Functorial Erkennen

Dario Dentamaro[†] and Fosco Loregian[‡]

Abstract. We outline a 'formal theory of scientific theories' rooted in the theory of profunctors; the category-theoretic asset stresses the fact that the scope of scientific knowledge is to build 'meaningful connections' (i.e. well-behaved adjunctions) between a linguistic object (a 'theoretical category' $\mathcal{T}$) and the world $\mathcal{W}$ said language ought to describe. Such a world is often unfathomable, and thus we can only resort to a smaller fragment of it in our analysis: this is the 'observational category' $\mathcal{O} \subseteq \mathcal{W}$. From this we build the category $[\mathcal{O}^{\text{op}}, \mathsf{Set}]$ of all possible displacements of observational terms $\mathcal{O}$. The self-duality of the bicategory of profunctors accounts for the fact that theoretical and observational terms can exchange their rôle without substantial changes in the resulting predictive-descriptive theory; this provides evidence for the idea that their separation is a mere linguistic convention; to every profunctor $\mathfrak{R}$ linking $\mathcal{T}$ and $\mathcal{O}$ one can associate an object $\mathcal{O} \uplus_{\mathfrak{R}} \mathcal{T}$ obtained *glueing* together the two categories and accounting for the mutual relations subsumed by $\mathfrak{R}$. Under mild assumptions, such an arrangement of functors, profunctors, and gluings provides a categorical interpretation for the 'Ramseyfication' operation, in a very explicit sense: in a scientific theory, if a computation entails a certain behaviour for the system the theory describes, then saturating its theoretical variables with actual observed terms, we obtain the entailment *in the world*.

Mathematics Subject Classification (2010). 18-03, 18B25, 18C10.

The second author was supported by the ESF funded Estonian IT Academy research measure (project 2014-2020.4.05.19-0001).

> [. . .] Le rôle essentiel [du travail de mathématicien] c'est cette transition qu'il y a entre ce quelque chose qui est écrit, qui parait incompréhensible, et les images mentales que l'on crée.
>
> A. Connes

1. Semantic conception of theories

The present work approaches a well-established problem in epistemology: how do we build representations of the world from perception? What is, if any, the relation between the two 'worlds', one depicted in our minds, and one on our fingertips? Sometimes, such a representation results in a faithful image of the perceived world (we call it 'science'); sometimes it doesn't (we call it 'superstition').

We hereby propose a sense in which science and superstition can be told apart using a mathematical theory, or even a mathematical object.

1.1. A convincing notion of theory: two dictionaries

Along the XXth century, there have been many attempts towards a formal definition of a scientific theory.

Examples are the *Wiener Kreis*' verificationist paradigm, and Neurath's theory of 'protocol statements', that gave an initial input towards the elaboration of a semantic framework for scientific theories and spurred the search for a pan-linguistic vision of philosophy of science [29].

The formal account in which –among others– Carnap [11] provided his notion of 'theory' is known in the literature as *syntactical conception of theories* or 'received view' [14, 13, 12]. Albeit the term 'semantic' is due to later developments, the field of epistemology that logical neopositivism started can legitimately be labelled a 'semantics of theories', because some of its features, if not the underlying ideology, are the same throughout the work of Carnap [10, 11], Beth [4], and Suppe [26].

Thus, a 'Wiener Kreis' theory' is understood as a structure $(F_{\mathcal{L}}, \mathcal{K})$ where $F_{\mathcal{L}}$ is a formal language, and $\mathcal{K}$ the totality of all its interpretations, or *models*.

The idea to separate further $F_{\mathcal{L}}$ into two 'vocabularies' $(\mathcal{T}, \mathcal{O})$ (intended, in modern terms, as two syntactic categories carved from two first order theories) first appears in Carnap; these are respectively the *pure* or $\mathcal{T}$*heoretical* terms, and the *applied* or $\mathcal{O}$*bservational* terms [10].

It is a commonly accepted belief –albeit rarely formalised– that scientific theories arise from some kind of tension between the theoretical and the observational world. Our aim here is to try and 'resolve the tension', acknowledging $\mathcal{T}, \mathcal{O}$ and their mutual relations as a concrete mathematical object, rooted in category theory [20, 23, 25, 18].

As elementary as it may seem, this idea seems fruitful to us: building on Carnap,

Remark. A reasonable notion of 'scientific theory' is a triple $\langle(\mathcal{T},\mathcal{O}),\mathcal{K}\rangle$ whose first two elements form the 'underlying logic' $F_{\mathcal{L}} = (\mathcal{T},\mathcal{O})$ and where $\mathcal{K}$ is a (possibly large) category of models or 'interpretations'.

This is, in fact, a familiar old idea for mathematicians, as the habit of identifying a sort of mathematical structure in a way that is independent from the cohort of its syntactic presentations, permeates classical universal algebra since the early work of Lawvere [16, 17] (see also [1, 7, 21] for applications to logic and other disciplines).

In the Carnapian –and in general the neopositivistic– account a theory can be expressed as a sentence formed by terms $\tau_1,\dots,\tau_k$ taken from both the dictionaries of $F_{\mathcal{L}}$.

In the Wiener Kreis paradigm, the formal specification of $\mathcal{O}$ is left unclear; Carnap [11] posits the existence of *correspondence rules* between $\mathcal{O}$ and $\mathcal{T}$, associating to each term o of $\mathcal{O}$, or O-term, its companion in $\mathcal{T}$, or the T-term τ derived from o. In general, Carnap holds that $\mathcal{O}\subset\mathcal{T}$, but at the same time he blurs the features of this identification of observational terms as 'types of T-terms'.

We can maintain a similar idea, just phrased in a slightly more precise way: we posit the existence of a function φ that translates O-terms into T-terms. So, a Wiener definition of a theory is a suitable set of pairs $\{\langle\tau,\varphi(\tau)\rangle \mid \tau\in\mathcal{T}\}\subseteq\mathcal{T}\times\mathcal{O}$, where $\varphi:\mathcal{T}\to\mathcal{O}$ is called a *translation function*. In this way, it is trivially true that all the terms of a theory are in the first dictionary.

Remark. A reasonable notion of scientific theory must take into account 'meaningful relations' between the observable world $\mathcal{O}$ and the theoretical world $\mathcal{T}$; the Carnapian request that there is a functional correspondence between the two is, however, too restrictive when $\mathcal{T},\mathcal{O}$ are thought as categories.

In fact, the set of pairs of a Carnap translation function φ is precisely the *graph* of φ, and not by chance: cf. our Corollary 2.7.[1]

Now, the neopositivistic current of epistemologists was the first to observe that one can build an observational version of a theory T following a procedure first outlined by Ramsey [24] and colloquially called *Ramseyfication* of a theory; the nature of this operation seems quite elusive to those approaching it: colloquially, it can be thought as the process of replacement of each observational term of a theory with a 'corresponding' theoretical term. The nature of this replacement, the syntactic domain of terms, and the sense

[1]In passing, it is worth to notice that Carnap's intuition fits even more nicely in our functorial framework: assume $\mathcal{T},\mathcal{O}$ exhibit some kind of structure, and that $i:\mathcal{O}\subseteq\mathcal{T}$ *as substructures* (e.g., assume that they are some sort of ordered sets, and that the order on $\mathcal{O}$ is induced by the inclusion); then, a *left* (resp., *right*) *translation function* φ_L (resp., φ_R), is a left (resp., right) adjoint for the inclusion $i:\mathcal{O}\hookrightarrow\mathcal{T}$. We will not expand further on this idea, but see Remark 5.10.

in which the process makes a theory $F_{\mathcal{L}}$ and its 'Ramseyfied' analogue $F_{\mathcal{L}}^{\mathrm{Ra}}$ equivalent are however quite elusive.

