in compositional terms. If this project is situated in terms of discussion restricted to the ontology of this model, that is fine. But if it is developed with a view to developing a true compositional account of the physical world it fails if we assume that QFT offers a better account of the world than NRQM, for as we have seen such an account of composition is only true relative to an engineer's model, and is not supported by QFT without idealization and approximation.

to the strong coupling and non-abelian nature of QCD as a gauge theory.[19] The strength of the coupling in QCD at this energy scale, based on the initial application of the Volkmann device in which isolated quark and gluon fields are purportedly identified, suggests caution in this *initial use* of a 'superposition-like' concept in the Volkmann device. This difficulty arises in addition to the use of iterative solution techniques, which is where the difficulty is often located.[20] It appears unlikely that one can to a good approximation abstract gluon and quark fields and partial states associated with them when they are strongly coupled,[21] so that it is a poor model of a nucleon to consider it as a bound state of quarks from a non-engineering perspective when the coupling is large.

However, at very high energies the coupling reduces so that QCD becomes a weakly coupled theory (Lancaster and Blundell 2014, 307-308; Bain 2000), a phenomenon known as 'asymptotic freedom'. In this case 'superposition' is an idealization supporting the ability to abstract quark and gluon fields and associated partial states to a good approximation, comparable with the asymptotic states of the weakly coupled Yukawa theory that we have considered. That is, in the Yukawa theory and in QED, it is at large length (low energy) scales that a restricted application of 'superposition' is possible to form a particle description of the theory as an idealized natural description, whereas in QCD, it is at the asymptotically short length scale / high energy that a particle description is possible as an idealized natural description. This means that at high energies it can be meaningful, as an engineer's approximation, to consider a nucleon as composed of quarks, with such a model supporting calculations and explanations (cf. Lancaster and Blundell 2014, 307-308).[22] The difficulty in both cases is that the descriptions associated with idealized asymptotic limits are often prolonged by semantic mimicry into general states. When the coupling is

[19] I only consider the issue of the coupling strength here.
[20] The problem is usually discussed in relation to the coupling being too strong to support an iterative calculation. I am saying that the problem runs much deeper than this, relating to the failure of the initial application of the Volkmann device to be even approximately true, even if there are 'workarounds' to support calculations.
[21] This procedure has a mathematical architecture in the gauge principle. As previously noted (§9.2), it merits further analysis to clarify the relationship between the gauge principle and the use of the Volkmann device in which the physical architecture is central.
[22] Bain (2000) briefly considers the role of asymptotic freedom in developing a particle notion in QCD as noted in §10.2.3.

weak such prolongation often offers a good model that supports approximate calculations, and explanations in some sense, even if the model is strictly false.[23]

As a final illustration of these ideas in the engineering approach I briefly consider the situation in nuclear modelling, where we have a context involving the analysis of bound states using a strongly coupled theory. Cook draws attention to the 'curious state' of nuclear physics:

> a great deal is known about the technology of nuclear energy, and yet our understanding of the nucleus itself is seemingly quite incomplete. More than 30 (!) nuclear models – based on strikingly different assumptions – are currently employed … Each provides some insight into nuclear structure or dynamics, but none can claim to be more than a partial truth, often in conflict with the partial truths offered by other models. …
> It is rather disconcerting … that some … unanswered questions include truly basic issues, such as the phase state of nuclear matter … the nature of nuclear force … and the nature of nucleons themselves. (2006, 5-6)

Drawing upon Cook and Greiner and Maruhn (1996), Morrison (2015) discusses the multiplicity of partially successful nuclear models in the context of her treatment of the role of inconsistent and contradictory models. She suggests that 'we are left in an epistemic quandary when trying to evaluate the realistic status of these nuclear models and the information they provide. We can't simply conclude that all the information extracted from the models is dubious, since some provides the very foundation on which a good deal of technological knowledge (and some theoretical knowledge) is based.' (2015, 191)

