NEW VARIABLE SAMPLING INTERVAL RUN SUM STANDARD DEVIATION AND RUN SUM MULTIVARIATE COEFFICIENT OF VARIATION CHARTS

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by

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LIST OF ABBREVIATIONS

ARL	Average run length
ARL_0	In-control average run length
ARL_1	Out-of-control average run length
ATS	Average time to signal
ATS_0	In-control average time to signal
ATS_1	Out-of-control average time to signal
CCC	Cumulative count of conforming
CDF	Cumulative distribution function
CL	Center line
CRL	Conforming run length
CV	Coefficient of variation
DEWMA	Double exponentially weighted moving average
EARL	Expected average run length
$EARL_0$	In-control expected average run length
$EARL_1$	Out-of-control expected average run length
EATS	Expected average time to signal
$EATS_0$	In-control expected average time to signal
$EATS_1$	Out-of-control expected average time to signal
EWMA	Exponentially weighted moving average
ESARL	Expected steady-state average run length
$ESARL_0$	In-control expected steady-state average run length
$ESARL_1$	Out-of-control expected steady-state average run length

ESATS	Expected steady-state average time to signal
$ESATS_0$	In-control expected steady-state average time to signal
$ESATS_1$	Out-of-control expected steady-state average time to signal
FIR	Fast initial response
FSI	Fixed sampling interval
FSR	Fixed sampling ratio
GLR	Generalized likelihood ratio
LCL	Lower control limit
MCUSUM	Multivariate cumulative sum
MDL	Median Line
MEWMA	Multivariate exponentially weighted moving average
PDF	Probability density function
RS	Run sum
SARL	Steady-state average run length
$SARL_0$	In-control steady-state average run length
$SARL_1$	Out-of-control steady-state average run length
SAS	Statistical analysis software
SATS	Steady-state average time to signal
$SATS_0$	In-control steady-state average time to signal
$SATS_1$	Out-of-control steady-state average time to signal
SPC	Statistical Process Control
SPRT	Sequential probability ratio test
SQC	Statistical Quality Control
tpm	Transition probability matrix

UCL	Upper control limit
VP	Variable Parameters
VSI	Variable sampling interval
VSR	Variable sampling rate
VSS	Variable sample size
VSSI	Variable sample size and sampling interval

LIST OF NOTATIONS

σ	Standard deviation
σ_0	In-control standard deviation
σ_1	Out-of-control standard deviation
γ	Coefficient of variation
γ^2	Coefficient of variation squared
μ	Population mean
n	Fixed sample size
d_0	Fixed sampling interval
d_1	Short sampling interval
d_2	Long sampling interval
S_i	Shewhart *S* chart's statistic
v	Degree of freedom of chi-square distribution
α	False alarm rate
k	Number of regions of the RS chart
C_m	Positive score for the m^{th} region of the RS *S*, VSI RS *S* and RS multivariate CV charts
$-C_m$	Negative score for the m^{th} region of the RS *S*, VSI RS *S* and RS multivariate CV charts
$F_{n-1}(\cdot)$	Cumulative distribution function of the chi-square random variable with $n-1$ degrees of freedom
U_i	Upper cumulative score for the RS *S*, VSI RS *S* and RS multivariate CV charts at sample *i*
U_0	Upper head-start value

L_i	Lower cumulative score for the RS *S*, VSI RS *S* and RS multivariate CV charts at sample *i*
L_0	Lower head-start value
H	Critical value for the RS *S*, VSI RS *S* and RS multivariate CV charts
$\Phi(\cdot)$	Cumulative distribution function for the standard normal random variable
$\theta_{\text{RS } S}$	Control limit constant of the RS *S* chart
p_m	Probability of S_i (of the RS *S* or VSI RS *S* chart) or probability of $\hat{\gamma}_\xi$ (of the RS multivariate CV chart) falling in the region $\left[\text{UCL}_{m-1}, \text{UCL}_m\right)$
p_{-m}	Probability of S_i (of the RS *S* or VSI RS *S* chart) or probability of $\hat{\gamma}_\xi$ (of the RS multivariate CV chart) falling in the region $\left(\text{LCL}_m, \text{LCL}_{m-1}\right]$
$\mathbf{e}^T$	Initial probability vector for the RS *S*, one-sided upward EWMA *S*, VSI RS *S* and RS multivariate CV charts
$\mathbf{I}$	Identity matrix
$\mathbf{1}$	Unity vector
$\mathbf{Q}$	Transition probability matrix for the transient states
Z_i	EWMA statistic at sample *i*
Z_0	Initial value of the EWMA statistic
Y_i	Logarithmic transformation of the process variance at sample *i*
λ	Smoothing constant of the EWMA *S* chart
$\theta_{\text{EWMA } S}$	Control limit constant of the EWMA *S* chart
Ω	Width of each subinterval on the two-sided EWMA *S* chart
p_{rs}	Transition probability from state *r* to state *s*

$G(\cdot, a, b)$	Cumulative distribution function of the gamma random variable with shape parameter a and scale parameter b
ψ	Number of subintervals of the one-sided upward EWMA S chart
δ	Width of each subinterval (except the first one) on the one-sided upward EWMA S chart
$\mathbf{e}_{\text{EWMA}}$	Initial probability vector for the two-sided EWMA S chart
$\mathbf{X}_{\xi}$	Multivariate observation at sample ξ
$\boldsymbol{\mu}$	Population mean vector
$\boldsymbol{\Sigma}$	Population covariance matrix
$N_{\varpi}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$	ϖ-variate normal distribution
ϖ	Number of quality characteristics monitored simultaneously for a multivariate process
γ	Population multivariate coefficient of variation
$\bar{\mathbf{X}}$	Sample mean vector
$\mathbf{S}$	Sample covariance matrix
$\hat{\gamma}$	Sample multivariate coefficient of variation
γ_0	In-control population multivariate coefficient of variation
γ_1	Out-of-control population multivariate coefficient of variation
$\hat{\gamma}_0$	In-control sample multivariate coefficient of variation
ϑ	Number of Phase-I multivariate sample CVs
ε	Non-centrality parameter
ε_0	Non-centrality parameter when the process is in-control
ε_1	Non-centrality parameter when the process is out-of-control
$H_{\hat{\gamma}}(\cdot \mid n, \varpi, \varepsilon)$	Cumulative distribution function of $\hat{\gamma}$ with n and ϖ degrees of freedom and non-centrality parameter ε

