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Metric measure spaces satisfying curvature-dimension bounds: geometric and analytic properties.

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Non tutti possiamo fare grandi cose,
ma possiamo fare piccole cose con grande amore.
— Madre Teresa di Calcutta

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Contents

Introduction

Motivated by a description of lower bounds on Ricci curvature not relying on any smoothness assumption (i.e. synthetic), the theory of metric spaces (X, d) endowed with a reference measure $\mathfrak{m}$ (i.e. *metric measure space*, or m.m.s. for short) has aroused great interest in the last twenty years. The latter theory has grown more and more, addressing several issues: the study of functional and geometric inequalities in structures which are very far from being Euclidean (therefore requiring new non-Riemannian tools), the description of the closure of classes of Riemannian manifolds under suitable geometric constraints, the stability of analytic and geometric properties of spaces.

This thesis is devoted to the study of geometric and structural properties of metric measure spaces satisfying curvature-dimension bounds; this will done by means of Optimal Transport.

Building on the metric structure of the space, optimal transport provides a natural way to introduce a geometric distance between probability measures, which reflects well the metric properties of the base space. In particular, the W_2-Wasserstein distance arises when considering the optimal transport problem with quadratic cost (L^2-optimal transport) and it is defined on the space $\mathcal{P}_2(X)$ of Borel probability measures on X with finite second moment

$$\mathcal{P}_2(X) = \left\{ \mathfrak{m} \in \mathcal{P}(X) : \int_X \mathsf{d}^2(x, x_0)\, \mathfrak{m}(dx) < +\infty, \text{for some } x_0 \in X \right\}.$$

In his pioneering paper [**67**], McCann pointed out the interest of convexity along constant speed geodesics of $\mathcal{P}_2(X)$ of integral functionals in Euclidean spaces, introducing the notion of *displacement convexity*, i.e. convexity along $\mathrm{Geo}(\mathcal{P}_2(X))$ (the set containing the geodesics of $\mathcal{P}_2(X)$). After the works of Cordero-Erausquin–McCann–Schmuckenshläger [**45**], Otto–Villani [**79**] and von Renesse–Sturm [**86**], it was realized that having Ricci curvature bounded from below by K in the smooth setting could be equivalently formulated synthetically as a displacement convexity property of an entropy functional along W_2-Wasserstein geodesics. This idea led Lott–Villani [**66**] and Sturm [**93, 94**], to propose the definition of $\mathsf{CD}(K, \infty)$; the latter being a dimension independent concept. In order to obtain more precise estimates, the strategy was to reinforce the curvature lower bound to a curvature-dimension condition $\mathsf{CD}(K, N)$, involving two real parameters K and N, $N \geq 1$ playing in some generalized sense the roles of a lower bound for the Ricci curvature and an upper bound for the dimension respectively. This resulted into a successful (and compatible with the classical one) synthetic definition of $\mathsf{CD}(K, N)$ for a complete and separable metric space (X, d) endowed with a locally-finite Borel reference measure $\mathfrak{m}$ (“metric-measure space”, or m.m.s.) [**66**],[**93, 94**].

The $\mathsf{CD}(K, N)$ condition is formulated in terms of displacement convexity of the Renyi entropy $S_N(\cdot|\mathfrak{m})$; the latter being defined on $\mathcal{P}_2(X)$ as follows

$$S_N(\mu|\mathfrak{m}) := -\int_X \rho^{-1/N}\, d\mu,$$

where ρ denotes the density of the absolutely continuous part of μ with respect to $\mathfrak{m}$ (see Definition 1.33). A Riemannian manifold (M, g) has a natural structure of metric measure space, when endowed with the Riemaniann distance induced by g and the volume measure. In this case, the curvature-dimension condition $\mathsf{CD}(K, N)$ will be satisfied if and only if $\dim(M) \leq N$

and $\mathrm{Ric}_M(\eta,\eta) \geq K|\eta|^2$ for all $\eta \in TM$. Beyond this consistency with the smooth case, another remarkable property about Curvature-Dimension condition is its stability under measured Gromov-Hausdorff convergence. Several geometric properties can be derived directly from the curvature-dimension condition; among them we recall the Bishop-Gromov theorem on the volume growth of concentric balls, the Bonnet-Myers theorem on the diameter of metric measure spaces with positive lower curvature bounds and Brunn Minkowski inequality [**94**]. The theory of curvature-dimension bounds has been extensively developed, leading to a better understanding of the geometry of m.m.s.'s by means of Optimal Transport.

A completely different approach to generalized curvature-dimension bounds was set forth in the pioneering work [**13**] of Bakry and Émery in the 1980's; the latter was introduced in the context of diffusion generators, having in mind primarily the setting of weighted Riemannian manifolds, i.e. smooth Riemannian manifolds whose volume measure has been multiplied by a smooth, positive and integrable density function. In the Bakry-Émery theory the starting point is Bochner-Lichnerowicz formula

$$\frac{1}{2}\Delta_g(|\nabla f|^2) - \langle \nabla f, \nabla \Delta_g f\rangle = |\mathrm{Hess} f|^2 + \mathrm{Ric}(\nabla f, \nabla f)$$

valid for Riemannian manifold. In the framework of Dirichlet forms and Γ-calculus there is still the possibility to write Bochner-Lichnerowicz formula in the weak form of an inequality, leading to the definition of the $\mathsf{BE}(K,N)$ condition.

We mention that Curvature-Dimension condition has been recently investigated also in the case in which the generalized dimension N is negative; we refer to [**77**] for an extension of the range of N to negative values in the curvature-dimension condition $\mathsf{CD}(K,N)$ for general m.m.s.'s and to [**71**] for a proof of isoperimetric, functional and concentration properties of n-dimensional weighted Riemannian manifolds satisfying the Curvature-Dimension condition $\mathsf{CD}(K,N)$ for $N \in (-\infty, 1)$.

One of the greatest achievements of the theory of curvature-dimension bounds is due to the introduction in [**7, 8, 46, 11**] of a more restrictive condition that still retains the stability properties under measured Gromov-Hausdorff convergence. One considers the Sobolev space $W^{1,2}(X)$ of functions on X and requires the latter, that is always a Banach space, to be a Hilbert space or, equivalently, the Laplace operator on X to be linear (infinitesimal Hilbertianity). The notion of a lower Ricci curvature bound coupled with this last Hilbertian condition is called Riemannian Curvature Dimension bound, RCD for short. Contrary to what happens in the $\mathsf{CD}(K,N)$ case, the $\mathsf{RCD}(K,N)$ condition does not allow for Finsler structures, which are known not to appear as limits of smooth manifolds with Ricci curvature bounds. Several results have been obtained in this setting, leading to an extensive literature [**7, 8, 10, 53, 11, 72, 55, 63, 25**]. In particular, we mention [**8**], [**10**] for a proof of the equivalence of $\mathsf{RCD}(K,\infty)$ and $\mathsf{BE}(K,\infty)$. Combining the results obtained in [**46**], [**11**] and [**35**], also the equivalence of $\mathsf{RCD}(K,N)$ and $\mathsf{BE}(K,N)$ with $N<\infty$ follows.

On the other hand, one can also consider a weaker variant of $\mathsf{CD}(K,N)$, namely the Measure Contraction Property $\mathsf{MCP}(K,N)$, independently introduced by Ohta in [**75**] and Sturm in [**94**]. Roughly, the idea is to only require the $\mathsf{CD}(K,N)$ condition to hold along any W_2-Wasserstein geodesic ending at a Dirac delta centered at any $o \in \mathrm{supp}(\mathfrak{m})$. Still retaining a weaker synthetic lower bound on the Ricci curvature, an upper bound on the dimension and stability in the measured Gromov-Hausdorff sense (see also [**76**] for further properties), $\mathsf{MCP}(K,N)$ includes a larger family of spaces than $\mathsf{CD}(K,N)$. It is now well known for instance that the Heisenberg group equipped with a left-invariant measure, which is the simplest sub-Riemannian structure, does not satisfy any form of $\mathsf{CD}(K,N)$ and does satisfy $\mathsf{MCP}(0,N)$ for a suitable choice of N, see [**65**]. It is worth mentioning that MCP was first investigated in Carnot groups in [**65, 87**], see also [**16**]. The theory of curvature-dimension bounds has indeed strongly influenced the research in sub-Riemannian geometry; in particular, we point out that interpolation inequalities à la Cordero-Erausquin–McCann–Schmuckenshläger [**45**] have been recently obtained, under suitable modifications, by Barilari and Rizzi [**17**] in the ideal sub-Riemannian setting and by Balogh, Kristály and Sipos [**15**] for the Heisenberg group.

An important turning point for the development of the theory was represented by the fine-tuning of a localization technique for nonsmooth spaces. The localization paradigm, initially developed by Payne–Weinberger [**82**], Gromov–Milman [**58**] and Kannan–Lovász–Simonovits [**60**], permitted to reduce various analytic and geometric inequalities to appropriate one-dimensional counterparts. The original approach by these authors was based on a bisection method, and thus inherently confined to $\mathbb{R}^n$. In 2015 [**64**], Klartag extended the localization paradigm to the weighted Riemannian setting, by disintegrating the reference measure $\mathfrak{m}$ on L^1-Optimal Transport geodesics associated to the inequality under study, and proving that the resulting conditional one-dimensional measures inherit the Curvature-Dimension properties of the underlying manifold. Nevertheless, Klartag's approach heavily made use of the smoothness of the ambient space; to overcome this difficulty, the strategy pursued in [**20**], [**30**], [**31**] was to use the structural properties of geodesics and of L^1-optimal transport in metric measure spaces.

Cavalletti and Mondino in [**36**] extended the localization paradigm to the framework of m.m.s.'s $(X, \mathsf{d}, \mathfrak{m})$ verifying the local version of $\mathsf{CD}(K,N)$ (namely $\mathsf{CD}_{loc}(K,N)$, Definition 1.36) for $N \in (1,\infty)$ coupled with an assumption on the behaviour of the geodesics (i.e. *essentially non-branching property*): the Curvature-Dimension information encoded in the W_2-geodesics was transferred to the individual rays along which a given W_1-geodesic evolves. This permitted to obtain several new results in the field: an isoperimetric inequality á la Levy-Gromov-Milman [**36**] (after a while obtained also in a quantitative form in [**34**]) and several functional inequalities such as the p-spectral gap (or equivalently the p-Poincaré inequality) for any $p \in [1,\infty)$, the log-Sobolev inequality and the Talagrand inequality [**37**].
The use of L^1-Optimal transport was then encoded in the $\mathsf{CD}^1(K,N)$ condition; the latter was introduced for the first time in [**35**] to solve the long standing problem of the so-called local-to-global property of $\mathsf{CD}(K,N)$. The approach of [**35**] to the local-to-global problem was to demonstrate that $\mathsf{CD}_{loc}(K,N)$ implies $\mathsf{CD}^1(K,N)$ and then that $\mathsf{CD}^1(K,N)$ implies $\mathsf{CD}(K,N)$.
Despite of all the results mentioned above, several fundamental questions have not an answer yet. The aim of this thesis is to contribute to the development of the theory of curvature-dimension bounds for metric measure spaces, exploiting the localization method. The next sections of this introduction will be devoted to the presentation of the main results of this thesis.

Isoperimetric inequality under Measure-Contraction Property

In this section we briefly introduce the results obtained in [**43**]; they will be presented with all the details in Chapter 2.

The isoperimetric problem is one of the most classical problems in mathematics; it addresses the following natural question: given a space X, find the minimal amount of area needed to enclose a fixed volume v. If the space X has a simple structure or has many symmetries the problem can be completely solved and the optimal shapes (i.e. the isoperimetric regions) can be explicitly described (e.g. Euclidean space and the sphere). In the general case however one cannot hope to obtain a complete solution to the problem and a comparison result is already completely satisfactory. Probably the most popular result in this direction is the Lévy-Gromov isoperimetric inequality [**57**, Appendix C] stating that if A is a (sufficiently regular) subset of a Riemannian manifold (M,g) of dimension $n = N$ and Ricci bounded below by $K > 0$, then

$$\frac{|\partial A|}{|M|} \geq \frac{|\partial B|}{|S|}, \tag{1}$$

where B is a spherical cap in the model sphere S, i.e. the N-dimensional round sphere with constant Ricci curvature equal to K, and $|M|, |S|, |\partial A|, |\partial B|$ denote the appropriate N or $N-1$ dimensional volume, and where B is chosen so that $|A|/|M| = |B|/|S|$.

The Lévy-Gromov isoperimetric inequality has been then extended to more general settings; the case $N = +\infty$, where the model space is Gaussian, was addressed by Bakry and Ledoux in [**14**]. For the scope of this presentation, the most relevant progress was the one obtained by E. Milman [**70**] for weighted Riemannian manifolds verifying the Curvature-Dimension condition $\mathsf{CD}(K,N)$ introduced in the 1980's by Bakry and Émery [**12, 13**]. E. Milman discovered a model isoperimetric profile $\mathcal{I}^{\mathsf{CD}}_{K,N,D}$ such that if a Riemannian manifold with density verifying $\mathsf{CD}(K,N)$ has diameter at most $D > 0$, then the isoperimetric profile function of the weighted manifold is bounded from below by $\mathcal{I}^{\mathsf{CD}}_{K,N,D}$. Finally the extension to the case $N < 0$ was obtained in [**71**].
Regarding the m.m.s.'s setting, it is clear that the volume of a Borel set A can be replaced by its $\mathfrak{m}$-measure, $\mathfrak{m}(A)$; for what concerns the boundary area of the smooth framework, it can be replaced by the Minkowski content of A

$$\mathfrak{m}^+(A) = \liminf_{\varepsilon \to 0} \frac{\mathfrak{m}(A^\varepsilon) - \mathfrak{m}(A)}{\varepsilon}, \tag{2}$$

where A^ε is the ε-enlargement of A given by $A^\varepsilon = \{x \in X : \mathsf{d}(x,A) < \varepsilon\}$. Minkowski content turns out to be a reasonable notion to consider in our setting: it coincides with the classic boundary area if the set is sufficiently smooth and its definition just involves the ambient measure and the distance. Building on the work by Klartag [**64**] and the localization paradigm, Cavalletti and Mondino [**36**] managed to extend Lévy-Gromov-Milman isoperimetric inequality to a class of essentially non-branching m.m.s.'s (see Definition 1.28) with $\mathfrak{m}(X) = 1$ and verifying the $\mathsf{CD}(K,N)$ condition; in particular [**36**] proves that

$$\mathfrak{m}^+(A) \geq \mathcal{I}^{\mathsf{CD}}_{K,N,D}(\mathfrak{m}(A)). \tag{3}$$

The isoperimetric inequality (3) is equivalent to the following inequality

$$\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v) \geq \mathcal{I}^{\mathsf{CD}}_{K,N,D}(v),$$

for all $v \in (0,1)$. Here $\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}$ denotes the isoperimetric profile function of the m.m.s. $(X,\mathsf{d},\mathfrak{m})$ and it is defined as follows

$$\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v) := \inf\{\, \mathfrak{m}^+(A) : A \subset X \text{ Borel}, \, \mathfrak{m}(A) = v\}.$$

In some cases, given a one-dimensional density h defined on the real interval (a,b) integrating to 1, we will adopt the shorter notation $\mathcal{I}_h$ to denote the isoperimetric profile function $\mathcal{I}_{((a,b),|\cdot|,h\mathcal{L}^1)}$.

Thanks to the influence of the theory of curvature-dimension bounds in the context of sub-Riemannian geometry, an increasing number of examples of spaces verifying MCP and not CD is currently at our disposal, e.g. the Heisenberg group, generalized H-type groups, the Grushin plane and Sasakian structures (for more details, see [**17**]). In all the previous examples a sharp isoperimetric inequality is not at disposal yet; due to lack of regularity of minimizers, sharp isoperimetric inequality has been proved just for subclasses of competitors having extra regularity or additional symmetries; in particular, Pansu Conjecture [**80**] is still unsolved. For more details we refer to [**74, 88, 89, 28**] and references therein. In this regard, we stress here that one of greatest advantages of the localization technique lies in the fact that it does not require any a priori regularity assumption on the optimizer.

In Chapter 2 we address the isoperimetric inequality à la Lévy-Gromov-Milman within the class of spaces verifying MCP. In particular, we identify a family of one-dimensional $\mathsf{MCP}(K,N)$-densities, each for every choice of K,N, volume v and diameter D, not verifying $\mathsf{CD}(K,N)$, and having optimal isoperimetric profile for the volume v; we thus denote the model isoperimetric profile at volume v by $\mathcal{I}_{K,N,D}(v)$ and obtain the following result:

Theorem 1. [Theorem 2.15] Let $K,N \in \mathbb{R}$ with $N>1$ and let $(X,\mathsf{d},\mathfrak{m})$ be an essentially non-branching m.m.s. verifying $\mathsf{MCP}(K,N)$ with $\mathfrak{m}(X)=1$ and having diameter less than D (where $D \leq \pi\sqrt{(N-1)/K}$ if $K>0$). For any $A \subset X$,

$$\mathfrak{m}^+(A) \geq \mathcal{I}_{K,N,D}(\mathfrak{m}(A)). \tag{4}$$

Moreover (4) is sharp, i.e. for each $v \in [0,1]$, K,N,D there exists a m.m.s. $(X,\mathsf{d},\mathfrak{m})$ with $\mathfrak{m}(X)=1$ and $A \subset X$ with $\mathfrak{m}(A)=v$ such that (4) is an equality.

We stress again that estimate (4) is sharp in the class of $\mathsf{MCP}(K,N)$ spaces as equality is attained on the 1-dimensional model densities.

Via localization paradigm for MCP-spaces (see Section 1.4 for details), following [**64, 36**], the proof of Theorem 1 is reduced to the proof of the corresponding statement in the one-dimensional setting. However, contrary to the CD framework, due to lack of any form of concavity, the isoperimetric problem for a general one-dimensional $\mathsf{MCP}(K,N)$-density seems to be out of reach. We instead directly exhibit, for each K,N,D and v, an optimal one-dimensional $\mathsf{MCP}(K,N)$-density, denoted by $h_{K,N,D,v}$ that will be optimal only for that choice of K,N,D and v.
In order to detect the family of one-dimensional densities $h_{K,N,D,v}$, we first prove the existence of a lower bound $f_{K,N,D}$ for any $\mathsf{MCP}(K,N)$ density integrating to 1. Then we define $h_{K,N,D,v}$ as the density touching the lower bound in a certain point (depending on the parameters K,N,D,v) and moving away from it in the steepest way allowed by the $\mathsf{MCP}(K,N)$ condition. Moreover, we show that the lower bound $f_{K,N,D}$ enjoys a rigidity property: if some $\mathsf{MCP}(K,N)$ density integrating to 1 touches the lower bound, then it must be equal to $h_{K,N,D,v}$ for a certain value of v.
Once the family of one-dimensional densities $h_{K,N,D,v}$ has been introduced, one can prove that:

$$\mathcal{I}_{K,N,D}(v) = \begin{cases} h_{K,N,D,v}(a_{K,N,D}(v)), & K \leq 0, \\ \min\limits_{D' \leq D} h_{K,N,D',v}(a_{K,N,D'}(v)), & K>0, \end{cases} \tag{5}$$

where $a_{K,N,D}(v)$ is the unique point of $[0,D]$ such that $\int_{[0,a_{K,N,D}(v)]} h_{K,N,D,v}(x)\,dx = v$; in particular

$$\mathcal{I}_{h_{K,N,D,v}}(v) = h_{K,N,D,v}(a_{K,N,D}(v)),$$

for all K,N,D and v. To explain (5), we underline that for each K,N,D and v, $h_{K,N,D,v}(a_{K,N,D}(v))$ is the optimal perimeter when minimization is constrained to all one-dimensional $\mathsf{MCP}(K,N)$-densities (integrating to 1) having support of *exactly* length D, see Theorem 2.7. Denoting the optimal value of the latter minimization problem by $\widetilde{\mathcal{I}}_{K,N,D}(v)$, the previous sentence reads as

$$\widetilde{\mathcal{I}}_{K,N,D}(v) = \mathcal{I}_{h_{K,N,D,v}}(v). \tag{6}$$

Hence (5) is a direct consequence of the following fact: $\widetilde{\mathcal{I}}_{K,N,D}(v)$ is strictly decreasing as a function of D only if $K \leq 0$, showing a remarkable difference with the CD-framework [**70**].

The geometric strength of Lévy-Gromov isoperimetric inequality lies in its rigidity property: if a Riemannian manifold verifies the equality case in (1) then it is isometric to the round sphere of the correct dimension [**57**]; if equality is attained in (3) and the metric measure space verifies the stronger $\mathsf{RCD}(K,N)$ condition (see [**7, 8, 46, 11, 53, 35**] and references therein), then it is isomorphic in the metric-measure sense to a spherical suspension (see [**36**] for details). At the present generality, i.e. the class of m.m.s.'s verifying $\mathsf{MCP}(K,N)$, competitors are less regular and a weaker rigidity is valid.

In particular, the proof of Theorem 2.7 is sufficiently stable to imply one-dimensional rigidity (Theorem 2.12), valid for each choice of K, N, D and v. Building on this and on the monotonicity in D of $\widetilde{\mathcal{I}}_{K,N,D}(v)$, we show that whenever $K \leq 0$ the optimal metric measure space has a product structure in a measure theoretic sense (see Theorem 2.16 for the precise result). Finally, we mention the recent paper by Han and Milman [**59**], where the 1-dimensional density $h_{K,N,D,1/2}$ is used to deduce a sharp Poincaré inequality in m.m.s.'s satisfying $\mathsf{MCP}(K,N)$ and having diameter upper bounded by $D \in (0,\infty)$.

Independence of synthetic Curvature Dimension conditions on transport distance exponent

In this section we introduce the results obtained in [**2**]; a comprehensive presentation of the latters will be given in Chapter 3.

The idea of Lott-Sturm-Villani synthetic approach is to analyse weighted convexity properties of the Renyi Entropy along geodesics in the space of probability measures endowed with the quadratic transportation distance. As the $\mathsf{CD}(K,N)$ condition for smooth manifolds is equivalent to a joint lower bound on the Ricci curvature and an upper bound on the dimension, it is a natural question to consider whether the squared-distance function plays a special role in the theory or not.