However, the difficulties encountered in modelling nuclei exemplify some of the issues raised in the analysis of complicated 'bound states' of strongly coupled non-abelian gauge theories. The role that nuclear models play might be better understood with reference to a Wilsonian 'theory façade' in which an 'engineer's approach' is implicitly adopted to form a patchwork structure of locally successful models, rather than in terms of 'partial "truths"' (as Morrison puts it). The diagnosis of the situation is that a nucleus is a complicated general interacting state of a strongly coupled theory,[24] for which we do not have a natural description because 'superposition' fails, even to a reasonable approximation in most contexts of nuclear modelling. This necessitates the construction of 'engineering models' to obtain 'good enough' results in limited contexts. The 'curious state' occurs perhaps because we expect the Volkmann

[23] One might ask how the concept of semantic mimicry is to be understood in an engineering approach, but this is not a question that I can pursue here beyond noting that semantic mimicry is associated with an unreflective realist interpretation.
[24] Moreover, more than one theory is needed – QCD and electroweak theory.

device to apply so that we can form a natural description of a nucleus in terms of, say, neutrons and protons – each with a finer quark and gluon structure, when this is not the case. Perhaps we have come to expect 'superposition' to be approximately true at the level of the structure of the nucleus, when it is not, and it is by semantic mimicry that we picture a nucleus as a bound state that is composed of rather than made from protons and neutrons, with such a naively realist picture of the nucleus leading to conceptual confusions.

This is to be contrasted with models of the electron configuration of atoms, for which the approximate applicability of superposition, and the approximate applicability of a non-relativistic model, indicate that NRQM is adequate for many purposes in this context, even if it is strictly false.

11.7 Summary

We have seen how 'in principle' knowledge of the eigenstates of the 4-momentum operator leads to the ability to model unstable particles. We have also related the $\hat{X}$ to the $\hat{X}_{in}$ species using Haag-Ruelle theory, and established a scattering theory based on the idealized asymptotic states in LSZ scattering theory. The Gell-Mann and Low theorem allows approximation of the Green's functions required using Dyson's expansion, after renormalization, giving approximate scattering amplitudes. Reliable calculations are supported in scattering theory without a usable natural description of interacting states.

The Gell-Mann and Low theorem can support calculations of the properties and behaviour of bounds states in weakly coupled theories even though QFT does not supply a fine-grained enough structure to support the identification of a natural description of any internal structure to bound states, owing to the failure of 'superposition' in the context of general interacting states. That is, QFT does not model bound states as composed of simple constituent particles or partial states – bound states are themselves simple elements in QFT. However, in the case of weakly coupled theories 'superposition' approximately holds and so it is a good 'engineer's approximation' to model a bound state of a weakly coupled theory as 'composed of' constituent particles even though this may evade a straightforwardly realist construal. In strongly coupled theories 'superposition' does not even hold approximately, which leads to confusion.

Chapter 12

Concluding remarks and suggestions for further research

12.1 Summary

We began by considering the patchwork structure and philosophical significance of the concept of superposition in classical physics as motivated by a metaphysically quietist Wilsonian framework, drawing in particular on and developing the work of Paul Volkmann (1896) as re-appropriated within this philosophical context. This involved the application of the concept of superposition within Fourier techniques interpreted in terms of 'Hilbert superposition', which enabled us to identify and characterize instances of 'semantic mimicry', in the slightly narrower sense that I outlined in §1.2.3, in the use of Fourier series and in the interpretation of various series expansion methods. In particular we saw in the classical context (chapters 2-5) how:

(1) There are two main patches of application of 'superposition' as primarily associated with either Peter Simons' approach (i), or Paul Volkmann's approach (ii). According to (i) components of a complicated phenomenon may be identified that have independent causal origins such that the components persist when combined according to a trace principle. We say that the phenomenon is the superposition of the components. According to (ii), one seeks to identify abstracted partial states and associated partial laws of a complicated phenomenon or system that completely characterize it and take the same form individually and in combination, whilst not stating the facts in combination. When this is accomplished we call the combination the 'superposition' of the partial states and / or laws. Traditionally, a linear form of combination is understood, although this may be generalized. The generalized form I referred to as the 'Volkmann device'. The importance of such decompositions is that the components, as either the partial states or laws, have physical salience in supporting explanations of, and counterfactual reasoning regarding, the behaviour of complicated

phenomena or systems. The Mill-Ramsey-Lewis 'best system' account, interpreted in a metaphysically neutral sense, was adopted to characterize the partial laws *as* laws in order to offer a criterion for acceptance of a decomposition obtained as a superposition, so that the representation obtained according to this procedure may be characterized as 'natural'. In some cases, for instance in the example of conjoined gravitational and electrostatic forces, (i) and (ii) both apply simultaneously to yield the same components.

