Combinatorial sums associated with balancing and Lucas-balancing polynomials

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Abstract

The aim of the paper is to use some identities involving binomial co-efficients to derive new combinatorial identities for balancing and Lucas-balancing polynomials. Evaluating these identities at specific points, we can also establish some combinatorial expressions for Fibonacci and Lucas numbers.

Keywords: Balancing polynomial, Lucas-balancing polynomial, balancing number, Fibonacci number, Lucas number.

MSC: 11B39, 11B83, 05A10

1. Introduction

Balancing polynomials $(B_n(x))_{n\geq 0}$ and Lucas-balancing polynomials $(C_n(x))_{n\geq 0}$ are defined for $x \in \mathbb{C}$ by the recurrences [17]

$$B_n(x) = 6xB_{n-1}(x) - B_{n-2}(x), \quad n \geq 2,$$

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with $B_0(x) = 0$, $B_1(x) = 1$ and

$$C_n(x) = 6xC_{n-1}(x) - C_{n-2}(x), \quad n \geq 2,$$

with $C_0(x) = 1$, $C_1(x) = 3x$.

(Lucas-) Balancing numbers and (Lucas-) balancing polynomials are related by $B_n = B_n(1)$ and $C_n = C_n(1)$. Sequences $(B_n)_{n \geq 0}$ and $(C_n)_{n \geq 0}$ are indexed in On-Line Encyclopedia of Integer Sequences [19] (see entries A001109 and A001541, respectively). The polynomials are interesting also due to their direct connection to Fibonacci numbers, Lucas numbers and Chebyshev and Legendre polynomials [7].

These polynomials have been introduced recently as an extension of the popular balancing and Lucas-balancing numbers B_n and C_n, respectively, as presented by Behera and Panda in [2].

Balancing polynomials (numbers) are members the Lucas sequence of the first kind defined by the recurrence relation $U_0 = 0$, $U_1 = 1$, $U_n = pU_{n-1} + qU_{n-2}$ ($n \geq 2$). Lucas-balancing polynomials (numbers) can also be defined using initial values $C_0(x) = 2$ and $C_1(x) = 6x$. In this case, Lucas-balancing polynomials will belong to the Lucas sequence of the second kind defined by $V_0 = 2$, $V_1 = p$, $V_n = pV_{n-1} + qV_{n-2}$ ($n \geq 2$). Such an approach would allow us to simplify some formulas, but would complicate our comparative analysis with articles where these polynomials are defined by initial values $C_0(x) = 1$ and $C_1(x) = 3x$.

Solving the recurrences routinely we get the following closed forms for polynomials $B_n(x)$ and $C_n(x)$ known as Binet formulas:

$$B_n(x) = \frac{\lambda^n(x) - \lambda^{-n}(x)}{\lambda(x) - \lambda^{-1}(x)}, \quad C_n(x) = \frac{\lambda^n(x) + \lambda^{-n}(x)}{2}, \tag{1.1}$$

where $\lambda(x) = 3x + \sqrt{9x^2 - 1}$.

Using (1.1), it is easy to see that

$$B_{2n}(x) = 2B_n(x)C_n(x), \quad n \geq 0. \tag{1.2}$$

Combinatorial expressions for balancing and Lucas-balancing polynomials are [3, 15]

$$B_n(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n-1-k}{k} (6x)^{n-1-2k}, \quad n \geq 1, \tag{1.3}$$

$$C_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} (6x)^{n-2k}, \quad n \geq 1. \tag{1.4}$$

The relations $B_n(-x) = (-1)^{n+1}B_n(x)$ and $C_n(-x) = (-1)^n C_n(x)$ follow from $\lambda(\pm x) = -\lambda^{-1}(\mp x)$.

Some examples of recent works involving balancing and Lucas-balancing polynomials conclude [7–9, 16].