In the language of category theory –and especially through our profunctorial approach– instead things become clearer: under mild assumptions on a diagram

$$\begin{tikzcd} & \mathcal{W} \arrow[dl, "N_\varphi"'] \arrow[dr, "N_\psi"] & \\ {[\mathcal{T}^{\mathrm{op}}, \mathsf{Set}]} \arrow[rr, "N_{\hat{R}}"'] & & {[\mathcal{O}^{\mathrm{op}}, \mathsf{Set}]} \end{tikzcd}$$

of categories and profunctors, if a 'deduction' entails a certain interaction or a certain behaviour for the theoretical category over the observational one, then there exists a particularly well behaved natural transformation

$$\varpi : N_{\hat{R}} \Rightarrow \langle \varphi/\psi \rangle$$

filling the triangle above; in non-mathematical terms, this means that the entailment of a theoretical prediction into an observed system (i.e. a term τ of type $\mathfrak{R}(T, O)$) yields an entailment $\varphi(T) \to \psi(O)$ *in the world*. The details of this construction, that we consider the heart of the paper, are contained in Remark 5.5, Definition 5.6, and heavily rely on the terminology introduced in Sections 2 and 3.

The next subsection offers a birds-eye view of the structure of the paper.

1.2. Our contribution

The first remarks that we made in the introductory subsection motivate at least our tentative definition for a 'pre-scientific' theory: it is some sort of correspondence of categories $R : \mathcal{T} \nrightarrow \mathcal{O}$, between an observational and a theoretical category.

Another important point throughout the above discussion however is that from a neo-positivistic stance the distinction between theoretical and empirical is purely formal.

This is not due to the hypothetical nature of the former (empirical laws can be hypothetical), but to the fact that the two kinds of law contain different types of terms, as first observed in [10]. This purports a purely linguistic approach to epistemological issues, that we want to take at the extreme.

In fact, our work pushes in this direction even more: the profunctorial formulation of scientific theories deletes even more forcefully any intrinsic distinction that might be between the observational and the theoretical/-linguistic structure of a theory.

In profunctorial terms, thanks to Corollary 2.7 and standard category-theoretic arguments, there is a direct counterpart *in the mathematical model* for the vanishing of the distinction between observational and theoretical.

First of all, the bicategory defined in Definition 2.1, taking from [9], is self-dual; this means that every profunctor $\mathfrak{R} : \mathcal{T} \nrightarrow \mathcal{O}$ admits a 'mirror image' $\mathfrak{R}^{\mathrm{op}} : \mathcal{O} \rightarrow\mapsto \mathcal{T}$;[2] second, and certainly more decisive a comment towards our thesis, as outlined in Remark 5.9 a generic profunctor $\mathfrak{R} : \mathcal{T} \rightarrow\mapsto \mathcal{O}$ yields the 'collage' of the observational and theoretical categories $\mathcal{T}, \mathcal{O}$ 'glued along' $\mathfrak{R}$; in simple terms, the collage of $\mathcal{T}, \mathcal{O}$ along $\mathfrak{R}$ is a new category $\mathcal{T} \uplus_p \mathcal{O}$, fitting in a span

$$\begin{tikzcd} & \mathcal{T} \uplus_p \mathcal{O} \arrow[dl] \arrow[dr] & \\ \mathcal{T} & & \mathcal{O} \end{tikzcd} \tag{1.1}$$

having suitable fibrational properties (cf. Definition 2.4 and in particular Remark 2.8) allowing to recover the theoretical and observational terms as 'lying over' $(T, O) \in \mathcal{T} \times \mathcal{O}$.

From this perspective, it seems obvious why we require the pair $(\mathcal{T}, \mathcal{O})$ to admit a profunctor in either direction; profunctors categorify the notion of 'meaningful relation' between structured high-level systems, i.e. two syntactic categories $\mathcal{T}, \mathcal{O}$ 'modelling' the environment to which we have access.

Proposing the fundamental features of a 'general theory of scientific theories' stated in terms of profunctors is the main contribution of the present work.

We conclude this introductory section with a paragraph discussing the 'nature" of the categories $\mathcal{T}, \mathcal{O}$, while surveying on the main arguments of the paper. As we already observed in our previous work, the problem of locating the syntactic objects embodying a linguistic theory can be easily solved from an experientialist stance: the world undeniably exists, and it is a sufficiently complex structure to contain the concrete building blocks of a formal system. We derive the primitive symbols of language from a portion of the world, complex enough to offer expressive power.

This problem, and its proposed solution, reflect how the categories $\mathcal{T}, \mathcal{O}$ are built. In our model the world is a (possibly large) category $\mathcal{W}$, unfathomable and given since the beginning of time, to which we can only access through *probe maps* (functors) $\varphi : \mathcal{L} \to \mathcal{W}$ (cf. Definition 4.3) representing small 'accessible' categories construed from parts of $\mathcal{W}$ that we can experience.

The request that $\mathcal{W}$ is 'sufficiently expressive' now translates into the request that as a category $\mathcal{W}$ contains enough traces of functors like φ; this (cf. Definition 4.2) translates formally in the request that any such φ admits a *colimit* (cf. [5, Ch. 2]) in $\mathcal{W}$.

When things are put in this perspective, a few remarks are in order:

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[2] This is reminiscent of the fact that, as observed in section 2, a relation has not a privileged domain of definition; clearly, the category Cat has a nontrivial involution given by oping a category, and this renders the auto-duality slightly more visible in the case of categorified relations (i.e., profunctors).

- This perspective allows to close the circle over the problem of representation of a world $\mathcal{W}$ in terms of a portion $\mathcal{T}$ to which we have hermeneutical access, and from which we have carved a language.

  In fact, such a representation happens through 'canvas functors' $\varphi : \mathcal{L} \to \mathcal{W}$ that, thanks to the cocompleteness property of $\mathcal{W}$, extend uniquely to representation functors $[\mathcal{L}^{\text{op}}, \mathsf{Set}] \leftrightarrows \mathcal{W}$.
- On the other hand, 'the world' as a whole is unknowable: instead of $\mathcal{W}$, we can access to an observational fragment $\mathcal{O}$, from which we recover, exploiting the cocompleteness of $[\mathcal{O}^{\text{op}}, \mathsf{Set}]$, a further representation $[\mathcal{L}^{\text{op}}, \mathsf{Set}] \leftrightarrows [\mathcal{O}^{\text{op}}, \mathsf{Set}]$. In general, this is all that can be said; such a picture is already capable of determining, by elementary means, an equivalence of categories (i.e., an equivalence of models) between the observational and the theoretical *nuclei* of $[\mathcal{T}^{\text{op}}, \mathsf{Set}] \leftrightarrows [\mathcal{O}^{\text{op}}, \mathsf{Set}]$: we discuss the matter in Definition 5.3, and Remark 5.9.
- Additional assumptions on the canvas $\varphi : \mathcal{L} \to \mathcal{W}$, however, can refine our analysis: we can infer that the totality of models $[\mathcal{L}^{\text{op}}, \mathsf{Set}]$ *contains a copy* of the world $\mathcal{W}$. In this precise sense, assuming what is outlined in the definition of science© in Definition 4.3, language prevails: the unfathomable world is a full subcategory of the class of all modes in which the language of $\mathcal{T}$ can be interpreted.
- Under very mild assumptions on the arrangement of functors

$$\begin{tikzcd}
 & \mathcal{W} \arrow[dl, "N_\varphi"'] \arrow[dr, "N_\psi"] & \\
{[\mathcal{T}^{\text{op}}, \mathsf{Set}]} \arrow[rr, "N_{\hat{R}}"'] & & {[\mathcal{O}^{\text{op}}, \mathsf{Set}]}
\end{tikzcd}$$

  (cf. Proposition 3.3) where φ, ψ are two canvases, respectively on the theoretical and observational side, we can find a natural 2-cell filling the triangle; this amounts to a 'concretisation' of the canvases (see Definition 5.6 and Remark 5.7) into an implication between (a trace that) the theoretical terms (left in the world via φ) and the observational terms (to which we have experimental access) in $\mathcal{W}$. This last sentence is 'the Ramsey sentence' that the canvases carve into the world, expressed in the internal language of $\mathcal{W}$.

