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Transport Theorem for Spaces and Subspaces of Arbitrary Dimensions

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Abstract: Using the apparatus of traditional differential geometry, the transport theorem is derived for the general case of a M-dimensional domain moving in a N-dimensional space, $M \leq N$. The interesting concepts of curvatures and normals are illustrated with well-known examples of lines, surfaces and volumes. The special cases where either the space or the moving subdomain are material are discussed. Then, the transport at hypersurfaces of discontinuity is considered. Finally, the general local balance equations for continuum of arbitrary dimensions with discontinuities are derived.

Keywords: hypersurfaces; discontinuities; convected coordinates

1. Introduction

The transport theorem is a fundamental theorem used in formulating the basic conservation and balance laws in continuum mechanics (mass, momentum, and energy), which are adopted from classical mechanics and thermodynamics where the system approach is normally followed. Analogous to the classical Reynolds transport in continuum mechanics, the surface transport theorem is essential in the study of thin films undergoing large deformations, in epitaxial growth and in the study of phase boundary evolution. It is also important in the modeling of a singular surface which carries a certain structure of its own as it migrates. There is a vast literature on transport theorem and many references can be found in [1–3].

Betounes formulated and proved the general transport theorem associated with the motion of an arbitrary p-dimensional submanifold in a n-dimensional semi-Riemannian manifold [4]. He used the language and notation of modern differential geometry on manifolds e.g., [5,6], which is inconvenient for engineering and physics applications. Here, we formulate and prove the theorem using the language and concepts of traditional differential geometry and tensor calculus (e.g., [7,8]). Moreover, we apply the transport theorem to hypersurfaces of discontinuities and discuss the applications in continuum mechanics.

Petryk and Mroz derived the expressions for the first- and second-time derivatives of integrals and functionals defined on volume and surface domains which vary in time [9]. Their result is more general then the classical transport theorem as it pertains to piecewise regular surfaces and contains the edge terms. Cermelli et al. proved a transport theorem for smooth surfaces which evolve with time in Euclidean space, expressed in terms of the parameter-independent derivatives [10]. Recently, Sequin et al. extended the 3D transport theorem to rough domains of integration [11].

The need for the transport theorem arises in different contexts and consequently requires different derivation methods. The space-time approach [12] was used in [13] to derive the transport theorem for moving surface in a moving 3D region. A general transport theorem for moving surfaces based on the theory of generalized derivatives in n-dimensional space is presented in [14].

Two interesting attempts to present a unified approach to the topic of continuum mechanics on arbitrarily moving domains are given in [15,16]. They point out that it is desirable to formulate the transport theorem in a single unified way by using the classical approach expressed in terms of standards quantities from differential geometry and explicitly displaying the features that are common to all submanifolds, regardless of their finite dimensions.

This paper is organized as follows: In Sections 2 and 3, we consider geometry and kinematics in higher dimensions with special emphasis on the definitions of curvature and normals. Section 4 contains the derivation of the generalized transport theorem. In Section 5, we illustrate the concepts with well-known examples of lines, surfaces and volumes. In Section 6, we consider dependence of parametrization, i.e., on the choice of coordinates. In Sections 7–9, we consider the cases where the space and/or the moving subdomain are material in the sense of material in continuum mechanics. In Section 10, we consider a moving domain with hypersurfaces of discontinuity. Finally, in Section 11, we use the transport theorem to formulate the general local balance equations for continuum of arbitrary dimensions with discontinuities.

2. Geometry of $V_M(t) \subseteq V_N$

Here and throughout i, j, k assume values 1 to N, the Greek letters α, β, γ, δ, Λ, Δ and Γ assume values from 1 to M; ρ, σ range from 1 to $M-1$, and π, τ range from 1 to $N-M$. The range and the summation convention will apply unless stated otherwise.

Consider a *N*-dimensional Riemannian space V_N with positive definite metric tensor g_{mn}, for an allowable coordinate system x^k, $k = 1,2,.., N$. We shall denote a typical point in V_N by $\boldsymbol{x}(x^k)$. Vectors

$$\boldsymbol{g}_k = \frac{\partial \boldsymbol{x}}{\partial x^k}$$

represent the basis in V_N. The reciprocal basis $\boldsymbol{g}^k$ is defined by $\boldsymbol{g}_j \cdot \boldsymbol{g}^k = \delta_j^k$, $j,k = 1,2,...,N$.

Let V_M be a subspace of V_N, i.e., $V_M \subseteq V_N$. Further, we assume that the Riemannian subspace V_M changes it position continuously with respect to time t in V_N. We emphasize it by writing $V_M(t)$. Let u^α $(\alpha = 1,2,...,M)$ be its intrinsic coordinate system:

$$V_M(t): \quad \mathbf{x} = \mathbf{x}(\mathbf{u}, t), \tag{1}$$

where $\mathbf{u}$ or u^α we denote a typical point of $V_M(t)$. It is possible to rewrite in component form, i.e.,

$$x^k = x^k(u^\alpha, t), \ \operatorname{rank}\left(\frac{\partial x^k}{\partial u^\alpha}\right) = M. \tag{2}$$

Equivalently,

$$V_M(t): \quad f^{(\pi)}(x^k, t) = 0, \quad \pi = 1,2,...,N-M, \tag{3}$$

where

$$\operatorname{rank}\left(\frac{\partial f^{(\pi)}}{\partial x^k}\right) = N - M. \tag{4}$$

To illustrate dual representations (2) and (3), consider the representations of $V_1(t)$ in E_3. The family of curves in E_3, given in the parametric form

$$x^1 = a(t)\cos u, \quad x^2 = a(t)\sin u, \quad x^3 = b(t)\sin 2u,$$

may be represented as the intersection of circular cylinders:

$$f^{(1)}(x^1, x^2; t) = (x^1)^2 + (x^2)^2 - a(t)^2 = 0,$$

and as hyperbolic paraboloids:

$$f^{(2)}(x^1, x^2, x^3; t) = \frac{b(t)}{a(t)^2} x^1 x^2 - x^3 = 0.$$

Each of the two representations has advantages and further we shall make use of both. While the representation (2) provides a convenient description of kinematics of $V_M(t)$, it is dependent on parametrization. On the other hand, the representation (3) is independent of parameterization, i.e., independent of the choice of intrinsic coordinate system u^α. It means that any transformation of intrinsic coordinate systems:

$$u^\alpha = u^\alpha(U^\Lambda, t), \qquad J(U^\Lambda, t) = \left| \frac{\partial u^\alpha}{\partial U^\Lambda} \right| \neq 0,$$

does not change the representation (3). Consequently, the vectors $\mathrm{grad} f^{(\pi)}$, $\pi = 1, 2, \ldots, N - M$, are independent of parameterization u^α. Moreover, they are linearly independent because of (4), and may be taken as the base in V_{N-M}, the complement of $V_M(t)$ in V_N.

Consider a different parametrization $\boldsymbol{U}$ such that $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{U}, t)$. Then,

$$V_M(t): \quad \boldsymbol{x} = \hat{\boldsymbol{x}}(\boldsymbol{U}, t) = \boldsymbol{x}(\boldsymbol{u}(\boldsymbol{U}, t), t). \tag{5}$$

From this, we have:

$$\boldsymbol{A}_\Gamma = \frac{\partial u^\alpha}{\partial U^\Gamma} \boldsymbol{a}_\alpha,$$

where

$$\boldsymbol{a}_\alpha(u^\gamma, t) = \frac{\partial \mathbf{x}}{\partial u^\alpha} = \left(\frac{\partial x^i}{\partial u^\alpha} \right) \text{ and } \boldsymbol{A}_\Gamma(U^\Delta, t) = \frac{\partial \hat{\mathbf{x}}}{\partial U^\Gamma} = \left(\frac{\partial \hat{x}^i}{\partial U^\Gamma} \right)$$

are the basis vectors of $V_M(t)$ with respect to coordinate systems u^α and U^Γ, respectively. Moreover, since $x^k(u^\alpha, t)$ in (2) satisfies (3), it follows that:

$$\frac{\partial f^{(\pi)}}{\partial x^k} \frac{\partial x^k}{\partial u^\alpha} = \mathrm{grad} f^{(\pi)} \cdot \boldsymbol{a}_\alpha = 0, \tag{6}$$

whence we conclude that $\mathrm{grad} f^{(\pi)}$ are orthogonal to $\boldsymbol{a}_\alpha$, and hence to any vector in $V_M(t)$. They are also orthogonal to covariant base vectors $\boldsymbol{a}^\alpha$ in $V_M(t)$ defined through the relations $\boldsymbol{a}^\alpha \cdot \boldsymbol{a}_\beta = \delta^\alpha_\beta$. Since the set of vectors $\boldsymbol{a}_\alpha$ and $\mathrm{grad} f^{(\pi)}$ are linearly independent, they may be taken as the base vectors of the space V_N. It is important to notice that unit vectors

$$\mathbf{n}^{(\pi)} = \mathrm{grad} f^{(\pi)} \Big/ \left| \mathrm{grad} f^{(\pi)} \right|$$

are also independent of parameterization. Moreover, from (8), it follows that:

$$\mathbf{n}^{(\pi)} \cdot \boldsymbol{a}_\alpha = 0. \tag{7}$$

Further, we make use of vectors $\mathbf{n}_{(\pi)}$, reciprocal to $\mathbf{n}^{(\pi)}$ in V_{N-M}, defined by:

$$\mathbf{n}^{(\pi)} \cdot \mathbf{n}_{(\sigma)} = \delta^\pi_\sigma, \quad \pi, \sigma = 1, \ldots., N - M.$$

Here, $\boldsymbol{n}^{(\pi)}$ and $\boldsymbol{n}_{(\pi)}$ are both orthogonal to $V_M(t)$, but neither set need not be orthonormal. In V_{N-M}, we may consider $\mathbf{n}^{(\pi)}$ (or $\boldsymbol{n}_{(\pi)}$) as non-holonomic basis. Let

$$n^{\pi\tau} = \boldsymbol{n}^{(\pi)} \cdot \boldsymbol{n}^{(\tau)}, \qquad n^{\pi\tau} = n^{\tau\pi}, \qquad \det(n^{\pi\tau}) \neq 0.$$

Then, upon defining $\boldsymbol{n}_{(\pi)} \cdot \boldsymbol{n}_{(\tau)} = n_{\pi\tau}$, it follows that:

$$n^{\pi\tau} n_{\tau\sigma} = \delta^{\pi}_{\sigma}, \quad \boldsymbol{n}_{(\pi)} = n_{\pi\tau} \boldsymbol{n}^{(\tau)}, \quad \mathbf{n}^{(\pi)} = n^{\pi\tau} \mathbf{n}_{(\tau)}.$$

Coefficients $n_{\pi\tau}$ and $n^{\pi\tau}$ may be used as a tool for raising and lowering indices for the quantities defined in the above defined non-holonomic basis. Note that $n^{\pi\tau}$ do not depend on parameterization of $V_M(t)$, and neither do the coefficients $n_{\tau\sigma}$, nor vectors $\boldsymbol{n}_{(\pi)}$.

