Newtonian and single layer potentials for the Stokes system with L^∞ coefficients and the exterior Dirichlet problem

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Dedicated to Professor H. Begehr on the occasion of his 80th birthday.

Abstract. A mixed variational formulation of some problems in L^2-based Sobolev spaces is used to define the Newtonian and layer potentials for the Stokes system with L^∞ coefficients on Lipschitz domains in $\mathbb{R}^3$. Then the solution of the exterior Dirichlet problem for the Stokes system with L^∞ coefficients is presented in terms of these potentials and the inverse of the corresponding single layer operator.

Mathematics Subject Classification (2010). Primary 35J25, 35Q35, 42B20, 46E35; Secondary 76D, 76M.

Keywords. Stokes system with L^∞ coefficients, Newtonian and layer potentials, variational approach, inf-sup condition, Sobolev spaces.

1. Introduction

Let $\mathbf{u}$ be an unknown vector field, π be an unknown scalar field, and $\mathbf{f}$ be a given vector field defined on an exterior Lipschitz domain $\Omega_- \subset \mathbb{R}^3$. Let also $\mathbb{E}(\mathbf{u})$ be the symmetric part of the gradient of $\mathbf{u}$, $\nabla\mathbf{u}$. Then the equations

$$\mathcal{L}_\mu(\mathbf{u},\pi) := \operatorname{div}(2\mu\mathbb{E}(\mathbf{u})) - \nabla\pi = \mathbf{f},\ \operatorname{div}\,\mathbf{u} = 0 \text{ in } \Omega_- \tag{1.1}$$

determine the *Stokes system* with a known viscosity coefficient $\mu \in L^\infty(\Omega_-)$. This linear PDE system describes the flows of viscous incompressible fluids, when the inertia of such a fluid can be neglected. The coefficient μ is related to the physical properties of the fluid (for further details we refer the reader to the books [45] and [23] and the references therein).

The methods of layer potential theory have a main role in the analysis of boundary value problems for elliptic partial differential equations (see, e.g., [13, 17, 30, 36, 39, 43, 48]). Fabes, Kenig and Verchota [21] obtained mapping properties of layer potentials for the constant coefficient Stokes system in L^p

spaces. Mitrea and Wright [43] have used various methods of layer potentials in the analysis of the main boundary problems for the Stokes system with constant coefficients in arbitrary Lipschitz domains in $\mathbb{R}^n$. The authors in [32] have obtained mapping properties of the constant coefficient Stokes layer potential operators in standard and weighted Sobolev spaces by exploiting results of singular integral operators. Gatica and Wendland [24] used the coupling of mixed finite element and boundary integral methods for solving a class of linear and nonlinear elliptic boundary value problems. The authors in [33] used the Stokes and Brinkman integral layer potentials and a fixed point theorem to show an existence result for a nonlinear Neumann-transmission problem for the Stokes and Brinkman systems with data in L^p, Sobolev, and Besov spaces (see also [34, 35]). All above results are devoted to elliptic boundary value problems with constant coefficients.

Potential theory plays also a main role in the study of elliptic boundary value problems with variable coefficients. Dindoš and Mitrea [19] have obtained well-posedness results in Sobolev spaces for Poisson problems for the Stokes and Navier-Stokes systems with Dirichlet condition on C^1 and Lipschitz domains in compact Riemannian manifolds by using mapping properties of Stokes layer potentials in Sobolev and Besov spaces. Chkadua, Mikhailov and Natroshvili [14] obtained direct segregated systems of boundary-domain integral equations for a mixed boundary value problem of Dirichlet-Neumann type for a scalar second-order divergent elliptic partial differential equation with a variable coefficient in an exterior domain of $\mathbb{R}^3$ (see also [13]). Hofmann, Mitrea and Morris [29] considered layer potentials in L^p spaces for elliptic operators of the form $L=-\mathrm{div}(A\nabla u)$ acting in the upper half-space $\mathbb{R}^n_+$, $n \geq 3$, or in more general Lipschitz graph domains, with an L^∞ coefficient matrix A, which is t-independent, and solutions of the equation $Lu=0$ satisfy interior De Giorgi-Nash-Moser estimates. They obtained a Calderón-Zygmund type theory associated to the layer potentials, and well-posedness results of boundary problems for the operator L in L^p and endpoint spaces.

Our variational approach is inspired by that developed by Sayas and Selgas in [46] for the constant coefficient Stokes layer potentials on Lipschitz boundaries, and is based on the technique of Nédélec [44]. Girault and Sequeira [26] obtained a well-posed result in weighted Sobolev spaces for the Dirichlet problem for the standard Stokes system in exterior Lipschitz domains of $\mathbb{R}^n$, $n = 2, 3$. Băcuţă, Hassell and Hsiao [8] developed a variational approach for the standard Brinkman single layer potential and used it in the analysis of the time dependent exterior Stokes problem with Dirichlet boundary condition in $\mathbb{R}^n$, $n = 2, 3$. Barton [7] constructed layer potentials for strongly elliptic differential operators in general settings by using the Lax-Milgram theorem, and generalized various properties of layer potentials for harmonic and second order elliptic equations. Brewster et al. in [9] have used a variational approach and a deep analysis to obtain well-posedness results for boundary problems of Dirichlet, Neumann and mixed type for higher order divergence-form elliptic equations with L^∞ coefficients in locally (ϵ, δ)-domains and in Besov and Bessel potential spaces. Choi and Lee [15] have studied the Dirichlet problem for the Stokes system with nonsmooth coefficients, and proved the unique solvability of the problem in Sobolev spaces on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ with a small Lipschitz constant when the coefficients have vanishing mean oscillations with respect to all variables. Choi and Yang [16] obtained the existence and pointwise bound of the fundamental solution for the Stokes system with measurable coefficients in $\mathbb{R}^n$, $n \geq 3$, whenever the weak solutions of the system are locally Hölder continuous. Alliot and Amrouche [3] have used a variational approach to obtain weak solutions for the exterior Stokes problem in weighted Sobolev spaces. Also, Amrouche and Nguyen [5] proved existence and uniqueness results in weighted Sobolev spaces for the Poisson problem with Dirichlet boundary condition for the Navier-Stokes system in exterior Lipschitz domains in $\mathbb{R}^3$.

The purpose of this work is to show the well-posedness result of the Poisson problem of Dirichlet type for the Stokes system with L^∞ coefficients in L^2-based Sobolev spaces on an exterior Lipschitz domain in $\mathbb{R}^3$. We use a variational approach that reduces this boundary value problem to a mixed variational formulation. A similar variational approach is used to define the Newtonian and layer potentials for the Stokes system with L^∞ coefficients on Lipschitz surfaces in $\mathbb{R}^3$, by using the weak solutions of some transmission problems in L^2-based Sobolev spaces. Finally, the mapping properties of these layer potentials are used to construct explicitly the solution of the exterior Dirichlet problem for the Stokes system with L^∞ coefficients. The analysis developed in this paper confines to the case $n = 3$, due to its practical interest, but the extension to the case $n \geq 3$ can be done with similar arguments.

2. Functional setting and useful results

Let $\Omega_+ := \Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, i.e., an open connected set whose boundary $\partial\Omega$ is locally the graph of a Lipschitz function. Assume that $\partial\Omega$ is connected. Let $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega}_+$ denote the exterior Lipschitz domain. Let $\mathring{E}_\pm$ denote the operators of extension by zero outside $\Omega_\pm$.

2.1. Standard L^2-based Sobolev spaces and related results

Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse defined on the the space of tempered distributions $\mathcal{S}^*(\mathbb{R}^3)$ (i.e., the topological dual of the space $\mathcal{S}(\mathbb{R}^3)$ of all rapidly decreasing infinitely differentiable functions on $\mathbb{R}^3$). The Lebesgue space of (equivalence classes of) measurable, square integrable functions on $\mathbb{R}^3$ is denoted by $L^2(\mathbb{R}^3)$, and by $L^\infty(\mathbb{R}^3)$ we denote the space of (equivalence classes of) essentially bounded measurable functions on $\mathbb{R}^3$. Let $H^1(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)^3$ denote the L^2-based Sobolev (Bessel potential) spaces

$$H^1(\mathbb{R}^3) := \left\{ f \in \mathcal{S}^*(\mathbb{R}^3) : \|f\|_{H^1(\mathbb{R}^3)} = \left\| \mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{1}{2}} \mathcal{F} f] \right\|_{L^2(\mathbb{R}^3)} < \infty \right\}, \quad (2.1)$$

$$H^1(\mathbb{R}^3)^3 := \{ f = (f_1, f_2, f_3) : f_j \in H^1(\mathbb{R}^3),\ j = 1, 2, 3 \}. \quad (2.2)$$

Now let Ω' be Ω_+, Ω_- or $\mathbb{R}^3$. We denote by $\mathcal{D}(\Omega') := C_0^\infty(\Omega')$ the space of infinitely differentiable functions with compact support in Ω', equipped with the inductive limit topology. Let $\mathcal{D}^*(\Omega')$ denote the corresponding space of distributions on Ω', i.e., the dual space of $\mathcal{D}(\Omega')$. Let us consider the space

$$H^1(\Omega') := \{f \in \mathcal{D}^*(\Omega') : \exists\, F \in H^1(\mathbb{R}^3) \text{ such that } F|_{\Omega'} = f\}\,, \tag{2.3}$$

where $\cdot|_{\Omega'}$ is the restriction operator to Ω'. The space $\widetilde{H}^1(\Omega')$ is the closure of $\mathcal{D}(\Omega')$ in $H^1(\mathbb{R}^3)$. This space can be also characterized as

$$\widetilde{H}^1(\Omega') := \left\{\widetilde{f} \in H^1(\mathbb{R}^3) : \operatorname{supp} \widetilde{f} \subseteq \overline{\Omega'}\right\}. \tag{2.4}$$

Similar to definition (2.2), $H^1(\Omega')^3$ and $\widetilde{H}^1(\Omega')^3$ are the spaces of vector-valued functions whose components belong to the scalar spaces $H^1(\Omega')$ and $\widetilde{H}^1(\Omega')$, respectively (see, e.g., [38]). The Sobolev space $\widetilde{H}^1(\Omega')$ can be identified with the closure $\mathring{H}^1(\Omega')$ of $\mathcal{D}(\Omega')$ in the norm of $H^1(\Omega')$ (see, e.g., [42, (3.11), (3.13)], [38, Theorem 3.33]). The space $\mathcal{D}(\overline{\Omega'})$ is dense in $H^1(\Omega')$, and the following spaces can be isomorphically identified (cf., e.g., [38, Theorem 3.14])

$$(H^1(\Omega'))^* = \widetilde{H}^{-1}(\Omega'), \quad H^{-1}(\Omega') = (\widetilde{H}^1(\Omega'))^*. \tag{2.5}$$

For $s \in [0,1]$, the Sobolev space $H^s(\partial\Omega)$ on the boundary $\partial\Omega$ can be defined by using the space $H^s(\mathbb{R}^2)$, a partition of unity and the pull-backs of the local parametrization of $\partial\Omega$, and $H^{-s}(\partial\Omega) = (H^s(\partial\Omega))^*$. All the above spaces are Hilbert spaces. For further properties of Sobolev spaces on bounded Lipschitz domains and Lipschitz boundaries, we refer to [1, 31, 38, 43, 47].

