Exhaustive generation of atomic combinatorial differential operators

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Outline

History lesson

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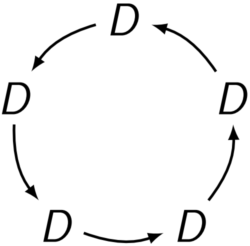
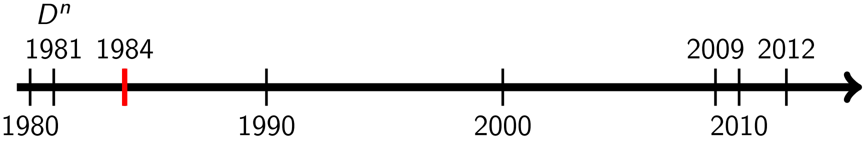
$$D \longrightarrow D \longrightarrow \cdots \longrightarrow D$$

$$D^n$$

History lesson



$G(D)$



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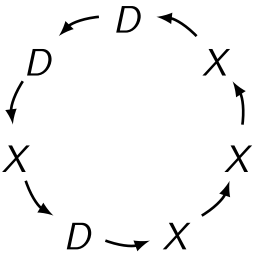
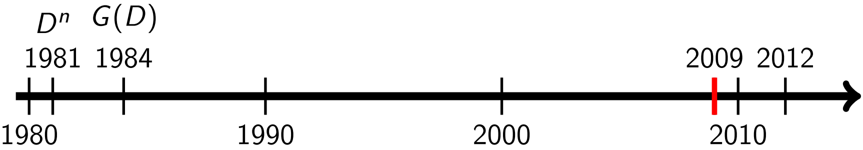


$\Omega(X, D)$



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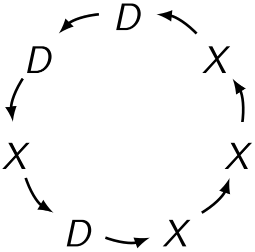

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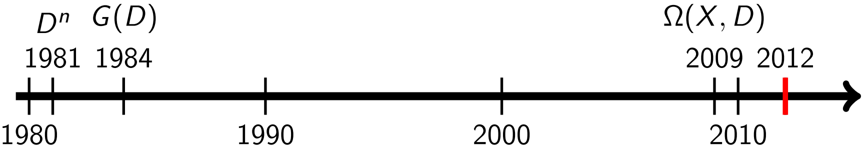


Goal : Study the irreducible components of $\Omega(X, D)$.

Outline

Species of structures

Definition

A *species of structures* is a functor

$$F : \mathbb{B} \longrightarrow \mathbb{B}$$

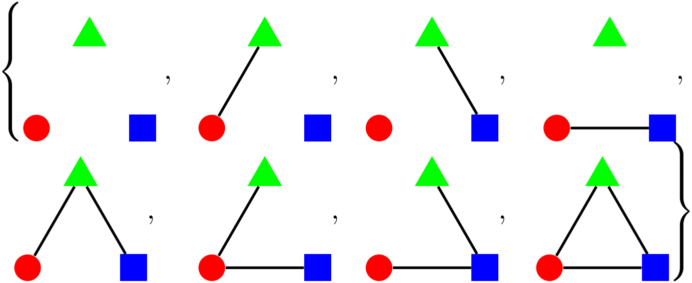
where $\mathbb{B}$ is the category of finite sets with bijections.

For any finite set U, the elements of $F[U]$ are called F-structures on the set U.

Example (Species of simple graphs)

Let $U = \{$●, ▲, ■$\}$ and $F := \mathcal{G}_3$. We have,

$$\mathcal{G}_3[U] =$$



Example (Cont.)

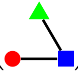
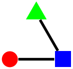
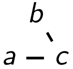
Moreover, for the bijection

$$\sigma : U \longrightarrow \{a, b, c\}$$

defined by

$$\sigma(\textcolor{red}{\bullet}) = a,\ \sigma(\textcolor{green}{\blacktriangle}) = b \text{ and } \sigma(\textcolor{blue}{\blacksquare}) = c,$$

we have

$\mathcal{G}_3[\sigma]\Bigg($  $\Bigg) := \sigma\Bigg($  $\Bigg) =$  .

Some basic combinatorial operations

Definition

The *sum* of F and G is a functor

$$(F+G) : \mathbb{B} \longrightarrow \mathbb{B}$$

where $(F+G)[U] := F[U] + G[U]$ (disjoint sum).