Among all the possible transport cost functions, the power distance costs, namely d^p with $p > 1$, are related to the geometry of the underlying space. Moreover, the latters have already appeared in the literature in the definition of the p-Wasserstein distance W_p that turns the space of probability measures with finite p^{th}-moments into a complete and separable metric space $(\mathcal{P}_p(X), W_p)$. Another natural setting for such spaces can also be seen in the case of doubly-degenerate diffusion dyanamics [**78**], [**1**].

Accordingly, the modified displacement convexity of the entropy functional can be considered with respect to W_p-geodesics – and this in turn furnishes a straightforward and legitimate extension of the definition of $\mathsf{CD}(K,N)$ condition proposed by Kell [**62**] and denoted by $\mathsf{CD}_p(K,N)$. The notation $\mathsf{CD}(K,N)$ will be reserved for the classical case $p = 2$. While Kell established the equivalence of all $\mathsf{CD}_p(K,N)$ in the smooth setting via the use of Ricci curvature, no previous results are known in the context of nonsmooth metric measure spaces.

In Chapter 3 we will attack this problem with a strategy relying on the $\mathsf{CD}^1(K,N)$ condition. More precisely, we will use the same point of view of [**35**] to link two different curvature dimension conditions: we will demonstrate the equivalence of $\mathsf{CD}_p(K,N)$ and $\mathsf{CD}_q(K,N)$ for a general m.m.s. $(X, \mathsf{d}, \mathfrak{m})$, for $1 < p, q$ and $K, N \in \mathbb{R}$ with $N > 1$, provided suitable restrictions are placed on X. In particular, we will require that $(X, \mathsf{d}, \mathfrak{m})$ is either non-branching or at least satisfies appropriate versions of the essentially non-branching condition of Definition 1.29. More specifically, we obtain the following results:

Theorem 2. [Theorem 3.44] Let $(X, \mathsf{d}, \mathfrak{m})$ be such that $\mathfrak{m}(X) = 1$. Assume it is p-essentially non-branching and verifies $\mathsf{CD}_p(K,N)$ for some $p > 1$. If $(X, \mathsf{d}, \mathfrak{m})$ is also q-essentially non-branching for some $q > 1$, then it verifies $\mathsf{CD}_q(K,N)$.

As we will extend the strategy used in [**35**] to powers other than $p = 2$, also the local-to-global property will be established for the $\mathsf{CD}_p(K,N)$.

Corollary 1. [Corollary 3.45] Fix any $p > 1$ and $K, N \in \mathbb{R}$ with $N > 1$. Let $(X, \mathsf{d}, \mathfrak{m})$ be a p-essentially non-branching metric measure space verifying $\mathsf{CD}_{p,loc}(K,N)$ from Definition

1.36 and such that (X, d) is a length space with $\mathrm{supp}(\mathfrak{m}) = X$ and $\mathfrak{m}(X) = 1$. Then $(X, \mathsf{d}, \mathfrak{m})$ verifies $\mathsf{CD}_p(K,N)$.

In Theorem 2 and Corollary 1 we are assuming $\mathfrak{m}(X) = 1$. This assumption is also used in [**35**] but we believe that it is most likely a purely technical assumption. At the moment, the main obstacle to the case of a general Radon measure $\mathfrak{m}$ is the lack of a canonical disintegration theorem once a "measurable" partition is given. For some preliminary results in this direction we refer to [**41**].

Another motivation to studying distance costs with powers other than $p = 2$ comes from the recent works of McCann [**68**] and Mondino-Suhr [**73**], where the authors analyze the relation between optimal transportation and timelike Ricci curvature bounds in the smooth Lorentzian setting. Analogously to the Riemannian setting, timelike Ricci curvature lower bounds can be equivalently characterised in terms of convexity properties of the Bolzmann-Shannon entropy functional along ℓ_p-geodesics of probability measures, where ℓ_p denotes the causal transport distance with exponent $p \in (0, 1]$. This point of view has been pushed forward in [**42**] and [**69**] where the authors proposed a synthetic formulation of the Strong Energy condition, denoted by $\mathsf{TCD}_p(K,N)$, which is valid for non-smooth Lorentzian spaces. Unlike the Riemmannian case, the Lorentzian setting does not have a distinguished p; and one of the next steps of the theory will be to address whether $\mathsf{TCD}_p(K,N)$ depends on p or not.

Displacement convexity of the Entropy and the distance cost Optimal Transportation

In this section we briefly introduce the results obtained in [**33**]; they will be presented with all the details in Chapter 4.

The formulation of an appropriate version of Ricci curvature lower bounds valid for possibly singular spaces has been a central topic of research for several years.

As we have observed, the theory by Lott-Villani [**66**] and Sturm [**93, 94**] is formulated in terms of displacement convexity of the Renyi entropy; in rough terms, a space will satisfy the $\mathsf{CD}(K,N)$ condition if the entropy evaluated along W_2-geodesics is more convex than the entropy evaluated along W_2-geodesics of the model space with constant curvature K and dimension N in an appropriate sense (see Definition 1.33). This approach had a huge impact, leading to the establishment of a rich and robust theory.
On the other hand, substantial recent advancements in the theory (localization paradigm and local-to-global property) have been obtained considering the different point of view of L^1-Optimal transport problems yielding a different curvature dimension condition $\mathsf{CD}^1(K,N)$ [**35**] formulated in terms of one-dimensional curvature properties of integral curves of Lipschitz maps. As a means to establish the local-to-global property for the curvature-dimension condition, it has been shown in [**35**] that a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ verifies $\mathsf{CD}(K,N)$ if and only if it satisfies $\mathsf{CD}^1(K,N)$, provided X is essentially non-branching (see Definition 1.28) and the total space has finite mass (i.e. $\mathfrak{m}(X) < \infty$).

It remained however unclear if the $\mathsf{CD}^1(K,N)$ condition could be equivalently formulated in terms of displacement convexity of the Entropy functional along W_1-geodesics.

In Chapter 4 we show that this is the case and the two approaches produce the same curvature-dimension condition, reconciling the two definitions; we report here the main result of the chapter:

Theorem 3. Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching metric measure space and further assume $\mathfrak{m}(X) = 1$. Then $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathsf{CD}^1(K,N)$ condition if and only if it satisfies the $\mathsf{CD}_1(K,N)$ condition.

The $\mathsf{CD}_1(K, N)$ condition is formulated, in analogy with the classical $\mathsf{CD}(K, N)$, as displacement convexity of the Renyi entropy along W_1-geodesics; its precise formulation is given in Definition 4.1.

Hence, combining this result with the one described in the previous section, we establish that for any $p \geq 1$, all the $\mathsf{CD}_p(K, N)$ conditions, when expressed in terms of displacement convexity, are equivalent, provided the space X satisfies the appropriate essentially non-branching condition.

CHAPTER 1

Preliminaries

In this chapter we recall some basic notions that we will use throughout the thesis. First of all, we will start collecting some concepts of the Optimal transport Theory; being these results classical, we will refer for proofs and in-depth analyses to [**6**], [**97**]. Next, we will briefly recall the synthetic notions of lower Ricci curvature bounds we will deal with; we will focus just on their main ideas and properties, referring to [**66**],[**93**],[**94**],[**75**] for further details. Finally, we will present a brief overview of L^1-optimal transport and localization technique, the latters being fundamental ingredients of the new results collected in this thesis.

1.1. The Optimal Transport Problem

1.1.1. Monge and Kantorovich formulation. Let X, Y be two complete and separable metric spaces, $T : X \to Y$ a Borel map and $\mu \in \mathcal{P}(X)$; the measure $T_\sharp\mu \in \mathcal{P}(Y)$, called the *push forward of* μ *through* T, is defined by

$$T_\sharp\mu(E) = \mu(T^{-1}(E)), \quad \forall E \subset Y, \text{ Borel.}$$

Let us fix a Borel cost function $c : X \times Y \to \mathbb{R} \cup \{+\infty\}$. The Monge version of the Optimal trasport problem can be stated as follows:

Problem 1 (Monge's Optimal Transport Problem). Let $\mu \in \mathcal{P}(X), \nu \in \mathcal{P}(Y)$. Minimize

$$T \mapsto \int_X c(x, T(x))\, d\mu(x)$$

among all transport maps T from μ to ν, i.e. all maps T such that $T_\sharp\mu = \nu$.

Depending on the choice of the cost function c, Monge's problem can be ill-posed: indeed, the existence of admissible maps is not a priori guaranteed and the constraint $T_\sharp\mu = \nu$ turns out to be not sequentially closed. To overcome such difficulties, Kantorovich proposed the following new version of the problem:

Problem 2 (Kantorovich's formulation). Minimize:

$$\pi \mapsto \int_{X\times Y} c(x, y) d\pi(x, y)$$

in the set $\mathrm{Adm}(\mu, \nu)$ of all transport plans $\pi \in \mathcal{P}(X \times Y)$ from μ to ν, i.e. the set of Borel Probability measures on $X \times Y$ such that $P^X_\sharp\pi = \mu$, $P^Y_\sharp\pi = \nu$, where P^X, P^Y are the projections from $X \times Y$ onto X and Y respectively.

This formulation has several advantages:

1. since at least $\mu \times \nu$ belongs to $Adm(\mu, \nu)$, the latter set is always not empty,
2. the set $Adm(\mu, \nu)$ is convex and compact with respect to the narrow topology in $\mathcal{P}(X \times Y)$ (see below for the definition) and the functional $\pi \mapsto \int c d\pi$ is linear,
3. to any transport map T from μ to ν, it is naturally associated the plan $\pi = (Id \times T)_\sharp\mu \in Adm(\mu, \nu)$.

DEFINITION 1.1. We say that a sequence $(\mu_n) \subset \mathcal{P}(X)$ narrowly converges to μ provided

$$\int \varphi d\mu_n \mapsto \int \varphi d\mu, \ \ \forall \varphi \in C_b(X),$$

$C_b(X)$ being the space of the continuous and bounded functions on X.

Existence of minimizers for Kantorovich's formulation of the transport problem now comes from a standard lower-semicontinuity and compactness argument:

THEOREM 1.2. *If the cost c is lower semicontinuous and bounded from below, there exists a minimizer for the Problem 2.*

We will denote by $\mathrm{Opt}(\mu, \nu)$ the set of optimal plans from μ to ν, i.e. the set of all minimizers for the Problem 2.

REMARK 1.3. It is possible to prove that if the cost c is continuous and μ is non atomic, then the infima of Monge and Kantorovich problems coincide [**50**], [**4**]. Hence, Kantorovich formulation can be seen as a "relaxation" of the Monge's Problem.

1.1.2. Necessary and sufficient optimality conditions. According to the previous results, one needs a way to detect optimal plans; in order to do so, it is useful to provide some characterizations of this notion. Let's start from the following definitions:

DEFINITION 1.4. (c-cyclical monotonicity) We say that $\Gamma \subset X \times Y$ is c-cyclically monotone if given $(x_i, y_i) \in \Gamma, 1 \leq i \leq N$, it holds:

$$\sum_{i=1}^{N} c(x_i, y_i) \leq \sum_{i=1}^{N} c(x_i, y_{\sigma(i)}),$$

for every permutation σ of $\{1, \dots, N\}$.

DEFINITION 1.5. (c-transform) Let $\psi : Y \to \mathbb{R} \cup \{\pm\infty\}$ be any function. Its *c-transform* $\psi^c : X \to \mathbb{R} \cup \{-\infty\}$ is defined as

$$\psi^c(x) := \inf_{y \in Y} c(x, y) - \psi(y).$$

Similarly, given $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$, its *$c$-transform* $\varphi^c : Y \to \mathbb{R} \cup \{\pm\infty\}$ is defined as

$$\varphi^c(y) := \inf_{x \in X} c(x, y) - \varphi(x).$$

DEFINITION 1.6. (c-concavity) We say that $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is c-concave if there exists $\psi : Y \to \mathbb{R} \cup \{-\infty\}$ such that $\varphi = \psi^c$. Similarly, $\psi : Y \to \mathbb{R} \cup \{-\infty\}$ is c-concave if there exists $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ such that $\psi = \varphi^c$.

DEFINITION 1.7. (c-superdifferential) Let $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ be a c-concave function. The c-superdifferential $\partial^c \varphi \subset X \times Y$ is defined as

$$\partial^c \varphi := \{(x, y) \in X \times Y : \varphi(x) + \varphi^c(y) = c(x, y)\}.$$

Moreover, we define the c-superdifferential $\partial^c \varphi(x)$ at $x \in X$ as the set of $y \in Y$ for which $(x, y) \in \partial^c \varphi$.

REMARK 1.8. Observe that $y \in \partial^c \varphi(x)$ if and only if holds

$$\begin{aligned} &\varphi(x) = c(x, y) - \varphi^c(y), \\ &\varphi(z) \leq c(x, y) - \varphi^c(y), \ \ \forall z \in X. \end{aligned}$$

In particular, it turns out that

$$\varphi(x) - c(x, y) \geq \varphi(z) - c(z, y), \ \forall z \in X.$$

From this it follows that the c-superdifferential of a c-concave function is always a c-cyclically monotone set; indeed, if $(x_i, y_i) \in \partial^c\varphi$, it holds

$$\sum_i c(x_i, y_i) = \sum_i \varphi(x_i) + \varphi^c(y_i) = \sum_i \varphi(x_i) + \varphi^c(y_{\sigma(i)}) \le \sum_i c(x_i, y_{\sigma(i)})$$

for any permutation σ of the indexes.

THEOREM 1.9 (Fundamental Theorem of Optimal Transport). *Assume that* $c : X \times Y \to \mathbb{R}$ *is continuous and bounded from below and let* $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$ *be such that*

$$c(x, y) \le a(x) + b(y), \tag{1.1}$$

for some $a \in L^1(\mu)$, $b \in L^1(\nu)$. *Moreover, let* $\pi \in$Adm(μ, ν). *Then the following are equivalent:*

(1) *the plan* π *is optimal,*
(2) *the set* supp(π) *is c-cyclically monotone,*
(3) *there exists a c-concave function* φ *such that* $\max\{\varphi, 0\} \in L^1(\mu)$ *and* supp$(\pi) \subset \partial^c\varphi$.

REMARK 1.10. In particular, it follows that the optimality depends only on the support of the plan.

1.1.3. Dual Problem.

PROBLEM 3. (Dual Problem) Let $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$. Maximize the value of the functional

$$\int_X \varphi(x)\, d\mu(x) + \int_Y \psi(y) d\,\nu(y),$$

among all functions $\varphi \in L^1(\mu), \psi \in L^1(\nu)$ such that

$$\varphi(x) + \psi(y) \le c(x, y),\ \forall x \in X, y \in Y. \tag{1.2}$$

THEOREM 1.11. *Let* $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$ *and* $c : X \times Y \to \mathbb{R}$ *be continuous and bounded from below. If the condition* (1.1) *holds, then*

$$\inf_{\pi \in Adm(\mu,\nu)} \int c(x, y)\, d\pi(x, y) = \sup_{\varphi,\psi} \int \varphi(x) d\mu(x) + \int \psi(y)\, d\nu(y)$$

where the supremum is taken among all φ, ψ *that satisfy* (1.2). *Moreover, the supremum of the Dual Problem is attained and the maximizing couple* (φ, ψ) *is of the form* (φ, φ^c) *for some c-concave function* φ.

DEFINITION 1.12. (Kantorovich potential) A c-concave function φ such that (φ, φ^c) is a maximizing pair for the dual problem 3 is called *c-concave Kantorovich potential* for the couple μ, ν.

1.1.4. Existence of Optimal Maps. So far the problem of existence of Optimal transport maps reduces in looking for Optimal plans π that are induced by a map T, i.e. such that $\pi = (Id, T)_\sharp\mu$. In this regard, it is useful the following characterization:

LEMMA 1.13. *Let* $\pi \in Adm(\mu, \nu)$. *The plan* π *is induced by a map if and only if there exists a* π*-measurable set* $\Gamma \subset X \times Y$ *on which* π *is concentrated, such that for* μ*-a.e.* x *there exists only one* $y = T(x)$ *for which* $(x, y) \in \Gamma$. *In this case* π *is induced by the map* T.

Hence, combining what has been deduced so far, the Optimal Transport Problem reduces to understand "how often" the c-superdifferential of a c-concave function is single valued. There is no general answer to this question but particular cases can be studied, such as:

(1) $X = Y = \mathbb{R}^d$ and $c(x, y) = |x - y|^2/2$,

(2) $X = Y = M$, where M is a Riemaniann manifold and $c(x,y) = \mathsf{d}(x,y)^2/2$, where d is the Riemaniann distance.

Concerning the first case, crucial is the following characterization of c-concavity and c-superdifferential:

LEMMA 1.14. *Let $\varphi : \mathbb{R}^d \mapsto \mathbb{R} \cup \{-\infty\}$. Then φ is c-concave if and only if $x \mapsto \bar{\varphi}(x) = |x|^2/2 - \varphi(x)$ is convex and lower semincontinuous. In this case, $y \in \partial^c\varphi(x)$ if and only if $y \in \partial^-\bar{\varphi}(x)$.*

Thus, in this setting, being concentrated on the c-superdifferential of a c-concave map turns out to be equivalent to being concentrated on the graph of the subdifferential of a convex function. The latter condition has been already studied in literature and can be related with the following notion.

DEFINITION 1.15. A set $E \subset \mathbb{R}^d$ is called *$c - c$ hypersurface* if, in a suitable system of coordinates, it is the graph of the difference of two real valued convex functions.

The following result holds true:

THEOREM 1.16. *Let $E \subset \mathbb{R}^d$. There exists a convex function $\bar{\varphi} : \mathbb{R}^d \mapsto \mathbb{R}$ such that E is contained in the set of points of non differentiability of $\bar{\varphi}$ if and only if E can be covered by countably many $c - c$ hypersurfaces.*

DEFINITION 1.17. A measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ is called *regular* if $\mu(E) = 0$ for any $c - c$ hypersurface $E \subset \mathbb{R}^d$.

Now we are ready to state the celebrated *Brenier Theorem*, which concerns the problem of existence and uniqueness of optimal maps. Actually, we state an improved version of the latter, anticipated in a footnote by Gangbo–McCann [**51**] and stated by Gigli in [**6**]. We refer to [**97**] for further bibliographical remarks.

THEOREM 1.18. *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ be such that $\int |x|^2 d\mu(x) < \infty$. Then the following are equivalent:*

(1) for every $\nu \in \mathcal{P}(\mathbb{R}^d)$ with $\int |x|^2 d\nu(x) < \infty$, there exists a unique transport plan from μ to ν and this plan is induced by a map T,
(2) μ is regular.

If one of the two equivalent conditions holds, the optimal map T is the gradient of a convex function.

REMARK 1.19. When $X = Y = \mathbb{R}^d$ and $c(x,y) = \theta(x-y)$ with θ strictly convex and μ absolutely continuous with respect to $\mathcal{L}^d$, duality methods yield that any optimal plan is induced by a transport map; thus, the optimal map exists and is unique (see [**26**], [**51**]).

We now spend few words concerning the case $X = Y = M$ with M smooth Riemannian manifold and $c(x,y) = \mathsf{d}(x,y)^2/2$. The latter case shares some similarities with the Euclidean one; indeed, the concepts of semiconvexity (i.e. second derivatives bounded from below) and semiconcavity also make sense on manifolds, since these properties can be read locally and change of coordinates are smooth. The main difference is in the fact that Lemma 1.14 doesn't hold anymore.

PROPOSITION 1.20. *Let M be a smooth, compact Riemann manifold without boundary. Let $\varphi : M \to \mathbb{R} \cup \{-\infty\}$ be a c-concave function not identically equal to $-\infty$. Then φ is Lipschitz, semiconcave and real valued. If $y \in \partial^c\varphi(x)$, then $\exp_x^{-1}(y) \subset -\partial^+\varphi(x)$. Conversely, if φ is differentiable at x, then $\exp_x(-\nabla\varphi(x)) \in \partial^c\varphi(x)$.*

DEFINITION 1.21. (Regular Measures in $\mathcal{P}(M)$) A measure $\mu \in \mathcal{P}(M)$ is called regular if it vanishes on the set of points of non-differentiability of ψ, for any semiconvex function $\psi : M \to \mathbb{R}$.

By Proposition 1.20 , it is possible to derive a result about existence of optimal transport maps on manifolds which closely resembles the Brenier Theorem.

THEOREM 1.22 (Mc Cann). *Let M be a smooth, compact Riemannian manifold without boundary and $\mu \in \mathcal{P}(X)$. The following are equivalent:*

(1) for every $\nu \in \mathcal{P}(M)$ there exists a unique transport plan from μ to ν and this plan is induced by a map T.
(2) μ is regular.

If one of the two equivalent conditions holds, the optimal map T can be written as $x \to \exp_x(-\nabla\varphi(x))$ for some c-concave function $\varphi : M \to \mathbb{R}$.

1.2. Geodesics and measures

A triple $(X, \mathsf{d}, \mathfrak{m})$ is called a *metric measure space* if (X, d) is a Polish space (i.e. a complete and separable metric space) and $\mathfrak{m}$ is a positive Radon measure over X. In what follows we will always deal with m.m.s. in which $\mathfrak{m}$ is a probability measure, i.e. $\mathfrak{m}(X) = 1$; we will denote with $\mathcal{P}(X)$ the space of all Borel probability measures over X.

DEFINITION 1.23. A curve $\gamma \in C([0,1], X)$ is called a *constant speed geodesic* if

$$\mathsf{d}(\gamma_s, \gamma_t) = |s-t|\mathsf{d}(\gamma_0, \gamma_1), \quad \forall s, t \in [0,1].$$

From now on the set of all constant speed geodesics will be denoted with $\mathrm{Geo}(X)$ while $\mathrm{e}_t : \mathrm{Geo}(X) \to \mathbb{R}$ will denote the *evaluation map* defined as follows

$$\mathrm{e}_t(\gamma) = \gamma_t.$$

DEFINITION 1.24. A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is called *geodesic* if, for any choice of $x, y \in X$, there exists $\gamma \in \mathrm{Geo}(X)$ with $\gamma_0 = x$, $\gamma_1 = y$.