A useful result for the next arguments is given below (see, e.g., [17], [31, Proposition 3.3]).

Lemma 2.1. *Assume that $\Omega := \Omega_+ \subset \mathbb{R}^3$ is a bounded Lipschitz domain with connected boundary $\partial\Omega$ and denote by $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega}$ the corresponding exterior domain. Then there exist linear and bounded trace operators $\gamma_\pm : H^1(\Omega_\pm) \to H^{\frac{1}{2}}(\partial\Omega)$ such that $\gamma_\pm f = f|_{\partial\Omega}$ for any $f \in C^\infty(\overline{\Omega}_\pm)$. These operators are surjective and have (non-unique) bounded linear right inverse operators $\gamma_\pm^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega_\pm)$.*

The jump of a function $u \in H^1(\mathbb{R}^3 \setminus \partial\Omega)$ across $\partial\Omega$ is denoted by $[\gamma(u)] := \gamma_+(u) - \gamma_-(u)$. For $u \in H^1_{\rm loc}(\mathbb{R}^3)$, $[\gamma(u)] = 0$. The trace operator $\gamma : H^1(\mathbb{R}^3) \to H^{\frac{1}{2}}(\partial\Omega)$ can be also considered and is linear and bounded[1].

If X is either an open subset or a surface in $\mathbb{R}^3$, then we use the notation $\langle\cdot,\cdot\rangle_X$ for the duality pairing of two dual Sobolev spaces defined on X.

2.2. Some weighted Sobolev spaces and related results

For a point $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, its distance to the origin is denoted by $|\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$. Let ρ denote the weight function

$$\rho(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{\frac{1}{2}}\,. \tag{2.6}$$

[1]The trace operators defined on Sobolev spaces of vector fields on $\Omega_\pm$ or $\mathbb{R}^3$ are also denoted by $\gamma_\pm$ and γ, respectively.

For $\lambda \in \mathbb{R}$, we consider the weighted space $L^2(\rho^\lambda;\mathbb{R}^3)$ given by

$$f \in L^2(\rho^\lambda;\mathbb{R}^3) \Longleftrightarrow \rho^\lambda f \in L^2(\mathbb{R}^3), \tag{2.7}$$

which is a Hilbert space when it is endowed with the inner product and the associated norm,

$$(f,g)_{L^2(\rho^\lambda;\mathbb{R}^3)} := \int_{\mathbb{R}^3} fg\rho^{2\lambda}dx,\ \ \|f\|^2_{L^2(\rho^\lambda;\mathbb{R}^3)} := (f,f)_{L^2(\rho^\lambda;\mathbb{R}^3)}. \tag{2.8}$$

We also consider the weighted Sobolev space

$$\begin{aligned}\mathcal{H}^1(\mathbb{R}^3) :&= \left\{f \in \mathcal{D}'(\mathbb{R}^3) : \rho^{-1}f \in L^2(\mathbb{R}^3),\ \nabla f \in L^2(\mathbb{R}^3)^3\right\}\\ &= \left\{f \in L^2(\rho^{-1};\mathbb{R}^3) : \nabla f \in L^2(\mathbb{R}^3)^3\right\},\end{aligned} \tag{2.9}$$

which is a Hilbert space with respect to the inner product

$$(f,g)_{\mathcal{H}^1(\mathbb{R}^3)} := (f,g)_{L^2(\rho^{-1};\mathbb{R}^3)} + (\nabla f,\nabla g)_{L^2(\mathbb{R}^3)^3} \tag{2.10}$$

and the associated norm

$$\|f\|^2_{\mathcal{H}^1(\mathbb{R}^3)} := \left\|\rho^{-1}f\right\|^2_{L^2(\mathbb{R}^3)} + \|\nabla f\|^2_{L^2(\mathbb{R}^3)^3} \tag{2.11}$$

(cf. [28]; see also [5]). The spaces $L^2(\rho^\lambda;\Omega_-)$ and $\mathcal{H}^1(\Omega_-)$ can be similarly defined, and $\mathcal{D}(\overline{\Omega}_-)$ is dense in $\mathcal{H}^1(\Omega_-)$ (see, e.g., [28, Theorem I.1], [27, Ch.1, Theorem 2.1]). The seminorm

$$|f|_{\mathcal{H}^1(\Omega_-)} := \|\nabla f\|_{L^2(\Omega_-)^3} \tag{2.12}$$

is equivalent to the norm of $\mathcal{H}^1(\Omega_-)$ defined as in (2.11), with Ω_- in place of $\mathbb{R}^3$ (see, e.g., [18, Chapter XI, Part B, §1, Theorem 1]). The weighted spaces $L^2(\rho^{-1};\Omega_+)$ and $\mathcal{H}^1(\Omega_+)$ coincide with the standard spaces $L^2(\Omega_+)$ and $H^1(\Omega_+)$, respectively (with equivalent norms).

Note that the result in Lemma 2.1 extends also to the weighted Sobolev space $\mathcal{H}^1(\Omega_-)$. Therefore, there exists a linear bounded *exterior trace operator*

$$\gamma_- : \mathcal{H}^1(\Omega_-) \to H^{\frac{1}{2}}(\partial\Omega), \tag{2.13}$$

which is also surjective (see [46, p. 69]). Moreover, the embedding of the space $H^1(\Omega_-)$ into $\mathcal{H}^1(\Omega_-)$ and Lemma 2.1 show the existence of a (non-unique) linear and bounded right inverse $\gamma_-^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to \mathcal{H}^1(\Omega_-)$ of operator (2.13) (see [32, Lemma 2.2], [40, Theorem 2.3, Lemma 2.6]).

Let $\mathring{\mathcal{H}}^1(\Omega_-) \subset \mathcal{H}^1(\Omega_-)$ denote the closure of $\mathcal{D}(\Omega_-)$ in $\mathcal{H}^1(\Omega_-)$. This space can be characterized as

$$\mathring{\mathcal{H}}^1(\Omega_-) = \left\{v \in \mathcal{H}^1(\Omega_-) : \gamma_- v = 0 \text{ on } \partial\Omega\right\} \tag{2.14}$$

(cf., e.g., [38, Theorem 3.33]). Also let $\widetilde{\mathcal{H}}^1(\Omega_-) \subset \mathcal{H}^1(\mathbb{R}^3)$ denote the closure of $\mathcal{D}(\Omega_-)$ in $\mathcal{H}^1(\mathbb{R}^3)$. This space can be also characterized as

$$\widetilde{\mathcal{H}}^1(\Omega_-) = \{u \in \mathcal{H}^1(\mathbb{R}^3) : \operatorname{supp} u \subseteq \overline{\Omega}_-\}, \tag{2.15}$$

and can be isomorphically identified with the space $\mathring{\mathcal{H}}^1(\Omega_-)$ via the extension by zero operator $\mathring{E}_-$, i.e., $\widetilde{\mathcal{H}}^1(\Omega_-)=\mathring{E}_-\mathring{\mathcal{H}}^1(\Omega_-)$ (cf., e.g., [38, Theorem 3.29 (ii)]). In addition, consider the spaces (see, e.g., [5, p. 44], [37, Theorem 2.4])

$$\mathcal{H}^{-1}(\mathbb{R}^3):=\big(\mathcal{H}^1(\mathbb{R}^3)\big)^*, \mathcal{H}^{-1}(\Omega_-):=\big(\widetilde{\mathcal{H}}^1(\Omega_-)\big)^*, \widetilde{\mathcal{H}}^{-1}(\Omega_-):=\big(\mathcal{H}^1(\Omega_-)\big)^*.$$

3. The conormal derivative operators for the Stokes system with L^∞ coefficients

In the sequel we assume that the viscosity coefficient μ of the Stokes system (1.1) belongs to $L^\infty(\mathbb{R}^3)$ and there exists a constant $c_\mu>0$, such that

$$c_\mu^{-1}\le\mu\le c_\mu \text{ a.e. in } \mathbb{R}^3. \tag{3.1}$$

Let $\mathbb{E}(\mathbf{u}):=\frac{1}{2}(\nabla\mathbf{u}+(\nabla\mathbf{u})^\top)$ be the strain rate tensor. If $(\mathbf{u},\pi)\in C^1(\overline{\Omega}_\pm)^3\times C^0(\overline{\Omega}_\pm)$, we can define the *classical* interior and exterior conormal derivatives (i.e., *the boundary traction fields*) for the Stokes system (1.1) with continuously differentiable viscosity coefficient μ by the well-known formula

$$\mathbf{t}_\mu^{c\pm}(\mathbf{u},\pi):=\gamma_\pm\left(-\pi\mathbb{I}+2\mu\mathbb{E}(\mathbf{u})\right)\boldsymbol{\nu}, \tag{3.2}$$

$\boldsymbol{\nu}$ being the outward unit normal to Ω_+, defined a.e. on $\partial\Omega$, and the symbol $\pm$ refers to the limit and conormal derivative from $\Omega_\pm$. Then for any function $\boldsymbol{\varphi}\in\mathcal{D}(\mathbb{R}^3)^3$, we obtain

$$\pm\left\langle\mathbf{t}_\mu^{c\pm}(\mathbf{u},\pi),\boldsymbol{\varphi}\right\rangle_{\partial\Omega}=2\langle\mu\mathbb{E}(\mathbf{u}),\mathbb{E}(\boldsymbol{\varphi})\rangle_{\Omega_\pm}-\langle\pi,\operatorname{div}\boldsymbol{\varphi}\rangle_{\Omega_\pm}+\langle\mathcal{L}_\mu(\mathbf{u},\pi),\boldsymbol{\varphi}\rangle_{\Omega_\pm}.$$

This formula suggests the following weak definition of the generalized conormal derivative for the Stokes system with L^∞ coefficients in the setting of L^2-weighted Sobolev spaces (cf., e.g., [17, Lemma 3.2], [32, Lemma 2.9], [34, Lemma 2.2], [40, Definition 3.1, Theorem 3.2], [43, Theorem 10.4.1]).

