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Abstract

We consider the problem of dividing one unit of an infinitely divisible object among a finite number of agents. We provide a characterization of all single-peaked domains on which the uniform rule is the unique division rule satisfying efficiency, strategy-proofness, and equal treatment of equals (ETE). We also provide a class of division rules satisfying these properties on the remaining single-peaked domains. Next, we consider non single-peaked domains and provide a characterization of all such domains on which the uniform rule satisfies efficiency, strategy-proofness, and ETE. We also show that under some mild richness conditions the uniform rule is the unique rule satisfying the mentioned properties on these domains. Finally, we provide a class of division rules satisfying efficiency, strategy-proofness, and ETE on the remaining non single-peaked domains. We conclude the paper by providing a wide range of applications to justify the usefulness of our results.

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1. INTRODUCTION

We consider the problem of dividing one unit of an infinitely divisible good among a finite number of agents. Each agent has a preference over his possible shares. A division rule decides a share for each agent at every collection of preferences of the agents.

A division rule is efficient if there is no other way to divide the good so that no one is worse off and someone is better off. It is strategy-proof if no agent can strictly benefit by misreporting his preferences. It satisfies equal treatment of equals (ETE) if whenever two agents have the same preference, their shares are equal. A division rule is anonymous if the identities of the agents do not play any role in the decision. Note that anonymity implies ETE.

A preference over the possible shares (that is, over the interval $[0,1]$) is called single-peaked if there is a most-preferred share, called the peak, such that as shares increase or decrease from that, preference declines. The collection of all such continuous preferences is called the maximal continuous single-peaked domain.[1] Sprumont (1991) shows that a division rule satisfies efficiency, strategy-proofness, and anonymity on the maximal continuous single-peaked domain if and only if it is the uniform rule. Later, Weymark (1999) generalizes this result for supersets of maximal continuous single-peaked domains, that is, domains that admit non-continuous single-peaked preferences in addition to *all* continuous single-peaked preferences.

The assumptions of maximality, as well as continuity, are somewhat restrictive for their practical applications. Maximality requires the presence of "extreme" preferences such as the ones where almost all shares on the left side of the peak are preferred to almost all on the right. Many well-known single-peaked domains such as Euclidean (and any of its variants) do not admit such preferences, and consequently the existing result does not apply to these domains.[2] On the other hand, continuity is a technical condition and we do not see any reason why agents' preferences should always be continuous. For instance, in the problem of dividing a task among some agents (teaching hours among faculties) an agent's preference with peak 0.3 might fall suddenly beyond 0.8 as he might find it totally impossible (or, unacceptable) to handle more than 0.8 amount of the task. In view of these observations, we intend to explore the structure of division rules when the assumptions of continuity and maximality on a domain are dropped.

We provide a condition on a single-peaked domain which implies that a division rule satisfies efficiency, strategy-proofness, and ETE if and only if it is the uniform rule. Our condition depends on the number of agents. We further show that under some mild richness assumption, it is both necessary and sufficient for a domain to ensure the property that a division on it satisfies efficiency, strategy-proofness, and ETE if and only if it is the uniform rule.

[1] Informally speaking, a preference is continuous if it can be represented by a continuous utility function.

[2] A single-peaked domain is called Euclidean if preference declines as Euclidean distance increases from the peak.

Although single-peakedness is quite plausible, there are situations where preferences do not exhibit this property. Stiglitz (1974) shows that the preference of a person from low-income group over educational subsidies has two peaks: one very low (so that the primary education is totally free) and one very high (so that the higher education is totally free). The point is, a moderate amount of subsidy is not helpful for such a person as he cannot afford to pay the remaining expenses. For another instance, consider the preference of a faculty member over different amounts of a fund. Suppose that the faculty wants to buy equipments using the fund and an equipment costs, say INR $1,00,000$. Then, his preferences will have multiple (local) peaks at multiples of $1,00,000$. Similarly, if a corporation or a promoter needs some minimum amount of land, say 1 acre, to setup a new factory, then, for instance, he might prefer both 5 acres and 6 acres of lands to 5.5 acres.

In view of the preceding discussion, we consider situations where agents' preferences are arbitrary, that is, not necessarily single-peaked. Let us call a domain possibility if there is a division rule on it satisfying efficiency, strategy-proofness, and ETE. In a seminal paper, Massó and Neme (2001) provide a characterization of all possibility domains.[3] The contribution of our paper over theirs is as follows. Firstly, to our understanding, knowing whether a domain is possibility or not might not be enough as, even if a domain is so, one does not know the structure of division rules satisfying efficiency, strategy-proofness, and ETE on it. Moreover, as we show in Example 6.1, the structure of such a division rule might indeed be quite complicated (not even tops-only) for its practical use. We resolve these issues by requiring that particularly the uniform rule satisfies all the mentioned desirable properties on the domains. Secondly, the results in Massó and Neme (2001) require that *all* continuous single-peaked preferences are present in the domain which seems to be a strong requirement, whereas we derive our results under a much weaker richness condition called regularity.

Motivated by the preceding discussion and the importance of the uniform rule, we provide a necessary and sufficient condition on a domain so that the uniform rule satisfies efficiency, strategy-proofness, and ETE on it. Furthermore, we show that under some mild richness condition, the uniform rule is the unique rule that satisfies these properties. It is worth mentioning that our result applies to domains which admit indifference (even) on the same side of the peak of a preference. Note that continuous single-peaked preferences too admit indifference, but only on the opposite sides of the peak of a preference. Thus, our consideration of weak preferences is non-trivial.

The uniform rule is introduced by Benassy (1982) as a strategy-proof rule, and is considered to be the most important rule for the division problem when agents have single-peaked preferences. This rule is studied extensively in the literature and several characterizations of it using properties such as monotonicity, consistency, maximality, etc., is available in the literature (see Thomson (1994), Thomson

[3] In a different paper, Massó and Neme (2004) provide a characterization of the maximal domain where a division rule satisfying efficiency, strategy-proofness, tops-onlyness, and continuity exists.

(1994), Thomson (1995), Otten et al. (1996), and de Frutos and Massó (1995) for details). Also, the uniform reallocation rule is used in exchange economies with two goods and fixed prices. In view of all these, we think our result establishing the full span of the uniform rule on both single-peaked and non-single-peaked domains complements the existing literature.

All our aforementioned results are centered around the uniform rule: they characterize domains on which the uniform rule satisfies efficiency, strategy-proofness, and ETE or it is the unique rule satisfying these properties. Questions arise as to (i) if the uniform rule does not satisfy efficiency, strategy-proofness, and ETE on a domain, then what type of rules will satisfy these properties, and (ii) if the uniform rule is not the unique rule satisfying these properties on a domain, then what other rules will satisfy these properties. Of course, these questions are worth investigating if the concerned domains are useful and the corresponding rules are "simple" enough for practical purposes.

In response to (i), we note that in order for the uniform rule to be efficient and strategy-proof on a domain, preferences in it must decline till the share $\frac{1}{n}$. As we have already mentioned, preferences often exhibit "double-peakedness" with one high peak and one low. Considering this, we consider domains where preferences with relatively high peak might have another local peak at a low level, and provide a class of division rules that satisfy efficiency, strategy-proofness, and ETE on such domains.

In response to (ii), we note that efficient, strategy-proof, and ETE division rules other than the uniform rule exist on a domain if it has the following property: there is some interval such that for all preferences with peaks in that interval, one particular boundary is preferred to the other. For instance, it may happen that for all preferences with peaks in the interval $(0.3, 0.4)$, the share 0.3 is preferred to the share 0.4. As we have explained earlier, such situations occur in land or fund division problems where a particular amount of land or fund is needed to set-up a factory or to buy an equipment. Therefore, we provide a class of rules other than the uniform rule which satisfy efficiency, strategy-proofness, and ETE on these domains.

We provide a wide range of applications of our results. We show that the uniform rule is the unique division rule satisfying efficiency, strategy-proofness, and ETE on most single-peaked domains that naturally occur, for instance, when agents have single-peaked utility functions that satisfy a convergence property or satisfy a translation property, or when the preferences of agents exhibit the well known single-crossing property (Saporiti (2009)). Simple examples of such preferences consist of Euclidean ones and its different variants where preference declines on different sides of the peak following different functional forms. Regarding applications of our results on non single-peaked domains, they provide the structure of division rules satisfying efficiency, strategy-proofness, and ETE on semi-single-peaked domains (Chatterji et al. (2013)) and partially single-peaked domains. It is worth mentioning that (i) for single-peaked domains, we do not assume preferences to be continuous (in contrast to Sprumont (1991), Ching (1994), Weymark (1999)), and (ii) for non single-peaked domains, as we have mentioned, we do not assume the presence of all continuous single-peaked preferences in the domain (in contrast to Massó and Neme (2001)). Instead, for both these cases, we only need *one* preference for every share as the peak that is continuous in an arbitrarily small neighborhood around the peak. We feel our weaker requirements expand the applicability of our results considerably. Altogether, we feel our paper enriches the literature of the classical division problem by establishing the full applicability of the well known uniform rule, as well as, by introducing new division rules for scenarios where the uniform rule "fails".

The rest of the paper is organized as follows. Section 2 introduces the model and basic definitions regarding domains. Section 3 introduces division rules and discusses their relevant properties. Section 4 presents a characterization of all single-peaked domains on which the uniform rule is the unique division rule satisfying efficiency, strategy-proofness, and ETE. Section 5 considers the remaining single-peaked domains and provides a class of division rules satisfying those properties on these domains. Section 6 considers non single-peaked domains and provides a necessary condition for the uniform rule to satisfy efficiency, strategy-proofness, and ETE. It further shows that the necessary condition is also sufficient (for the same purpose) under a mild richness condition. Section 7 considers the non single-peaked domains on which the uniform rule does not satisfy efficiency, strategy-proofness, and ETE, and provides a class of division rules on these domains satisfying the mentioned properties. Section 8 provides applications of our results to several well-known domains.

2. DOMAINS AND THEIR PROPERTIES

Let $N = \{1,\dots,n\}$ be a set of agents who must share one unit of some perfectly divisible good. Each agent $i \in N$ has a preference R_i over his possible shares which is a complete and transitive binary relation on $[0,1]$. Throughout this paper we assume that each R_i has a unique top-ranked share $\tau(R_i)$, also called the peak of R_i. For all $x, y \in [0,1]$, xR_iy means consuming a quantity x of the good is, from i's viewpoint, at least as good as consuming a quantity y. Strict preference of R_i is denoted by P_i, indifference by I_i. A preference R_i is *continuous* if for each $x \in [0,1]$, $\{y \in [0,1] \mid yR_ix\}$ and $\{y \in [0,1] \mid xR_iy\}$ are closed sets. A preference is **locally continuous around the peak** if there exists $\varepsilon > 0$ such that for all each $x \in (\tau(R_i) - \varepsilon, \tau(R_i) + \varepsilon)$, the sets $\{y \in [0,1] \mid yR_ix\}$ and $\{y \in [0,1] \mid xR_iy\}$ are closed. Clearly, local continuity is much weaker than continuity as the former requires continuity only on an arbitrarily small neighborhood around the peak.

We denote a collection of preferences (henceforth, will be referred to as a domain) by $\mathscr{D}$. We let $R_N = (R_i)_{i \in N} \in \mathscr{S}^n$ denote the announced preferences (also called a profile) of all agents and R_{-i} denote $(R_j)_{j \in N \setminus i}$ for $i \in N$. For a profile R_N, we define $\tau(R_N) = (\tau(R_1), \dots, \tau(R_n))$ as the collection of peaks at the profile R_N. For a profile R_N and $S \subseteq N$, by $T(R_S)$ we denote $\sum\limits_{i \in S} \tau(R_i)$, i.e., the sum of peaks of the agents in S at the profile R_N. We let $\mathscr{S}^n_+ = \{R_N \in \mathscr{S}^n \mid T(R_N) \geq 1\}$ denote the profiles where the total demand is at least 1 and let $\mathscr{S}^n_- = \{R_N \in \mathscr{S}^n \mid T(R_N) < 1\}$ denote the profiles where the total demand is at most 1.[4]

A preference R_i is **single-peaked** if there exists $\tau(R_i) \in [0,1]$, called the peak of R_i, such that for all $x, y \in [0,1]$

$$[\tau(R_i) < x < y] \text{ or } [y < x < \tau(R_i)] \implies [\tau(R_i) P_i x P_i y].$$

Thus, a preference is single-peaked if it declines as one goes far away from its peak (in one particular direction). Throughout this paper we denote by $\mathscr{S}$ a set of single-peaked preferences.

A single-peaked preference R_i is called **scaled Euclidean** if there exist positive numbers κ_1, κ_2 such that for all $x < \tau(R_i) < y$, $\kappa_1(\tau(R_i) - x) < \kappa_2(y - \tau(R_i))$ implies xP_iy, and $\kappa_1(\tau(R_i) - x) > \kappa_2(y - \tau(R_i))$ implies yP_ix. A scaled Euclidean preference is called **Euclidean** if $\kappa_1 = \kappa_2$.

All the domains we consider in this paper are assumed to be *regular*: for all $x \in [0,1]$, there exists a single-peaked preference $R \in \mathscr{S}$ with $\tau(R) = x$ that is locally continuous around the peak.

3. Division rules and their properties

Let Δ_n be the set $\{(x_1, \ldots, x_n) \in [0,1]^n \mid \sum_{i=1}^{n} x_i = 1\}$ of all divisions of the good among n agents. A division rule f is a function $f : \mathscr{S}^n \to \Delta_n$. In other words, a division rule decides a division of the good at every given profile. For a division rule f, a profile R_N, and an agent $i \in N$, we denote by $f_i(R_N)$ the share of agent i at the profile R_N by the rule f. Below, we mention some desirable properties of a division function.

A division rule is efficient if its outcome cannot be modified in a way so that everybody is weakly better off and somebody is strictly better off, that is, for every other divisions, there will be some agent who is worse off.

Definition 3.1. A division rule $f : \mathscr{S}^n \to \Delta_n$ is **efficient** if for all $R_N \in \mathscr{S}^n$ and all $x \in \Delta_n \setminus f(R_N)$, there exists $i \in N$ such that $f_i(R_N) P_i x_i$.

Note that if preferences are single-peaked, then efficiency says that if the total demand at a profile, i.e., the sum of the peaks at that profile, is weakly less than the total available amount 1 (or weakly bigger than that), then each agent will receive a share that is weakly bigger than (or weakly lesser than) his peak.

Strategy-proofness ensures that if an agent misreports his preferences, then he will not get a share that is strictly preferred for him/her.

Definition 3.2. A division rule $f : \mathscr{S}^n \to \Delta_n$ is **strategy-proof** if for all $i \in N$, all $R_N \in \mathscr{S}^n$, and all

[4] By the total demand at a profile R_N, we mean the amount $T(R_N)$.

$R_i' \in \mathscr{S}$, we have

$$f_i(R_N) R_i f_i(R_i', R_{-i}).$$

Equal treatment of equals says that if two agents have the same preference, they will get the same share of the good.

Definition 3.3. A division rule $f : \mathscr{S}^n \to \Delta_n$ satisfies **equal treatment of equals (ETE)** if for all $i, j \in N$ and all $R_N \in \mathscr{S}^n$, we have

$$[R_i = R_j] \implies [f_i(R_N) = f_j(R_N)].$$

Next, we introduce the notion of the uniform rule (Benassy (1982)).

Definition 3.4. A division rule $u : \mathscr{S}^n \to \Delta_n$ is called the **uniform rule** if for all $R_N \in \mathscr{S}^n$ and all $i \in N$,

$$u_i(R_N) = \begin{cases} \min\{\tau(R_i), \lambda(R_N)\} \text{ if } R_N \in \mathscr{S}^n_+, \text{ and} \\ \max\{\tau(R_i), \mu(R_N)\} \text{ if } R_N \in \mathscr{S}^n_-, \end{cases}$$

where $\lambda(R_N) \geq 0$ solves the equation $\sum_{i \in N} \min\{\tau(R_i), \lambda(R_N)\} = 1$ and $\mu(R_N) \geq 0$ solves the equation $\sum_{i \in N} \max\{\tau(R_i), \mu(R_N)\} = 1$.

REMARK 3.1. The uniform rule is monotonic, that is, as an agent (unilaterally) moves his peak in some direction, his shares also move in that direction. More formally, for all $R_N \in \mathscr{S}^n$, all $i \in N$, all $R_i' \in \mathscr{S}$, $\tau(R_i) \leq \tau(R_i') \implies u_i(R_N) \leq u_i(R_i', R_{-i})$.

In what follows, we explain how the outcome of the uniform rule is computed at different profiles. Consider a profile $R_N = (R_1, \ldots, R_5)$ with $\tau(R_N) = (0.2, 0.1, 0, 0.6, 0.5)$. Note that the total demand at R_N is more than 1. For a "cut-off" $\lambda(R_N) \in [0,1]$, consider the following allocation vector: if some agent's peak is more than $\lambda(R_N)$ then he receives $\lambda(R_N)$, otherwise he receives his peak. For instance, if $\lambda(R_N) = 0.4$, then we obtain the following allocation vector $(0.2, 0.1, 0, 0.4, 0.4)$. Note that this vector is not a division as the total share is more than 1. So, keep decreasing the cut-off so that the total share becomes 1. In this example, this happens when $\lambda(R_N) = 0.35$ giving the division $(0.2, 0.1, 0, 0.35, 0.35)$. The uniform rule says that the outcome at R_N must be the division $(0.2, 0.1, 0, 0.35, 0.35)$. For profiles with total demand less than 1, the uniform rule follows a symmetrically opposite procedure of increasing the shares of agents to a cut-off to attain the total share 1.

4. A CHARACTERIZATION OF SINGLE-PEAKED DOMAINS FOR THE UNIFORM RULE

In Sprumont (1991), it is shown that if a domain contains all continuous single-peaked preferences, then a division rule satisfies efficiency, strategy-proofness, and anonymity if and only if it is the uniform rule.[5] Later, Ching (1994) derives the same result by replacing anonymity with ETE. In this section, we characterize all single-peaked domains on which the above-mentioned result holds. Our characterization depends on the number of agents. We distinguish cases accordingly.

4.1 THE CASE OF TWO AGENTS

We present a condition on a domain, called *Condition U for* 2 *agents*, that we use in our characterization result. It says that for every interval (x,y) not containing the point $\frac{1}{2}$, there is a preference with peak in that interval such that the boundary point of the interval that is closer to $\frac{1}{2}$ is strictly preferred to the other one, that is, if $(x,y) \subseteq [0,\frac{1}{2}]$ then y is preferred to x, and if $(x,y) \subseteq [\frac{1}{2},1]$ then x is preferred to y according to that preference.

Definition 4.1. A domain $\mathscr{S}$ satisfies **Condition U for** 2 **agents** if

(i) for all intervals $(x,y) \subseteq [0,\frac{1}{2}]$ there exists $R \in \mathscr{S}$ with $\tau(R) \in (x,y)$ such that yPx, and

(ii) for all intervals $(x,y) \subseteq [\frac{1}{2},1]$ there exists $R \in \mathscr{S}$ with $\tau(R) \in (x,y)$ such that xPy.

Our next theorem provides a characterization of all domains on which every division rule for two agents satisfying efficiency, strategy-proofness, and ETE is the uniform rule.

Theorem 4.1. *(i) Suppose a single-peaked domain $\mathscr{S}$ satisfies Condition U for* 2 *agents. Then, a division rule $f : \mathscr{S}^2 \to \Delta_n$ satisfies efficiency, strategy-proofness, and ETE if and only if it is the uniform rule.*
(ii) Suppose a single-peaked domain $\mathscr{S}$ does not satisfy Condition U for 2 *agents. Then, there is a division rule $f : \mathscr{S}^2 \to \Delta_n$ other than the uniform rule that satisfies efficiency, strategy-proofness, and ETE.*

The proof of Theorem 4.1 is relegated to Appendix A; we provide a brief sketch here. "If" part of (i) of Theorem 4.1: First we argue that the share of an agent cannot change unless he changes his peak. This is because, if an agent does not change his peak, by efficiency his share cannot go to the other side of the peak, now by strategy-proofness, it must remain the same. Since there are two agents, this means the rule will be peaks-only that is, will depend only on the peaks. In view of this, for the remaining discussion we denote a profile by its peaks.