Structure of the paper. Section 2 and 3 outline the mathematical background we need throughout the work; the focus is not on proofs, but we refrain from delivering a terse account of the mathematical paraphernalia without any intuition. Section 4 introduces our main notions: a canvas, i.e. a functor $\varphi : \mathcal{L} \to \mathcal{W}$ representing a small category in a 'world', a big category $\mathcal{W}$; a theory, and a science, i.e. a well-behaved canvas. Sections 5 and 6 will conclude the discussion; we propose some vistas for future investigation.

2. Profunctors and the Grothendieck construction

There are two possible ways to define a *relation* R between two sets A, B:

R1) a relation R is a subset of the cartesian product $A \times B$;
R2) a relation R is a function $A \times B \to \{0, 1\}$.

This notion of 'relation between A and B' is inherently symmetric, in the sense that such R can be regarded both as a kind of 'botched' map $A \rightsquigarrow B$ or $B \rightsquigarrow A$: a relation between A and B is equally a function $B \to PA$ or $A \to PB$.

Because of this, every relation R between sets A, B gives rise to a *Galois connection*

$$^{R}(\_) : PA^{\text{op}} \leftrightarrows PB : (\_)^{R} \tag{2.1}$$

between the power-sets $PA = 2^A$ and $PB = 2^B$: the set $U \subset A$ goes to the set ^{R}U of all b such that $(a, b) \in R$ for all $a \in U$; in an exactly symmetric way, a set $V \subseteq B$ goes to the set

$$V^R = \{a \in A \mid (a, b) \in R, \forall b \in V\}. \tag{2.2}$$

Unwinding the definition, it is easy to verify that $V \subseteq {}^{R}U$ if and only if $U \subseteq V^R$, if and only if $U \times V \subseteq R$, so the adjunction rules of [5, 3.1.6m] are satisfied.

Now, using a process known as 'categorification' [2], we can replace a *two-valued* relation $R : A \times B \to \{0, 1\}$ with a *set-valued* functor $\mathcal{A}^{\text{op}} \times \mathcal{B} \to \mathsf{Set}$ between two (small) categories $\mathcal{A}, \mathcal{B}$.[1] In this perspective, the truth value for the proposition aRb='$(a, b) \in R$' is considered too poor an information about R (it is also usually wise to approach mathematics rejecting two-valued logic); because of this, instead of the mere truth value of the proposition aRb we consider the *type* of all *proofs* that aRb. This point of view will be re-introduced along section 5.

More precisely, we can give the following definition.

Definition 2.1 (Profunctor). Let $\mathcal{A}, \mathcal{B}$ be two small categories; a *profunctor* $\mathfrak{R} : \mathcal{A} \looparrowright \mathcal{B}$ is, by definition, a functor $\mathcal{A}^{\text{op}} \times \mathcal{B} \to \mathsf{Set}$; we define the *bicategory of profunctors* Prof having

P1) objects the small categories $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$;
P2) 1-cells the profunctors $\mathfrak{R} : \mathcal{A} \looparrowright \mathcal{B}$, and composition law between $\mathcal{A} \xrightarrow{\mathfrak{R}} \mathcal{B} \xrightarrow{\mathfrak{P}} \mathcal{C}$ given by the assignment[2]

$$\mathfrak{P} \diamond \mathfrak{R} : (A, C) \mapsto \int^{B} \mathfrak{R}(A, B) \times \mathfrak{P}(B, C) \tag{2.3}$$

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[1]The reason why the category $\mathcal{A}$ is twisted with an '$^{\text{op}}$' functor is that we want to bestow the hom functor $\hom_{\mathcal{A}} : \mathcal{A}^{\text{op}} \times \mathcal{A} \to \mathsf{Set}$ with the rôle of identity 'profunctor'; in the categorification perspective, hom plays the rôle of the diagonal relation $R = \Delta : A \to A \times A$. The category of sets (i.e., of discrete categories) has no nontrivial involution on objects, so in the case of sets the opping operation is hidden.

[2]See [6, 6.2.10] for the definition: representing a profunctor as a matrix of sets, this universal construction is the matrix product whose (A, C)-entry is the generalised sum $\sum_B \mathfrak{R}(A, B) \times \mathfrak{P}(B, C)$ modded out for a certain equivalence relation.

P3) the identity 1-cell is the hom functor $\hom_{\mathcal{A}} : \mathcal{A}^{\text{op}} \times \mathcal{A} \to \mathsf{Set}$;

P4) 2-cells $\mathfrak{R} \Rightarrow \mathfrak{R}'$ the natural transformations $\alpha :$
$$\begin{tikzcd} \mathcal{A}^{\text{op}} \times \mathcal{B} \arrow[r, bend left, "\mathfrak{R}"] \arrow[r, bend right, "\mathfrak{R}'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{Set} \end{tikzcd}.$$

This formalises the above intuition that $\mathfrak{R}(A, B)$ is the *type* whose terms are all proofs that $(A, B) \in \mathcal{A}^{\text{op}} \times \mathcal{B}$ are in a 'generalised relation' $\mathfrak{R}$. This intuition agrees with the fact that a profunctor between discrete categories is precisely a relation between the sets that those two categories are.

Starting from here, one can build a rich and expressive theory; for our purposes, we are contempt with a careful analysis of the analogue of R2 and (2.1) above: the latter is the scope of section 3, we now concentrate on describing a ubiquitous technical tool in category theory, called *Grothendieck construction*, suited to categorify the equivalence between R1 and R2.

Every function is a particular relation: this means that the category of sets and functions embed into the category of relations and that a relation is just a function satisfying a suitable rigidity request; something similar happens to profunctors, and we will freely employ the notation that the following definition sets up throughout the paper (cf. for example Proposition 2.9, Remark 3.4, Definition 5.6).

Definition 2.2 (The upper and lower image of a functor). Let $F : \mathcal{A} \to \mathcal{B}$ be a functor; we define

IM1) the *upper image* F^* of F is Prof to be the functor
$$\mathcal{B}(F, 1) : \mathcal{A}^{\text{op}} \times \mathcal{B} \to \mathsf{Set} : (A, B) \mapsto \mathcal{B}(FA, B) \tag{2.4}$$

IM2) the *lower image* F_* of F is Prof to be the functor
$$\mathcal{B}(1, F) : \mathcal{B}^{\text{op}} \times \mathcal{A} \to \mathsf{Set} : (B, A) \mapsto \mathcal{B}(B, FA) \tag{2.5}$$

The correspondence $F \mapsto F^*$ is a functor, covariant on 1-cells and contravariant on 2-cells; the correspondence $F \mapsto F_*$ is a functor, contravariant on 1-cells, and covariant on 2-cells.

2.1. Grothendieck construction

Each profunctor $\mathfrak{R} : \mathcal{A} \rightarrow\!\!\!\!\!\!\mapsto \mathcal{B}$ can be realised as a suitable 'fibration' $p_{\mathfrak{R}} : \mathcal{E} \to \mathcal{A}^{\text{op}} \times \mathcal{B}$, that in turn uniquely determines $\mathfrak{R}$. We now recall a few basic definitions.