Then, vectors $\boldsymbol{g}^k$ at the points $\boldsymbol{x}$ of $V_M(t)$ may be decomposed as:

$$\boldsymbol{g}^k = x^k{}_{,\alpha} \boldsymbol{a}^{\alpha} + n^k_{(\pi)} \mathbf{n}^{(\pi)}; \quad x^k{}_{,\alpha} = \frac{\partial x^k}{\partial u^{\alpha}} = \boldsymbol{g}^k \cdot \boldsymbol{a}_{\alpha}, \quad n^k_{(\pi)} = \boldsymbol{g}^k \cdot \mathbf{n}_{(\pi)}.$$

This relation is of crucial importance for decomposition of any tensor quantity defined on $V_M(t)$ into *tangent* components in $V_M(t)$ and *normal* components in V_{N-M}.

We will often make use of the relation between metric tensors of $V_M(t)$ defined in two coordinate systems u^{α} and U^{Λ}, i.e., the relation

$$A_{\Lambda\Delta}(U^{\Gamma}, t) = g_{kl} \frac{\partial \hat{x}^k}{\partial U^{\Delta}} \frac{\partial \hat{x}^l}{\partial U^{\Delta}} = a_{\gamma\delta} \frac{\partial u^{\gamma}}{\partial U^{\Lambda}} \frac{\partial u^{\delta}}{\partial U^{\Delta}}. \tag{8}$$

Then,

$$A = A(U^{\Lambda}, t) = J^2 a, \qquad A(U^{\Lambda}, t) = \det(A_{\Gamma\Delta}) > 0. \tag{9}$$

The metric tensor $a_{\alpha\beta}(t)$ and the volume element $dv_{(M)}(t)$ of $V_M(t)$ are given by the expressions

$$a_{\alpha\beta}(t) = g_{kl} \frac{\partial x^k}{\partial u^{\alpha}} \frac{\partial x^l}{\partial u^{\beta}}, \qquad a(t) = \det(a_{\alpha\beta}) > 0, \qquad dv_{(M)}(t) = a^{1/2} d\tau_{(M)}(t),$$

where $d\tau_{(M)}(t) = \left| d_{(a)} u^{\alpha} \right|$ is the *extension* [8].

3. Kinematics of $V_M(t) \subseteq V_N$

Making use of (3), we obtain:

$$\frac{\partial f^{(\pi)}}{\partial t} + \mathrm{grad} f^{(\pi)} \cdot \mathbf{v} = 0,$$

where $\boldsymbol{v} = d\boldsymbol{x}/dt$. By d/dt we denote the time derivative along $u^{\alpha} = const$. Note that:

$$V^{(\pi)} = \boldsymbol{v} \cdot \boldsymbol{n}^{(\pi)} = -\frac{1}{\left| \mathrm{grad} f^{(\pi)} \right|} \frac{\partial f^{(\pi)}}{\partial t}, \tag{10}$$

represents the scalar-normal velocity in $\mathbf{n}^{(\pi)}$ direction which is independent of parameterization. Then, we can write:

$$\boldsymbol{v} = v^{\alpha} \boldsymbol{a}_{\alpha} + V^{(\pi)} \boldsymbol{n}_{(\pi)} = \boldsymbol{v}_{\tan} + V^{(\pi)} \boldsymbol{n}_{(\pi)}, \tag{11}$$

where

$$\boldsymbol{v}_{\tan} = v^{\alpha} \boldsymbol{a}_{\alpha}$$

denotes the component of $\boldsymbol{v}$ in $V_M(t)$. This notation for vectors in $V_M(t)$ is used throughout the paper. Obviously, it is dependent on parameterization since $\boldsymbol{a}_{\alpha}$ are.

Note that $V^{(\pi)} \mathbf{n}_{(\pi)}$ is independent of parameterization, so that $\boldsymbol{v}$ depends on parameterizations only through $\boldsymbol{v}_{\tan}$. This property of $\boldsymbol{v}_{\tan}$ is very important for writing various forms of transport theorem.

We will consider the coordinate system U^{Λ} as a convected system in $V_M(t)$. In continuum mechanics, the terms material or Lagrangian coordinates are used for convected coordinates, since the material particles can be labeled by these coordinates and, as such, they do not change their values during the motion of the material body. Note that, for a moving non-material domain, the choice of convected coordinates is arbitrary. However, once chosen, they remain fixed. The final result-the transport theorem-will be expressed in terms independent of the choice of coordinates.

At any time t, we consider the relation

$$u^{\alpha} = u^{\alpha}(U^{\Lambda},t), \qquad J(U^{\Lambda},t) = \left|\frac{\partial u^{\alpha}}{\partial U^{\Lambda}}\right| \neq 0,$$

as the coordinate transformation between the convected coordinate system U^{Λ} and intrinsic coordinate system u^{α}. Further, we consider

$$\boldsymbol{V} = \frac{\partial \widehat{\boldsymbol{x}}}{\partial t}$$

as the velocity of $\boldsymbol{x} \in V_M(t)$ in V_N along the path of convected coordinates U^{Λ}, i.e., $U^{\Lambda} = C^{\Lambda}$, where C^{Λ} are some constants. Generally, we denote by $\delta/\delta t$ the time derivative of any quantity $\boldsymbol{T}(U^{\Lambda},t)$, while keeping $U^{\Lambda} = C^{\Lambda}$. Of course,

$$\frac{\delta \boldsymbol{T}(U^{\Lambda},t)}{\delta t} = \frac{\partial \boldsymbol{T}(U^{\Lambda},t)}{\partial t}.$$

In particular: $\delta\widehat{\boldsymbol{x}}/\delta t = \partial\widehat{\boldsymbol{x}}/\partial t$.

From (5), we obtain the relation between velocities $\boldsymbol{v}$ and $\boldsymbol{V}$ of the point $\boldsymbol{x} \in V_M(t)$:

$$\boldsymbol{V} = \frac{\partial \widehat{\boldsymbol{x}}}{\partial t} = \frac{\delta \boldsymbol{x}}{\delta t} = \frac{d\boldsymbol{x}}{dt} + \frac{\partial \boldsymbol{x}}{\partial u^{\alpha}}\frac{\partial u^{\alpha}}{\partial t} = \boldsymbol{v} + \frac{\partial u^{\alpha}}{\partial t}\boldsymbol{a}_{\alpha}. \tag{12}$$

From (12), (10), and (7), we conclude that:

$$\mathrm{V}^{(\pi)} = \boldsymbol{V}\cdot\boldsymbol{n}^{(\pi)} = \mathbf{v}\cdot\boldsymbol{n}^{(\pi)} = -\frac{1}{\left|\mathrm{grad} f^{(\pi)}\right|}\frac{\partial f^{(\pi)}}{\partial t}, \tag{13}$$

thus proving that the scalar-normal velocity in $\mathbf{n}^{(\pi)}$ direction is independent of parameterization.

Instead of (13), a more compact representation is given by:

$$\boldsymbol{V} = \mathrm{V}^{(\pi)}\mathbf{n}_{(\pi)} + \boldsymbol{V}_{\tan}; \quad \boldsymbol{V}_{\tan} = \left(v^{\alpha} + \frac{\partial u^{\alpha}}{\partial t}\right)\boldsymbol{a}_{\alpha}. \tag{14}$$

Alternatively,

$$\boldsymbol{V}_{\tan} = V^{\Lambda}\boldsymbol{A}_{\Lambda}; \qquad V^{\Lambda} = \left(v^{\alpha} + \frac{\partial u^{\alpha}}{\partial t}\right)\frac{\partial U^{\Lambda}}{\partial u^{\alpha}}.$$

We may also write:

$$\boldsymbol{V} = \mathrm{V}_{(\pi)}\mathbf{n}^{(\pi)} + \boldsymbol{V}_{\tan}; \quad \mathrm{V}_{(\pi)} = n_{\pi\tau}\mathrm{V}^{(\tau)}.$$

In particular, for the hypersurface ($M = N-1$): $n^{11} = n_{11} = 1$, and $\mathrm{V}_{(1)} = \mathrm{V}^{(1)}$.

4. Generalized Transport Theorem in V_N

Let $\varphi(\boldsymbol{x},t)$ be defined on $R_M(t)$. We want to calculate

$$\frac{\delta}{\delta t}\int_{R_M(t)} \varphi(\boldsymbol{x},t)\,dv_{(M)}$$

along the path of convected coordinates U^Λ. Since $d_{(a)}u^\alpha = \frac{\partial u^\alpha}{\partial U^\Lambda} d_{(a)}U^\Lambda$, then according to [8] (p. 262):

$$dv_{(M)} = \sqrt{a}\, d\tau_{(M)} = \sqrt{a}\left|d_{(a)}u^\alpha\right| = \sqrt{A}\left|d_{(a)}U^\Lambda\right|. \tag{15}$$

Next, making use of (5), we write:

$$\varphi(\boldsymbol{x},t) = \varphi\Big(\boldsymbol{x}\big(\boldsymbol{u}(\boldsymbol{U},t),t\big),t\Big) = \widehat{\varphi}(\boldsymbol{U},t).$$

Thus, in view of this and (15) we have:

$$\frac{\delta}{\delta t}\int_{R_M(t)} \varphi(\boldsymbol{x},t)dv_{(M)} = \int_{R_M} \frac{\delta}{\delta t}(\widehat{\varphi}\sqrt{A})\left|d_{(a)}U^\alpha\right| = \int_{R_M}\left[\frac{\partial\widehat{\varphi}}{\partial t}\sqrt{A} + \widehat{\varphi}\frac{\partial}{\partial t}\sqrt{A}\right]\left|d_{(a)}U^\alpha\right|, \tag{16}$$

where R_M is defined with respect to convected coordinates U^Λ, and therefore the integration with respect to U^Λ is independent of t. Now, we show in Appendix A, that:

$$\frac{\partial}{\partial t}\sqrt{A} = \frac{1}{2\sqrt{A}}\frac{\partial A}{\partial t} = \frac{1}{2}\sqrt{A}A^{\Lambda\Delta}\,\frac{\partial A_{\Lambda\Delta}}{\partial t} = \sqrt{A}\left(\mathrm{div}_{V_M}\mathbf{V}_{\tan} - \mathrm{V}^{(\pi)}K_{(\pi)}\right), \tag{17}$$

where

$$\mathrm{div}_{V_M}\mathbf{V}_{\tan} = A^{\Lambda\Delta}\mathrm{V}_{\Lambda},_{\Delta} = a^{\alpha\beta}\mathrm{V}_{\alpha},_{\beta},$$

and

$$K_{(\pi)} = A^{\Lambda\Delta}\Omega_{\Lambda\Delta|(\pi)},$$

where $\Omega_{\Lambda\Delta|(\pi)}$ is defined in Appendix A.