Fixed $p \geq 1$, we will denote with the symbol $\mathcal{P}_p(X)$ the space of probability measures with finite p-moment, i.e.

$$\mathcal{P}_p(X) = \{\mathfrak{m} \in \mathcal{P}(X) : \int_X \mathsf{d}^p(x, x_0)\, \mathfrak{m}(dx) < +\infty, \text{for some } x_0 \in X\}.$$

The subspace of $\mathfrak{m}$-absolutely continuous probability measures will be denoted by $\mathcal{P}_p(X, \mathsf{d}, \mathfrak{m})$. The space $\mathcal{P}_p(X)$ will be endowed with the L^p-Wasserstein distance W_p defined by

$$W_p^p(\mu_0, \mu_1) = \inf_\pi \int_{X\times X} \mathsf{d}^p(x,y)\pi(dxdy), \tag{1.3}$$

where the infimum is taken in the class of all probability measures in $\mathcal{P}(X \times X)$ with first and second marginal given by μ_0 and μ_1 respectively. As (X, d) is a complete and separable metric space, so is $(\mathcal{P}_p(X), W_p)$. Also, it is known that (X, d) is geodesic if and only if $(\mathcal{P}_p(X), W_p)$ is geodesic.
The following theorem holds true:

THEOREM 1.25 (Theorem 3.10,[**6**]). *If (X, d) is geodesic, $(\mathcal{P}_p(X), W_p)$ is geodesic too. Moreover, the following are equivalent:*

(1) $[0,1] \ni t \mapsto \mu_t \in \mathcal{P}_p(X)$ is a geodesic;
(2) $\exists\, \nu \in \mathcal{P}(\mathrm{Geo}(X))$ s.t. $(\mathrm{e}_0, \mathrm{e}_1)_\sharp \nu$ realizes the minimum in (1.3), $\mu_t = \mathrm{e}_{t\sharp}\nu$.

REMARK 1.26. The measures $\nu \in \mathcal{P}(\mathrm{Geo}(X))$ verifying (2) are called dynamical optimal plans; the set containing all dynamical optimal plans from μ_0 to μ_1 is denoted by $\mathrm{OptGeo}_p(\mu_0, \mu_1)$. Notice that if $\nu \in \mathrm{OptGeo}_p(\mu_0, \mu_1)$, then also $(\mathrm{e}_t, \mathrm{e}_s)_\sharp \nu$ is p-optimal between its marginals.

DEFINITION 1.27. A set $A \subset \mathrm{Geo}(X)$ is called *a set of non-branching geodesics* if for any $\gamma^1, \gamma^2 \in A$

$$\exists \bar{t} \in (0,1) : \gamma^1(s) = \gamma^2(s),\ \forall s \in [0,\bar{t}] \implies \gamma^1(t) = \gamma^2(t),\ \forall t \in [0,1].$$

Finally we recall the classical definition of essentially non-branching. This notion has been firstly introduced in [**85**] and considers only the case $p = 2$.

DEFINITION 1.28 (Essentially non-branching). Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s.. We say that $(X, \mathsf{d}, \mathfrak{m})$ is W_2-essentially non-branching if for any $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathsf{d}, \mathfrak{m})$ any element of $\mathrm{OptGeo}_2(\mu_0, \mu_1)$ is concentrated on a set of non-branching geodesics.

The previous notion has been later generalized in [**61**], resulting in the following:

DEFINITION 1.29 (p-essentially non branching). $(X, \mathsf{d}, \mathfrak{m})$ is called *p-essentially non-branching* if for all $\mu_0, \mu_1 \in \mathcal{P}_p(X, \mathsf{d}, \mathfrak{m})$, any $\nu \in \mathrm{OptGeo}_p(\mu_0, \mu_1)$ is concentrated on a Borel non-branching set $G \subset \mathrm{Geo}(X)$, in agreement with the terminology of [**85**] when $p = 2$.

Being p-essentially non branching for a m.m.s. is strictly related to the uniqueness of optimal plans for the L^p transport; in order to state more precisely the result we refer to, we need to introduce the so-called *qualitatively non-degenerate property* [**61**, Lemma 5.14]. The latter asserts that for each ball $B_R(x_0)$, there is a ratio $f(t) \in (0,1]$ with $\limsup_{t\to 0} f(t) > 1/2$ which bounds the decrease in measure whenever any Borel set $A \subset B_R(x_0)$ is contracted a fraction t of the distance towards any $x \in B_R(x_0)$:

$$\mathfrak{m}(\mathrm{e}_t(G)) \geq f(t)\mathfrak{m}(\mathrm{e}_0((G))) \tag{1.4}$$

for $G = (\mathrm{e}_0 \times \mathrm{e}_1)^{-1}(A \times \{x\})$. The following theorem holds true:

THEOREM 1.30 ([**61**]). *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space with $\mathfrak{m}$ qualitatively non-degenerate. Then the following properties are equivalent:*

1. *$(X, \mathsf{d}, \mathfrak{m})$ is p-essentially non-branching;*
2. *for every $\mu_0, \mu_1 \in \mathcal{P}_p(X)$ with $\mu_0 \ll \mathfrak{m}$ there is a unique $\nu \in \mathrm{OptGeo}_p(\mu_0, \mu_1)$. Moreover, the p-optimal coupling $(\mathrm{e}_0, \mathrm{e}_1)_\sharp \nu$ is induced by a transport map and every interpolation $\mu_t = (\mathrm{e}_t)_\sharp \nu$, $t \in [0,1)$, is absolutely continuous with respect to $\mathfrak{m}$.*

1.3. Several notions of synthetic Ricci curvature bounds

Starting from the pioneering papers of Lott-Villani [**66**] and Sturm [**93**],[**94**], synthetic and abstract notions of lower Ricci curvature bounds were introduced in the class of complete and separable metric spaces (X, d) endowed with a locally finite Borel measure $\mathfrak{m}$.

The first attempt was done prescribing a certain convexity property of an entropy functional along W_2-Wasserstein geodesics, leading in this way to the well-known definition of the Curvature Dimension condition $\mathsf{CD}(\mathsf{K}, \mathsf{N})$ that we now briefly recall.
Given a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ and $N \in \mathbb{R}, N \geq 1$, we define the *Renyi entropy functional* $S_N(\cdot|\mathfrak{m}) : \mathcal{P}_2(X, \mathsf{d}) \to \mathbb{R}$ as follows

$$S_N(\mu|\mathfrak{m}) := -\int_X \rho^{-1/N}\, d\mu,$$

where ρ denotes the density of the absolutely continuous part of μ with respect to $\mathfrak{m}$.

DEFINITION 1.31 ($\sigma_{K,N}$-coefficients). For every $K, N \in \mathbb{R}$ with $N \geq 1$, we set

$$D_{K,N} := \begin{cases} \frac{\pi}{\sqrt{K/N}} & K > 0\ ,\ N < \infty \\ +\infty & \text{otherwise} \end{cases};$$

in addition, given $t \in [0,1]$ and $0 \le \theta < D_{K,N}$, we define the so called *distortion coefficients* $\sigma^{(t)}_{K,N}(\theta)$ as follows:

$$\sigma^{(t)}_{K,N}(\theta) := \begin{cases} \infty & \text{if } K\theta^2 \ge N\pi^2, \\ \frac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } 0 < K\theta^2 < N\pi^2, \\ t & \text{if } K\theta^2 < 0 \text{ and } N = 0, \text{ or if } K\theta^2 = 0, \\ \frac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} & \text{if } K\theta^2 \le 0 \text{ and } N > 0. \end{cases}$$

Definition 1.32 ($\tau_{K,N}$-coefficients). Given $K \in \mathbb{R}$, $N \in (1,\infty]$ and $(t,\theta) \in [0,1] \times \mathbb{R}_+$, define:

$$\tau^{(t)}_{K,N}(\theta) := t^{\frac{1}{N}} \sigma^{(t)}_{K,N-1}(\theta)^{1-\frac{1}{N}}.$$

When $N = 1$, set $\tau^{(t)}_{K,1}(\theta) = t$ if $K \le 0$ and $\tau^{(t)}_{K,1}(\theta) = +\infty$ if $K > 0$.

We are now in a position to give the following :

Definition 1.33. Given two numbers $K, N \in \mathbb{R}$ with $N \ge 1$ we say that a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfies the *curvature-dimension condition* $\mathsf{CD}(K,N)$ if and only if for each pair of $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathsf{d}, \mathfrak{m})$ there exist an optimal coupling π of $\mu_0 = \rho_0\mathfrak{m}$ and $\mu_1 = \rho_1\mathfrak{m}$ and a W_2-geodesic $\{\mu_t\}$ interpolating the two such that

$$S_{N'}(\mu_t|\mathfrak{m}) \le -\int_{X\times X} \big[\tau^{(1-t)}_{K,N'}(\mathsf{d}(x,y))\rho_0^{-1/N'}(x) + \tau^{(t)}_{K,N'}(\mathsf{d}(x,y))\rho_1^{-1/N'}(y)\big]\pi(dx,dy) \tag{1.5}$$

for all $t \in [0,1]$ and all $N' \ge N$.

For Riemannian manifolds, the curvature-dimension condition $\mathsf{CD}(K,N)$ will be satisfied if and only if $\dim(M) \le N$ and $\mathrm{Ric}_M(\eta,\eta) \ge K|\eta|^2$ for all $\eta \in TM$. Beyond this consistency with the smooth case, another fundamental property about Curvature-dimension condition that is worth to mention is its stability under measured Gromov-Hausdorff convergence.

Remark 1.34. A relevant case for our purposes (due to the crucial use of the localization technique) is the one of one-dimensional spaces $(X, \mathsf{d}, \mathfrak{m}) = (I, |\cdot|, h\mathcal{L}^1)$, where $I \subset \mathbb{R}$ is an interval, $h \in L^1(I)$ and positive. In this case the density h has to satisfy

$$\left(h^{1/(N-1)}\right)'' + \frac{K}{N-1}h^{1/(N-1)} \le 0, \tag{1.6}$$

in the sense of distribution. In particular, by (1.6), there exists a (locally Lipschitz) continuous representative of h - that we shall continue to denote by h - and it satisfies

$$h((1-t)R_0 + tR_1)^{\frac{1}{N-1}} \ge \sigma^{(1-t)}_{K,N-1}(R_1 - R_0)h(R_0)^{\frac{1}{N-1}} + \sigma^{(t)}_{K,N-1}(R_1 - R_0)h(R_1)^{\frac{1}{N-1}},$$

for any $R_0, R_1 \in I$, $R_0 \le R_1$, and $t \in [0,1]$.

One can also prescribe the convexity inequality (1.5) to hold along a W_p-geodesic, getting to the more general definition of $\mathsf{CD}_p(K,N)$.

Definition 1.35. Given two numbers $K, N \in \mathbb{R}$ with $N \ge 1$ we say that a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathsf{CD}_p(K,N)$ if and only if for each pair of $\mu_0, \mu_1 \in \mathcal{P}_p(X, \mathsf{d}, \mathfrak{m})$ there exist an optimal coupling π of $\mu_0 = \rho_0\mathfrak{m}$ and $\mu_1 = \rho_1\mathfrak{m}$ and a W_p-geodesic $\{\mu_t\}$ interpolating the two such that

$$S_{N'}(\mu_t|\mathfrak{m}) \le -\int_{X\times X} \big[\tau^{(1-t)}_{K,N'}(\mathsf{d}(x,y))\rho_0^{-1/N'}(x) + \tau^{(t)}_{K,N'}(\mathsf{d}(x,y))\rho_1^{-1/N'}(y)\big]\pi(dx,dy), \tag{1.7}$$

for all $t \in [0,1]$ and all $N' \ge N$.

When we omit the subscript p from $\mathsf{CD}_p(K,N)$, we tacitly mean the classical $p=2$, as introduced independently by Lott-Villani in [**66**] and Sturm in [**93, 94**].

As a natural curvature notion, $\mathsf{CD}_p(K,N)$ has a local version that is denoted by $\mathsf{CD}_{p,loc}(K,N)$.

Definition 1.36 ($\mathsf{CD}_{p,loc}(K,N)$). Given $K,N\in\mathbb{R}$ with $N\geq 1$, $(X,\mathsf{d},\mathfrak{m})$ is said to satisfy $\mathsf{CD}_{p,loc}(K,N)$ if for any $o\in\mathrm{supp}(\mathfrak{m})$, there exists a neighborhood $X_o\subset X$ of o, so that for all $\mu_0,\mu_1\in\mathcal{P}_p(X,\mathsf{d},\mathfrak{m})$ supported in X_o, there exists $\nu\in\mathrm{OptGeo}_p(\mu_0,\mu_1)$ so that for all $t\in[0,1]$, $\mu_t:=(\mathrm{e}_t)_\sharp\nu\ll\mathfrak{m}$, and for all $N'\geq N$, (1.7) holds.

Note that $(\mathrm{e}_t)_\sharp\nu$ is not required to be supported in X_o for intermediate times $t\in(0,1)$ in the latter definition.

We will also consider a weak variant of the curvature-dimension condition encapsulating generalized Ricci curvature lower bounds coupled with generalized dimension upper bounds, namely the *measure contraction property* $\mathsf{MCP}(K,N)$ [**75**].

Definition 1.37 ($\mathsf{MCP}(K,N)$). A m.m.s. $(X,\mathsf{d},\mathfrak{m})$ is said to satisfy $\mathsf{MCP}(K,N)$ if for any $o\in\mathrm{supp}(\mathfrak{m})$ and $\mu_0\in\mathcal{P}_2(X,\mathsf{d},\mathfrak{m})$ of the form $\mu_0=\frac{1}{\mathfrak{m}(A)}\mathfrak{m}\llcorner_A$ for some Borel set $A\subset X$ with $0<\mathfrak{m}(A)<\infty$, there exists $\nu\in\mathrm{OptGeo}(\mu_0,\delta_o)$ such that:

$$\frac{1}{\mathfrak{m}(A)}\mathfrak{m}\geq(\mathrm{e}_t)_\sharp\big(\tau_{K,N}^{(1-t)}(\mathsf{d}(\gamma_0,\gamma_1))^N\nu(d\gamma)\big)\quad\forall t\in[0,1]. \tag{1.8}$$

If $(X,\mathsf{d},\mathfrak{m})$ is a m.m.s. verifying $\mathsf{MCP}(K,N)$, then $(\mathrm{supp}(\mathfrak{m}),\mathsf{d})$ is Polish, proper and it is a geodesic space. With no loss in generality for our purposes we will assume that $X=\mathrm{supp}(\mathfrak{m})$. Many additional results on the structure of W_2-geodesics can be obtained just from the MCP condition together with the essentially non-branching assumption (see [**38**]).

To conclude, referring to [**75, 94**] for more general results, we report the following important fact [**75**, Theorem 3.2]: if (M,g) is n-dimensional Riemannian manifold with $n\geq 2$, the m.m.s. (M,d_g,vol_g) verifies $\mathsf{MCP}(K,n)$ if and only if $Ric_g\geq Kg$, where d_g is the geodesic distance induced by g and vol_g is the volume measure.

If $(X,\mathsf{d},\mathfrak{m})=(I,|\cdot|,h\mathcal{L}^1)$, it is a standard fact that the m.m.s. $(I,|\cdot|,h\mathcal{L}^1)$ verifies $\mathsf{MCP}(K,N)$ if and only if the non-negative Borel function h has a continuous representative (still denoted by h) which satisfies the following inequality:

$$h(tx_1+(1-t)x_0)\geq\sigma_{K,N-1}^{(1-t)}(|x_1-x_0|)^{N-1}h(x_0), \tag{1.9}$$

for all $x_0,x_1\in I$ and $t\in[0,1]$, see for instance [**20**, Theorem 9.5]. We will call h an $\mathsf{MCP}(K,N)$-density. Inequality (1.9) implies several known properties that we recall for readers convenience. To write them in a unified way, we define for $\kappa\in\mathbb{R}$ the function $s_\kappa:[0,+\infty)\to\mathbb{R}$ (on $[0,\pi/\sqrt{\kappa})$ if $\kappa>0$)

$$s_\kappa(\theta):=\begin{cases}(1/\sqrt{\kappa})\sin(\sqrt{\kappa}\theta) & \text{if } \kappa>0,\\ \theta & \text{if } \kappa=0,\\ (1/\sqrt{-\kappa})\sinh(\sqrt{-\kappa}\theta) & \text{if } \kappa<0.\end{cases} \tag{1.10}$$

For the moment we confine ourselves to the case $I=(a,b)$ with $a,b\in\mathbb{R}$; hence (1.9) implies (actually is equivalent to)

$$\left(\frac{s_{K/(N-1)}(b-x_1)}{s_{K/(N-1)}(b-x_0)}\right)^{N-1}\leq\frac{h(x_1)}{h(x_0)}\leq\left(\frac{s_{K/(N-1)}(x_1-a)}{s_{K/(N-1)}(x_0-a)}\right)^{N-1}, \tag{1.11}$$

for $x_0\leq x_1$. In particular, h is locally Lipschitz in the interior of I and continuous up to the boundary. The next lemma was stated and proved in [**35**, Lemma A.8] under the CD condition; as the proof only uses $\mathsf{MCP}(K,N)$ we report it in this more general version.

LEMMA 1.38. *Let h denote a $\mathsf{MCP}(K,N)$ density on a finite interval (a,b), $N \in (1,\infty)$, which integrates to 1. Then:*

$$\sup_{x\in(a,b)} h(x) \le \frac{1}{b-a}\begin{cases} N & K \ge 0 \\ (\int_0^1 (\sigma_{K,N-1}^{(t)}(b-a))^{N-1}dt)^{-1} & K < 0 \end{cases}. \tag{1.12}$$

In particular, for fixed K and N, h is uniformly bounded from above as long as $b-a$ is uniformly bounded away from 0 (and from above if $K<0$).

As one would expect, $\mathsf{CD}_p(K,N)$ implies $\mathsf{MCP}(K,N)$; it is sufficient indeed to repeat the same argument presented in [**35**, Lemma 6.11] for the case $p=2$ noticing that W_p-geodesics ending in a delta measure are also W_2-geodesics. When coupled with essentially non-branching condition, MCP yields nice properties for W_p-geodesics. In particular, MCP implies the qualitatively non-degenerate property, thus Theorem 1.30 holds true.

REMARK 1.39. It is worth also recalling that the local version of $\mathsf{CD}(K,N)$, denoted by $\mathsf{CD}_{loc}(K,N)$ is known to imply $\mathsf{MCP}(K,N)$ provided the (X,d) is a non-branching length space, see [**44**]. Since any $\mathsf{CD}_{p,loc}(K,N)$ gives the very same information when considered for Wasserstein geodesics arriving at a Dirac mass, we can conclude that the same argument of [**44**] implies that $\mathsf{CD}_{p,loc}(K,N)$ implies $\mathsf{MCP}(K,N)$, provided (X,d) is a non-branching length space.

Moreover, it has already been observed and used in literature that the non-branching assumption can be weakened to essentially non-branching for $p=2$: the non-branching property in [**44**] was used to get a partition of X formed of all geodesics arriving at the same point $o \in X$ and subsequently to unsure uniqueness of a dynamical optimal plan connection μ_0 to μ_1 with $\mu_0 \ll \mathfrak{m}$. Both properties can be deduced from p-essentially non-branching together with Theorem 1.30; for more details see Section 3.4.1. Hence we will take for granted that given any $p>1$, a metric measure spaces satisfying $\mathsf{CD}_{p,loc}(K,N)$ and being a p-essentially non-branching length space also verifies $\mathsf{MCP}(K,N)$.

1.4. L^1-Optimal Transport and Localization Technique

To be precise, the original formulation of Monge's transport problem handled with the cost $c(x,y) = |x-y|$ on $\mathbb{R}^d$; the latter case is quite different from the one just discussed and needs a separate discussion. Indeed, due to the lack of strict convexity of the cost, in this setting it is typically not true that optimal plans are unique or that they are induced by maps. For example, considering on $\mathbb{R}$ any two probability measures μ, ν such that μ is concentrated on the negative half-line and ν on the positive one, it turns out that any admissible plan is optimal for the cost $c(x,y) = |x-y|$.

Nevertheless, even in this case existence of optimal maps can be shown but, in order to find them, one has to use a selection principle. The first attempt to solve this problem came with the work of Sudakov [**95**], who claimed to have a solution for any distance cost function induced by a norm. Sudakov's approach consisted in using a disintegration principle to reduce the d-dimensional problem to a family of problems on $\mathbb{R}$; anyway, the original argument presented in [**95**] was flawed and needed to be fixed. This was done in [**4**] in the case of the Euclidean distance; meanwhile several proofs of existence of optimal maps were proposed ([**48**],[**96**],[**27**]).

1.4.1. Sudakov's approach to Monge Problem.

Besides being interesting in its own, Sudakov's approach to Monge problem permits to figure out how L^1-Optimal Transport naturally yields a reduction of the problem to a family of one-dimensional problems. For this reason we will briefly present here its main ideas working in the Euclidean setting; we refer to [**32**] for futher details.

First of all it is crucial to note that, when the cost c is given by the distance, the notion of c-concavity is equivalent to 1-Lipschitz continuity. Thus, for a plan π being optimal is equivalent to ask that $\pi(\Gamma) = 1$ where $\Gamma = \{(x, y) \in \mathbb{R}^{2d} : \varphi(x) - \varphi(y) = |x - y|\}$ for some 1-Lipschitz function φ. Almost by definition, the set Γ is $|\cdot|$-cyclically monotone and whenever $(x, y) \in \Gamma$, we have that $(z_s, z_t) \in \Gamma$ for any $s \leq t$ where $z_s := (1 - s)x + sy$ for any $s \in [0, 1]$. Hence, Γ produces a family of disjoint lines of $\mathbb{R}^d$ along which the optimal transportation moves. Roughly speaking, $\mathbb{R}^d$ can be decomposed up to a set of measure zero as $\mathcal{T} \cup Z$ where Z is the set of points not moved by the optimal transport problem and $\mathcal{T}$, the so called *transport set*, is such that

$$\mathcal{T} = \cup_{\alpha \in Q} X_\alpha$$

where X_α are disjoint lines and Q is a set of indices. Using this partition of the space, it is possible to obtain via *Disintegration Theorem* (see subsection 1.4.2) a corresponding decomposition of marginal measures:

$$\mu_0 = \int_Q \mu_{0,\alpha} \mathfrak{q}(d\alpha), \qquad \mu_1 = \int_Q \mu_{1,\alpha} \mathfrak{q}(d\alpha);$$

where $\mathfrak{q}$ is a Borel probability measure over the set of indices $Q \subset \mathbb{R}^d$. If Q enjoys a measurability condition, the conditional measures $\mu_{0,\alpha}$ and $\mu_{1,\alpha}$ are such that $\mu_{0,\alpha}(X_\alpha) = \mu_{1,\alpha}(X_\alpha) = 1$ for $\mathfrak{q}$-a.e. $\alpha \in Q$. Having done this, one can construct an optimal transport map first considering an optimal map T_α associated to the transport of $\mu_{0,\alpha}$ to $\mu_{1,\alpha}$ and then defining the transport map T as T_α on each X_α. In this way the original Monge Problem has been reduced to a family of one-dimensional problems.