Definition 2.3. Let $\mathcal{C}$ be an ordinary category, and let $W : \mathcal{C} \to \mathsf{Set}$ be a functor; the *category of elements* $\mathcal{C} \int W$ of W is the category which results from the pullback
$$\begin{tikzcd} \mathcal{C} \int W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Set}_* \arrow[d, "U"] \\ \mathcal{C} \arrow[r, "W"'] & \mathsf{Set} \end{tikzcd} \tag{2.6}$$
where $U : \mathsf{Set}_* \to \mathsf{Set}$ is the forgetful functor which sends a pointed set to its underlying set.

More explicitly, $\mathcal{C} \int W$ has objects the pairs $(C \in \mathcal{C}, u \in WC)$, and morphisms $(C, u) \to (C', v)$ those $f \in \mathcal{C}(C, C')$ such that $W(f)(u) = v$.

Definition 2.4 (Discrete fibration). A *discrete fibration* of categories is a functor $G : \mathcal{E} \to \mathcal{C}$ with the property that for every object $E \in \mathcal{E}$ and every arrow $p : C \to GE$ in $\mathcal{C}$ there is a unique $q : E' \to E$ 'over p', i.e. such that $Gq = p$.

Taking as morphisms between discrete fibrations the morphisms in $\mathsf{Cat}/\mathcal{C}$, we can define the category $\mathrm{DFib}(\mathcal{C})$ of discrete fibrations *over* $\mathcal{C}$.

Proposition 2.5. The category of elements $\mathcal{C} \int W$ of a functor $W : \mathcal{C} \to \mathsf{Set}$ comes equipped with a canonical *discrete fibration* to the domain of W, which we denote $\Sigma : \mathcal{C} \int W \to \mathcal{C}$, defined forgetting the distinguished element $u \in WC$.

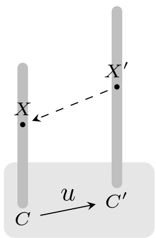
With this terminology at hand, we can consider the category of elements of a functor $F : \mathcal{C} \to \mathsf{Set}$; this sets up a functor from $\mathsf{Cat}(\mathcal{C}, \mathsf{Set})$ to the category of discrete fibrations over $\mathcal{C}$: the Grothendieck construction asserts that this is assignment sets up an equivalence of categories.

Theorem 2.6. There is an equivalence of categories

$$\mathsf{Cat}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}) \to \mathrm{DFib}(\mathcal{C}) \tag{2.7}$$

defined by the correspondence sending $F \in \mathsf{Cat}(\mathcal{C}, \mathsf{Set})$ to its *fibration of elements* $\Sigma_F : \mathcal{C} \int F \to \mathcal{C}$.

The inverse correspondence sends a discrete fibration $\Phi : \mathcal{E} \to \mathcal{C}$ to the functor whose action on objects and morphisms is depicted in the following image: an object $C \in \mathcal{C}$ goes to the fiber $\Phi^{-1}C$ in $\mathcal{E}$, that since Φ is a discrete fibration is a discrete subcategory of $\mathcal{E}$, hence a set; a morphism $u : C \to C'$ defines a function $\Phi^{-1}C' \to \Phi^{-1}C$: the object $X' \in \Phi^{-1}C'$ goes to the (unique) object X in the fiber over C, that is the domain of the arrow v such that $\Phi v = u$.



There is of course a similar correspondence for *covariant* functors; the situation is conveniently depicted by the table

name	**variance**	**condition**
fibration	$\mathcal{C} \to \mathsf{Set}$	$\begin{array}{ccc} X & \to & X' \\ \vdots & & \vdots \\ pX & \xrightarrow[f]{} & C' \end{array}$
opfibration	$\mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$	$\begin{array}{ccc} X & \to & X' \\ \vdots & & \vdots \\ C & \xrightarrow[f]{} & pX' \end{array}$

Corollary 2.7. Given a profunctor $\mathfrak{R} : \mathcal{A} \rightarrow\!\!\!\!\!+ \mathcal{B}$, regarded as a functor $R : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathsf{Set}$, we can consider the category of elements $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \int R$; this is often called the *collage* or the *graph* of R. In this case, we denote the category $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \int R$ as $\mathcal{A} \uplus_R \mathcal{B}$, to stress the intuition that R prescribes a way to glue together two categories $\mathcal{A}, \mathcal{B}$ specifying a set of 'fake' arrows $R(A, B)$ that consistently interact with the arrows in $\mathcal{A}, \mathcal{B}$ (compare this with Definition 5.1 below).

Remark 2.8. The above definition deserves to be expanded a little more: from Definition 2.3 we get that the category $\mathcal{A} \uplus_R \mathcal{B}$ results as the category whose objects are those of the disjoint union $\mathcal{A}_o \sqcup \mathcal{B}_o$, and where the hom-set $\mathcal{A} \uplus_R \mathcal{B}(X, Y)$ is equal to

- C1) $\mathcal{A}(A, A')$ if $(X, Y) = (A, A')$ is a pair of objects in $\mathcal{A}$;
- C2) $\mathcal{B}(B, B')$ if $(X, Y) = (B, B')$ is a pair of objects in $\mathcal{B}$;
- C3) $R(A, B)$ if $X = A$ is an object of $\mathcal{A}$, and $Y = B$ is an object of $\mathcal{B}$;
- C4) empty in every other case.

From this definition, it is evident that every profunctor $\mathfrak{R} : \mathcal{A} \rightarrow\!\!\!\!\!+ \mathcal{B}$ gives rise via its fibration of elements to a span of categories

$$\begin{tikzcd} & \mathcal{A} \uplus_{\mathfrak{R}} \mathcal{B} \arrow[dl] \arrow[dr] & \\ \mathcal{A}^{\mathrm{op}} & & \mathcal{B} \end{tikzcd} \tag{2.8}$$

Thus, we have obtained a concrete model for a category that realises the generalised relation between $\mathcal{A}, \mathcal{B}$; the structure $\mathcal{A} \uplus_R \mathcal{B}$ is 'carved' from $\mathcal{A}, \mathcal{B}$ separately, starting from (semi-)free relations witnessing the fact that $\mathfrak{R}$ connects $\mathcal{A}, \mathcal{B}$ in a weak way. For example, if $\mathfrak{R} : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathsf{Set}$ is the empty functor, then $\mathcal{A} \uplus_R \mathcal{B}$ is just he disjoint union of $\mathcal{A}, \mathcal{B}$; and if $\mathfrak{R}$ is the functor constant at the singleton set, then $\mathcal{A} \uplus_R \mathcal{B}$ is the *join* of $\mathcal{A}, \mathcal{B}$, i.e. the category $\mathcal{A} \coprod \mathcal{B}$ where exactly a single new morphism is added between each and every object of $\mathcal{A}$ and of $\mathcal{B}$ (but not in the opposite direction).

As a final remark, we observe that the extremely rich features of the Grothendieck construction can be at least partly explained resorting to a dual construction for $\mathcal{C} \int F$: more in detail,

Proposition 2.9. The collage construction of Corollary 2.7 enjoys the following universal property: the category $\mathcal{A} \uplus_R \mathcal{B}$ fits into a cospan

$$\mathcal{A} \xrightarrow{J} \mathcal{A} \uplus_R \mathcal{B} \xleftarrow{K} \mathcal{B} \tag{2.9}$$

where both functors J, K are the obvious embeddings, and there exists a canonical natural transformation $\gamma : K_* \diamond \mathfrak{R} \Rightarrow J_*$ which is initial among all these: this means that given any other arrangement of profunctors

$$
\begin{tikzcd}
\mathcal{A} \arrow[dr, "J_*"] \arrow[drr, bend left, "\mathfrak{P}"] \arrow[dd, "\mathfrak{R}"'] & & \\
\gamma\Uparrow & \mathcal{A} \uplus_R \mathcal{B} \arrow[r, dotted, "\mathfrak{U}"] & \mathcal{C} \\
\mathcal{B} \arrow[ur, "K_*"'] \arrow[urr, bend right, "\mathfrak{Q}"'] & &
\end{tikzcd}
\tag{2.10}
$$

like in this diagram of solid arrows, there exists a unique profunctor $\mathfrak{U} : \mathcal{A} \uplus_R \mathcal{B} \rightarrow\!\!\!\!\!\!\mapsto \mathcal{C}$ such that $\mathfrak{U} \diamond J_* = \mathfrak{P}$, $\mathfrak{U} \diamond K_* = \mathfrak{Q}$, and $\mathfrak{U} * \gamma = \alpha$.