Consider a profile such that both the peaks are more than or equal to $\frac{1}{2}$. The uniform rule would give the outcome $(\frac{1}{2},\frac{1}{2})$ at this profile, so assume for contradiction that some agent receives less than $\frac{1}{2}$. However, then he will misreport his preference as the one the other agent has, and by ETE, he will get $\frac{1}{2}$. Since the domain is single-peaked, he will prefer $\frac{1}{2}$ to his original share which was less, and thus

[5] A division rule if anonymous if agents' identities do not play any role in deciding the outcome. More formally, a division rule $f : \mathscr{S}^n \to \Delta_n$ is anonymous if for all permutations $\sigma : N \to N$ and for all $R_N \in \mathscr{S}^n$, we have $f_i(R_N) = f_{\sigma(i)}(R_N^{\sigma})$ where $R_N^{\sigma} = (R_{\sigma(1)},\ldots,R_{\sigma(n)})$.

will manipulate, a contradiction. Note that just single-peaked structure of the domain is sufficient for this argument. Similarly, it can be shown that when both the agents have peaks less than $\frac{1}{2}$, the outcome will be $(\frac{1}{2},\frac{1}{2})$, same as the uniform rule. Now, consider a profile (p_1,p_2) such that $p_1 < \frac{1}{2}$ and $p_2 > \frac{1}{2}$. Assume WLOG that $p_1+p_2 < 1$. It can be verified that the outcome of the uniform rule at such a profile is $(1-p_2,p_2)$. By efficiency and single-peakedness, agent 2 cannot receive an amount that is less than his peak. So, assume for contradiction that $f_2(p_1,p_2) > p_2$. Suppose that agent 2 continuously moves his peak towards left till the outcome changes. Suppose the "boundary" at which the outcome changes is p_2^*. Mathematically, p_2^* is the infimum of all the peaks of agent 2 such that the outcome does not change from $f_2(p_1,P_2)$. We claim that $p_2^* \geq \frac{1}{2}$. Assume for contradiction that $p_2^* < \frac{1}{2}$. Since $p_1 < \frac{1}{2}$, by our earlier argument, the outcome at all the profiles where agent 1 has peak p_1 and 2 has the peak in the interval $[p_2^*,\frac{1}{2}]$, must be $(\frac{1}{2},\frac{1}{2})$. However, this is a contradiction to the fact that p_2^* is the infimum over the peaks of agent 2 such that the outcome does not change.

Next, we argue that the share of agent 2 either remains the same (i.e. $f(p_1,P_2)$) or "jump" to p_2^* when his peak is p_2^*. To see this, first note that by efficiency the outcome cannot move to the left of p_2^*. If the outcome goes to the right of $f_2(p_1,p_2)$, then by single-peakedness agent 2 will manipulate by moving to p_2. Suppose that the outcome comes closer (but, remains strictly on the left) to p_2^*. If agent 2 misreports his peak as slightly more than p_2^* (less than the outcome), by strategy-proofness the outcome has to remain the same. Since this outcome is different from the original outcome $f_2(p_1,p_2)$, this contradicts the fact that p_2^* is the infimum of the peaks of agent 2 such that the outcome does not change.

Now, we distinguish two cases: (a) $f_2(p_1,p_2^*) = p_2^*$, and (b) $f_2(p_1,p_2^*) = f_2(p_1,p_2)$. For Case (a), we apply Condition U for 2 agents over the interval $[p_2^*, f_2(p_1,p_2)]$ (as we have argued, this interval is above $\frac{1}{2}$ as $p_2^* \geq \frac{1}{2}$), and get hold of a preference $\tilde{R}$ such that $p_2^*\tilde{P}f_2(p_1,p_2)$. By construction, the share of agent 2 when he has preference $\tilde{R}$ is $f_2(p_1,p_2)$, and hence, he will manipulate by misreporting his sincere preference $\tilde{R}$ as any preference with p_2^* as peak. In Case (b), note that since the domain is regular, by local continuity there is a preference $\hat{R}$ with peak p_2^* such that for some $x < p_2^*$ (arbitrarily close to p_2^*), we have $x\hat{P}f_2(p_1,p_2)$. By the definition of p_2^* and by the assumption of Case (b), if agent 2 misreports his preference as one with x as the peak, then he will get a share in $[x,p_2^*)$ and manipulate at (p_1,p_2^*). This proves that $f_2(p_1,p_2) = p_2$ as required by the uniform rule. The proof for other cases follows by using similar argument.

4.2 The case of more than two agents

We use a condition, called *Condition U for n agents*, where $n > 2$, for our characterization. It is a stricter version of Condition U for 2 agents. Firstly (and somewhat naturally), it modifies (i) and (ii) of Condition U for 2 agents by replacing $\frac{1}{2}$ with $\frac{1}{n}$. Secondly, it additionally imposes two other conditions that are, in a sense, partial complements of (i) and (ii) in Condition U for 2 agents. Recall that (i) of the said condition says that for every subset (x,y) of $\big[0,\frac{1}{2}\big]$, there is a preference with the peak in that interval according to which y is preferred to x. Part (iii) of Condition U for n agents requires that for such intervals (now subsets of $\big[0,\frac{1}{n}\big]$), there is another preference according to which x is preferred to y. In a similar manner, (iv) of Condition U for n agents is kind of the complement of (ii) of Condition U for 2 agents with some additional modification: in contrast to (ii), (iv) is imposed only on the intervals that are subsets of $\big[\frac{1}{n},\frac{1}{2}\big]$. For such intervals (x,y), it requires that there is a preference with peak in that interval according to which y is preferred to x.

Note that combining (i) and (iii), and (ii) and (iv) in Condition U for n agents, it follows that for every interval (x,y) such that either $(x,y)\subseteq[0,\frac{1}{n}]$ or $(x,y)\subseteq[\frac{1}{n},\frac{1}{2}]$, there are two preferences with the peaks in that interval such that preference over x and y is reversed in those two preferences. Note that apart from the said implication, Condition (ii) additionally imposes some restrictions on intervals that are subsets of $\big[\frac{1}{n},1\big]$.

Definition 4.2. A domain $\mathscr{S}$ satisfies **Condition U for n agents**, where $n>2$, if

(i) for all intervals $(x,y)\subseteq\big[0,\frac{1}{n}\big]$, there exists $R\in\mathscr{S}$ with $\tau(R)\in(x,y)$ such that yPx,

(ii) for all intervals $(x,y)\subseteq\big[\frac{1}{n},1\big]$, there exists $R\in\mathscr{S}$ with $\tau(R)\in(x,y)$ such that xPy,

(iii) for all intervals $(x,y)\subseteq\big[0,\frac{1}{n}\big]$, there exists $R\in\mathscr{S}$ with $\tau(R)\in(x,y)$ such that xPy, and

(iv) for all intervals $(x,y)\subseteq\big[\frac{1}{n},\frac{1}{2}\big]$, there exists $R\in\mathscr{S}$ with $\tau(R)\in(x,y)$ such that yPx.

Note that Condition U is satisfied if, for instance, the domain is Euclidean (or even scaled Euclidean).

Our next theorem presents a characterization of all domains on which every division rule for more than two agents satisfying efficiency, strategy-proofness, and ETE is the uniform rule.

Theorem 4.2. *(i) Suppose $n>2$ and let a single-peaked domain $\mathscr{S}$ satisfy Condition U for n agents. Then, a division rule $f:\mathscr{S}^n\to\Delta_n$ satisfies efficiency, strategy-proofness, and ETE if and only if it is the uniform rule.*

(ii) Suppose $n>2$ and let a single-peaked domain $\mathscr{S}$ do not satisfy Condition U for n agents. Then, there is a division rule $f:\mathscr{S}^n\to\Delta_n$ other than the uniform rule that satisfies efficiency, strategy-proofness, and ETE.

The proof of Theorem 4.2 is relegated to Appendix B. The idea of this proof is somewhat similar to that of Theorem 4.1, but much more involved. Firstly, note that the peaks-only property does not follow so straightforwardly for this case. Furthermore, ETE was much stronger for two agents since an agent can unilaterally deviate to the preference of the other agent, and can enforce ETE. However, for n agents this is not possible. We leave the details for the formal proof.

5. THE STRUCTURE OF DIVISION RULES ON SINGLE-PEAKED DOMAINS THAT VIOLATE CONDITION U

In Section 4, we have presented the structure of division rules satisfying efficiency, strategy-proofness, ETE on domains that satisfy condition U. In this section, we do this for the remaining domains. As before, we treat the cases of 2 agents and more than 2 agents separately as our results vary over these cases.

5.1 THE CASE OF TWO AGENTS

Note that a domain violates Condition U for 2 agents if there is an interval (x,y) not containing the point $\frac{1}{2}$ such that for each preferences with the peak in that interval, the boundary point of the interval that is farther away from $\frac{1}{2}$ is weakly preferred to the other one. Below, we present this observation formally.

Observation 5.1. *A domain $\mathscr{S}$ violates Condition U for 2 agents on an interval (x,y) with $\frac{1}{2} \notin (x,y)$ if for all $R \in \mathscr{S}$, $\tau(R) \in (x,y)$ implies*

(i) xRy if $(x,y) \subseteq [0,\frac{1}{2}]$, and

(ii) yRx if $(x,y) \subseteq [\frac{1}{2},1]$.

To ease our presentation, for two subsets A and B of $[0,1]$, we write $A < B$ to mean that each element of A is less than each element of B, that is, $a < b$ for all $a \in A$ and all $b \in B$. Similarly, for a number x and an interval (a,b), we write $x < (a,b)$ to mean that $x < a$. We use similar notations without further explanation.

To help the reader, we first consider domains that violates Condition U for 2 agents only on two intervals (x,y) and (w,z), where $(x,y) < \frac{1}{2} < (w,z)$. We present the notion of *adjusted uniform rules for 2 agents* on such domains. These rules behave like the uniform rule at every profile except a few where they adjust the outcome of the uniform rule by giving some lesser preferred amount to some particular agent i_0. These profiles are those where (i) the total demand (that is, the sum of the peaks) is at least 1 and agent i_0's peak is lies the interval $[x,y)$, or (ii) the total demand is at most 1 and agent i_0's peak lies in the interval $(w,z]$. In Case (i), if $x+\tau(R_j) \geq 1$ then agent i_0 gets x and the other agent j gets the rest, and if $x+\tau(R_j) < 1$ then agent j gets his peak and agent i_0 gets the rest. In Case (ii), if $z+\tau(R_j) \leq 1$ then agent i_0 gets z and agent j gets the rest, and if $z+\tau(R_j) > 1$ then agent j gets his peak and agent i_0 gets the rest.

Note that in both Case (i) and Case (ii), agent i_0 would get his peak and agent j would get the rest by the uniform rule. Thus, these rules are in a sense negatively biased towards the agent i_0 relative to the uniform rule. For ease of presentation, we just mention the outcome share of one agent, that of the other agent is the remaining share.

Definition 5.1. A division rule $f : \mathscr{S}^2 \to \Delta_2$ is an **adjusted uniform rule for 2 agents with respect to intervals** (x,y) and (w,z) if there exists an agent $i_0 \in N$, such that

(i) for all $(R_1,R_2) \in \mathscr{S}_+^2$ with $\tau(R_{i_0}) \in [x,y)$, we have for $j \neq i_0$

(a) $x + \tau(R_j) \geq 1 \implies f_j(R_N) = 1 - x$,

(b) $x + \tau(R_j) < 1 \implies f_j(R_N) = \tau(R_j)$,

(ii) for all $(R_1,R_2) \in \mathscr{S}_-^2$ with $\tau(R_{i_0}) \in (w,z]$, we have for $j \neq i_0$

(a) $z + \tau(R_j) \leq 1 \implies f_j(R_N) = 1 - z$,

(b) $z + \tau(R_j) > 1 \implies f_j(R_N) = \tau(R_j)$, and

(iii) for all other profiles $(R_1,R_2) \in \mathscr{S}^2$, $f(R_1,R_2) = u(R_1,R_2)$.

We are now ready to present our rules for the domains that violate Condition U for 2 agents over multiple intervals. For intervals $0 < (x_1,y_1) < \cdots < (x_{k_1},y_{k_1}) < \frac{1}{2} < (w_1,z_1) < \cdots < (w_{k_2},z_{k_2}) < 1$, we say a domain violates Condition U for 2 agents on these intervals if each interval in this collection satisfies the corresponding condition (based on whether it is less than or bigger than $\frac{1}{2}$) in Observation 5.1.

In what follows, we present a general class of division rules on domains that violate Condition U for 2 agents on multiple intervals. These rules treat each interval below $\frac{1}{2}$ and each interval above $\frac{1}{2}$ in the same way as adjusted uniform rules presented above treat the intervals (x,y) and (w,z), respectively.

Definition 5.2. Let $(x_1,y_1),\ldots,(x_{k_1},y_{k_1})$ and $(w_1,z_1),\ldots,(w_{k_2},z_{k_2})$ be a collection of intervals such that $0 < (x_1,y_1) < \cdots < (x_{k_1},y_{k_1}) < \frac{1}{2} < (w_1,z_1) < \cdots < (w_{k_2},z_{k_2}) < 1$. A division rule $f : \mathscr{S}^2 \to \Delta_2$ is an **adjusted uniform rule for 2 agents with respect to intervals** $(x_1,y_1),\ldots,(x_{k_1},y_{k_1})$ and $(w_1,z_1),\ldots,(w_{k_2},z_{k_2})$ if there exists an agent $i_0 \in N$ such that

(i) for all $(R_1,R_2) \in \mathscr{S}_+^2$ for which there exists $r \in \{1,\ldots,k_1\}$ with $\tau(R_{i_0}) \in [x_r,y_r)$, we have for all $j \neq i_0$

(a) $x_r + \tau(R_j) \geq 1 \implies f_j(R_N) = 1 - x_r$,

(b) $x_r + \tau(R_j) < 1 \implies f_j(R_N) = \tau(R_j)$,

(ii) for all $(R_1,R_2) \in \mathscr{S}_-^2$ for which there exists $s \in \{1,\ldots,k_2\}$ with $\tau(R_{i_0}) \in (w_s,z_s]$, we have for all $j \neq i_0$

(a) $z_s + \tau(R_j) \leq 1 \implies f_j(R_N) = 1 - z_s$,

(b) $z_s + \tau(R_j) > 1 \implies f_j(R_N) = \tau(R_j)$, and

(iii) for all other profiles $(R_1,R_2) \in \mathscr{S}^2$, $f(R_1,R_2) = u(R_1,R_2)$.

Clearly, adjusted uniform rules are different from the uniform rule. Our next theorem says that adjusted uniform rule for 2 agents with respect to intervals $(x_1,y_1),\ldots,(x_{k_1},y_{k_1})$ and $(w_1,z_1),\ldots,(w_{k_2},z_{k_2})$ rules satisfy efficiency, strategy-proofness, and ETE on a domain that violates Condition U for 2 agents on intervals $(x_1,y_1),\ldots,(x_{k_1},y_{k_1})$ and $(w_1,z_1),\ldots,(w_{k_2},z_{k_2})$, where $0<(x_1,y_1)<\cdots<(x_{k_1},y_{k_1})<\frac{1}{2}<(w_1, z_1)<\cdots<(w_{k_2},z_{k_2})<1$.

Theorem 5.1. *Let* $(x_1,y_1),\ldots,(x_{k_1},y_{k_1})$ *and* $(w_1,z_1),\ldots,(w_{k_2},z_{k_2})$ *be a collection of intervals such that* $0<(x_1,y_1)<\cdots<(x_{k_1},y_{k_1})<\frac{1}{2}<(w_1,z_1)<\cdots<(w_{k_2},z_{k_2})<1$ *and let* $\mathscr{S}$ *be a single-peaked domain that violates Condition U for* 2 *agents on these intervals. Then, every adjusted uniform rule for 2 agents satisfies efficiency, strategy-proofness, and ETE.*

The proof of Theorem 5.1 is relegated to Appendix C.

5.2 THE CASE OF n AGENTS

As we have mentioned earlier, (i) and (ii) of Condition U for n agents are suitable adaptation of (i) and (ii) of Condition U for 2 agents (with $\frac{1}{2}$ being replaced by $\frac{1}{n}$). Thus, if a domain violates any of these conditions, then suitably modified (for n agents) versions of adjusted uniform rules will satisfy efficiency, strategy-proofness, and ETE. For the sake of completeness, we present these rules below. As before, to help the reader, we first present these rules for the case where a domain violates (i) and (ii) on just two intervals (x,y) and (w,z) such that $0<(x,y)<\frac{1}{n}<(w,z)<1$.

To describe the rules formally, we introduce a generalized version of the uniform rule. While the uniform rule divides 1 amount of the good amongst all the agents, a *generalized uniform rule* does the same for arbitrary amount of the good amongst arbitrary subsets of agents. It has a similar formulation as the uniform rule.

To ease the presentation, we introduce the following notations. For an amount $x\in[0,1]$ of the good and a subset $\bar{N}=\{1,\ldots,|\bar{N}|\}\subseteq N$ of agents, we denote by $\Delta^x_{|\bar{N}|}$ the set of all divisions of the amount x among the agents in $\bar{N}$, that is, $\Delta^x_{|\bar{N}|}=\{(x_1,\ldots,x_{|\bar{N}|})\in[0,1]^{|\bar{N}|}\mid \sum_{i\in\bar{N}}x_i=x\}$.

Definition 5.3. For $\bar{N}\subseteq N$ and $x\in[0,1]$, a division rule $u^{(x,\bar{N})}:\mathscr{S}^{|\bar{N}|}\to\Delta^x_{|\bar{N}|}$ is the **generalized uniform rule for** $(x,\bar{N})$ if for all $R_{\bar{N}}\in\mathscr{S}^{|\bar{N}|}$ and all $i\in\bar{N}$,

$$u_i^{(x,\bar{N})}(R_{\bar{N}})=\begin{cases}\min\{\tau(R_i),\lambda(R_{\bar{N}})\} \text{ if } \sum_{i\in\bar{N}}\tau(R_i)\geq x, \text{ and}\\ \max\{\tau(R_i),\mu(R_{\bar{N}})\} \text{ if } \sum_{i\in\bar{N}}\tau(R_i)<x,\end{cases}$$

where $\lambda(R_{\bar{N}})\geq 0$ solves the equation $\sum_{i\in\bar{N}}\min\{\tau(R_i),\lambda(R_{\bar{N}})\}=x$ and $\mu(R_{\bar{N}})\geq 0$ solves the equation $\sum_{i\in\bar{N}}\max\{\tau(R_i),\mu(R_{\bar{N}})\}=x$.

Note that when $x = 1$ and $\bar{N} = N$, the rule $u^{(x,\bar{N})}$ coincides with the uniform rule u.

We are now ready to present our rules. As before, we only specify the shares of $n-1$ agents in an outcome, the remaining agent gets the remaining share. An *adjusted uniform rule for n agents* behaves in the same manner as an adjusted uniform rule for 2 agents with the modification that the shares of the agents other than the "particular agent" i_0 are decided by using a generalized uniform rule.

Definition 5.4. A division rule $f : \mathscr{S}^n \to \Delta_n$ is an **adjusted uniform rule for n agents with respect to intervals** (x,y) and (w,z), where $0 < (x,y) < \frac{1}{n} < (w,z) < 1$, if there exists an agent $i_0 \in N$, such that

(i) for all $R_N \in \mathscr{S}^n_+$ with $\tau(R_{i_0}) \in [x,y)$ and $\tau(R_j) \geq y$ for all $j \neq i_0$, we have

(a) $x + T(R_{N\setminus i}) \geq 1 \implies f_j(R_N) = u_j^{1-x}(R_{N\setminus i_0})$ for all $j \neq i_0$,

(b) $x + T(R_{N\setminus i_0}) < 1 \implies f_j(R_N) = \tau(R_j)$ for all $j \neq i_0$,

(ii) for all $R_N \in \mathscr{S}^n_-$ with $\tau(R_{i_0}) \in (w,z]$ and $\tau(R_j) \leq w$ for all $j \neq i_0$, we have

(a) $z + T(R_{N\setminus i_0}) \leq 1 \implies f_j(R_N) = u_j^{1-z}(R_{N\setminus i_0})$ for all $j \neq i_0$,

(b) $z + T(R_{N\setminus i}) > 1 \implies f_j(R_N) = \tau(R_j)$ for all $j \neq i_0$, and

(iii) for all other profiles $R_N \in \mathscr{S}^n$, $f(R_N) = u(R_N)$.

We now present the notion of adjusted uniform rules for n agents for the general case where a domain violates (i) and (ii) of Condition U for n agents on multiple intervals.