Definition 3.1. Let $\mu\in L^\infty(\mathbb{R}^3)$ satisfy assumption (3.1). Let

$$\boldsymbol{\mathcal{H}}^1(\Omega_\pm,\mathcal{L}_\mu):=\Big\{(\mathbf{u}_\pm,\pi_\pm,\tilde{\mathbf{f}}_\pm)\in\mathcal{H}^1(\Omega_\pm)^3\times L^2(\Omega_\pm)\times\widetilde{\mathcal{H}}^{-1}(\Omega_\pm)^3: \boldsymbol{\mathcal{L}}_\mu(\mathbf{u}_\pm,\pi_\pm)=\tilde{\mathbf{f}}_\pm|_{\Omega_\pm} \text{ and div } \mathbf{u}_\pm=0 \text{ in } \Omega_\pm\Big\}. \tag{3.3}$$

Then define the conormal derivative operator $\mathbf{t}_\mu^\pm:\boldsymbol{\mathcal{H}}^1(\Omega_\pm,\boldsymbol{\mathcal{L}}_\mu)\to H^{-\frac{1}{2}}(\partial\Omega)^3$,

$$\boldsymbol{\mathcal{H}}^1(\Omega_\pm,\boldsymbol{\mathcal{L}}_\mu)\ni(\mathbf{u}_\pm,\pi_\pm,\tilde{\mathbf{f}}_\pm)\longmapsto\mathbf{t}_\mu^\pm(\mathbf{u}_\pm,\pi_\pm;\tilde{\mathbf{f}}_\pm)\in H^{-\frac{1}{2}}(\partial\Omega)^3, \tag{3.4}$$

$$\pm\left\langle\mathbf{t}_\mu^\pm(\mathbf{u}_\pm,\pi_\pm;\tilde{\mathbf{f}}_\pm),\Phi\right\rangle_{\partial\Omega}:=2\langle\mu\mathbb{E}(\mathbf{u}_\pm),\mathbb{E}(\gamma_\pm^{-1}\Phi)\rangle_{\Omega_\pm} -\langle\pi_\pm,\operatorname{div}(\gamma_\pm^{-1}\Phi)\rangle_{\Omega_\pm}+\langle\tilde{\mathbf{f}}_\pm,\gamma_\pm^{-1}\Phi\rangle_{\Omega_\pm},\ \forall\,\Phi\in H^{\frac{1}{2}}(\partial\Omega)^3, \tag{3.5}$$

where $\gamma_\pm^{-1}:H^{\frac{1}{2}}(\partial\Omega)^3\to\mathcal{H}^1(\Omega_\pm)^3$ is a (non-unique) bounded right inverse of the trace operator $\gamma_\pm:\mathcal{H}^1(\Omega_\pm)^3\to H^{\frac{1}{2}}(\partial\Omega)^3$.

We use the simplified notation $\mathbf{t}_\mu^\pm(\mathbf{u}_\pm,\pi_\pm)$ for $\mathbf{t}_\mu^\pm(\mathbf{u}_\pm,\pi_\pm;\mathbf{0})$. The following assertion can be proved similar to [41, Theorem 5.3], [32, Lemma 2.9].

Lemma 3.2. *Let $\mu \in L^\infty(\mathbb{R}^3)$ satisfy assumption (3.1). Then for all $\mathbf{w}_\pm \in \mathcal{H}^1(\Omega_\pm)^3$ and $(\mathbf{u}_\pm, \pi_\pm, \tilde{\mathbf{f}}_\pm) \in \boldsymbol{\mathcal{H}}^1(\Omega_\pm, \boldsymbol{\mathcal{L}}_\mu)$ the following identity holds*

$$\pm\left\langle \mathbf{t}_\mu^\pm(\mathbf{u}_\pm, \pi_\pm; \tilde{\mathbf{f}}_\pm), \gamma_\pm \mathbf{w}_\pm \right\rangle_{\partial\Omega} = 2\langle \mu\mathbb{E}(\mathbf{u}_\pm), \mathbb{E}(\mathbf{w}_\pm)\rangle_{\Omega_\pm} - \langle \pi_\pm, \operatorname{div}\mathbf{w}_\pm\rangle_{\Omega_\pm} + \langle \tilde{\mathbf{f}}_\pm, \mathbf{w}_\pm\rangle_{\Omega_\pm}. \quad (3.6)$$

Let γ denote the trace operator from $\mathcal{H}^1(\mathbb{R}^3)^3$ to $H^{\frac{1}{2}}(\partial\Omega)^3$ (cf., e.g., [40, Theorem 2.3, Lemma 2.6], [8, (2.2)]). For $(\mathbf{u}_\pm, \pi_\pm, \tilde{\mathbf{f}}_\pm) \in \boldsymbol{\mathcal{H}}^1(\Omega_\pm, \boldsymbol{\mathcal{L}}_\mu)$, let

$$\mathbf{u} := \mathring{E}_+\mathbf{u}_+ + \mathring{E}_-\mathbf{u}_-,\ \pi := \mathring{E}_+\pi_+ + \mathring{E}_-\pi_-,\ \mathbf{f} := \tilde{\mathbf{f}}_+ + \tilde{\mathbf{f}}_- \quad (3.7)$$

$$[\mathbf{t}_\mu(\mathbf{u}, \pi; \mathbf{f})] := \mathbf{t}_\mu^+(\mathbf{u}_+, \pi_+; \tilde{\mathbf{f}}_+) - \mathbf{t}_\mu^-(\mathbf{u}_-, \pi_-; \tilde{\mathbf{f}}_-). \quad (3.8)$$

Moreover, if $\mathbf{f} = \mathbf{0}$, we define

$$[\mathbf{t}_\mu(\mathbf{u}, \pi)] := [\mathbf{t}_\mu(\mathbf{u}, \pi; \mathbf{0})] = \mathbf{t}_\mu^+(\mathbf{u}_+, \pi_+) - \mathbf{t}_\mu^-(\mathbf{u}_-, \pi_-). \quad (3.9)$$

Then Lemma 3.2 leads to the following result.

Lemma 3.3. *Let $\mu \in L^\infty(\mathbb{R}^3)$ satisfy assumption (3.1). Also let $(\mathbf{u}_\pm, \pi_\pm, \tilde{\mathbf{f}}_\pm) \in \boldsymbol{\mathcal{H}}^1(\Omega_\pm, \boldsymbol{\mathcal{L}}_\mu)$ and let $(\mathbf{u}, \pi, \mathbf{f})$ be defined as in (3.7). Then for all $\mathbf{w} \in \mathcal{H}^1(\mathbb{R}^3)^3$, the following formula holds*

$$\left\langle [\mathbf{t}_\mu(\mathbf{u}, \pi; \mathbf{f})], \gamma\mathbf{w} \right\rangle_{\partial\Omega} = 2\langle \mu\mathbb{E}(\mathbf{u}_+), \mathbb{E}(\mathbf{w})\rangle_{\Omega_+} + 2\langle \mu\mathbb{E}(\mathbf{u}_-), \mathbb{E}(\mathbf{w})\rangle_{\Omega_-} - \langle \pi, \operatorname{div}\mathbf{w}\rangle_{\mathbb{R}^3} + \langle \mathbf{f}, \mathbf{w}\rangle_{\mathbb{R}^3}. \quad (3.10)$$

We also need the following particular case of Lemma 3.3 when $\mathbf{f} = \mathbf{0}$.

Lemma 3.4. *Let $\mu \in L^\infty(\mathbb{R}^3)$ satisfy assumption (3.1). Also let $(\mathbf{u}_\pm, \pi_\pm, \mathbf{0}) \in \boldsymbol{\mathcal{H}}^1(\Omega_\pm, \boldsymbol{\mathcal{L}}_\mu)$. Let $\mathbf{u}$ and π be defined as in formula (3.7). Then for all $\mathbf{w} \in \mathcal{H}^1(\mathbb{R}^3)^3$,*

$$\left\langle [\mathbf{t}_\mu(\mathbf{u}, \pi)], \gamma\mathbf{w} \right\rangle_{\partial\Omega} = 2\langle \mu\mathbb{E}(\mathbf{u}_+), \mathbb{E}(\mathbf{w})\rangle_{\Omega_+} + 2\langle \mu\mathbb{E}(\mathbf{u}_-), \mathbb{E}(\mathbf{w})\rangle_{\Omega_-} - \langle \pi, \operatorname{div}\mathbf{w}\rangle_{\mathbb{R}^3}. \quad (3.11)$$

4. Newtonian and single layer potentials for the Stokes system with L^∞ coefficients

Recall that the function $\mu \in L^\infty(\mathbb{R}^3)$ satisfies conditions (3.1). Next, we define the Newtonian and single layer potentials for the L^∞ coefficient Stokes system (1.1).

4.1. Variational solution of the variable-coefficient Stokes system in $\mathbb{R}^3$.

First we show the following useful well-posedness result.

Lemma 4.1. *Let $a_\mu(\cdot,\cdot) : \mathcal{H}^1(\mathbb{R}^3)^3 \times \mathcal{H}^1(\mathbb{R}^3)^3 \to \mathbb{R}$ and $b(\cdot,\cdot) : \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3) \to \mathbb{R}$ be the bilinear forms given by*

$$a_\mu(\mathbf{u}, \mathbf{v}) := 2\langle \mu\mathbb{E}(\mathbf{u}), \mathbb{E}(\mathbf{v})\rangle_{\mathbb{R}^3},\ \forall\, \mathbf{u}, \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^3)^3, \quad (4.1)$$

$$b(\mathbf{v}, q) := -\langle \operatorname{div}\mathbf{v}, q\rangle_{\mathbb{R}^3},\ \forall\, \mathbf{v} \in \mathcal{H}^1(\mathbb{R}^3)^3,\ \forall\, q \in L^2(\mathbb{R}^3). \quad (4.2)$$

Also let $\boldsymbol{\ell}:\mathcal{H}^1(\mathbb{R}^3)^3\to\mathbb{R}$ *be a linear and bounded map. Then the mixed variational formulation*

$$\begin{cases} a_\mu(\mathbf{u},\mathbf{v})+b(\mathbf{v},p)=\boldsymbol{\ell}(\mathbf{v}),\ \forall\ \mathbf{v}\in\mathcal{H}^1(\mathbb{R}^3)^3,\\ b(\mathbf{u},q)=0,\ \forall\, q\in L^2(\mathbb{R}^3)\end{cases} \tag{4.3}$$

is well-posed. Hence, (4.3) *has a unique solution* $(\mathbf{u},p)\in\mathcal{H}^1(\mathbb{R}^3)^3\times L^2(\mathbb{R}^3)$ *and there exists a constant* $C=C(c_\mu)>0$ *such that*

$$\|\mathbf{u}\|_{\mathcal{H}^1(\mathbb{R}^3)^3}+\|p\|_{L^2(\mathbb{R}^3)}\le C\|\boldsymbol{\ell}\|_{\mathcal{H}^{-1}(\mathbb{R}^3)^3}. \tag{4.4}$$

Proof. By using conditions (3.1) and definition (2.11) of the norm of the weighted Sobolev space $\mathcal{H}^1(\mathbb{R}^3)$ we obtain that

$$\begin{aligned}|a_\mu(\mathbf{u},\mathbf{v})|&\le 2c_\mu\|\mathbb{E}(\mathbf{u})\|_{L^2(\mathbb{R}^3)^{3\times 3}}\|\mathbb{E}(\mathbf{v})\|_{L^2(\mathbb{R}^3)^{3\times 3}}\\ &\le 2c_\mu\|\mathbf{u}\|_{\mathcal{H}^1(\mathbb{R}^3)^3}\|\mathbf{v}\|_{\mathcal{H}^1(\mathbb{R}^3)^3},\ \forall\ \mathbf{u},\mathbf{v}\in\mathcal{H}^1(\mathbb{R}^3)^3.\end{aligned} \tag{4.5}$$

Moreover, by using the Korn type inequality for functions in $\mathcal{H}^1(\mathbb{R}^3)^3$,

$$\|\operatorname{grad}\mathbf{v}\|_{L^2(\mathbb{R}^3)^{3\times 3}}\le 2^{\frac12}\|\mathbb{E}(\mathbf{v})\|_{L^2(\mathbb{R}^3)^{3\times 3}} \tag{4.6}$$