3. Nerve and realisations

We start by recalling the universal property of the category of presheaves over $\mathcal{C}$: let $\mathcal{C}$ be a small category, $\mathcal{W}$ a cocomplete category; then, precomposition with the Yoneda embedding $y_\mathcal{C} : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \mathsf{Set}]$ determines a functor

$$
\mathsf{Cat}([\mathcal{C}^{\text{op}}, \mathsf{Set}], \mathcal{W}) \xrightarrow{- \circ y_\mathcal{C}} \mathsf{Cat}(\mathcal{C}, \mathcal{W}), \tag{3.1}
$$

that restricts a functor $G : [\mathcal{C}^{\text{op}}, \mathsf{Set}] \to \mathcal{W}$ to act only on representable functors, confused with objects of $\mathcal{C}$, thanks to the fact that $y_\mathcal{C}$ is fully faithful. We then have that

Theorem 3.1.

YE1) The universal property of the category $[\mathcal{C}^{\text{op}}, \mathsf{Set}]$ amounts to the existence of a left adjoint $\mathrm{Lan}_{y_\mathcal{C}}$ to precomposition, that has invertible unit (so, the left adjoint is fully faithful).

This means that $\mathsf{Cat}(\mathcal{C}, \mathcal{W})$ is a full subcategory of $\mathsf{Cat}([\mathcal{C}^{\text{op}}, \mathsf{Set}], \mathcal{W})$. Moreover

YI1) The essential image of $\mathrm{Lan}_{y_\mathcal{C}}$ consists of those $F : [\mathcal{C}^{\text{op}}, \mathsf{Set}] \to \mathcal{W}$ that preserve all colimits.

YI2) If $\mathcal{W} = [\mathcal{E}^{\text{op}}, \mathsf{Set}]$, this essential image is equivalent to the subcategory of left adjoints $F : [\mathcal{C}^{\text{op}}, \mathsf{Set}] \to [\mathcal{E}^{\text{op}}, \mathsf{Set}]$.

As a consequence of this,

Definition 3.2 (Nerve and realisation contexts). Any functor $F : \mathcal{C} \to \mathcal{W}$ from a small category $\mathcal{C}$ to a (locally small) *cocomplete* category $\mathcal{W}$ is called a *nerve-realisation context* (a NR *context* for short).

Given a NR context F, we can prove the following result:

Proposition 3.3 (Nerve-realisation paradigm). The left Kan extension of F along the Yoneda embedding $y_\mathcal{C} : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \mathsf{Set}]$, i.e. the functor

$$
L_F = \mathrm{Lan}_{y_\mathcal{C}} F : [\mathcal{C}^{\text{op}}, \mathsf{Set}] \to \mathcal{W} \tag{3.2}
$$

is a left adjoint, $L_F \dashv N_F$. L_F is called the *$\mathcal{W}$-realisation functor* or the *Yoneda extension* of F, and its right adjoint the *$\mathcal{W}$-coherent nerve.*

Proof. From a straightforward computation, it follows that if we define $N_F(D)$ to be $C \mapsto \mathcal{W}(FC, D)$, this last set becomes canonically isomorphic to $[\mathcal{C}^{\text{op}}, \mathsf{Set}](P, N_F(D))$. We can thus denote $\mathcal{W}(F, 1)$ the functor $N_F : D \mapsto \lambda C.\mathcal{W}(FC, D)$. □

Now, let's review the way in which a profunctorial analogue of (2.1) can be obtained: Proposition 3.3 yields that a functor

$$\mathfrak{R} : \mathcal{A}^{\text{op}} \times \mathcal{B} \to \mathsf{Set} \tag{3.3}$$

whose mate under the adjunction $\mathsf{Cat}(\mathcal{A}^{\text{op}} \times \mathcal{B}, \mathsf{Set}) \cong \mathsf{Cat}(\mathcal{B}, [\mathcal{A}^{\text{op}}, \mathsf{Set}])$ is a functor

$$\hat{R} : \mathcal{B} \to \mathsf{Cat}(\mathcal{A}^{\text{op}}, \mathsf{Set}) \tag{3.4}$$

determines a NR paradigm, and thus gives rise to a pair of adjoint functor

$$\mathrm{Lan}_{y_\mathcal{B}} \hat{R} : \mathsf{Cat}(\mathcal{B}^{\text{op}}, \mathsf{Set}) \leftrightarrows \mathsf{Cat}(\mathcal{A}^{\text{op}}, \mathsf{Set}) : [\mathcal{A}^{\text{op}}, \mathsf{Set}](\hat{R}, 1). \tag{3.5}$$

Remark 3.4. Note that, given a functor $F : \mathcal{A} \to \mathcal{B}$, the functor $N_F = \mathcal{B}(F, 1)$ coincides with the lower image of F into Prof, described in Definition 2.2.

We have just laid down all the terminology needed to prove that

Proposition 3.5. There is an equivalence of categories between $\mathsf{Prof}(\mathcal{A}, \mathcal{B})$ and the category of colimit preserving functors $\mathsf{Cat}(\mathcal{B}^{\text{op}}, \mathsf{Set}) \to \mathsf{Cat}(\mathcal{A}^{\text{op}}, \mathsf{Set})$.

4. Theories and models

In this section, we exploit the terminology established before.

Definition 4.1 (Theory). A *theory* $\mathcal{L}$ is the syntactic category $\mathcal{T}_L$ (cf. [15, II.11]) of a type theory L.

The reader interested in how the construction of $\mathcal{T}_L$ goes can take from [15] as a standard reference or [1] for a shorter survey, in the simple case L admits product and function types.

Definition 4.2 (World, Yuggoth). A *world* is a large category $\mathcal{W}$; a *Yuggoth*[1] is a world that, as a category, admits all small colimits.

Definition 4.3 (Canvas, science). Given a theory $\mathcal{L}$ and a world $\mathcal{W}$, a $\mathcal{L}$-*canvas* of $\mathcal{W}$ is a functor

$$\mathcal{L} \xrightarrow{\varphi} \mathcal{W}. \tag{4.1}$$

A canvas $\varphi : \mathcal{L} \to \mathcal{W}$ is a *science*© if φ is a dense functor.

Remark 4.4. The NR paradigm exposed in Definition 3.2 now entails that given a canvas $\varphi : \mathcal{L} \to \mathcal{W}$

[1] *Yuggoth* (also *Iukkoth*, or Yuggoth in Chtuvian language) is an enormous trans-Neptunian planet whose orbit is perpendicular to the ecliptic plane of the solar system. A Yuggoth is a world so big to inspire a sense of unfathomable awe.