Definition 5.5. Let $(x_1,y_1),\ldots,(x_{k_1},y_{k_1})$ and $(w_1,z_1),\ldots,(w_{k_2},z_{k_2})$ be a collection of intervals such that $0 < (x_1,y_1) < \cdots < (x_{k_1},y_{k_1}) < \frac{1}{n} < (w_1,z_1) < \cdots < (w_{k_2},z_{k_2}) < 1$. A division rule $f : \mathscr{S}^n \to \Delta_n$ is an **adjusted uniform rule for n agents with respect to intervals** $(x_1,y_1),\ldots,(x_{k_1},y_{k_1})$ and $(w_1,z_1),\ldots,(w_{k_2},z_{k_2})$ if there exists an agent $i_0 \in N$, such that

(i) for all $R_N \in \mathscr{S}^n_+$ such that there exists $r \in \{1,\ldots,k_1\}$ with $\tau(R_{i_0}) \in [x_r,y_r)$ and $\tau(R_j) \geq y_r$ for all $j \neq i_0$, we have

(a) $x_r + T(R_{N\setminus i_0}) \geq 1 \implies f_j(R_N) = u_j^{1-x_r}(R_{N\setminus i_0})$ for all $j \neq i_0$,

(b) $x_r + T(R_{N\setminus i_0}) < 1 \implies f_j(R_N) = \tau(R_j)$ for all $j \neq i_0$,

(ii) for all $R_N \in \mathscr{S}^n_-$ such that there exists $s \in \{1,\ldots,k_2\}$ with $\tau(R_{i_0}) \in (w_s,z_s]$ and $\tau(R_j) \leq w_s$ for all $j \neq i_0$, we have

(a) $z_s + T(R_{N\setminus i_0}) \leq 1 \implies f_j(R_N) = u_j^{1-z_s}(R_{N\setminus i_0})$ for all $j \neq i_0$,

(b) $z_s + T(R_{N\setminus i_0}) > 1 \implies f_j(R_N) = \tau(R_j)$ for all $j \neq i_0$, and

(iii) for all other profiles $R_N \in \mathscr{S}^n$, $f(R_N) = u(R_N)$.

Our next theorem says that adjusted uniform rules satisfy efficiency, strategy-proofness, and ETE on a domain that violates (i) and (ii) of Condition U for n agents.

Theorem 5.2. *Let $(x_1,y_1),\dots,(x_{k_1},y_{k_1})$ and $(w_1,z_1),\dots,(w_{k_2},z_{k_2})$ be a collection of intervals such that $0 < (x_1,y_1) < \cdots < (x_{k_1},y_{k_1}) < \frac{1}{n} < (w_1,z_1) < \cdots < (w_{k_2},z_{k_2}) < 1$ and let $\mathscr{S}$ be a single-peaked domain that violates (i) and (ii) of Condition U for n agents on these intervals. Then, every adjusted uniform rule for n agents satisfies efficiency, strategy-proofness, and ETE.*

The proof of Theorem 5.2 is relegated to Appendix D.

Next, we investigate what happens if a domain violates (iii) or (iv) of Condition U for n agents. Note that a domain violates (iii) or (iv) if either there is an interval $(x,y) \subseteq [0,\frac{1}{n}]$ such that y is weakly preferred to x for every preference with peak in that interval, or there is an interval $(x,y) \subseteq [\frac{1}{n},\frac{1}{2}]$ such that x is weakly preferred to y for every preference with peak in that interval.

Observation 5.2. *A domain $\mathscr{S}$ is said to violate (iii) or (iv) of Condition U for n agents on an interval $(x,y) \subseteq [0,\frac{1}{2}]$ with $\frac{1}{n} \notin (x,y)$ if for all $R \in \mathscr{S}$, $\tau(R) \in (x,y)$ implies*

(i) yRx if $(x,y) \subseteq [0,\frac{1}{n}]$, and

(ii) xRy if $(x,y) \subseteq [\frac{1}{n},\frac{1}{2}]$.

In what follows, we present a class of division rules on domains that violate (iii) and (iv) of Condition U for n agents. With slight abuse of terminologies, we keep calling them adjusted uniform rules with respect to the concerned intervals. For simplicity, we present them for the case where there are exactly two intervals (x,y) and (w,z) with $0 < (x,y) < \frac{1}{n} < (w,z) < \frac{1}{2}$ on which (iii) or (iv) of Condition U for n agents is violated. Versions of these rules for other cases (that is, when the said condition is violated on multiple intervals) can be obtained in a similar way as we have done in Definition 5.5.

We explain the behaviour of *adjusted* uniform rules for n agents* with respect to an interval $(x,y) < \frac{1}{n}$, the behaviour of the same with respect to an interval $(w,z) > \frac{1}{n}$ is similar. Such a rule is based on a collection of parameters: $0 \le (x_1,y_1) < (x,y) < x_0 \le 1$ such that $(n-2)x_0 + x + y_1 = (n-2)x_0 + y + x_1 = 1$, and two particular agents who we denote by i_0 and j_0. Note that the structure of the collection of parameters implies that we can divide the good by giving each agent other than i_0 and j_0 a share x_0, and agents i_0 and j_0 shares either x and y_1, or y and x_1. An adjusted* uniform rule for n agents coincides with the uniform rule at all profiles except a few as follows. Consider an arbitrary profile with total demand at most 1 such that agents other than i_0, j_0 have peaks at x_0 and agent i_0 has peak in the interval (x,y). Adjusted* uniform rule for n agents says that (a) if agent j_0's peak lies in the interval (x_1,y_1), then everybody except agent i_0 will get their peaks, and (b) if agent j's peak is less than or equal to x_1, then everybody except agents i_0 and j_0 will get their peaks, and agent j_0 will get x_1. Note that for the uniform rule, agent i_0 would get his peak in both the cases. Thus, adjusted uniform rules are negatively biased towards some particular agent, who, in our case, is i_0.

Definition 5.6. Let (x,y) and (w,z) be two intervals such that $0 < (x,y) < \frac{1}{n} < (w,z) < \frac{1}{2}$. A division rule $f : \mathscr{S}^n \to \Delta_n$ is an **adjusted* uniform rule for n agents with respect to (x,y) and (w,z)** if there are

(a) $x_0, (x_1,y_1), w_0, (w_1,z_1)$ with

  (a) $(x_1,y_1) < (x,y) < x_0$ and $w_0 < (w,z) < (w_1,z_1)$,

  (b) $(n-2)x_0+x+y_1 = (n-2)x_0+y+x_1 = 1$ and $(n-2)w_0+w+z_1 = (n-2)w_0+z+w_1 = 1$, and

(b) two particular agents i_0 and j_0

such that

(i) for all $R_N \in \mathscr{S}^n_-$ with $\tau(R_k) = x_0$ for all $k \in N \setminus \{i_0, j_0\}$ and $\tau(R_{i_0}) \in (x,y)$, we have

  (a) $\tau(R_{j_0}) \in (x_1,y_1) \implies f_l(R_N) = \tau(R_l)$ for all $l \neq i_0$, and

  (b) $\tau(R_{j_0}) \leq x_1 \implies f_l(R_N) = \tau(R_l)$ for all $l \in N \setminus \{i_0, j_0\}$ and $f_{j_0}(R_N) = x_1$,

(ii) for all $R_N \in \mathscr{S}^n_+$ with $\tau(R_k) = w_0$ for all $k \in N \setminus \{i_0, j_0\}$ and $\tau(R_{i_0}) \in (w,z)$, we have

  (a) $\tau(R_{j_0}) \in (w_1,z_1) \implies f_l(R_N) = \tau(R_l)$ for all $l \neq i_0$, and

  (b) $\tau(R_{j_0}) \geq z_1 \implies f_l(R_N) = \tau(R_l)$ for all $l \in N \setminus \{i_0, j_0\}$ and $f_{j_0}(R_N) = z_1$, and

(iii) for all other $R_N \in \mathscr{S}^n$, we have $f(R_N) = u(R_N)$.

REMARK 5.1. The outcomes of an adjusted* uniform rule for n agents and the uniform rule can only differ at the profiles where either (i) or (ii) in Definition 5.6 is satisfied. Moreover, even on those profiles, the two rules can differ only over the shares of agents i_0 and j_0.

Our next theorem says that adjusted* uniform rules for n agents satisfy efficiency, strategy-proofness, and ETE on a domain that violates (iii) and (iv) of Condition U for n agents.

Theorem 5.3. *Let (x,y) and (w,z) be two intervals such that $0 < (x,y) < \frac{1}{n} < (w,z) < \frac{1}{2}$ and let $\mathscr{S}$ be a single-peaked domain that violates (iii) and (iv) of Condition U for n agents on these intervals. Then, every adjusted* uniform rule for n agents with respect to (x,y) and (w,z) satisfies efficiency, strategy-proofness, and ETE.*

The proof of Theorem 5.3 is relegated to Appendix E.

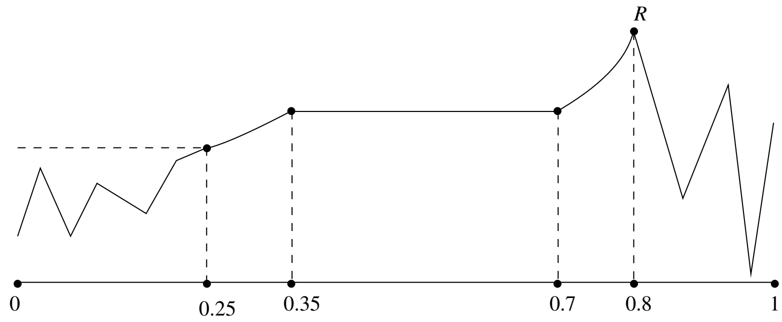

Figure 1: A pictorial illustration of the feebly single-peaked preference in Example 6.1

6. A CHARACTERIZATION OF NON SINGLE-PEAKED DOMAINS FOR THE UNIFORM RULE

In Section 4, we have provided a necessary and sufficient condition on a single-peaked domain so that the uniform rule is the only division rule satisfying efficiency, strategy-proofness, and ETE. Question arises as to whether single-peakedness is necessary for the uniform rule to satisfy these properties. We show that the answer is "no" , that is, the uniform rule can satisfy these properties even on non-single-peaked domains. In what follows, we provide a necessary condition on an arbitrary domain so that the uniform rule satisfies efficiency, strategy-proofness, and ETE. We further show that if a domain contains all preferences satisfying our necessary condition, then the uniform rule is the unique rule satisfying efficiency, strategy-proofness, and ETE.

Massó and Neme (2001) provide a characterization of possibility domains as feebly single-plateaued domains. The following example shows that the structure of division rules satisfying efficiency, strategy-proofness, ETE on a possibility domain might be quite complicated, even when the plateau is restricted to be a peak.

Example 6.1. Let $N = \{1,\dots,4\}$ and let $\mathscr{D}$ be a domain containing exactly one non single-peaked preference $\bar{R}$ with $\tau(\bar{R}) = 0.8$ such that (i) $y\bar{R}x$ for all $0.25 \leq x < y \leq 0.8$, (ii) $y\bar{I}x$ if and only if $x, y \in [0.35, 0.7]$, and (iii) $0.25\bar{P}w$ for all $w < 0.25$. See Figure 1 for a pictorial illustration of this preference. It can be verified that the domain $\mathscr{D}$ is a feebly single-plateaued domain (as defined in Massó and Neme (2001)), and hence, by Massó and Neme (2001) there exists a division rule that admits efficiency, strategy-proofness, and ETE.

First, we show that the uniform rule violates efficiency on $\mathscr{D}$. Consider the preference profile (R_1, R_2, R_3, R_4) where R_i is a single-peaked preference for all $i \in \{1,2,3\}$ and $R_4 = \bar{R}$. Let $\tau(R_1) = \tau(R_2) = 0.1$ and $\tau(R_3) = 0.8$. By the definition of the uniform rule, $u(R_N) = (0.1, 0.1, 0.4, 0.4)$. Let $\mathbf{y} = (0.1, 0.1, 0.45, 0.35)$. Since $0.35I_4 0.4$ and $0.45P_3 0.4$, we have $y_i R_i u_i(R_N)$ for all $i \in N$ and $y_3 P_3 u_3(R_N)$. This shows the uniform rule u is not efficient.

Now, we provide a division rule $f : \mathscr{D}^4 \to \Delta_4$ satisfying efficiency, strategy-proofness, and ETE to show that the structure of such rules is quite complicated. Consider the division rule $f : \mathscr{D}^4 \to \Delta_4$ as given below

(i) if $T(R_N) > 1$ and there exists $i \in N$ such that $u_i(R_N) > 0.35$, then

$$f_j(R_N) = \begin{cases} \min\{u_j(R_N) + \lambda(R_N), \tau(R_j)\} \text{ if } R_j \neq \bar{R}, \\ \min\{0.35, u_j(R_N) - \lambda(R_N)\} \text{ if } R_j = \bar{R}, \end{cases}$$

where $\lambda(R_N) > 0$ is such that $\sum_{j \in N} f_j(R_N) = 1$, and

(ii) for all other profiles R_N, $f(R_N) = u(R_N)$. □

It is worth mentioning that the division rule f presented above does not even satisfy the tops-only property.[6] The domain under consideration violates single-peakedness for exactly one preference; more complicated division rules can be constructed for feebly single-peaked domains that violate single-peakedness for more preferences. This constitutes the main motivation for the analysis of this section.

A domain (not necessarily single-peaked) satisfies *Condition N* if the following holds: (i) If the peak of a preference is bigger than $\frac{1}{n}$, then preference weakly declines as one moves from the peak towards the left direction until $\frac{1}{2}$ (if $\frac{1}{2}$ is on the left of the peak), it strictly declines after that until $\frac{1}{n}$, and all the shares that are less than $\frac{1}{n}$ are ranked below $\frac{1}{n}$. (ii) If the peak is smaller than $\frac{1}{n}$, then preference strictly declines until $\frac{1}{2}$, and and all the shares that are more than $\frac{1}{n}$ are ranked below $\frac{1}{n}$. Note that there is no restriction on the relative ordering of two shares if they are less than $\frac{1}{n}$ or bigger than the peak for case (i), and if they are more than $\frac{1}{n}$ or smaller than the peak for case (ii). Further note that Condition N does not put any restrictions on the preferences with peak $\frac{1}{n}$. Chatterji et al. (2013) introduce the notion of semi-single-peaked domains. It can be verified that Condition N is a generalization of semi single-peakedness for (weak) preferences.

Definition 6.1. A domain $\mathscr{D}$ satisfies **Condition N** if for all $R \in \mathscr{D}$ the following holds:

(i) $\tau(R) > \frac{1}{n}$ implies that for all x, y, z with $z < \frac{1}{n} \leq x < y \leq \tau(R)$,

  (a) if $x < \frac{1}{2}$, we have $yPxPz$,

  (b) if $x \geq \frac{1}{2}$, we have $yRxPz$, and

(ii) $\tau(R) < \frac{1}{n}$ implies that for all x, y, z with $\tau(R) \leq y < x \leq \frac{1}{n} < z$, we have $yPxPz$.

Our next theorem says that a domain must satisfy Condition N in order for the uniform rule to satisfy efficiency, strategy-proofness, and ETE.

[6] A division rule $f : \mathscr{D}^n \to \Delta_n$ is tops-only if for all $R_N, R'_N \in \mathscr{D}^n$, $[\tau(R_i) = \tau(R'_i)$ for all $i \in N] \implies [f(R_N) = f(R'_N)]$.

Theorem 6.1. *The uniform rule satisfies efficiency, strategy-proofness, and ETE on a domain if and only if it satisfies Condition N.*

The proof of Theorem 6.1 is relegated to Appendix F.

Theorem 6.2 shows that if a domain satisfies an additional richness condition, then the uniform rule is the unique rule that satisfies efficiency, strategy-proofness, and ETE on a domain satisfying Condition N. Thus, Theorem 6.2 generalizes Theorem 4.1 and Theorem 4.2 for arbitrary (non-single-peaked) domains. A domain $\mathscr{D}$ is minimally rich if for all $x,y,z \in [0,1]$ with $x<y<z$, there exist single-peaked $R,R' \in \mathscr{D}$ with $\tau(R)=\tau(R')=y$ such that xPz and $zP'x$.

Theorem 6.2. *Let $\mathscr{D}$ be a minimally rich domain satisfying Condition N. Then, a division rule $f:\mathscr{D}^n \to \Delta_n$ satisfies efficiency, strategy-proofness, and ETE if and only if it is the uniform rule.*

The proof of Theorem 6.2 is relegated to Appendix G.

7. The structure of division rules on non single-peaked domains that violate Condition N

In this section, we investigate the structure of efficient, strategy-proof, and ETE divisions rules for domains on which the uniform rule does not satisfy those properties, that is, domains which violate Condition N. There are many ways a domain can violate Condition N, we consider the one that we find important for practical applications.[7]

In what follows, we introduce the notion of local-peaked domains. To ease our presentation, we say that a preference R strictly (or weakly) declines till a point z if $z<\tau(R)$ implies $xPyPz$ (or $xRyRz$) for all $z<y<x<\tau(R)$, and if $\tau(R)<z$ implies $xPyPz$ (or $xRyRz$) for all $z>y>x>\tau(R)$. Recall that Condition N implies that every preference weakly declines till the share $\frac{1}{n}$. For local-peaked domains, every preference satisfies this property except (possibly) the ones with relatively high peak (more than $(1-\frac{1}{n})$) which might violate this by giving the share $\frac{1}{n}$ some "special" preference: it strictly declines towards left till a point $p^* > (1-\frac{1}{n})$, and the share $\frac{1}{n}$ dominates all shares on the left of p^*. Thus, the share $\frac{1}{n}$ acts like a "local peak" in such a preference. In Figures 2 and 3, we provide two instances of such a preference.

We further assume that when the peak of a preference is relatively low (on the left of $\frac{1}{n}$), then the share $\frac{1}{n}$ strictly dominates any share on the left of the peak, which, again, says that the share $\frac{1}{n}$ gets some special preference.

Definition 7.1. A domain $\mathscr{D}$ is a local-peaked domain if for all $R \in \mathscr{D}$,

[7] See Section 1 for practical relevance of such domains.

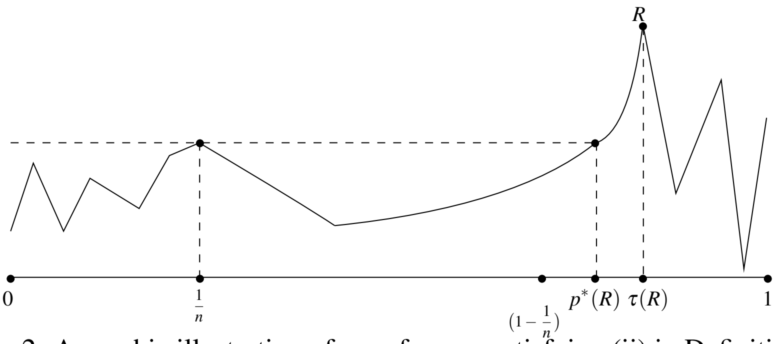

Figure 2: A graphic illustration of a preference satisfying (ii) in Definition 7.1

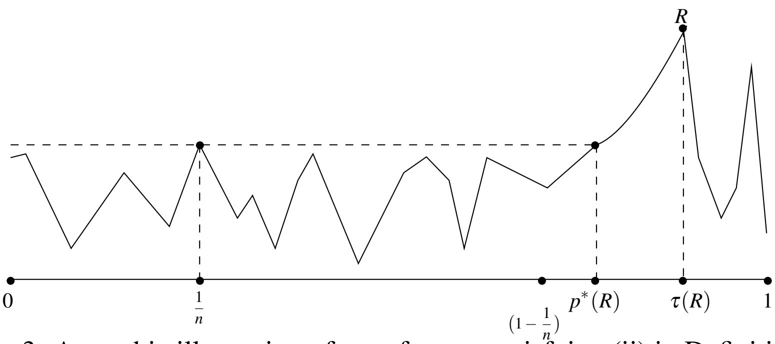

Figure 3: A graphic illustration of a preference satisfying (ii) in Definition 7.1

(i) $\tau(R) \leq 1 - \frac{1}{n}$ implies that R satisfies Condition N,

(ii) $\tau(R) > 1 - \frac{1}{n}$ implies that either R satisfies Condition N or there exists $p^*(R) \in \big((1 - \frac{1}{n}), \tau(R)\big)$ such that R strictly declines till $p^*(R)$, $\frac{1}{n}Px$ for all $x < p^*(R)$, and $p^*(R)I\frac{1}{n}$, and

(iii) $\tau(R) < \frac{1}{n}$ implies $\frac{1}{n}Px$ for all $x < \tau(R)$.