$H_F\left(\cdot \mid \varpi, n-\varpi, \varepsilon \right)$	Cumulative distribution function of the non-central F random variable with ϖ and $n - \varpi$ degrees of freedom and non-centrality parameter ε
$H_{\hat{\gamma}}^{-1}\left(\cdot \mid n, \varpi, \varepsilon \right)$	Inverse cumulative distribution function of $\hat{\gamma}$ with n and ϖ degrees of freedom and non-centrality parameter ε
$H_F^{-1}\left(\cdot \mid \varpi, n-\varpi, \varepsilon \right)$	Inverse cumulative distribution function of the non-central F random variable with ϖ and $n-\varpi$ degrees of freedom and non-centrality parameter ε
$\theta_{\text{VSI RS } S}$	Control limit constant of the VSI RS S chart
$\text{VSIRS}_S\left\{k, \theta_{\text{VSI RS } S}, d_1, d_2, \left\{C_1, C_2, ..., C_k\right\}\right\}$	VSI RS S chart
D_{S_i}	Length of the sampling interval between samples S_{i-1} and S_i
G	Switching constant
$\mathbf{P}$	Transition probability matrix with the absorbing state
$\mathbf{d}$	Vector of sampling intervals
$\boldsymbol{\beta}$	Modified cyclical steady state probability vector
$\boldsymbol{\pi}$	Cyclical steady state probability vector
d_t	t^{th} entry of $\mathbf{d}$
β_t	t^{th} entry of $\boldsymbol{\beta}$
π_t	t^{th} entry of $\boldsymbol{\pi}$
$\tau_{\min}$	Lower bound of the shift in the shift interval $\left(\tau_{\min}, \tau_{\max}\right)$
$\tau_{\max}$	Upper bound of the shift in the shift interval $\left(\tau_{\min}, \tau_{\max}\right)$
t_f	Initial sampling interval
$\theta_{\text{RS MCV}}$	Control limit constant of the RS multivariate CV chart
$RS_{\hat{\gamma}_\xi}\left(k, \theta_{\text{RS MCV}}, \left\{C_1, C_2, ..., C_k\right\}\right)$	Upward RS multivariate CV chart

$RS_{\hat{\gamma}_\xi}\left(k, \theta_{\text{RS MCV}}, \{-C_1, -C_2, ..., -C_k\}\right)$	Downward RS multivariate CV chart
p_0	Probability of $\hat{\gamma}_\xi$ falling in the region below MDL $\left(\text{or UCL}_0\right)$
p_{-0}	Probability of $\hat{\gamma}_\xi$ falling in the region above MDL $\left(\text{or LCL}_0\right)$
$\acute{\eta}$	Number of Phase-I univariate sample CVs
X_i	Univariate observation at sample *i*

CARTA-CARTA BAHARU SISIHAN PIAWAI HASIL TAMBAH LARIAN DENGAN SELANG PENSEMPELAN BERUBAH DAN PEKALI VARIASI MULTIVARIAT HASIL TAMBAH LARIAN

ABSTRAK

Dalam Kawalan Proses Berstatistik (SPC), teknik carta kawalan ialah kaedah yang berkesan untuk menyelesaikan isu-isu kualiti dalam industri pembuatan dan perkhidmatan. Carta-carta *R* dan *S* sering digunakan untuk memantau varians proses dalam industri kerana carta-carta tersebut mempunyai rekabentuk yang mudah dan kepekaan yang tinggi terhadap anjakan besar. Walau bagaimanapun, carta-carta tersebut tidak peka terhadap anjakan kecil dan serderhana dalam varians proses. Sebaliknya, carta-carta yang lebih canggih, seperti carta purata bergerak berpemberat eksponen (EWMA) *S* dan carta hasiltambah longgokan (CUSUM) *S* adalah sangat berkesan untuk mengesan anjakan kecil dalam varians proses. Walau bagaimanapun, kebanyakan pengamal kualiti tidak menggunakan carta-carta sedemikian dalam aplikasi sebenar kerana rekabentuknya yang rumit. Di atas kelemahan ini, pendekatan selang pensempelan berubah (VSI) digabungkan dengan carta hasil tambah larian (RS) *S* bagi mencadangkan suatu carta yang berkesan dan mudah untuk mengesan anjakan kecil, sederhana dan besar dalam varians proses. Selain itu, pekali variasi (CV) merupakan suatu ciri kualiti yang penting untuk diambil kira apabila min dan sisihan piawai proses adalah tidak malar, walaupun proses masih berada dalam kawalan. Sesetengah penyelidik telah memperkenalkan carta-carta CV univariat untuk memantau CV proses. Walau bagaimanapun, masih terdapat kekurangan dalam kesusasteraan berkaitan dengan carta CV multivariat. Sebenarnya, hanya terdapat satu carta dalam kesusasteraan carta-carta multivariat untuk memantau CV multivariat.

Bagi mengatasi kekurangan ini, kaedah RS digunakan untuk mencadangkan suatu carta CV multivariat yang lebih cekap. Objektif-objektif utama tesis ini adalah untuk mencadangkan (i) carta VSI RS *S* dan (ii) carta CV multivariat RS. Prosedur pengoptimuman diperkenalkan untuk mengira parameter optimum dan skor untuk carta-carta yang dicadangkan berdasarkan model rantai Markov. Prosedur pengoptimuman ini membolehkan para pengamal kualiti untuk merekabentuk carta terbaik yang dapat mengesan anjakan proses dengan kelajuan yang paling tinggi. Tambahan pula, rekabentuk carta-carta yang dicadangkan melibatkan dua situasi, iaitu apabila saiz anjakan yang tepat boleh ditentukan dan apabila ia adalah tidak diketahui. Pembinaan dan pelaksanaan carta-carta yang dicadangkan akan ditunjukkan dengan contoh-contoh ilustrasi berdasarkan set-set data sebenar. Prestasi carta-carta yang dicadangkan diukur dan dibandingkan dengan carta-carta yang sedia ada, untuk kes-kes keadaan sifar dan keadaan mantap. Secara amnya, hasil perbandingan menunjukkan bahawa carta-carta yang dicadangkan mempunyai prestasi yang lebih baik daripada carta-carta yang sedia ada.