The geometric meaning of $K_{(\pi)}$ is evident from

$$K_{(\pi)} = -\boldsymbol{A}^\Lambda\cdot\frac{\partial\mathbf{n}_{(\pi)}}{\partial U^\Lambda} = -\mathrm{Div}_{V_M(t)}\mathbf{n}_{(\pi)} = -\boldsymbol{a}^\alpha\cdot\frac{\partial\mathbf{n}_{(\pi)}}{\partial u^\alpha} = -\mathrm{div}_{V_M(t)}\mathbf{n}_{(\pi)}, \tag{18}$$

which is independent of parameterization. Further,

$$\begin{aligned}\frac{\partial}{\partial t}\widehat{\varphi}(\boldsymbol{U},t) &= \frac{\delta}{\delta t}\varphi(\mathbf{x},t) = \frac{\partial\varphi}{\partial t} + \mathrm{grad}\varphi\cdot\mathbf{V} = \frac{\partial\varphi}{\partial t} + \mathrm{grad}\varphi\cdot\left(\mathrm{V}^\Lambda\boldsymbol{A}_\Lambda + \mathrm{V}^{(\pi)}\mathbf{n}_{(\pi)}\right)\\ &= \frac{\partial\varphi}{\partial t} + \mathrm{V}^\Lambda\varphi,_\Lambda + \mathrm{V}^{(\pi)}\mathrm{grad}\varphi\cdot\mathbf{n}_{(\pi)} = \frac{\partial\varphi}{\partial t} + \mathbf{V}_{\tan}\cdot\mathrm{Grad}_{\mathrm{V_M}}\varphi + \mathrm{V}^{(\pi)}\frac{\partial\varphi}{\partial n_{(\pi)}},\end{aligned}$$

where

$$\mathrm{Grad}_{\mathrm{V_M}}\varphi = \varphi,_\Lambda\boldsymbol{A}^\Lambda \quad\text{and}\quad \frac{\partial\varphi}{\partial n_{(\pi)}} = \mathrm{grad}\varphi\cdot\mathbf{n}_{(\pi)}.$$

The expression

$$\overset{\square}{\varphi} = \frac{\partial\varphi}{\partial t} + \mathrm{V}^{(\pi)}\frac{\partial\varphi}{\partial n_{(\pi)}}$$

represents the normal time derivative of φ following convected $V_M(t)$. Clearly, $\overset{\square}{\varphi}$ is independent of parameterization.

Remark 1. *To justify its name, we write* $\mathrm{V}^{(\pi)}\mathbf{n}_{(\pi)}$ *in a unique compact invariant form as:*

$$\mathrm{V_n}\,\mathbf{n} = \mathrm{V}^{(\pi)}\mathbf{n}_{(\pi)},$$

where $\mathbf{n}$ *is a unit vector orthogonal to* $R_M(t)$. *Then, we have:*

$$\mathrm{V_n} = \sqrt{\mathrm{V}^{(\pi)}\mathrm{V}^{(\rho)}\mathbf{n}_{(\pi)}\cdot\mathbf{n}_{(\rho)}} = \sqrt{n_{\pi\rho}\mathrm{V}^{(\pi)}\mathrm{V}^{(\rho)}} \ ,$$

and $\mathbf{n} = \mathrm{V}^{(\pi)}\mathbf{n}_{(\pi)} \big/ \mathrm{V_n}$. *Accordingly, we have simple expression for* $\mathbf{V}$: $\mathbf{V} = \mathbf{V}_{\tan} + \mathrm{V_n}\mathbf{n}$. *Since* $\mathbf{V}_{\tan}\cdot\mathbf{n} = 0$ *, we have* $\mathrm{V_n} = \mathbf{V}\cdot\mathbf{n}$ *, which is of the form of classical expression for the scalar normal-velocity of the surface evolving in three dimensional Euclidean space. Then:*

$$\overset{\square}{\varphi} = \frac{\partial\varphi}{\partial t} + \mathrm{V}^{(\pi)}\frac{\partial\varphi}{\partial \mathrm{n}_{(\pi)}} = \frac{\partial\varphi}{\partial t} + \mathrm{V}^{(\pi)}\mathrm{grad}\varphi\cdot\mathbf{n}_{(\pi)} = \frac{\partial\varphi}{\partial t} + \mathrm{V_n}\mathrm{grad}\varphi\cdot\mathbf{n} \, .$$

The case $N = 3$ *has been discussed previously in [10,15].*

Remark 2. *Making use of* $\boldsymbol{a}^{\alpha} = \dfrac{\partial u^{\alpha}}{\partial U^{\boldsymbol{\Gamma}}}\boldsymbol{A}^{\boldsymbol{\Gamma}}$ *, it easy to see that:*

$$\mathrm{Grad}_{\mathrm{V_M}}\varphi = \varphi,_{\boldsymbol{\Gamma}}\boldsymbol{A}^{\boldsymbol{\Gamma}} = \varphi,_{\alpha}\boldsymbol{a}^{\alpha} = \mathrm{grad}_{\mathrm{V_M}(t)}\varphi, \qquad \mathrm{Div}_{\mathrm{V_M}}\mathbf{V}_{\tan} = \mathrm{div}_{\mathrm{V_M}(t)}\mathbf{V}_{\tan}\, .$$

Therefore,

$$\frac{\partial}{\partial t}\widehat{\varphi}(\boldsymbol{U},t) = \frac{\delta}{\delta t}\varphi(\mathbf{x},t) = \overset{\square}{\varphi} + \mathbf{V}_{\tan}\cdot\mathrm{Grad}_{\mathrm{V_M}}\varphi = \overset{\square}{\varphi} + \mathbf{V}_{\tan}\cdot\mathrm{grad}_{\mathrm{V_M}}\varphi \, . \tag{19}$$

After substituting (18) and (19) into (16), we obtain:

$$\begin{aligned} &\frac{\delta}{\delta t}\int_{R_M(t)}\varphi(\boldsymbol{x},t)\,dv_{(M)} \\ &= \int_{R_M}\left[\left(\overset{\square}{\varphi} + \mathbf{V}_{\tan}\cdot\mathrm{Grad}_{\mathrm{V_M}}\varphi\right) + \widehat{\varphi}\left(\mathrm{Div}_{\mathrm{V_M}}\mathbf{V}_{\tan} - \mathrm{V}^{(\pi)}K_{(\pi)}\right)\right]\sqrt{A}\left|d_{(\beta)}U^{\alpha}\right| \\ &= \int_{R_M(t)}\left[\left(\overset{\square}{\varphi} + \mathbf{V}_{\tan}\cdot\mathrm{grad}_{\mathrm{V_M}(t)}\varphi\right) + \varphi\left(\mathrm{div}_{\mathrm{V_M}(t)}\mathbf{V}_{\tan} - \mathrm{V}^{(\pi)}K_{(\pi)}\right)\right]dv_{(M)} \, . \end{aligned}$$

However,

$$\mathbf{V}_{\tan}\cdot\mathrm{grad}_{\mathrm{V_M}(t)}\varphi + \varphi\mathrm{div}_{\mathrm{V_M}(t)}\mathbf{V}_{\tan} = \mathrm{div}_{\mathrm{V_M}(t)}(\varphi\mathbf{V}_{\tan})\, ,$$

so that

$$\frac{\delta}{\delta t}\int_{R_M(t)}\varphi(\boldsymbol{x},t)dv_{(M)} = \int_{R_M(t)}\left[\left(\overset{\square}{\varphi} - \varphi\mathrm{V}^{(\pi)}K_{(\pi)}\right) + \mathrm{div}_{\mathrm{V_M}(t)}(\varphi\boldsymbol{V}_{\tan})\right]dv_{(M)} \, . \tag{20}$$

Further, using divergence theorem we obtain (when the boundary consists of "material" points defined by convected coordinates, see Appendix B):

$$\frac{\delta}{\delta t}\int_{R_M(t)}\varphi(\boldsymbol{x},t)dv_{(M)} = \int_{R_M(t)}\left(\overset{\square}{\varphi} - \varphi\mathrm{V}^{(\pi)}K_{(\pi)}\right)dv_{(M)} + \int_{\partial R_M(t)}\varphi\boldsymbol{V}_{\tan}\cdot\boldsymbol{\nu}_{(M-1)}dv_{(M-1)}\, , \tag{21}$$

where $\boldsymbol{\nu}_{(M-1)}$ is normal to $\partial R_M(t)$ but tangent to $R_M(t)$. The scalar normal velocity of $\partial R_M(t) \subset R_M(t)$,

$$\mathrm{V}_{\nu} = \boldsymbol{V}_{\tan}\cdot\boldsymbol{\nu}_{(M-1)}\, ,$$

is intrinsic to the motion of $\partial R_M(t)$. (See [10] for the case $N = 3$). Finally, the transport theorem with respect to convective coordinate reads:

$$\frac{\delta}{\delta t}\int\limits_{R_M(t)}\varphi(\boldsymbol{x},t)dv_{(M)}=\int\limits_{R_M(t)}\left(\overset{\square}{\varphi}-\varphi\mathrm{V}^{(\pi)}K_{(\pi)}\right)dv_{(M)}+\int\limits_{\partial R_M(t)}\varphi\,\mathrm{V}_{\nu}dv_{(M-1)} \tag{22}$$

We emphasize that the first integral is intrinsic to $R_M(t)$, while the second integral is intrinsic with respect to $\partial R_M(t)$. Since the first integral on the right-hand side of (22) is invariant to any parameterization we may equally apply it for parameters u^{α}. In this case, the scalar normal velocity of $\partial R_M(t)$ with respect to parameters u^{α}, i.e., $\boldsymbol{v}_{|\partial R_M(t)}\cdot\boldsymbol{\nu}_{(M-1)}$ needs to be used (Appendix B). Then, we write the generalized transport theorem with respect to non-convective coordinates u^{α} in the form:

$$\frac{\delta}{\delta t}\int\limits_{R_M(t)}\varphi(\boldsymbol{x},t)dv_{(M)}=\int\limits_{R_M(t)}\left(\overset{\square}{\varphi}-\varphi\mathrm{V}^{(\pi)}K_{(\pi)}\right)dv_{(M)}+\int\limits_{\partial R_M(t)}\varphi\boldsymbol{v}_{|\partial R_M(t)}\cdot\boldsymbol{\nu}_{(M-1)}dv_{(M-1)} \tag{23}$$

5. Examples

In this section, we consider familiar special cases and motivate the subsequent analysis of material domains and propagating discontinuity fronts. The section ends with the analysis of the capillary flow problem, which encompasses many of the special cases.

In all the cases below, the transport theorems holds also for the coordinates u^{α}, i.e., (22), when we substitute $\delta/\delta t$ with d/dt and $\mathrm{V}_{\nu}=\mathbf{V}\cdot\boldsymbol{\nu}=\mathbf{V}_{\tan}\cdot\boldsymbol{\nu}$, with $\boldsymbol{v}_{|\partial R_M(t)}\cdot\boldsymbol{\nu}_{(M-1)}$, as defined in Appendix B.

5.1. 3D Domain Moving in 3D Space

$M=N=3$, $K=0$ and $\mathrm{V}^{(0)}=0$. Let $R_3(t)=V(t)$, $\partial R_3(t)=\partial V(t)$, $dV_{(3)}=dV$ and $dV_{(2)}=da$. Then, from (22):

$$\frac{d}{dt}\int\limits_{V(t)}\varphi(\boldsymbol{x},t)\,dV=\int\limits_{V(t)}\frac{\partial\varphi}{\partial t}dV+\int\limits_{\partial V(t)}\varphi\,\mathrm{V}_{\nu}da\,,$$

where $\mathrm{V}_{\nu}=\mathbf{V}_{\tan}\cdot\boldsymbol{\nu}$; $\boldsymbol{\nu}$ is unit vector in $V(t)$ orthogonal to $\partial V(t)$ [1,10,12,13].