Definition 7.2. A division rule $f : \mathscr{D}^n \to \Delta_n$ is called partially uniform rule if for all $R_N \in \mathscr{D}^n$,

(i) if $T(R_N) > 1$ and $u_i(R_N) < p^*(R_i)$ for all $i \in N$ such that R_i satisfies (ii) of Definition 7.1, then $f(R_N) = u(\bar{R}_N)$ where $\tau(\bar{R}_i) = \frac{1}{n}$ if R_i satisfies (ii) of Definition 7.1 and $\bar{R}_i = R_i$ otherwise, and

(ii) $f(R_N) = u(R_N)$ otherwise.

Theorem 7.1. *Let $\mathscr{D}$ be a local-peaked domain. Then, the partially uniform rule satisfies efficiency, strategy-proofness, and ETE.*

The proof of this theorem is relegated to Appendix H.

8. Applications

In this section we apply our results to analyse the structure of division rules satisfying efficiency, strategy-proofness, and ETE on domains that arise naturally in practical scenarios. We assume that all the domains we consider in this section are regular. Throughout this section, whenever we say that a domain satisfies Condition U we mean it for any number of agents (that is, both 2 agents and more than 2 agents).

8.1 PREFERENCES GIVEN BY UTILITY FUNCTIONS

In this subsection we consider scenarios where agents have specific (single-peaked) utility functions. The main message of this subsection is that Condition U is satisfied for most utility functions of economic interest leaving the uniform rule as the unique rule satisfying efficiency, strategy-proofness, ETE.

A (normalized) utility function is a function $f : [0,1] \to [0,1]$. For any property of a preference, we say a utility function satisfies it if the preference it represents satisfies the same. We follow similar terminologies for domains, for instance, we say a class $\mathscr{F}$ of utility functions is Euclidean if the class of preferences it represents is Euclidean. We now present some general instances where Condition U is satisfied. Throughout this subsection, we denote a single-peaked utility function with peak at x by f^x.

8.1.1 UTILITY FUNCTIONS SATISFYING A CONVERGENCE PROPERTY

Suppose that a class of single-peaked utility functions $\mathscr{F}$ satisfies the following convergence property: for every sequence $\{x_n\}$ in $[0,1]$, x_n converges to x implies that there exists a sequence $\{f^{x_n}\} \in \mathscr{F}$ such that f^{x_n} converges to f^x pointwise.[8] Let $\mathscr{F}^c$ be a set of utility functions satisfying the above-mentioned convergence property. The following proposition shows that it satisfies Condition U.

Proposition 8.1. *The class of utility functions $\mathscr{F}^c$ satisfies Condition U.*

Proof: Consider two points $x, y \in [0,1]$ such that $x < y$. We show that there exists a utility function $f^z : [0,1] \to [0,1]$ such that $z \in (x,y)$ and $f^z(x) < f^z(y)$. The existence of a utility function $f^w : [0,1] \to [0,1]$ such that $w \in (x,y)$ and $f^w(x) > f^w(y)$ follows by using similar arguments. Consider a sequence $\{y_n\}$ of real numbers such that $\{y_n\}$ converges to y and $x < y_n < y$ for all $n \in \mathbb{N}$. By the definition of $\mathscr{F}^c$, there exists a sequence of utility functions f^{y_n} such that f^{y_n} converges to f^y pointwise. Since the peak of f^y is y, we have $f^y(y) > f^y(x)$, and hence, there must exist $\hat{n} \in \mathbb{N}$, large enough, such that $f^{y_{\hat{n}}}(y) > f^{y_{\hat{n}}}(x)$. Because $y_{\hat{n}} \in (x,y)$, this completes the proof of the proposition. ■

It follows from Theorem 4.1 and 4.2 that the uniform rule is the unique division rule satisfying efficiency, strategy-proofness, and ETE on $\mathscr{F}^c$.

8.1.2 UTILITY FUNCTIONS SATISFYING A TRANSLATION PROPERTY

Let $\mathscr{F}^0$ be a finite class of single-peaked utility functions $f^0 : \mathbb{R} \to [0,1]$ with peak at 0. For $x \in [0,1]$ and $f^0 \in \mathscr{F}^0$, define $f^x : [0,1] \to [0,1]$ as $f^x(y) = f^0(y-x)$ for all $y \in [0,1]$. In other words, f^x translates f^0

[8] A sequence of functions $f_n : [0,1] \to [0,1]$ converges pointwise to a function $f : [0,1] \to [0,1]$ if for all $x \in [0,1]$, $f_n(x) \to f(x)$.

by shifting the peak of f^0 from 0 to x and maintaining the same functional structure as f^0. For instance, if f^0 is Euclidean (with peak at 0), then f^x is also Euclidean but with peak at x. Let $\mathscr{F}^T$ be a subset of $\cup_{x\in[0,1]}\mathscr{F}^x$. The following proposition shows that it satisfies Condition U.

Proposition 8.2. *The class of utility functions $\mathscr{F}^T$ satisfies Condition U.*

Proof: We show that for all $x,y \in [0,1]$ with $x < y$, there exist $w,z \in (x,y)$ and $\hat{f}^w, \hat{f}^z \in \mathscr{F}^T$ such that $\hat{f}^w(x) > \hat{f}^w(y)$ and $\hat{f}^z(y) > \hat{f}^z(x)$. We show the existence of z and $\hat{f}^z$; the same for w and $\hat{f}^w$ follows using similar arguments.

Let $\bar{\mathscr{F}}^0$ be the subset of $\mathscr{F}^0$ consisting of all the utility functions in $\mathscr{F}^0$ that are locally continuous around the peak. Since $\mathscr{F}^T$ is regular, for every $s \in [0,1]$, there exists $\bar{f}^s \in \mathscr{F}^T$ such that its translation $\bar{f}^0$ to shift the peak to 0 is in $\bar{\mathscr{F}}^0$. Let $u = \dfrac{x+y}{2}$. By the finiteness of $\bar{\mathscr{F}}^0$ and the local continuity around the peak property of all $f^0 \in \bar{\mathscr{F}}^0$, there must exist $\varepsilon > 0$ such that $\bar{f}^0(\varepsilon) > \bar{f}^0(x-u)$ for all $\bar{f}^0 \in \bar{\mathscr{F}}^0$. Take $z \in (x,y)$ such that $z \geq u$ and $z + \varepsilon > y$. By the regularity of $\mathscr{F}^T$, there exists $\hat{f}^z \in \mathscr{F}^T$ such that $\hat{f}^0 \in \bar{\mathscr{F}}^0$. Since $\hat{f}^0(\varepsilon) > \hat{f}^0(x-u)$, single-peakedness of $\hat{f}^0$ implies $\hat{f}^0(y-z) > \hat{f}^0(x-z)$. Therefore, by definition, $\hat{f}^z(y) > \hat{f}^z(x)$. This completes the proof of the proposition. ■

Euclidean or scaled Euclidean utility functions are straightforward examples of $\mathscr{F}^T$ satisfying Condition U. Moreover, one can (suitably) consider different types of utility functions for different peaks maintaining Condition U. It follows from Theorem 4.1 and 4.2 that the uniform rule is the unique division rule satisfying efficiency, strategy-proofness, and ETE on $\mathscr{F}^T$.

8.1.3 Most single-peaked domains satisfy Condition U

In this subsection, we emphasize that most single-peaked domains satisfy Condition U and thereby the uniform rule is the unique rule satisfying efficiency, strategy-proofness, and ETE on them. In what follows, we present an example of a domain that violates Condition U to clarify that such a violation can happen only under complicated structure of preferences.

Example 8.1. For simplicity consider the case where for each $w \in [0,1]$ there is a unique scaled Euclidean utility function f^w with (scale) parameters $\kappa_1(w)$ and $\kappa_2(w)$. Thus, different peaks can have different values of κ's. Let the class of such utility functions be $\mathscr{F}$. Suppose this class $\mathscr{F}$ does not satisfy Condition U. Then, there must exist $x,y \in [0,1]$ with $x < y$ such that for all $z \in (x,y)$, we have, for instance, $f^z(x) \geq f^z(y)$. By the definition of scaled Euclidean utility functions, this means $\kappa_1(z)(z-x) \leq \kappa_2(z)(y-z)$, which means $\dfrac{\kappa_1(z)}{\kappa_2(z)} \leq \dfrac{(y-z)}{(z-x)}$. For z arbitrarily close to y, this requires $\dfrac{\kappa_1(z)}{\kappa_2(z)}$ to be arbitrarily close to zero. However, by the definition of single-peakedness, the slopes κ's cannot be zero or infinity. Therefore, violation of Condition U requires preferences to be arbitrarily close to being flat but not flat. To our understanding, such preferences are not "realistic".

8.2 Single-crossing domains

Single-crossing preferences are important for their practical usefulness (see Saporiti (2009) for details).

Definition 8.1. A domain $\mathscr{D}$ satisfies the single-crossing property if for all $x, y \in [0,1]$ with $x < y$, there exists $\kappa_{xy} \in [0,1]$ such that for all preferences $R \in \mathscr{D}$, $\tau(R) < \kappa_{xy}$ implies xPy, and $\tau(R) > \kappa_{xy}$ implies yPx.

Note that by the definition of κ_{xy}, we must have $\kappa_{xy} \in [x,y]$. Let us call a domain strictly single-crossing if $\kappa_{xy} \in (x,y)$ for all $x, y \in [0,1]$. It is easy to verify that strict single-crossing domains are subsets of single-peaked domains and satisfy Condition U. Domains with nice functional structure such as (scaled) Euclidean, etc., are simple examples of single-crossing domains. It is worth mentioning that there are single-crossing domains with much complicated structure. It follows from Theorem 4.1 and 4.2 that the uniform rule is the unique division rule satisfying efficiency, strategy-proofness, and ETE on a strict single-crossing domain.

8.3 Semi-single-peaked domains

The notion of semi-single-peaked domains is introduced by Chatterji et al. (2013). A domain $\mathscr{D}$ is semi-single-peaked with respect to a threshold $\kappa \in [0,1]$, if (i) every preference $R \in \mathscr{D}$ is single-peaked over the interval $[\tau(R), \kappa]$, and (ii) each point located beyond the threshold is less preferred than the threshold, that is, $\kappa \in [\tau(R), x]$ implies $\kappa P x$. It can be verified that a semi-single-peaked domain satisfies Condition N if and only if the threshold κ is $\frac{1}{n}$. In what follows, we generalize the notion of semi single-peaked domains for two thresholds.

A generalized semi-single-peaked domain $\mathscr{D}$ involves two thresholds $\kappa_1 < \kappa_2$ and requires that preferences are semi-single-peaked with respect to each of these thresholds, that is, (i) every preference $R \in \mathscr{D}$ is single-peaked over the interval $[\tau(R), \kappa_2]$ if $\tau(R) < \kappa_1$, over the interval $[\kappa_1, \kappa_2]$ if $\tau(R) \in [\kappa_1, \kappa_2]$, and over the interval $[\kappa_1, \tau(R)]$ if $\tau(R) > \kappa_2$, and (ii) each point located beyond the threshold κ_i is less preferred than the threshold κ_i, that is, $\kappa_i \in [\tau(R), x]$ implies $\kappa_i P x$ for all $i \in \{1,2\}$, and (iii) there are no $\kappa_1' < \kappa_1$ and $\kappa_2' > \kappa_2$ such that (i) and (ii) hold. One can think of a generalized semi-single-peaked domain with more than two thresholds $\kappa_1, \ldots, \kappa_l$ but such a domain will be reduced to one with two thresholds $\min\{\kappa_1, \ldots, \kappa_l\}$ and $\max\{\kappa_1, \ldots, \kappa_l\}$. Our next proposition provides a characterization of semi-single-peaked domains satisfying Condition N.

Proposition 8.3. *A generalized semi-single-peaked domain $\mathscr{D}$ with thresholds $\kappa_1 < \kappa_2$ satisfies Condition N if and only if $\kappa_1 \leq \frac{1}{n} \leq \kappa_2$.*

Proof: (If part) Let $\mathscr{D}$ be a generalized semi-single-peaked domain with thresholds $\kappa_1 < \kappa_2$ such that $\kappa_1 \leq \frac{1}{n} \leq \kappa_2$. We show that $\mathscr{D}$ satisfies Condition N. Without loss of generality, let $R \in \mathscr{D}$ be such that

$\tau(R) > \frac{1}{n}$. This in particular means $\tau(R) > \kappa_1$. By the definition of generalized semi-single-peaked domains, if $\tau(R) \leq \kappa_2$ then R is single-peaked over the interval $[\kappa_1, \kappa_2]$, and if $\tau(R) > \kappa_2$ then R is single-peaked over the interval $[\kappa_1, \tau(R)]$. Moreover, each point located on the left of the threshold κ_1 is less preferred to the threshold κ_1 according to R. Combining all these observations, we have (i) for all $x, y \in [0,1]$ with $\kappa_1 \leq x < y \leq \tau(R)$, we have yPx, and (ii) for all $z < \kappa_1$, we have $\kappa_1 Pz$. Since $\kappa_1 \leq \frac{1}{n} < \tau(R)$, this implies R satisfies Condition N. This completes the proof of the if part of the proposition.

(Only-if part) Let $\mathscr{D}$ be a generalized semi-single-peaked domain with threshold $\kappa_1 < \kappa_2$ that satisfies Condition N. We show that $\kappa_1 \leq \frac{1}{n} \leq \kappa_2$. Assume for contradiction $\frac{1}{n} < \kappa_1$. Recall that by the definition of generalized intermediate domains, there is no $\kappa_1' < \kappa_1$ such that $\mathscr{D}$ is generalized semi-single-peaked with thresholds κ_1' and κ_2. This implies that for all $\kappa_1' < \kappa_1$, there exists $R \in \mathscr{D}$ such that R violates either (i) or (ii) with respect to κ_1' and κ_2. Let $\kappa_1' = \frac{1}{n}$ and $R \in \mathscr{D}$ be a preference that violates either (i) or (ii). If R violates (i), then as $\frac{1}{n} < \kappa_1 < \kappa_2$, by the definition of generalized semi-single-peaked domains, we have (i) $\tau(R) > \kappa_1'$, and (ii) xPy for some x, y with $\frac{1}{n} \leq x < y \leq \tau(R)$. This means $\mathscr{D}$ violates Condition N. If R violates (ii), then by the definition of generalized semi-single-peaked domains, (i) we have $\tau(R) > \kappa_1'$, and (ii) $xP\frac{1}{n}$ for some x with $x < \frac{1}{n} \leq \tau(R)$, which is again a violation of Condition N. This completes the proof of the only-if part of the proposition. ■

By Theorem 6.2 and Proposition 8.3, it follows that the uniform rule is the unique rule that satisfies efficiency, strategy-proofness, and ETE on any minimally rich generalized semi-single-peaked domain with thresholds κ_1 and κ_2 where $\kappa_1 \leq \frac{1}{n} \leq \kappa_2$.

8.4 Partially single-peaked domain

The notion of *partially single-peaked* domains is introduced in Achuthankutty and Roy (2017). As the name suggests, these domains violate single-peakedness over a subinterval of $[0,1]$ and satisfies it over the remaining alternatives. We consider a slightly more general notion of partially single-peaked domains. Let $\underline{x}$ and $\bar{x}$ be two fixed points with $\underline{x} < \bar{x}$.

Definition 8.2. A domain $\mathscr{D}$ *satisfies single-peakedness outside* $[\underline{x}, \bar{x}]$ if for all $R \in \mathscr{D}$, all $u \notin (\underline{x}, \bar{x})$, and all $v \in X$,

$$\big[v < u \leq \tau(R) \text{ or } \tau(R) \leq u < v\big] \text{ implies } uPv.$$

Definition 8.3. A domain $\mathscr{D}$ *violates single-peakedness over* $[\underline{x}, \bar{x}]$ if there exist $\tilde{R} \equiv \underline{x}y\cdots, \tilde{R}' \equiv \bar{x}z\cdots \in \tilde{\mathscr{D}}$ such that $\big[y \in (\underline{x}, \bar{x}] \text{ and } z \in [\underline{x}, \bar{x})\big]$.

Definition 8.4. A domain $\mathscr{D}$ is called $[\underline{x}, \bar{x}]$-*partially single-peaked* if it satisfies single-peakedness outside $[\underline{x}, \bar{x}]$ and violates it over $[\underline{x}, \bar{x}]$.

Proposition 8.4. *No partially single-peaked domain satisfies Condition N.*

Proof: Let $\mathscr{D}$ be a $[\underline{x},\overline{x}]$-partially single-peaked domain. We show that $\mathscr{D}$ violates Condition N. Suppose $\frac{1}{n} < \overline{x}$. By the definition of partially single-peaked domains, there exists $\tilde{R}' \in \mathscr{D}$ with $\tilde{R}' \equiv \overline{x}z\cdots$ such that $z \in [\underline{x},\overline{x})$. Take $w \in (z,\overline{x})$ such that $w > \frac{1}{n}$. By the definition of $\tilde{R}'$ we have $z\tilde{P}'w$. Combining all these observations, we have $z < w < \tau(\tilde{R}')$, $w > \frac{1}{n}$, and $z\tilde{P}'w$, which is a violation of Condition N. Now, suppose $\overline{x} \leq \frac{1}{n}$. This implies $\underline{x} < \frac{1}{n}$. By the definition of partially single-peaked domains, there exists $\tilde{R} \in \mathscr{D}$ with $\tilde{R} \equiv \underline{x}y\cdots$ such that $y \in (\underline{x},\overline{x}]$. Take $w \in (\underline{x},y)$ such that $w < \frac{1}{n}$. By the definition of $\tilde{R}$ we have $y\tilde{P}w$. Combining all these observations, we have $\tau(\tilde{R}) < w < y$, $w < \frac{1}{n}$, and $y\tilde{P}w$, which is a violation of Condition N. This completes the proof of the proposition. ■

By Theorem 6.1 and Proposition 8.4, it follows that the uniform rule does not satisfy efficiency, strategy-proofness, and ETE on partially single-peaked domains.

A. PROOF OF THEOREM 4.1

Proof of Part (i): Let $\mathscr{S}$ be a regular single-peaked domain satisfying Condition U for 2 agents. We show that a division rule $f : \mathscr{S}^2 \to \Delta_2$ satisfies efficiency, strategy-proofness, and ETE if and only if it is the uniform rule. To ease the presentation of the proof, by R^x we denote a preference with peak x.

("If" part) The "if" part of the theorem follows from Theorem 6.1 of this paper, where we show that the uniform rule satisfies efficiency, strategy-proof, ETE on a domain satisfying a condition called Condition N. It is straightforward to verify that Condition U for 2 agents implies Condition N.

("Only if" part) Let $\mathscr{S}$ be a domain satisfying Condition U for 2 agents and let $f : \mathscr{S}^2 \to \Delta_2$ be a division rule that satisfies efficiency, strategy-proofness, and ETE. We show that for all $(R_1,R_2) \in \mathscr{S}^2$, $f(R_1,R_2) = u(R_1,R_2)$. Consider a profile $(R_1,R_2) \in \mathscr{S}^2$. We distinguish the following cases:

Case 1: Suppose $\max\{\tau(R_1),\tau(R_2)\} \leq \frac{1}{2}$ or $\min\{\tau(R_1),\tau(R_2)\} \geq \frac{1}{2}$.

We only prove for the case $\max\{\tau(R_1),\tau(R_2)\} \leq \frac{1}{2}$. The proof of the remaining case $\min\{\tau(R_1),\tau(R_2)\} \geq \frac{1}{2}$ follows by using similar arguments. By the definition of the uniform rule, $u(R_1,R_2) = (\frac{1}{2},\frac{1}{2})$. Note that if $\tau(R_1) = \tau(R_2) \leq \frac{1}{2}$, then by strategy-proofness, efficiency, and ETE, we have $f(R_1,R_2) = (\frac{1}{2},\frac{1}{2})$. We claim that if $\tau(R_1) < \tau(R_2) = \frac{1}{2}$, then $f(R_1,R_2) = (\frac{1}{2},\frac{1}{2})$. Suppose not. By efficiency we must have $f_2(R_1,R_2) > \frac{1}{2}$. However, this means agent 2 will manipulate at (R_1,R_2) via R_1 as $f_2(R_1,R_1) = \frac{1}{2}$, a contradiction. Similarly, we have $f(R_1,R_2) = (\frac{1}{2},\frac{1}{2})$ when $\tau(R_2) < \tau(R_1) = \frac{1}{2}$. Now, suppose $\tau(R_1) < \frac{1}{2}$ and $\tau(R_2) < \frac{1}{2}$. If $f(R_1,R_2) \neq (\frac{1}{2},\frac{1}{2})$, then there exists $i \in \{1,2\}$ such that $f_i(R_1,R_2) > \frac{1}{2}$. WLG assume $i = 1$. However, then agent 1 will manipulate via R_2 since $f(R_2,R_2) = (\frac{1}{2},\frac{1}{2})$ and by single-peakedness $\frac{1}{2}P_1 f_1(R_1,R_2)$.