The aim of the paper is to derive new combinatorial identities for polynomials $B_n(x)$ and $C_n(x)$. Evaluating these identities at specific points, we can also establish some interesting combinatorial identities as special cases, especially those with Fibonacci and Lucas numbers.

2. Combinatorial identities using Waring's formulas

Our first result provides two combinatorial identities for balancing and Lucas-balancing polynomials involving binomial coefficients.

Theorem 2.1. *Let $m \geq 0$. Then*

$$B_{(n+1)m}(x) = B_m(x) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} (2C_m(x))^{n-2k}, \quad n \geq 0, \qquad (2.1)$$

$$C_{nm}(x) = \frac{n}{2} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} (2C_m(x))^{n-2k}, \quad n \geq 1. \qquad (2.2)$$

Proof. We combine the Binet formulas (1.1) with the following combinatorial formulas

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} (XY)^k (X+Y)^{n-2k} = \frac{X^{n+1} - Y^{n+1}}{X - Y} \qquad (2.3)$$

and

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (XY)^k (X+Y)^{n-2k} = X^n + Y^n. \qquad (2.4)$$

To get (2.1), set $X = \lambda^m(x)$ and $Y = \lambda^{-m}(x)$ in (2.3). Formula (2.1) is the immediate result when replacing n by $n-1$. To get (2.2) apply the same argument to the formula (2.4). $\qquad \square$

Remark 2.2. Formulas (2.3) and (2.4) are well-known in combinatorics and called Waring's (sometimes Girard-Waring's) formulas. In [12] the reader will find some interesting remarks about the history and the use of these formulas and their generalizations. The proof of these formulas can be seen, for example, in [4].

In view of (1.2), formulas (2.1) and (2.2) can be written entirely in terms of balancing polynomials $B_n(x)$. Special cases of (2.1) and (2.1) for $m = 1$ are formulas (1.3) and (1.4), respectively.

Setting $x = 1$ in (2.1), we immediately get

$$B_{mn} = B_m \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n-1-k}{k} (2C_m)^{n-1-2k}.$$

This result appears as Theorem 3.2 in [18]. Similarly, setting $x = 1$ in (2.2) yields

$$C_{mn} = \frac{n}{2} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} (2C_m)^{n-2k}. \qquad (2.5)$$

Special cases of (2.5) are

$$C_n = \frac{n}{2} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} 6^{n-2k}, \qquad (2.6)$$

$$C_{2n} = \frac{n}{2} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} 34^{n-2k},$$

and so on. Formula (2.6) may be found in [15]. More expressions of this kind can be found in [10].

Next we are going to present some consequences of the above results to combinatorial sums involving Fibonacci numbers F_n and Lucas numbers L_n. Recall that both sequences satisfy the same recurrence relation $u_n = u_{n-1} + u_{n-2}$ for $n \geq 2$, but with initial conditions $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$ (sequences A000045 and A000032 in [19], respectively).

Balancing and Lucas-balancing polynomials are linked to Fibonacci and Lucas numbers via

$$B_n\left(\frac{L_{2q}}{6}\right) = \frac{F_{2qn}}{F_{2q}}, \quad C_n\left(\frac{L_{2q}}{6}\right) = \frac{L_{2qn}}{2}, \qquad (2.7)$$

and

$$B_n\left(\frac{L_{2q+1}}{6}i\right) = \frac{F_{(2q+1)n}}{F_{2q+1}} i^{n-1}, \quad C_n\left(\frac{L_{2q+1}}{6}i\right) = \frac{L_{(2q+1)n}}{2} i^n, \qquad (2.8)$$

where q is an integer and i is the imaginary unit; see [7].

Formulas (2.7) and (2.8), coupled with Theorem 2.1 above, yield the following results, which are known.