(cf., e.g., [46, (2.2)]) and since the seminorm

$$|g|_{\mathcal{H}^1(\mathbb{R}^3)}:=\|\nabla g\|_{L^2(\mathbb{R}^3)^3} \tag{4.7}$$

is a norm in $\mathcal{H}^1(\mathbb{R}^3)^3$ equivalent to the norm defined by (2.11) (see, e.g., [18, Chapter XI, Part B, §1, Theorem 1]), there exists a constant $c_1>0$ such that

$$\begin{aligned}a_\mu(\mathbf{u},\mathbf{u})\ge 2c_\mu^{-1}\|\mathbb{E}(\mathbf{u})\|^2_{L^2(\mathbb{R}^3)^{3\times 3}}&\ge c_\mu^{-1}\|\nabla\mathbf{u}\|^2_{L^2(\mathbb{R}^3)^{3\times 3}}\\ &\ge c_\mu^{-1}c_1\|\mathbf{u}\|^2_{\mathcal{H}^1(\mathbb{R}^3)^3},\ \forall\,\mathbf{u}\in\mathcal{H}^1(\mathbb{R}^3)^3.\end{aligned} \tag{4.8}$$

Inequalities (4.5) and (4.8) show that $a_\mu(\cdot,\cdot):\mathcal{H}^1(\mathbb{R}^3)^3\times\mathcal{H}^1(\mathbb{R}^3)^3\to\mathbb{R}$ is a bounded and coercive bilinear form. Moreover, since the divergence operator

$$\operatorname{div}:\mathcal{H}^1(\mathbb{R}^3)^3\to L^2(\mathbb{R}^3) \tag{4.9}$$

is bounded, then the bilinear form $b(\cdot,\cdot):\mathcal{H}^1(\mathbb{R}^3)^3\times L^2(\mathbb{R}^3)\to\mathbb{R}$ is bounded as well. In addition, the operator in (4.9) is surjective (cf. [2, Proposition 2.1], [46, Proposition 2.4]) and also

$$\begin{aligned}\mathcal{H}^1_{\operatorname{div}}(\mathbb{R}^3)^3&:=\left\{\mathbf{w}\in\mathcal{H}^1(\mathbb{R}^3)^3:\operatorname{div}\mathbf{w}=0\right\}\\ &=\left\{\mathbf{w}\in\mathcal{H}^1(\mathbb{R}^3)^3:b(\mathbf{w},q)=0,\ \forall\,q\in L^2(\mathbb{R}^3)\right\}.\end{aligned}$$

In addition, the operator in (4.9) is surjective (cf. [2, Proposition 2.1], [46, Proposition 2.4]), and hence the operator

$$-\operatorname{div}:\mathcal{H}^1(\mathbb{R}^3)^3/\mathcal{H}^1_{\operatorname{div}}(\mathbb{R}^3)^3\to L^2(\mathbb{R}^3)$$

is an isomorphism. Then by Lemma A.2(ii) the bounded bilinear form $b(\cdot,\cdot):\mathcal{H}^1(\mathbb{R}^3)^3\times L^2(\mathbb{R}^3)\to\mathbb{R}$ satisfies the inf-sup condition (A.7). Hence, there exists $\beta_0\in(0,\infty)$ such that

$$\inf_{q\in L^2(\mathbb{R}^3)\setminus\{0\}}\ \sup_{\mathbf{w}\in\mathcal{H}^1(\mathbb{R}^3)^3\setminus\{\mathbf{0}\}}\frac{b(\mathbf{w},q)}{\|\mathbf{w}\|_{\mathcal{H}^1(\mathbb{R}^3)^3}\|q\|_{L^2(\mathbb{R}^3)}}\ge\beta_0. \tag{4.10}$$

By applying Theorem A.4, with $X = \mathcal{H}^1(\mathbb{R}^3)^3$, $M = L^2(\mathbb{R}^3)$, $V = \mathcal{H}^1_{\rm div}(\mathbb{R}^3)^3$, we conclude that the mixed variational formulation (4.3) has a unique solution $(\mathbf{u}, p) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$ and there exists a constant $C = C(c_\mu) > 0$ such that $(\mathbf{u}, p)$ satisfies inequality (4.4). □

Next we use the result of Lemma 4.1 in order to show the well-posedness of the L^∞ coefficient Stokes system in the space $\mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$ (see also [2, Theorem 3] for the constant-coefficient case).

Theorem 4.2. *Let $\mu \in L^\infty(\mathbb{R}^3)$ satisfy conditions (3.1). Then the L^∞ coefficient Stokes system*

$$\begin{cases} \nabla\pi - \mathrm{div}\,(2\mu\mathbb{E}(\mathbf{u})) = \boldsymbol{\ell}, & \boldsymbol{\ell} \in \mathcal{H}^{-1}(\mathbb{R}^3)^3, \\ \mathrm{div}\,\mathbf{u} = 0, & \textit{in } \mathbb{R}^3, \end{cases} \tag{4.11}$$

has a unique solution $(\mathbf{u}, p) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$, and there exists a constant $C_0 = C_0(c_\mu) > 0$ such that

$$\|\mathbf{u}\|_{\mathcal{H}^1(\mathbb{R}^3)^3} + \|p\|_{L^2(\mathbb{R}^3)} \le C_0\|\boldsymbol{\ell}\|_{\mathcal{H}^{-1}(\mathbb{R}^3)^3}. \tag{4.12}$$

Proof. Note that the Stokes system (4.11) is equivalent to the variational problem (4.3) as follows from the density of $\mathcal{D}(\mathbb{R}^3)^3$ in the space $\mathcal{H}^1(\mathbb{R}^3)^3$ (cf., e.g., [28], [46, Proposition 2.1]). Then the well-posedness result of the Stokes system with L^∞ coefficients (4.11) follows from Lemma 4.1. □

4.2. Newtonian potential for the Stokes system with L^∞ coefficients

The well-posedness of problem (4.11) allows us to define the *Newtonian potential for the Stokes system with L^∞ coefficients* as follows.

Definition 4.3. For any $\boldsymbol{\ell} \in \mathcal{H}^{-1}(\mathbb{R}^3)^3$, we define the *Newtonian velocity and pressure potentials* for the Stokes system with L^∞ coefficients as

$$\boldsymbol{\mathcal{N}}_{\mu;\mathbb{R}^3}\boldsymbol{\ell} := -\mathbf{u},\ \ \mathcal{Q}_{\mu;\mathbb{R}^3}\boldsymbol{\ell} := -\pi, \tag{4.13}$$

where $(\mathbf{u}, \pi) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)$ is the unique solution of problem (4.11) with the given datum $\boldsymbol{\ell}$.

Moreover, the well-posedness of problem (4.11) yields the continuity of the above operators as stated in the following assertion (cf. also [32, Lemma A.3] for $\mu = 1$).

Lemma 4.4. *The Newtonian velocity and pressure potential operators*

$$\boldsymbol{\mathcal{N}}_{\mu;\mathbb{R}^3} : \mathcal{H}^{-1}(\mathbb{R}^3)^3 \to \mathcal{H}^1(\mathbb{R}^3)^3,\ \ \mathcal{Q}_{\mu;\mathbb{R}^3} : \mathcal{H}^{-1}(\mathbb{R}^3)^3 \to L^2(\mathbb{R}^3) \tag{4.14}$$

are linear and continuous.

4.3. Single layer potential for the Stokes system with L^∞ coefficients

For a given $\boldsymbol{\varphi} \in H^{-\frac{1}{2}}(\partial\Omega)^3$, we now consider the following transmission problem for the Stokes system with L^∞ coefficients

$$\begin{cases} \mathrm{div}\,(2\mu\mathbb{E}(\mathbf{u}_{\boldsymbol{\varphi}})) - \nabla\pi_{\boldsymbol{\varphi}} = \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \partial\Omega, \\ \mathrm{div}\,\mathbf{u}_{\boldsymbol{\varphi}} = 0 & \text{in } \mathbb{R}^3 \setminus \partial\Omega, \\ [\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol{\varphi}}, \pi_{\boldsymbol{\varphi}})] = \boldsymbol{\varphi} & \text{on } \partial\Omega, \end{cases} \tag{4.15}$$

and show that this problem has a unique solution $(\mathbf{u}_{\boldsymbol\varphi},\pi_{\boldsymbol\varphi})\in\mathcal{H}^1(\mathbb{R}^3)^3\times L^2(\mathbb{R}^3)$ (cf. also [46, Proposition 5.1] for $\mu=1$). Note that the membership of $\mathbf{u}_{\boldsymbol\varphi}$ in $\mathcal{H}^1(\mathbb{R}^3)^3$ implies the transmission condition

$$[\gamma(\mathbf{u}_{\boldsymbol\varphi})]=\mathbf{0} \text{ on } \partial\Omega\,, \tag{4.16}$$

and the first equation in (4.15) implies also that the jump $[\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol\varphi},\pi_{\boldsymbol\varphi})]$ is well defined.

Theorem 4.5. *Let $\boldsymbol\varphi\in H^{-\frac12}(\partial\Omega)^3$ be given. Then the transmission problem (4.15) has the following equivalent mixed variational formulation: Find $(\mathbf{u}_{\varphi},\pi_{\boldsymbol\varphi})\in\mathcal{H}^1(\mathbb{R}^3)^3\times L^2(\mathbb{R}^3)$ such that*

$$\begin{cases} 2\langle\mu\mathbb{E}(\mathbf{u}_{\boldsymbol\varphi}),\mathbb{E}(\mathbf{v})\rangle_{\mathbb{R}^3}-\langle\pi_{\boldsymbol\varphi},\operatorname{div}\mathbf{v}\rangle_{\mathbb{R}^3}=\langle\boldsymbol\varphi,\gamma\mathbf{v}\rangle_{\partial\Omega},\ \forall\,\mathbf{v}\in\mathcal{H}^1(\mathbb{R}^3)^3,\\ \langle\operatorname{div}\mathbf{u}_{\boldsymbol\varphi},q\rangle_{\mathbb{R}^3}=0,\ \forall\,q\in L^2(\mathbb{R}^3).\end{cases} \tag{4.17}$$

Moreover, problem (4.17) is well-posed. Hence (4.17) has a unique solution $(\mathbf{u}_{\boldsymbol\varphi},\pi_{\boldsymbol\varphi})\in\mathcal{H}^1(\mathbb{R}^3)^3\times L^2(\mathbb{R}^3)$, and there exists a constant $C=C(c_\mu)$ such that

$$\|\mathbf{u}_{\boldsymbol\varphi}\|_{\mathcal{H}^1(\mathbb{R}^3)^3}+\|\pi_{\boldsymbol\varphi}\|_{L^2(\mathbb{R}^3)}\le C\|\boldsymbol\varphi\|_{H^{-\frac12}(\partial\Omega)^3}. \tag{4.18}$$

Proof. The equivalence between the transmission problem (4.15) and the variational problem (4.17) follows from the density of the space $\mathcal{D}(\mathbb{R}^3)^3$ in $\mathcal{H}^1(\mathbb{R}^3)^3$ and formula (3.11), while the well-posedness of the variational problem (4.17) is an immediate consequence of Lemma 4.1 with the linear and continuous form $\boldsymbol\ell:\mathcal{H}^1(\mathbb{R}^3)^3\to\mathbb{R}$ given by

$$\boldsymbol\ell(\mathbf{v}):=\langle\boldsymbol\varphi,\gamma\mathbf{v}\rangle_{\partial\Omega}=\langle\gamma^*\boldsymbol\varphi,\mathbf{v}\rangle_{\mathbb{R}^3},\ \forall\,\mathbf{v}\in\mathcal{H}^1(\mathbb{R}^3)^3,$$

and hence $\boldsymbol\ell=\gamma^*\boldsymbol\varphi$, where $\gamma^*:H^{-\frac12}(\partial\Omega)^3\to\mathcal{H}^{-1}(\mathbb{R}^3)^3$ is the adjoint of the trace operator $\gamma:\mathcal{H}^1(\mathbb{R}^3)^3\to H^{\frac12}(\partial\Omega)^3$. □

Theorem 4.5 leads to the following definition (cf. [46, p. 75] for $\mu=1$).