Case 2: Suppose $\max\{\tau(R_1),\tau(R_2)\} > \frac{1}{2}$ and $\min\{\tau(R_1),\tau(R_2)\} < \frac{1}{2}$.

WLG assume that $\max\{\tau(R_1),\tau(R_2)\} = \tau(R_1)$ and $\min\{\tau(R_1),\tau(R_2)\} = \tau(R_2)$. Suppose $\tau(R_1) + \tau(R_2) < 1$. By the definition of the uniform rule, we have $u(R_1,R_2) = (\tau(R_1), 1-\tau(R_1))$. Assume for contradiction that $u(R_1,R_2) \neq f(R_1,R_2)$. This means by efficiency $f_1(R_1,R_2) > \tau(R_1)$. Note that by efficiency and strategy-proofness, we have for all $y \in [0,1]$ and all $R^y, R_1^y \in \mathscr{S}$, $f_1(R^y,R_2) = f_1(R_1^y,R_2)$. Consider the set $\{y \in [0,1] \mid f_1(R_1,R_2) = f_1(R^y,R_2)\}$. Let $x = \inf\{y \in [0,1] \mid f_1(R_1,R_2) = f_1(R^y,R_2)\}$. Note that by strategy-proofness, we have for all $y \in (x, f_1(R_1,R_2)]$, $f_1(R^y,R_2) = f_1(R_1,R_2)$. Since $f_1(R_1,R_2) > \tau(R_1)$, we have $x < f_1(R_1,R_2)$ and by Case 1, $x > \frac{1}{2}$. Suppose $x \in \{y \in [0,1] \mid f_1(R_1, R_2) = f_1(R^y,R_2)\}$, i.e., $f_1(R^x,R_2) = f_1(R_1,R_2)$. Note that since $\mathscr{S}$ is a regular domain, there exists a preference R with $\tau(R) = x$ such that R is locally continuous around the peak. This implies there exists z such that $z < x$ and $zPf_1(R_1,R_2)$. By efficiency and the fact that $f_1(R^x,R_2) = f_1(R_1,R_2)$, we have $f_1(R^z,R_2) \in [z,x)$. This means agent 1 manipulates at (R,R_2) via R^z, a contradiction. Thus, $x \notin \{y \in [0, 1] \mid f_1(R_1,R_2) = f_1(R^y,R_2)\}$. By efficiency, we have $f_1(R^x,R_2) = x$. Since $x > \frac{1}{2}$ by Condition U for 2 agents, there exists a preference R' with $\tau(R') \in (x, f_1(R_1,R_2))$ such that $xP'f_1(R_1,R_2)$. But this means f is manipulable at (R',R_2) via R^x as $f_1(R^y,R_2) = f_1(R_1,R_2)$ for all $y \in (x,\tau(R_1)]$. Thus $f_1(R_1, R_2) = \tau(R_1)$ and $f_2(R_1,R_2) = 1-\tau(R_1)$, which in turn implies that $f(R_1,R_2) = u(R_1,R_2)$. The proof for the case $\tau(R_1)+\tau(R_2) > 1$ follows in a similar way. This completes the proof of the "only if" part.

Proof of Part (ii): Part (ii) of the theorem follows from Theorem 5.1. ■

B. Proof of Theorem 4.2

Proof of Part (i): Let $n \geq 3$ and let $\mathscr{S}$ be a regular single-peaked domain satisfying Condition U for n agents. We show that a division rule $f : \mathscr{S}^n \to \Delta_n$ satisfies efficiency, strategy-proofness, and ETE if and only if it is the uniform rule.

("If" part) The "if" part of the theorem follows from Theorem 6.1 of this paper, where we show that the uniform rule satisfies efficiency, strategy-proof, ETE on a domain satisfying a condition called Condition N. It is straightforward to verify that Condition U for 2 agents implies Condition N.

("Only if" part) We first prove a lemma.

Lemma B.1. *(i) Let $R_N \in \mathscr{S}^n$ and $i \in N$ be such that $f_i(R_N) < \tau(R_i)$. Further let $R'_i \in \mathscr{S}$ be such that $f_i(R_N) \leq \tau(R'_i)$. Then, $f_i(R'_i,R_{-i}) \leq \frac{1}{2}$ implies $f_i(R_N) = f_i(R'_i,R_{-i})$.*

(ii) Let $R_N \in \mathscr{S}^n$ and $i \in N$ be such that $f_i(R_N) > \tau(R_i)$. Further let $R'_i \in \mathscr{S}$ be such that $f_i(R_N) \geq \tau(R'_i)$. Then, $f_i(R_N) = f_i(R'_i,R_{-i})$.

Proof: We show that (i) holds and the proof of (ii) follows using similar arguments. Since $f_i(R_N) < \tau(R_i)$, by efficiency $f_j(R_N) \leq \tau(R_j)$ for all $j \in N$, and hence, $T(R_N) > 1$. Similarly, as $f_i(R_N) \leq \tau(R_i')$ and $f_j(R_N) \leq \tau(R_j)$ for all $j \neq i$, we have $T(R_i', R_{-i}) \geq \sum_{j=1}^{n} f_j(R_N) = 1$. This means by efficiency, $f_i(R_i', R_{-i}) \leq \tau(R_i')$.

Assume for contradiction $f_i(R_i', R_{-i}) \leq \frac{1}{2}$ but $f_i(R_i', R_{-i}) \neq f_i(R_N)$. This together with strategy-proofness imply, $f_i(R_N) < \tau(R_i) < f_i(R_i', R_{-i}) \leq \tau(R_i')$. Let $x = \sup\{y \in [0,1] \mid f_i(R^y, R_{-i}) = f_i(R_N)\}$. This is well defined since by strategy-proofness and efficiency, $f_i(R^y, R_{-i}) = f_i(\bar{R}^y, R_{-i})$ for all $y \in [0, 1]$, and all $R^y, \bar{R}^y \in \mathscr{S}$. Note that since $f_i(R_N) < \tau(R_i) < f_i(R_i', R_{-i}) \leq \frac{1}{2}$, $f_i(R_N) < x < \frac{1}{2}$. Suppose $x \in \{y \in [0,1] \mid f_i(R^y, R_{-i}) = f_i(R_N)\}$. Let R be a locally continuous around the peak preference with $\tau(R) = x$. Such a preference exists as $\mathscr{S}$ is regular. This means there exists z such that $x < z$ and $zPf_i(R_N)$. Consider $R^z \in \mathscr{S}$. By efficiency, $f_i(R^z, R_{-i}) \in (x, z]$. This means agent i manipulates at (R, R_{-i}) via R^z, a contradiction, and hence, $x \notin \{y \in [0,1] \mid f_i(R^y, R_{-i}) = f_i(R_N)\}$. This together with efficiency imply $f_i(R, R_{-i}) = x$. Note that by strategy-proofness for all $y \in [f_i(R_N), x)$, $f_i(R^y, R_{-i}) = f_i(R_N)$. Since $\mathscr{S}$ satisfies Condition U for n agents and $x < \frac{1}{2}$, there exists a preference $\hat{R}$ with (i) $\tau(\hat{R}) \in (f_i(R_N), x)$ if either $(f_i(R_N), x) \subseteq (0, \frac{1}{n})$ or $(f_i(R_N), x) \subseteq (\frac{1}{n}, \frac{1}{2})$ such that $x\hat{P}f_i(R_N)$, or (ii) $\tau(\hat{R}) \in (\frac{1}{n}, x)$ if $f_i(R_N) < \frac{1}{n} < x$ such that $x\hat{P}\frac{1}{n}$. In both the cases agent i manipulates at R_N via R, a contradiction, and hence, $f_i(R_N) = f_i(R_i', R_{-i})$. ■

Let $R_N \in \mathscr{S}^n$. If $T(R_N) = 1$, then by efficiency, $\tau(R_i) = f_i(R_N) = g_i(R_N)$ for all $i \in N$. Suppose $T(R_N) > 1$. Note that by efficiency, $f_i(R_N) \leq \tau(R_i)$ and $g_i(R_N) \leq \tau(R_i)$ for all $i \in N$. WLG assume that $\tau(R_1) \leq \cdots \leq \tau(R_n)$. Assume for contradiction $f(R_N) \neq g(R_N)$. If $R_N = (R_n, \ldots, R_n)$, then by ETE $f_i(R_N) = g_i(R_N)$ for all $i \in N$, a contradiction. Therefore, assume $R_N \neq (R_n, \ldots, R_n)$. We complete the proof in the following steps.

Step 1: Since $f(R_N) \neq g(R_N)$, there exists $i \in N$ such that $f_i(R_N) < g_i(R_N) \leq \tau(R_i)$. Let $R_i' = R_n$. We show $f_i(R_i', R_{-i}) < g_i(R_i', R_{-i})$. If $i = n$, then there is nothing to show. Suppose $i \neq n$. First we show $f_i(R_i', R_{-i}) \leq \frac{1}{2}$. Since $R_i' = R_n$ at (R_i', R_{-i}), by ETE we have $f_i(R_i', R_{-i}) = f_n(R_i', R_{-i})$, which in turn implies that $f_i(R_i', R_{-i}) \leq \frac{1}{2}$. By Lemma B.1, this yields $f_i(R_N) = f_i(R_i', R_{-i})$. By the definition of the uniform rule, $\tau(R_i) \leq \tau(R_i')$ implies $g_i(R_N) \leq g_i(R_i', R_{-i})$. Combining all these observations, we get $f_i(R_i', R_{-i}) < g_i(R_i', R_{-i})$. If $(R_i', R_{-i}) = (R_n, \ldots, R_n)$, then by ETE we have a contradiction. Suppose $(R_i', R_{-i}) \neq (R_n, \ldots, R_n)$. We proceed to Step 2.

Step 2: Since $f_k(R_i', R_{-i}) < g_k(R_i', R_{-i})$ for all $k \in \{i, n\}$, there exists $j \notin \{i, n\}$ such that $g_j(R_i', R_{-i}) < f_j(R_i', R_{-i})$. By efficiency, $g_j(R_i', R_{-i}) < f_j(R_i', R_{-i}) \leq \tau(R_j)$. Let $R_j' = R_n$. Strategy-proofness of f implies $f_j(R_i', R_{-i}) \leq f_j(R_i', R_j', R_{-\{i,j\}})$. By the definition of the uniform rule, $g_j(R_i', R_{-i}) = g_j(R_i', R_j', R_{-\{i,j\}})$ since $g_j(R_i', R_{-i}) < \tau(R_j) < \tau(R_j')$. Combining all these observations, we get $g_j(R_i', R_j',$

$R_{-\{i,j\}}) < f_j(R'_i, R'_j, R_{-\{i,j\}})$. If $(R'_i, R'_j, R_{-\{i,j\}}) = (R_n, \ldots, R_n)$, then by ETE we have a contradiction, otherwise we apply Step 1 to $(R'_i, R'_j, R_{-\{i,j\}})$.

Since N is finite and at every step we change the preference of a new agent by R_n, eventually it will lead to a contradiction.

Proof of Part (ii): Part (ii) of the theorem follows from Theorem 5.2 and Theorem 5.3. ■

C. Proof of Theorem 5.1

Proof: Let $(x_1, y_1), \ldots, (x_{k_1}, y_{k_1})$ and $(w_1, z_1), \ldots, (w_{k_2}, z_{k_2})$ be a collection of intervals such that $0 < (x_1, y_1) < \cdots < (x_{k_1}, y_{k_1}) < \frac{1}{2} < (w_1, z_1) < \cdots < (w_{k_2}, z_{k_2}) < 1$. Suppose that $\mathscr{S}$ is a single-peaked domain that violates Condition U for 2 agents on these intervals and f is an adjusted uniform rule for 2 agents with respect to intervals $(x_1, y_1), \ldots, (x_{k_1}, y_{k_1})$ and $(w_1, z_1), \ldots, (w_{k_2}, z_{k_2})$. Let i_0 be the particular agent mentioned in Definition 5.2. It is left to the reader to check that f satisfies efficiency and ETE. We establish strategy-proofness of f by considering different cases with respect to R_N.

Suppose part (a) of condition (i) of Definition 5.2 holds, i.e., there exists $r \in \{1, \ldots, k_1\}$ such that $\tau(R_{i_0}) \in [x_r, y_r)$ and $\tau(R_{i_0}) + \tau(R_j) \geq x_r + \tau(R_j) \geq 1$. This implies $f_{i_0}(R_{i_0}, R_j) = x_r$ and $f_j(R_{i_0}, R_j) = 1 - x_r$. Consider $R'_j \in \mathscr{S}$. If $1 - x_r \leq \tau(R'_j)$, then by definition $f_j(R_{i_0}, R'_j) = 1 - x_r$, so agent j cannot manipulate. If $1 - \tau(R_{i_0}) \leq \tau(R'_j) < 1 - x_r$, then by the definition of adjusted uniform rule for 2 agents, $f_j(R_{i_0}, R'_j) = \tau(R'_j)$. Since $\tau(R'_j) < 1 - x_r$ and $\tau(R_j) \geq 1 - x_r = f_j(R_{i_0}, R_j)$, agent j cannot manipulate. Let $R'_{i_0} \in \mathscr{S}$. If $\tau(R'_{i_0}) \geq \tau(R_{i_0})$, then by the definition of adjusted uniform rule for 2 agents, $f_1(R'_{i_0}, R_j) \geq y_r$. As $\mathscr{S}$ violates Condition U for 2 agents, we have $x_r R_{i_0} y_r$. So agent i_0 cannot manipulate. If $x_r \leq \tau(R'_{i_0}) < \tau(R_{i_0})$ then $f_{i_0}(R'_{i_0}, R_j) = x_r$, so agent i_0 cannot manipulate. If $y_{r-1} \leq \tau(R'_{i_0}) < x_r$, then either $f_{i_0}(R'_{i_0}, R_j) = \tau(R'_{i_0})$ or $f_{i_0}(R'_{i_0}, R_j) = 1 - \tau(R_j)$. In both these cases, agent i_0 cannot manipulate since $\tau(R'_{i_0}) < x_r$ and $1 - \tau(R_j) \leq x_r$. If $\tau(R'_{i_0}) \in [x_{r-1}, y_r)$, then either $f_{i_0}(R'_{i_0}, R_j) = x_{r-1}$ or $f_{i_0}(R'_{i_0}, R_j) = 1 - \tau(R_j)$. Again, in both these cases, agent i_0 cannot manipulate since $x_{r-1} < x_r$ and $1 - \tau(R_j) \leq x_r$. Using similar arguments, it follows that agent i_0 cannot manipulate if $\tau(R'_{i_0}) \leq x_{r-1}$.

Suppose part (b) of condition (i) of Definition 5.2 holds, i.e., there exists $r \in \{1, \ldots, k_1\}$ with $\tau(R_{i_0}) \in [x_r, y_r)$ and $x_r + \tau(R_j) < 1 \leq \tau(R_{i_0}) + \tau(R_j)$. By the definition of adjusted uniform rule for 2 agents, $f_{i_0}(R_{i_0}, R_j) = 1 - \tau(R_j) > x_r$ and $f_j(R_{i_0}, R_j) = \tau(R_j)$. Agent j cannot manipulate as his share is equal to his peak $\tau(R_j)$. Consider $R'_{i_0} \in \mathscr{S}$. If $x_r \leq \tau(R'_{i_0}) < y_r$, then by the definition of adjusted uniform rule for 2 agents, $f_{i_0}(R_{i_0}, R_j) = 1 - \tau(R_j)$ and agent i_0 cannot manipulate. If $y_r \leq \tau(R'_{i_0}) < x_{r+1}$, then by the definition of adjusted uniform rule for 2 agents, $f_{i_0}(R'_{i_0}, R_j) = \tau(R'_{i_0})$, which implies $f_{i_0}(R'_{i_0}, R_j)$. As $\mathscr{S}$ violates Condition U for 2 agents, we have $x_r R_{i_0} y_r$ and $f_{i_0}(R_{i_0}, R_j) P_{i_0} x_r$. So agent i_0 cannot manipulate. If $x_{r+1} \leq \tau(R'_{i_0}) < y_{r+1}$, then $f_{i_0}(R'_{i_0}, R_j) = x_{r+1} > y_r$, and by the same argument as presented above, agent i_0 cannot manipulate. Using similar arguments, it follows that agent i_0 cannot manipulate if $\tau(R'_{i_0}) \geq y_{r+1}$.

If $\tau(R'_{i_0}) < x_r$, then by the definition of adjusted uniform rule for 2 agents, $f_{i_0}(R_{i_0}, R_j) = 1 - \tau(R_j)$, and hence agent i_0 cannot manipulate.

Using similar arguments as for condition (i), it follows that no agent can manipulate f if condition (ii) of Definition 5.2 holds.

Suppose condition (iii) of Definition 5.2 holds. Then, $f(R_{i_0}, R_j) = u(R_{i_0}, R_j)$. Note that agent j cannot manipulate as no matter how he misreports his preferences, f will continue to follow the uniform rule, and the uniform rule is not manipulable. We proceed to show that agent i_0 cannot manipulate. Consider $R'_{i_0} \in \mathscr{S}$. Note that agent i_0 can manipulate only if (R'_{i_0}, R_j) satisfies either condition (i) or condition (ii). We distinguish the following cases.

Case 1: Suppose either $\tau(R_{i_0}), \tau(R_j) \geq \frac{1}{2}$ or $\tau(R_{i_0}), \tau(R_j) \leq \frac{1}{2}$.

By the definition of adjusted uniform rule for 2 agents, we have $f_k(R_{i_0}, R_j) = \frac{1}{2}$ for all $k \in \{i_0, j\}$. Let R'_{i_0} be such that (R'_{i_0}, R_j) satisfies condition (i), i.e., $\tau(R'_{i_0}) \in [x_r, y_r)$ for some $r \in \{1, \ldots, k_1\}$ and $(R'_{i_0}, R_j) \in \mathscr{S}^2_+$. Then, either $f_{i_0}(R'_{i_0}, R_j) = x_r$ or $f_{i_0}(R'_{i_0}, R_j) = 1 - \tau(R_j)$, which implies $f_{i_0}(R'_{i_0}, R_j) \leq \frac{1}{2}$. So, agent i_0 cannot manipulate.

Case 2: Suppose $\tau(R_{i_0}) < \frac{1}{2}$ and $\tau(R_j) \geq \frac{1}{2}$.

By condition (iii), $\tau(R_{i_0}) \in [y_r, x_{r+1})$ for some $r \in \{1, \ldots, k_1\}$, and either $f_{i_0}(R_{i_0}, R_j) = \tau(R_{i_0})$ or $f_{i_0}(R_{i_0}, R_j) = 1 - \tau(R_j)$. If $f_{i_0}(R_{i_0}, R_j) = \tau(R_{i_0})$, then agent i_0 cannot manipulate. So, assume $f_{i_0}(R_{i_0}, R_j) = 1 - \tau(R_j)$, which implies $\tau(R_{i_0}) + \tau(R_j) < 1$. Let R'_{i_0} be such that (R'_{i_0}, R_j) satisfies condition (i), i.e., $\tau(R'_{i_0}) \in [x_r, y_r)$ for some $r \in \{1, \ldots, k_1\}$ and $(R'_{i_0}, R_j) \in \mathscr{S}^2_+$. If $x_r + \tau(R_j) \geq 1$, then $f_{i_0}(R'_{i_0}, R_j) = x_r$, and if $x_r + \tau(R_j) < 1$, then $f_{i_0}(R'_{i_0}, R_j) = 1 - \tau(R_j)$. If $f_{i_0}(R'_{i_0}, R_j) = 1 - \tau(R_j)$, then agent i_0 cannot manipulate as $f_{i_0}(R_{i_0}, R_j) = 1 - \tau(R_j)$. On the other hand, if $x_r + \tau(R_j) \geq 1$, then $\tau(R_{i_0}) < 1 - \tau(R_j) \leq x_r$, and hence agent i_0 cannot manipulate.

Case 3: Suppose $\tau(R_{i_0}) > \frac{1}{2}$ and $\tau(R_j) \leq \frac{1}{2}$.