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ABSTRACT

In Statistical Process Control (SPC), the control charting technique is an effective method to solve quality issues in manufacturing and service industries. The *R* and *S* charts are commonly used to monitor the process variance in industries due to the charts' simplicity and high sensitivity toward large shifts. However, these charts are not sensitive toward small and moderate shifts in the process variance. On the other hand, the more sophisticated charts, such as the exponentially weighted moving average (EWMA) *S* chart and the cumulative sum (CUSUM) *S* chart are very effective in detecting small changes in the process variance. However, most quality practitioners do not adopt these charts in real applications due to their design complexity. In view of this setback, the variable sampling interval (VSI) approach is incorporated into the run sum (RS) *S* chart, in order to suggest an effective, yet a simple chart, for detecting small, moderate and large shifts in the process variance. Apart from that, the coefficient of variation (CV) is an important quality characteristic to take into account when the process mean and standard deviation are not constant, even though the process is in-control. Some researchers have introduced univariate CV charts to monitor the process CV. However, there is a scarcity in the literature concerning the multivariate CV chart. In fact, there is only one chart in the existing literature of multivariate charts, for monitoring the multivariate CV. To circumvent this limitation, the RS method is adopted to suggest a more efficient multivariate CV chart. The main objectives of this thesis are to propose the (i) VSI RS *S* chart and (ii) RS multivariate

CV chart. Optimization procedures are introduced to compute the optimal parameter and scores of the proposed charts, based on the Markov chain model. These optimization procedures will enable quality practitioners to design the best chart that detects a desired process shift at the quickest speed. In addition, the design of the proposed charts involves two situations, i.e. when the exact shift size can be specified and when it is unknown. The construction and implementation of the proposed charts are demonstrated with illustrative examples, based on real datasets. The performances of the proposed charts are evaluated and compared with the existing charts, for the zero state and steady state cases. In general, the comparative studies show that the proposed charts perform better than their existing counterparts.

CHAPTER 1

INTRODUCTION

1.1 An Overview on Statistical Quality Control Charts

Quality improvement is vital in manufacturing and service industries to drive business growth, sustaining competitiveness and ensuring success. The eight dimensions of quality emphasized by Garvin (1987) are reliability, durability, serviceability, performance, features, aesthetics, conformance to standards and perceived quality. Practitioners need to identify and understand the characteristics that are affecting the quality of products produced or services provided. There are two factors, called the common causes and assignable causes that induce variability in all processes, leading to quality deterioration. The common causes of variation are inherent in the production and are unavoidable. On the other hand, assignable causes of variation can be eliminated by identifying the sources, such as machine wear out, human errors and defective raw materials (Montgomery, 2009).

Statistical Process Control (SPC) is used to reduce variability in processes and to ensure that products or services produced are of high quality. SPC is also known as the magnificent seven as it consists of seven powerful statistical tools that can assist practitioners to detect changes in a process. These seven statistical tools are the check sheet, cause-and-effect diagram, stem-and-leaf diagram, scatter diagram, histogram, Pareto chart and control chart (Gupta and Walker, 2007). These statistical tools are easy to implement and are suitable to shop floor personnel who may have little knowledge on statistics, yet providing significant results in quality improvement. SPC techniques have been widely applied in manufacturing and service industries to raise a company’s profit by reducing the number of nonconforming products produced.

A control chart is the most popular technique among all the statistical tools in SPC that are used to monitor a process. A basic control chart is a diagram that consists of the upper and lower control limits, center line and sample points. The history of control charts started with the introduction of the first control chart by Walter A. Shewhart in 1924. Subsequently, researchers developed other types of control charts to meet a variety of quality issues encountered in different industries. Control charts can be classified into two categories, which are variable control charts and attribute control charts. A variable chart is used to monitor variable quality characteristics that can be measured in a continuous scale, such as width, volume and height. Some examples of variable control charts are $\bar{X}$, S and R charts. On the other hand, an attribute chart is used to monitor attribute quality characteristics, such as the number of defective items. Commonly used attribute charts are the p, c and u-charts.

Control charting methods have also been extended to monitor multivariate quality characteristics. For instance, the multivariate exponentially weighted moving average (MEWMA), multivariate cumulative sum (MCUSUM) and Hotelling's χ^2 charts have received great attention from practitioners and researchers worldwide. This multivariate type charts can be used to monitor two or more quality characteristics in a process simultaneously, in order to meet new quality improvement requirements.

Two distinct phases of process monitoring, i.e. Phase I and Phase II, are encountered in the use of control charts for process monitoring. In Phase I, a historical dataset is taken to compute trial control limits. These trial control limits are suitable for monitoring a future process in Phase II, if the Phase I historical dataset shows that the process is statistically in-control. In Phase II, the control chart is used to monitor a future process by comparing the plotted sample statistics in Phase II with the trial control limits established in Phase I.

Statistical control charts are widely used by practitioners in various industries to guard against process deterioration. For instance, an engineer can use a control chart to detect assignable causes in production lines while an investment broker may apply control charting techniques to identify the presence of abnormal market behavior in the stock market. It is noteworthy that statistical control charts enable the user to identify assignable causes quickly and to prevent the process from operating in an out-of-control condition.

1.2 An Overview on *S* Type Charts

Nowadays, quality improvement strategies in manufacturing and service industries not only require process monitoring of the mean but also the variability. This is because a large number of nonconforming items will be produced when a process is operating with a high variability. On the contrary, a low process variability environment can improve process capability (Acosta-Mejia *et al*., 1999). Therefore, several *S* type control charts have been developed by researchers, in order to reduce process variability. A more advanced chart was proposed by Crowder and Hamilton (1992) which was the exponentially weighted moving average (EWMA) *S* chart. The EWMA *S* chart involves a logarithmic transformation of the sample variance. It was shown that the EWMA *S* chart can detect small increasing shifts in the process standard deviation quicker than the usual range chart or S^2 chart. Subsequently, Klein (2000) proposed three modified *S*-charts that outperform the traditional Shewhart *S* chart, in detecting a shift in the standard deviation. Huang and Chen (2005) introduced the synthetic *S* chart that consists of an *S* chart and a conforming run length (CRL) chart to monitor the process dispersion more effectively. A variable sampling interval (VSI) scheme was also incorporated into the synthetic *S* chart to further enhance its performance. Shu and Jiang (2008) modified the traditional EWMA *S* chart by truncating negative normalized observations to zero to propose a new EWMA *S* chart. The comparative results show that the new EWMA *S* chart is superior to the traditional EWMA *S* chart, in monitoring the process standard deviation. The *k*-of-*k* runs rule was incorporated into the standard *S* chart by Acosta-Mejia and Pignatiello (2009) to improve the standard *S* chart's sensitivity towards small shifts in the process standard deviation. This was followed by Antzoulakos and Rakitzis (2010) who applied the general runs rules schemes on the Shewhart *S* chart to further enhance its performance in monitoring the process standard deviation. Furthermore, recommendations were also provided on the choice of the optimal runs rules scheme, based on a specified shift size where a quick detection is deemed important.