- If $\mathcal{W}$ is a world, we obtain a *representation* functor
$$\mathcal{W} \longrightarrow [\mathcal{L}^{\mathrm{op}}, \mathsf{Set}]; \tag{4.2}$$
this means: given a canvas φ of the world, the latter leaves an image on the canvas.
- If in addition $\mathcal{W}$ is a Yuggoth, we obtain a NR-adjunction
$$\mathcal{W} \rightleftarrows [\mathcal{L}^{\mathrm{op}}, \mathsf{Set}]; \tag{4.3}$$
this has to be interpreted as: if $\mathcal{W}$ is sufficiently expressive, then models of the theory that explains $\mathcal{W}$ through φ can be used to acquire a two-way knowledge. Phenomena have a theoretical counterpart in $[\mathcal{L}^{\mathrm{op}}, \mathsf{Set}]$ via the nerve; theoretical objects strive to describe phenomena via their realisation.
- If an $\mathcal{L}$-canvas $\varphi : \mathcal{L} \to \mathcal{W}$ is a science$^{©}$, 'the world' is a full subcategory of the class of all modes in which 'language' can create interpretation.

Remark 4.5. The terminology is chosen to inspire the following idea in the reader: science strives to define *theories* that allow for the creation of world representations; said representations are descriptive when there is dialectic opposition between world and models; when such representation is faithful, we have reduced 'the world' to a piece of the models created to represent it.

The tongue-in-cheek here is: a science in the usual sense of the world can never attain the status of a science$^{©}$, if not potentially; this because all of its attempts at describing the world are partial. But if the chosen language is powerful enough, even a small fragment of it can result in a 'free linguistic category' (precisely $[\mathcal{L}^{\mathrm{op}}, \mathsf{Set}]$) that is large enough to encompass $\mathcal{W}$ completely.

In this perspective, attempts to generate scientific knowledge are the attempts of

- recognizing the world $\mathcal{W}$ as a sufficiently expressive object for it to contain phenomena and information;
- carve a language L, if necessary from a small subset of $\mathcal{C}$, that is sufficiently 'compact', but also sufficiently expressive for its syntactic category to admit a representation into the world;
- obtaining an *adjunction* between $\mathcal{W}$ and models of the worlds obtained as models of the syntactic theory $\mathcal{L}$; this is meant to generate models starting from observed phenomena, and to predict new phenomena starting from models;
- obtaining that 'language is a dense subset of the world', by this meaning that the adjunction outlined above is sufficiently well-behaved to describe the world as a fragment of the semantic interpretations obtained from $\mathcal{L}$.

It is evident that there is a tension between two opposite feature that $\mathcal{L}$ must exhibit; it has to be not too large to remain tractable, but on the other hand, it must be large enough to be able to speak about 'everything' it aims to describe.

Regarding our definition of science©, we can't help but admit we had the following definition in mind [3, 2.1], regarded with a pair of category theorist' goggles:

Definition ([3, 2.1]). A *scientific theory* $\mathcal{T}$ consists of a formal structure F and a class of interpretations M_i, shortly denoted as $\mathcal{T} = \langle F, M_i \mid i \in I\rangle$. The structure F consists of its won right of

- a language $\mathcal{L}$, in which it is possible to formulate propositions. If $\mathcal{L}$ is fully formalised, it will consist of a finite set of symbols, and a finite set of rules to determine which expressions are well-formed. This is commonly called *technical language*;
- A set A of 'axioms' or 'postulates' in $\mathcal{L}^\star$;
- A *logical apparatus* R, whose elements are rules of inference and logical axioms, allowing to prove propositions.

The language of category theory allows for a refined rephrasing of the previous definition: we say that a $\mathcal{S}$*-scientific theory* is the following arrangement of data:

ST1) a formal language $\mathcal{L}$;
ST2) the syntactic category $T_{\mathcal{L}}$, obtained as in [15, II.11];
ST3) the category of functors $[T_{\mathcal{C}}, \mathcal{S}]$, whose codomain is a Yuggoth.

More than often, our theories will be Set-scientific: in such case we just omit the specification of the semantic Yuggoth, and call them *scientific theories*.

Since the category $[T_{\mathcal{C}}, \mathsf{Set}]$ determines $\mathcal{L}$ and $T_{\mathcal{L}}$ completely, up to Cauchy-completion [8], we can see that the triple $(\mathcal{L}, T_{\mathcal{L}}, [T_{\mathcal{L}}, \mathsf{Set}])$ can uniquely be recovered from its model category $[T_{\mathcal{C}}, \mathsf{Set}]$. We thus comply to the additional abuse of notation to call 'scientific theory' the category $[T_{\mathcal{L}}, \mathsf{Set}]$ for some $T_{\mathcal{L}}$.

So, a 'coherent correspondence linking expressions of $\mathcal{F}$ with semantic expressions' boils down to a functor; this is compatible with [3, 2.1], and in fact an improvement (the mass of results in category theory become readily available to speak about –scientific– theories; not to mention that the concept of 'formal structure' is never rigorously defined throughout [3]).

Let us consider two categories $\mathcal{O}, \mathcal{T}$, respectively the *observational* and the *theoretical*. Even though their origin is never examined further, it is fruitful to think that $\mathcal{O}, \mathcal{T} \subseteq \mathcal{W}$, i.e. that they are 'carved' from the world, building respectively on the tangible experience (for $\mathcal{O}$) and a linguistic structure (for $\mathcal{L}$).

If $\mathcal{W}$ is a Yuggoth each pair of canvases

$$\mathcal{O} \xrightarrow{\psi} \mathcal{W} \xleftarrow{\varphi} \mathcal{T} \tag{4.4}$$

gives rise, according to (4.2), to representations

$$\begin{tikzcd}[\mathcal{O}^{\mathrm{op}}, \mathsf{Set}] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{W} \arrow[l, shift left, "N_\psi"] \arrow[r, shift right, "N_\varphi"'] \arrow[r, phantom, "\perp"] & {[\mathcal{T}^{\mathrm{op}}, \mathsf{Set}]} \arrow[l, shift right]\end{tikzcd} \tag{4.5}$$

The leftmost category is the category we have experimental access, starting from the fragment $\mathcal{O} \subseteq \mathcal{W}$ we can observe. The rightmost category is the category of symbols we can speak of, trying to reproduce the observed behaviour.

Definition 4.6. We refine the terminology introduced above to speak of a *theoretical* (resp., a *observational*) *science*, assuming that φ (resp., ψ) is a science©.

Assuming that $\varphi : \mathcal{T} \to \mathcal{W}$ is a theoretical science©, now, the representation functor $\mathcal{W} \to [\mathcal{T}^{\mathrm{op}}, \mathsf{Set}]$ above acquires a left adjoint.

5. The tension between observational and theoretical

When working with categorified relations, it is unnatural and somewhat restrictive to take into account a two-element set for the possible values a proposition(al function) '$(a, b) \in R$' can assume; instead we would like to consider an entire *space* of such values, or rather a type of proofs that $(a, b) \in R$ is true. Again, this idea is best appreciated when thinking that the same proposition

```
(n : Nat) -> (m : Nat) ->  n + m = m + n
```

when encoded in any (sufficiently strongly-typed) DSL, can be interpreted as either the *proposition* 'given n and m natural numbers, their sum is a commutative operation' or as the *type* `n + m` $\equiv$ `m + n` whose elements are the proofs that $n + m$ is in fact equal to $m + n$.

This intuition is based on the well-known proportion

$$\text{truth values} : \text{proposition} = \text{section} : \text{presheaf}$$

inspired by the 'proposition as types' paradigm. In simple terms, categorifying a proposition $P : X \to \{0, 1\}$ that can or cannot hold for an element x of a set X, we shall marry the constructive church and say that there is an entire *type* PC, image of an object $C \in \mathcal{C}$ under a functor $P : \mathcal{C} \to \mathsf{Set}$, whose *terms* are the *proofs* that PC holds true. This is nothing but the propositions-as-types philosophy, in (not so much) disguise: [27, 28, 22]

The important point for us is that the dialectical tension between observational and theoretical can be faithfully represented through profunctor theory; one can think of propositional functions as relations $(x, y) \in R$ if and only if the pair x, y renders φ true; we use this idea, suitably adapted to our purpose and categorified. This very natural extension of propositional calculus, pushed to its limit, yields the following reformulation of the 'tension between observational and theoretical'.