(2) Application of 'superposition', as it is prolonged, is in some senses 'promiscuous' but not arbitrary. It is determined by the context of application, as it may be associated with different patches of application regarding how the physical significance of the components is to be construed. Cases (i) and (ii) above might be regarded as different patches of application of superposition. The applicability of the concept leading to different decompositions simultaneously according to (i) and (ii) in the same physical situation is one way in which the components of a superposition may be underdetermined, but not arbitrary.

(3) Fourier techniques are associated with natural descriptions of certain kinds of physical phenomena modelled by linear partial differential equations subject to boundary conditions, via 'Hilbert superposition'. There are two aspects to Fourier techniques – first, an eigenfunction representation of differential operators obtained via the separation variables that enables the construction of 'simple solutions' to the original partial differential equation from which general solutions can be built according to Hilbert superposition; secondly, decomposition of the initial or boundary condition into the relevant eigenfunctions or modes. The two different aspects of Fourier techniques represent two different kinds of applications of 'superposition' according to (ii) above;

(4) The applicability of the concept of superposition depends on the linearity of the system or PDE modelling it.

(5) The architecture of 'superposition' becomes more intricate and complicated as one encounters different methods of linear systems analysis, such as Laplace transform techniques for example. These

> exemplify borderline cases of the application of the concept that might be taken to indicate either further (sub)patches of application, or misapplication of the concept. Correspondingly, the interpretative dilemmas for the realist are magnified.

It was primarily through Courant and Hilbert's textbook (1924) and the Fourier analysis of vibrating classical systems that the concept of superposition naturally entered quantum physics as Fourier techniques were extended to quantum phenomena, as we saw in chapters 6-7. This led to a new quantum patch of application of 'superposition' in continuity with its classical usage, but with novel interpretation of the semantic architecture of the concept via Born's rule. We saw that it was these aspects of continuity of the architecture of 'superposition' from classical to quantum physics that were central to the analysis of QFT that I present. Complications arise from an additional underdetermination of the components of a superposition in NRQM in that eigenstate decompositions of different (non-commuting) observables give rise to different applications of 'superposition' to the same quantum state. This complication does not arise in QFT as eigenstates may be chosen that are simultaneous eigenstates of the Hamiltonian, momentum and permutation operators. Moreover, these important aspects of continuity that are used to diagnose the misapplication of 'superposition' and the conceptual confusions that arise in QFT are prior to the 'measurement problem', which therefore need not be considered here.

We saw in chapter 8 that 'superposition' applies straightforwardly to the construction of free QFTs and their Fock space structures. A particle interpretation is supported as a natural description of the system modelled, even if some interpretative questions remain regarding how we are to understand the composition of *N*-quanta states. What is crucial here is the application of the 'Volkmann device' to allow the identification of fields and corresponding partial states so as to allow the state to be characterized according to distinct particle types, where the partial states then support a Fock representation to give a natural description of the system in terms of particle types, numbers and states using Fourier techniques. That this is possible is indicated by the linearity of both the (free) uncoupled wave equations associated with the fields and Schrödinger's equation characterizing the evolution of the state for which the (time

independent) eigenfunction solutions can be chosen to be simultaneous eigenfunctions of 4-momentum and relevant permutation operators.

The problems arise when one seeks to introduce an interaction (chapters 9-10). Implicit appeal is made to the Volkmann device again using either an explicitly perturbative introduction of the interaction, or the gauge principle, which introduces the interaction as an 'effective perturbation'. However, the Volkmann device fails in both cases upon the introduction of an interaction. But the failure of the Volkmann device, and its consequences, generally go unnoticed. The overall state evolves linearly according to Schrödinger's equation as before, but the decomposition of the overall state into partial states associated with coupled fields is no longer supported. The non-linearity of the coupled field equations prevents the use of Fourier techniques to establish a physically meaningful Fock space structure to the partial and overall states, quite apart from considerations from Haag's theorem. This means that the choice of fields is not a natural one for interacting theories, and that a particle description is not available for interacting QFTs. A natural description of the state exists in principle, but cannot be stated explicitly or related to any chosen fields other than in the asymptotic, idealized free contexts. Renormalization is the marker and partial remedy for the failure of the Volkmann device that allows empirically adequate calculations to be 'engineered'.