Definition

The *product* of F and G is a functor

$$(FG) : \mathbb{B} \longrightarrow \mathbb{B}$$

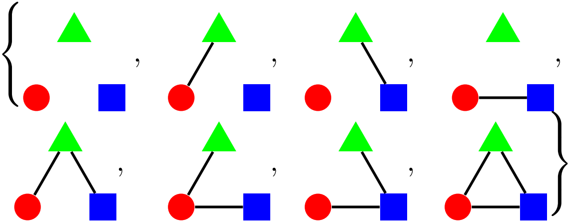
where

$$(FG)[U] := \sum_{(U_1, U_2)} F[U_1] \times G[U_2]$$

and $U_1 + U_2 = U$ (disjoint sum).

Example (Sum)

$\mathcal{G}_3[U] =$



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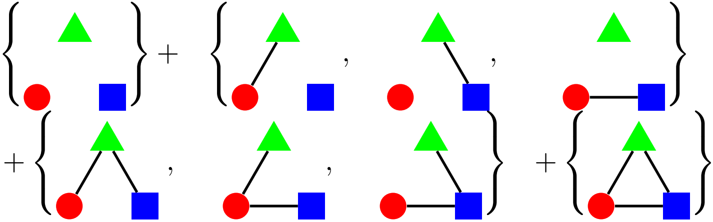

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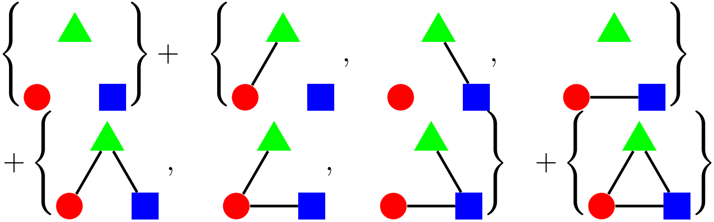
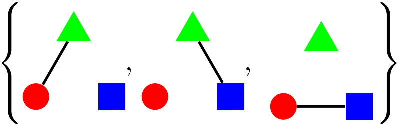


$$\mathcal{G}_3 \simeq \mathcal{G}_3^0 + \mathcal{G}_3^1 + \mathcal{G}_3^2 + \mathcal{G}_3^3$$

Example (Product)

$\mathcal{G}_3^1[U] =$ 

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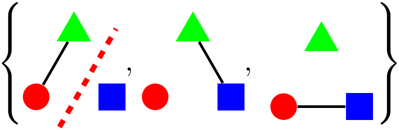
$\mathcal{G}_3^1[U] =$ 

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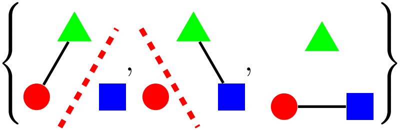

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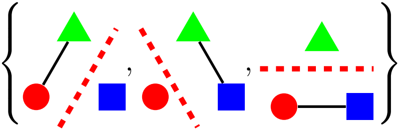

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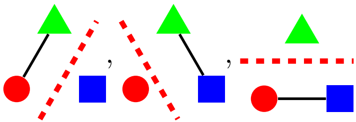
$$\mathcal{G}_3^1[U] = \left\{ \quad \right\}$$



$$\mathcal{G}_3^1 \simeq XE_2$$

Molecular and atomic species

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A species M is *molecular* if and only if

$$M \simeq F + G \Longrightarrow F \simeq 0 \text{ or } G \simeq 0.$$

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A molecular species $A \neq 1$ is *atomic* if and only if

$$A \simeq FG \Longrightarrow F \simeq 1 \text{ or } G \simeq 1.$$

For example, $\mathcal{G}_3$ is not molecular (and therefore not atomic) whilst XE_2 is molecular non-atomic. Furthermore, X and E_2 are both atomic.

Characterization of molecular species

We have $M(X)$ molecular and $H \leq \mathbb{S}_n$ stabilizer of a $M(X)$-structure.