Definition 5.1. Let $\mathcal{T}, \mathcal{O}$ be two small categories, dubbed respectively the *theoretical* and the *observational* settings. A *Ramsey map* is merely a profunctor

$$\mathfrak{K} : \mathcal{T}^{\mathrm{op}} \nrightarrow \mathcal{O} \tag{5.1}$$

or, spelled out completely, a functor $\mathfrak{K} : \mathcal{T} \times \mathcal{O} \to \mathsf{Set}$.

Example 5.2. Every functor $F : \mathcal{A} \to \mathcal{B}$ gives rise to a profunctor $F_* := \mathcal{B}(1, F) : \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Set}$ and a profunctor $F^* := \mathcal{B}(F, 1) : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathsf{Set}$ as in Proposition 3.3; the two functors are mutually adjoint, $F^* \dashv F_*$, see [6, 6.2]. This yield an example of what we call *representable* Ramsey maps.

Definition 5.3 (Observational and theoretical nucleus). Let $\mathfrak{R} : \mathcal{T}^{\mathrm{op}} \times \mathcal{O} \to \mathsf{Set}$ be a Ramsey map, and $\hat{R} : \mathcal{O} \to [\mathcal{T}^{\mathrm{op}}, \mathsf{Set}]$ the associated canvas. Let

$$\mathrm{Lan}_{y_{\mathcal{O}}} \hat{R} : [\mathcal{O}^{\mathrm{op}}, \mathsf{Set}] \leftrightarrows [\mathcal{T}^{\mathrm{op}}, \mathsf{Set}] : N_{\hat{R}} \tag{5.2}$$

be the adjunction between presheaf categories determined by virtue of Proposition 3.5. Let us consider the equivalence of categories between the fix-points of the monad $T = N_{\hat{R}} \circ \mathrm{Lan}_{y_{\mathcal{O}}} \hat{R}$ and the comonad $S = \mathrm{Lan}_{y_{\mathcal{O}}} \hat{R} \circ N_{\hat{R}}$.

This is the equivalence between the *observational nucleus* $Fix(T) \subseteq [\mathcal{O}^{\mathrm{op}}, \mathsf{Set}]$ and the *theoretical nucleus* $Fix(S) \subseteq [\mathcal{T}^{\mathrm{op}}, \mathsf{Set}]$.

Remark 5.4. Observational nucleus and theoretical nucleus always form equivalent categories; the tension in creating a satisfying image of reality as it is observed oscillates between the desire to enlarge as much as possible the subcategory of $[\mathcal{O}^{\mathrm{op}}, \mathsf{Set}]$ with which our theoretical model is equivalent, where we can have access to $\mathcal{T}, [\mathcal{T}^{\mathrm{op}}, \mathsf{Set}]$ only.

The following remark shows how new structure comes 'almost for free' when things are interpreted this way.

Assume $\varphi : \mathcal{T} \to \mathcal{W}$ and $\psi : \mathcal{O} \to \mathcal{W}$ are canvases, $\mathfrak{R}$ is a Ramsey map, and $\mathrm{Lan}_{y_{\mathcal{O}}} \hat{R}$ the functor corresponding to $\mathfrak{R}$ under the construction in (5.2); in this notation, we can state a tight condition of compatibility between the theory identified by (φ, ψ) and the Ramsey map $\mathfrak{R}$. We employ freely the presence of adjunction

$$L_\varphi \dashv N_\varphi \qquad L_\psi \dashv N_\psi \qquad L_{\hat{R}} \dashv N_{\hat{R}}. \tag{5.3}$$

Remark 5.5 (Inducing an hermeneutics). Consider the diagram

$$\begin{tikzcd} & \mathcal{W} \arrow[dl, "N_\varphi"'] \arrow[dr, "N_\psi"] & \\ {[\mathcal{T}^{\mathrm{op}}, \mathsf{Set}]} \arrow[rr, shift right, "N_{\hat{R}}"'] & & {[\mathcal{O}^{\mathrm{op}}, \mathsf{Set}]} \arrow[ll, shift right, "L_{\hat{R}}"'] \end{tikzcd} \tag{5.4}$$

given by the theoretical and observational nerves, plus the Ramsey adjunction mentioned above.

We seek sufficient conditions in order for (5.4) to be filled by a suitable natural transformation $\omega : N_{\hat{\mathfrak{R}}} \circ N_\varphi \Rightarrow N_\psi$: such a 2-cell will force a tameness property on the system described by the two canvases (φ, ψ): this is made precise by the following

Definition 5.6 (Fundamental cell, Hermeneutics). In a display of categories like (4.4) we say that

- A *fundamental cell* is a natural transformation $\omega : N_{\hat{\mathfrak{R}}} \circ N_\varphi \Rightarrow N_\psi$;

- we say that in the world $\mathcal{W}$ *hermeneutics is possible* if the right extension $\langle\varphi/\psi\rangle := \mathrm{Ran}_{N_\varphi} N_\psi$ exists *as a functor* (note that it always exists as a profunctor, but this might not be representable).

If hermeneutics is possible in $\mathcal{W}$, and $R : \mathcal{T} \not\to \mathcal{O}$ is a Ramsey map, any fundamental cell induces a natural transformation

$$\varpi : N_{\hat{R}} \Rightarrow \langle\varphi/\psi\rangle \tag{5.5}$$

obtained exploiting the universal property of $\langle\varphi/\psi\rangle$.

If the right extension is representable in the sense above, this amounts to a higher type map (in the sense of the internal language of a closed category) comparing 'generalised formulas' of kind $\mathfrak{R} \Rightarrow \mathcal{W}(\varphi, \psi)$.

Remark 5.7. If we follow the customary practice to identify a morphism of a category as an entailment between sequents in a deductive system, it is easy to see that the condition that the possibility of hermeneutics captures is that we can embody sequents of the form $[\![T \vdash O]\!] \in \mathfrak{R}(T, O)$ in the internal language of $\mathcal{W}$; more precisely, if we think of $\mathfrak{R}(T, O)$ as the type of all proofs that some theoretical terms describe an observational phenomenon, then the map ϖ above can be represented as the higher order entailment relation between $[\![T \vdash O]\!]$ and the entailment $\varphi(T) \to \psi(O)$ *valid in* $\mathcal{W}$:

$$\frac{\varpi_{T,O} : \mathfrak{R}(T, O) \to \mathcal{W}(\varphi(T), \psi(O))}{[\![T \vdash O]\!] \Vdash (\Phi[T] \vdash \Psi[O])} \tag{5.6}$$

where $\Phi[T]$ is a shorthand for $\varphi[\vec{x}/T]$, the context of premises saturated by the theoretical terms, and same for $\Psi[O]$, the context of deductions saturated by the observational terms.

All in all, the map $\varpi_{T,O}$ exhibits a *witness* of the expressibility of the entailment '$[\![T \vdash o]\!]$' in the world $\mathcal{W}$, through the Ramsey map.

More is true: the presence of a fundamental cell means that we can find a way to assert that the entailment $T \to O$ is actually embodied in the world by an entailment $\varphi(T) \to \psi(O)$ in the internal language of $\mathcal{W}$.