On the other hand, the familiar form of the transport theorem for the material body $V(t)$ and the material field $\Phi(\boldsymbol{x},t)$ given per unit mass [$\varphi(\boldsymbol{x},t)=\rho(\boldsymbol{x},t)\Phi(\boldsymbol{x},t)$ where $\rho(\boldsymbol{x},t)$ is mass density], reads:

$$\frac{d}{dt}\int\limits_{V(t)}\varphi(\boldsymbol{x},t)dV=\frac{d}{dt}\int\limits_{V(t)}\rho\Phi dV=\int\limits_{V(t)}\rho\frac{\partial\Phi}{\partial t}dV\,. \tag{24}$$

The volume $V(t)$ in (24) is the material volume and the motion of the each material point $\boldsymbol{v}(\boldsymbol{x},t)$ is fully defined as the motion of point mass $\rho(\boldsymbol{x},t)dV$.

5.2. Surface Moving in 3D Space

$M=2$; $N=3$. Let: $R_2(t)=A(t)$ with unit normal $\boldsymbol{n}$, $K_{(1)}=K=-\mathrm{div}_{A(t)}\boldsymbol{n}$. Then, from (22), with $dv_{(1)}=ds$, we have:

$$\frac{d}{dt}\int\limits_{A(t)}\varphi(\boldsymbol{x},t)dA=\int\limits_{A(t)}\left(\frac{\partial\varphi}{\partial t}+V_n\boldsymbol{n}\cdot\mathrm{grad}\varphi-\varphi VK\right)dA+\oint\limits_{\partial A(t)}\varphi V_{\nu}ds\,, \tag{25}$$

5.3. Line Moving in a 3D Space

$M=1$, $N-M=2$. Let $s(t)$ $(s_1<s<s_2)$ be the arc length. Equivalently, the intersection of surfaces $S^{(\pi)}(t)$: $f^{(\pi)}(x^k,t)=0$, $\pi=1,2,$ defines $s(t)$. We denote by $\boldsymbol{n}_{(\pi)}$ and

$K_{(\pi)} = -\mathrm{div}_{R_1(t)}\boldsymbol{n}_{(\pi)}$ the unit normal to $S^{(\pi)}(t)$ and its total curvature. $\mathrm{V}^{(\pi)}$ is defined through (13). Then, from (22), we have:

$$\frac{d}{dt}\int\limits_{s(t)} \varphi(\boldsymbol{x},t)\,ds = \int\limits_{s(t)} \left(\frac{\partial\varphi}{\partial t} + \mathrm{grad}\varphi\cdot \mathrm{V}^{(\pi)}\mathbf{n}_{(\pi)} - \varphi V^{(\pi)}K_{(\pi)}\right)ds + \varphi V_\nu\Big|_{s_1}^{s_2}.$$

5.4. Closed Line Moving in a 3D Space

Define the unit tangent, normal, and binormal, $\boldsymbol{t}$, $\boldsymbol{m}$ and $\boldsymbol{b}$, in the standard way, the curvature κ ($K^{(\pi)}\boldsymbol{n}_{(\pi)} = \kappa\boldsymbol{m}$) and $\mathrm{V_n}\,\mathbf{n} = \mathrm{V}^{(\pi)}\mathbf{n}_{(\pi)} = (V_m\boldsymbol{m} + V_b\boldsymbol{b})$. Then:

$$\frac{d}{dt}\oint\limits_{s(t)} \varphi(\boldsymbol{x},t)ds = \oint\limits_{s(t)} \left(\frac{\partial\varphi}{\partial t} + (V_m\boldsymbol{m} + V_b\boldsymbol{b})\cdot\mathrm{grad}\varphi - \varphi V_m\kappa\right)ds\,.$$

As a special case, we compute the change of the total line energy arising from the line energy density $\varphi(\boldsymbol{x},t) \equiv \sigma = \text{constant}$:

$$\frac{d}{dt}\oint\limits_{s(t)} \sigma ds = -\sigma\oint\limits_{s(t)} V_m\kappa\, ds\,. \tag{26}$$

In anticipation of the next example, we establish the relationship between the normal line velocity $V_m\boldsymbol{m}$ and one of the surfaces. Let the surface with the normal $\boldsymbol{n}^{(2)}$ be a material surface, moving with the velocity $\boldsymbol{w}^{(2)}$. The line then glides on this surface with the velocity $\xi\mathbf{v}^{(2)}$ relative to the surface (2). The unit normal-tangent vectors point outside the respective surfaces:

$$\mathbf{v}^{(1)} = -\boldsymbol{n}^{(1)}\times\boldsymbol{t}; \quad \mathbf{v}^{(2)} = \boldsymbol{n}^{(2)}\times\boldsymbol{t}\,.$$

The normal component of the line velocity the normal-tangent component relative to the surface (1) are then:

$$V_m = \left[\boldsymbol{w}^{(2)} + \xi\mathbf{v}^{(2)}\right]\cdot\boldsymbol{m}; \quad \mathrm{V}_{\mathbf{v}} = \left[\boldsymbol{w}^{(2)} + \xi\mathbf{v}^{(2)}\right]\cdot\mathbf{v}^{(1)}\,. \tag{27}$$

5.5. Interface Energies for Liquid Drop on a Solid Surface

The free liquid–gas interface $a(t)$ has the unit normal $\boldsymbol{n}^{(1)}$ pointing outside the liquid and interface energy γ. The drop slides on the solid (rigid) surface $A(t)$ with the unit normal $\boldsymbol{n}^{(2)}$ pointing inside the solid and outside the liquid. The difference between solid-liquid and solid-gas interface energies is $\Delta\gamma$. The velocities of two surfaces are denoted $\boldsymbol{w}^{(1)}$ and $\boldsymbol{w}^{(2)}$. The total interface energy can be written as:

$$\Gamma = \int\limits_{a(t)} \gamma da + \int\limits_{A(t)} \Delta\gamma dA\,.$$

The rate of change of the 1st integral is given by (25). The boundary of the 2nd domain $\partial A(t)$ now represents the line of discontinuity, so that the rate of change of the 2nd integral is

$$\frac{d}{dt}\int\limits_{A(t)} \Delta\gamma dA = \oint\limits_{s(t)} \Delta\gamma\xi ds\,. \tag{28}$$

Then, from (25) and (28):

$$\frac{d\Gamma}{dt} = -\gamma\int\limits_{a(t)} \boldsymbol{w}^{(1)}\cdot\boldsymbol{n}^{(1)}K da + \gamma\oint\limits_{s(t)} \left[\boldsymbol{w}^{(2)} + \xi\boldsymbol{v}^{(2)}\right]\cdot\boldsymbol{v}^{(1)}ds + \Delta\gamma\oint\limits_{s(t)} \xi ds\,, \tag{29}$$

where K is twice the mean curvature of the surface $a(t)$.

5.6. Capillary Flow

We consider an incompressible liquid flowing over a rigid solid and the surrounded by a gas with negligible viscosity and mass density, so that, without loss of generality, we assume the uniform vanishing pressure in the gas. The motion of the solid surface $\boldsymbol{w}^{(2)}$ is prescribed and the triple line glides on the solid surface with the relative velocity $\xi\boldsymbol{v}^{(2)}$.

The total energy of the system can be written as the sum of the bulk energy (kinetic + gravitational potential), the interface energies and the line energy:

$$E = \int_{V(t)} \left[\frac{1}{2}\rho\boldsymbol{v}\cdot\boldsymbol{v} + \rho\psi(\boldsymbol{x})\right] dV + \int_{a(t)} \gamma da + \int_{A(t)} \Delta\gamma dA + \oint_{s(t)} \sigma ds\,,$$

where $\psi(\boldsymbol{x})$ is the gravitational potential. Although the liquid–gas interface is not a material surface, the normal component of its velocity is identical to corresponding component of the material velocity of the fluid:

$$\boldsymbol{w}^{(1)}\cdot\boldsymbol{n}^{(1)} = \boldsymbol{v}\cdot\boldsymbol{n}^{(1)}\,.$$

Using (24), (26), (27) and (29), the rate of change of the total energy is

$$\begin{aligned}\frac{dE}{dt} &= \int_{v(t)} \rho\left[\frac{d\boldsymbol{v}}{dt} + \mathrm{grad}\psi\right]\cdot\boldsymbol{v}dv - \int_{a(t)} \gamma K\boldsymbol{n}^{(1)}\cdot\boldsymbol{v}da \\ &+ \oint_{s(t)} \left[\gamma\boldsymbol{v}^{(2)}\cdot\boldsymbol{v}^{(1)} + \Delta\gamma - \sigma\kappa\boldsymbol{v}^{(2)}\cdot\boldsymbol{m}\right]\xi ds + \oint_{s(t)} \gamma\boldsymbol{w}^{(2)}\cdot\boldsymbol{v}^{(1)}ds,\end{aligned} \tag{30}$$

where we have taken into account that for the rigid body motion $\boldsymbol{w}^{(2)}$:

$$\oint_{s(t)} \kappa\boldsymbol{w}^{(2)}\cdot\boldsymbol{m}ds = 0\,.$$

The last term in (30) is the correction arising from the rigid body motion of the whole assembly. To illustrate that point, consider a uniform translation $\boldsymbol{w}^{(2)} = const.$ The last term in (30) is then:

$$\gamma\boldsymbol{w}^{(2)}\cdot\oint_{s(t)} \boldsymbol{v}^{(1)}ds = \gamma\boldsymbol{w}^{(2)}\cdot\int_{a(t)} K\boldsymbol{n}^{(1)}da\,.$$

The liquid–gas surface contribution in (30) (for pure translation of the solid substrate) can then be written as:

$$-\int_{a(t)} \gamma K\boldsymbol{n}^{(1)}\cdot(\boldsymbol{v} - \boldsymbol{w}^{(2)})da\,.$$

The energy balance requires that the total power input P be equal to the sum of the rate of total energy and dissipation rate D. We include the incompressibility condition, $\mathrm{div}(\boldsymbol{v}) = 0$, with the Lagrange multiplier field p which is recognized as the pressure in the fluid:

$$P = \int_{A(t)} \boldsymbol{T}\cdot\boldsymbol{v}dA = \frac{dE}{dt} + D - \int_{v(t)} p\,\mathrm{div}(\boldsymbol{v})dv\,, \tag{31}$$

where $\boldsymbol{T}$ represents the traction vector exerted by the solid on the liquid. We assume, without loss of generality, that the pressure in the (inviscid) gas vanishes, so that the traction on $a(t)$ vanishes.