By condition (iii), $\tau(R_{i_0}) \in [z_l, w_{l+1})$ for some $l \in \{1, \ldots, k_2\}$, and either $f_{i_0}(R_{i_0}, R_j) = \tau(R_{i_0})$ or $f_{i_0}(R_{i_0}, R_j) = 1 - \tau(R_j)$. If $f_{i_0}(R_{i_0}, R_j) = \tau(R_{i_0})$, then agent i_0 cannot manipulate. So, assume $f_{i_0}(R_{i_0}, R_j) = 1 - \tau(R_j)$, which implies $\tau(R_{i_0}) + \tau(R_j) > 1$. Let R'_{i_0} be such that (R'_{i_0}, R_j) satisfies condition (ii), i.e., $\tau(R'_{i_0}) \in [w_s, z_s)$ for some $s \in \{1, \ldots, k_2\}$ and $(R'_{i_0}, R_j) \in \mathscr{S}^2_-$. If $z_s + \tau(R_j) \leq 1$, then $f_{i_0}(R'_{i_0}, R_j) = z_s$, and if $z_s + \tau(R_j) > 1$, then $f_{i_0}(R'_{i_0}, R_j) = 1 - \tau(R_j)$. If $f_{i_0}(R'_{i_0}, R_j) = 1 - \tau(R_j)$, then agent i_0 cannot manipulate as $f_{i_0}(R_{i_0}, R_j) = 1 - \tau(R_j)$. On the other hand, if $z_s + \tau(R_j) \leq 1$, then $\tau(R_{i_0}) > 1 - \tau(R_j) \geq z_s$, and hence agent i_0 cannot manipulate.

Since Cases 1,2, and 3 are exhaustive, it follows that agent i_0 cannot manipulate f if condition (iii) holds. ■

D. Proof of the Theorem 5.2

Proof: Let $(x_1,y_1),\dots,(x_{k_1},y_{k_1})$ and $(w_1,z_1),\dots,(w_{k_2},z_{k_2})$ be a collection of intervals such that $0<(x_1, y_1)<\cdots<(x_{k_1},y_{k_1})<\frac{1}{n}<(w_1,z_1)<\cdots<(w_{k_2},z_{k_2})<1$. Suppose $\mathscr{S}$ is a single-peaked domain that violates (i) and (ii) of Condition U for n agents on these intervals and f is an adjusted uniform rule for n agents with respect to intervals $(x_1,y_1),\dots,(x_{k_1},y_{k_1})$ and $(w_1,z_1),\dots,(w_{k_2},z_{k_2})$. Let $i_0\in N$ be the particular agent mentioned in Definition 5.5. We first show that f satisfies efficiency and ETE. Since the uniform rule satisfies efficiency and ETE, we need to show efficiency and ETE only for the profiles where f differs from the uniform rule.

Consider $R_N\in\mathscr{S}^n$ such that part (a) of condition (i) of Definition 5.5 holds. This means $R_N\in\mathscr{S}^n_+$ and for some $r\in\{1,\dots,k_1\}$, $\tau(R_{i_0})\in[x_r,y_r)$ and $\tau(R_j)\geq y_r$ for all $j\neq i_0$, and $x_r+T(R_{N\setminus i_0})\geq 1$. By the definition of adjusted uniform rule for n agents, $f_{i_0}(R_N)=x_r$ and $f_j(R_N)=u_j^{1-x_r}(R_{N\setminus i_0})$ for all $j\neq i_0$. Since $T(R_{N\setminus i_0})\geq 1-x_r$, by the definition of the uniform rule, $u_j^{1-x_r}(R_{N\setminus i_0})\leq\tau(R_j)$ for all $j\neq i_0$. Moreover, as $\tau(R_{i_0})\in[x_r,y_r)$ and $f_{i_0}(R_N)=x_r$, we have $f_{i_0}(R_N)\leq\tau(R_{i_0})$. Combining these two observations, we obtain $f_j(R_N)\leq\tau(R_j)$ for all $j\in N$ which implies f is efficient at R_N. Suppose $R_k=R_l$ for some $k,l\in N$. As $\tau(R_{i_0})\in[x_r,y_r)$ and $\tau(R_j)\geq y_r$ for all $j\neq i_0$, we have $k,l\in N\setminus i_0$. By the definition of adjusted uniform rule for n agents, this implies $f_k(R_N)=u_k^{1-x_r}(R_{N\setminus i_0})$ and $f_l(R_N)=u_l^{1-x_r}(R_{N\setminus i_0})$. Since the uniform rule satisfies ETE, $R_k=R_l$ implies $u_k^{1-x_r}(R_{N\setminus i_0})=u_l^{1-x_r}(R_{N\setminus i_0})$, and hence, $f_k(R_N)=f_l(R_N)$. Therefore, f satisfies ETE at R_N.

Consider $R_N\in\mathscr{S}^n$ such that part (b) of condition (i) of Definition 5.5 holds, i.e., $R_N\in\mathscr{S}^n_+$, and for some $r\in\{1,\dots,k_1\}$, $\tau(R_{i_0})\in[x_r,y_r)$, $\tau(R_j)\geq y_r$ for all $j\neq i_0$, and $x_r+T(R_{N\setminus i_0})<1$. By the definition of the adjusted uniform rule for n agents, this means $f_j(R_N)=\tau(R_j)$ for all $j\neq i_0$. Since $R_N\in\mathscr{S}^n_+$, we have $\tau(R_{i_0})\geq 1-T(R_{N\setminus i_0})=f_{i_0}(R_N)$. This shows f is efficient at R_N. Suppose $R_k=R_l$ for some $k,l\in N$. Note that $k,l\neq i_0$ as by the definition of adjusted uniform rule for n agents, $\tau(R_{i_0})\in[x_r,y_r)$, $\tau(R_j)\geq y_r$ for all $j\neq i_0$. As $f_j(R_N)=\tau(R_j)$ for all $j\neq i_0$, this implies $f_k(R_N)=f_l(R_N)$. Therefore, f satisfies ETE at R_N.

Using similar arguments, it follows that f satisfies ETE for the profiles satisfying condition (ii) of Definition 5.5.

Now, we show that f is strategy-proof. We show this by considering different cases with respect to R_N.

Case 1: Suppose part (a) of condition (i) of Definition 5.5 holds, i.e., $R_N\in\mathscr{S}^n_+$, and for some $r\in\{1, \dots,k_1\}$, $\tau(R_{i_0})\in[x_r,y_r)$, $\tau(R_j)\geq y_r$ for all $j\neq i_0$, and $x_r+T(R_{N\setminus i})\geq 1$. We show that no agent can manipulate at this profile.

First consider agent i_0. Let $R'_{i_0}\in\mathscr{S}$. By the definition of adjusted uniform rules for n agents, if $\tau(R'_{i_0})\in[x_r,y_r)$, then $f(R'_{i_0},R_{-i_0})=f(R_N)=x_r$. Suppose $\tau(R'_{i_0})\in[y_r,x_{r+1})$. Again, by the definition of the mentioned rules, we have $f_{i_0}(R'_{i_0}, R_{-i_0}) = u_{i_0}(R'_{i_0}, R_{-i_0}) = \tau(R'_{i_0})$. Since $T(R'_{i_0}, R_{-i_0}) \geq T(R_N) \geq 1$ and $\tau(R'_{i_0}) \in [y_r, x_{r+1})$ where $[y_r, x_{r+1}) < \frac{1}{n}$, by the definition of the uniform rule, we have $u_{i_0}(R'_{i_0}, R_{-i_0}) = \tau(R'_{i_0}) \geq y_r$. Because $\mathscr{S}$ violates part (i) of Condition U for n agents, we have $x_r R_{i_0} y_r$. Since $\tau(R_{i_0}) \leq y_r \leq \tau(R'_{i_0})$, by single-peakedness this means $x_r R_{i_0} \tau(R'_{i_0})$. Therefore, agent i_0 cannot manipulate. Suppose $\tau(R'_{i_0}) \in [x_{r+1}, y_{r+1})$. By the definition of adjusted uniform rule for n agents, this means $f_{i_0}(R'_{i_0}, R_{-i_0})$ is either x_{r+1} or $\tau(R'_{i_0})$. In any case, $f_{i_0}(R'_{i_0}, R_{-i_0}) \geq x_{r+1}$. Since $x_{r+1} \geq y_r$, by using the arguments in the preceding paragraph, agent i_0 cannot manipulate. The fact that agent i_0 cannot manipulate when $\tau(R'_{i_0}) \geq y_r$ follows by using similar arguments. Now, suppose $\tau(R'_{i_0}) < x_r$. If $T(R'_{i_0}, R_{-i_0}) > 1$, by efficiency of f, $f_{i_0}(R'_{i_0}, R_{-i_0}) \leq \tau(R'_{i_0}) < x_r$. Since $\tau(R'_{i_0}) < x_r \leq \tau(R_{i_0})$ and $f_{i_0}(R_N) = x_r$, by single-peakedness, agent i_0 cannot manipulate. If $T(R'_{i_0}, R_{-i_0}) \leq 1$, by the definition of f, $f(R'_{i_0}, R_{-i_0}) = u(R'_{i_0}, R_{-i_0})$. By the definition of the uniform rule, this means $f_{i_0}(R'_{i_0}, R_{-i_0}) \leq 1 - T(R_{N\setminus i_0})$. Since $x_r \geq 1 - T(R_{N\setminus i_0})$, this implies $f_{i_0}(R'_{i_0}, R_{-i_0}) \leq x_r$. Using similar arguments as before we can show that agent i_0 cannot manipulate.

Now, consider an agent $j \neq i_0$. Since $f_j(R_N) = u_j^{1-x_r}(R_{N\setminus i_0})$ and $T(R_{N\setminus i_0}) \geq 1 - x_r$, by the definition of the uniform rule, either $f_j(R_N) = \tau(R_j)$ or $f_j(R_N) < \tau(R_j)$ and $f_j(R_N) \geq \frac{1-x_r}{n-1}$. If $f_j(R_N) = \tau(R_j)$, then agent j cannot manipulate. So, assume $\tau(R_j) \geq f_j(R_N) \geq \frac{1-x_r}{n-1}$. Since $x_r < \frac{1}{n}$, this implies $\tau(R_j) \geq f_j(R_N) \geq \frac{1}{n}$. Consider $R'_j \in \mathscr{S}$. If $\tau(R'_j) \geq y_r$, then by the definition of f, $f_j(R'_j, R_{-j}) = f_j(R_N)$. So, agent j cannot manipulate. If $\tau(R'_j) < y_r$, then by the definition of f, $f_j(R'_j, R_{-j}) = u_j(R'_j, R_{-j})$. Since $\tau(R'_j) < y_r \leq \frac{1}{n}$, by the definition of the uniform rule, $f_j(R'_j, R_{-j}) \leq \frac{1}{n}$, which means agent j cannot manipulate.

Case 2: Suppose part (b) of condition (i) of Definition 5.5 holds, i.e., $R_N \in \mathscr{S}^n_+$, and for some $r \in \{1, \dots, k_1\}$, $\tau(R_{i_0}) \in [x_r, y_r)$, $\tau(R_j) \geq y_r$ for all $j \neq i_0$, and $x_r + T(R_{N\setminus i_0}) < 1$.

By the definition of f, $f_j(R_N) = \tau(R_j)$ for all $j \neq i_0$. Since $f_j(R_N) = \tau(R_j)$ for all $j \neq i_0$, agents other than i_0 cannot manipulate. By the definition of f, $f_{i_0}(R_N) = 1 - T(R_{N\setminus i_0}) \geq x_r$. Consider $R'_{i_0} \in \mathscr{S}$. If $\tau(R'_{i_0}) \in [x_r, y_r)$, then by the definition of f, $f_{i_0}(R'_{i_0}, R_{-i_0}) = f_{i_0}(R_N)$ and agent i_0 cannot manipulate. If $\tau(R'_{i_0}) \geq y_r$, then by following our previous arguments it can be shown that $f_{i_0}(R'_{i_0}, R_{-i_0}) \geq y_r$. Because $\mathscr{S}$ violates condition (i) of Definition 4.2, we have $x_r R_{i_0} y_r$. Since $x_r \leq 1 - T(R_{N\setminus i_0}) \leq \tau(R_{i_0}) \leq y_r$, by single-peakedness, $[1 - T(R_{N\setminus i_0})] R_{i_0} y_r$. Therefore, agent i_0 cannot manipulate. If $\tau(R'_{i_0}) < x_r$, then using similar arguments as in the previous paragraph, we can show that agent i_0 can not manipulate.

The fact that no agent can manipulate f when parts (a) and (b) of Condition (ii) of Definition 5.5 hold follows by using similar arguments. This completes the proof of the theorem. ■

E. PROOF OF THEOREM 5.3

Proof: Let (x,y) and (w,z) be two intervals such that $0 < (x,y) < \frac{1}{n} < (w,z) < \frac{1}{2}$ and let $\mathscr{S}$ be a single-peaked domain that violates (iii) and (iv) of Condition U for n agents on these intervals. Consider an adjusted* uniform rule for n agents with respect to (x,y) and (w,z). Let i_0 and j_0 be the two particular agents as mentioned in Definition 5.6. We first show that f satisfies efficiency and ETE. As the uniform rule satisfies efficiency and ETE, by Remark 5.1, it is sufficient to check efficiency and ETE for the profiles where either condition (i) or condition (ii) of Definition 5.6 is satisfied. Let $R_N \in \mathscr{S}^n$ be such that part (a) of condition (i) of Definition 5.6 is satisfied, i.e., $R_N \in \mathscr{S}^n_-$ with $\tau(R_k) = x_0$ for all $k \in N \setminus \{i_0, j_0\}$, $\tau(R_{i_0}) \in (x,y)$, and $\tau(R_{j_0}) \in (x_1,y_1)$. By the definition of f, $f_l(R_N) = \tau(R_l)$ for all $l \neq i_0$ and $f_{i_0}(R_N) = 1 - T(R_{N\setminus i_0})$. Since $R_N \in \mathscr{S}^n_-$, $\tau(R_{i_0}) \leq 1 - T(R_{N\setminus i_0})$. This means $\tau(R_l) \leq f_l(R_N)$ for all $l \in N$ which implies f is efficient at R_N. Suppose $R_k = R_l$ for some $k,l \in N$. Since $(x_1,y_1) < (x,y) < x_0$, it must be that $k,l \in N \setminus \{i_0, j_0\}$. By the definition of f, $f_s(R_N) = x_0$ for all $s \in N \setminus \{i_0, j_0\}$, which implies $f_k(R_N) = f_l(R_N)$, and hence f satisfies ETE at R_N. The same arguments hold for the profiles satisfying part (b) of condition (i) and condition (ii) in Definition 5.6.

To show strategy-proofness f, we distinguish the following cases with respect to R_N.

Case 1: Suppose part (a) of condition (i) of Definition 5.6 holds, i.e., $R_N \in \mathscr{S}^n_-$ with $\tau(R_k) = x_0$ for all $k \in N \setminus \{i_0, j_0\}$, $\tau(R_{i_0}) \in (x,y)$, and $\tau(R_{j_0}) \in (x_1,y_1)$.

By the definition of adjusted* uniform rule for n agents, all agents except agent i_0 get shares equal to their peaks, so they cannot manipulate. Consider the agent i_0. He gets $f_{i_0}(R_N) = 1-(n-2)x_0 - \tau(R_{j_0})$. Note that as $R_N \in \mathscr{S}^n_-$, by efficiency of f, $f_{i_0}(R_N) = 1-(n-2)x_0 - \tau(R_{j_0}) \geq \tau(R_{i_0})$. Again, as $(n-2)x_0 + y + x_1 = 1$, we have $f_{i_0}(R_N) = 1-(n-2)x_0 - \tau(R_{j_0}) \leq 1-(n-2)x_0 - x_1 = y$. So, $\tau(R_{i_0}) \leq f_{i_0}(R_N) \leq y$. Consider $R'_{i_0} \in \mathscr{S}$. If $\tau(R'_{i_0}) \in (x,y)$ and $(R'_{i_0}, R_{-i_0}) \in \mathscr{S}^n_-$, then by the definition of adjusted* uniform rule for n agents, we have $f_{i_0}(R_N) = f_{i_0}(R'_{i_0}, R_{-i_0})$, and hence agent i_0 cannot manipulate. If $\tau(R'_{i_0}) \in (x,y)$ and $(R'_{i_0}, R_{-i_0}) \in \mathscr{S}^n_+$, then by the definition of adjusted* uniform rule for n agents, $f_{i_0}(R'_{i_0}, R_{-i_0}) = \tau(R'_{i_0}) \geq 1-(n-2)x_0 - \tau(R_{j_0}) = f_{i_0}(R_N)$. Since $f_{i_0}(R_N) \geq \tau(R_{i_0})$, by single-peakedness, agent i_0 cannot manipulate. If $\tau(R'_{i_0}) \geq y$, then $(R'_{i_0}, R_{-i_0}) \in \mathscr{S}^n_+$, and by the definition of f, $f_{i_0}(R'_{i_0}, R_{-i_0}) \geq y$. As $y \geq f_{i_0}(R_N) \geq \tau(R_{i_0})$, by single-peakedness, agent i_0 cannot manipulate. If $\tau(R'_{i_0}) \leq x$, then by the definition of f, $f(R'_{i_0}, R_{-i_0}) = u(R'_{i_0}, R_{-i_0})$. Since $(R'_{i_0}, R_{-i_0}) \in \mathscr{S}^n_-$, this means by the definition of the uniform rule, $f_k(R'_{i_0}, R_{-i_0}) = x_0$ for all $k \in N \setminus \{i_0, j_0\}$. Moreover, as $(n-2)x_0 + x + y_1 = 1$ and $y_1 < x$, we have by the definition of uniform rule that $f_{i_0}(R'_{i_0}, R_{-i_0}) \leq x$. Since $\mathscr{S}$ violates condition (iii) Condition U for n agents on (x,y), we have $yR_{i_0}x$. As $\tau(R_{i_0}) \leq f_{i_0}(R_N) \leq y$, combining all these observations together, we get $f_{i_0}(R_N) R_{i_0} f_{i_0}(R'_{i_0}, R_{-i_0})$. This implies agent i_0 cannot manipulate.

Case 2: Suppose part (b) of condition (i) of 5.6 holds, i.e., $R_N \in \mathscr{S}^n_-$ with $\tau(R_k) = x_0$ for all $k \in N \setminus \{i_0, j_0\}$, $\tau(R_{i_0}) \in (x,y)$, and $\tau(R_{j_0}) \leq x_1$.

By the definition of adjusted* uniform rule for n agents, all agents except agent i_0 and j_0 get shares equal to their peaks, so they cannot manipulate. Using similar arguments as in Case 1, we can show that agent i_0 cannot manipulate. Consider $R'_{j_0} \in \mathscr{S}$. If $\tau(R'_{j_0}) \leq x_1$, then by the definition of adjusted* uniform rule for n agents, $f_{j_0}(R_N) = f_{j_0}(R'_{j_0}, R_{-j_0})$ and agent j_0 cannot manipulate. If $\tau(R'_{j_0}) \in (x_1, y_1)$, then by the definition of adjusted* uniform rule for n agents, $f_{j_0}(R'_{j_0}, R_{-j_0}) = \tau(R'_{j_0})$, and as $\tau(R_{j_0}) \leq x_1 < \tau(R'_{j_0})$, agent j_0 cannot manipulate. If $y_1 \leq \tau(R_{j_0}) \leq \frac{1}{n}$, then $(R'_{j_0}, R_{-j_0}) \in \mathscr{S}^n_+$, and by the definition of adjusted* uniform rule for n agents, $f_{j_0}(R'_{j_0}, R_{-j_0}) = \tau(R'_{j_0})$, which implies agent j_0 cannot manipulate. If $\tau(R_{j_0}) \geq \frac{1}{n}$, then by the definition of adjusted* uniform rule for n agents , $f_{j_0}(R'_{j_0}, R_{-j_0}) \geq \frac{1}{n}$, and hence agent j_0 cannot manipulate.

Case 3: Suppose condition (ii) of Definition 5.6 holds. The fact that f is strategy-proof in this case follows by using similar arguments as in Case 1 and Case 2.

Case 4: Suppose condition (iii) of Definition 5.6 holds, i.e., $f(R_N) = u(R_N)$.