Corollary 2.3. *Let* $m \geq 0$. *Then*

$$F_{2m(n+1)} = F_{2m} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} L_{2m}^{n-2k}, \quad n \geq 0, \qquad (2.9)$$

$$F_{(2m+1)(n+1)} = F_{2m+1} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} L_{2m+1}^{n-2k}, \quad n \geq 0, \qquad (2.10)$$

$$L_{2mn} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} L_{2m}^{n-2k}, \quad n \geq 1, \qquad (2.11)$$

$$L_{(2m+1)n} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} L_{2m+1}^{n-2k}, \quad n \geq 1. \tag{2.12}$$

The above results are rediscoveries of known identities. Formulas (2.9) and (2.10) we can united as a single formula [13]

$$F_{m(n+1)} = F_m \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{k(m+1)} \binom{n-k}{k} L_m^{n-2k}, \quad n, m \geq 0. \tag{2.13}$$

Also, formulas (2.11) and (2.12) may be written in the same manner as follows [13]

$$L_{mn} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{k(m+1)} \frac{n}{n-k} \binom{n-k}{k} L_m^{n-2k}, \quad n \geq 1, \ m \geq 0. \tag{2.14}$$

Since $L_s = F_{2s}/F_s$, formulas (2.13) and (2.14) can be written entirely in terms of Fibonacci numbers.

Specific examples of (2.13) and (2.14) include the following combinatorial Fibonacci and Lucas identities:

$$F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k}, \tag{2.15}$$

$$F_{2n} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n-1-k}{k} 3^{n-2k-1}, \tag{2.16}$$

$$F_{3n} = 2 \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} 4^{n-2k-1}, \tag{2.17}$$

$$L_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k},$$

$$L_{2n} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} 3^{n-2k},$$

$$L_{3n} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} 4^{n-2k},$$

and so on. All identities in our list are know. For instance, identity (2.15) appears as equation (1) in [11] and again as equation (5.1) in [5]. Identity (2.16) is equation (2) in [11] and stated slightly differently equation (5.10) in [5].

3. Combinatorial identities using Jennings' formulas

Theorem 3.1. *For* $m, n \geq 0$, *we have*

$$\frac{B_{(2n+1)m}(x)}{2n+1} = \sum_{k=0}^{n} \binom{n+k}{2k} \frac{(36x^2-4)^k}{2k+1} B_m^{2k+1}(x), \tag{3.1}$$

$$\frac{C_{(2n+1)m}(x)}{2n+1} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{2k} \frac{4^k}{2k+1} C_m^{2k+1}(x). \tag{3.2}$$

Proof. The following identities are from Jennings [14, Lemmas (i) and (ii)]:

$$\sum_{k=0}^{n} \frac{2n+1}{2k+1} \binom{n+k}{2k} \left(\frac{z^2-1}{z}\right)^{2k} = \frac{z^{2(n+1)} - z^{-2n}}{z^2-1}, \tag{3.3}$$

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{2n+1}{2k+1} \binom{n+k}{2k} \left(\frac{z^2+1}{z}\right)^{2k} = \frac{z^{2(n+1)} + z^{-2n}}{z^2+1}. \tag{3.4}$$

To get (3.1), set $z = X/Y$ in (3.3) to derive at

$$\sum_{k=0}^{n} \frac{2n+1}{2k+1} \binom{n+k}{2k} (XY)^{n-k} (X-Y)^{2k+1} = X^{2n+1} - Y^{2n+1}.$$

Now, we can insert $X = \lambda^m(x)$ and $Y = \lambda^{-m}(x)$, and the statement follows. To get (3.2) apply the same argument to formula (3.4). $\qquad\square$

Note that identity (3.3) also appears in [1] to prove some Fibonacci identities.