The lack of a natural field characterization and particle description of interacting states has important implications (chapter 11). For instance, according to QFT 'bound states', which are ubiquitous in nature, cannot be endowed with any internal structure so that we cannot describe the composition of such states in QFT. We also do not have a truly adequate description of unstable particles in QFT, which are also ubiquitous in nature. Together, these observations indicate the severe limitations of our ability to describe or understand nature at the (near) fundamental level. This is not to say that interacting systems at this level do not have any internal structure, rather, the claim is that interacting systems at the (near) fundamental level do not have any internal structure *according to QFT*, at least as the theory stands, which is widely regarded as our most successful (near) fundamental theory. In other words, I do not exclude the possibility that according to some future theory it may be

possible to analyse the internal structure of the (near fundamental) interacting states that we currently model with QFT.

We saw that whilst there is no particle description, or natural description of general interacting QFTs available in any of the free fields, interacting fields, or idealized asymptotic free fields, nonetheless by the use of LSZ scattering theory, the Gell-Mann and Low theorem, and Dyson's expansion, after renormalization calculations at the near fundamental level may be supported, even if a description is unavailable (chapter 11). The kind of knowledge that QFT offers is more comparable with the kind of knowledge that an engineer provides rather than that which the realist or metaphysician seeks.

The kind of 'engineering perspective' that I have sought to outline dovetails with Wilson's analyses of various physical systems in the sense that it draws attention to the kinds of descriptive opportunities that are available, or that we can construct in certain situations by suitable approximations and idealizations, whilst being cognisant of the possibilities of semantic mimicry as they may occur in the interpretation of various solution techniques especially as coupled with an avoidance of 'Theory T' type interpretations of the theories and models used. As in Wilson's approach, an 'engineering approach' stops short of advocating either a realist commitment to the entities of some given model and associated theory of a physical system, or a 'merely' instrumentalist interpretation of the model, where both poles might be said to arise from semantic mimicry.

We noted that despite these severe difficulties, in limited contexts in weakly coupled theories, in the context of 'engineering models' QFT supplies approximate descriptions of interacting states in terms of a particle concept, although it is important to be cognizant of the limitations inherent in such an approximate description, and it is not clear what stance the realist ought to adopt toward such 'description'. Moreover, the approximate particle model of interacting states in weakly coupled QFTs, such as used when modelling the electron configuration of an atom, might owe more to the framework of NRQM than QFT, where QFTs are used in special circumstances to provide 'corrections' to such models, as for instance in the Lamb shift of the spectrum of the hydrogen

atom. Such modelling practice might be best interpreted within an engineering perspective, reflecting an instance of a Wilsonian theory façade.

12.2 Future work and implications of the thesis

I have only considered one, and rather elementary, approach to QFT, neglecting for instance the now more popular path-integral approach as well as the wave-functional approach (cf. Hatfield 1992). Might one expect a different conclusion within these approaches? It seems not. For Hatfield demonstrates the equivalence of the approaches in free theories whilst noting that to develop an interacting theory in each approach requires some form of perturbative addition to the free theory, the architecture of which is different in each case. In my terms, this means that each approach depends on a selection of fields that make the same implicit, initial and problematic appeal to the Volkmann device that leads to nonlinear, perturbative (in my sense) coupled expressions that require recourse to approximation techniques, and renormalization, in interacting cases.

Moreover, I have not considered gauge theory or renormalization group techniques beyond a few comments, although I suggested that neither framework is able to bypass the conceptual difficulties identified that arise form the failure of the Volkmann device. However, further study is merited to situate more precisely the techniques in relation to the Volkmann device and the compensation for its failure, especially in an engineering perspective.