Schoonhoven *et al*. (2011) developed a chart with estimated process parameters that is robust against contamination to monitor the process variance. Besides that, Rakitzis and Antzoulakos (2011) presented one-sided adaptive *S* charts that are effective in monitoring both increasing and decreasing shifts in the process variance. They considered the VSI, variable sample size (VSS) and variable sample size and sampling interval (VSSI) schemes, both with and without runs rules. The CUSUM and EWMA charts are well-known for their efficiency in detecting small and moderate process shifts. Abbas *et al*. (2013) combined both the CUSUM and EWMA techniques into a single new control chart, called the CS-EWMA chart, where the latter prevails over the two former charts in detecting increasing and decreasing shifts in the process variance. Kuo and Lee (2013) introduced several different adaptive schemes for the Shewhart *S* chart to improve its sensitivity towards small increases in the process standard deviation. The simple random sampling and double sampling techniques were applied by Ahmad *et al*. (2013) to variance type charts to monitor the process variance more effectively. Ahmad *et al*. (2013) conducted their study using auxiliary characteristics and in the presence of contaminated data. Rakitzis and Antzoulakos (2014) incorporated signaling and switching rules into the one-sided VSI *S* chart, in order for the chart to outdo more advanced charts, like the VSI synthetic *S*, VSI CUSUM-*S* and VSI EWMA-ln S^2 charts, in detecting small shifts in the process variance. Recently, the synthetic double sampling *S* chart was proposed by Lee and Khoo (2017) to monitor increasing shifts in the process variance more effectively.

1.3 An Overview on Variable Sampling Interval Charts

Traditional charts are designed based on the fixed sampling ratio (FSR) scheme in monitoring a process. Thus, all samples taken from a process are of fixed sample size *n* and each sample is taken after a fixed sampling interval d_0. In order to further enhance the performance of traditional charts, researchers started to design control charts based on the variable sampling rate (VSR) scheme. The most commonly used VSR features on control charts are the VSI, VSS and VSSI methods as a function of prior sample information. Many research works have shown that the VSI, VSS and VSSI schemes have significantly reduced the average time to detect process changes. The VSI $\bar{X}$ chart was proposed by Reynolds *et al*. (1988), where it was found to be superior to the standard fixed sampling interval (FSI) $\bar{X}$ chart. The usual VSI policy involves two different sampling intervals which are the short sampling interval d_1 and the long sampling interval d_2. The next sample is taken after a short sampling interval if the current sample shows that a process change is likely to occur. This allows practitioners to detect assignable causes at a shorter time period so that rework and scrap costs can be reduced. On the other hand, the next sample is taken after a long sampling interval if the current sample does not show any sign of a potential process change. This means that the production line is not interrupted when the process is in-control. In practice, process changes usually occur after the production line has been running for some time even though it may also happen at the beginning of the process. Therefore, Stoumbos *et al*. (2001) presented the steady state optimal VSI charts and suggested that only two sampling intervals are considered. Göb *et al*. (2006) studied the process distribution of the time to failure under the Weibull distribution for the VSI Shewhart charts.

An adaptive VSI CUSUM chart was proposed by Luo *et al*. (2009), where the chart is robust against unknown shift sizes and it performs substantially better than the traditional FSI chart in monitoring a range of mean shifts. In addition, Li and Wang (2010) introduced an EWMA chart supplemented with the VSI scheme, where the chart is robust and effective in detecting various types of shifts, which include intercept shifts, slope shifts and standard deviation shifts. In the existing literature, the sequential probability ratio test (SPRT) chart is well-known for its ability to detect moderate mean shifts quickly as compared to the more advanced CUSUM chart. Ou *et al*. (2011) applied the VSI scheme on the SPRT chart to further enhance its detection effectiveness. The VSI $\bar{X}$ chart with estimated process parameters was proposed by Zhang *et al*. (2012).

Recently, Lee *et al*. (2015) combined the CRL chart with the VSI $\bar{X}$ chart to propose the VSI synthetic $\bar{X}$ chart that is superior to the VSI $\bar{X}$ chart in detecting moderate and large mean shifts. Liu *et al*. (2015) presented the Phase II VSI nonparametric EWMA chart that is efficient in monitoring a wide range of shifts regardless of the underlying process distribution. Patil and Shirke (2015) introduced the VSI policy on the economic design of the moving average $\bar{X}$ chart to monitor a process having a non-normal quality characteristic. It was shown that the proposed chart results in a substantial percentage of savings in cost as compared to the chart's FSI counterpart. Peng *et al*. (2015) proposed the VSI generalized likelihood ratio (GLR) chart that is effective in monitoring mean shifts regardless of the shift size. The VSI feature has also been extended to attribute type control charts to monitor the number of nonconforming items in a process. For example, Lee and Khoo (2015) incorporated both the VSI and runs rules schemes into the cumulative count of conforming (CCC) chart.

1.4 An Overview on Run Sum Charts

The run sum control charting technique is a special technique that involves partitioning the in-control region of a chart into several regions to enhance the chart's sensitivity. The run sum concept involves assigning score to each of the regions and then the cumulative score is computed by summing up all the scores based on the regions where the sequence of past samples fall into till the current sample. The process of computing the cumulative score continues until the cumulative score reaches or exceeds a critical value, where an out-of-control signal is triggered. Davis *et al*. (1990) studied the performance of the run sum chart and concluded that the chart is superior to the Shewhart chart with the common runs rules. Davis *et al*. (1994) introduced a model which consists of the fast initial response (FIR) feature and score vectors that improves the competitiveness of the run sum chart. Ho and Case (1994) extended the run sum chart based on the economic criterion to jointly monitor the process mean and variance more effectively. Champ and Rigdon (1997) highlighted that by adding more regions and scores, the run sum $\bar{X}$ chart can be made more sensitive than the CUSUM and EWMA charts. As both process location and variability are important factors that affect the production quality, Aguirre-Torres and Reyes-López (1999) presented the

RS charts that are able to monitor both the sample mean $\left(\bar{X}\right)$ and sample range (R) statistics simultaneously.

In practice, a process mean that changes linearly over time can also lead to process deterioration. Davis and Krehbiel (2002) compared the performance of the Shewhart charts with supplementary runs rules to that of run sum charts when the process mean changes linearly over time and concluded that the latter surpasses the former. Acosta-Mejia and Pignatiello (2010) proposed the RS *R* chart that is superior to *R* charts with runs rules, in monitoring the process dispersion. In certain situations, practitioners need to monitor two or more related quality characteristics simultaneously. In view of this, Khoo *et al*. (2013) proposed the RS Hotelling's χ^2 charts with and without the fast initial response (FIR) feature that are more efficient than the χ^2 chart with runs rules and the synthetic χ^2 chart. Sitt *et al*. (2014) introduced the RS *t* chart that provides sufficient robustness against estimation errors in the process standard deviation while Acosta-Mejia and Rincon (2014) presented a simplified version of the RS scheme which is a one-parameter continuous RS chart. Chew *et al*. (2015) applied the VSI policy to enhance the RS $\bar{X}$ chart's sensitivity towards a mean shift in the process. Lastly, the RS *S* chart was proposed by Rakitzis and Antzoulakos (2016) to monitor the process variability more effectively.