Definition

1. $$M(X)[U] \simeq \frac{X^n}{H}[U] = \{\lambda H \mid \lambda : [n] \tilde{\to} U,\ \text{bijection}\}$$
   for any finite set U, where $[n] := \{1, 2, \ldots, n\}$ and $\lambda H = \{\lambda \circ h \mid h \in H\}$.
2. $$\frac{X^n}{H_1} \simeq \frac{X^m}{H_2} \Longleftrightarrow \left\{ \begin{array}{ll} & n = m \\ \text{and} & \\ & H_1 \text{ Conj } H_2 \text{ in } \mathbb{S}_n \end{array} \right.$$

Characterization of molecular species

We have $M(X,T)$ molecular and $H \leq \mathbb{S}_{m,n}$ stabilizer of a $M(X,T)$-structure.

Definition

1. $$M(X,T)[U] \simeq \frac{X^m T^n}{H}[U] = \{\lambda H \mid \lambda : [m+n] \tilde{\to} U, \text{ bijection}\}$$

   for any finite multiset U, where $[m+n] := \{1,2,\ldots,m+n\}$ and $\lambda H = \{\lambda \circ h \mid h \in H\}$.

2. $$\frac{X^{m_1} T^{n_1}}{H_1} \simeq \frac{X^{m_2} T^{n_2}}{H_2} \Longleftrightarrow \left\{ \begin{array}{ll} & (m_1, n_1) = (m_2, n_2) \\ \text{and} & \\ & H_1 \text{ Conj } H_2 \text{ in } \mathbb{S}_{m,n} \end{array} \right.$$

Remark

$\mathbb{S}_{m,n} \leq \mathbb{S}_{m+n}$ permutes the set

$$\{1, 2, \ldots, m, m+1, m+2, \ldots, m+n\}.$$

For example,

$(1\ 3\ 2)(4\ 5) \in \mathbb{S}_{3,2}$

$(1\ 2)(5\ 9)(4\ 7) \in \mathbb{S}_{3,7}$

$(1\ 4) \in \mathbb{S}_{5,8}$

$\mathrm{Id}_{\mathbb{S}_{m+n}} \in \mathbb{S}_{m,n}$.

Outline

Operator D

Definition

The *derivative* of F is a functor

$$DF : \mathbb{B} \longrightarrow \mathbb{B}$$

where $DF[U] := F[U^+]$, with $U^+ = U \cup \{*\}$.

$$D^n F = \begin{cases} F & \text{if} \quad n = 0 \\ DD^{n-1}F & \text{if} \quad n \geq 1 \end{cases}$$

Example (Linear orders)

We have $DL \simeq L \cdot L \simeq L^2$. For example, let $U = \{a, c, g, m, o, s\}$.

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$$g \longrightarrow a \longrightarrow s \longrightarrow c \longrightarrow o \longrightarrow m \in L[U].$$

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We have $DL \simeq L \cdot L \simeq L^2$. For example, let $U = \{a, c, g, m, o, s\}$.

$$\left(g \longrightarrow a \longrightarrow s \,,\; c \longrightarrow o \longrightarrow m \right) \in L^2[U]$$

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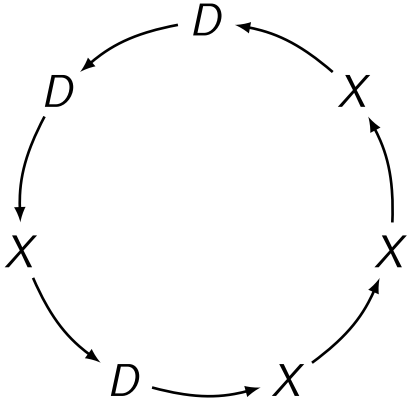
Can we generalize this idea?



$=$?

Definition

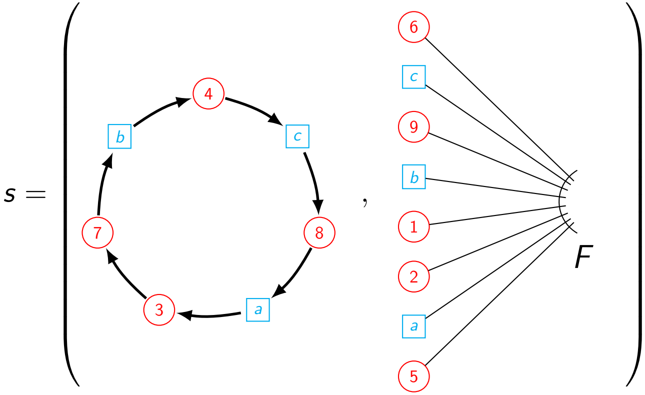
The *partial cartesian product with respect to* T of $\Omega_1(X, T)$ and $\Omega_2(X, T)$ is a functor