Anyway, Monge Problem can be actually stated and solved, in a much more general framework. See [**29**] for a complete Sudakov approach to Monge problem when the Euclidean distance is replaced by any strictly convex norm and [**21**] where any norm is considered. More generally, given two Borel probability measures μ_0, μ_1 over a complete and separable metric space (X, d), the notion of transport map still makes sense and the optimality condition can be naturally formulated using the distance d as a cost function; the strategy proposed in the Euclidean case can be adopted as well. It is important to notice that, besides the regularity of μ_0, the regularity of the ambient space plays a crucial role. More precisely, together with the localization of the Monge problem to X_α, it should come a localization of the regularity of the space; this is the case when the metric space (X, d) is endowed with a reference probability measure $\mathfrak{m}$ and the resulting metric measure space verifies a weak Ricci curvature lower bound. We will investigate this topic in the subsection 1.4.3.

1.4.2. Disintegration theorem. Here we provide the version of the Disintegration Theorem we will use (for a self-contained approach and a proof, we refer to [**19**]).

Let $(X, \mathscr{F}, \mathfrak{m})$ be a measure space. Given a function $\mathfrak{Q} : X \to Q$, it is possible to construct another measure space $(Q, \mathscr{Q}, \mathfrak{q})$ where $\mathfrak{q} = \mathfrak{Q}_\sharp \mathfrak{m}$ and $\mathscr{Q}$ is the biggest σ-algebra on Q for which $\mathfrak{Q}$ is measurable i.e.

$$C \in \mathscr{Q} \iff \mathfrak{Q}^{-1}(C) \in \mathscr{F}.$$

This general scheme fits well with the following situation: given a measure space $(X, \mathscr{F}, \mathfrak{m})$ and a partition $\{X_\alpha\}_{\alpha \in Q}$ of X, then Q is the set of indices and the quotient map $\mathfrak{Q} : X \to Q$ is such that

$$\alpha = \mathfrak{Q}(x) \iff x \in X_\alpha.$$

Hence, following the previous scheme, it is possible to consider the quotient measure space $(Q, \mathscr{Q}, \mathfrak{q})$.

Definition 1.40. A *disintegration* of $\mathfrak{m}$ consistent with $\mathfrak{Q}$ is a map $Q \ni \alpha \mapsto \mathfrak{m}_\alpha \in \mathcal{P}(X, \mathscr{F})$ such that the following hold:

(1) $\forall B \in \mathscr{F}$, $Q \ni \alpha \mapsto \mathfrak{m}_\alpha(B)$ is $\mathfrak{q}$-measurable;
(2) $\forall B \in \mathscr{F}$, $\forall C \in \mathscr{Q}$, $\mathfrak{m}(B \cap \mathfrak{Q}^{-1}(C)) = \int_C \mathfrak{m}_\alpha(B)\, \mathfrak{q}(d\alpha)$.

The measures $\mathfrak{m}_\alpha$ are called *conditional probabilities*.

Definition 1.41. A disintegration is called *strongly consistent with respect to* $\mathfrak{Q}$ if in addition for $\mathfrak{q}$-a.e. $\alpha \in Q$, $\mathfrak{m}_\alpha(X - \mathfrak{Q}^{-1}(\alpha)) = 0$.

Theorem 1.42. *(Theorem A.7, Proposition A.9 of* [**19**]*) Let $(X, \mathscr{F}, \mathfrak{m})$ be a countably generated probability space. Take $\{X_\alpha\}_{\alpha\in Q}$ partition of X and let $\mathfrak{Q}$ be the quotient map. Then the measure space $(Q, \mathscr{Q}, \mathfrak{q})$ obtained as before is essentially countably generated and there exists a unique disintegration $\alpha \mapsto \mathfrak{m}_\alpha$ consistent with $\mathfrak{Q}$.*
Moreover, the disintegration is strongly consistent if and only if there exists a $\mathfrak{m}$-section $S \in \mathscr{F}$ of the partition such that the quotient σ-algebra $\mathscr{S}$ obtained pushing forward the σ-algebra $\mathscr{F}$ on S via the map $x \in X_\alpha \mapsto x_\alpha \in S \cap X_\alpha$ contains $\mathcal{B}(S)$.

We now recall all the notions needed to clarify the statement of the Theorem 1.42. In the measure space $(Q, \mathscr{Q}, \mathfrak{q})$, the σ-algebra $\mathscr{Q}$ is *essentially countably generated* if, by definition, there exists a countable family of sets $Q_n \subset Q$ such that for any $C \in \mathscr{Q}$ there exists $\hat{C} \in \hat{\mathscr{Q}}$ such that $\mathfrak{q}(C \,\Delta\, \hat{C}) = 0$, where $\hat{\mathscr{Q}}$ is the σ-algebra generated by $\{Q_n\}_{n\in\mathbb{N}}$. A set S is a section for the partition $X = \cup_\alpha X_\alpha$ if for any $\alpha \in Q$ there exists a unique $x_\alpha \in S \cap X_\alpha$; a set S is called a $\mathfrak{m}$-section if there exists $Y \subset X$ with $\mathfrak{m}(X \setminus Y) = 0$ such that the partition $Y = \cup_\alpha (X_\alpha \cap Y)$ has section S.

Remark 1.43. For what concerns uniqueness, it is understood in the following sense: given $\alpha \mapsto \mathfrak{m}^1_\alpha$ and $\alpha \mapsto \mathfrak{m}^2_\alpha$ two disintegrations consistent with respect to $\mathfrak{Q}$ (hence $\mathfrak{q}$ is fixed) then $\mathfrak{m}^1_\alpha = \mathfrak{m}^2_\alpha$ for $\mathfrak{q}$-a.e. $\alpha \in Q$.

1.4.3. Transport Set. To any 1-Lipschitz function $u : X \to \mathbb{R}$ can be naturally associated a d-cyclically monotone set Γ_u defined in the following way:

$$\Gamma_u := \{(x, y) \in X \times X : u(x) - u(y) = \mathsf{d}(x, y)\}.$$

Thus Γ_u can be interpreted as the set of couples between which u has maximal slope. We write $x \geq_u y$ if and only if $(x, y) \in \Gamma_u$; the 1-Lipschitz condition on u implies $\geq_u$ is a partial-ordering. Accordingly, we call Γ_u *transport ordering.* Moreover, we define the *transport relation* R_u and the *transport set* $\mathcal{T}_u$ in the following way:

$$R_u := \Gamma_u \cup \Gamma_u^{-1}\ ,\ \mathcal{T}_u := P_1(R_u \setminus \{x = y\}), \tag{1.13}$$

where $\{x = y\}$ denotes the diagonal $\{(x, y) \in X^2 : x = y\}$, P_i the projection onto the i-th component and $\Gamma_u^{-1} = \{(x, y) \in X \times X : (y, x) \in \Gamma_u\}$.
Since u is 1-Lipschitz, Γ_u, Γ_u^{-1} and R_u are closed sets, and so are $\Gamma_u(x)$ and $R_u(x)$ (recall that $\Gamma_u(x) = \{y \in X\ ;\ (x, y) \in \Gamma_u\}$ and similarly for $R_u(x)$). Consequently $\mathcal{T}_u$ is a projection of a Borel set and hence it is analytic; it follows that it is universally measurable, and in particular, $\mathfrak{m}$-measurable [**92**].

The transport "flavor" of the previous definitions can be seen in the next property that is immediate to verify: for any $\gamma \in \mathrm{Geo}(X)$ such that $(\gamma_0, \gamma_1) \in \Gamma_u$, then

$$(\gamma_s, \gamma_t) \in \Gamma_u, \quad \forall\ 0 \leq s \leq t \leq 1.$$

Finally, recall the definition of the sets of *forward and backward branching points* introduced in [**30**]:

$$A_{+,u} := \{x \in \mathcal{T}_u : \exists\, z, w \in \Gamma_u(x), (z, w) \notin R_u\},$$
$$A_{-,u} := \{x \in \mathcal{T}_u : \exists\, z, w \in \Gamma_u(x)^{-1}, (z, w) \notin R_u\}.$$

Once branching points are removed, one obtains the *non-branched transport set* and the *non-branched transport relation*,

$$\mathcal{T}_u^b := \mathcal{T}_u \setminus (A_{+,u} \cup A_{-,u}), \quad R_u^b := R_u \cap (\mathcal{T}_u^b \times \mathcal{T}_u^b); \tag{1.14}$$

the following was obtained in [**30**] and highlights the motivation to remove branching points.

PROPOSITION 1.44. *The set of transport rays $R_u^b \subset X \times X$ is an equivalence relation on the set $\mathcal{T}_u^b$.*

Noticing that once we fix $x \in \mathcal{T}_u^b$, for any choice of $z, w \in R_u(x)$, there exists $\gamma \in \mathrm{Geo}(X)$ such that

$$\{x, z, w\} \subset \{\gamma_s : s \in [0, 1]\},$$

it is not hard to deduce that each equivalence class is a geodesic. The next step consists in proving that branching happens on rays with zero $\mathfrak{m}$-measure. Already from the statement of this property, it is clear that some regularity assumption on $(X, \mathsf{d}, \mathfrak{m})$ should play a role. Indeed, in [**30**], it was proved that under $\mathsf{RCD}(K, N)$ condition the measure of the sets of branching points is zero. As observed several times in the literature, the proof only requires all optimal plans to be induced by maps, and so the same argument works for any $p > 1$:

THEOREM 1.45 (Negligibility of forward and backward branching points). *Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. such that for any $\mu_0, \mu_1 \in \mathcal{P}_p(X)$ with $\mu_0 \ll \mathfrak{m}$ any optimal transference plan for W_p is concentrated on the graph of a function. Then*

$$\mathfrak{m}(A_{+,u}) = \mathfrak{m}(A_{-,u}) = 0. \tag{1.15}$$

From Theorem 1.30, the p-essentially non-branching hypothesis implies that for every $\mu_0, \mu_1 \in \mathcal{P}_p(X)$ with $\mu_0 \ll \mathfrak{m}$ there exists a unique p-optimal plan and it is induced by a map. Hence, the assumptions of Theorem 1.45 are satisfied, and therefore the equation (1.15) holds true for any $u : X \to \mathbb{R}$ 1-Lipschitz function.

Once this partition of the transport set made of equivalence classes is established, one wants to deduce a corresponding decomposition of the ambient measure $\mathfrak{m}$ restricted to $\mathcal{T}_u^b$; Disintegration Theorem will be the appropriate technical tool to use. In order to apply it, one needs to build a $\mathfrak{m}$-measurable quotient map $\mathfrak{Q}$ for the equivalence relation R_u^b over $\mathcal{T}_u^b$; for its construction, by now classical, we refer to [**36**]. It is worth stressing that the quotient set will be identified with a subset of $\mathcal{T}_u^b$ containing a point for each equivalence class, i.e. for each geodesic forming $\mathcal{T}_u^b$. In particular, there will be a $\mathfrak{m}$-measurable quotient set $Q \subset \mathcal{T}_u^b$, image of $\mathfrak{Q}$. The Disintegration Theorem then implies the following disintegration formula:

$$\mathfrak{m}\llcorner_{\mathcal{T}_u^b} = \int_Q \mathfrak{m}_\alpha \mathfrak{q}(d\alpha), \tag{1.16}$$

with $\mathfrak{q} = \mathfrak{Q}_\sharp \mathfrak{m}\llcorner_{\mathcal{T}_u^b}$ and for $\mathfrak{q}$-a.e. $\alpha \in Q$ we have $\mathfrak{m}_\alpha \in \mathcal{P}(X)$, $\mathfrak{m}_\alpha(X \setminus X_\alpha) = 0$, where we have used the notation X_α to denote the equivalence class of the element $\alpha \in Q$ (indeed $X_\alpha = R(\alpha)$). Observe that from the $\mathfrak{m}$-measurability of $\mathfrak{Q}$ it follows that $\mathfrak{q}$ is a Borel measure.

REMARK 1.46. It is worth mentioning here that the map $Q \ni \alpha \mapsto \mathfrak{m}_\alpha \in \mathcal{P}(X)$ is essentially unique (meaning that any two maps for which (1.16) holds true have to coincide $\mathfrak{q}$-a.e.) thanks to the assumption $\mathfrak{m}(X) = 1$, while $\mathfrak{m}_\alpha(X \setminus X_\alpha) = 0$ (also called strongly consistency of the disintegration) is a consequence of the existence a $\mathfrak{m}$-measurable quotient map $\mathfrak{Q}$.

In order to deduce regularity properties of the conditional measures $\mathfrak{m}_\alpha$ it will be necessary to make some assumptions on the geometry of the space [**20**], [**30**], [**31**]. For this purpose, it will be useful to introduce the following *ray map* $g : \mathrm{Dom}\,(g) \subset Q \times \mathbb{R} \to \mathcal{T}_u^b$, defined as follows:

$$\begin{aligned}\mathrm{graph}(g) :=& \big\{(\alpha,t,x) \in Q \times [0,+\infty) \times \mathcal{T}_u^b : (\alpha,x) \in \Gamma_u, \mathsf{d}(\alpha,x) = t\big\} \\ &\cup \big\{(\alpha,t,x) \in Q \times (-\infty,0] \times \mathcal{T}_u^b : (x,\alpha) \in \Gamma_u, \mathsf{d}(x,\alpha) = t\big\}.\end{aligned} \tag{1.17}$$

The ray map g enjoys several properties already obtained in [**30**, Proposition 5.4]:

(1) g is a Borel map;
(2) $t \mapsto g(\alpha,t)$ is an isometry. If $s,t \in \mathrm{Dom}\,(g(\alpha,\cdot))$ with $s \le t$, then $(g(\alpha,s), g(\alpha,t)) \in \Gamma_u$;
(3) $\mathrm{Dom}\,(g) \ni (\alpha,t) \mapsto g(\alpha,t)$ is bijective on $\mathfrak{Q}^{-1}(Q) \subset \mathcal{T}_u^b$.

First of all, observe that $\mathrm{Dom}\,(g(\alpha,\cdot))$ is a convex subset of $\mathbb{R}$ (i.e. an interval), for any $\alpha \in Q$. Moreover, since $t \mapsto g(\alpha,t)$ is an isometry, it holds $\mathcal{H}^1 \llcorner_{\{g(\alpha,t):t\in\mathbb{R}\}} = g(\alpha,\cdot)_\sharp \mathcal{L}^1$.

Using the ray map g one can prove that $\mathfrak{q}$-almost every conditional measure $\mathfrak{m}_\alpha$ is absolutely continuous with respect to the 1-dimensional Hausdorff measure considered on the ray passing through α, provided $(X,\mathsf{d},\mathfrak{m})$ enjoys weak curvature properties. In order to present such topics, we will follow the approach of [**32**] that collects results spread across [**20**], [**30**], [**31**], [**36**].

First of all, given a compact set $C \subset X$, we define the *t-translation* C_t of C by

$$C_t := g(\{(\alpha, s+t) : (\alpha,s) \in g^{-1}(C)\})$$

The following general property holds true:

Theorem 1.47 (Theorem 5.7, [**20**]). *Assume that for any compact $C \subset \mathcal{T}_u^b$ with $\mathfrak{m}(C) > 0$, it holds $\mathfrak{m}(C_t) > 0$ for a set of $t \in \mathbb{R}$ with $\mathcal{L}^1$-positive measure. Then, for $\mathfrak{q}$-a.e. $\alpha \in Q$ the conditional measure $\mathfrak{m}_\alpha$ is absolutely continuous with respect to $g(\alpha,\cdot)_\sharp \mathcal{L}^1$.*

Observe that, since $C \subset \mathcal{T}_u^b$ is compact, $g^{-1}(C) \subset Q \times \mathbb{R}$ is σ-compact and the same holds true for $\{(\alpha, s+t) : (\alpha,s) \in g^{-1}(C)\}$. Since

$$C_t = P_3(\mathrm{graph}(g) \cap \{(\alpha, s+t) : (\alpha,s) \in g^{-1}(C)\} \times \mathcal{T}_u^b),$$

it follows that C_t is σ-compact. Moreover, being the set $\{(t,x) \in \mathbb{R} \times \mathcal{T}_u^b : x \in C_t\}$ Borel, the map $t \mapsto \mathfrak{m}(C_t)$ turns out to be Borel.

We stress that the assumption of Theorem 1.47 deals with the regularity of the set Γ_u (therefore with the Monge problem itself) and not with the smoothness of the space.
Under $\mathsf{MCP}(K,N)$, the assumption of Theorem 1.47 is satisfied, hence every conditional measure $\mathfrak{m}_\alpha$ is absolutely continuous with respect to $g(\alpha,\cdot)_\sharp \mathcal{L}^1$; we will denote with h_α its density. In particular, it holds that

$$\{t \in \mathrm{Dom}\,(g(\alpha,\cdot)) : h(\alpha,t) > 0\} = \mathrm{Dom}\,(g(\alpha,\cdot));$$

furthermore, such set is convex and the map $t \mapsto h(\alpha,t)$ is locally Lipschitz continuous.

The last result we want to state concerns the property shared by the conditional probabilities of inheriting the synthetic Ricci curvature bounds of the ambient space.
We will now sketch the strategy used in [**36**] to prove the localization of the $\mathsf{CD}(K,N)$ condition. Actually, it is enough to assume the space to verify such a lower bound only locally to obtain globally the synthetic Ricci curvature lower bound on almost every 1-dimensional metric measure space. Since under the essentially non-branching assumption $\mathsf{CD}_{loc}(K,N)$ implies $\mathsf{MCP}(K,N)$, all the properties deduced until now still hold true. In [**36**, Lemma 3.9] it is shown that the quotient set Q, labelling the various transport rays, can be covered by a countable collection of sets $\{Q_i\}_{i\in I}$ where each Q_i is contained in a rational level set of u; for each Q_i one can consider the transport of one uniform measure to another, of possibly

differing size, along the transport rays of u. More specifically, the countable decomposition is constructed to provide for each i a uniform subinterval

$$(a_0, a_1) \subset \mathrm{dom}(g(\alpha, \cdot)) \quad \text{for all } \alpha \in Q_i$$

as well as real numbers $A_0, A_1 \in (a_0, a_1)$ and $L_0, L_1 \in (0, \infty)$ such that

$$A_0 + L_0 < A_1 \text{ and } A_1 + L_1 < a_1.$$

This allows us to consider the measures

$$\mu_0 = \int_{Q_i} g(\alpha, \cdot)_\sharp \left(\frac{1}{L_0} \mathcal{L}^1_{[A_0, A_0+L_0]}(dt) \right) \mathfrak{q}(d\alpha), \qquad \mu_1 = \int_{Q_i} g(\alpha, \cdot)_\sharp \left(\frac{1}{L_1} \mathcal{L}^1_{[A_1, A_1+L_1]}(dt) \right) \mathfrak{q}(d\alpha).$$

Transporting these measures allows to get the concavity information of the density h_α of $\mathfrak{q}$-a.e. $\mathfrak{m}_\alpha$ from the entropic concavity (1.5) asserted by $\mathsf{CD}(K, N)$. We collect here the main results concerning the localization in spaces satisfying $\mathsf{CD}(K, N)$.

Theorem 1.48. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching metric measure space verifying the $\mathsf{CD}_{loc}(K, N)$ condition for some $K \in \mathbb{R}, N \in [1, \infty)$ and let $u : X \mapsto \mathbb{R}$ be a 1-Lipschitz function. Then the corresponding decomposition of the space in rays $\{X_\alpha\}_{\alpha \in Q}$ produces a disintegration $\{\mathfrak{m}_\alpha\}_{\alpha \in Q}$ of $\mathfrak{m}$ such that for $\mathfrak{q}$-a.e. $\alpha \in Q$*

$$(Dom\,(g(\alpha, \cdot)), |\cdot|, h_\alpha \mathcal{L}^1) \quad \textit{verifies} \quad \mathsf{CD}(K, N).$$

Localization for $\mathsf{MCP}(K, N)$ was, partially and in a different form, already known in 2009, see [**20**, Theorem 9.5], for non-branching m.m.s.. The case of essentially non-branching m.m.s.'s and an effective reformulation (after the work of Klartag [**64**]) has been recently discussed in [**41**, Section 3] to which we refer for all the missing details (see in particular [**41**, Theorem 3.5]). For future references we collect in the next statement the main results we will use:

Theorem 1.49. *If $(X, \mathsf{d}, \mathfrak{m})$ is an essentially non-branching m.m.s. with* $\mathrm{supp}(\mathfrak{m}) = X$ *and satisfying* $\mathsf{MCP}(K, N)$*, for some $K \in \mathbb{R}, N \in (1, \infty)$, then, for any 1-Lipschitz function $u : X \to \mathbb{R}$, the non-branching transport set $\mathcal{T}_u^b$ associated with u admits a disjoint family of unparametrized geodesics $\{X_\alpha\}_{\alpha \in Q}$ such that $\mathfrak{m}(\mathcal{T}_u^b \setminus \cup_\alpha X_\alpha) = 0$ and the corresponding disintegration of $\mathfrak{m}$ is as follows*

$$\mathfrak{m}\llcorner_{\mathcal{T}_u^b} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha), \qquad \mathfrak{q}(Q) = 1, \qquad \mathfrak{q}\text{−a.e. } \mathfrak{m}_\alpha(X) = \mathfrak{m}_\alpha(X_\alpha) = 1. \tag{1.18}$$

Moreover, $\mathfrak{q}$-a.e. $\mathfrak{m}_\alpha$ is a Radon measure with $\mathfrak{m}_\alpha = h_\alpha \mathcal{H}^1\llcorner_{X_\alpha} \ll \mathcal{H}^1\llcorner_{X_\alpha}$ and $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ verifies $\mathsf{MCP}(K, N)$.