1.5 An Overview on Coefficient of Variation Charts

Nowadays, process monitoring is becoming more complicated and challenging. As a result, control charting techniques have undergone various modifications and extensions in order to meet the present industrial requirements. In certain scenarios, practitioners not only need to monitor the process mean μ and standard deviation σ but also the coefficient of variation (CV). The need to monitor the CV arises when the process mean and standard deviation are not constant but the process is still deemed as in-control. In other words, the process standard deviation σ is a function of the process mean μ, i.e. $\sigma = \gamma\mu$. Therefore, the aim of monitoring the CV is to maintain a constant CV, $\gamma = \frac{\sigma}{\mu}$, while providing flexibility for the mean and standard deviation to vary. In line with this aim, Kang *et al*. (2007) proposed a CV chart to monitor the process CV. A shortcoming of this CV chart is its inability to detect small shifts in the CV quickly. Thus, a CV-EWMA (CV-exponentially weighted moving average) chart was introduced by Hong *et al*. (2008), for a quicker detection of small shifts in the CV. Instead of monitoring the CV itself, Castagliola *et al*. (2011) developed an EWMA-γ^2 chart that monitors the coefficient of variation squared, i.e. γ^2, where this chart performs generally better than the CV-EWMA chart in detecting a shift in the CV.

The double exponentially weighted moving average-CV (DEWMA-CV) chart was presented by Hong *et al*. (2011a) to outdo the CV-EWMA chart in detecting small CV shifts. Hong *et al.* (2011b) also proposed the generally weighted moving average-CV (GWMA-CV) chart that is superior to both the CV-EWMA and DEWMA-CV charts. Subsequently, Calzada and Scariano (2013) proposed the synthetic CV chart that is more efficient than the standard CV chart and the former also surpasses the EWMA-γ^2 chart in detecting large shifts in the CV. Castagliola *et al*. (2013a) introduced the runs rule CV chart and suggested the best runs rule scheme based on the desired shift size in the CV. The VSI scheme was incorporated into the CV chart by Castagliola *et al*. (2013b) in proposing the VSI CV chart that is more efficient than the FSI CV chart. Zhang *et al*. (2014) modified the EWMA-γ^2 chart to propose a new EWMA chart to monitor the CV, where the latter prevails over the former in the detection of shifts in the CV. A method was developed by Park *et al*. (2014) to enhance the CV-EWMA chart's precision and accuracy.

Traditional control charts are designed based on the assumption of infinite production horizon and may not be suitable in the current industrial setting involving flexible manufacturing. To circumvent this problem, Castagliola *et al*. (2015a) developed the one-sided Shewhart-type charts for short production runs to monitor the CV. The VSS approach is an adaptive policy which requires the next sample having a large sample size to be taken if the current sample shows that a process shift is likely to occur, while requiring the next sample with a small sample size to be taken if the current sample does not show any sign of a potential process change. Castagliola *et al*. (2015b) incorporated the VSS scheme into the Shewhart chart to monitor the CV more effectively. Furthermore, Amdouni *et al*. (2015) modified the VSS CV chart to enable it to monitor the CV in finite production horizon. Lastly, Yeong *et al*. (2015) developed the first multivariate CV chart that is able to monitor two or more related quality characteristics simultaneously. This contribution enables the process CV of two or more correlated variables to be jointly monitored and the research by Yeong *et al*. (2015) has extended the monitoring of the process CV from the univariate case to the multivariate case.

1.6 Problem Statements and Research Motivations

In process monitoring, the monitoring of the process variance is a vital factor to take into consideration besides the monitoring of the process mean. This is particularly true when dealing with increasing shifts in the process variance which result in process deterioration. The RS *S* chart proposed by Rakitzis and Antzoulakos (2016) is a simple and straightforward approach that can be easily implemented by practitioners, yet having the desired feature of being more sensitive than the Shewhart S charts with and without runs rules. In view of the advantages of the RS S chart, the aim of this thesis is to further enhance the sensitivity of this chart by incorporating the VSI feature.

In addition, it is crucial to monitor the process coefficient of variation when the process standard deviation is directly proportional to the process mean. For instance, (i) a clinical surgeon relates the standard deviation to the mean amount of chemical in a patient's blood, which varies among patients (Kang *et al*., 2007) and (ii) a dealer from an investment bank relates the volatility of the return on an asset to the expected value of the return in order to calculate the risk of a particular stock (Sharpe, 1994). It is meaningless to monitor the process mean or variance separately in such situations because both are not constants. In the literature, various control charts were proposed to monitor the univariate process CV. However, until now only one control chart is available to monitor the multivariate process CV, where this chart was proposed by Yeong *et al*. (2015). A main drawback of this multivariate CV chart is its lack of sensitivity toward small and moderate shifts in the multivariate process CV. To circumvent this problem, the RS multivariate CV chart is proposed in this thesis so that a more powerful chart for monitoring the multivariate CV is available in the literature.

1.7 Objectives of the Study

The main objectives of this thesis are as follows:

(i) To develop the VSI RS S chart that is more effective in detecting shifts in the process variance than existing charts available in the literature.

(ii) To develop the multivariate RS CV chart that is more sensitive than the existing chart for monitoring the multivariate process CV.

1.8 Organization of the Thesis

This thesis consists of 5 chapters and the organization of the thesis is given in this section. Chapter 1 begins with an introduction on the background of Statistical Quality Control (SQC) and an overview on various control charts. Brief overviews on the *S*, VSI, RS and CV charts are provided in Chapter 1. This is followed by an explanation on the problem statements and research motivations, as well as the objectives of the study.

Chapter 2 provides a discussion on the performance measures of control charts, which include the average run length (ARL), steady-state ARL (SARL), average time to signal (ATS), steady-state ATS (SATS), expected average run length (EARL), expected steady-state ARL (ESARL), expected average time to signal (EATS) and expected steady-state ATS (ESATS). Chapter 2 also presents a detailed review on the related univariate *S* type control charts that are considered in Section 3.5 and the existing multivariate CV chart which is considered in Section 4.5. The *S* type charts reviewed here are the Shewhart *S*, RS *S* and EWMA *S* charts.

Chapter 3 gives an overview and a procedure to implement the proposed VSI RS *S* chart. A detailed optimal design procedure of the proposed chart is also discussed in this chapter. Consequently, a comparison of the performances of the proposed VSI RS *S* chart and existing *S* type charts, namely, the Shewhart *S*, RS *S* and EWMA *S* charts is explained in this chapter. This is followed by an illustrative example that demonstrates the use of the proposed chart in practice.

The proposed RS multivariate CV chart is presented in Chapter 4. The construction of the proposed chart together with the computation of the chart's optimal parameters are explained in this chapter. A performance comparison between the proposed RS multivariate CV chart and the existing multivariate CV chart is also given in Chapter 4. The last section of this chapter provides an illustrative example to show the procedure to implement the proposed chart in practice.