$$\Omega_1(X, T) \times_T \Omega_2(X, T) : \mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{B}$$

where, for any finite two-set (U, V) of sort X and T respectively, a $\Omega_1(X, T) \times_T \Omega_2(X, T)$-structure s is a pair $s = (s_1, s_2)$ where $s_1 \in \Omega_1[U_1, V]$ and $s_2 \in \Omega_2[U_2, V]$ with $U_1 \cup U_2 = U$ and $U_1 \cap U_2 = \varnothing$.

Example

$\mathcal{C}(X+T) \times_T F(X+T)$-structure on (U, V) with $U = \{1, 2, \ldots, 9\}$ (sort X) and $V = \{a, b, c\}$ (sort T).



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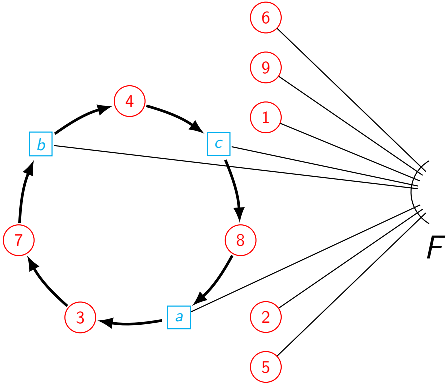


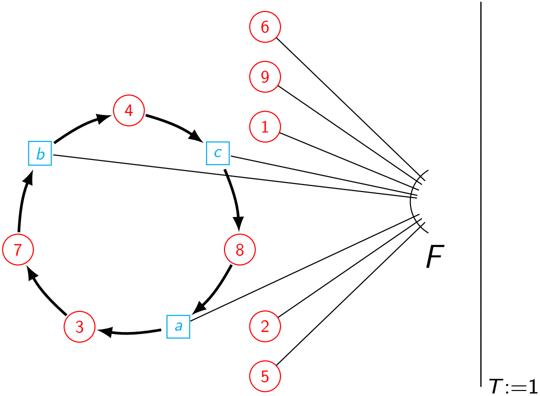
Definition

Let $\Omega(X, T)$ and $F(X)$ be two-sort and one-sort species respectively. One defines $\Omega(X, D)F(X)$ by

$$\Omega(X, D)F(X) := \Omega(X, T) \times_T F(X + T)\,|_{T:=1}\ .$$

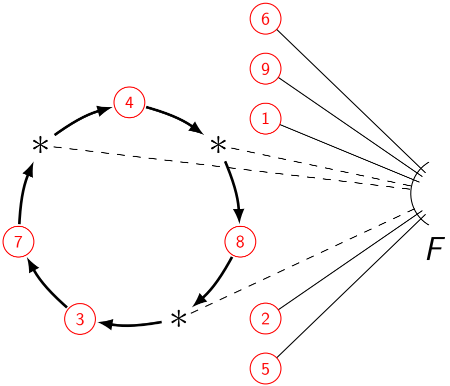
Example ($\mathcal{C}(X+D)F(X)$-structure on (U,V))

By definition, $\mathcal{C}(X+T)\times_T F(X+T)|_{T:=1}$-structure on U



Example ($\mathcal{C}(X+D)F(X)$-structure on (U,V))

By definition, $\mathcal{C}(X+T)\times_T F(X+T)|_{T:=1}$-structure on U



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Any species can be uniquely decomposed as a sum of products of atomic species (Y.-N. Yeh (1985)).

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$$=$$

It is enough to consider only atomic differential operators to study all differential operators.