Corollary 3.2. *For* $n \geq 0$,

$$\sum_{k=0}^{n} \binom{n+k}{2k} \frac{(-4)^k}{2k+1} = \frac{(-1)^n}{2n+1}.$$

Proof. Set $x = 0$ in (3.1) and use

$$B_n(0) = \begin{cases} 0, & \text{if } n \text{ is even}, \\ (-1)^{\frac{n-1}{2}}, & \text{if } n \text{ is odd}. \end{cases} \qquad\square$$

Corollary 3.3. *For* $n, m \geq 0$,

$$B_{(2n+1)m} = (2n+1) \sum_{k=0}^{n} \binom{n+k}{2k} \frac{32^k}{2k+1} B_m^{2k+1},$$

$$C_{(2n+1)m} = (2n+1) \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{2k} \frac{4^k}{2k+1} C_m^{2k+1}.$$

Proof. Set $x = 1$ in (3.1) and (3.2), respectively. $\qquad\square$

Corollary 3.4. *For* $n, m \geq 0$,

$$F_{2m(2n+1)} = (2n+1) \sum_{k=0}^{n} \binom{n+k}{2k} \frac{5^k}{2k+1} F_{2m}^{2k+1}, \qquad (3.5)$$

$$F_{(2m+1)(2n+1)} = (2n+1)(-1)^n \sum_{k=0}^{n} \binom{n+k}{2k} \frac{(-5)^k}{2k+1} F_{2m+1}^{2k+1}. \qquad (3.6)$$

Proof. Insert $x = L_{2q}/6$ and $x = iL_{2q+1}/6$ in (3.1), use (2.7) and (2.8), and simplify using $L_n^2 = 5F_n^2 + (-1)^n 4$. $\qquad\square$

Remark 3.5. Equations (3.5) and (3.6) are rediscoveries of Theorem 1 in [14].

4. Combinatorial identities using Toscano's identity

Theorem 4.1. *For* $n \geq 1$ *and* $m \geq 0$, *we have the following combinatorial identity:*

$$2^{2n-1} C_m^{2n}(x) = \sum_{k=1}^{n} \binom{2n-k-1}{n-1} 2^k C_m^k(x) C_{mk}(x). \qquad (4.1)$$

Proof. Combine the Binet formula for $C_n(x)$ with combinatorial identity

$$\sum_{k=1}^{n} \binom{2n-k-1}{n-1} (X^k + Y^k) \left(\frac{XY}{X+Y} \right)^{n-k} = (X+Y)^n,$$

which have been proved in [20] by Toscano. $\qquad\square$

Setting $x = 1$ in (4.1) immediately gives the next relation.

Corollary 4.2. *For* $n \geq 1$ *and* $m \geq 0$,

$$2^{2n-1} C_m^{2n} = \sum_{k=1}^{n} \binom{2n-k-1}{n-1} 2^k C_m^k C_{mk}.$$

The next two identities are special instances of the previous corollary for $m = 0$ and $m = 1$, respectively:

$$\sum_{k=1}^{n} \binom{2n-k-1}{n-1} 2^k = 2^{2n-1}$$

and

$$2 \sum_{k=1}^{n} \binom{2n-k-1}{n-1} 6^k C_k = 36^n.$$

Focusing on Lucas numbers we obtain the following known combinatorial identities [6].

Corollary 4.3. *For $n \geq 1$ and $m \geq 0$, Lucas numbers satisfy*

$$L_{2m}^{2n} = \sum_{k=1}^{n} \binom{2n-k-1}{n-1} L_{2m}^{k} L_{2mk},$$

and

$$L_{2m+1}^{2n} = \sum_{k=1}^{n} (-1)^{n-k} \binom{2n-k-1}{n-1} L_{2m+1}^{k} L_{(2m+1)k}.$$

The next evaluation are consequences of Corollary 4.3:

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{2n-k-1}{n-1} L_k = 1,$$

$$\sum_{k=1}^{n} \binom{2n-k-1}{n-1} \frac{L_{2k}}{3^{2n-k}} = 1,$$

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{2n-k-1}{n-1} \frac{L_{3k}}{4^{2n-k}} = 1,$$

$$\sum_{k=1}^{n} \binom{2n-k-1}{n-1} \frac{L_{4k}}{7^{2n-k}} = 1.$$

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