Definition 4.6. For any $\boldsymbol\varphi\in H^{-\frac12}(\partial\Omega)^3$, we define the *single layer velocity and pressure potentials* for the Stokes system with L^∞ coefficients (1.1) as

$$\mathbf{V}_{\mu;\partial\Omega}\boldsymbol\varphi:=\mathbf{u}_{\boldsymbol\varphi},\ \mathcal{Q}^s_{\mu;\partial\Omega}\boldsymbol\varphi:=\pi_{\boldsymbol\varphi}, \tag{4.19}$$

and the *potential operators* $\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}:H^{-\frac12}(\partial\Omega)^3\to H^{\frac12}(\partial\Omega)^3$ and $\mathbf{K}^*_{\mu;\partial\Omega}:H^{-\frac12}(\partial\Omega)^3\to H^{-\frac12}(\partial\Omega)^3$ as

$$\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}\boldsymbol\varphi:=\gamma\mathbf{u}_{\boldsymbol\varphi},\ \mathbf{K}^*_{\mu;\partial\Omega}\boldsymbol\varphi:=\frac12\left(\mathbf{t}^+_\mu(\mathbf{u}_{\boldsymbol\varphi},\pi_{\boldsymbol\varphi})+\mathbf{t}^-_\mu(\mathbf{u}_{\boldsymbol\varphi},\pi_{\boldsymbol\varphi})\right), \tag{4.20}$$

where $(\mathbf{u}_{\boldsymbol\varphi},\pi_{\boldsymbol\varphi})$ is the unique solution of problem (4.15) in $\mathcal{H}^1(\mathbb{R}^3)^3\times L^2(\mathbb{R}^3)$.

The next result shows the continuity of single layer velocity and pressure potential operators for the variable coefficient Stokes system (cf. [46, Proposition 5.2], [32, Lemma A.4, (A.10), (A.12)] and [43, Theorem 10.5.3] in the case $\mu=1$).

Lemma 4.7. *The following operators are linear and continuous*

$$\mathbf{V}_{\mu;\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^3 \to \mathcal{H}^1(\mathbb{R}^3)^3,\ \ \mathcal{Q}^s_{\mu;\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^3 \to L^2(\mathbb{R}^3), \tag{4.21}$$

$$\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^3 \to H^{\frac{1}{2}}(\partial\Omega)^3,\ \mathbf{K}^*_{\mu;\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^3 \to H^{-\frac{1}{2}}(\partial\Omega)^3. \tag{4.22}$$

Proof. The continuity of operators (4.21) and (4.22) follows from the well-posedness of the transmission problem (4.15) and Definition 4.6. □

The next result yields the jump relations of the single layer potential and its conormal derivative across $\partial\Omega$ (see also [46, Proposition 5.3] for $\mu = 1$).

Lemma 4.8. *Let $\boldsymbol{\varphi}\in H^{-\frac{1}{2}}(\partial\Omega)^3$. Then almost everywhere on $\partial\Omega$,*

$$[\gamma\mathbf{V}_{\mu;\partial\Omega}\boldsymbol{\varphi}] = \mathbf{0}, \tag{4.23}$$

$$\left[\mathbf{t}_\mu\left(\mathbf{V}_{\mu;\partial\Omega}\boldsymbol{\varphi}, \mathcal{Q}^s_{\mu;\partial\Omega}\boldsymbol{\varphi}\right)\right]=\boldsymbol{\varphi},\ \mathbf{t}^{\pm}_\mu\left(\mathbf{V}_{\mu;\partial\Omega}\boldsymbol{\varphi}, \mathcal{Q}^s_{\mu;\partial\Omega}\boldsymbol{\varphi}\right)=\pm\frac{1}{2}\boldsymbol{\varphi}+\mathbf{K}^*_{\mu;\partial\Omega}\boldsymbol{\varphi}. \tag{4.24}$$

Proof. Formulas (4.23) and (4.24) follow from Definition 4.6 and the transmission condition in (4.16), as well as the transmission condition in the third line of (4.15). □

Let $\mathbb{R}\boldsymbol{\nu} = \{c\boldsymbol{\nu} : c \in \mathbb{R}\}$. Let $\mathrm{Ker}\,\{T : X \to Y\} := \{x \in X : T(x) = 0\}$ denote the null space of the map $T : X \to Y$.

We next obtain the main properties of the single layer potential operator (cf., e.g., [43, Theorem 10.5.3], and [8, Proposition 3.3(c)] and [46, Proposition 5.4] for $\mu = 1$ and $s \in [0,\infty)$).

Lemma 4.9. *The following properties hold*

$$\mathbf{V}_{\mu;\partial\Omega}\boldsymbol{\nu} = \mathbf{0}\ \ in\ \mathbb{R}^3,\ \mathcal{Q}^s_{\mu;\partial\Omega}\boldsymbol{\nu} = -\chi_{\Omega_+} \tag{4.25}$$

$$\mathrm{Ker}\left\{\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega} : H^{-\frac{1}{2}}(\partial\Omega)^3 \to H^{\frac{1}{2}}(\partial\Omega)^3\right\} = \mathbb{R}\boldsymbol{\nu}, \tag{4.26}$$

$$\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}\boldsymbol{\varphi} \in H^{\frac{1}{2}}_{\boldsymbol{\nu}}(\partial\Omega)^3,\ \forall\,\boldsymbol{\varphi} \in H^{-\frac{1}{2}}(\partial\Omega)^3, \tag{4.27}$$

where $\chi_{\Omega_+}=1$ in Ω_+, $\chi_{\Omega_+}=0$ in Ω_-, and

$$H^{\frac{1}{2}}_{\boldsymbol{\nu}}(\partial\Omega)^3 := \left\{\boldsymbol{\phi} \in H^{\frac{1}{2}}(\partial\Omega)^3 : \langle\boldsymbol{\nu},\boldsymbol{\phi}\rangle_{\partial\Omega} = 0\right\}. \tag{4.28}$$

Proof. First, we consider the transmission problem (4.15) with the datum $\boldsymbol{\varphi}=\boldsymbol{\nu}\in H^{-\frac{1}{2}}(\partial\Omega)^3$. Then the solution of this problem is given by

$$(\mathbf{u}_{\boldsymbol{\nu}}, \pi_{\boldsymbol{\nu}}) = \left(\mathbf{0}, -\chi_{\Omega_+}\right) \in \mathcal{H}^1(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3). \tag{4.29}$$

Indeed, the pair $(\mathbf{u}_{\boldsymbol{\nu}}, \pi_{\boldsymbol{\nu}})$ satisfies the equations and the transmission condition in (4.15), as well as the transmission condition (4.16), and, in view of formula (3.11) and the divergence theorem,

$$\langle[\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol{\nu}}, \pi_{\boldsymbol{\nu}})], \gamma\mathbf{v}\rangle_{\partial\Omega} = -\langle\pi_{\boldsymbol{\nu}}, \mathrm{div}\,\mathbf{v}\rangle_{\mathbb{R}^3} = \langle\boldsymbol{\nu}, \gamma\mathbf{v}\rangle_{\partial\Omega},\ \forall\,\mathbf{v} \in \mathcal{D}(\mathbb{R}^3)^3. \tag{4.30}$$

Then by formula (2.3), Lemma 2.1, the dense embedding of the space $\mathcal{D}(\mathbb{R}^3)^3$ in $\mathcal{H}^1(\mathbb{R}^3)^3$, and the above equality, we obtain that $\langle[\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol{\nu}}, \pi_{\boldsymbol{\nu}})], \Phi\rangle_{\partial\Omega} = \langle\boldsymbol{\nu}, \Phi\rangle_{\partial\Omega}$ for any $\Phi \in H^{\frac{1}{2}}(\partial\Omega)^3$. Hence, $[\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol{\nu}}, \pi_{\boldsymbol{\nu}})] = \boldsymbol{\nu}$, as asserted. Then

Definition 4.6 implies relations (4.25). Moreover, $\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}\boldsymbol{\nu}=\mathbf{0}$, i.e., $\mathbb{R}\boldsymbol{\nu}\subseteq \mathrm{Ker}\big\{\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}:H^{-\frac{1}{2}}(\partial\Omega)^3\to H^{\frac{1}{2}}(\partial\Omega)^3\big\}$.

Now let $\boldsymbol{\varphi}_0\in\mathrm{Ker}\left\{\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}:H^{-\frac{1}{2}}(\partial\Omega)^3\to H^{\frac{1}{2}}(\partial\Omega)^3\right\}$. Let $(\mathbf{u}_{\boldsymbol{\varphi}_0},\pi_{\boldsymbol{\varphi}_0})=\left(\mathbf{V}_{\mu;\partial\Omega}\boldsymbol{\varphi}_0,\mathcal{Q}^s_{\mu;\partial\Omega}\boldsymbol{\varphi}_0\right)\in\mathcal{H}^1(\mathbb{R}^3)^3\times L^2(\mathbb{R}^3)$ be the unique solution of problem (4.15) with datum $\boldsymbol{\varphi}_0$. Since $\gamma\mathbf{u}_{\boldsymbol{\varphi}_0}=\mathbf{0}$ on $\partial\Omega$, formula (3.11) yields that

$$0=\langle[\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol{\varphi}_0},\pi_{\boldsymbol{\varphi}_0})],\gamma\mathbf{u}_{\boldsymbol{\varphi}_0}\rangle_{\partial\Omega}=a_\mu\left(\mathbf{u}_{\boldsymbol{\varphi}_0},\mathbf{u}_{\boldsymbol{\varphi}_0}\right), \tag{4.31}$$

and hence $\mathbf{u}_{\boldsymbol{\varphi}_0}=\mathbf{0}$, $\pi_{\boldsymbol{\varphi}_0}=c\chi_{\Omega_+}$ in $\mathbb{R}^3$, where $c\in\mathbb{R}$. In view of formula (3.11),

$$\langle[\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol{\varphi}_0},\pi_{\boldsymbol{\varphi}_0})],\gamma\mathbf{w}\rangle_{\partial\Omega}=-\langle\pi_{\boldsymbol{\varphi}_0},\operatorname{div}\mathbf{w}\rangle_{\mathbb{R}^3}=-c\langle\boldsymbol{\nu},\gamma\mathbf{w}\rangle_{\partial\Omega},\ \forall\,\mathbf{w}\in\mathcal{D}(\mathbb{R}^3)^3,$$

and, thus, $\boldsymbol{\varphi}_0=[\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol{\varphi}},\pi_{\boldsymbol{\varphi}_0})]=-c\boldsymbol{\nu}$. Hence, formula (4.26) follows.