If after a computation we find that a cannonball will follow a parabolic trajectory, the cannonball fired in the actual world is to be found at the point we predicted, even though there is no such thing as 'a parabola' in the physical world. (Parabolas, and for that matter, all geometric figures, arise as abstractions of a bundle of recurrent perceptions)

Such assumptions imply that "hermeneutics is possible", in the very sense of the word: we can interpret linguistic facts about the world, and derivations in the former system correspond to variations in the latter.

Remark 5.8. There is nothing in their mere syntactical presentation allowing to tell apart the observational and the theoretical category; this can be justified with the fact that the bicategory Prof of Definition 2.1 is endowed with a canonical self-involution, exchanging the rôle of domain and codomain of 1-cells, and thus of the theoretical and observational category $\mathcal{T}, \mathcal{O}$.

This is perhaps of some help in solving the conundrum posed by the existence of ‘fictional objects’. Sherlock Holmes clearly is the object of a theoretical category. Gandhi is the object of an observational category. But as linguistic objects they can’t be told apart completely; they can be at most separated by a profunctor embedding the former in a realistic counterpart of a fictitious model (that is, for example, the Reichenbach falls), and representing the latter as part of a fictional model (for example, as part of a movie directed by R. Attenborough).

We can surely discuss what is the ontological status of each such object. If it is clear that in the universe of Conan Doyle, an individual named Sherlock Holmes lives at 221b Baker Street, it is also clear that it ‘projects’ its existence in the actual world $\mathcal{W}^{@}$; undoubtedly there are relations between Conan Doyle’s Sherlock Holmes and its shadow in $\mathcal{W}^{@}$; it is possible to rephrase their relations in terms of the syntactic categories presenting/describing the two universes, in the way that we have sketched. For a related topic, see the notion of metakosmial accessibility between worlds or modal semantics of the narrative objects [19]; as interesting as the topic may seem, we refrain to go further in this analysis, leaving the stage open for future discussion.

The question deserves a deeper analysis: Attenborough’s Gandhi isn’t exactly an object inside $\mathcal{W}^{@}$, but instead of an accessible sub-world $\mathcal{U}_{\mathrm{G}} \subseteq \mathcal{W}^{@}$ that works as a canvas; it might be that many well-tested approaches to the theory of modal relations might become more streamlined when expressed in our language: fictional worlds are just *particular ways* to build canvases and representations thereof.

Remark 5.9. The clearest possible sense in which the profunctorial approach ‘resolves’ the tension between observational and theoretical is that the Grothendieck construction associated to a profunctor $\mathfrak{R} : \mathcal{T} \looparrowright \mathcal{O}$ yields a category where the two ‘worlds’, one carved from perception, and the other concocted from language, live harmoniously together. All in all said tension is just an incarnation of the tension between speakable and unspeakable: given a Ramsey map $\mathfrak{R} : \mathcal{T} \looparrowright \mathcal{O}$, the equivalence between its theoretical and observational nuclei is an equivalence between the speakable (a subclass of $[\mathcal{T}^{\mathrm{op}}, \mathsf{Set}]$), with the observable (a subclass of $[\mathcal{O}^{\mathrm{op}}, \mathsf{Set}]$); what lies outside this equivalence in the latter category is observable but ‘unspeakable’ in the strongest possible sense.

Remark 5.10 (Ramseyfication and translation functors). Assume that there exists an adjunction

$$F : \mathcal{O} \rightleftarrows \mathcal{T} : G \tag{5.7}$$

between the theoretical and the observable categories. Following Carnap, we might assume that $F : \mathcal{O} \hookrightarrow \mathcal{T}$, and thus G is a right translation functor for $(\mathcal{T}, \mathcal{O})$.

In these assumptions, given a Ramsey map $\mathfrak{K} : \mathcal{T} \looparrowright \mathcal{O}$ the function term

$$(O, X) \mapsto \mathfrak{K}(O, X) \tag{5.8}$$

can be pre-composed with F obtaining

$$(O, O') \mapsto \mathfrak{K}(FO, O'). \tag{5.9}$$

We say that a translation adjunction (F, G) is '$\mathfrak{K}$-admissible' relative to a Ramsey map $\mathfrak{R}$ (denoted $F \dashv_{\mathfrak{K}} G$) when there is a natural isomorphism $\mathfrak{K}(F, 1) \cong \mathfrak{K}(1, G)$.

The property of $\mathfrak{K}$-admissibility for a pair of functors is in general difficult to assess; nevertheless, there are interesting properties for the relation $F \dashv_{\mathfrak{K}} G$: for example

Theorem 5.11. Let $F : \mathcal{A} \leftrightarrows \mathcal{B} : G$ be a pair of functors in opposite directions; let $\mathfrak{K} : \mathcal{B} \looparrowright \mathcal{A}$ be a profunctor; if $F \dashv_{\mathfrak{K}} G$, then there is a 'genuine' adjunction

$$F^e : \mathcal{A} \uplus_{\mathfrak{K}} \mathcal{B} \leftrightarrows \mathcal{A} \uplus_{\mathfrak{K}} \mathcal{B} : G^e \tag{5.10}$$

'extended' to the category of elements of $\mathfrak{K}$.

6. Towards a universal notion of theory

This concluding section wraps up the various topics touched along the paper.

As we have seen in Remark 5.9, profunctor theory is just a mathematization of the well-known tenet that epistemology is a relational theory: scientific theories are but well-behaved adjunctions between the part of the world that we want to model (this part doesn't have to be physical), the part of the world to which we have experimental access, and the linguistic paraphernalia that we use to represent the latter in terms of the former.

Theories can't be told apart in terms of their objects of study; instead, they can be classified in terms of the web of relations that they entertain together with other theories/categories.

This entails that

- There is no substantial difference between the syntactic categories $\mathcal{T}, \mathcal{O} \subset \mathcal{W}$, i.e. between observative and theoretical terms. Far from being a step back towards an efficient representation of reality, this elegantly gets rid of the early gawky attempts towards a 'naturalisation of epistemology', originally thought to even happen *inside syntax*.

  No theory can exit language; this does not mean that a theory isn't telling something about the world: instead, theories –and metatheories about the world– are linguistic objects above all else. How this linguistic practice unravels, on the other hand, is too loose to be functional; it is, instead, relational.
- Being able to exchange the rôles of $\mathcal{T}, \mathcal{O}$ is reflected in the model in the property of every profunctor (i.e. Ramsey map) $\mathfrak{R} : \mathcal{T} \looparrowright \mathcal{O}$ to be 'swapped' into a Ramsey map $\mathfrak{R}^{\mathrm{op}} : \mathcal{O} \looparrowright \mathcal{T}$. The observational and theoretical categories bear this name as a result of nothing but arbitrary labelling. This gives way to all sorts of fruitful interpretations: it becomes possible to label as 'sciences' sufficiently expressive descriptions

of possible worlds (few would object that the complicated hierarchy of sub-worlds in which Eä is divided, or the narration by which it became the world as we know it, form a 'science'), or the –strictly speaking– unobservable phenomena that occur in Physics as well as in theology.

Being 'scientific' is thus not a property of the object we want to describe; instead, 'scientificity' is a measure of the faithfulness of described phenomena in a 'world' $\mathcal{W}$, and of the ability of descriptions to cast predictions on the behaviour of the system. This is akin to scientific practice: if quantum mechanics gave more correct predictions about the world accepting that uncertainty is induced by an *ogbanje*, physicists would study Igbo mythology instead of functional analysis.

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Dario Dentamaro†
†Università degli Studi di Firenze,
Dipartimento di Matematica e Informatica
e-mail: dario.dentamaro@stud.unifi.it

Fosco Loregian‡
‡Tallinn University of Technology,
Institute of Cybernetics, Akadeemia tee 15/2,
12618 Tallinn, Estonia
e-mail: fosco.loregian@taltech.ee