Since the uniform rule is strategy-proof, an agent k can manipulate at R_N via R'_k if (R'_k, R_{-k}) satisfies either condition (i) or condition (ii) of Defintion 5.6. By Remark 5.1, it is sufficient to check strategy-proofness for agents i_0 and j_0. Let $R_N \in \mathscr{S}^n$ be such that $\tau(R_k) = x_0$ for all $k \in N \setminus \{i_0, j_0\}$, $\tau(R_{i_0}) \in (x, y)$, and $\tau(R_{j_0}) \geq y_1$. Since $(n-2)x_0 + x + y_1 = 1$, this means $R_N \in \mathscr{S}^n_+$. If $\tau(R_{j_0}) \leq \frac{1}{n}$, then $f_{j_0}(R_N) = \tau(R_{j_0})$. So, agent j_0 cannot manipulate. If $\tau(R_{j_0}) > \frac{1}{n}$, then $f_{j_0}(R_N) \geq \frac{1}{n}$. Since $\tau(R'_{j_0}) \leq y_1$, we have $f_{j_0}(R'_{j_0}, R_{-j_0}) < y_1$, and hence, agent j_0 cannot manipulate. Let $R_N \in \mathscr{S}^n_-$ be such that $\tau(R_k) = x_0$ for all $k \in N \setminus \{i_0, j_0\}$ and $\tau(R_j) < y_1$. Since condition (iii) of Definition 5.6 holds, $\tau(R_{i_0}) \notin (x, y)$. If $\tau(R_{i_0}) \geq y$ and $R_N \in \mathscr{S}^n_-$, then by the definition of adjusted* uniform rule for n agents, $f_k(R_N) = x_0$ for all $k \in N \setminus \{i_0, j_0\}$. As $(n-2)x_0 + x_1 + y = 1$ and $x_1 < y$, we have $f_{i_0}(R_N) = \tau(R_{i_0})$, and hence agent i_0 cannot manipulate. Suppose $\tau(R_{i_0}) \geq y$ and $R_N \in \mathscr{S}^n_+$. If $\tau(R_{i_0}) \leq \frac{1}{n}$, then we have $f_{i_0}(R_N) = \tau(R_{i_0})$, on the other hand, if $\tau(R_{i_0}) > \frac{1}{n}$, then we have $f_{i_0}(R_N) \geq \frac{1}{n}$. In both the cases, agent i_0 cannot manipulate since $\tau(R'_{i_0}) \in (x, y)$ implies $f_{i_0}(R'_{i_0}, R_{-i_0}) \in (x, y)$. If $\tau(R_{i_0}) = x$, then $R_N \in \mathscr{S}^n_-$ and $f_k(R_N) = x_0$ for all $k \in N \setminus \{i_0, j_0\}$. Since $(n-2)x_0 + y_1 + x = 1$ and $\tau(R_{j_0}) < y_1$, by the definition of adjusted* uniform rule for n agents, $f_{i_0}(R_N) = x$ and agent i_0 cannot manipulate. If $\tau(R_{i_0}) < x$, then by monotonicity of the uniform rule, we know $f_{i_0}(R_N) < x$, and hence agent i_0 cannot manipulate as $\tau(R'_{i_0}) \in (x, y)$ implies $f_{i_0}(R'_{i_0}, R_{-i_0}) \in (x, y)$. Using similar arguments, it follows that each of the agents i_0 and j_0 cannot manipulate via some R'_l such that (R'_l, R_{-l}) satisfies condition (ii) of Definition 5.6 for all $l \in \{i_0, j_0\}$. ■

F. Proof of Theorem 6.1

Proof: ("If" part) Let $\mathscr{D}$ be a domain satisfying Condition N and let $u : \mathscr{D}^n \to \Delta_n$ be the uniform rule. Note that u satisfies ETE by definition. We show that u satisfies efficiency and strategy proofness.

Efficiency: Suppose u is not efficient and there exists $R_N \in \mathscr{D}^n$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in \Delta_n$ such that

$x_i R_i u_i(R_N)$ for all $i \in N$ and $x_i P_i f_i(R_N)$ for some $i \in N$. Without loss of generality, assume $T(R_N) > 1$. By the definition of the uniform rule, all agents i with $\tau(R_i) \leq \frac{1}{n}$ get shares equal to their peaks, and each agent i with $\tau(R_i) > \frac{1}{n}$ gets a share between $\frac{1}{n}$ and $\tau(R_i)$. Since there exists $i \in N$ such that $x_i P_i u_i(R_N)$, this implies $\frac{1}{n} < \tau(R_i)$. Moreover, as $\mathscr{D}$ satisfies Condition N, it must be that $u_i(R_N) < x_i \leq \tau(R_i)$. This implies there exists $j \in N$ such that $x_j < u_j(R_N)$. If $\tau(R_j) \leq \frac{1}{n}$, then by the definition of the uniform rule, $u_j(R_N) = \tau(R_j)$, and hence, $u_j(R_N) P_j x_j$, which is a contradiction. So $\tau(R_j) > \frac{1}{n}$. By the definition of the uniform rule, this means $\frac{1}{n} \leq u_j(R_N) \leq \tau(R_j)$ and $x_j < u_j(R_N)$. However, since $\mathscr{D}$ satisfies Condition N, this is a contradiction to $x_j R_j u_j(R_N)$. This proves that u satisfies efficiency.

Strategy-proofness: Let $R_N \in \mathscr{D}^n$ be such that $T(R_N) > 1$. We show that no agent can manipulate at R_N. By the definition of the uniform rule, for an agent i with $\tau(R_i) \leq \frac{1}{n}$, we have $u_i(R_N) = \tau(R_i)$. So, such an agent i cannot manipulate. Consider an agent i with $\tau(R_i) > \frac{1}{n}$. By the definition of the uniform rule, $\frac{1}{n} \leq u_i(R_N) \leq \tau(R_i)$. If $u_i(R_N) = \tau(R_i)$, then agent i cannot manipulate. So, assume $u_i(R_N) < \tau(R_i)$. Consider $R_i' \in \mathscr{D}$. If $\tau(R_i') \in [u_i(R_N), 1]$, then by the definition of the uniform rule, we have $u_i(R_N) = u_i(R_i', R_{-i})$, and hence, agent i cannot manipulate. Suppose $\tau(R_i') < u_i(R_N)$. By the monotonicity of the uniform rule, this means $u_i(R_i', R_{-i}) < u_i(R_N)$. Since $\frac{1}{n} \leq u_i(R_N) < \tau(R_i)$ and $\mathscr{D}$ satisfies Condition N, we have $u_i(R_N) P_i x$ for all $x < u_i(R_N)$. This means agent i cannot manipulate at R_N via R_i'. This proves that u is strategy-proof.

("Only if" part) Let $\mathscr{D}$ be a regular domain such that the uniform rule $u : \mathscr{D}^n \to \Delta_n$ satisfies efficiency, strategy-proofness, and ETE. We prove that $\mathscr{D}$ satisfies Condition N. We show that (i) of Condition N holds; the fact that (ii) of Condition N also holds follows from similar arguments.

Assume for contradiction that there exists $R \in \mathscr{D}$ with $\tau(R) > \frac{1}{n}$ such that R does not satisfy condition (i) of Condition N. We distinguish the following cases.

Case 1: For some x, y with $x, y \in [\frac{1}{n}, \frac{1}{2}]$ and $x < y \leq \tau(R)$, we have xRy.
Let $\bar{R} \in \mathscr{D}$ be a single-peaked preference with 1 as the peak. Consider $R_N \in \mathscr{D}^n$ where $R_1 = R$, $R_2 = \bar{R}$, $\tau(R_3) = 1 - \frac{(n-3)}{n} - 2y$, and $\tau(R_i) = \frac{1}{n}$ for all $i > 3$. Since $1 - \frac{(n-3)}{n} - 2y \leq \frac{1}{n}$, by the definition of the uniform rule, $f_1(R_N) = f_2(R_N) = y$, $f_3(R_N) = 1 - \frac{(n-3)}{n} - 2y$, and $f_i(R_N) = \frac{1}{n}$ for all $i > 3$. Let $\boldsymbol{q} = (q_1, \ldots, q_n) \in \Delta_n$ be such that $q_1 = x$, $q_2 = 2y - x$, and $q_i = f_i(R_N)$ for all $i \geq 3$. Since xRy and $\bar{R}$ is single-peaked with 1 as the peak, we have $q_1 R_1 f_1(R_N)$, $q_2 P_2 f_2(R_N)$, and $q_i I_i f_i(R_N)$ for all $i \geq 3$. However, this is a contradiction to efficiency.

Case 2: For some x, y with $x, y \in [\frac{1}{2}, 1]$ and $x < y \leq \tau(R)$, we have xPy.
Consider $R_N \in \mathscr{D}^n$ where $R_1 = R$, $\tau(R_2) = 1 - y$, and $\tau(R_i) = 0$ for all $i > 2$. Since $T(R_N) = \tau(R_1) + 1 - y \geq 1$, by the definition of the uniform rule, $f_1(R_N) = y$, $f_2(R_N) = 1 - y$, and $f_i(R_N) = 0$ for all $i > 2$.

Let $R'_1 \in \mathscr{D}$ be such that $\tau(R'_1) = x$. Since $T(R'_1, R_{-i}) = 1 - y + x < 1$, by the definition of the uniform rule, $f_1(R'_1, R_{-1}) = x$. However, this means agent 1 manipulates at R_N via R'_1, a contradiction.

Case 3: For some $x \in [0, \frac{1}{n}]$, we have $xP\frac{1}{n}$.

Consider $R_N \in \mathscr{D}^n$ such that $R_1 = R$ and $\tau(R_i) = \frac{1}{n}$ for all $i > 1$. Since $T(R_N) > 1$, by the definition of the uniform rule, $f_i(R_N) = \frac{1}{n}$ for all $i \in N$. Let $R'_1 \in \mathscr{D}$ be such that $\tau(R'_1) = x$. Since $T(R'_1, R_{-1}) < 1$, by the definition of uniform rule, $f_1(R'_1, R_{-1}) = x$. However, this means agent 1 manipulates at R_N via R'_1, a contradiction. This completes the proof of the "only if" part of the theorem. ■

G. Proof of Theorem 6.2

Proof: Let $\mathscr{D}$ be a minimally rich single-peaked domain satisfying Condition N. We show that a division rule $f : \mathscr{D}^n \to \Delta_n$ satisfies efficiency, strategy-proofness, and ETE if and only if it is the uniform rule.

("If" part) The "if" part of the theorem follows from the "if" part of Theorem 6.1.

("Only if" part) Let us denote the set of all single-peaked preferences in $\mathscr{D}$ by $\mathscr{S}_0$. Since $\mathscr{D}$ is minimally rich, for every $x, y, z \in [0,1]$ with $x < y < z$ we have $R, R' \in \mathscr{S}_0$ such that (i) $\tau(R) = \tau(R') = y$ and (ii) xPz and $zP'x$. We show that $f(R_N) = u(R_N)$ for all $R_N \in \mathscr{D}^n$ where u is the uniform rule. Let $R_N \in \mathscr{D}^n$ and let us denote by $N(R_N)$ the set $\{i \in N \mid R_i \in \mathscr{D} \setminus \mathscr{S}_0\}$. We prove the "only if" part by using induction on $|N(R_N)|$. We consider the case $|N(R_N)| = 0$ as the base case.

Base case Suppose $|N(R_N)| = 0$. This means all agents have single-peaked preferences. Since $\mathscr{S}_0 \subseteq \mathscr{D}$, the proof for this case follows from Sprumont (1991).

Induction step Suppose the theorem holds for all $R_N \in \mathscr{D}^n$ with $|N(R_N)| = p$ for some $p \geq 0$. We show that the theorem holds for all $R_N \in \mathscr{D}^n$ with $|N(R_N)| = p+1$. Consider $R_N \in \mathscr{D}^n$ with $|N(R_N)| = p+1$ and $i \in N(R_N)$. Let $\hat{R}_i \in \mathscr{S}_0$ be such that $\tau(R_i) = \tau(\hat{R}_i)$. Note that since $|N(\hat{R}_i, R_{-i})| = p$, by the induction hypothesis, $f(\hat{R}_i, R_{-i}) = u(\hat{R}_i, R_{-i})$. So, to prove the theorem, it is enough to show that $f(R_N) = f(\hat{R}_i, R_{-i})$. We prove for the case where $T(R_N) \geq 1$. The proof for the case where $T(R_N) < 1$ follows from similar arguments. We first prove a few claims.

Claim 1: $f_i(R_N) = f_i(\hat{R}_i, R_{-i})$.

Since $T(\hat{R}_i, R_{-i}) \geq 1$, by the definition of the uniform rule, if $\tau(\hat{R}_i) < \frac{1}{n}$, then $u_i(\hat{R}_i, R_{-i}) = \tau(\hat{R}_i)$, and if $\tau(\hat{R}_i) \geq \frac{1}{n}$, then $\frac{1}{n} \leq u_i(\hat{R}_i, R_{-i}) \leq \tau(\hat{R}_i)$. Since $f(\hat{R}_i, R_{-i}) = u(\hat{R}_i, R_{-i})$, by strategy-proofness, $\tau(\hat{R}_i) < \frac{1}{n}$ implies $f_i(R_N) = f_i(\hat{R}_i, R_{-i})$. Suppose $\tau(\hat{R}_i) \geq \frac{1}{n}$ and $f_i(R_N) \neq f_i(\hat{R}_i, R_{-i})$. Suppose further that $f_i(\hat{R}_i, R_{-i}) < f_i(R_N) \leq \tau(R_i)$. Since $\hat{R}_i$ is single-peaked, this implies agent i manipulates at $(\hat{R}_i, R_{-i})$ via R_i, a contradiction. Now suppose $f_i(R_N) < f_i(\hat{R}_i, R_{-i}) \leq \tau(R_i)$. If $f_i(R_N) < \frac{1}{2}$, then by Condition N, $f_i(\hat{R}_i, R_{-i}) P_i f_i(R_N)$, and hence, agent i manipulates at R_N via $\hat{R}_i$, a contradiction. Suppose $f_i(R_N) \geq \frac{1}{2}$, which by Condition N means $f_i(\hat{R}_i, R_{-i}) R_i f_i(R_N)$. If $f_i(\hat{R}_i, R_{-i}) P_i f_i(R_N)$, then agent i manipulates at R_N via $\hat{R}_i$, a contradiction. So, suppose $f_i(\hat{R}_i, R_{-i}) I_i f_i(R_N)$. Then, since $f_i(\hat{R}_i, R_{-i}) > \frac{1}{2}$, by the definition of the uniform rule, for all $j \neq i$, we have $f_j(\hat{R}_i, R_{-i}) = \tau(R_j)$. As $f_i(R_N) < f_i(\hat{R}_i, R_{-i})$, this implies there exists $k \neq i$ such that $f_k(R_N) > \tau(R_k)$. However, this contradicts efficiency of f as $f_j(\hat{R}_i, R_{-i}) R_j f_j(R_N)$ for all $j \in N$ and $f_k(\hat{R}_i, R_{-i}) P_k f_k(R_N)$. Therefore, $\tau(R_i) < f_i(R_N)$. This implies if $f_i(\hat{R}_i, R_{-i}) = \tau(R_i)$, then agent i manipulates at R_N via $\hat{R}_i$. Combining all these observations, we get $f_i(\hat{R}_i, R_{-i}) < \tau(R_i) < f_i(R_N)$. Let $\bar{R}_i \in \mathscr{D}$ be such that (i) $\bar{R}_i \in \mathscr{S}_0$, (ii) $\tau(R_i) = \tau(\bar{R}_i)$, and (iii) $f_i(R_N) \bar{P}_i f_i(\hat{R}_i, R_{-i})$. Such a preference exists as $\mathscr{D}$ is minimally rich. Note that $|N(\bar{R}_i, R_{-i})| = p$, and hence, $f_i(\hat{R}_i, R_{-i}) = f_i(\bar{R}_i, R_{-i})$. However, this means agent i manipulates at $(\bar{R}_i, R_{-i})$ via R_i, a contradiction. This completes the proof of Claim 1. □

Claim 2: $f_j(R_N) = f_j(\hat{R}_i, R_{-i})$ for all $j \in N(R_N)$.

By Claim 1, $f_i(R_N) = f_i(\hat{R}_i, R_{-i})$. So, consider $j \in N(R_N) \setminus i$. Let $\hat{R}_j \in \mathscr{S}_0$ be such that $\tau(R_j) = \tau(\hat{R}_j)$. Note that by the induction hypothesis, $f(\hat{R}_j, R_{-j}) = u(\hat{R}_j, R_{-j})$ and $f(\hat{R}_i, R_{-i}) = u(\hat{R}_i, R_{-i})$. Since $\tau(\hat{R}_i) = \tau(R_i)$ and $\tau(\hat{R}_j) = \tau(R_j)$, by the definition of the uniform rule, $u(\hat{R}_i, R_{-i}) = u(\hat{R}_j, R_{-j})$. This means $f(\hat{R}_i, R_{-i}) = f(\hat{R}_j, R_{-j})$, and hence, to prove Claim 2 it is enough to show that $f_j(R_N) = f_j(\hat{R}_j, R_{-j})$ where $j \in N(R_N)$. However, this follows by using similar arguments as in the proof of Claim 1. This completes the proof of Claim 2. □

Note that since by our induction hypothesis we have $f(\hat{R}_i, R_{-i}) = u(\hat{R}_i, R_{-i})$, by the definition of the uniform rule, $f_i(\hat{R}_i, R_{-i}) \leq \tau(\hat{R}_i)$. This, together with Claim 1 and the fact that $\tau(R_i) = \tau(\hat{R}_i)$, imply $f_i(R_N) \leq \tau(R_i)$. Since $f(\hat{R}_j, R_{-j}) = u(\hat{R}_j, R_{-j})$ where $\hat{R}_j \in \mathscr{S}_0$ and $\tau(\hat{R}_j) = \tau(R_j)$, using Claim 2 and arguments similar to the above, it follows that $f_j(R_N) \leq \tau(R_j)$ for all $j \in N(R_N)$. In the following claim we show that the same happens for the agents outside $N(R_N)$ as well.

Claim 3: $f_l(R_N) \leq \tau(R_l)$ for all $l \in N \setminus N(R_N)$.

Assume for contradiction that there exists $l \in N \setminus N(R_N)$ such that $f_l(R_N) > \tau(R_l)$. Since $T(R_N) \geq 1$, this means there exists $j \neq l$ such that $f_j(R_N) < \tau(R_j)$. Suppose $j \notin N(R_N)$. Let $\varepsilon > 0$ be such that $f_l(R_N) - \varepsilon \geq \tau(R_l)$ and $f_j(R_N) + \varepsilon \leq \tau(R_j)$. Since $j \notin N(R_N)$, we have $R_l, R_j \in \mathscr{S}_0$. This means $f_l(R_N) - \varepsilon P_l f_l(R_N)$ and $f_j(R_N) + \varepsilon P_j f_j(R_N)$. Consider $x \in \Delta_n$ such that $x_j = f_j(R_N) + \varepsilon$, $x_l = f_l(R_N) - \varepsilon$, and $x_k = f_k(R_N)$ for all $k \neq j, l$. However, since $x_k R_k f_k(R_N)$ for all $k \in N$ and $x_k P_k f_k(R_N)$ for all $k \in \{j, l\}$, this is a contradiction to efficiency of f. Suppose $j \in N(R_N)$. This means $f_j(R_N) < \tau(R_j)$. Since $f_j(R_N) = f_j(\hat{R}_i, R_{-i})$ (by Claim 2) and $f_j(\hat{R}_i, R_{-i}) = u_j(\hat{R}_i, R_{-i})$ (by the induction hypothesis), $T(R_N) > 1$ and $f_j(R_N) < \tau(R_j)$ together with the definition of the uniform rule imply $\frac{1}{n} \leq f_j(R_N)$. By Condition N, this implies that for all $\varepsilon > 0$ such that $f_j(R_N) + \varepsilon \leq \tau(R_j)$, we have $f_j(R_N) + \varepsilon P_j f_j(R_N)$. Now, using similar arguments as in the case of $j \notin N(R_N)$, it can be shown that this leads to a contradiction. This completes the proof of Claim 3. □

We now complete the proof of induction step by showing $f_j(R_N) = f_j(\hat{R}_i, R_{-i})$ for all $j \notin N(R_N)$.

Suppose not. Without loss of generality, assume that $R_n = \max_{j \notin N(R_N)} R_j$. Note that by Claim 2, $f_j(R_N) = f_j(\hat{R}_i, R_{-i})$ for all $j \in N(R_N)$. This means if $R_l = R_n$ for all $l \notin N(R_N)$, then by ETE, $f_j(R_N) = f_j(\hat{R}_i, R_{-i})$ for all $j \notin N(R_N)$, a contradiction. Therefore, assume that $R_j \neq R_n$ for some $j \notin N(R_N)$. We proceed in few steps.