We now recall the definition of the $\mathsf{CD}^1(K, N)$ condition introduced in [**35**] and based on the localization of Ricci curvature lower bounds along integral curves associated to the gradient of a 1-Lipschitz function.

Definition 1.50. ($\mathsf{CD}^1_u(K, N)$ when $\mathrm{supp}(\mathfrak{m}) = X$) Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space such that $\mathrm{supp}(\mathfrak{m}) = X$ and $\mathfrak{m}(X) = 1$. Let us consider $K, N \in \mathbb{R}$, $N > 1$ and let $u : (X, \mathsf{d}) \to \mathbb{R}$ be a 1-Lipschitz function. We say that $(X, \mathsf{d}, \mathfrak{m})$ satisfies the CD^1_u condition if there exists a family $\{X_\alpha\}_{\alpha \in Q} \subset X$ such that :

(1) There exists a disintegration of $\mathfrak{m}\llcorner_{\mathcal{T}_u}$ on $\{X_\alpha\}_{\alpha \in Q}$:

$$\mathfrak{m}\llcorner_{\mathcal{T}_u} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha), \;\; \text{where } \mathfrak{m}_\alpha(X_\alpha) = 1, \;\; \text{for } \mathfrak{q}\text{-a.e. } \alpha \in Q.$$

(2) For $\mathfrak{q}$-a.e. $\alpha \in Q$, X_α is a *transport ray* for Γ_u.
(3) For $\mathfrak{q}$-a.e. $\alpha \in Q$, $\mathfrak{m}_\alpha$ is supported on X_α.

(4) For $\mathfrak{q}$-a.e. $\alpha \in Q$, the metric measure space $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ satisfies $\mathsf{CD}(K, N)$.

Following [**35**, Definition 7.7], a maximal chain R in $(X, \mathsf{d}, \leq_u)$ is called a *transport ray* if it is isometric to a closed interval I in $(\mathbb{R}, |\cdot|)$ of positive (possibly infinite) length.

Remark 1.51 (The assumption $\mathfrak{m}(X) = 1$). It is worth mentioning here that the assumption $\mathfrak{m}(X) = 1$ is most probably purely technical. In the framework of general Radon measure, Disintegration Theorem does not furnish a unique family of conditional measures and one has to consider an additional normalization function; for additional details we refer to [**41**] where a localization of synthetic lower Ricci curvature bounds has been obtained also for general Radon measure.

Remark 1.52. It is well known that the last condition of Definition 1.50 is equivalent to ask $\mathfrak{m}_\alpha \sim h_\alpha \mathcal{L}^1 \llcorner_{[0,|X_\alpha|]}$ where $|X_\alpha|$ denotes the length of the transport ray X_α ($\sim$ means up to isometry of the space) and h_α satisfies (1.6); hence Remark 1.34 applies.

Finally, we will say that the metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfies $\mathsf{CD}^1_{Lip}(K, N)$ if $(\mathrm{supp}(\mathfrak{m}), \mathsf{d}, \mathfrak{m})$ verifies $\mathsf{CD}^1_u(K, N)$ for all 1-Lipschitz functions $u : (\mathrm{supp}(\mathfrak{m}), \mathsf{d}) \to \mathbb{R}$ and satisfies $\mathsf{CD}^1(K, N)$ if $(\mathrm{supp}(\mathfrak{m}), \mathsf{d}, \mathfrak{m})$ verifies $\mathsf{CD}^1_u(K, N)$ where u is a signed distance function defined as follows: given a continuous function $f : (X, \mathsf{d}) \to \mathbb{R}$ such that $\{f = 0\} \neq \emptyset$, the function

$$d_f : X \to \mathbb{R}, \qquad d_f(x) := \mathrm{dist}(x, \{f = 0\})\mathrm{sgn}(f), \tag{1.19}$$

is called the signed distance function (from the zero-level set of f). Notice that d_f is 1-Lipschitz on $\{f \geq 0\}$ and $\{f \leq 0\}$. If (X, d) is a length space, then d_f is 1-Lipschitz on the entire X.

CHAPTER 2

Isoperimetric inequality under Measure-Contraction property

In this Chapter we address the isoperimetric inequality à la Lévy-Gromov-Milman within the class of metric measure spaces verifying MCP. In particular, we identify a family of one-dimensional $\mathsf{MCP}(K,N)$-densities, each for every choice of K, N, volume v and diameter D, not verifying $\mathsf{CD}(K,N)$, and having optimal isoperimetric profile for the volume v. Denoting the model isoperimetric profile at volume v and the Minkowski content defined in (2) by $\mathcal{I}_{K,N,D}(v)$ and $\mathfrak{m}^+$ respectively, we will prove the following result:

THEOREM 2.1 (Theorem 2.15). *Let $K, N \in \mathbb{R}$ with $N > 1$ and let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching m.m.s. verifying $\mathsf{MCP}(K,N)$ with $\mathfrak{m}(X) = 1$ and having diameter less than D (with $D \leq \pi\sqrt{(N-1)/K}$ if $K > 0$). For any $A \subset X$,*

$$\mathfrak{m}^+(A) \geq \mathcal{I}_{K,N,D}(\mathfrak{m}(A)). \tag{2.1}$$

Moreover (2.1) is sharp, i.e. for each $v \in [0,1]$, K, N, D there exists a m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ with $\mathfrak{m}(X) = 1$ and $A \subset X$ with $\mathfrak{m}(A) = v$ such that (2.1) is an equality.

Via localization paradigm for MCP-spaces (see Section 1.4 for details), following [**64, 36**], the proof of Theorem (2.1) is reduced to the proof of the corresponding statement in the one-dimensional setting. For this purpose, Section 2.1 will be devoted to a careful analysis of one-dimensional $\mathsf{MCP}(K,N)$ spaces while Section 2.2 will be dedicated to the solution of the isoperimetric problem for a one-dimensional density h verifying $\mathsf{MCP}(K,N)$ for some $K, N \in \mathbb{R}$ and $N > 1$. Finally, in Section 2.3 a proof of Theorem (2.1) will be provided.

2.1. One-dimensional analysis

In this section we will introduce a family of $\mathsf{MCP}(K,N)$ densities, depending on four parameters, that will be the model one-dimensional isoperimetric densities.
Let us consider a one-dimensional density h verifying $\mathsf{MCP}(K,N)$ for some $K, N \in \mathbb{R}$ and $N > 1$; without loss of generality we can assume h to be defined over $[0, D']$ (recall that $D' \leq D_{K,N-1} = \pi\sqrt{(N-1)/K}$, whenever $K > 0$). Observe that the case $K > 0$ and $D' = \pi\sqrt{(N-1)/K}$ is trivial as (1.11) forces the density to coincide with the model density $\sin^{N-1}(t)$ (that in particular is also a $\mathsf{CD}(K,N)$-density).

PROPOSITION 2.1 (Lower Bound). *Define the following strictly positive function*

$$f_{K,N,D'}(x) := \left(\int_{(0,x)} \left(\frac{s_{K/(N-1)}(D'-y)}{s_{K/(N-1)}(D'-x)} \right)^{N-1} dy + \int_{(x,D')} \left(\frac{s_{K/(N-1)}(y)}{s_{K/(N-1)}(x)} \right)^{N-1} dy \right)^{-1}$$

for $x \in (0, D')$ and equal 0 for $x = 0, D'$. Then

i) $f_{K,N,D'}$ is strictly increasing over $(0, D'/2)$;
ii) $f_{K,N,D'}(x) = f_{K,N,D'}(D'-x)$;
iii) if $h : [0, D'] \to \mathbb{R}$ is an $\mathsf{MCP}(K,N)$-density integrating to 1, then $h(x) \geq f_{K,N,D'}(x)$.

PROOF. The second claim is straightforward to check. For the first one, being $f_{K,N,D'}$ a smooth function, strictly positive in $(0, D')$, it will be enough to show that $f'_{K,N,D'}(x) = 0$ has no solution for $x \in (0, D'/2)$. Imposing $f'_{K,N,D'}(x) = 0$ is equivalent to

$$\frac{s'_{K/(N-1)}(D'-x)}{s^N_{K/(N-1)}(D'-x)} \int_{(0,x)} s^{N-1}_{K/(N-1)}(D'-y)\,dy = \frac{s'_{K/(N-1)}(x)}{s^N_{K/(N-1)}(x)} \int_{(x,D')} s^{N-1}_{K/(N-1)}(y)\,dy,$$

that can be rewritten as

$$\frac{s'_{K/(N-1)}(D'-x)}{s^N_{K/(N-1)}(D'-x)} \int_{(D'-x,D')} s^{N-1}_{K/(N-1)}(y)\,dy = \frac{s'_{K/(N-1)}(x)}{s^N_{K/(N-1)}(x)} \int_{(x,D')} s^{N-1}_{K/(N-1)}(y)\,dy.$$

Since $D' - x \geq x$, the previous identity implies

$$\frac{|s'_{K/(N-1)}(D'-x)|}{s^N_{K/(N-1)}(D'-x)} > \frac{|s'_{K/(N-1)}(x)|}{s^N_{K/(N-1)}(x)}. \tag{2.2}$$

For $K = 0$, (2.2) becomes

$$\frac{1}{(D'-x)^N} > \frac{1}{x^N},$$

giving a contradiction. For negative $K = -(N-1)$ (the other negative cases follow similarly) (2.2) implies

$$\frac{\cosh(D'-x)}{\sinh(D'-x)^N} > \frac{\cosh(x)}{\sinh(x)^N}$$

forcing

$$\frac{\cosh(D'-x)}{\sinh(D'-x)} > \left(\frac{\sinh(D'-x)}{\sinh(x)}\right)^{N-1} \frac{\cosh(x)}{\sinh(x)} > \frac{\cosh(x)}{\sinh(x)},$$

giving a contradiction with monotonicity of tanh. Finally, for $K = N-1$, (2.2) becomes

$$\frac{\cos(D'-x)}{\sin(D'-x)^N} > \frac{\cos(x)}{\sin(x)^N}, \qquad \operatorname{sgn}(\cos(D'-x)) = \operatorname{sgn}(\cos(x));$$

the second identity implies that $x < D'-x < \pi/2$ or $\pi/2 < x < D'-x$. The second case would imply that $D' > 2x > \pi$ giving a contradiction. Hence we are left with $x < D' - x < \pi/2$:

$$1 > \frac{\cos(D'-x)}{\cos x} > \left(\frac{\sin(D'-x)}{\sin x}\right)^N,$$

giving a contradiction.

The third claim follows simply observing that (1.11) gives

$$\begin{aligned}
1 &= \int_{(0,x)} h(y)\,dy + \int_{(x,D')} h(y)\,dy \\
&\leq \frac{h(x)}{s^{N-1}_{K/(N-1)}(D'-x)} \int_{(0,x)} s^{N-1}_{K/(N-1)}(D'-y)\,dy + \frac{h(x)}{s^{N-1}_{K/(N-1)}(x)} \int_{(x,D')} s^{N-1}_{K/(N-1)}(y)\,dy,
\end{aligned}$$

and the claim is proved. □

Starting from the lower bound of Proposition 2.1, we define a distinguished family of $\mathsf{MCP}(K,N)$ densities, depending on four parameters, that will be the model one-dimensional isoperimetric density:

$$h^a_{K,N,D'}(x) := f_{K,N,D'}(a) \begin{cases} \left(\dfrac{s_{K/(N-1)}(D'-x)}{s_{K/(N-1)}(D'-a)}\right)^{N-1}, & x \leq a, \\[2ex] \left(\dfrac{s_{K/(N-1)}(x)}{s_{K/(N-1)}(a)}\right)^{N-1}, & x \geq a. \end{cases} \tag{2.3}$$

Notice that $h^{D'-a}_{K,N,D'}(D'-x) = h^a_{K,N,D'}(x)$ and

$$h^a_{K,N,D'}(zD'/D'') = h^{aD''/D'}_{(D'/D'')^2K,N,D''}(z),$$

showing that it will not be restrictive to assume for some of the next proofs $K = N-1$ or $K = -(N-1)$, letting D' vary.

COROLLARY 2.2 (Rigidity of lower bound). *Let $h : [0, D'] \to \mathbb{R}$ be a $\mathsf{MCP}(K,N)$-density integrating to 1. Assume $h(y) = f_{K,N,D'}(y)$ for some $y \in (0, D')$; then $h = h^y_{K,N,D'}$.*

PROOF. From the proof of Proposition 2.1, point *iii*), and (1.11) one deduces that

$$h(x) = h(y) \begin{cases} \left(\dfrac{s_{K/(N-1)}(D'-x)}{s_{K/(N-1)}(D'-a)} \right)^{N-1}, & x \le a, \\ \\ \left(\dfrac{s_{K/(N-1)}(x)}{s_{K/(N-1)}(a)} \right)^{N-1}, & x \ge a. \end{cases}$$

The claim then follows. □

To avoid cumbersome notation, the dependence of $h^a_{K,N,D'}$ on K, N, D' will be omitted and we will use h_a.

LEMMA 2.3. *For every $a \in (0, D')$, the function h_a integrates to 1 and it is an $\mathsf{MCP}(K,N)$-density.*

PROOF. Each h_a has by definition integral 1. To check $\mathsf{MCP}(K,N)$ it will be enough to verify that the inequality (1.11) is satisfied.

We start observing that the function

$$\frac{s_{K/(N-1)}(D' - \cdot\,)}{s_{K/(N-1)}(\cdot)} \tag{2.4}$$

is decreasing in $[0, D']$; this will be proved showing its first derivative to be negative:

$$\frac{s'_{K/(N-1)}(D'-a)}{s_{K/(N-1)}(a)} + \frac{s_{K/(N-1)}(D'-a)s'_{K/(N-1)}(a)}{s^2_{K/(N-1)}(a)} \ge 0.$$

The previous inequality is straightforward for $K \le 0$; for $K > 0$, assuming without loss of generality $K = N-1$, it reduces to $\sin(a)\cos(D'-a) + \sin(D'-a)\cos(a) = \sin(D') \ge 0$, that is always verified with the strict inequality except for the trivial case $D' = \pi$ (where the function (2.4) is identically equal to one).
Using the result just obtained, we are able to check (1.11) distinguishing three cases.
If $x_0 \le x_1 \le a$:

$$\left(\frac{s_{K/(N-1)}(D'-x_1)}{s_{K/(N-1)}(D'-x_0)} \right)^{N-1} = \frac{h_a(x_1)}{h_a(x_0)} \le \left(\frac{s_{K/(N-1)}(x_1)}{s_{K/(N-1)}(x_0)} \right)^{N-1}.$$

If $a \le x_0 \le x_1$:

$$\left(\frac{s_{K/(N-1)}(D'-x_1)}{s_{K/(N-1)}(D'-x_0)} \right)^{N-1} \le \frac{h_a(x_1)}{h_a(x_0)} = \left(\frac{s_{K/(N-1)}(x_1)}{s_{K/(N-1)}(x_0)} \right)^{N-1}.$$

If $x_0 \le a \le x_1$:

$$\frac{h_a(x_1)}{h_a(x_0)} = \left(\frac{s_{K/(N-1)}(x_1)}{s_{K/(N-1)}(a)} \right)^{N-1} \cdot \left(\frac{s_{K/(N-1)}(D'-a)}{s_{K/(N-1)}(D'-x_0)} \right)^{N-1};$$

using again the fact that (2.4) is decreasing, we get the claim. □

LEMMA 2.4. *For every choice of K, N and D', except the case in which $K > 0$ and $D' = \pi\sqrt{(N-1)/K}$, the density h_a defined in* (2.3) *does not verify* $\mathsf{CD}(K,N)$.

PROOF. By Remark 1.52, a non-negative Borel function h defined on an interval $I \subset \mathbb{R}$ is called a $\mathsf{CD}(K,N)$ density if for every $t \in [0,1]$ and for all $x_0, x_1 \in I$ such that $x_0 < x_1$, it holds:

$$h((1-t)x_0 + tx_1)^{\frac{1}{N-1}} \geq \sigma^{(1-t)}_{K,N-1}(x_1 - x_0)h(x_0)^{\frac{1}{N-1}} + \sigma^{(t)}_{K,N-1}(x_1 - x_0)h(x_1)^{\frac{1}{N-1}}. \tag{2.5}$$

In order to prove our claim we will discuss several cases.
If $K = 0$, the inequality (2.5) simply reduces to the concavity of $h^{\frac{1}{N-1}}$. We will prove now that (2.5) fails for the density $h_a(\cdot)$ exactly for convex combinations that give out the point a. Pick $x_0 < a < x_1$ and let $t \in (0,1)$ be such that $a = (1-t)x_0 + tx_1$. It follows that

$$\begin{aligned}(1-t)h_a(x_0)^{\frac{1}{N-1}} + th_a(x_1)^{\frac{1}{N-1}} &= f_{0,N,D'}(a)^{\frac{1}{N-1}}\left[(1-t)\left(\frac{D'-x_0}{D'-a}\right) + t\left(\frac{x_1}{a}\right)\right] \\ &> f_{0,N,D'}(a)^{\frac{1}{N-1}} = h_a(a)^{\frac{1}{N-1}},\end{aligned}$$

hence(2.5) is not satisfied.
If $K \neq 0$, we argue as follows. Since $a = (1-t)x_0 + tx_1$, it should be $t = \frac{a-x_0}{x_1-x_0}$ and $1 - t = \frac{x_1-a}{x_1-x_0}$. Hence, we can rewrite the second member of the inequality (2.5) in this form

$$f_{K,N,D'}(a)^{\frac{1}{N-1}}\left[\frac{s_{K/(N-1)}(x_1-a)}{s_{K/(N-1)}(x_1-x_0)}\cdot\frac{s_{K/(N-1)}(D'-x_0)}{s_{K/(N-1)}(D'-a)} + \frac{s_{K/(N-1)}(a-x_0)}{s_{K/(N-1)}(x_1-x_0)}\cdot\frac{s_{K/(N-1)}(x_1)}{s_{K/(N-1)}(a)}\right]; \tag{2.6}$$

using now that (2.4) is a strictly decreasing function, we get that the quantity above is strictly greater than

$$f_{K,N,D'}(a)^{\frac{1}{N-1}}\left[\frac{s_{K/(N-1)}(x_1-a)}{s_{K/(N-1)}(x_1-x_0)}\cdot\frac{s_{K/(N-1)}(x_0)}{s_{K/(N-1)}(a)} + \frac{s_{K/(N-1)}(a-x_0)}{s_{K/(N-1)}(x_1-x_0)}\cdot\frac{s_{K/(N-1)}(x_1)}{s_{K/(N-1)}(a)}\right]. \tag{2.7}$$

If $K < 0$, assuming without loss of generality that $K = -(N-1)$, we get that (2.7) can be rewritten in the following way

$$f_{-(N-1),N,D'}(a)^{\frac{1}{N-1}}\left[\frac{\sinh(x_1-a)\sinh(x_0) + \sinh(a-x_0)\sinh(x_1)}{\sinh(a)\sinh(x_1-x_0)}\right] = f_{-(N-1),N,D'}(a)^{\frac{1}{N-1}},$$

by straightforward computations. Arguing in the same way in the case $K > 0$ (assuming as usual that $K = N-1$), we get that (2.7) can be rewritten in this form

$$f_{N-1,N,D}(a)^{\frac{1}{N-1}}\left[\frac{\sin(x_1-a)\sin(x_0) + \sin(a-x_0)\sin(x_1)}{\sin(a)\sin(x_1-x_0)}\right] = f_{N-1,N,D}(a)^{\frac{1}{N-1}}.$$

Hence the claim follows also in this case. □

2.2. One-dimensional isoperimetric inequality

To properly formulate the one-dimensional minimization problem, let us consider the following set of probabilities

$$\widetilde{\mathcal{F}}_{K,N,D'} = \{\mu \in \mathcal{P}(\mathbb{R}) : \mu = h_\mu \mathcal{L}^1, h_\mu : [0,D'] \to \mathbb{R}, \quad \mathsf{MCP}(K,N) \text{ density}\},$$

and consider the following "restricted" minimization: for each $v \in (0,1)$

$$\widetilde{\mathcal{I}}_{K,N,D'}(v) := \inf\{\mu^+(A) : A \subset [0,D'], \mu(A) = v, \mu \in \widetilde{\mathcal{F}}_{K,N,D'}\}.$$

The term "restricted" is motivated by the choice of fixing the domain of the $\mathsf{MCP}(K,N)$ densities. For the "unrestricted" one-dimensional minimization we will adopt the classical notation

$$\mathcal{I}_{K,N,D}(v) := \inf\{\mu^+(A) : A \subset [0,D], \mu(A) = v, \mu \in \mathcal{F}_{K,N,D}\}, \tag{2.8}$$

where $\mathcal{F}_{K,N,D} = \cup_{D' \leq D} \widetilde{\mathcal{F}}_{K,N,D'}$.

The final claim will be to prove that each h_a is a minimum of the isoperimetric problem for the volume equal to $\int_{(0,a)} h_a(x)\,dx$. We will therefore show that each volume $v \in (0,1)$ is reached in this manner.

Lemma 2.5. *The map*

$$(0,D) \ni a \longmapsto v(a) := \int_{(0,a)} h_a(x)\,dx \ \in (0,1),$$

is invertible.

Proof. It will be convenient to rewrite the function in the following way

$$v(a) = \frac{f_{K,N,D'}(a)}{s^{N-1}_{K/(N-1)}(D'-a)} \int_{(0,a)} s^{N-1}_{K/(N-1)}(D'-x)\,dx \tag{2.9}$$

implying differentiability. Given the strict monotonicity of the integral with respect to the variable a, it is sufficient to prove that also the other factor is an increasing function. Since

$$\left(\frac{s^{N-1}_{K/(N-1)}(D'-a)}{f_{K,N,D'}(a)}\right)' = \left[\left(\frac{s_{K/(N-1)}(D'-a)}{s_{K/(N-1)}(a)}\right)^{N-1}\right]' \int_{(a,D')} s^{N-1}_{K/(N-1)}(x)\,dx,$$

it follows that the previous derivative has the same sign of the derivative of (2.4), thus it is non positive and the claim follows. □

Hence for each K, N, D' it is possible to define the inverse map of $v(a)$ from Lemma 2.5:

$$(0,1) \ni v \longmapsto a_{K,N,D'}(v) \in (0,D'),$$

with $a_{K,N,D'}(v)$ the unique element such that

$$\int_{(0,a_{K,N,D'}(v))} h_{a_{K,N,D'}(v)}(x)\,dx = v. \tag{2.10}$$

For ease of notation we will prefer in few places the shorter notation a_v to denote $a_{K,N,D'}(v)$.