Now let $\boldsymbol{\varphi}\in H^{-\frac{1}{2}}(\partial\Omega)^3$. By using the first formula in (4.20), we obtain for any $\boldsymbol{\varphi}\in H^{-\frac{1}{2}}(\partial\Omega)^3$ that $\langle\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}\boldsymbol{\varphi},\boldsymbol{\nu}\rangle_{\partial\Omega}=\langle\gamma\mathbf{u}_{\boldsymbol{\varphi}},\boldsymbol{\nu}\rangle_{\partial\Omega}=\langle\operatorname{div}\mathbf{u}_{\boldsymbol{\varphi}},1\rangle_\Omega=0$, where $\mathbf{u}_{\boldsymbol{\varphi}}=\mathbf{V}_{\mu;\partial\Omega}\boldsymbol{\varphi}$. Thus, we get relation (4.27). □

Next we use the notation $[\![\cdot]\!]$ for the equivalence classes of the quotient space $H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\boldsymbol{\nu}$. Thus, any $[\![\boldsymbol{\varphi}]\!]\in H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\boldsymbol{\nu}$ can be written as $[\![\boldsymbol{\varphi}]\!]=\boldsymbol{\varphi}+\mathbb{R}\boldsymbol{\nu}$, where $\boldsymbol{\varphi}\in H^{-\frac{1}{2}}(\partial\Omega)^3$.

Exploiting properties (4.26) and (4.27), we now show the following invertibility result (cf. [43, Theorem 10.5.3], [8, Proposition 3.3(d)], [46, Proposition 5.5] for $\mu=1$ and $\alpha\geq 0$ constant).

Lemma 4.10. *The following operator is an isomorphism*

$$\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}:H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\nu\to H^{\frac{1}{2}}_{\boldsymbol{\nu}}(\partial\Omega)^3. \tag{4.32}$$

Proof. We use arguments similar to those for Proposition 5.5 in [46]. First, Lemma 4.7 and the membership relation (4.27) imply that the linear operator in (4.32) is continuous. We show that this operator is also $H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\boldsymbol{\nu}$-elliptic, i.e., that there exists a constant $c=c(\partial\Omega)>0$ such that

$$\langle\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}[\![\boldsymbol{\varphi}]\!],[\![\boldsymbol{\varphi}]\!]\rangle_{\partial\Omega}\geq c\,\|[\![\boldsymbol{\varphi}]\!]\|^2_{H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\boldsymbol{\nu}},\ \forall\,[\![\boldsymbol{\varphi}]\!]\in H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\boldsymbol{\nu}. \tag{4.33}$$

Let $[\![\boldsymbol{\varphi}]\!]\in H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\boldsymbol{\nu}$. Thus, $[\![\boldsymbol{\varphi}]\!]=\boldsymbol{\varphi}+\mathbb{R}\boldsymbol{\nu}$, where $\boldsymbol{\varphi}\in H^{-\frac{1}{2}}(\partial\Omega)^3$. In view of formula (3.11), Definition 4.6, relations (4.26), (4.27), and inequality (4.8),

$$\begin{aligned}\langle\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}([\![\boldsymbol{\varphi}]\!]),[\![\boldsymbol{\varphi}]\!]\rangle_{\partial\Omega}&=\langle\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}(\boldsymbol{\varphi}),\boldsymbol{\varphi}\rangle_{\partial\Omega}=\langle\gamma\mathbf{u}_{\boldsymbol{\varphi}},[\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol{\varphi}},\pi_{\boldsymbol{\varphi}})]\rangle_{\partial\Omega}\\&=a_\mu(\mathbf{u}_{\boldsymbol{\varphi}},\mathbf{u}_{\boldsymbol{\varphi}})\geq c_\mu^{-1}\|\mathbf{u}_{\boldsymbol{\varphi}}\|^2_{H^1(\mathbb{R}^3)^3},\end{aligned} \tag{4.34}$$

where $\mathbf{u}_{\boldsymbol{\varphi}}=\mathbf{V}_{\mu;\partial\Omega}\boldsymbol{\varphi}$ and $\pi_{\boldsymbol{\varphi}}=\mathcal{Q}^s_{\mu;\partial\Omega}\boldsymbol{\varphi}$. Now we use the property that the trace operator

$$\gamma:\mathcal{H}^1_{\mathrm{div}}(\mathbb{R}^3)^3\to H^{\frac{1}{2}}_{\boldsymbol{\nu}}(\partial\Omega)^3 \tag{4.35}$$

is surjective having a bounded right inverse $\gamma^{-1}:H^{\frac{1}{2}}_{\boldsymbol{\nu}}(\partial\Omega)^3\to\mathcal{H}^1_{\mathrm{div}}(\mathbb{R}^3)^3$ (cf., e.g., [46, Proposition 4.4]). Hence, for any $\boldsymbol{\Phi}\in H^{\frac{1}{2}}_{\boldsymbol{\nu}}(\partial\Omega)^3$, we have that $\mathbf{w}=\gamma^{-1}\boldsymbol{\Phi}\in\mathcal{H}^1_{\mathrm{div}}(\mathbb{R}^3)^3$. Then there exists $c'\equiv c'(\partial\Omega)\in(0,\infty)$ such that

$$|\langle[\![\boldsymbol{\varphi}]\!],\boldsymbol{\Phi}\rangle_{\partial\Omega}|=|\langle\boldsymbol{\varphi},\boldsymbol{\Phi}\rangle_{\partial\Omega}|=|\langle[\mathbf{t}_\mu(\mathbf{u}_{\boldsymbol{\varphi}},\pi_{\boldsymbol{\varphi}})],\gamma\mathbf{w}\rangle_{\partial\Omega}|=|a_\mu(\mathbf{u}_{\boldsymbol{\varphi}},\mathbf{w})| \tag{4.36}$$

$$\leq 2c_\mu \|\mathbf{u}_{\boldsymbol{\varphi}}\|_{\mathcal{H}^1(\mathbb{R}^3)^3} \|\gamma^{-1}\mathbf{\Phi}\|_{\mathcal{H}^1(\mathbb{R}^3)^3} \leq 2c_\mu c' \|\mathbf{u}_{\boldsymbol{\varphi}}\|_{\mathcal{H}^1(\mathbb{R}^3)^3} \|\mathbf{\Phi}\|_{H^{\frac{1}{2}}(\partial\Omega)^3},$$

where the first equality in (4.36) follows from the relation $[\![\boldsymbol{\varphi}]\!] = \boldsymbol{\varphi} + \mathbb{R}\boldsymbol{\nu}$ and the membership of $\mathbf{\Phi}$ in $H^{\frac{1}{2}}_{\boldsymbol{\nu}}(\partial\Omega)^3$, the second equality follows from Definition 4.6, and the third equality is a consequence of formula (3.11). Since the space $H^{\frac{1}{2}}_{\boldsymbol{\nu}}(\partial\Omega)^3$ is the dual of the space $H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\nu$, formula (4.36) yields that

$$\| [\![\boldsymbol{\varphi}]\!] \|_{H^{-\frac{1}{2}}(\partial\Omega)^3/\mathbb{R}\boldsymbol{\nu}} \leq 2c_\mu c' \|\mathbf{u}_{\boldsymbol{\varphi}}\|_{\mathcal{H}^1(\mathbb{R}^3)^3}. \tag{4.37}$$

Then by (4.34) and (4.37) we obtain inequality (4.33), and the Lax-Milgram lemma yields that operator (4.32) is an isomorphism. □

Remark 4.11. The fundamental solution of the constant-coefficient Stokes system in $\mathbb{R}^3$ is well known and leads to the construction of Newtonian and boundary layer potentials via the integral approach (see, e.g., [17, 36, 43, 48]). In view of Theorems 4.2 and 4.5, the Newtonian and single layer potentials provided by the variational approach (in the case $\mu=1$) coincide with classical ones expressed in terms of the fundamental solution, since they satisfy the same boundary value problems (4.11) and (4.15), respectively (see also [46, Proposition 5.1] for $\mu=1$). The assumption $\mu = 1$ is a particular case of a more general case of L^∞ coefficients analyzed in this paper. We also note that an alternative approach, reducing various boundary value problems for variable-coefficient elliptic partial differential equations to *boundary-domain integral equations*, by employing the explicit parametrix-based integral potentials, was explored in, e.g., [12, 13, 14].

5. Exterior Dirichlet problem for the Stokes system with L^∞ coefficients

In this section we analyze the exterior Dirichlet problem for the Stokes system with L^∞ coefficients

$$\begin{cases} \operatorname{div}(2\mu\mathbb{E}(\mathbf{u})) - \nabla\pi = \mathbf{f} & \text{in } \Omega_-, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega_-, \\ \gamma_-\mathbf{u} = \boldsymbol{\phi} & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

with given data $(\mathbf{f}, \boldsymbol{\phi}) \in \mathcal{H}^{-1}(\Omega_-)^3 \times H^{\frac{1}{2}}(\partial\Omega)^3$.

5.1. Variational approach

First, we use a variational approach and show that problem (5.1) has a unique solution $(\mathbf{u}, \pi) \in \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-)$ (see also [26, Theorem 3.4] and [3, Theorem 3.16] for the constant-coefficient Stokes system).

Theorem 5.1. *Assume that $\mu \in L^\infty(\Omega_-)$ satisfies conditions (3.1). Then for all given data $(\mathbf{f}, \boldsymbol{\phi}) \in \mathcal{H}^{-1}(\Omega_-)^3 \times H^{\frac{1}{2}}(\partial\Omega)^3$ the exterior Dirichlet problem for the L^∞ coefficient Stokes system (5.1) is well posed. Hence problem (5.1)* *has a unique solution* $(\mathbf{u},\pi)\in\mathcal{H}^1(\Omega_-)^3\times L^2(\Omega_-)$ *and there exists a constant* $C\equiv C(\partial\Omega;c_\mu)>0$ *such that*

$$\|\mathbf{u}\|_{\mathcal{H}^1(\Omega_-)^3}+\|\pi\|_{L^2(\Omega_-)}\le C\left(\|\mathbf{f}\|_{\mathcal{H}^{-1}(\Omega_-)^3}+\|\boldsymbol{\phi}\|_{H^{\frac12}(\partial\Omega)^3}\right). \tag{5.2}$$

Proof. First, we note that the density of the space $\mathcal{D}(\Omega_-)^3$ in $\widetilde{\mathcal{H}}^1(\Omega_-)^3$ implies that the exterior Dirichlet problem (5.1) has the following equivalent variational formulation: Find $(\mathbf{u},\pi)\in\mathcal{H}^1(\Omega_-)^3\times L^2(\Omega_-)$ such that

$$\begin{cases} 2\langle\mu\mathbb{E}(\mathbf{u}),\mathbb{E}(\tilde{\mathbf{v}})\rangle_{\Omega_-}-\langle\pi,\operatorname{div}\tilde{\mathbf{v}}\rangle_{\Omega_-}=-\langle\mathbf{f},\tilde{\mathbf{v}}\rangle_{\Omega_-},\ \forall\,\tilde{\mathbf{v}}\in\widetilde{\mathcal{H}}^1(\Omega_-)^3,\\ \langle\operatorname{div}\mathbf{u},q\rangle_{\Omega_-}=0,\ \forall\,q\in L^2(\Omega_-),\\ \gamma_-(\mathbf{u})=\boldsymbol{\phi}\text{ on }\partial\Omega.\end{cases} \tag{5.3}$$