The flow at the re-entrant corner where the two surfaces intersect is singular (Taylor 1960). Within the sharp interface model the typical solution is to allow slip in some vicinity of the triple line [17]. Thus, at the solid surface we allow for slip, but not separation/penetration:

$$\boldsymbol{n}^{(2)} \cdot (\boldsymbol{v} - \boldsymbol{w}^{(2)}) = 0 \,. \tag{32}$$

The dissipation includes viscous dissipation in the bulk and the dissipation at the triple line:

$$\mathcal{D} = = \int_{v(t)} \boldsymbol{\tau} : \mathrm{grad}(\boldsymbol{v}) dv + \oint_{s(t)} Q \xi ds \,, \tag{33}$$

where $\boldsymbol{\tau}$ is the viscous stress (deviatoric for incompressible fluid: $\mathrm{tr}(\boldsymbol{\tau}) = 0$) and Q is the power conjugate of the triple line glide. Although we allow the slip at the solid surface, the dissipation (33) does not include this slip, owing to the choice of the external power input: $\boldsymbol{T} \cdot \boldsymbol{v}$. Had we represented the external power as $\boldsymbol{T} \cdot \boldsymbol{w}^{(2)}$, the slip dissipation would have to be included. We will shortly see that the triple line dissipation is necessary if the experimentally observed difference between advancing and receding triple lines is to be described. The 2nd law of thermodynamics for isothermal processes reduces to the requirement that dissipation be positive. Assuming that the constitutive law for the fluid satisfies this requirement on its own (as it does for Newtonian fluids), it follows that the triple line force Q must have the same sign as the triple line glide ξ . The simplest linear constitutive law is $Q = C\xi \quad (C > 0)$.

Upon substitution of (30) and (33) into (31) and some manipulation, we obtain the power balance equation:

$$\begin{aligned}
&\int_{A(t)} \left[\boldsymbol{T} - \boldsymbol{n}^{(2)} \cdot (\tau - p\boldsymbol{I}) \right] \cdot \boldsymbol{v} dA \\
&= \int_{v(t)} \left[\rho \frac{d\boldsymbol{v}}{dt} + \rho \mathrm{grad}\psi - \mathrm{div}(\tau - p\boldsymbol{I}) \right] \cdot \boldsymbol{v} dv + \int_{a(t)} \left[\boldsymbol{n}^{(1)} \cdot (\tau - p\boldsymbol{I}) - \gamma K \boldsymbol{n}^{(1)} \right] \cdot \boldsymbol{v} da \\
&+ \oint_{s(t)} \left[\gamma \boldsymbol{v}^{(2)} \cdot \boldsymbol{v}^{(1)} + \Delta\gamma - \sigma\kappa \boldsymbol{v}^{(2)} \cdot \boldsymbol{m} + Q \right] \xi ds + \oint_{s(t)} \gamma \boldsymbol{w}^{(2)} \cdot \boldsymbol{v}^{(1)} ds.
\end{aligned} \tag{34}$$

The simplest method for deriving the strong form of governing equations is to formulate the weak form directly via the Principle of virtual power (PVP) [18,19]. Application of the PVP yields the following governing equations and boundary conditions:

- The natural boundary condition on $A(t)$:

$$\mathrm{T} = \boldsymbol{n}^{(2)} \cdot (\tau - p\boldsymbol{I}) \,.$$

- The governing (Cauchy) equations of motion in $v(t)$:

$$\rho \frac{d\boldsymbol{v}}{dt} + \rho \mathrm{grad}\psi - \mathrm{div}(\tau - p\boldsymbol{I}) = 0 \,.$$

- The capillary jump in normal stress across $a(t)$: $\tau - p\boldsymbol{I} = \gamma K \boldsymbol{I}$. The deviatoric nature of the viscous stress τ then implies that on $a(t)$:

$$\tau = 0; \quad p = -\gamma K \,.$$

The sign in the pressure jump condition is the consequence of the choice of $\boldsymbol{n}^{(1)}$ as outward normal to the liquid surface.

- The contact angle condition: $\gamma \cos\theta + \Delta\gamma + \sigma\kappa \cos\beta + Q = 0$; with the contact angle θ : $\mathbf{v}^{(2)} \cdot \mathbf{v}^{(1)} = \cos\theta$; and the angle between the line normal and inside tangent $(-\boldsymbol{v}^{(2)})$: $-\mathbf{v}^{(2)} \cdot \boldsymbol{m} = \cos\beta$. This condition can be written with reference to the equilibrium contact angle:

$$Q = \gamma(\cos\theta_0 - \cos\theta); \qquad \cos\theta_0 = -\frac{\Delta\gamma + \sigma\kappa \cos\beta}{\gamma} \,. \tag{35}$$

In relation to (35) we note that, in the absence of line energy ($\sigma = 0$), the equilibrium contact angle satisfies the standard condition

$$\cos\theta_0 = -\frac{\Delta\gamma}{\gamma} = \frac{\gamma_{solid/gas} - \gamma_{solid/liquid}}{\gamma}.$$

Moreover, from (35) for the advancing contact line

$$\xi > 0 \Rightarrow Q > 0 \Rightarrow \theta > \theta_0.$$

Conversely, for the receding contact line the contact angle is smaller than the equilibrium contact angle. This is consistent with experimental observations [20]. Moreover, failure to include the triple line dissipation implies that the contact angle is always equal to the equilibrium one [21], which is in direct contradiction to experimental results.

Finally, we note that to complete the formulation, a definition of the slip constitutive law on $A(t)$ is needed. This is beyond the scope of the current paper. We only note that such definition will not change any of the derived equations.

6. Transport Theorem Depending on Parametrization

In some cases, it is useful to use transport theorem in terms depending on parameterization. This can be done simply if we substitute (19), now written as:

$$\frac{\delta}{\delta t}\varphi(\mathbf{x},t) = \overset{\square}{\varphi} + \mathbf{V}_{\tan}\cdot \mathrm{grad}_{V_M}\varphi$$

into (20). It is convenient to put it in more familiar form:

$$\begin{aligned}&\frac{\delta}{\delta t}\int\limits_{R_M(t)}\varphi(\boldsymbol{x},t)dv_{(M)}\\ &= \int\limits_{R_M(t)}\left[\frac{\delta\varphi(\boldsymbol{x},t)}{\delta t} - \mathbf{V}_{\tan}\cdot \mathrm{grad}_{V_M}\varphi + \varphi\mathbf{V}_{\tan}\mathrm{div}_{V_M(t)} - \varphi \mathrm{V}^{(\pi)}K_{(\pi)}\right]dv_{(M)}\end{aligned}$$

However: $-\mathbf{V}_{\tan}\cdot \mathrm{grad}_{V_M(t)}\varphi + \mathrm{div}_{V_M(t)}\left(\varphi\mathbf{V}_{\tan}\right) = \varphi\mathrm{div}_{V_M(t)}\mathbf{V}_{\tan}$, so that

$$\frac{\delta}{\delta t}\int\limits_{R_M(t)}\varphi(\boldsymbol{x},t)dv_{(M)} = \int\limits_{R_M(t)}\left[\frac{\delta}{\delta t}\varphi + \varphi\left(\mathrm{div}_{V_M(t)}\mathbf{V} - \mathrm{V}^{(\pi)}K_{(\pi)}\right)\right]dv_{(M)}$$

Further,

$$\begin{aligned}&-\mathrm{V}^{(\pi)}K_{(\pi)} = \mathrm{V}^{(\pi)}\mathrm{div}_{V_M(t)}\boldsymbol{n}_{(\pi)} = \mathrm{div}_{V_M(t)}\left(\mathrm{V}^{(\pi)}\boldsymbol{n}_{(\pi)}\right) - \mathrm{grad}_{V_M(t)}\mathrm{V}^{(\pi)}\cdot\boldsymbol{n}_{(\pi)} =\\ &= \mathrm{div}_{V_M(t)}\left(\mathrm{V}^{(\pi)}\boldsymbol{n}_{(\pi)}\right).\end{aligned}$$

Hence, $\mathrm{div}_{V_M(t)}\mathbf{V}_{\tan} - \mathrm{V}^{(\pi)}K_{(\pi)} = \mathrm{div}_{V_M(t)}\left(\mathbf{V}_{\tan} + \mathrm{V}^{(\pi)}\boldsymbol{n}_{(\pi)}\right) = \mathrm{div}_{V_M(t)}\mathbf{V}$, so that

$$\frac{\delta}{\delta t}\int\limits_{R_M(t)}\varphi(\boldsymbol{x},t)dv_{(M)} = \int\limits_{R_M(t)}\left[\frac{\delta\varphi(\boldsymbol{x},t)}{\delta t} + \varphi\,\mathrm{div}_{V_M(t)}\mathbf{V}\right]dv_{(M)}.$$

7. Migrating $V_M(t)$ in *N*-Dimensional Fluid

The transport relation given by (23) permits us to consider several particular cases of importance in continuum mechanics in a unified way. We consider *N*-dimensional fluid in analogy to 3-dimensional fluid. First, we denote by $\mathbf{w}(\mathbf{x},t)$ the velocity of the fluid. We may write it as:

$$\mathbf{w} = \mathrm{W}^{(\pi)}\mathbf{n}_{(\pi)} + \mathbf{w}_{\tan},$$

where

$$\mathbf{w}_{\tan} = \mathrm{w}^{\alpha}\boldsymbol{a}_{\alpha}\,.$$

Then, $\boldsymbol{v}_r = \boldsymbol{v} - \mathbf{w}$ represent is the relative velocities of motion of $V_M(t)$ with respect to the fluid. We may decompose it as:

$$\boldsymbol{v}_r = \boldsymbol{v} - \mathbf{w} = (\mathrm{V}^{(\pi)} - W^{(\pi)})\mathbf{n}_{(\pi)} + \boldsymbol{v}_{\tan} - \mathbf{w}_{\tan}\,.$$

Now, looking at the transport theorem the only term which may be influenced by motion of the fluid is $\boldsymbol{v}_{|\partial R_M(t)} \cdot \boldsymbol{\nu}_{(M-1)}$. For definiteness we write:

$$\boldsymbol{v}_r = \boldsymbol{v}_{|\partial R_M(t)} - \mathbf{w} \quad \text{and} \quad V^{mig}_{|\partial R_M(t)} = \boldsymbol{v}_r \cdot \boldsymbol{\nu}_{(M-1)},$$

for the relative velocities of motion and normal migrational velocity of $\partial R_M(t)$ with respect to the fluid. Then, since $\mathbf{n}_{(\pi)} \cdot \boldsymbol{\nu}_{(M-1)} = 0$,

$$V^{mig}_{|\partial R_M(t)} = \boldsymbol{v}_r \cdot \boldsymbol{\nu}_{(M-1)} = \boldsymbol{v}_{|\partial R_M(t)} \cdot \boldsymbol{\nu}_{(M-1)} - \mathbf{w}_{\tan} \cdot \boldsymbol{\nu}_{(M-1)},$$

and hence:

$$\boldsymbol{v}_{|\partial R_M(t)} \cdot \boldsymbol{\nu}_{(M-1)} = V^{mig}_{|\partial R_M(t)} + \mathbf{w}_{\tan} \cdot \boldsymbol{\nu}_{(M-1)}$$

Upon substituting this into (23), we obtain:

$$\frac{d}{dt}\int_{R_M(t)} \varphi(\boldsymbol{x},t)\,dv_{(M)} =$$
$$\int_{R_M(t)} \left(\overset{\square}{\varphi} - \varphi \mathrm{V}^{(\pi)} K_{(\pi)}\right) dv_{(M)} + \int_{\partial R_M(t)} \varphi \left(V^{mig}_{|\partial R_M(t)} + \mathbf{w}_{\tan} \cdot \boldsymbol{\nu}_{(M-1)}\right) dv_{(M-1)}.$$