Remark 2.6. The function $v \mapsto a_v$ enjoys a simple symmetry property: by definition we have that

$$\begin{aligned} 1 - v &= \frac{f_{K,N,D'}(a_v)}{s^{N-1}_{K/(N-1)}(a_v)} \int_{(a_v,D')} s^{N-1}_{K/(N-1)}(x)\,dx \\ &= \frac{f_{K,N,D'}(D'-a_v)}{s^{N-1}_{K/(N-1)}(a_v)} \int_{(0,D'-a_v)} s^{N-1}_{K/(N-1)}(D'-x)\,dx \\ &= v(D'-a_v), \end{aligned}$$

where the last identity follows from (2.9). Since there exists a unique value $a_{1-v} \in (0,D')$ such that $v(a_{1-v}) = 1-v$, it turns out that $a_{1-v} = D' - a_v$.

The first main result of this chapter is the following explicit formula for $\widetilde{\mathcal{I}}_{K,N,D'}$.

THEOREM 2.7. *For each volume $v \in (0,1)$, it holds*

$$\widetilde{\mathcal{I}}_{K,N,D'}(v) = f_{K,N,D'}(a_{K,N,D'}(v)).$$

In particular, since $f_{K,N,D'}(a_{K,N,D'}(v)) = h_{a_{K,N,D'}(v)}(a_{K,N,D'}(v))$, the lower bound is attained.

For the proof of Theorem 2.7 it will be useful to consider the function $A_{K,N,D'} : [0, D') \to [0,\infty)$ defined as follows:

$$A_{K,N,D'}(a) := \frac{v(a)}{f_{K,N,D'}(a)} = \int_{(0,a)} \left(\frac{s_{K/(N-1)}(D'-x)}{s_{K/(N-1)}(D'-a)} \right)^{N-1} dx. \tag{2.11}$$

We will use that $[0, D') \ni a \mapsto A_{K,N,D'}(a)$ is increasing; we postpone the proof of this fact at the end of the section. From the symmetric property of a_v observed few lines above, we obtain the analogous one for $A_{K,N,D'}$:

$$\frac{1-v}{A_{K,N,D'}(D'-a_v)} = \frac{v(D'-a_v) f_{K,N,D'}(D'-a_v)}{v(D'-a_v)} = f_{K,N,D'}(a_v). \tag{2.12}$$

PROOF OF THEOREM 2.7. Fix $K, N, D' \in \mathbb{R}$ with $N > 1$ and any $v \in (0,1)$. Consider h_{a_v} and $h_{a_{1-v}}$ and notice that

$$\int_{(0,a_v)} h_{a_v}(x)\, dx = \int_{(a_{1-v},D')} h_{a_{1-v}}(x)\, dx = v$$

and

$$h_{a_v}(a_v) = f_{K,N,D'}(a_v) = f_{K,N,D'}(a_{1-v}) = h_{a_{1-v}}(a_{1-v}),$$

where the second equality follows from $a_{1-v} = D' - a_v$ and the symmetric property of $f_{K,N,D'}$. Hence it is enough to show that for any $\mathsf{MCP}(K,N)$ density $h : [0, D'] \to [0,\infty)$, the following inequality is valid

$$\mathcal{I}_h(v) \geq f_{K,N,D'}(a_{K,N,D'}(v)).$$

In the one-dimensional setting, taking the lowest possible Minkowski content or the lowest possible perimeter with respect to h makes no difference (see [**39**, Corollary 3.2]). Hence fix any h as above and a set E of finite perimeter with respect to $h\mathcal{L}^1$. It follows that, up to a Lebesgue negligible set, $E = \cup_{i\in\mathscr{I}}[a_i, b_i] \subseteq [0, D']$, where $\mathscr{I} \subseteq \mathbb{N}$ is a set of indices, so that (see [**39**, Proposition 3.1])

$$\mathsf{P}_h(E) = \sum_i h(a_i) + h(b_i),$$

where P_h denotes the perimeter with respect to h. First notice that if any a_i, b_i is in the interval having as boundary points a_v and $D' - a_v$, the claim is proved

$$h(x) \geq f_{K,N,D'}(x) \geq \inf_{y\in[a_v,D'-a_v]} f_{K,N,D'}(y) = f_{K,N,D'}(a_v);$$

the same chain of inequalities is valid if $2a_v \geq D'$. So for each $i \in \mathscr{I}$, points $a_i, b_i \notin (a_v, D'-a_v)$ if $a_v \leq D'/2$, or $a_i, b_i \notin (D'-a_v, a_v)$ if $a_v \geq D'/2$.

It is convenient to assume with no loss in generality that $a_v \leq D' - a_v$ and consider the following subsets of indices

$$\mathscr{I}_1 := \{i \in \mathscr{I} : a_i \geq D' - a_v\}, \qquad \mathscr{I}_2 := \{i \in \mathscr{I} : b_i \leq a_v\};$$

notice that $\mathscr{I}_1 \cap \mathscr{I}_2 = \emptyset$.

Case 1. $\mathscr{I} = \mathscr{I}_1$.
Then

$$v = \sum_{i\in\mathscr{I}} \int_{a_i}^{b_i} h(y)\,dy \le \sum_{i\in\mathscr{I}} h(a_i) \int_{a_i}^{D'} \left(\frac{s_{K/(N-1)}(y)}{s_{K/(N-1)}(a_i)}\right)^{N-1} dy$$
$$= \sum_{i\in\mathscr{I}} h(a_i)A(D'-a_i) \le A(a_v)\sum_{i\in\mathscr{I}} h(a_i).$$

Hence, we get

$$\sum_{i\in\mathscr{I}}(h(a_i)+h(b_i)) \ge \sum_{i\in\mathscr{I}} h(a_i) \ge \frac{v}{A(a_v)} = f_{K,N,D'}(a_v).$$

Case 2. $\mathscr{I} = \mathscr{I}_2$.
It holds true

$$v = \sum_{i\in\mathscr{I}} \int_{a_i}^{b_i} h(y)dy \le \sum_{i\in\mathscr{I}} h(b_i) \int_{a_i}^{b_i} \left(\frac{s_{K/(N-1)}(D'-y)}{s_{K/(N-1)}(D'-b_i)}\right)^{N-1} dy$$
$$\le \sum_{i\in\mathscr{I}} h(b_i)A(b_i)$$
$$\le A(a_v)\sum_{i\in\mathscr{I}} h(b_i),$$

for the increasing monotonicity of the function $A(\cdot)$.

Case 3. $\mathscr{I} \ne \mathscr{I}_1 \cup \mathscr{I}_2$.
There exists $i \in \mathscr{I}$ such that $a_i \le a_v, D' - a_v \le b_i$. Then

$$1 - v \le \int_0^{a_i} h(y)\,dy + \int_{b_i}^{D'} h(y)\,dy$$
$$\le h(a_i)\int_0^{a_i} \left(\frac{s_{K/(N-1)}(D'-y)}{s_{K/(N-1)}(D'-a_i)}\right)^{N-1} dy + h(b_i)\int_{b_i}^{D'} \left(\frac{s_{K/(N-1)}(y)}{s_{K/(N-1)}(b_i)}\right)^{N-1} dy$$
$$= h(a_i)A(a_i) + h(b_i)A(D'-b_i)$$
$$\le A(D'-a_v)[h(a_i)+h(b_i)],$$

proving the claim.

Case 4. $\mathscr{I} = \mathscr{I}_1 \cup \mathscr{I}_2$.
We use the estimates of **Case 2.** for $\mathscr{I}_1$ and the ones in **Step 1.** for $\mathscr{I}_2$, so:

$$v = \sum_{i\in\mathscr{I}} \int_{a_i}^{b_i} h(y)\,dy = \sum_{i\in\mathscr{I}_1} \int_{a_i}^{b_i} h(y)\,dy + \sum_{j\in\mathscr{I}_2} \int_{a_j}^{b_j} h(y)\,dy \le A(a_v)(\sum_{i\in\mathscr{I}_1} h(a_i) + \sum_{j\in\mathscr{I}_2} h(b_j)).$$

Hence, the claim is proved also in this class. □

LEMMA 2.8. *The function* $A_{K,N,D'}(\cdot)$ *is strictly increasing on* $[0, D')$.

PROOF. If we are in the case $K = 0$, we get that

$$A_{0,N,D'}(a) = \int_{(0,a)} \left(\frac{D'-x}{D'-a}\right)^{N-1} dx$$

and so $A_{0,N,D'}(\cdot)$ is trivially increasing. If $K < 0$, without loss of generality we can assume $K = -(N-1)$. In this case we have

$$A_{-(N-1),N,D'}(a) = \int_{(0,a)} \left(\frac{\sinh(D'-x)dx}{\sinh(D'-a)}\right)^{N-1} dx$$

and so again we get the claim by the monotonicity of the hyperbolic sine. If $K > 0$, we can directly deal with the case $D' < \pi\sqrt{(N-1)/K}$. Assuming $K = N-1$, we can rewrite (2.11) in the following way:

$$A_{N-1,N,D'}(a) = \int_{(0,a)} \left(\frac{\sin(D'-x)}{\sin(D'-a)}\right)^{N-1} dx.$$

For sure this function is increasing for $a \in [D'-\pi/2, D')$ by the monotonicity of $\sin(D'-\cdot)$; so, if $D' \leq \pi/2$, we are done. If this is not the case, i.e. $D' > \pi/2$, we have to prove that the same result holds in $[0, D'-\pi/2)$. Computing the first derivative we obtain that

$$\begin{aligned} A'_{N-1,N,D'}(a) &= 1 + (N-1)\frac{\cos(D'-a)}{\sin^N(D'-a)}\int_{(0,a)} \sin^{N-1}(D'-x)\,dx \\ &= 1 + \frac{N-1}{\tan(D'-a)} A_{N-1,N.D'}(a); \end{aligned} \tag{2.13}$$

so $A(\cdot)$ is solution of a differential equation. In order to prove that $A(\cdot)$ is an increasing function, we will check that its first derivative is positive, i.e.

$$A_{N-1,N,D'}(a) \leq -\frac{\tan(D'-a)}{N-1} := g(a),\ \forall a \in [0, D'-\pi/2).$$

For $a = 0$ we have $A_{N-1,N,D'}(a) = 0$ and $g(a) = -\frac{\tan D'}{N-1} > 0$, hence the inequality at the initial point holds true. In order to prove that it holds for every $a \in [0, D'-\pi/2)$, we will check that g verifies the following differential inequality:

$$g'(a) > 1 + \frac{N-1}{\tan(D'-a)} \cdot g(a).$$

Since the choice of g makes the second member identically equals to zero, it is sufficient to prove that $g'(a) > 0$ for every $a \in [0, D-\pi/2)$. This trivially holds true since

$$g'(a) = \frac{1}{(N-1)\cos^2(D'-a)} > 0.$$

Hence, the claim follows also in this case. □

We now analyse the dependence of $\widetilde{\mathcal{I}}_{K,N,D'}(v)$ on the diameter.

LEMMA 2.9. *Fix $N, D' > 0$ and $v \in (0,1)$.*

- *if $K \leq 0$, the map $D' \mapsto \widetilde{\mathcal{I}}_{K,N,D'}(v)$ is strictly decreasing;*
- *if $K > 0$, the map $D' \mapsto D'\widetilde{\mathcal{I}}_{K,N,D'}(v)$ is non-decreasing;*

PROOF. Given any $\mathsf{MCP}(K,N)$ density h with domain $[0,D']$ and any other D'', defining $g(x) := \frac{D'}{D''}h(\frac{D'x}{D''})$, for each $x \in [0,D'']$, one easily gets that g is an $\mathsf{MCP}(K',N)$ with domain $[0,D'']$ and $K' = K(D'/D'')^2$. Moreover for any $A \subset [0,D']$,

$$\mathsf{P}_g\left(A\frac{D''}{D'}\right) = \frac{D'}{D''}\mathsf{P}_h(A),$$

where P_g is the perimeter with respect to g and P_h the one with respect to h. Assume h is the optimal density and A the optimal set, one gets

$$\widetilde{\mathcal{I}}_{K',N,D''} \leq \frac{D'}{D''}\widetilde{\mathcal{I}}_{K,N,D'}.$$

Hence if $K \leq 0$ and $D'' \geq D'$: $\widetilde{\mathcal{I}}_{K,N,D'} \geq \frac{D''}{D'}\widetilde{\mathcal{I}}_{K,N,D''} \geq \widetilde{\mathcal{I}}_{K,N,D''}$; if $K > 0$ and $D' \geq D''$: $D'\widetilde{\mathcal{I}}_{K,N,D'} \geq D''\widetilde{\mathcal{I}}_{K,N,D''}$. The claim follows. □

We then obtain straightforwardly the next fact.

COROLLARY 2.10. *The one-dimensional isoperimetric profile function* (2.8) *has the following representation:*

$$\mathcal{I}_{K,N,D}(v) = \begin{cases} f_{K,N,D}(a_{K,N,D}(v)) & \text{if } K \leq 0, \\ \inf_{D' \leq D} f_{K,N,D'}(a_{K,N,D'}(v)) & \text{if } K > 0. \end{cases} \tag{2.14}$$

REMARK 2.11. In the case $K > 0$ we expect the map $D \mapsto f_{K,N,D'}(a_{K,N,D'}(v))$ to be strictly convex as some explicit calculations for particular choices of v would suggest. However at the moment we cannot conclude the existence of a unique minimizer $\bar{D} = \bar{D}(K,N,D,v) < D$ representing $\mathcal{I}_{K,N,D}(v)$ in the case $K > 0$. This in turn affects rigidity of the equality case of the isoperimetric inequality in the regime $K > 0$.

2.2.1. One-dimensional rigidity. Building on Corollary 2.2, we prove that the one-dimensional isoperimetric inequality obtained in Theorem 2.7 is rigid.

THEOREM 2.12. *Let $h : [0, D'] \to \mathbb{R}$ be a $\mathsf{MCP}(K,N)$ density which integrates to 1. Assume there exists $v \in (0,1)$ such that $\mathcal{I}_h(v) = \widetilde{\mathcal{I}}_{K,N,D'}(v)$. Then either $h = h_{a_v}$ or $h = h_{a_{1-v}}$.*

PROOF. Assume the existence of a sequence of sets $E_i \subset [0, D']$ so that

$$\int_{E_i} h(x)\,dx = v, \qquad \lim_{i\to\infty} (h\mathcal{L}^1\llcorner_{[0,D']})^+(E_i) = \widetilde{\mathcal{I}}_{K,N,D'}(v).$$

Then one can find a sequence of sets having perimeter with respect to h converging to $\widetilde{\mathcal{I}}_{K,N,D'}(v)$ still with volume v. By lower-semicontinuity we deduce the existence of a set $\cup_{i\in\mathscr{I}}[a_i, b_i]$ of volume v such that

$$\sum_i h(a_i) + h(b_i) = f_{K,N,D'}(a_{K,N,D'}(v)).$$

We then proceed as in the proof of Theorem 2.7.
In the **Case 1.**, $\mathscr{I} = \mathscr{I}_1$, the first chain of inequalities yields that $\cup_{i\in\mathscr{I}}[a_i, b_i] = [a_1, D']$ and strict monotonicity of $A_{K,N,D'}$ implies that $D' - a_1 = a_v$. The second chain of inequalities then implies

$$h(D' - a_v) = f_{K,N,D'}(a_{K,N,D'}(v)) = f_{K,N,D'}(D' - a_{K,N,D'}(v)).$$

Corollary 2.2 yields $h = h_{D'-a_v}$ and the set $\cup_{i\in\mathscr{I}}[a_i, b_i] = [D' - a_v, D']$. Equality in **Case 2.**, $\mathscr{I} = \mathscr{I}_2$, implies, repeating the same argument, that $h = h_{a_v}$ and the set $\cup_{i\in\mathscr{I}}[a_i, b_i] = [0, a_v]$. Equality in **Case 3.** cannot be achieved: the chain of inequality implies that $\cup_{i\in\mathscr{I}}[a_i, b_i] = [a_1, b_1]$ and $a_1 = a_v$ and $b_1 = D' - a_v$; coupled with the chain of inequality implies

$$f_{K,N,D'}(a_v) = h(a_v) + h(D' - a_v) \geq 2f_{K,N,D'}(a_v),$$

giving a contradiction. The same argument implies that also equality in **Case 4.** cannot be achieved. □

Exploiting Lemma 2.9, in the case $K \leq 0$ one can obtain the following stronger rigidity

COROLLARY 2.13. *Let $h : [0, D'] \to \mathbb{R}$ be a $\mathsf{MCP}(K,N)$ density which integrates to 1 with $K \leq 0$. Assume there exists $v \in (0,1)$ such that $\mathcal{I}_h(v) = \mathcal{I}_{K,N,D}(v)$ with $D' \leq D$. Then $D = D'$ and either $h = h_{a_v}$ or $h = h_{a_{1-v}}$.*

PROOF. Lemma 2.9 forces $D' = D$ and then Theorem 2.12 applies. □

To conclude we present another application of one-dimensional rigidity. Since $\mathsf{CD}(K,N) \subset \mathsf{MCP}(K,N)$, we already know that $\widetilde{\mathcal{I}}_{K,N,D}(v) \leq \widetilde{\mathcal{I}}^{\mathsf{CD}}_{K,N,D}(v)$. We can now prove that the inequality is always strict, with the exception of a single case.

Corollary 2.14. *For every choice of K, N and D (with $D < \pi\sqrt{(N-1)/K}$, if $K > 0$), it holds*

$$\widetilde{\mathcal{I}}_{K,N,D}(v) < \mathcal{I}^{\mathsf{CD}}_{K,N,D}(v).$$

In particular, $\mathcal{I}_{K,N,D}(v) < \mathcal{I}^{\mathsf{CD}}_{K,N,D}(v)$.

Proof. Suppose by contradiction the existence of K, N, D, v such that $\widetilde{\mathcal{I}}_{K,N,D}(v) = \mathcal{I}^{\mathsf{CD}}_{K,N,D}(v)$. As proved in [**70**](see Corollary 1.4)

$$\mathcal{I}^{\mathsf{CD}}_{K,N,D}(v) = \widetilde{\mathcal{I}}^{\mathsf{CD}}_{K,N,D}(v),$$

and there exists (see [**70**, Corollary A.3]) a $\mathsf{CD}(K,N)$-density, and therefore an $\mathsf{MCP}(K,N)$-density g defined on $[0,D]$ and integrating to 1 such that $\mathcal{I}_{([0,D],g)}(v) = \mathcal{I}^{\mathsf{CD}}_{K,N,D}(v)$. As observed in the Theorem 2.12, this would force the density g to be exactly h_{a_v} or $h_{a_{1-v}}$ contradicting Lemma 2.4. The final claim simply follows observing that $\inf_{D'\leq D} \widetilde{\mathcal{I}}_{K,N,D'}(v) \leq \mathcal{I}_{K,N,D}(v)$. □

2.3. Isoperimetric inequality

We now deduce Theorem 2.1 from the one-dimensional results of Theorem 2.7 and Lemma 2.9 via localization techniques (Theorem 1.49). Notice that the second part of Theorem 2.1 will then follow by Theorem 2.7.

Theorem 2.15. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching metric measure space with $\mathfrak{m}(X) = 1$ and $\operatorname{diam}(X) \leq D$ (where $D \leq \pi\sqrt{(N-1)/K}$, if $K > 0$). If $(X, \mathsf{d}, \mathfrak{m})$ satisfies $\mathsf{MCP}(K,N)$ for some $K \in \mathbb{R}, N \in [1,\infty)$, then*

$$\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v) \geq \mathcal{I}_{K,N,D}(v), \quad \forall v \in [0,1]$$

where $\mathcal{I}_{K,N,D}$ is explicitly given in (2.14).

Even though the proof is a standard consequence of localization, we present it below for readers' convenience.

Proof. Fix $v \in (0,1)$ and let $A \subset X$ be a Borel set with $\mathfrak{m}(A) = v$. Define the $\mathfrak{m}$-measurable function $f := \chi_A - v$ having zero integral with respect to $\mathfrak{m}$, and study the L^1-Optimal Transport problem from $\mu_0 := f^+\mathfrak{m}$ to $\mu_1 := f^-\mathfrak{m}$, where $f^{\pm}$ denotes the positive and the negative part of f respectively. The associated Kantorovich potential u has $|\nabla u| = 1$ $\mathfrak{m}$-a.e. implying by Theorem 1.49 the existence of a family of unparametrized geodesics $\{X_\alpha\}_{\alpha\in Q}$ (of length at most D) such that $\mathfrak{m}(X \setminus \cup_\alpha X_\alpha) = 0$ and

$$\mathfrak{m} = \int_Q \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha), \qquad \mathfrak{q} - a.e.\ \mathfrak{m}_\alpha(X) = \mathfrak{m}_\alpha(X_\alpha) = 1;$$

moreover $\mathfrak{m}_\alpha = h_\alpha \mathcal{H}^1\llcorner_{X_\alpha}$ and h_α is a $\mathsf{MCP}(K,N)$-density. From the localization of the constraint, it follows that for $\mathfrak{q}$-a.e. $\mathfrak{m}_\alpha(A) = \mathfrak{m}(A) = v$. Hence

$$\begin{aligned}
\mathfrak{m}^+(A) &= \liminf_{\varepsilon\to 0} \frac{\mathfrak{m}(A^\varepsilon) - \mathfrak{m}(A)}{\varepsilon} \\
&\geq \liminf_{\varepsilon\to 0} \int_Q \frac{\mathfrak{m}_\alpha((A\cap X_\alpha)^\varepsilon) - \mathfrak{m}_\alpha(A)}{\varepsilon}\, \mathfrak{q}(d\alpha), \\
&\geq \int_Q \mathfrak{m}^+_\alpha(A\cap X_\alpha)\, \mathfrak{q}(d\alpha) \\
&\geq \int_Q \mathcal{I}_{K,N,D}(v)\, \mathfrak{q}(d\alpha) \\
&= \mathcal{I}_{K,N,D}(v).
\end{aligned}$$

□

In the case $K \leq 0$, one-dimensional rigidity (Theorem 2.12) implies the following measure rigidity.