Computation of $\frac{X^m D^k}{K} \frac{X^n}{H}$

Theorem (G. Labelle & C. Lamathe 2009)

For any subgroups $H \leq \mathbb{S}_n$ and $K \leq \mathbb{S}_{m,k}$, we have

(i) $\frac{X^m D^k}{K} \frac{X^n}{H} = \frac{X^m T^k}{K} \times_T \frac{(X+T)^n}{H} \,|_{T:=1}$,

(ii) $\frac{(X+T)^n}{H} = \sum_{k=0}^{n} \sum_{\omega \,\underline{\in}\, \mathbb{S}_{n-k,k} \backslash \mathbb{S}_n / H} \frac{X^{n-k} T^k}{\omega H \omega^{-1} \cap \mathbb{S}_{n-k,k}}$,

(iii) $\frac{X^a T^k}{A} \times_T \frac{X^b T^k}{B} = \sum_{\tau \,\underline{\in}\, (\pi_2 A) \backslash \mathbb{S}_k / (\pi_2 B)} \frac{X^{a+b} T^k}{A \times_{\mathbb{S}_k} B^\tau}$,

(iv) $\left[\frac{X^a T^k}{A}\right]_{T:=1} = \frac{X^a}{\pi_1 A}$,

where $\omega \,\underline{\in}\, \mathbb{S}_{n-k,k} \backslash \mathbb{S}_n / H$ means that ω runs through a system of representatives of the double cosets $H_1 \sigma H_2$, $\sigma \in \mathbb{S}_n$;
$\pi_i G = \{g_i \in \mathbb{S}_{n_i} \mid (g_1, g_2) \in G\}$, $G \leq \mathbb{S}_{n_1,n_2}$;
$B^\tau = (\mathrm{Id}, \tau) B (\mathrm{Id}, \tau^{-1})$; $A \times_{\mathbb{S}_k} B$ is the fibered product (pullback) of A by B over $\mathbb{S}_k$.

Outline

Lattice of set partitions

Recall that the set of partitions of $\{1, 2, \ldots, m\}$ forms a complete lattice with

$$\hat{0} = \{\{1\}, \{2\}, \ldots, \{m\}\} \text{ (finest partition)}$$

and

$$\hat{1} = \{\{1, 2, \ldots, m\}\} \text{ (coarsest partition).}$$

The partition $\sup(p1, p2, \ldots, p_k)$ is the finest partition which is coarser than each of the p_i's.

Example

Let

$$\begin{aligned} p_1 &= \{\{1,3\},\{2\},\{4,5\}\} \\ p_2 &= \{\{1,2\},\{3\},\{4,5\}\} \\ p_3 &= \{\{1\},\{2,3\},\{4,5\}\} \end{aligned}$$

then,

$$\sup(p_1, p_2, p_3) = \{\{1,2,3\},\{4,5\}\}.$$

A few more definitions

Let $g \in \mathbb{S}_m$ and $s \subseteq \{1, 2, \ldots, m\}$.

1. $\hat{g}$ is the partition of $\{1, 2, \ldots, m\}$ obtained by replacing each cycle of g by the corresponding set,
2. $g_s^*(x) := \begin{cases} g(x) & \text{if } x \in s \\ x & \text{otherwise} \end{cases}$

Example

Take $g = (2\ 5)(4\ 6\ 1)(3)(7)(8) \in \mathbb{S}_8$ and $s = \{2, 3, 5, 8\}$.

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2. $g_s^* = [g_s^*(1), g_s^*(2), g_s^*(3), g_s^*(4), g_s^*(5), g_s^*(6), g_s^*(7), g_s^*(8)]$

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2. $g_s^*(x) := \begin{cases} g(x) & \text{if } x \in s \\ x & \text{otherwise} \end{cases}$

Example

Take $g = (2\ 5)(4\ 6\ 1)(3)(7)(8) \in \mathbb{S}_8$ and $s = \{2, 3, 5, 8\}$.

1. $\hat{g} = \{\{2, 5\}, \{4, 6, 1\}, \{3\}, \{7\}, \{8\}\}$
2. $g_s^* = [1, g(2), g_s^*(3), g_s^*(4), g_s^*(5), g_s^*(6), g_s^*(7), g_s^*(8)]$

A few more definitions

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Example

Take $g = (2\ 5)(4\ 6\ 1)(3)(7)(8) \in \mathbb{S}_8$ and $s = \{2, 3, 5, 8\}$.