Finally, a summary drawn from the findings in Chapters 3 and 4 is given in Chapter 5. Apart from that, Chapter 5 also highlights the main contributions of this thesis, as well as provides suggestions for future research.

The last part of the thesis contains references and appendices. Numerous optimization (for computing optimal parameters) programs written in MATLAB and Statistical Analysis Software (SAS) for the VSI RS *S* and RS multivariate CV charts are given in Appendices A and B, respectively. In addition, programs written for competing charts, such as the Shewhart *S*, RS *S*, EWMA *S* and multivariate CV charts are provided in Appendices C, D, E and F, respectively.

CHAPTER 2

A DISCUSSION ON PERFORMANCE MEASURES AND RELATED CONTROL CHARTS

2.1 Introduction

Statistical Quality Control chart is one of the most popular tools used in manufacturing and service industries to guard against process deterioration, in order to improve quality. It is important for practitioners to select the best control chart that gives the quickest speed in detecting process changes, based on desired circumstances. In view of this, the efficiency of the control charts studied in this thesis is evaluated based on performance measures under various conditions. A detailed discussion on the performance measures is given in Section 2.2.

Section 2.3 presents an overview on the related univariate S type control charts that are used in the performance comparison with the proposed chart in Chapter 3. This includes the Shewhart S, run sum S (RS S) and EWMA S charts that are reviewed in detail in Sections 2.3.1, 2.3.2 and 2.3.3, respectively. On the other hand, the multivariate CV chart which is the sole existing CV chart for multivariate process, in the literature, is reviewed in Section 2.4, where it is compared with the proposed chart in Chapter 4.

2.2 Control Charts' Performance Measures

In this section, several performance measures are used to assess and compare the effectiveness of control charts in detecting process shifts. A control chart is said to be superior if it is able to react to process changes at the shortest time. The performance measures considered in this study are ARL, SARL, ATS, SATS, EARL, ESARL, EATS and ESATS. Note that, the common performance criteria for the FSI type charts are based on ARL, SARL, EARL and ESARL while the performance measures for the VSI type charts include ATS, SATS, EATS and ESATS. When the competing charts have the same in-control value, in terms of a certain performance measure, the superior chart has the smallest out-of-control value.

2.2.1 Average Run Length (ARL) and Steady-State Average Run Length (SARL)

The performance measures to evaluate the efficiency and effectiveness of the FSI type charts are ARL and SARL, for the zero state and steady state cases, respectively. The ARL denotes the average number of samples that are plotted on a control chart until the chart triggers an out-of-control alarm (Montgomery, 2009). Note that the definition of ARL assumes that process changes occur at the beginning of the process, however, the majority of process changes happen at a random time after process monitoring activity is initiated. Hence, the SARL, which is defined as the "expected value" of the number of samples taken from the time the process shifts, that occurs at a random time, until the chart signals an out-of-control, is used to assess the performance of the chart under the steady state condition. When the process is in-control, the average number of samples taken until the chart issues a false alarm is denoted as ARL_0 and SARL_0, for the zero state and steady state cases, respectively. On the other hand, the average number of samples required by the chart to signal when the process is out-of-control is denoted as ARL_1 and SARL_1, for the zero state and steady state cases, respectively. Therefore, it is desirable to have a large value of ARL_0 or SARL_0 to reduce unnecessary inspection work and a small value of ARL_1 or SARL_1 to identify assignable causes quickly.

2.2.2 Average Time to Signal (ATS) and Steady-State Average Time to Signal (SATS)

For VSI type charts, the common performance measures are the ATS for the zero state case and SATS for the steady state case. Unlike FSI type charts whose sampling intervals are fixed, the VSI type charts adopt different sampling interval lengths based on information obtained from the current sample. Therefore, it is more appropriate to measure the amount of time required by a VSI type chart to signal. The definition for the ATS is given as the expected amount of time from the beginning of the process monitoring activity until an out-of-control signal is triggered by the chart while the SATS is the steady state ATS (similar to the SARL explanation in Section 2.2.1). The in-control ATS and out-of-control ATS are denoted as ATS_0 and ATS_1, respectively, for the zero state case. In addition, the SATS_0 and SATS_1 are used to represent the in-control SATS and out-of-control SATS, respectively, for the steady state case. An ideal control chart has the largest ATS_0 or SATS_0 value and the smallest ATS_1 or SATS_1 value.

2.2.3 Expected Average Run Length (EARL) and Expected Steady-State Average Run Length (ESARL)

In an ideal situation, the size of the process shift where a quick detection is important can be specified in advance, so that the performance measures discussed in Sections 2.2.1 and 2.2.2 can be used to evaluate the efficiency of the charts. However, this situation may not be the case in real practice and is considered as too restrictive. The process shift size which is a random quantity that follows an unknown stochastic model and lack of historical data are the main reasons practitioners are unable to specify shift sizes precisely (Castagliola *et al*., 2011). In view of this, EARL and

ESARL are used to assess the performance of control charts when the shift size is unknown. The definition of EARL is given as the expected value of the ARL which is integrated over a density function of the shift size, $f_{\tau}\left(\tau\right)$ (see Chapters 3 and 4) while ESARL is the steady state EARL. The in-control EARL and ESARL are denoted as EARL_0 and ESARL_0, respectively, while the out-of-control EARL and ESARL are denoted as EARL_1 and ESARL_1, respectively. Note that $\mathrm{EARL}_0 = \mathrm{ARL}_0$ while $\mathrm{ESARL}_0 = \mathrm{SARL}_0$ when there is no process shift. A control chart is superior when it has a larger value of EARL_0 or ESARL_0 and a smaller value of EARL_1 or ESARL_1.

2.2.4 Expected Average Time to Signal (EATS) and Expected Steady-State Average Time to Signal (ESATS)

For VSI type charts, the EATS and ESATS enable practitioners to evaluate the performance of the charts, for the zero state and steady state cases, respectively, when the shift size where a quick detection is important cannot be specified in advance. The EATS is the expected value of ATS while ESATS is the expected value of SATS, where both involve integrals of the density function of the shift size over a range of possible shifts (between minimum and maximum shifts). The EATS_0 and ESATS_0 are used to denote the in-control EATS and ESATS, respectively. On the other hand, the EATS_1 and ESATS_1 are used to denote the out-of-control EATS and ESATS, respectively. For in-control situations, $\mathrm{EATS}_0 = \mathrm{ATS}_0$ and $\mathrm{ESATS}_0 = \mathrm{SATS}_0$. A VSI type chart that has a large value of EATS_0 or ESATS_0 and a small value of EATS_1 or ESATS_1 compared with the other charts, is said to be more efficient.