THEOREM 2.16. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an essentially non-branching metric measure space satisfying* $\mathsf{MCP}(K, N)$ *for* $K \leq 0$, $N \in [1, \infty)$ *with* $\mathfrak{m}(X) = 1$ *and* $\mathrm{diam}\,(X) \leq D$.

If there exists $v \in (0,1)$ *such that* $\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v) = \mathcal{I}_{K,N,D}(v)$, *then* $\mathrm{diam}\,(X) = D$, *there exist a measure space* $(Q, \mathfrak{q})$ *and a measurable isomorphism between* $(0, D) \times Q$ *and* $X' \subset X$ *with* $\mathfrak{m}(X') = 1$.

Moreover, the measure $\mathfrak{m}$ *admits the following representation*

$$\mathfrak{m} = \int_Q h_\alpha \, \mathcal{H}^1 \llcorner_{X_\alpha} \mathfrak{q}(d\alpha),$$

and $\mathfrak{q}$*-a.e.,* $h_\alpha = h_{a_{K,N,D}(v)}$ *or* $h_\alpha = h_{a_{K,N,D}(1-v)}$.

PROOF. We will prove that X has diameter D. Arguing by contradiction, let us suppose that there exists $\varepsilon > 0$ such that $\mathrm{diam}\,(X) = D - \varepsilon$. From (2.14), $K \leq 0$ and Lemma 2.9, for any $v \in (0,1)$ the function $\mathcal{I}_{K,N,D}(v)$ is strictly decreasing in D. Hence, there exists $\eta > 0$ such that

$$\mathcal{I}_{K,N,D'}(v) \geq \mathcal{I}_{K,N,D}(v) + \eta, \quad \forall D' \in (0, D - \varepsilon].$$

Let $A \subset X$ be such that $\mathfrak{m}(A) = v$ and $\mathfrak{m}^+(A) \leq \mathcal{I}_{K,N,D}(v) + \eta/2$. Arguing as in the proof of Theorem 2.15, we get that

$$\begin{aligned}
\mathcal{I}_{K,N,D}(v) + \eta/2 \geq \mathfrak{m}^+(A) &\geq \int_Q \mathfrak{m}^+_\alpha(A \cap X_\alpha)\, \mathfrak{q}(d\alpha) \\
&\geq \int_Q \widetilde{\mathcal{I}}_{K,N,|\mathrm{supp}\, h_\alpha|}(v)\, \mathfrak{q}(d\alpha) \\
&\geq \mathcal{I}_{K,N,D}(v) + \eta
\end{aligned}$$

where the last inequality is due to the fact that $\mathrm{supp}(h_\alpha)$ is isometric to a geodesic X_α of (X, d) and hence $|\mathrm{supp}\, h_\alpha| \leq D - \varepsilon$ and from $K \leq 0$ together with Lemma 2.9. Thus the contradiction is obtained.

The same argument implies that $|\mathrm{supp}(h_\alpha)| = D$ for $\mathfrak{q}$-a.e. α and

$$\mathcal{I}_{K,N,D}(v) = \widetilde{\mathcal{I}}_{K,N,|\mathrm{supp}\, h_q|}(v);$$

the claim follows from the one-dimensional rigidity obtained in Corollary 2.13. □

In the case $K > 0$ (even in the maximal diameter case $D = \pi\sqrt{(N-1)/K}$) rigidity properties are still not clear. As discussed in Remark 2.11, a first step in this direction would be the identification of the possibly unique optimal 1-dimensional diameter realizing the infimum in the definition of $\mathcal{I}_{K,N,D}(v)$ for $K > 0$.

CHAPTER 3

Independence of synthetic Curvature Dimension conditions on transport distance exponent

As we had the chance to see, the Curvature-Dimension condition $\mathsf{CD}(K,N)$ is formulated in terms of a modified displacement convexity of an entropy functional along W_2-Wasserstein geodesics. In this chapter we will show that the choice of the squared-distance function as transport cost does not influence the theory. In particular, denoting with $\mathsf{CD}_p(K,N)$ the analogous condition obtained choosing as transport cost the distance function raised to the power $p > 1$, we will show that all $\mathsf{CD}_p(K,N)$ are equivalent conditions provided suitable restrictions are placed on the ambient space. Following [**35**], the trait-d'union between all the seemingly unrelated $\mathsf{CD}_p(K,N)$ conditions will be again the L^1-optimal transport problem. As a consequence, also the local-to-global property of $\mathsf{CD}_p(K,N)$ will be established. We state now the main results of the Chapter:

THEOREM 3.1 (Theorem 3.44). *Let $(X, \mathsf{d}, \mathfrak{m})$ be such that $\mathfrak{m}(X) = 1$. Assume it is p-essentially non-branching and verifies $\mathsf{CD}_p(K,N)$ for some $p > 1$. If $(X, \mathsf{d}, \mathfrak{m})$ is also q-essentially non-branching for some $q > 1$, then it verifies $\mathsf{CD}_q(K,N)$.*

As we will extend the strategy used in [**35**] to powers other than $p = 2$, also the local-to-global property will be established for the $\mathsf{CD}_p(K,N)$.

COROLLARY 3.1 (Corollary 3.45). *Fix any $p > 1$ and $K, N \in \mathbb{R}$ with $N > 1$. Let $(X, \mathsf{d}, \mathfrak{m})$ be a p-essentially non-branching metric measure space verifying $\mathsf{CD}_{p,loc}(K,N)$ from Definition 1.36 and such that (X, d) is a length space with* $\mathrm{supp}(\mathfrak{m}) = X$ *and $\mathfrak{m}(X) = 1$. Then $(X, \mathsf{d}, \mathfrak{m})$ verifies $\mathsf{CD}_p(K,N)$.*

We now briefly describe the structure of the chapter, providing a general outline of the approach we will use. In Section 3.1 we fix the notation and the terminology we will use for the rest of the chapter. Section 3.2 is devoted to a careful analysis of Kantorovich potentials and their evolution via the Hopf-Lax semigroup with a general exponent $p > 1$. The aim is to obtain information about the time derivative of the t-propagated s-Kantorovich potential Φ_s^t as defined in Section 3.2.5. This quantity is crucial for the Jacobian factor that appears when comparing interpolant measures, μ_t, between measures μ_0 and μ_1 along a transport geodesic at two times. To achieve this goal, Sections 3.2.1 - 3.2.3 are dedicated to a detailed study of the regularity properties of the Hopf-Lax transform. In particular we establish second order regularity for the Hopf-Lax transform of a Kantorovich potential as well as a few identities related to the positional information stored in a Kantorovich potential. From here, Section 3.2.4 demonstrates, through a delicate argument, third order temporal regularity of time propagated Kantorovich potentials along transport geodesics, leading to the fundamental Theorem 3.26.

In Section 3.3 we show that a local version of $\mathsf{CD}_p(K,N)$ implies $\mathsf{CD}^1(K,N)$ in the version reported in Theorem 3.31; this is done passing curvature information from the total space down to the L^1-transport rays.

Finally, in Section 3.4 we obtain a complete equivalence of all $\mathsf{CD}_p(K,N)$ (Theorem 3.44) and each of them also enjoys the local-to-global property (Corollary 3.45). The first step in this direction is to transfer the curvature properties along transport geodesics back to the total space through q-Wasserstein geodesics and hence proving that an enhanced version of $\mathsf{CD}^1_{Lip}(K,N)$ implies $\mathsf{CD}_q(K,N)$. This will be done by proving, in the terminology of [**35**], an "LY"-decomposition for the densities ρ_t of the q-Wasserstein geodesic μ_t (see Theorem 3.43). More precisely, this "LY"-decomposition provides a factorization of the ratio ρ_t/ρ_s into two factors: the first one — denoted by L — is a concave function taking into account only the one dimensional distortion due to the volume stretching in the direction of the geodesic. The second factor is denoted by Y and contains the volume distortion in the transversal directions. To achieve this goal we first use the Disintegration theorem from Section 1.4 to represent $\mathfrak{m}$ as an average of measures that live on L^1-transport geodesics for the signed distance to any given level set of a p-Kantorovich potential. In this disintegration of $\mathfrak{m}$ we follow the evolution of a specific collection of Kantorovich geodesics. More specifically, we fix $a \in \mathbb{R}$ and $s \in (0,1)$, and consider q-Kantorovich geodesics γ which satisfy $\varphi_s(\gamma_s) = a$, where φ_s is the evolved Kantorovich potential for the q-Wasserstein geodesic. We denote such geodesics by $G_{a,s}$ and we disintegrate $\mathfrak{m}$ over $\{\gamma_t \colon \gamma \in G_{a,s}\}_{t\in[0,1]}$ to obtain

$$\mathfrak{m}\llcorner_{\mathrm{e}_{[0,1]}(G_{a,s})} = \int_{[0,1]} \mathfrak{m}^{a,s}_t \, \mathcal{L}^1(dt),$$

Then we compare this to a disintegration of $\mathfrak{m}$ over $\{\varphi_s^{-1}(a)\}_{a\in\mathbb{R}}$ on the time t evaluation of a sufficiently large set of Kantorovich geodesics denoted by G. Specifically, we obtain

$$\mathfrak{m}\llcorner_{\mathrm{e}_t(G)} = \int_{\varphi_s(\mathrm{e}_s(G))} \mathfrak{m}^t_{a,s} \mathcal{L}^1(da)$$

This leads to two measures, $\mathfrak{m}^{a,s}_t$ and $\mathfrak{m}^t_{a,s}$, that live on $\mathrm{e}_t(G_{a,s})$. In Section 3.4.3 we compare these two disintegrations to deduce that $\mathfrak{m}^{a,s}_t$ and $\mathfrak{m}^t_{a,s}$ differ only by $\partial_t \Phi^t_s$. This information is used in Section 3.4.4 to deduce the Jacobian factor between $\rho_t(\gamma_t)$ and $\rho_s(\gamma_s)$; the latter step allows us to conclude the desired "LY" decomposition. Once the "LY" decomposition is at our disposal, we can invoke [**35**] to conclude that the space satisfies $\mathsf{CD}_q(K,N)$.

3.1. Preliminaries and notations

In order to carry out a third order analysis of Kantorovich potentials, we will frequently use incremental ratios over arbitrary subsets of $\mathbb{R}$; in this section we will fix notation and terminology needed to do so.

3.1.1. Derivatives. For a function $g : A \to \mathbb{R}$ defined on a subset $A \subset \mathbb{R}$, we denote its upper and lower derivatives at a point $t_0 \in A$ which is an accumulation point of A by:

$$\frac{\overline{d}}{dt} g(t_0) = \limsup_{A \ni t \to t_0} \frac{g(t) - g(t_0)}{t - t_0} \ , \ \frac{d}{\underline{dt}} g(t_0) = \liminf_{A \ni t \to t_0} \frac{g(t) - g(t_0)}{t - t_0}.$$

We will say that g is differentiable at t_0 iff $\frac{d}{dt} g(t_0) := \frac{\overline{d}}{dt} g(t_0) = \frac{d}{\underline{dt}} g(t_0) \in \mathbb{R}$. This is a slightly more general definition of differentiability than the traditional one which requires t_0 to be an interior point of A.

Remark 3.1. Note that there are only a countable number of isolated points in A, so a.e. point in A is an accumulation point. In addition, it is clear that if $t_0 \in B \subset A$ is an accumulation point of B and g is differentiable at t_0, then $g|_B$ is also differentiable at t_0 with the same derivative. In particular, if g is a.e. differentiable on A then $g|_B$ is also a.e. differentiable on B and the derivatives coincide.

Remark 3.2. Denote by $A_1 \subset A$ the subset of density one points of A (which are in particular accumulation points of A). By Lebesgue's Density Theorem $\mathcal{L}^1(A \setminus A_1) = 0$. If

$g : A \to \mathbb{R}$ is locally Lipschitz, consider any locally Lipschitz extension $\hat{g} : \mathbb{R} \to \mathbb{R}$ of g. Then it is easy to check that for $t_0 \in A_1$, g is differentiable in the above sense at t_0 if and only if $\hat{g}$ is differentiable at t_0 in the usual sense, in which case the derivatives coincide. In particular, as $\hat{g}$ is a.e. differentiable on $\mathbb{R}$, it follows that g is a.e. differentiable on A_1 and hence on A, and it holds that $\frac{d}{dt} g = \frac{d}{dt} \hat{g}$ a.e. on A.

If $f : I \to \mathbb{R}$ is a convex function on an open interval $I \subset \mathbb{R}$, it is a well-known fact that the left and right derivatives $f^{\prime,-}$ and $f^{\prime,+}$ exist at every point in I and that f is locally Lipschitz. In particular, f is differentiable at a given point if and only if the left and right derivatives coincide there. Denoting by $D \subset I$ the differentiability points of f in I, it is also well-known that $I \setminus D$ is at most countable. Consequently, any point in D is an accumulation point, and we may consider the differentiability in D of $f' : D \to \mathbb{R}$ as defined above.

We will recall the following classical one-dimensional result about twice differentiability a.e. of convex functions on $\mathbb{R}^n$. The result extends to locally semi-convex and semi-concave functions as well; recall that a function $f : I \to \mathbb{R}$ is called semi-convex (semi-concave) if there exists $C \in \mathbb{R}$ so that $I \ni x \mapsto f(x) + Cx^2$ is convex (concave).

Lemma 3.3 (Second Order Differentiability of Convex Function). *Let $f : I \to \mathbb{R}$ be a convex function on an open interval $I \subset \mathbb{R}$, and let $\tau_0 \in I$ and $\Delta \in \mathbb{R}$. Then the following statements are equivalent:*

(1) f is differentiable at τ_0, and if $D \subset I$ denotes the subset of differentiability points of f in I, then $f' : D \to \mathbb{R}$ is differentiable at τ_0 with:

$$(f')'(\tau_0) := \lim_{D \ni \tau \to \tau_0} \frac{f'(\tau) - f'(\tau_0)}{\tau - \tau_0} = \Delta.$$

(2) The right derivative $f^{\prime,+} : I \to \mathbb{R}$ is differentiable at τ_0 with $(f^{\prime,+})'(\tau_0) = \Delta$.
(3) The left derivative $f^{\prime,-} : I \to \mathbb{R}$ is differentiable at τ_0 with $(f^{\prime,-})'(\tau_0) = \Delta$.
(4) f is differentiable at τ_0 and has the following second order expansion there:

$$f(\tau_0 + \varepsilon) = f(\tau_0) + f'(\tau_0)\varepsilon + \Delta \frac{\varepsilon^2}{2} + o(\varepsilon^2) \text{ as } \varepsilon \to 0.$$

In this case, f is said to have a second Peano derivative at τ_0.

For a locally semi-convex or semi-concave function f, we will say that f is twice differentiable at τ_0 if any (all) of the above equivalent conditions hold for some $\Delta \in \mathbb{R}$, and we will write $(\frac{d}{d\tau})^2|_{\tau=\tau_0} f(\tau) = \Delta$.
Finally, we recall the following slightly different version of the second order differential.

Definition 3.4 (Upper and lower second Peano derivatives). Given an open interval $I \subset \mathbb{R}$ and a function $f : I \to \mathbb{R}$ which is differentiable at $\tau_0 \in I$, we define its upper and lower second Peano derivatives at τ_0, denoted $\overline{\mathcal{P}}_2 f(\tau_0)$ and $\underline{\mathcal{P}}_2 f(\tau_0)$ respectively, by:

$$\overline{\mathcal{P}}_2 f(\tau_0) := \limsup_{\varepsilon \to 0} \frac{h(\varepsilon)}{\varepsilon^2} \geq \liminf_{\varepsilon \to 0} \frac{h(\varepsilon)}{\varepsilon^2} =: \underline{\mathcal{P}}_2 f(\tau_0), \tag{3.1}$$

where:

$$h(\varepsilon) := 2(f(\tau_0 + \varepsilon) - f(\tau_0) - \varepsilon f'(\tau_0)). \tag{3.2}$$

We say that f has a second Peano derivative at τ_0 iff $\overline{\mathcal{P}}_2 f(\tau_0) = \underline{\mathcal{P}}_2 f(\tau_0) \in \mathbb{R}$.

Lemma 3.5. *Given an open interval $I \subset \mathbb{R}$ and a locally absolutely continuous function $f : I \to \mathbb{R}$ which is differentiable at $\tau_0 \in I$, we have:*

$$\underline{\frac{d}{dt}} f'(\tau_0) \leq \underline{\mathcal{P}}_2 f(\tau_0) \leq \overline{\mathcal{P}}_2 f(\tau_0) \leq \frac{\overline{d}}{dt} f'(\tau_0).$$

3.1.2. Notations. Given a subset $D \subset X \times \mathbb{R}$, we denote its sections by:

$$D(t) := \{x \in X \; ; \; (x,t) \in D\} \;\; , \;\; D(x) := \{t \in \mathbb{R} \; ; \; (x,t) \in D\} \, .$$

Given a subset $G \subset \mathrm{Geo}(X)$, we denote by $\mathring{G} := \left\{\gamma|_{(0,1)} \; ; \; \gamma \in G\right\}$ the corresponding open-ended geodesics on $(0,1)$. For a subset of (closed or open) geodesics $\tilde{G}$, we denote:

$$Im(\tilde{G}) := \left\{(x,t) \in X \times \mathbb{R} \; ; \; \exists \gamma \in \tilde{G} \; , \; t \in \mathrm{Dom}(\gamma) \; , \; x = \gamma_t\right\} . \tag{3.3}$$

3.2. Hopf-Lax transform with exponent p

In this section we review the basic properties of the Hopf-Lax transform in a metric measure space setting with a general exponent $p > 1$. Some of following properties are well-known for the case $p = 2$; we will omit the proofs for general p whenever they follow the same line of reasoning as the corresponding one for $p = 2$. The main references for most of the definitions and proofs will be [**7, 8, 35**]; further developments related to ours may also be found in [**5**] [**56**] [**52**] [**18**] and their references.

Remark 3.6. As motivation for the needed properties of the metric measure space Hopf-Lax transform we remind the reader of the relationship between the Hopf-Lax transform and the Eulerian point of view of optimal transport. We also provide a comparison between the results found in this chapter to those familiar from Euclidean space.

We illustrate the main relationship for the case $(\mathbb{R}^n, d)$ with d the Euclidean distance, and the cost function $c(x,y) = \frac{d(x,y)^p}{p}$ where $p > 1$. Recall that, as shown in [**9**, Theorem 8.3.1], in the Eulerian view of optimal transport the Wasserstein distance can be interpreted as the minimizing energy related to the problem

$$\begin{cases} \rho_t + \nabla \cdot (\rho v) = 0 & \text{in } \mathbb{R}^n \times (0,1) \\ \rho(\cdot, 0) = \rho_0 & \text{in } \mathbb{R}^n \\ \rho(\cdot, 1) = \rho_1 & \text{in } \mathbb{R}^n \end{cases} \tag{3.4}$$

where ρ, v are the distribution of mass and the velocity at position x at time t respectively [**9**, Theorem 8.3.1]. By choosing $v = DH(\nabla \varphi)$, where in our case $H(w) = |w|^{p'}/p'$ and φ is a solution to the Hamilton-Jacobi equation

$$\begin{cases} \partial_t \varphi + H(\nabla \varphi) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ \varphi(x,0) = \varphi_0(x) & \text{for } x \in \mathbb{R}^n \end{cases} \tag{3.5}$$

where φ_0 is a Kantorovich potential for the optimal transport problem and p' is the real number satisfying $\frac{1}{p} + \frac{1}{p'} = 1$. That is, p' is the Hölder dual of p. The method of characteristics gives a solution to the Hamilton-Jacobi equation for a convex Hamiltonian H [**47**]; furthermore, this solution can be expressed by the Hopf-Lax formula

$$\varphi(x,t) = \inf_{y \in \mathbb{R}^n} \left\{ \varphi_0(y) + tL\left(\frac{x-y}{t}\right) \right\}$$

where the Lagrangian L is defined by

$$L(z) = \inf_{w \in \mathbb{R}^n} \left\{ z \cdot w - H(w) \right\} .$$

In our case, the Lagrangian is explicitly computed as $L(v) = \frac{|v|^p}{p}$, hence

$$\varphi(x,t) = \inf_{y \in \mathbb{R}^n} \left\{ \varphi_0(y) + \frac{|x-y|^p}{p t^{p-1}} \right\} . \tag{3.6}$$

Finally, it turns out that, at least in the context of smooth manifolds, the spatial gradient of φ behaves as one would expect. Specifically, it turns out that

$$\nabla\varphi(x) = \frac{|x-y|^{p-2}(x-y)}{t^{p-1}},$$

where y is chosen to be a minimizer in the Hopf-Lax infimum (3.6). Hence,

$$\frac{|\nabla\varphi(x,t)|^{p'}}{p'} = \frac{(p-1)|x-y|^p}{pt^p} \tag{3.7}$$

Note that, due to (3.7), (3.5) and (3.6) can be compared to conclusion 3 of Theorem 3.9 and Corollary 3.17 respectively. In particular, notice that the expression found in (3.7) depends on x only through its distance to the minimizing value y. This should be compared to Definition 3.8. With the above in mind we are now ready to present the details of the nonsmooth case.

In the following sections, we will only consider the cost function $c = \mathsf{d}^p/p$ on $X \times X$. Hence, in this case the c-transform of a function $\psi : X \to \mathbb{R} \cup \{\pm\infty\}$ is defined as the following (upper semi-continuous) function:

$$\psi^c(x) = \inf_{y\in X} \frac{\mathsf{d}(x,y)^p}{p} - \psi(y).$$

In these sections, we only assume that (X, d) is a **proper geodesic metric space**. (Here proper refers to the requirement that closed balls are compact.)