1. $\hat{g} = \{\{2, 5\}, \{4, 6, 1\}, \{3\}, \{7\}, \{8\}\}$
2. $g_s^* = (2\ 5)$

Algorithm

Require: $M = \frac{X^m D^n}{H}$, where H is generated by $\{g_1, g_2, \ldots, g_i\}$.
Ensure: `true`, if M is atomic and `false`, otherwise.

```
Construct the list of partitions ĝ_1, ĝ_2, ..., ĝ_i of {1, ..., m + n};
Construct the partition p = sup(ĝ_1, ĝ_2, ..., ĝ_i);
for k from 1 to |p| − 1, do
    for each k-subset {c_1, c_2, ..., c_k} ⊆ p, do
        c = ⋃_{1≤i≤k} c_i;
        if ∀g ∈ {g_1, g_2, ..., g_i}, c is stable under g and g*_c ∈ H, then
            return false.
        end if
    end for
end for
return true.
```

Example

Require: $M = \frac{X^2 D^3}{\langle (1\ 2),(4\ 5) \rangle}$, where H is generated by $\{g_1 = (1\ 2), g_2 = (4\ 5)\}$.

Ensure: `true`, if M is atomic and `false`, otherwise.

Example

Require: $M = \frac{X^2 D^3}{\langle (1\ 2),(4\ 5)\rangle}$, where H is generated by $\{g_1 = (1\ 2), g_2 = (4\ 5)\}$.

Ensure: `true`, if M is atomic and `false`, otherwise.

1: Construct the list of partitions $\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_i$ of $\{1, \ldots, m+n\}$;

Example

Require: $M = \frac{X^2 D^3}{\langle (1\ 2),(4\ 5)\rangle}$, where H is generated by $\{g_1 = (1\ 2), g_2 = (4\ 5)\}$.

Ensure: `true`, if M is atomic and `false`, otherwise.

1: $(\hat{g}_1 = \{\{1,2\},\{3\},\{4\},\{5\}\}, \hat{g}_2 = \{\{1\},\{2\},\{3\},\{4,5\}\})$;

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Ensure: `true`, if M is atomic and `false`, otherwise.

1: $(\hat{g}_1 = \{\{1,2\},\{3\},\{4\},\{5\}\}, \hat{g}_2 = \{\{1\},\{2\},\{3\},\{4,5\}\})$;

2: Construct the partition $p = \sup(\hat{g}_1, \hat{g}_2)$;

Example

Require: $M = \frac{X^2 D^3}{\langle (1\ 2),(4\ 5)\rangle}$, where H is generated by $\{g_1 = (1\ 2), g_2 = (4\ 5)\}$.

Ensure: `true`, if M is atomic and `false`, otherwise.

1: $(\hat{g}_1 = \{\{1,2\},\{3\},\{4\},\{5\}\}, \hat{g}_2 = \{\{1\},\{2\},\{3\},\{4,5\}\});$

2: $p = \{\{1,2\},\{3\},\{4,5\}\};$

Example

Require: $M = \frac{X^2 D^3}{\langle (1\ 2),(4\ 5)\rangle}$, where H is generated by $\{g_1 = (1\ 2), g_2 = (4\ 5)\}$.

Ensure: `true`, if M is atomic and `false`, otherwise.

$(\hat{g}_1 = \{\{1,2\},\{3\},\{4\},\{5\}\}, \hat{g}_2 = \{\{1\},\{2\},\{3\},\{4,5\}\});$

$p = \{\{1,2\},\{3\},\{4,5\}\};$

for k from 1 to $3-1$, **do**

end for

Example

Require: $M = \frac{X^2 D^3}{\langle (1\ 2),(4\ 5)\rangle}$, where H is generated by $\{g_1 = (1\ 2), g_2 = (4\ 5)\}$.

Ensure: `true`, if M is atomic and `false`, otherwise.

$(\hat{g}_1 = \{\{1,2\},\{3\},\{4\},\{5\}\}, \hat{g}_2 = \{\{1\},\{2\},\{3\},\{4,5\}\})$;

$p = \{\{1,2\},\{3\},\{4,5\}\}$;

for $k = 1$ **do**

  for each k-subset $\{c_1, c_2, \ldots, c_k\} \subseteq p$, **do**

  end for

end for

Example

Require: $M = \frac{X^2 D^3}{\langle (1\ 2),(4\ 5) \rangle}$, where H is generated by $\{g_1 = (1\ 2), g_2 = (4\ 5)\}$.

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$(\hat{g}_1 = \{\{1,2\},\{3\},\{4\},\{5\}\}, \hat{g}_2 = \{\{1\},\{2\},\{3\},\{4,5\}\})$;

$p = \{\{1,2\},\{3\},\{4,5\}\}$;

for $k = 1$ **do**

  for $\{c_1 = \{1,2\}\} \subseteq p$, **do**

  end for

end for

Example

Require: $M = \frac{X^2 D^3}{\langle (1\ 2),(4\ 5)\rangle}$, where H is generated by $\{g_1 = (1\ 2), g_2 = (4\ 5)\}$.