Next, we use the divergence theorem, in the form

$$\int_{\partial R_M(t)} \varphi \mathbf{w}_{\tan} \cdot \boldsymbol{\nu}_{(M-1)} dv_{(M-1)} = \int_{R_M(t)} \mathrm{div}_{\mathrm{V}_M(t)}(\varphi \mathbf{w}_{\tan}) dv_{(M)}\,,$$

to obtain the transport theorem for migrating $V_M(t)$ in the fluid, in the form

$$\begin{aligned} &\frac{d}{dt}\int_{R_M(t)} \varphi(\boldsymbol{x},t) dv_{(M)} \\ &= \int_{R_M(t)} \left(\overset{\square}{\varphi} - \varphi \mathrm{V}^{(\pi)} K_{(\pi)} + \mathrm{div}_{V_M}(\varphi \mathbf{w}_{\tan})\right) dv_{(M)} + \int_{\partial R_M(t)} \varphi V^{mig}_{|\partial R_M(t)} dv_{(M-1)}. \end{aligned} \tag{36}$$

8. Migrating Normal Time Derivative $V_M(t)$ in the Fluid

The normal time derivative of φ following the convected $V_M(t)$

$$\overset{\square}{\varphi} = \frac{\partial \varphi}{\partial t} + \mathrm{V}^{(\pi)} \frac{\partial \varphi}{\partial \mathrm{n}_{(\pi)}}$$

has to be modified for the migration $V_M(t)$ through the fluid. We start from (36), rewritten as:

$$\frac{d}{dt}\int_{R_M(t)} \varphi(\boldsymbol{x},t) dv_{(M)} =$$
$$\int_{R_M(t)} \left(\overset{\square}{\varphi} - \varphi \mathrm{V}^{(\pi)} K_{(\pi)} + \mathrm{grad}_{V_M(t)}\varphi \cdot \mathbf{w}_{\tan} + \varphi \mathrm{div}_{V_M(t)} \mathbf{w}_{\tan}\right) dv_{(M)} + \int_{\partial R_M(t)} \varphi V^{mig}_{\partial R_M(t)} dv_{(M)}.$$

Taking into account:

$$\mathrm{grad}\varphi = \partial\varphi_{,k}\boldsymbol{g}^k = \varphi_{,k}\left(x^k_{,\alpha}\boldsymbol{a}^\alpha + n^k_{(\pi)}\mathbf{n}^{(\pi)}\right) = \mathrm{grad}_{V_M(t)}\varphi + \frac{\partial\varphi}{\partial n_{(\pi)}}\mathbf{n}^{(\pi)},$$

it follows: $\mathrm{grad}_{V_M(t)}\varphi\cdot\mathbf{w}_{\tan} = \mathrm{grad}\varphi\cdot\mathbf{w}_{\tan}$ (since $\mathbf{n}^{(\pi)}\cdot\mathbf{w}_{\tan} = 0$). Therefore,

$$\overset{\square}{\varphi} + \mathrm{grad}_{V_M(t)}\varphi\cdot\mathbf{w}_{\tan} = \frac{\partial\varphi}{\partial t} + \mathrm{V}^{(\pi)}\frac{\partial\varphi}{\partial \mathrm{n}_{(\pi)}} + \mathrm{grad}\varphi\cdot\mathbf{w}_{\tan} =$$
$$= \frac{\partial\varphi}{\partial t} + \mathrm{grad}\varphi\cdot\left(\mathrm{V}^{(\pi)}\mathbf{n}_{(\pi)} + \mathbf{w}_{\tan}\right) = \frac{\partial\varphi}{\partial t} + \mathrm{grad}\varphi\cdot\boldsymbol{V},$$

where

$$\boldsymbol{V} = \mathrm{V}^{(\pi)}\mathbf{n}_{(\pi)} + \mathbf{w}_{\tan}$$

represents the migrational normal velocity for $V_M(t)$ computed relative to the fluid. Indeed, relative to the fluid:

$$\boldsymbol{V} - \mathbf{w} = (\mathrm{V}^{(\pi)} - \mathrm{w}^{(\pi)})\mathbf{n}_{(\pi)}.$$

Define the normal time derivative of $\varphi(\boldsymbol{x},t)$ following $V_M(t)$ and corresponding to the migrational normal velocity field $\boldsymbol{V}$:

$$\varphi^{\oplus} = \frac{\partial\varphi}{\partial t} + \mathrm{grad}\varphi\cdot\boldsymbol{V} = \overset{\square}{\varphi} + \mathrm{grad}_{V_M(t)}\varphi\cdot\mathbf{w}_{\tan}.$$

The corresponding transport theorem reads:

$$\frac{d}{dt}\int_{R_M(t)}\varphi(\boldsymbol{x},t)dv_{(M)}$$
$$= \int_{R_M(t)}\left(\varphi^{\oplus} - \varphi\mathrm{V}^{(\pi)}K_{(\pi)} + \varphi\,\mathrm{div}_{V_M}\mathbf{w}_{\tan}\right)dv_{(M)} + \int_{\partial R_M(t)}\varphi V^{mig}_{\partial R_M(t)}dv_{(M-1)}.$$

9. Material $V_M(t)$ in *N*-Dimensional Fluid

Assume that $V_M(t)$ is material, i.e., convected with the fluid (in analogy with material domains of 3 or fewer dimensions). Then, fluid velocity $\mathbf{w}$ is the material velocity for $V_M(t)$, i.e., $\mathrm{V} = \mathbf{w}$, the migrational normal velocity field for $V_M(t)$ coincides with the fluid velocity, $\mathbf{w} = \mathrm{V}^{(\pi)}\mathbf{n}_{(\pi)} + \mathbf{w}_{\tan}$, $\frac{\delta}{\delta t}\varphi(\mathbf{x},t) = \varphi^{\oplus}$, $V^{mig}_{\partial R_M(t)} = 0$. Then, transport theorem reads:

$$\frac{\delta}{\delta t}\int_{R_M(t)}\varphi(\boldsymbol{x},t)\,dv_{(M)} = \int_{R_M(t)}\left[\frac{\delta\varphi(\boldsymbol{x},t)}{\delta t} - \varphi(\mathbf{w}\cdot\mathbf{n}^{(\pi)})K_{(\pi)} + \varphi\mathrm{div}_{V_M(t)}\mathbf{w}_{\tan}\right]dv_{(M)}.$$

Then, we note that:

$$-(\mathbf{w}\cdot\mathbf{n}^{(\pi)})K_{(\pi)} = (\mathbf{w}\cdot\mathbf{n}^{(\pi)})\mathrm{div}_{V_M(t)}\mathbf{n}_{(\pi)} =$$
$$= \mathrm{div}_{V_M(t)}(\mathbf{w}\cdot\mathbf{n}^{(\pi)})\mathbf{n}_{(\pi)} - \mathbf{n}_{(\pi)}\cdot\mathrm{grad}_{V_M(t)}(\mathbf{w}\cdot\mathbf{n}^{(\pi)}) = \mathrm{div}_{V_M(t)}(\mathbf{w}\cdot\mathbf{n}^{(\pi)})\mathbf{n}_{(\pi)},$$

so that:

$$-(\mathbf{w}\cdot\mathbf{n}^{(\pi)})K_{(\pi)} + \mathrm{div}_{V_M(t)}\mathbf{w}_{\tan} = \mathrm{div}_{V_M(t)}\left[(\mathbf{w}\cdot\mathbf{n}^{(\pi)})\mathbf{n}_{(\pi)} + \mathbf{w}_{\tan}\right] = \mathrm{div}_{V_M(t)}\mathbf{w}.$$

Therefore, the transport theorem becomes:

$$\frac{\delta}{\delta t}\int\limits_{R_M(t)}\varphi(\boldsymbol{x},t)\,dv_{(M)}=\int\limits_{R_M(t)}\left[\frac{\delta}{\delta t}\varphi(\boldsymbol{x},t)+\varphi\mathrm{div}_{V_M(t)}\mathbf{w}\right]dv_{(M)}\ .$$

As a special case, we consider $V_M(t)$ is material but $R_M(t)$ is not. Then, $V^{mig}_{\partial R_M(t)}\neq 0$, and,

$$\frac{d}{dt}\int\limits_{R_M(t)}\varphi(\boldsymbol{x},t)\,dv_{(M)}=\int\limits_{R_M(t)}\left[\frac{d\varphi(\boldsymbol{x},t)}{dt}+\varphi\mathrm{div}_{V_M(t)}\mathbf{w}\right]dv_{(M)}+\int\limits_{\partial R_M(t)}\varphi V^{mig}_{\partial R_M(t)}dv_{(M-1)}\,.$$

10. Surfaces of Discontinuity

Examples of such singular surfaces are the interfaces within heterogeneous media, shock waves in gases, vortex sheets in fluids separating a main stream from the dead water, and cracks in solids. Moving surfaces of discontinuity are often related to waves. In these cases, the transport theorem has to be modified in order to include the influence of surfaces of discontinuity $\boldsymbol{\Sigma}(t)$ on

$$\frac{\delta}{\delta t}\int\limits_{v(t)}\varphi(\boldsymbol{x},t)dv\,.$$

We generalize the transport theorem for the integral

$$\frac{\delta}{\delta t}\int\limits_{R_M(t)}\varphi(\boldsymbol{x},t)\,dv_{(M)}$$

to a region $R_M(t)$ containing a hypersurface $\boldsymbol{\Sigma}_{M-1}(t)$ across which $\varphi(\boldsymbol{x},t)$ may suffer a jump discontinuity.

Definition 1. *An oriented smooth surface $\boldsymbol{\Sigma}_{M-1}(t)$ in a region $R_M(t)$ is called a singular hypersurface relative to a field $\varphi(\boldsymbol{x},t)$ defined on $R_M(t)$, if $\varphi(\boldsymbol{x},t)$ is smooth in $R_M\setminus\boldsymbol{\Sigma}_{M-1}$ and suffers a jump discontinuity across $\boldsymbol{\Sigma}_{M-1}(t)$. The jump of $\varphi(\boldsymbol{x},t)$ is defined as:*

$$[\![\varphi]\!]=\varphi^{+}-\varphi^{-}$$

where φ^{+} and φ^{-} are the limits from the two sides of $\boldsymbol{\Sigma}_{M-1}(t)$, in the sub-regions of $R_M(t)$ designated as $R_M^{+}(t)$ and $R_M^{-}(t)$.