Next, we consider $\mathbf{u}_0\in\mathcal{H}^1(\Omega_-)^3$ such that

$$\begin{cases}\operatorname{div}\mathbf{u}_0=0 & \text{in }\Omega_-,\\ \gamma_-\mathbf{u}_0=\boldsymbol{\phi} & \text{on }\partial\Omega.\end{cases} \tag{5.4}$$

Particularly, we can choose $\mathbf{u}_0$ as the solution of the Dirichlet problem for a constant-coefficient Brinkman system

$$\begin{cases}(\triangle-\alpha\mathbb{I})\mathbf{u}_0-\nabla\pi_0=0,\ \operatorname{div}\mathbf{u}_0=0 & \text{in }\Omega_-,\\ \gamma_-\mathbf{u}_0=\boldsymbol{\phi} & \text{on }\partial\Omega,\end{cases} \tag{5.5}$$

where $\alpha>0$ is an arbitrary constant. The solution is given by the double layer potential

$$\mathbf{u}_0=\mathbf{W}_{\alpha;\partial\Omega}\left(\frac12\mathbb{I}+\mathbf{K}_{\alpha;\partial\Omega}\right)^{-1}\boldsymbol{\phi}, \tag{5.6}$$

where $\mathbf{K}_{\alpha;\partial\Omega}:H^{\frac12}(\partial\Omega)^3\to H^{\frac12}(\partial\Omega)^3$ is the corresponding Brinkman double-layer boundary potential operator. Note that

$$(\mathbf{W}_\alpha\mathbf{h})_j(\mathbf{x}):=\int_{\partial\Omega}S^\alpha_{ij\ell}(\mathbf{x},\mathbf{y})\nu_\ell(\mathbf{y})h_i(\mathbf{y})d\sigma_{\mathbf{y}}. \tag{5.7}$$

The explicit form of the kernel $S^\alpha_{ij\ell}(\mathbf{x},\mathbf{y})$ can be found in [48, (2.14)-(2.18)] and [36, Section 3.2.1].

In addition, the operator $\frac12\mathbb{I}+\mathbf{K}_{\alpha;\partial\Omega}:H^{\frac12}(\partial\Omega)^3\to H^{\frac12}(\partial\Omega)^3$ is an isomorphism (see, e.g., [39, Proposition 7.1]), and $\mathbf{u}_0$ belongs to the space $H^1(\Omega_-)^3$ (cf., e.g., [32, Lemma A.8]) and satisfies (5.5), and hence (5.4). Moreover, the embedding $H^1(\Omega_-)^3\subset\mathcal{H}^1(\Omega_-)^3$ shows that $\mathbf{u}_0$ belongs also to the space $\mathcal{H}^1(\Omega_-)^3$ (see also [26, Lemma 3.2, Remark 3.3]).

Then with the new variable $\mathring{\mathbf{u}}:=\mathbf{u}-\mathbf{u}_0\in\mathring{\mathcal{H}}^1(\Omega_-)^3$, the variational problem (5.3) reduces to the following mixed variational formulation (c.f. Problem (Q) in p. 324 of [26] for the constant-coefficient Stokes system): Find $(\mathring{\mathbf{u}},\pi)\in\mathring{\mathcal{H}}^1(\Omega_-)^3\times L^2(\Omega_-)$ such that

$$\begin{cases}a_{\mu;\Omega_-}(\mathring{\mathbf{u}},\mathbf{v})+b_{\Omega_-}(\mathbf{v},\pi)=\mathfrak{F}_{\mu;\mathbf{u}_0}(\mathbf{v}),\ \forall\,\mathbf{v}\in\mathring{\mathcal{H}}^1(\Omega_-)^3,\\ b_{\Omega_-}(\mathring{\mathbf{u}},q)=0,\ \forall\,q\in L^2(\Omega_-),\end{cases} \tag{5.8}$$

where $a_{\mu;\Omega_-} : \mathring{\mathcal{H}}^1(\Omega_-)^3 \times \mathring{\mathcal{H}}^1(\Omega_-)^3 \to \mathbb{R}$ and $b_{\Omega_-} : \mathring{\mathcal{H}}^1(\Omega_-)^3 \times L^2(\Omega_-) \to \mathbb{R}$ are the bilinear forms given by

$$a_{\mu;\Omega_-}(\mathbf{w}, \mathbf{v}) := 2\langle \mu \mathbb{E}(\mathbf{w}), \mathbb{E}(\mathbf{v})\rangle_{\Omega_-}, \ \forall\, \mathbf{v}, \mathbf{w} \in \mathring{\mathcal{H}}^1(\Omega_-)^3, \tag{5.9}$$

$$b_{\Omega_-}(\mathbf{v}, q) := -\langle \operatorname{div} \mathbf{v}, q\rangle_{\Omega_-}, \ \forall\, \mathbf{v} \in \mathring{\mathcal{H}}^1(\Omega_-)^3, q \in L^2(\Omega_-), \tag{5.10}$$

and $\mathfrak{F}_{\mu;\mathbf{u}_0} : \mathring{\mathcal{H}}^1(\Omega_-)^3 \to \mathbb{R}$ is the linear form given by

$$\mathfrak{F}_{\mu;\mathbf{u}_0}(\mathbf{v}) := -\Big(\langle \mathbf{f}, \mathring{E}_-\mathbf{v}\rangle_{\Omega_-} + 2\langle \mu\mathbb{E}(\mathbf{u}_0), \mathbb{E}(\mathbf{v})\rangle_{\Omega_-}\Big), \ \forall\, \mathbf{v} \in \mathring{\mathcal{H}}^1(\Omega_-)^3. \tag{5.11}$$

Here we took into account that the spaces $\mathring{\mathcal{H}}^1(\Omega_-)^3$ and $\widetilde{\mathcal{H}}^1(\Omega_-)^3$ can be identified through the isomorphism $\mathring{E}_- : \mathring{\mathcal{H}}^1(\Omega_-)^3 \to \widetilde{\mathcal{H}}^1(\Omega_-)^3$. Note that

$$\begin{aligned}\mathring{\mathcal{H}}^1_{\operatorname{div}}(\Omega_-)^3 :&= \Big\{\mathbf{v} \in \mathring{\mathcal{H}}^1(\Omega_-)^3 : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega_-\Big\}\\ &= \Big\{\mathbf{v} \in \mathring{\mathcal{H}}^1(\Omega_-)^3 : b_{\Omega_-}(\mathbf{v}, q) = 0, \ \forall\, q \in L^2(\Omega_-)\Big\}.\end{aligned} \tag{5.12}$$

Now, formula (2.11), inequality (3.1) and the Hölder inequality yield that

$$\begin{aligned}|a_{\mu;\Omega_-}(\mathbf{v}_1, \mathbf{v}_2)| &\le 2c_\mu \|\mathbb{E}(\mathbf{v}_1)\|_{L^2(\Omega_-)^{3\times 3}} \|\mathbb{E}(\mathbf{v}_2)\|_{L^2(\Omega_-)^{3\times 3}}\\ &\le 2c_\mu \|\mathbf{v}_1\|_{\mathcal{H}^1(\Omega_-)^3} \|\mathbf{v}\|_{\mathcal{H}^1(\Omega_-)^3}, \ \forall\ \mathbf{v}_1,\, \mathbf{v}_2 \in \mathring{\mathcal{H}}^1(\Omega_-)^3.\end{aligned} \tag{5.13}$$

Moreover, the formula

$$2\|\mathbb{E}(\mathbf{v})\|^2_{L^2(\Omega_-)^{3\times 3}} = \|\operatorname{grad} \mathbf{v}\|^2_{L^2(\Omega_-)^{3\times 3}} + \|\operatorname{div} \mathbf{v}\|^2_{L^2(\Omega_-)}, \forall\, \mathbf{v} \in \mathcal{D}(\Omega_-)^3 \tag{5.14}$$

(cf., e.g., the proof of Corollary 2.2 in [46]), and the density of the space $\mathcal{D}(\Omega_-)^3$ in $\mathring{\mathcal{H}}^1(\Omega_-)^3$ show that the same formula holds also for any function in $\mathring{\mathcal{H}}^1(\Omega_-)^3$. Therefore, we obtain the following Korn type inequality

$$\|\operatorname{grad} \mathbf{v}\|_{L^2(\Omega_-)^{3\times 3}} \le 2^{\frac{1}{2}} \|\mathbb{E}(\mathbf{v})\|_{L^2(\Omega_-)^{3\times 3}}, \ \forall\, \mathbf{v} \in \mathring{\mathcal{H}}^1(\Omega_-)^3. \tag{5.15}$$

Then by using inequality (5.15), the equivalence of seminorm (2.12) to the norm (2.11) in the space $\mathcal{H}^1(\Omega_-)^3$, and assumption (3.1) we deduce that there exists a constant $C = C(\Omega_-) > 0$ such that

$$\begin{aligned}\|\mathbf{u}\|^2_{\mathcal{H}^1(\Omega_-)^3} &\le C\|\operatorname{grad} \mathbf{u}\|^2_{L^2(\Omega_-)^{3\times 3}} \le 2C\|\mathbb{E}(\mathbf{u})\|^2_{L^2(\Omega_-)^{3\times 3}}\\ &\le 2Cc_\mu \|\mu\mathbb{E}(\mathbf{u})\|^2_{L^2(\Omega_-)^{3\times 3}} = 2Cc_\mu a_{\mu;\Omega_-}(\mathbf{u}, \mathbf{u}), \forall\, \mathbf{u} \in \mathring{\mathcal{H}}^1(\Omega_-)^3,\end{aligned}$$

and accordingly that

$$a_{\mu;\Omega_-}(\mathbf{u}, \mathbf{u}) \ge \frac{1}{2Cc_\mu} \|\mathbf{u}\|^2_{\mathcal{H}^1(\Omega_-)^3}, \ \forall\, \mathbf{u} \in \mathring{\mathcal{H}}^1(\Omega_-)^3. \tag{5.16}$$