Ensure: `true`, if M is atomic and `false`, otherwise.

$(\hat{g}_1 = \{\{1,2\},\{3\},\{4\},\{5\}\}, \hat{g}_2 = \{\{1\},\{2\},\{3\},\{4,5\}\});$

$p = \{\{1,2\},\{3\},\{4,5\}\};$

for $k = 1$ **do**

  for $\{c_1 = \{1,2\}\} \subseteq p$, **do**

    $c = \bigcup_{1 \leq i \leq k} c_i;$

  end for

end for

Example

Require: $M = \frac{X^2 D^3}{\langle (1\ 2),(4\ 5)\rangle}$, where H is generated by $\{g_1 = (1\ 2), g_2 = (4\ 5)\}$.

Ensure: `true`, if M is atomic and `false`, otherwise.

$(\hat{g}_1 = \{\{1,2\},\{3\},\{4\},\{5\}\}, \hat{g}_2 = \{\{1\},\{2\},\{3\},\{4,5\}\});$

$p = \{\{1,2\},\{3\},\{4,5\}\};$

for $k = 1$ **do**

  for $\{c_1 = \{1,2\}\} \subseteq p$, **do**

    $c = c_1$

  end for

end for

Example

Require: $M = \frac{X^2 D^3}{\langle (1\ 2),(4\ 5)\rangle}$, where H is generated by $\{g_1 = (1\ 2), g_2 = (4\ 5)\}$.

Ensure: `true`, if M is atomic and `false`, otherwise.

$(\hat{g}_1 = \{\{1,2\},\{3\},\{4\},\{5\}\}, \hat{g}_2 = \{\{1\},\{2\},\{3\},\{4,5\}\})$;
$p = \{\{1,2\},\{3\},\{4,5\}\}$;
for $k = 1$ **do**
    for $\{c_1 = \{1,2\}\} \subseteq p$, **do**
        $c = c_1$
        if $\forall g \in \{g_1, g_2\}$, c is stable under g and $g_c^* \in H$, **then**

        end if
    end for
end for

Example

Require: $M = \frac{X^2 D^3}{\langle (1\ 2),(4\ 5) \rangle}$, where H is generated by $\{g_1 = (1\ 2), g_2 = (4\ 5)\}$.

Ensure: `true`, if M is atomic and `false`, otherwise.

$(\hat{g}_1 = \{\{1,2\},\{3\},\{4\},\{5\}\}, \hat{g}_2 = \{\{1\},\{2\},\{3\},\{4,5\}\})$;

$p = \{\{1,2\},\{3\},\{4,5\}\}$;

for $k = 1$ **do**

  for $\{c_1 = \{1,2\}\} \subseteq p$, **do**

    $c = c_1$

    if $g_1(c) = c$, $g_2(c) = c$ and ${g_1}^*_c = (1\ 2) \in H$, ${g_2}^*_c = \mathrm{Id} \in H$ **then**

    end if

  end for

end for

Example

Require: $M = \frac{X^2 D^3}{\langle (1\ 2), (4\ 5) \rangle}$, where H is generated by $\{g_1 = (1\ 2), g_2 = (4\ 5)\}$.

Ensure: `true`, if M is atomic and `false`, otherwise.

$(\hat{g}_1 = \{\{1,2\},\{3\},\{4\},\{5\}\}, \hat{g}_2 = \{\{1\},\{2\},\{3\},\{4,5\}\});$

$p = \{\{1,2\},\{3\},\{4,5\}\};$

for $k = 1$ **do**

  for $\{c_1 = \{1,2\}\} \subseteq p$, **do**

    $c = c_1$

    if $g_1(c) = c$, $g_2(c) = c$ and ${g_1}^*_c = (1\ 2) \in H$, ${g_2}^*_c = \mathrm{Id} \in H$ **then**

      return `false`.