2.3 Univariate *S* Type Control Charts

In practice, monitoring process variance is vital to reduce the number of non-conforming items produced. Hence, univariate *S* type charts were introduced to monitor the variance of a single quality characteristic in the process. This section gives an overview of three univariate *S* type charts that are used in the comparison with the proposed chart in Chapter 3, namely, the Shewhart *S*, RS *S* and EWMA *S* charts.

2.3.1 Shewhart *S* Chart

Suppose that there exist independent and identically distributed samples, each of size *n*. Then the i^{th} (for $i = 1, 2, \ldots$) sample is represented by $\{X_{i1}, X_{i2,}, \ldots, X_{in}\}$. The Shewhart *S* chart requires the S_i statistic, which is the standard deviation of the i^{th} sample to be plotted on the chart. The computation of the i^{th} sample mean and sample standard deviation are given as follows:

$$\bar{X}_i = \frac{1}{n}\sum_{j=1}^{n} X_{ij} \tag{2.1}$$

and

$$S_i = \sqrt{\frac{1}{n-1}\sum_{j=1}^{n}\left(X_{ij} - \bar{X}_i\right)^2}, \tag{2.2}$$

respectively. Here, the successive samples are assumed to be independent and X_{ij} is assumed to follow the normal $N\left(\mu, \sigma_1^2\right)$ distribution, where $\sigma_1 = \tau\sigma_0$, σ_1 is the process standard deviation and σ_0 is the in-control standard deviation. If $\tau = 1$, the process is considered as in-control, where $\sigma_1 = \sigma_0$; while $\tau > 1$ indicates an increase in the process variance (process deterioration) and $0 < \tau < 1$ indicates a decrease in the process variance (process improvement).

Ryan (2000) suggested the use of probability limits for the Shewhart *S* chart because the distribution of the sample standard deviation, *S*, is skewed. The upper and lower probability limits of the two-sided Shewhart *S* chart are

$$\mathrm{UCL} = \sigma_0 \sqrt{\frac{\chi^2_{n-1;1-\alpha/2}}{n-1}} \tag{2.3a}$$

and

$$\mathrm{LCL} = \sigma_0 \sqrt{\frac{\chi^2_{n-1;\alpha/2}}{n-1}}\,, \tag{2.3b}$$

respectively, where $\chi^2_{v,w}$ denotes the 100*w*-th percentile of the chi-square distribution with *v* degrees of freedom and α is the false alarm rate, i.e. $\alpha = \frac{1}{\mathrm{ARL}_0}$ and ARL_0 is a pre-specified in-control average run length. Due to the skewness of the *S* distribution, it is more appropriate to use the median of the *S* distribution to represent the median line (MDL) of the Shewhart *S* chart, where

$$\mathrm{MDL} = \sigma_0 \sqrt{\frac{\chi^2_{n-1;0.5}}{n-1}}\,. \tag{2.3c}$$

Note that LCL < MDL < UCL. The *S* chart works by plotting sample standard deviations over a horizon of time and an out-of-control signal is triggered by the chart when a plotted point falls beyond the chart's limits.

It is vital to detect an increasing shift in the process variance as it indicates process deterioration while a decreasing shift in the process variance indicates process improvement which is not a concern to practitioners. For this reason, besides the two-sided *S* chart, the one-sided upward *S* chart is also discussed here. The one-sided upward *S* chart has the UCL only, which is given as

$$\mathrm{UCL} = \sigma_0 \sqrt{\frac{\chi^2_{n-1;1-\alpha}}{n-1}} \tag{2.4}$$

and the MDL which is computed based on Equation (2.3c).

The performance of the Shewhart S chart is evaluated based on the ARL and the formula to compute the ARL is given as follows:

$$\text{ARL} = \frac{1}{p}, \tag{2.5}$$

where p is the probability in which a sample standard deviation, S_i falls beyond the chart's limits. The formulae to compute p, for the two-sided and the one-sided upward S charts are given as (Rakitzis and Antzoulakos, 2016)

$$p = 1 - F_{n-1}\left(\frac{(n-1)\text{UCL}^2}{(\tau\sigma_0)^2}\right) + F_{n-1}\left(\frac{(n-1)\text{LCL}^2}{(\tau\sigma_0)^2}\right) \tag{2.6}$$

and

$$p = 1 - F_{n-1}\left(\frac{(n-1)\text{UCL}^2}{(\tau\sigma_0)^2}\right), \tag{2.7}$$

respectively, where $F_{n-1}(\,\cdot\,)$ represents the cumulative distribution function (cdf) of a chi-square random variable with $(n-1)$ degrees of freedom. It is noted that $\tau = 1$ (no process standard deviation shift) and $\tau \neq 1$ (a shift in the process standard deviation occurs) give the in-control and out-of-control conditions, respectively, in Equations (2.6) and (2.7).

2.3.2 Run Sum *S* Chart

The two-sided RS S chart was proposed by Rakitzis and Antzoulakos (2016). The chart involves partitioning the regions above and below the MDL, each into k regions. The two-sided RS S chart consists of k UCL above the MDL, namely $\text{UCL}_1 < \text{UCL}_2 < \ldots < \text{UCL}_k$ and k LCL below the MDL, namely $\text{LCL}_k < \text{LCL}_{k-1} < \ldots < \text{LCL}_1$, where $\text{UCL}_k = \infty$ and $\text{LCL}_k = 0$. Each region above the MDL, i.e. $\left[\mathrm{UCL}_{m-1}, \mathrm{UCL}_m\right)$ is assigned with a positive score C_m, for $m = 1, 2, \ldots, k$ while each region below the MDL, i.e. $\left(\mathrm{LCL}_m, \mathrm{LCL}_{m-1}\right]$ is assigned with a negative score $-C_m$, for $m = 1, 2, \ldots, k$. The scores are set to satisfy the constraint $0 \le C_1 \le \ldots \le C_{k-1} \le C_k$ and $\mathrm{UCL}_0 = \mathrm{LCL}_0 = \mathrm{MDL}$.

The two-sided RS *S* chart involves plotting the cumulative scores U_i and L_i, where the computations are given as follows:

$$U_i = \begin{cases} 0 & \text{if} \quad S_i < \mathrm{MDL} \\ U_{i-1} + C\left(S_i\right) & \text{if} \quad S_i \ge \mathrm{MDL} \end{cases} \tag{2.8a}$$

and

$$L_i = \begin{cases} 0 & \text{if} \quad S_i > \mathrm{MDL} \\ L_{i-1} + C\left(S_i\right) & \text{if} \quad S_i \le \mathrm{MDL} \end{cases}, \tag{2.8b}$$

for $i = 1, 2, \ldots$, while both $U_0\left(\ge 0\right)$ and $L_0\left(\le 0\right)$ are the initial values. It is noted that a head-start feature can be activated by setting $U_0 > 0$ and $L_0 < 0$, so that the cumulative scores are nearer to the critical value, *H*, in order to enhance the chart's sensitivity towards an early process change. For simplicity, the no head-start feature is considered in this thesis, i.e. $U_0 = 0$ and $L_0 = 0$. This is because the head-start feature will fade away in the steady state case. When the statistic S_i falls in the region $\left[\mathrm{UCL}_{m-1}, \mathrm{UCL}_m\right)$, the score function is assigned to a score C_m, i.e. $C\left(S_i\right) = C_m$. Similarly, when the statistic S_i falls in the region $\left(\mathrm{LCL}_m, \mathrm{LCL}_{m-1}\right]$, the score function is assigned to a score $-C_m$, i.e. $C\left(S_i\right) = -C_m$. The two-sided RS *S* chart contains a critical value *H*, where the chart signals an out-of-control situation when $U_i \ge H$ or $L_i \le -H$.