In view of inequalities (5.13) and (5.16) it follows that the bilinear form $a_{\mu;\Omega_-}(\cdot,\cdot) : \mathring{\mathcal{H}}^1(\Omega_-)^3 \times \mathring{\mathcal{H}}^1(\Omega_-)^3 \to \mathbb{R}$ is bounded and coercive. Moreover, arguments similar to those for inequality (5.13) imply that the bilinear form $b_{\Omega_-}(\cdot,\cdot) : \mathring{\mathcal{H}}^1(\Omega_-)^3 \times L^2(\Omega_-) \to \mathbb{R}$ and the linear form $\mathfrak{F}_{\mu;\mathbf{u}_0} : \mathring{\mathcal{H}}^1(\mathbb{R}^3)^3 \to \mathbb{R}$ given by (5.10) and (5.11), are also bounded. Since the operator

$$\operatorname{div} : \mathring{\mathcal{H}}^1(\Omega_-)^3 \to L^2(\Omega_-) \tag{5.17}$$

is surjective (cf., e.g., [26, Theorem 3.2]), then by Lemma A.2, the bounded bilinear form $b_{\Omega_-}(\cdot,\cdot) : \mathring{\mathcal{H}}^1(\Omega_-)^3 \times L^2(\Omega_-) \to \mathbb{R}$ satisfies the inf-sup condition

$$\inf_{q\in L^2(\Omega_-)\setminus\{0\}} \sup_{\mathbf{v}\in\mathring{\mathcal{H}}^1(\Omega_-)^3\setminus\{\mathbf{0}\}} \frac{b_{\Omega_-}(\mathbf{v},q)}{\|\mathbf{v}\|_{\mathring{\mathcal{H}}^1(\Omega_-)^3}\|q\|_{L^2(\Omega_-)}} \geq \beta_D \tag{5.18}$$

with some constant $\beta_D > 0$ (cf. [26, Theorem 3.3]). Then Theorem A.4 (with $X = \mathring{\mathcal{H}}^1(\Omega_-)^3$ and $M = L^2(\Omega_-)$) implies that the variational problem (5.8) has a unique solution $(\mathring{\mathbf{u}},\pi) \in \mathring{\mathcal{H}}^1(\Omega_-)^3 \times L^2(\Omega_-)$. Moreover, the pair $(\mathbf{u},\pi) = (\mathring{\mathbf{u}} + \mathbf{u}_0,\pi) \in \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-)$, where $\mathbf{u}_0 \in \mathcal{H}^1(\Omega_-)^3$ satisfies relations (5.4), is the unique solution of the mixed variational formulation (5.3) and depends continuously on the given data $(\mathbf{f},\boldsymbol{\phi}) \in \mathcal{H}^{-1}(\Omega_-)^3 \times H^{\frac{1}{2}}(\partial\Omega)^3$. The equivalence between the variational problem (5.3) and the exterior Dirichlet problem (5.1) shows that problem (5.1) is also well-posed, as asserted. □

5.2. Potential approach

Theorem 5.1 asserts the well-posedness of the exterior Dirichlet problem for the Stokes system with L^∞ coefficients. However, if the given data $(\mathbf{f},\boldsymbol{\phi})$ belong to the space $\mathcal{H}^{-1}(\Omega_-)^3 \times H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\partial\Omega)^3$, then the solution can be expressed in terms of the Newtonian and single layer potential and of the inverse of the single layer operator as follows (cf. [26, Theorem 3.4] for $\mu > 0$ constant, [22, Theorem 10.1] and [37, Theorem 5.1] for the Laplace operator).

Theorem 5.2. *If $\mathbf{f} \in \mathcal{H}^{-1}(\Omega_-)^3$ and $\boldsymbol{\phi} \in H_{\boldsymbol{\nu}}^{\frac{1}{2}}(\partial\Omega)^3$ then the exterior Dirichlet problem (5.1) has a unique solution $(\mathbf{u},\pi) \in \mathcal{H}^1(\Omega_-)^3 \times L^2(\Omega_-)$, given by*

$$\mathbf{u} = \boldsymbol{\mathcal{N}}_{\mu;\mathbb{R}^3}(\tilde{\mathbf{f}})|_{\Omega_-} + \mathbf{V}_{\mu;\partial\Omega}\left(\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}^{-1}(\boldsymbol{\phi} - \gamma_-(\boldsymbol{\mathcal{N}}_{\mu;\mathbb{R}^3}(\tilde{\mathbf{f}})))\right), \tag{5.19}$$

$$\pi = \mathcal{Q}_{\mu;\mathbb{R}^3}(\tilde{\mathbf{f}})|_{\Omega_-} + \mathcal{Q}^s_{\mu;\partial\Omega}\left(\boldsymbol{\mathcal{V}}_{\mu;\partial\Omega}^{-1}(\boldsymbol{\phi} - \gamma_-(\boldsymbol{\mathcal{N}}_{\mu;\mathbb{R}^3}(\tilde{\mathbf{f}})))\right) \text{ in } \Omega_-, \tag{5.20}$$

where $\tilde{\mathbf{f}}$ is an extension of $\mathbf{f}$ to an element of $\mathcal{H}^{-1}(\mathbb{R}^3)^3$.

Proof. The result follows from Definition 4.3 and Lemmas 4.7, 4.8, 4.10. □

Appendix A. Mixed variational formulations and their well-posedness property

Here we make a brief review of well-posedness results due to Babuška [6] and Brezzi [10] for mixed variational formulations related to bounded bilinear forms in reflexive Banach spaces. We follow [20, Section 2.4], [11], [25, §4].

Let X and $\mathcal{M}$ be reflexive Banach spaces, and let X^* and $\mathcal{M}^*$ be their dual spaces. Let $a(\cdot,\cdot) : X \times X \to \mathbb{R}$, $b(\cdot,\cdot) : X \times \mathcal{M} \to \mathbb{R}$ be bounded bilinear forms. Then we consider the following abstract mixed variational formulation.

For $f \in X^*$, $g \in \mathcal{M}^*$ given, find a pair $(u,p) \in X \times \mathcal{M}$ such that

$$\begin{cases} a(u,v) + b(v,p) & = f(v), \quad \forall\, v \in X, \\ b(u,q) & = g(q), \quad \forall\, q \in \mathcal{M}. \end{cases} \tag{A.1}$$

Let $A : X \to X^*$ be the bounded linear operator defined by

$$\langle Av, w\rangle = a(v, w),\ \forall\, v, w \in X, \tag{A.2}$$

where $\langle\cdot,\cdot\rangle := {}_{X^*}\langle\cdot,\cdot\rangle_X$ is the duality pairing of the dual spaces X^* and X. We also use the notation $\langle\cdot,\cdot\rangle$ for the duality pairing ${}_{\mathcal{M}^*}\langle\cdot,\cdot\rangle_{\mathcal{M}}$. Let $B : X \to \mathcal{M}^*$ and $B^* : \mathcal{M} \to X^*$ be the bounded linear and transpose operators given by

$$\langle Bv, q\rangle = b(v, q),\ \ \langle v, B^*q\rangle = \langle Bv, q\rangle,\ \forall\, v \in X,\ \forall\, q \in \mathcal{M}. \tag{A.3}$$

In addition, we consider the spaces

$$V := \operatorname{Ker} B = \{v \in X : b(v, q) = 0,\ \ \forall\, q \in \mathcal{M}\}\,, \tag{A.4}$$

$$V^{\perp} := \{T \in X^* : \langle T, v\rangle = 0,\ \ \forall\, v \in V\}\,. \tag{A.5}$$

Then the following well-posedness result holds (cf., e.g., [20, Theorem 2.34]).

Theorem A.1. *Let X and $\mathcal{M}$ be reflexive Banach spaces, $f \in X^*$ and $g \in \mathcal{M}^*$, and $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ and $b(\cdot,\cdot) : X \times \mathcal{M} \to \mathbb{R}$ be bounded bilinear forms. Let V be the subspace of X defined by* (A.4)*. Then the variational problem* (A.1) *is well-posed if and only if $a(\cdot,\cdot)$ satisfies the conditions*

$$\begin{cases} \exists\ \lambda > 0 \ \ \textit{such that} \ \ \inf\limits_{u\in V\setminus\{0\}} \sup\limits_{v\in V\setminus\{0\}} \dfrac{a(u,v)}{\|u\|_X\|v\|_X} \geq \lambda, \\ \{v \in V : a(u, v) = 0,\ \ \forall\ u \in V\} = \{0\}, \end{cases} \tag{A.6}$$

and $b(\cdot,\cdot)$ satisfies the inf-sup (*Ladyzhenskaya-Babuška-Brezzi*) *condition,*

$$\exists\, \beta > 0 \ \ \textit{such that} \ \ \inf_{q\in\mathcal{M}\setminus\{0\}} \sup_{v\in X\setminus\{0\}} \frac{b(v,q)}{\|v\|_X\|q\|_{\mathcal{M}}} \geq \beta. \tag{A.7}$$

Moreover, there exists a constant C depending on β, λ and the norm of $a(\cdot,\cdot)$, such that the unique solution $(u, p) \in X \times \mathcal{M}$ of (A.1) *satisfies the inequality*

$$\|u\|_X + \|p\|_{\mathcal{M}} \leq C\left(\|f\|_{X^*} + \|g\|_{\mathcal{M}^*}\right). \tag{A.8}$$

In addition, we have (see [20, Theorem A.56, Remark 2.7], [4, Theorem 2.7]).

Lemma A.2. *Let $X, \mathcal{M}$ be reflexive Banach spaces. Let $b(\cdot,\cdot) : X \times \mathcal{M} \to \mathbb{R}$ be a bounded bilinear form. Let $B : X \to \mathcal{M}^*$ and $B^* : \mathcal{M} \to X^*$ be the operators defined by* (A.3)*, and let $V = \operatorname{Ker} B$. Then the following results are equivalent:*

(*i*) *There exists a constant $\beta > 0$ such that $b(\cdot,\cdot)$ satisfies condition* (A.7).

(*ii*) *$B : X/V \to \mathcal{M}^*$ is an isomorphism and $\|Bw\|_{\mathcal{M}^*} \geq \beta\|w\|_{X/V}$ for any $w \in X/V$.*

(*iii*) *$B^* : \mathcal{M} \to V^{\perp}$ is an isomorphism and $\|B^*q\|_{X^*} \geq \beta\|q\|_{\mathcal{M}}$ for any $q \in \mathcal{M}$.*

Remark A.3. Let X be a reflexive Banach space and V be a closed subspace of X. If a bounded bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is *coercive* on V, i.e., there exists a constant $c_a > 0$ such that

$$a(w, w) \geq c_a\|w\|_X^2,\ \ \forall\, w \in V, \tag{A.9}$$

then the conditions (A.6) are satisfied as well (see, e.g., [20, Lemma 2.8]).

The next result known as the *Babuška-Brezzi theorem* is the version of Theorem A.1 for Hilbert spaces (see [6], [10, Theorems 0.1, 1.1, Corollary 1.2]).

Theorem A.4. *Let X and $\mathcal{M}$ be two real Hilbert spaces. Let $a(\cdot,\cdot):X\times X\to\mathbb{R}$ and $b(\cdot,\cdot):X\times\mathcal{M}\to\mathbb{R}$ be bounded bilinear forms. Let $f\in X^*$ and $g\in\mathcal{M}^*$. Let V be the subspace of X defined by* (A.4)*. Assume that $a(\cdot,\cdot):V\times V\to\mathbb{R}$ is coercive and that $b(\cdot,\cdot):X\times\mathcal{M}\to\mathbb{R}$ satisfies the inf-sup condition* (A.7)*. Then the variational problem* (A.1) *is well-posed.*

Acknowledgements

The research has been supported by the grant EP/M013545/1: "Mathematical Analysis of Boundary-Domain Integral Equations for Nonlinear PDEs" from the EPSRC, UK. Part of this work was done in April/May 2018, when M. Kohr visited the Department of Mathematics of the University of Toronto. She is grateful to the members of this department for their hospitality.

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