    end if

  end for

end for

Partial results

$m = 8,\ n = 0$ (130 operators)
$X^8D^0/\langle(1\ 2)(3\ 4)(5\ 6)(7\ 8)\rangle$
$X^8D^0/\langle(1\ 2)(3\ 4)(5\ 6)(7\ 8),(1\ 3)(2\ 4)(5\ 7)(6\ 8)\rangle$
$X^8D^0/\langle(5\ 6)(7\ 8),(1\ 2)(3\ 4)(5\ 7\ 6\ 8)\rangle$
$X^8D^0/\langle(5\ 6)(7\ 8),(1\ 2)(3\ 4)(5\ 7)(6\ 8)\rangle$
$X^8D^0/\langle(1\ 2)(3\ 4)(5\ 6)(7\ 8),(1\ 3\ 2\ 4)(5\ 7\ 6\ 8)\rangle$
$X^8D^0/\langle(5\ 6)(7\ 8),(1\ 2)(3\ 4)(7\ 8)\rangle$
$X^8D^0/\langle(3\ 4)(5\ 6)(7\ 8),(1\ 2)(5\ 7)(6\ 8)\rangle$
$X^8D^0/\langle(3\ 4\ 5)(6\ 7\ 8),(1\ 2)(3\ 6)(4\ 8)(5\ 7)\rangle$
$X^8D^0/\langle(3\ 4\ 5)(6\ 7\ 8),(1\ 2)(4\ 5)(7\ 8)\rangle$
$X^8D^0/\langle(3\ 4\ 5)(6\ 7\ 8),(1\ 2)(3\ 6)(4\ 7)(5\ 8)\rangle$
$X^8D^0/\langle(1\ 2)(3\ 4)(5\ 6)(7\ 8),(1\ 3)(2\ 4)(5\ 7)(6\ 8),(1\ 5)(2\ 6)(3\ 7)(4\ 8)\rangle$
$X^8D^0/\langle(5\ 6)(7\ 8),(3\ 4)(7\ 8),(1\ 2)(7\ 8)\rangle$
$\vdots$

Partial results (cont.)

$m = 6,\ n = 2$ (46 operators)
$X^6D^2/\langle(3\ 4)(5\ 6), (1\ 2)(3\ 5)(4\ 6)(7\ 8)\rangle$
$X^6D^2/\langle(3\ 4)(5\ 6)(7\ 8), (1\ 2)(3\ 5)(4\ 6)\rangle$
$X^6D^2/\langle(1\ 2\ 3)(4\ 5\ 6), (1\ 4)(2\ 6)(3\ 5)(7\ 8)\rangle$
$X^6D^2/\langle(2\ 3)(5\ 6)(7\ 8), (1\ 2)(4\ 5)(7\ 8)\rangle$
$X^6D^2/\langle(1\ 2)(3\ 4)(5\ 6)(7\ 8), (1\ 3\ 5)(2\ 4\ 6)\rangle$
$X^6D^2/\langle(5\ 6)(7\ 8), (3\ 4)(7\ 8), (1\ 2)(7\ 8)\rangle$
$X^6D^2/\langle(5\ 6)(7\ 8), (3\ 4)(7\ 8), (1\ 2)(3\ 5)(4\ 6)(7\ 8)\rangle$
$X^6D^2/\langle(5\ 6), (3\ 4), (1\ 2)(3\ 5)(4\ 6)(7\ 8)\rangle$
$X^6D^2/\langle(3\ 4)(5\ 6), (3\ 5\ 4\ 6)(7\ 8), (1\ 2)(5\ 6)\rangle$
$X^6D^2/\langle(5\ 6)(7\ 8), (3\ 4)(7\ 8), (1\ 2)(3\ 5)(4\ 6)\rangle$
$X^6D^2/\langle(3\ 4)(5\ 6), (3\ 5\ 4\ 6)(7\ 8), (1\ 2)(5\ 6)(7\ 8)\rangle$
$X^6D^2/\langle(5\ 6)(7\ 8), (1\ 2)(3\ 4), (1\ 3)(2\ 4)(7\ 8)\rangle$
$\vdots$

Future perspectives

- Algorithm analysis
- Implementation of the theorem in Sage
- Study the factorization of molecular operators
- Develop methods to study $\Omega(X, D)F(X) \simeq G(X)$
- Links between $\Omega(X, D)$ and physics
- Study partial differential operators of the form $\Omega\left(X_1, X_2, \ldots, X_k, \frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \ldots, \frac{\partial}{\partial X_k}\right)$

Thank you!