The MDL of the two-sided RS *S* chart can be computed using Equation (2.3c). Rakitzis and Antzoulakos (2016) suggested the control limits of the two-sided RS *S* chart as follows:

$$\mathrm{UCL}_m = \sigma_0 \sqrt{\frac{\chi^2_{n-1,\Phi(m\theta_{\mathrm{RS}\,S})}}{n-1}} \tag{2.9a}$$

and

$$\mathrm{LCL}_m = \sigma_0 \sqrt{\frac{\chi^2_{n-1,1-\Phi(m\theta_{\mathrm{RS}\,S})}}{n-1}} \tag{2.9b}$$

where $\Phi(\ \cdot\ )$ is the standard normal random variable and $\theta_{\mathrm{RS}\,S}$ is a parameter computed based on a pre-specified ARL_0 value.

Let p_m denote the probability of S_i falling in the region $\left[\mathrm{UCL}_{m-1}, \mathrm{UCL}_m\right)$ and p_{-m} denote the probability of S_i falling in the region $\left(\mathrm{LCL}_m, \mathrm{LCL}_{m-1}\right]$, for *m* = 1, 2, …, *k*. Rakitzis and Antzoulakos (2016) recommended the computations for both p_m and p_{-m} as follows:

$$p_m = F_{n-1}\left(\frac{(n-1)\,\mathrm{UCL}_m^2}{(\tau\sigma_0)^2}\right) - F_{n-1}\left(\frac{(n-1)\,\mathrm{UCL}_{m-1}^2}{(\tau\sigma_0)^2}\right) \tag{2.10a}$$

and

$$p_{-m} = F_{n-1}\left(\frac{(n-1)\,\mathrm{LCL}_{m-1}^2}{(\tau\sigma_0)^2}\right) - F_{n-1}\left(\frac{(n-1)\,\mathrm{LCL}_m^2}{(\tau\sigma_0)^2}\right), \tag{2.10b}$$

for *m* = 1, 2, …, *k*.

Similar to Section 2.3.1 for the Shewhart *S* chart, the one-sided upward RS *S* chart is included in this thesis so that a performance comparison with the proposed chart can be conducted. The one-sided upward RS *S* chart can be easily constructed by adopting the characteristics above the MDL and excluding all the characteristics below the MDL of the two-sided RS S chart. Thus, the one-sided upward RS S chart involves only the cumulative score U_i, positive scores $0 \le C_1 \le ... \le C_{k-1} \le C_k$, upper control limits $\text{UCL}_1 < \text{UCL}_2 < ... < \text{UCL}_k$ and the probability of S_i falling in the region $\left[\text{UCL}_{m-1}, \text{UCL}_m\right)$, i.e. p_m, computed based on Equation (2.10a). When the statistic S_i falls below the MDL, the cumulative score U_i is reset to zero and P(S_i < MDL) is given as (Rakitzis and Antzoulakos, 2016)

$$p_0 = F_{n-1}\left(\frac{(n-1)\text{MDL}^2}{(\tau\sigma_0)^2}\right). \tag{2.11}$$

The Markov chain technique introduced by Champ and Ridgon (1997) is used to compute the ARL of the two-sided and one-sided upward RS S charts and the formula is given as follows:

$$\text{ARL} = \mathbf{e}^T (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{1}, \tag{2.12}$$

where $\mathbf{e}^T = (1, 0, ..., 0)$ is the initial probability vector with the first entry equal to unity and zeros elsewhere, **I** is an identity matrix, **1** is a vector with all entries equal to unity and **Q** denotes the transition probability matrix (tpm) for the transient states. The computation of the transition probabilities together with the tpm using the Markov chain method is provided in Chapter 3.

2.3.3 EWMA *S* Chart

The EWMA smoothing technique is considered as an advanced method that greatly enhances a control chart’s sensitivity towards small and medium shifts. The EWMA S chart’s statistics involve a logarithmic transformation of the quality characteristic and a smoothing constant is applied to it. For instance, the one-sided upward EWMA *S* chart proposed by Crowder and Hamilton (1992) requires plotting the following statistics:

$$Z_i = \max\left\{\ln\left(\sigma_0^2\right), (1-\lambda) Z_{i-1} + \lambda Y_i\right\}, \tag{2.13}$$

where $i = 1, 2, \ldots;$ σ_0 denotes the in-control process standard deviation, $Z_0 = \ln\left(\sigma_0^2\right)$, $Y_i = \ln\left(\frac{S_i^2}{\sigma_0^2}\right)$ and λ denotes the smoothing parameter, where $0 < \lambda \le 1$. Suppose that the random samples are normally distributed with the sample size *n*. Then the sample variance S_i^2 follows a gamma distribution with the shape parameter $a = \frac{(n-1)}{2}$ and scale parameter $b = \frac{2\sigma^2}{(n-1)}$. The mean and variance of Y_i are given as follows (Crowder and Hamilton, 1992):

$$\mathrm{E}\left(Y_i\right) = \ln\left(\sigma_0^2\right) - \frac{1}{n-1} - \frac{1}{3(n-1)^2} + \frac{2}{15(n-1)^4} \tag{2.14}$$

and

$$\mathrm{Var}\left(Y_i\right) = \frac{2}{n-1} + \frac{2}{(n-1)^2} + \frac{4}{3(n-1)^3} - \frac{16}{15(n-1)^5} \tag{2.15}$$

The use of symmetrical control limits of the two-sided EWMA *S* chart were recommended by Shu and Jiang (2008) after transforming S_i^2 to $\ln\left(\frac{S_i^2}{\sigma_0^2}\right)$. Thus, the two-sided EWMA *S* chart involves plotting Z_i that can be computed using Equation (2.13) based on the following limits:

$$\mathrm{UCL} = \mathrm{E}\left(Y_i\right) + \theta_{\mathrm{EWMA}\,S}\sqrt{\frac{\lambda}{2-\lambda}\mathrm{Var}\left(Y_i\right)} \tag{2.16a}$$

and