Alternatively, we might consider more significantly different approaches to QFT such as the algebraic approach (AQFT) often preferred by philosophers, or indeed various string theories. From the perspective that I have sought to establish, the crucial question to ask is whether or not these approaches can either support the application of the Volkmann device, or circumvent the problems associated with its failure in such a way as to support a natural description of the quantum system considered, rather than simply empirically adequate results. This would appear necessarily to be required for AQFT or string theory to offer a significant descriptive advantage, or the ability to support physical explanations of interactions and interaction processes, even if there may be other benefits to AQFT and string theory.

The failure to be able to represent or describe nature at the (near) fundamental level according to one of our best scientific theories has important implications for various metaphysical projects, especially those predicated on fundamentality, the concept of reduction or reductive accounts in which fundamental level physics and the entities, properties and laws that it is assumed to supply play a key role. Moreover, our inability to represent nature at the (near) fundamental level has implications for how we understand matter and the concept of matter interacting at the (near) fundamental level, and indeed what constitutes 'fundamentality',[1] and we remain largely ignorant of the nature of properties such as charge.

12.3 In conclusion

The development of the 'superposition principle' in science may be traced to Galileo, and the principle may claim a pivotal role in the Scientific Revolution and in the development and progress of science and technology from the early modern period into the 20th-century. That is, the ability to analyse a complicated phenomenon or system in terms of abstractable simple components such that partial laws associated with the simple components (as partial states) take the same form individually and in combination without stating the facts in combination appears foundational to the scientific method and analysis, as well as to the development of technology, whether or not one adopts a realist stance to the components and laws associated with superposition. This is to re-appropriate Volkmann's understanding of the development of science and the role that the isolation/superposition process plays within it (1896; 1900; 1910), which has perhaps not received the philosophical attention that it merits. The usage of the concept of superposition, whether explicitly or implicitly, was associated with the study of linear phenomena, or phenomena that supported linear models for their analysis in the 19th century in particular. Problems arose

[1] I have not attempted analysis of 'fundamentality', for which see e.g. Tahko (2018). The point is that my analysis does not support the kinds of assumptions or claims often made regarding QFT in relation to 'the fundamental', however defined (cf. e.g. McKenzie 2017), and that the relationship between 'superposition' and the concept of fundamentality merits further research.

when non-linear phenomena began to be modelled, as occurred in the early 20th century, for which superposition fails, leading to unexpected results.[2]

We have seen that implicitly scientists and philosophers in the 20th century, apart from Heitler (1936) perhaps, intuitively expected to be able to 'prolong' use of 'superposition', or the Volkmann device, from the analysis of (linear) classical macroscopic phenomena, especially in the 18th-19th century, to the (near) fundamental level in QFT. However, we appear to have reached the limits of the application of 'superposition' in QFT, with various philosophical dilemmas that arise being capable of diagnosis in terms of the unrecognized failure of 'superposition'. This – the failure of the Volkmann device and associated 'superposition' – is a, if not the, fundamental conceptual problem of QFT. The consequence is that we do not know what entities (or indeed relations) characterize the (near) fundamental level, so we cannot identify the basic constituents of matter or explain their interactions at this level for instance.

The conceptual difficulties that arise in QFT might be characterized in Wilsonian terms as the tacit prolongation of the superposition concept where it is assumed to apply, by semantic mimicry, but where it doesn't, so that one falsely believes that one has a natural (or even metaphysical) description of the world, having selected the correct 'isolation centres' or independent entities at the near fundamental level, when this is not the case.

It is not clear whether the identification of such 'isolation centres' or even a different conceptual framework that would support such analysis is ultimately within our conceptual abilities, perhaps via some non-linear superposition principle, or if the task will forever elude us so that our understanding of Nature will always be more like an engineer's than a metaphysician's. In the mean time the failure of the Volkmann device is manifested in the need for renormalization, which offers a limited pragmatic accommodation for the failure of 'superposition', which is a key conceptual problem of QFT quite apart from any ignorance on our part of the physics at very short length scales. Or, to put it another way, our ignorance begins before we think it does.

[2] See for instance Simmons (2017, 572-574) for brief historical comments on the unexpected consequences of the behaviour of electrical systems modeled by the nonlinear van der Pol equation, for which 'superposition' fails.

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