Step 1. Since $f_j(R_N) \neq f_j(\hat{R}_i, R_{-i})$ for some $j \notin N(R_N)$, there exists $k \in N \setminus N(R_N)$ such that $f_k(R_N) < f_k(\hat{R}_i, R_{-i})$. By the induction hypothesis, $f_k(\hat{R}_i, R_{-i}) = u_k(\hat{R}_i, R_{-i})$, and hence, by the definition of the uniform rule, $f_k(\hat{R}_i, R_{-i}) \leq \tau(R_k)$. Combining all these observations, we get $f_k(R_N) < f_k(\hat{R}_i, R_{-i}) \leq \tau(R_k)$. Let $R'_k = R_n$. Note that for the profile $(\hat{R}_i, R'_k, R_{-\{i,k\}})$, $|N(\hat{R}_i, R'_k, R_{-\{i,k\}})| = p$ and hence by the induction hypothesis $f(\hat{R}_i, R'_k, R_{-\{i,k\}}) = u(\hat{R}_i, R'_k, R_{-\{i,k\}})$. By the definition of the uniform rule, this, together with the fact that $f_k(\hat{R}_i, R_{-i}) \leq \tau(R_k)$, implies $f_k(\hat{R}_i, R_{-i}) \leq f(\hat{R}_i, R'_k, R_{-\{i,k\}})$. Since $\tau(R_k) \leq \tau(R'_k)$ and $f_k(R_N) < \tau(R_k)$, we claim $f_k(R_N) = f_k(R'_k, R_{-k})$. Suppose not. By strategy-proofness, this means $f_k(R_N) < \tau(R_k) < f_k(R'_k, R_{-k})$. Let $\bar{R}_k \in \mathscr{S}_0$ be such that $\tau(R_k) = \tau(\bar{R}_k)$ and $f_k(R'_k, R_{-k}) \bar{P}_k f_k(R_N)$. Such a preference exists as $\mathscr{D}$ is minimally rich. Since $\tau(R_k) = \tau(\bar{R}_k)$ and $|N(\bar{R}_k, R_{-k})| = p$, using similar arguments as in Claim 3, it can be shown that $f_k(\bar{R}_k, R_{-k}) \leq \tau(\bar{R}_k)$. By strategy-proofness, this means $f_k(R_k, R_{-k}) = f_k(\bar{R}_k, R_{-k})$. However, since $f_k(R'_k, R_{-k}) \bar{P}_k f_k(R_N)$, this means agent k manipulates at $(\bar{R}_k, R_{-k})$ via R'_k, a contradiction. Therefore, $f_k(R_N) = f_k(R'_k, R_{-k})$. Combining all these observations, we get $f_k(R'_k, R_{-k}) < f_k(\hat{R}_i, R'_k, R_{-\{i,k\}})$. Note that for the profile $(\hat{R}_i, R'_k, R_{-\{i,k\}})$, $|N(\hat{R}_i, R'_k, R_{-\{i,k\}})| = p$, and for the profile (R'_k, R_{-k}), $|N(R'_k, R_{-i})| = p+1$. By using similar arguments as in the proof of Claim 2, this implies $f_j(R'_k, R_{-k}) = f_j(\hat{R}_i, R'_k, R_{-\{i,k\}})$ for all $j \in N(R'_k, R_{-k})$. Therefore, if $R_j = R_n$ for all $j \notin N(R'_k, R_{-k})$, then by ETE, we have $f_j(R'_k, R_{-i}) = f_j(\hat{R}_i, R'_k, R_{-\{i,k\}})$ for all $j \notin N(R'_k, R_{-k})$, which is a contradiction since $f_k(R'_k, R_{-k}) < f_k(\hat{R}_i, R'_k, R_{-\{i,k\}})$. So, assume that $R_j \neq R_n$ for some $j \notin N(R'_k, R_{-k}) \cup k$. We proceed to Step 2.

Step 2. Since $f_k(R'_k, R_{-i}) < f_k(\hat{R}_i, R'_k, R_{-\{i,k\}})$, there exists $l \in N \setminus N(R_N)$ such that $f_l(\hat{R}_i, R'_k, R_{-\{i,k\}}) < f_l(R'_k, R_{-i})$. As $f_j(R'_k, R_{-k}) = f_j(\hat{R}_i, R'_k, R_{-\{i,k\}})$ for all $j \in N(R'_k, R_{-k})$ and $|N(R'_k, R_{-i})| = p+1$, using similar arguments as in the proof of Claim 3, we can show that $f_j(R'_k, R_{-k}) \leq \tau(R_j)$ for all $j \notin N(R_N)$. Combining all these observations, we get $f_l(\hat{R}_i, R'_k, R_{-\{i,k\}}) < f_l(R'_k, R_{-k}) \leq \tau(R_l)$. Let $R'_l = R_n$. Since $|N(\hat{R}_i, R'_k, R_{-\{i,k\}})| = |N(\hat{R}_i, R'_k, R'_l, R_{-\{i,k,l\}})| = p$, we have $f(\hat{R}_i, R'_k, R_{-\{i,k\}}) = u(\hat{R}_i, R'_k, R_{-\{i,k\}})$ and $f(\hat{R}_i, R'_k, R'_l, R_{-\{i,k,l\}}) = u(\hat{R}_i, R'_k, R'_l, R_{-\{i,k,l\}})$. This, together with the fact that, $f_l(\hat{R}_i, R'_k, R_{-\{i,k\}}) < \tau(R_l) \leq \tau(R'_l)$ implies $f(\hat{R}_i, R'_k, R_{-\{i,k\}}) = f(\hat{R}_i, R'_k, R'_l, R_{-\{i,k,l\}})$. Since $R_l, R'_l \in \mathscr{S}_0$, by strategy-proofness, $\tau(R_l) \leq \tau(R'_l)$ implies $f_l(R'_k, R_{-k}) \leq f_l(R'_k, R'_l, R_{-\{k,l\}})$. Since $f_l(\hat{R}_i, R'_k, R_{-\{i,k\}}) < f_l(R'_k, R_{-k})$, we have $f_l(\hat{R}_i, R'_k, R'_l, R_{-\{i,k,l\}}) < f_l(R'_k, R'_l, R_{-\{k,l\}})$. Note that $|N(\hat{R}_i, R'_k, R'_l, R_{-\{i,k,l\}})| = p$ and $|N(R'_k, R'_l, R_{-\{k,l\}})| = p+1$. Combining these observations, and using similar arguments as in the proof of Claim 1 and Claim 2, we obtain $f_j(R'_k, R'_l, R_{-\{k,l\}}) = f_j(\hat{R}_i, R'_k, R'_l, R_{-\{i,k,l\}})$ for all $j \in N(R'_k, R'_l, R_{-\{k,l\}})$. This means if $R_j = R_n$ for all $j \notin N(R'_k, R'_l, R_{-\{k,l\}})$, then by ETE we have $f_j(R'_k, R'_l, R_{-\{k,l\}}) =$

$f_j(\hat{R}_i, R'_k, R'_l, R_{-\{i,k,l\}})$ for all $j \notin N(R'_k, R'_l, R_{-\{k,l\}})$, a contradiction to $f_l(\hat{R}_i, R'_k, R'_l, R_{-\{i,k,l\}}) < f_l(R'_k, R'_l, R_{-\{k,l\}})$. Therefore, assume that $R_j \neq R_n$ for some $j \notin N(R'_k, R'_l, R_{-\{k,l\}}) \cup \{k,l\}$. We apply Step 1 to $(\hat{R}_i, R'_k, R'_l, R_{-\{i,k,l\}})$ and $(R'_k, R'_l, R_{-\{k,l\}})$.

Since N is finite and at every step we change the preference of a new agent by R_n, eventually it will lead to a contradiction. This completes the proof of the "only if" part of the theorem. ∎

H. Proof of Theorem 7.1

Proof: Let $\mathscr{D}$ be a local-peaked domain and $f : \mathscr{D}^n \to \Delta_n$ be the partially uniform rule. We show that f satisfies efficiency, strategy-proofness, and ETE.

Efficiency: Consider $R_N \in \mathscr{D}^n$. Suppose R_N is such that $f(R_N) = u(R_N)$, i.e., either $T(R_N) \leq 1$, or $T(R_N) > 1$ and there does not exist $j \in N$ such that R_j satisfies condition (i) in Definition 7.1 and $u_j(R_N) < p^*(R_j)$. If R_j satisfies Condition N for all $j \in N$, then by Theorem 6.1, $f(R_N)$ is efficient. Note that by the definition of the partially uniform rule, if $f(R_N) = u(R_N)$, then there can be at most one agent $i \in N$ such that R_i satisfies condition (i) in Definition 7.1. We proceed through distinguishing few cases.

Case 1: Suppose $T(R_N) \leq 1$. Then, by the definition of the partially uniform rule, $f_i(R_N) = \tau(R_i)$, and for all $j \neq i$, $\tau(R_j) \geq \frac{1}{n}$ implies $f_j(R_N) = \tau(R_j)$ and $\tau(R_j) < \frac{1}{n}$ implies $f_j(R_N) \leq \tau(R_j)$. Since each R_j satisfies Condition N, $f(R_N)$ is efficient.

Case 2: Suppose that $T(R_N) > 1$.

Case 2.a: Suppose further that for some $k \in N$, $u_k(R_N) \geq p^*(R_k)$. Since $\tau(R_i) > p^*(R_i)$ and $p^*(R_i) > (1 - \frac{1}{n})$, by the definition of the uniform rule, we have $k = i$. This implies for all $j \neq i$, $f_j(R_N) = \tau(R_j)$, and hence, $f(R_N)$ is efficient.

Case 2.b: Suppose further that $u_j(R_N) < p^*(R_j)$ for all $j \in N$ with R_j satisfying condition (i) in Definition 7.1. If each R_j satisfies Condition N, then $f(R_N)$ is efficient. Suppose there exists $i \in N$ such that R_i satisfies condition (i) in Definition 7.1.

Case 2.b.i: Suppose $T(\bar{R}_N) \leq 1$. This means for all $j \in N$ with $\tau(R_j) \geq \frac{1}{n}$, $f_j(R_N) = \tau(\bar{R}_j)$, and for all $j \in N$ with $\tau(R_j) < \frac{1}{n}$, $\tau(R_j) \leq f_j(R_N)$. Assume for contradiction that there exists $\boldsymbol{x} = (x_1, \ldots, x_n) \in \Delta_n$ such that $x_j R_j f_j(R_N)$ for all $j \in N$ and $x_k P_k f_k(R_N)$ for some $k \in N$. Combining all these observations, we have either R_k satisfies condition (i) in Definition 7.1 or $\tau(R_k) < \frac{1}{n}$. If R_k satisfies condition (i) in Definition 7.1, then x_k must be greater than $p^*(R_k)$, and hence, for all $j \neq k$, $x_j < \frac{1}{n}$. Since $u_k(R_N) < p^*(R_k)$, this implies there exists $l \in N$ such that $x_l < u_l(R_N) \leq \tau(R_l)$. If R_l satisfies condition (i) in Definition 7.1, then it is a contradiction $x_l R_l f_l(R_N)$ as $f_l(R_N) = \frac{1}{n}$ and $x_l < \frac{1}{n}$. If $\tau(R_l) < \frac{1}{n}$, then $\frac{1}{n} \geq f_l(R_N) > \tau(R_l)$, which is a contradiction $x_l R_l f_l(R_N)$ as by Definition 7.1, $\tau(R_l) \leq \frac{1}{n}$ implies $\frac{1}{n} P_l x_l$. If $\tau(R_l) > \frac{1}{n}$ and R_l satisfies Condition N, then $f_l(R_N) = \tau(R_l)$, which is a contradiction to

$x_l R_l f_l(R_N)$. If $\tau(R_k) < \frac{1}{n}$, then it must be that $\tau(R_k) \leq x_k < f_k(R_N) \leq \frac{1}{n}$. This means there exists $l \in N$ such that $x_l > f_l(R_N)$. If $\tau(R_l) < \frac{1}{n}$, then $f_l(R_N) P_l x_l$, a contradiction to $x_l R_l f_l(R_N)$. If $\tau(R_l) \geq \frac{1}{n}$ and R_l satisfies condition (i) of Definition 7.1, then we can come up with a contradiction to $x_l R_l f_l(R_N)$ by using similar arguments as above. If $\tau(R_l) \geq \frac{1}{n}$ and R_l satisfies Condition N, then $f_l(R_N) = \tau(R_N)$, and hence, $f_l(R_N) P_l x_l$, a contradiction to $x_l R_l f_l(R_N)$.

Case 2.b.ii: Suppose $T(\bar{R}_N) > 1$. This means for all $j \in N$ with $\tau(R_j) \leq \frac{1}{n}$, we have $f_j(R_N) = \tau(\bar{R}_j)$, and for all $j \in N$ with $\tau(R_j) > \frac{1}{n}$, we have $f_j(R_N) \leq \tau(R_j)$, and $f_i(R_N) = \frac{1}{n}$ for all $i \in N$ such that condition (i) of Definition 7.1 is satisfied. Assume for contradiction that there exists $\boldsymbol{x} = (x_1, \ldots, x_n) \in \Delta_n$ such that $x_j R_j f_j(R_N)$ for all $j \in N$ and $x_k P_k f_k(R_N)$ for some $k \in N$. Combining all these observations, it follows that either R_k satisfies condition (i) in Definition 7.1 or R_k satisfies Condition N with $\tau(R_k) > \frac{1}{n}$. If R_k satisfies condition (i) in Definition 7.1, then we can show a contradiction to $x_j R_j f_j(R_N)$ for all $j \in N$ by using similar arguments as in Case 2.b.i. If R_k satisfies Condition N with $\tau(R_k) > \frac{1}{n}$, then it must be that $\frac{1}{n} \leq f_k(R_N) < x_k \leq \tau(R_k)$. This implies there exists $l \in N$ such that $x_l < f_l(R_N)$. Since $f_j(R_N) \leq \tau(R_j)$ for all $j \in N$ and $f_i(R_N) = \frac{1}{n}$ for each R_i satisfying condition (i) in Definition 7.1, this yields $f_l(R_N) P_l x_l$, a contradiction to $x_j R_j f_j(R_N)$ for all $j \in N$.

Since Cases 1 and 2 are exhaustive, it follows that f is efficient.

Strategy-proofness: Consider $R_N \in \mathscr{D}^n$ and an arbitrary agent i. We show that i cannot manipulate at R_N. Consider $R'_i \in \mathscr{D}$. Suppose $R_N \in \mathscr{D}^n$ is such that $f(R_N) = u(R_N)$, i.e., either $T(R_N) \leq 1$, or $T(R_N) \geq 1$ and there does not exist $j \in N$ such that R_j satisfies condition (i) in Definition 7.1 and $u_j(R_N) < p^*(R_j)$. We distinguish these cases in the following.

Case 1: Suppose $T(R_N) \leq 1$. By the definition of the partially uniform rule, for all $j \in N$, $f_j(R_N) \leq \tau(R_j)$ and $f_j(R_N) < \tau(R_j)$ imply $\tau(R_j) < \frac{1}{n}$. Suppose $\tau(R_i) < \frac{1}{n}$ and $f_i(R_N) < \tau(R_i)$. By the definition of the partially uniform rule, agent i can manipulate only if (R'_i, R_{-i}) is such that $T(R'_i, R_{-i}) > 1$ and $u_j(R'_i, R_{-i}) < p^*(R_j)$ for all $j \in N$ with R_j satisfying condition (i) in Definition 7.1. Since $T(R'_i, R_{-i}) > 1$, by the definition of the uniform rule, this implies $\tau(R'_i) > f_i(R_N)$, and hence, either $f_i(R'_i, R_{-i}) \geq \tau(R'_i)$ or $f_i(R'_i, R_{-i}) = \frac{1}{n}$. Since R_i satisfies Condition N, this implies agent i cannot manipulate at R_N via R'_i.

Case 2: Suppose $T(R_N) \geq 1$. If there is $j \in N$ such that R_j satisfies condition (i) in Definition 7.1, then by using similar arguments as in Case 1, we can show that no agent can manipulate. Suppose for some $j \in N$ with R_j satisfying condition (i) in Definition 7.1, we have $u_j(R_N) \geq p^*(R_j)$. This means all agents except j get shares equal to their peaks, and $p^*(R_j) \leq f_j(R_N) < \tau(R_j)$. Suppose $i = j$. Then, agent i can change the outcome only by choosing $\tau(R'_i) < f_i(R_N)$, which implies $f_i(R'_i, R_{-i}) < f_i(R_N)$. However, by the definition of local-peaked domains, $f_i(R_N) P_i x$ for all $x < f_i(R_N)$, and hence, agent i cannot manipulate.

Suppose $R_N \in \mathscr{D}^n$ such that $T(R_N) > 1$ and for all $j \in N$ with R_j satisfying condition (i) in Definition

7.1, we have $u_j(R_N) < p^*(R_j)$. If i is such that R_i satisfies condition (i) of Definition 7.1, then $f_i(R_N) = \frac{1}{n}$. Since $u_j(R_N) \leq p^*(R_j)$ for all $j \in N$ with R_j satisfying condition (i) of Definition 7.1, we have $u_i(R_N) < p^*(R_i)$. By the definition of the uniform rule, agent i can only change his (uniform) share by choosing $\tau(R'_i) < u_i(R_N)$. However, this implies $u_i(R'_i, R_{-i}) \leq u_i(R_N)$, and hence, agent i cannot manipulate. Suppose $i \in N$ is such that R_i satisfies Condition N. Assume $T(\bar{R}_N) \leq 1$. If $\tau(R_i) \geq \frac{1}{n}$, by the definition of partially uniform rule this means $f_i(R_N) = \tau(R_i)$ and hence, agent i cannot manipulate. If $\tau(R_i) < \frac{1}{n}$ then $\tau(R_i) \leq f_i(R_N) \leq \frac{1}{n}$. Assume $\tau(R_i) < f_i(R_N)$. If R'_i satisfies condition (i) of Definition 7.1, then $f_i(R'_i, R_{-i}) = \frac{1}{n}$. However, since $\tau(R_i) \leq f_i(R_N) \leq \frac{1}{n}$, agent i cannot manipulate in this case. If R'_i satisfies Condition N and $\tau(R'_i) \geq \tau(R_i)$, then $T(R'_i, R_{-i}) > 1$ and for all $j \in N$ with R_j satisfying condition (i) in Definition 7.1, we have $u_j(R_N) < p^*(R_j)$. By the definition of the partially uniform rule, this implies $f(R'_i, R_{-i}) = u(\bar{R}'_i, \bar{R}_{-i})$. Since $\tau(\bar{R}'_i) \geq \tau(\bar{R}_i)$, by the definition of the uniform rule, $u(\bar{R}'_i, \bar{R}_{-i})_i \geq u_i(\bar{R}_N)$, and hence, agent i cannot manipulate. If R'_i is such that $\tau(R'_i) < \tau(R_i)$, $T(R'_i, R_{-i}) > 1$, and for all $j \in N$ with R_j satisfying condition (i) of Definition 7.1, we have $u_j(R_N) < p^*(R_j)$, then by the definition of the partially uniform rule, $f_i(R_N) = f_i(R'_i, R_{-i})$. Hence, agent i cannot manipulate. If $T(R'_i, R_{-i}) > 1$, and for all $j \in N$ with R_j satisfying condition (i) in Definition 7.1, we have $u_j(R'_i, R_{-i}) > p^*(R_j)$, then by the definition of the partially uniform rule, $f_i(R'_i, R_{-i}) = \tau(R'_i)$. Since $\tau(R'_i) < \tau(R_i) < f_i(R_N) \leq \frac{1}{n}$ and R_i satisfies Condition N with $\frac{1}{n} P_i x$ for all x with $x < \tau(R_i)$, agent i cannot manipulate. If R'_i is such that $\tau(R'_i) < \tau(R_i)$ and $T(R'_i, R_{-i}) \leq 1$, then, as $T(R_N) > 1$, by the definition of the uniform rule, $\tau(R'_i) \leq u_i(R'_i, R_{-i}) < \tau(R_i)$. This means $f_i(R'_i, R_{-i}) < \tau(R_i)$, and hence, agent i cannot manipulate. If $T(\bar{R}_N) > 1$, then using similar arguments it follows that agent i cannot manipulate. This shows that f is strategy-proof.

ETE: Note that at any profile R_N, the outcome of f is defined by the outcome of u at some profile, which we have denoted by $\bar{R}_N$. Moreover, if $R_i = R_j$ for some $i, j \in N$, then the description of $\bar{R}_N$ implies that $\bar{R}_i = \bar{R}_j$. Therefore, since the uniform rule satisfies ETE, it follows that the partially uniform rule also satisfies ETE. ■

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