Let $\boldsymbol{\Sigma}_{M-1}(t):\chi(u^{\alpha},t)=0$. Denote by $\boldsymbol{\mu}_{(M-1)}$ its outward unit normal and by $\boldsymbol{\omega}$ velocity of its points. Then, $\omega_{\mu}=\boldsymbol{\omega}\cdot\boldsymbol{\mu}_{(M-1)}$ is the normal speed of $\boldsymbol{\Sigma}_{M-1}(t)$ positive when pointing into $R_M^{+}(t)$. Upon applying (21) to the two regions $R_M^{+}(t)$ and $R_M^{-}(t)$, bounded by $\boldsymbol{\Sigma}_{M-1}(t)$, and, respectively, by $\partial R_M^{+}(t)$ and $\partial R_M^{-}(t)$, we write the transport theorem as:

$$\frac{\delta}{\delta t}\int\limits_{R_M(t)}\varphi(\boldsymbol{x},t)dv_{(M)}=\int\limits_{R_M(t)}\left(\overset{\square}{\varphi}-\varphi\mathrm{V}^{(\pi)}K_{(\pi)}\right)dv_{(M)}$$
$$+\int\limits_{\partial R_M(t)}\varphi\,\mathbf{V}_{\tan}\cdot\boldsymbol{\nu}_{(M-1)}dv_{(M-1)}-\int\limits_{\boldsymbol{\Sigma}_{M-1}(t)}[\![\varphi]\!]\boldsymbol{\omega}\cdot\boldsymbol{\mu}_{(M-1)}dv_{(M-1)},$$

or, using the Green–Gauss theorem, as:

$$\frac{\delta}{\delta t}\int_{R_M(t)}\varphi(\boldsymbol{x},t)dv_{(M)}=\int_{R_M(t)}\left(\overset{\square}{\varphi}-\varphi \mathrm{V}^{(\pi)}K_{(\pi)}+\mathrm{div}_{V_M(t)}(\varphi\boldsymbol{V})\right)dv_{(M)}$$
$$+\int_{\boldsymbol{\Sigma}_{M-1}(t)}[\![\varphi(\boldsymbol{V}-\boldsymbol{\omega})]\!]\cdot\boldsymbol{\mu}_{(M-1)}dv_{(M-1)}. \tag{37}$$

11. General Balance Laws

The basic laws of mechanics in 3D can all be expressed, in general, in the following form,

$$\frac{d}{dt}\int_{P_t}\psi\, dv=\int_{\partial P_t}\boldsymbol{\Phi}_\psi\cdot\mathbf{n}da+\int_{P_t}\sigma_\psi\, dv \tag{38}$$

for any bounded regular subregion P of the body $\mathscr{B}$, and the vector field $\mathbf{n}$, the outward unit normal to the boundary of the region P_t in the current configuration. The quantities ψ and σ_ψ are tensor fields of order m, and $\boldsymbol{\Phi}_\psi$ is a tensor field of order $m+1$.

The relation (38) – the general balance of ψ in integral form – asserts that the rate of increase of the quantity ψ in a part P of a body is affected by the inflow of ψ through the boundary of P and the production of ψ within P. $\boldsymbol{\Phi}_\psi$ is the flux of ψ, and σ_ψ is the source of ψ. In general, the source σ_ψ may include external and internal sources.

We state general balance laws for $R_M(t)\subseteq V_N$ containing a hypersurface $\boldsymbol{\Sigma}_{M-1}(t)$ across which $\varphi(\boldsymbol{x},t)$ may suffer a jump discontinuity. Inspired by the above balance law, we write:

$$\frac{\delta}{\delta t}\int_{R_M(t)}\varphi(\boldsymbol{x},t)dv_{(M)}=\int_{\partial R_M(t)}\boldsymbol{\Phi}_\varphi\cdot\boldsymbol{n}_{\partial R_M(t)}dv_{(M-1)}+\int_{R_M(t)}\sigma_\varphi dv_{(M)}, \tag{39}$$

where $\boldsymbol{n}_{\partial R_M(t)}$ is the outward unit normal to the boundary ∂R_M of R_M, while $\boldsymbol{\Phi}_\varphi$ and σ_φ are the flux and the source of φ.

From (39), we can obtain the local balance equation at a regular point, as well at a singular point. First, we apply the Green–Gauss theorem to the first integral on the right side of (39):

$$\int_{\partial R_M(t)}\boldsymbol{\Phi}_\varphi\cdot\boldsymbol{n}_{\partial R_M(t)}dv_{(M-1)}=$$
$$\int_{R_M(t)}\mathrm{div}_{V_M(t)}\boldsymbol{\Phi}_\varphi dv_{(M)}+\int_{\boldsymbol{\Sigma}_{M-1}(t)}[\![\boldsymbol{\Phi}_\varphi]\!]\cdot\boldsymbol{\mu}_{(M-1)}dv_{(M-1)}. \tag{40}$$

Then, after substituting (40) and (37) into (39) we obtain:

$$\int_{R_M(t)}\left(\overset{\square}{\varphi}-\varphi \mathrm{V}^{(\pi)}K_{(\pi)}+\mathrm{div}_{V_M(t)}(\varphi\boldsymbol{V})\right)dv_{(M)}+\int_{\boldsymbol{\Sigma}_{M-1}(t)}[\![\varphi(\boldsymbol{V}-\boldsymbol{\omega})]\!]\cdot\boldsymbol{\mu}_{(M-1)}dv_{(M-1)}$$
$$=\int_{R_M(t)}\mathrm{div}_{V_M(t)}\boldsymbol{\Phi}_\varphi\, dv_{(M)}+\int_{\boldsymbol{\Sigma}_{M-1}(t)}[\![\boldsymbol{\Phi}_\varphi]\!]\cdot\boldsymbol{\mu}_{(M-1)}dv_{(M-1)}+\int_{R_M(t)}\sigma_\varphi dv_{(M)},$$

and then,

$$\int_{R_M(t)}\left(\overset{\square}{\varphi}-\varphi \mathrm{V}^{(\pi)}K_{(\pi)}+\mathrm{div}_{V_M(t)}(\varphi\boldsymbol{V}-\boldsymbol{\Phi}_\varphi)-\sigma_\varphi\right)dv_{(M)}$$
$$+\int_{\boldsymbol{\Sigma}_{M-1}(t)}[\![\varphi(\boldsymbol{V}-\boldsymbol{\omega})-\boldsymbol{\Phi}_\varphi]\!]\cdot\boldsymbol{\mu}_{(M-1)}dv_{(M-1)}=0.$$

Finally, following the standard procedure, the local balance law reads:

$$\overset{\square}{\varphi} - \varphi \mathrm{V}^{(\pi)} K_{(\pi)} + \mathrm{div}_{V_M(t)}(\varphi \boldsymbol{V} - \boldsymbol{\Phi}_\varphi) - \sigma_\varphi = 0 \quad \text{in } R_M(t),$$

with the jump condition:

$$[\![\varphi(\boldsymbol{V} - \boldsymbol{\omega}) - \boldsymbol{\Phi}_\varphi]\!] \cdot \boldsymbol{\mu}_{(M-1)} = 0 \quad \text{on } \boldsymbol{\Sigma}_{M-1}(t) \quad \text{on } \boldsymbol{\Sigma}_{M-1}(t).$$

12. Summary and Discussion

To formulate the transport theorem, we define the elements of geometry and kinematics of a *M*-dimensional domain $R_M(t)$ moving in a *N*-dimensional space V_N $(M \le N)$ which are independent of parametrization, i.e., independent of the choice of the choice of coordinates intrinsic to $V_M(t)$. The geometry is characterized by $N-M$ normals $\boldsymbol{n}^{(\pi)}$, which—together with the arbitrary basis in $R_M(t)$—form a basis in V_N. The relevant components of the curvature of $R_M(t)$ are characterized by the mean curvature normal [4] ($\boldsymbol{K}_n = K_{(\pi)} \boldsymbol{n}^{(\pi)}$, where $K_{(\pi)} = -\mathrm{div}_{V_M(t)} \mathbf{n}_{(\pi)}$. The geometry of the boundary of $R_M(t)$, denoted ∂R_M, is characterized by the unit vector $\boldsymbol{\nu}_{(M-1)}$, which is orthogonal to ∂R_M, but tangent to $R_M(t)$.

Regarding the kinematics of $R_M(t)$, only the normal velocity, $\boldsymbol{V}_n = \mathrm{V}^{(\pi)} \mathbf{n}_{(\pi)}$ is independent of parametrization. The motion of the boundary is characterized by a single scalar component – the scalar normal velocity $\mathrm{V}_\nu = \boldsymbol{V}_{\tan} \cdot \boldsymbol{\nu}_{(M-1)}$. The final component is the normal time derivative independent of parametrization:

$$\overset{\square}{\varphi} = \left. \frac{\partial \varphi(x^k, t)}{\partial t} \right|_{x^k} + \boldsymbol{V}_n \cdot grad \varphi .$$

The transport theorem can then be written as:

$$\frac{d}{dt} \int_{R_M(t)} \varphi(\boldsymbol{x}, t) dv_{(M)} = \int_{R_M(t)} (\overset{\square}{\varphi} - \varphi \boldsymbol{V}_n \cdot \boldsymbol{K}) dv_{(M)} + \int_{\partial R_M(t)} \varphi \mathrm{V}_\nu dv_{(M-1)} .$$

The transport theorem has been applied to the moving *N*-dimensional fluid with a migrating subdomain $R_M(t)$, non-material and material, as well as to the moving surfaces of discontinuity. Finally, the general form of balance laws for *N*-dimensional continuum is formulated.

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Appendix A

To derive (17), we begin with

$$\frac{\partial}{\partial t} \sqrt{A} = \frac{1}{2} \sqrt{A} A^{\Lambda\Delta} \frac{\partial A_{\Lambda\Delta}}{\partial t}, \tag{A1}$$

where $A^{\Lambda\Delta} A_{\Delta\Gamma} = \delta_\Gamma^\Lambda$. It is convenient to write (8) as $A_{\Lambda\Delta} = \boldsymbol{A}_\Gamma \cdot \boldsymbol{A}_\Delta$. Then:

$$\frac{\partial A_{\Lambda\Delta}}{\partial t} = \frac{\partial}{\partial t} \left(\boldsymbol{A}_\Lambda \cdot \boldsymbol{A}_\Delta \right) = \frac{\partial \boldsymbol{A}_\Lambda}{\partial t} \cdot \boldsymbol{A}_\Delta + \boldsymbol{A}_\Lambda \cdot \frac{\partial \boldsymbol{A}_\Delta}{\partial t} = \frac{\partial \boldsymbol{V}}{\partial U^\Lambda} \cdot \boldsymbol{A}_\Delta + \boldsymbol{A}_\Lambda \cdot \frac{\partial \boldsymbol{V}}{\partial U^\Delta} \tag{A2}$$

where, according to (12), we have

$$\frac{\partial \boldsymbol{A}_{\Gamma}}{\partial t}=\frac{\partial}{\partial t}\frac{\partial \hat{\mathbf{x}}}{\partial U^{\Gamma}}=\frac{\partial}{\partial U^{\Gamma}}\frac{\partial \hat{\mathbf{x}}}{\partial t}=\frac{\partial \boldsymbol{V}}{\partial U^{\Gamma}} .$$

Thus,

$$\frac{\partial \boldsymbol{V}}{\partial U^{\Lambda}}\cdot \boldsymbol{A}_{\Delta}=\frac{\partial \boldsymbol{V}\cdot \boldsymbol{A}_{\Delta}}{\partial U^{\Lambda}}-\boldsymbol{V}\cdot\frac{\partial \boldsymbol{A}_{\Delta}}{\partial U^{\Lambda}} \tag{A3}$$

Next,

$$\frac{\partial \boldsymbol{A}_{\Delta}}{\partial U^{\Lambda}}=\begin{Bmatrix}\Gamma\\ \Delta\Lambda\end{Bmatrix}\boldsymbol{A}_{\Gamma}+\Omega_{\Lambda\Delta|(\pi)}\mathbf{n}^{(\pi)} \tag{A4}$$

and

$$\boldsymbol{V}\cdot \boldsymbol{A}_{\Delta}=\mathrm{V}_{\Delta} . \tag{A5}$$

Formally, we may write (A4) as

$$\boldsymbol{A}_{\Delta},_{\Lambda}=\Omega_{\Lambda\Delta|(\pi)}\mathbf{n}^{(\pi)}$$

where

$$\Omega_{\Lambda\Delta|(\pi)}=\Omega_{\Delta\Lambda|(\pi)} .$$

here, $\Omega_{\Lambda\Delta|(\pi)}$ are the components of symmetric 2nd order tensor for each value of π [22]. Alternatively: $\boldsymbol{A}_{\Delta},_{\Lambda}=\Omega^{(\pi)}_{\Lambda\Delta}\mathbf{n}_{(\pi)}$, where $\Omega^{(\pi)}_{\Lambda\Delta}=n^{\pi\tau}\Omega_{(\tau)\Lambda\Delta}$.

Next, we calculate (A3) and obtain

$$\frac{\partial \boldsymbol{V}}{\partial U^{\Lambda}}\cdot \boldsymbol{A}_{\Delta}=\frac{\partial \mathrm{V}_{\Delta}}{\partial U^{\Lambda}}-\begin{Bmatrix}\Gamma\\ \Delta\Lambda\end{Bmatrix}\mathrm{V}_{\Delta}-\Omega_{\Lambda\Delta|(\pi)}\mathrm{V}^{(\pi)}=\mathrm{V}_{\Delta},_{\Lambda}-\Omega_{\Lambda\Delta|(\pi)}\mathrm{V}^{(\pi)},$$

where we have made use of (A4) and (A5). Therefore (A2) is given by

$$\frac{\partial A_{\Lambda\Delta}}{\partial t}=\mathrm{V}_{\Delta},_{\Lambda}+\mathrm{V}_{\Lambda},_{\Delta}-2\mathrm{V}^{(\pi)}\Omega_{\Lambda\Delta|(\pi)} .$$

Finally,

$$\frac{\partial}{\partial t}\sqrt{A}=\frac{1}{2\sqrt{A}}\frac{\partial A}{\partial t}=\frac{1}{2}\sqrt{A}A^{\Lambda\Delta}\frac{\partial A_{\Lambda\Delta}}{\partial t}=\sqrt{A}\left(\mathrm{div}_{\mathrm{V}_M}\mathbf{V}_{\tan}-\mathrm{V}^{(\pi)}K_{(\pi)}\right).$$

Next, we use (A4) and obtain

$$\Omega_{\Lambda\Delta|(\pi)}=\frac{\partial \boldsymbol{A}_{\Delta}}{\partial U^{\Lambda}}\cdot \mathbf{n}_{(\pi)}=-\boldsymbol{A}_{\Delta}\cdot\frac{\partial \mathbf{n}_{(\pi)}}{\partial U^{\Lambda}} .$$

Hence,

$$\begin{aligned}K_{\pi}&=A^{\Lambda\Delta}\Omega_{\Lambda\Delta|(\pi)}=-A^{\Lambda\Delta}\boldsymbol{A}_{\Delta}\cdot\frac{\partial \mathbf{n}_{(\pi)}}{\partial U^{\Lambda}}=-\boldsymbol{A}^{\Lambda}\cdot\frac{\partial \mathbf{n}_{(\pi)}}{\partial U^{\Lambda}}=\\ &=-\mathrm{Div}_{V_M(t)}\mathbf{n}_{(\pi)}=-\boldsymbol{a}^{\alpha}\cdot\frac{\partial \mathbf{n}_{(\pi)}}{\partial u^{\alpha}}=-\mathrm{div}_{V_M(t)}\mathbf{n}_{(\pi)} ,\end{aligned}$$

which is independent of parameterization.

Appendix B. Elements of Geometry and Kinematics of $\partial R_M(t)$

So far, we have considered some elements of geometry and kinematics of $V_M(t)$. Now we let $R_M(t)$ be a finite region in $V_M(t)$, bounded by the closed $(M-1)$ space $\partial R_M(t)$. We first consider some elements of geometry and kinematics of $\partial R_M(t)$ whose "material" points are defined by convected coordinates. Its parametric equations in convected coordinates are

$$U^{\Lambda} = U^{\Lambda}(\xi^{\rho}) \quad \text{or} \quad \Phi(U^{\Lambda}) = 0\,.$$

Then, $\mathrm{Grad}\Phi$ and its unit normal,

$$\boldsymbol{\mu}_{(M-1)} = \frac{\mathrm{Grad}\Phi}{\left|\mathrm{Grad}\Phi\right|},$$

are independent of any parametrization of ξ^{ρ} in $\partial R_M(t)$. Note that the unit vector $\boldsymbol{\mu}_{(M-1)}$ is in $R_M(t)$ and orthogonal to $\partial R_M(t)$. In view of (5) the same boundary $\partial R_M(t)$ is defined in $V_N(t)$ by $\boldsymbol{x} = \hat{\boldsymbol{x}}(\boldsymbol{U}(\boldsymbol{\xi}), t)$. Then,

$$\frac{\partial \hat{\boldsymbol{x}}}{\partial t}_{|\boldsymbol{\xi}} = \frac{\partial \hat{\boldsymbol{x}}}{\partial t} = \mathbf{V}\,,$$

so that

$$\frac{\partial \hat{\boldsymbol{x}}}{\partial t}_{|\boldsymbol{\xi}} \cdot \boldsymbol{\mu}_{(M-1)} = \mathbf{V} \cdot \boldsymbol{\mu}_{(M-1)}$$

at the points of $\partial R_M(t)$ defined by (A1). From (14), having in mind that $\mathbf{n}^{(\pi)} \cdot \boldsymbol{\mu}_{(M-1)} = 0$, this can be written, as

$$\frac{\partial \hat{\boldsymbol{x}}}{\partial t}_{|\boldsymbol{\xi}} \cdot \boldsymbol{\mu}_{(M-1)} = \mathbf{V}_{\tan} \cdot \boldsymbol{\mu}_{(M-1)},$$

which is independent of any parametrization of ξ^{ρ}. It represents the speed of propagation of $\partial R_M(t)$ while its vector of displacement is given by

$$\boldsymbol{V}_{\partial R_M(t)} = \left(\frac{\partial \hat{\boldsymbol{x}}}{\partial t}_{|\boldsymbol{\xi}} \cdot \boldsymbol{\mu}_{(M-1)}\right)\boldsymbol{\mu}_{(M-1)} = \left(\boldsymbol{\mu}_{(M-1)} \otimes \boldsymbol{\mu}_{(M-1)}\right)\frac{\partial \hat{\boldsymbol{x}}}{\partial t}_{|\boldsymbol{\xi}}\,.$$

When the coordinate system u^{α} that is not convected, determination of the speed of movement of the border is more complex. Then,

$$\partial R_M(t) : u^{\alpha} = u^{\alpha}(\xi^{\rho}, t), \quad \mathrm{rank}\left(\frac{\partial u^{\alpha}}{\partial \xi^{\rho}}\right) = M-1, \quad \rho = 1, \ldots, M-1 \tag{A6}$$

or

$$\partial R_M(t) : \psi\left(u^{\alpha}, t\right) = 0\,. \tag{A7}$$

Moreover, $\mathrm{grad}_{R_M(t)}\psi$ is vector in $R_M(t)$ normal to $\partial R_M(t)$, which is independent of the choice of coordinates ξ^{ρ} intrinsic to $\partial R_M(t)$. The same holds for its unit vector

$$\boldsymbol{\nu}_{(M-1)} = \frac{\mathrm{grad}_{R_M(t)}\psi}{\left|\mathrm{grad}_{R_M(t)}\psi\right|}$$

Let $u^{\alpha} = u^{\alpha}(c^{\rho}, t)$ be the path of point $\mathbf{u}(u^{\alpha})$ in $R_M(t)$ when $\xi^{\rho} = c^{\rho}$. Then

$$\frac{\partial u^{\alpha}}{\partial t}_{|\xi^{\rho} = c^{\rho}} \boldsymbol{a}_{\alpha}$$

represents the velocity of points of the path. From (A6) and (A7) we obtain

$$\frac{\partial}{\partial t}\psi + \mathrm{grad}_{R_M(t)}\psi \cdot \left.\frac{\partial \mathbf{u}}{\partial t}\right|_{\xi^\rho} = 0$$

whence, in view of (19),

$$\left.\frac{\partial \mathbf{u}}{\partial t}\right|_{\xi^\rho} \cdot \boldsymbol{\nu}_{(M-1)} = -\frac{\partial \psi}{\partial t} \Big/ \left|\mathrm{grad}_{R_M(t)}\psi\right|$$

The expression

$$\left.\frac{\partial \mathbf{u}}{\partial t}\right|_{\xi^\rho} \cdot \boldsymbol{\nu}_{(M-1)}$$

is the speed of propagation of $\partial R_M(t)$ in $R_M(t)$. It is clearly independent of any parameterization with respect to intrinsic coordinates ξ^ρ. Its displacement vector $\boldsymbol{v}_{\partial R_M(t)}$ is given by

$$\boldsymbol{v}_{\partial R_M(t)} = \mathrm{v}_{\partial R_M(t)}\boldsymbol{\nu}_{(M-1)} = \left(\left.\frac{\partial \mathbf{u}}{\partial t}\right|_{\xi^\rho} \cdot \boldsymbol{\nu}_{(M-1)}\right)\boldsymbol{\nu}_{(M-1)}.$$

Thus far, we have considered the kinematics of $\partial R_M(t)$ as a subspace of $R_M(t)$. Now, upon substitution of (18) into (1), we obtain representation of $\partial R_M(t)$ in V_N:

$$\mathbf{x} = \mathbf{x}(u^\alpha, t) = \mathbf{x}\left[u^\alpha(\xi^\rho, t), t\right].$$

Then,

$$\boldsymbol{v}\big|_{\partial R_M(t)} = \left.\frac{\partial \mathbf{x}}{\partial t}\right|_{\xi^\rho = c^\rho} = \frac{\partial \mathbf{x}}{\partial t} + \left.\frac{\partial \mathbf{x}}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial t}\right|_{\xi^\rho = c^\rho} = \mathbf{v} + \left.\frac{\partial \mathbf{u}}{\partial t}\right|_{\xi^\rho}$$

represents the velocity of the points of the paths $\xi^\rho = c^\rho$ in V_N. Hence,

$$\boldsymbol{v}\big|_{\partial R_M(t)} \cdot \boldsymbol{\nu}_{(M-1)} = \mathbf{v} \cdot \boldsymbol{\nu}_{(M-1)} + \boldsymbol{v}_{\partial R_M(t)}$$

From (13), we have $\boldsymbol{v} \cdot \boldsymbol{\nu}_{(M-1)} = \boldsymbol{v}_{\tan} \cdot \boldsymbol{\nu}_{(M-1)}$. Then, since $\mathbf{n}_{(\pi)} \cdot \boldsymbol{\nu}_{(M-1)} = 0$, we have

$$\boldsymbol{v}\big|_{\partial R_M(t)} \cdot \boldsymbol{\nu}_{(M-1)} = \boldsymbol{v}_{\tan} \cdot \boldsymbol{\nu}_{(M-1)} + \mathrm{v}_{\partial R_M(t)}.$$

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