3.2.1. General definitions.

Definition 3.7 (Hopf-Lax transform). Let $f : X \to \mathbb{R} \cup \{\pm\infty\}$ be not identically $+\infty$ and $t > 0$, $p > 1$. The Hopf-Lax transform $Q_t f : X \to \mathbb{R} \cup \{-\infty\}$ is defined in the following way:

$$Q_t f(x) := \inf_{y\in X} \frac{\mathsf{d}(x,y)^p}{pt^{p-1}} + f(y). \tag{3.8}$$

Trivially either $Q_t f \equiv -\infty$ or $Q_t f(x)$ is finite for all $x \in X$. Indeed, if $Q_t f(\bar{x}) \in \mathbb{R}$ for some $\bar{x} \in X$ and $t > 0$, then $Q_s f(x) \in \mathbb{R}$ for all $x \in X$ and $0 < s \leq t$. Hence, defining

$$t_*(f) := \sup\{t > 0 : Q_t f \not\equiv -\infty\},$$

where we set $t_*(f) = 0$ if the supremum is over an empty set, it holds that $Q_t f(x) \in \mathbb{R}$ for every $x \in X, t \in (0, t_*(f))$. Moreover, we set $Q_0 f := f$. The definition of $Q_t f$ can be extended to negative times $t < 0$ by setting:

$$Q_t f(x) = -Q_{-t}(-f)(x) = \sup_{y\in X} -\frac{\mathsf{d}(x,y)^p}{p(-t)^{p-1}} + f(y), \quad t < 0. \tag{3.9}$$

If (X, d) is a length space (and in particular, if it is geodesic), the Hopf-Lax transform is in fact a semi-group on $[0, \infty)$:

$$Q_{s+t} f = Q_s \circ Q_t f \quad \forall t, s \geq 0.$$

Being the infimum of continuous functions in (t, x), the map $(0,\infty) \times X \ni (t,x) \mapsto Q_t f(x)$ is upper semi-continuous. Moreover, by definition $[0,\infty) \ni t \mapsto Q_t f(x)$ is monotone non-increasing; hence, it is continuous from the left.

We now define the *distance progressed* as the length of the geodesic segment in X along which information propagates from the initial values to (t, x); this geodesic plays the role of a characteristic curve. Since we are modeling optimal transport, shocks do not form before unit time has elapsed [**98**].

DEFINITION 3.8. (Distance progressed $D_f^{\pm}$). Given $f : X \to \mathbb{R} \cup \{+\infty\}$ not identically $+\infty$, we define

$$D_f^+(x,t) := \sup \limsup_{n\to+\infty} \mathsf{d}(x,y_n) \geq \inf \liminf_{n\to+\infty} \mathsf{d}(x,y_n) =: D_f^-(x,t)$$

where the supremum and the infimum are taken on the set of minimizing sequences $\{y_n\}_{n\in\mathbb{N}}$ in the definition of Hopf-Lax transform. Using a diagonal argument, it is possible to show that the supremum and infimum are attained, though they may differ in the presence of shocks.

For $p = 2$, the following properties were established in [**7**, Chapter 3]. For a proof adopted to a similar framework we refer to [**35**, Section 3.2].

THEOREM 3.9 (Hopf-Lax solution to metric space Hamilton-Jacobi equations). *For any metric space* (X, d) *the following properties hold:*

1. *Both functions* $D_f^{\pm}(x,t)$ *are locally finite on* $X \times (0, t_*(f))$ *and* $(x,t) \mapsto Q_t f(x)$ *is locally Lipschitz there.*
2. *The map* $(x,t) \mapsto D_f^+(x,t)$ $\big((x,t) \mapsto D_f^-(x,t)\big)$ *is upper (lower) semi-continuous on* $X \times (0, t_*(f))$.
3. *For every* $x \in X$,

$$\partial_t^{\pm} Q_t f(x) = -\frac{(p-1)D_f^{\pm}(x,t)^p}{pt^p}, \qquad \forall\ t \in (0, t_*(f)),$$

where ∂_t^- *and* ∂_t^+ *denote the left and right partial derivatives respectively. In particular, the map* $(0, t_*(f)) \ni t \mapsto Q_t f(x)$ *is locally Lipschitz and locally semi-concave. Moreover, it is differentiable at* $t \in (0, t_*(f))$ *if and only if* $D_f^+(x,t) = D_f^-(x,t)$.

The next property will be used several times throughout the chapter; we include a proof for the readers' convenience.

LEMMA 3.10 (Hopf-Lax attainment). *Let* X *be a proper metric space,* $f : X \to \mathbb{R}$ *a lower semi-continuous function, and* $t_*(f) > 0$. *For fixed* $x \in X$ *and* $t \in (0, t_*(f))$, *there exist* $y_t^{\pm} \in X$ *so that*

$$Q_t f(x) = \frac{\mathsf{d}(x, y_t^{\pm})^p}{pt^{p-1}} + f(y_t^{\pm}). \tag{3.10}$$

Moreover, the following holds: $\mathsf{d}(x, y_t^{\pm}) = D_f^{\pm}(x,t)$.

PROOF. Let $\{y_t^{\pm,n}\}$ be a minimizing sequence such that

$$Q_t f(x) = \lim_{n\to\infty} \frac{\mathsf{d}(x, y_t^{\pm,n})^p}{pt^{p-1}} + f(y_t^{\pm,n}) \text{ and } D_f^{\pm}(x,t) = \lim_{n\to\infty} \mathsf{d}(x, y_t^{\pm,n})$$

By local finiteness of $D_f^{\pm}$, it follows that $D_f^{\pm}(x,t) < R$ for some $R < \infty$. The properness of the space X guarantees that the closed geodesic ball $B_R(x)$ is compact, hence $\{y_t^{\pm,n}\}$ admits a subsequence converging to $\{y_t^{\pm}\}$. Using the lower semi-continuity of f, we get:

$$Q_t f(x) = \inf_{y\in X} \frac{\mathsf{d}(x,y)^p}{pt^{p-1}} + f(y) = \min_{y\in B_R(x)} \frac{\mathsf{d}(x,y)^p}{pt^{p-1}} + f(y) = \frac{\mathsf{d}(x,y_t^{\pm})^p}{pt^{p-1}} + f(y_t^{\pm}).$$

Hence, the claim holds true. □

LEMMA 3.11 (Time monotonicity of distance progressed). *Let* X *be a proper metric space and let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a lower semi-continuous function. Then, for every* $x \in X$, *both functions* $(0, t^*(f)) \ni t \mapsto D_f^{\pm}(x,t)$ *are monotone non-decreasing and coincide except where they have jump discontinuities.*

PROOF. Since trivially $D_f^- \le D_f^+$, it is sufficient to prove that

$$D_f^+(x,s) \le D_f^-(x,t), \qquad 0 < s < t < t^*(f)$$

in order to conclude. By Lemma 3.10, there exist y_s^+, y_t^- such that

$$\frac{\mathsf{d}(x,y_s^+)^p}{ps^{p-1}} + f(y_s^+) = Q_s(f)(x) \le \frac{\mathsf{d}(x,y_t^-)^p}{ps^{p-1}} + f(y_t^-),$$
$$\frac{\mathsf{d}(x,y_t^-)^p}{pt^{p-1}} + f(y_t^-) = Q_t(f)(x) \le \frac{\mathsf{d}(x,y_s^+)^p}{pt^{p-1}} + f(y_s^+).$$

Summing the two, we get

$$\mathsf{d}(x,y_s^+)^p \cdot \left(\frac{1}{s^{p-1}} - \frac{1}{t^{p-1}}\right) \le \mathsf{d}(x,y_t^-)^p \cdot \left(\frac{1}{s^{p-1}} - \frac{1}{t^{p-1}}\right).$$

Since the Lemma 3.10 also guarantees that $\mathsf{d}(x,y_t^-) = D_f^-(x,t)$ and $\mathsf{d}(x,y_s^+) = D_f^+(x,s)$, the claim follows. □

3.2.2. Intermediate-time Kantorovich potentials.

DEFINITION 3.12. (Interpolating Intermediate-Time Kantorovich Potentials). Given a Kantorovich potential $\varphi : X \to \mathbb{R}$, the interpolating p-Kantorovich potential at time $t \in [0,1]$, $\varphi_t : X \to \mathbb{R}$ is defined for all $t \in [0,1]$ by:

$$\varphi_t(x) := Q_{-t}(\varphi) = -Q_t(-\varphi). \tag{3.11}$$

Note that $\varphi_0 = \varphi$, $\varphi_1 = -\varphi^c$, and:

$$-\varphi_t(x) = \inf_{y \in X} \frac{\mathsf{d}^p(x,y)}{pt^{p-1}} - \varphi(y) \quad \forall t \in (0,1].$$

Applying the previous general properties of the Hopf-Lax semi-group we directly obtain that

1. $(x,t) \mapsto \varphi_t(x)$ is lower semi-continuous on $X \times (0,1]$ and continuous on $X \times (0,1)$,
2. For every $x \in X$, $[0,1] \ni t \mapsto \varphi_t(x)$ is monotone non-decreasing and continuous on $(0,1]$.

We also recall the following terminology: given a Kantorovich potential $\varphi : X \to \mathbb{R}$, $\gamma \in \mathrm{Geo}(X)$ is called a (φ,p)-*Kantorovich geodesic* if

$$\varphi(\gamma_0) + \varphi^c(\gamma_1) = \frac{\mathsf{d}(\gamma_0,\gamma_1)^p}{p} = \frac{\ell(\gamma)^p}{p}. \tag{3.12}$$

The set of all Kantorovich geodesics will be denoted with G_φ; the upper semi-continuity of φ and φ^c implies that G_φ is a closed subset of $\mathrm{Geo}(X)$.

Using the modified triangular inequality

$$\mathsf{d}(x,y)^p \le \frac{\mathsf{d}(x,z)^p}{t^{p-1}} + \frac{\mathsf{d}(z,y)^p}{(1-t)^{p-1}}, \tag{3.13}$$

valid for every choice of $x,y,z \in X$, we may conclude that along (φ,p)-Kantorovich geodesics, φ_t is affine in time, and it verifies the following expression:

$$\varphi_t(\gamma_t) = (1-t)\frac{\mathsf{d}(\gamma_0,\gamma_1)^p}{p} - \varphi^c(\gamma_1). \tag{3.14}$$

This result easily implies the following corollary.

COROLLARY 3.13. *Let γ be a (φ,p)-Kantorovich geodesic. Then, for any $s,r\in(0,1)$, we have:*

$$\varphi_s(\gamma_s)-\varphi_r(\gamma_r)=(r-s)\frac{\mathsf{d}(\gamma_0,\gamma_1)^p}{p}. \tag{3.15}$$

LEMMA 3.14. *Let x,y,z be points in X and let $t\in(0,1)$. If*

$$\frac{\mathsf{d}(x,y)^p}{pt^{p-1}}-\varphi(y)=\varphi^c(z)-\frac{\mathsf{d}(x,z)^p}{p(1-t)^{p-1}}, \tag{3.16}$$

then x is a t-intermediate point between y and z with

$$\mathsf{d}(y,z)=\frac{\mathsf{d}(x,y)}{t}=\frac{\mathsf{d}(x,z)}{1-t}. \tag{3.17}$$

Moreover there exists a (φ,p)-Kantorovich geodesic $\gamma:[0,1]\to X$ with $\gamma_0=y$, $\gamma_t=x,\gamma_1=z$.

PROOF. By definition of c-transform, from the assumption (3.16) it follows that

$$\frac{\mathsf{d}(x,y)^p}{pt^{p-1}}+\frac{\mathsf{d}(x,z)^p}{p(1-t)^{p-1}}=\varphi(y)+\varphi^c(z)\le\frac{\mathsf{d}(y,z)^p}{p}.$$

Hence, the equality holds since the reverse inequality is trivially satisfied by (3.13). In particular, requiring the equality in the Hölder inequality implies that

$$\frac{\mathsf{d}(x,z)^p}{(1-t)^p}=\mathsf{d}(y,z)^p=\frac{\mathsf{d}(x,y)^p}{t^p}. \tag{3.18}$$

So the concatenation $\gamma:[0,1]\to X$ of any constant speed geodesic $\gamma^1:[0,t]\to X$ between x and y with any constant speed geodesic $\gamma^2:[t,1]\to X$ between y and z so that $\gamma_0=y$, $\gamma_t=x$, $\gamma_1=z$ must be a constant speed geodesic itself by the triangle inequality. In particular also

$$\varphi(y)+\varphi^c(z)\le\frac{\mathsf{d}(y,z)^p}{p}.$$

must hold as equality, impling γ to be a (φ,p)-Kantorovich geodesic. □

In what follows, forward and backward evolution via the Hopf-Lax semi-group will permit us to obtain regularity properties and key estimates on the intermediate-time Kantorovich potential. However, it is immediate to show by inspecting the definitions that we always have

$$Q_{-s}\circ Q_sf\le f\text{ on }X\quad\forall s>0;$$

note that for $f=-\varphi$ where φ is a Kantorovich potential, we do have equality for $s=1$, and in fact for all $s\in[0,1]$; for $f=Q_t(-\varphi)$, $t\in(0,1)$ and $s=1-t$, we can only assert an *inequality*

$$(\varphi^c)_{1-t}=Q_{-(1-t)}\circ Q_1(-\varphi)\le Q_t(-\varphi)=-\varphi_t\text{ on }X, \tag{3.19}$$

and equality need not hold at every point of X.

DEFINITION (Time-Reversed Interpolating Potential). Given a Kantorovich potential $\varphi:X\to\mathbb{R}$, define the time-reversed interpolating Kantorovich potential at time $t\in[0,1]$, $\bar\varphi_t:X\to\mathbb{R}$, as:

$$\bar\varphi_t:=-(\varphi^c)_{1-t}=Q_{1-t}(-\varphi^c)=-Q_{-(1-t)}\circ Q_{1-t}(-\varphi_t).$$

Note that $\bar\varphi_0=\varphi$, $\bar\varphi_1=-\varphi^c$, and:

$$\bar\varphi_t(x)=\inf_{y\in X}\frac{\mathsf{d}^p(x,y)}{p(1-t)^{p-1}}-\varphi^c(y)\quad\forall t\in[0,1).$$

Note that, since any Kantorovich potential φ is upper semi-continuous, Lemma 3.10 applies to $f=-\varphi$.

LEMMA 3.15 (Relating forward to reverse evolution of potentials). *The following properties hold true:*

(1) $\varphi_0 = \bar\varphi_0 = \varphi$ *and* $\varphi_1 = \bar\varphi_1 = -\varphi^c$*;*
(2) For all $t \in [0,1]$*,* $\varphi_t \le \bar\varphi_t$*;*
(3) For any $t \in (0,1)$*,* $\varphi_t(x) = \bar\varphi_t(x)$ *if and only if* $x \in \mathrm{e}_t(G_\varphi)$*.*

PROOF. Point *1.* is a trivial consequence of the definitions. Also *2.* is straightforward, since

$$\bar\varphi_t := Q_{1-t}(-\varphi^c) = -Q_{-(1-t)} \circ Q_{1-t}(-\varphi_t) \ge \varphi_t.$$

To demonstrate *3.*, let us consider a point $x = \gamma_t$ with $\gamma \in G_\varphi$ and use the following notation $\ell(\gamma) = \mathsf{d}(\gamma_0, \gamma_1)$ for length. Applying Corollary 3.13 to γ with $s = 0$ and $r = t$ we get

$$\varphi(\gamma_0) - \varphi_t(\gamma_t) = t\frac{\ell(\gamma)^p}{p},$$

while applying the same result to $\gamma^c \in G_{\varphi^c}$, the time reversed curve, with $s = 1$, and $r = (1-t)$ we obtain

$$\begin{aligned} -\varphi(\gamma_0) - \varphi^c_{1-t}(\gamma_t) &= (\varphi^c)_1(\gamma^c_1) - (\varphi^c)_{1-t}(\gamma^c_{1-t}) \\ &= -t\frac{\ell(\gamma^c)^p}{p} = -t\frac{\ell(\gamma)^p}{p}. \end{aligned}$$

Summing the two identities, it follows that $\varphi_t(\gamma_t) = -(\varphi^c)_{1-t}(\gamma_t) = \bar\varphi_t(\gamma_t)$.

For the other implication, let us assume that for some $x \in X$, $t \in (0,1)$ $\varphi_t(x) = -(\varphi^c)_{1-t}(x)$. Applying Lemma 3.10 to the lower semi-continuous functions $-\varphi$ and $-\varphi^c$, it turns out that there exist $y_t, z_t \in X$ such that

$$\begin{aligned} -\varphi_t(x) = Q_t(-\varphi)(x) &= \frac{\mathsf{d}(x, y_t)^p}{pt^{p-1}} - \varphi(y_t), \\ \varphi_t(x) = Q_{1-t}(-\varphi^c)(x) &= \frac{\mathsf{d}(x, z_t)^p}{pt^{p-1}} - \varphi^c(z_t). \end{aligned}$$

Summing the two equations, we get that

$$\frac{\mathsf{d}(x, y_t)^p}{pt^{p-1}} - \varphi(y_t) = \varphi^c(z_t) - \frac{\mathsf{d}(x, z_t)^p}{p(1-t)^{p-1}},$$

so we are in position to apply Lemma 3.14, obtaining the claim. □

Motivated by Lemma 3.15 we will also consider the following set:

$$D(\mathring{G}_\varphi) = \{(x,t) \in X \times (0,1)\ ;\ \varphi_t(x) = \bar\varphi_t(x)\}\,; \tag{3.20}$$

which is a closed subset of $X \times (0,1)$.

3.2.3. First and Second Order inequalities. Let us now introduce the speed along which each characteristic is traversed; since the particles move freely, this coincides with the total length of the characteristic, which is why the same functions are called length functions ℓ_t in [**35**]. To emphasize the dynamic point of view, we shall also refer to $(p-1)\ell_t^p/p = (\ell_t^{p-1})^{p'}/p'$ as the *energy*, though it is really the energy per unit mass transported.

DEFINITION 3.16 (Speed functions $\ell_t^\pm, \bar\ell_t^\pm$). Given a Kantorovich potential $\varphi : X \to \mathbb{R}$, define the speed functions $\ell_t^\pm, \bar\ell_t^\pm$ as follows:

$$\ell_t^\pm(x) := \frac{D^\pm_{-\varphi}(x,t)}{t}, \quad \bar\ell_t^\pm(x) := \frac{D^\pm_{-\varphi^c}(x, 1-t)}{1-t}, \quad (x,t) \in X \times (0,1).$$

Let us mention that we will shortly see that if $x=\gamma_t$ with $\gamma\in G_\varphi$ and $t\in(0,1)$, then:

$$\ell_t^+(x)=\ell_t^-(x)=\bar{\ell}_t^+(x)=\bar{\ell}_t^-(x)=\ell(\gamma).$$

In particular, all (φ,p)-Kantorovich geodesics having x as their t-midpoint have necessarily the same length. For $\tilde{\ell}\in\{\ell,\bar{\ell}\}$, we define the set:

$$D_{\tilde{\ell}}:=\{(x,t)\in X\times(0,1):\tilde{\ell}_t^+(x)=\tilde{\ell}_t^-(x)\}. \tag{3.21}$$

On $D_{\tilde{\ell}}$ we set $\tilde{\ell}_t(x):=\tilde{\ell}_t^-(x)=\tilde{\ell}_t^+(x)$. Recalling that $\varphi_t=-Q_t(-\varphi)$ and $\bar{\varphi}_t=Q_{1-t}(-\varphi^c)$, we can apply Theorem 3.9 to deduce the following:

Corollary 3.17 (Time semi-continuity of speeds). *Let $\varphi:X\to\mathbb{R}$ denote a p-Kantorovich potential. Then:*

1. *Choosing $\tilde{\ell}\in\{\ell,\bar{\ell}\}$ and $\tilde{\varphi}\in\{\varphi,\bar{\varphi}\}$ correspondingly, $\tilde{\ell}_t^\pm(x)$ are locally finite on $X\times(0,1)$, and $(x,t)\mapsto\tilde{\varphi}_t(x)$ is locally Lipschitz there.*
2. *For $\tilde{\ell}\in\{\ell,\bar{\ell}\}$ the map $(x,t)\mapsto\tilde{\ell}_t^+(x)$ ($(x,t)\mapsto\tilde{\ell}_t^-(x)$) is upper (lower) semi-continuous on $X\times(0,1)$. In particular, $D_{\tilde{\ell}}\subset X\times(0,1)$ is Borel and $(x,t)\mapsto\tilde{\ell}_t(x)$ is continuous on $D_{\tilde{\ell}}$.*
3. *For every $x\in X$ we have:*
$$\partial_t^\pm\varphi_t(x)=\frac{(p-1)\ell_t^\pm(x)^p}{p},\quad \partial_t^\pm\bar{\varphi}_t(x)=\frac{(p-1)\bar{\ell}_t^\pm(x)^p}{p}\quad\forall t\in(0,1).$$
In particular, for $\tilde{\ell}\in\{\ell,\bar{\ell}\}$ and the corresponding $\tilde{\varphi}\in\{\varphi,\bar{\varphi}\}$, the map $(0,1)\ni t\mapsto\tilde{\varphi}_t(x)$ is locally Lipschitz, and it is differentiable at $t\in(0,1)$ iff $t\in D_{\tilde{\ell}}(x)$, the set on which both maps $(0,1)\ni t\mapsto\tilde{\ell}_t^\pm(x)$ coincide. $D_{\tilde{\ell}}(x)$ is precisely the set of continuity points of both maps, and thus coincides with $(0,1)$ with at most countably exceptions.

All four maps $(0,1)\ni t\mapsto t\ell_t^\pm(x)$ and $(0,1)\ni t\mapsto(t-1)\bar{\ell}_t^\pm(x)$ are monotone non-decreasing; in particular, both $D_\ell(x)\ni t\mapsto\ell_t^p(x)$ and $D_{\bar{\ell}}(x)\ni t\mapsto\bar{\ell}_t^p(x)$ are differentiable a.e.. From monotonicity it is straightforward to deduce

$$\underline{\partial}_t\ell_t(x)\geq-\frac{1}{t}\ell_t(x)\quad\forall t\in D_\ell(x),$$

as well as a similar estimate for $\bar{\ell}_t$. In particular, the following estimates holds true (see [**35**, Corollary 3.10]).

Corollary 3.18 (Energies are locally Lipschitz in time). *The following estimates hold true for every $x\in X$:*

$$\underline{\partial}_t\frac{\ell_t^p(x)}{p}\geq-\frac{1}{t}\ell_t^p(x),\qquad\forall t\in D_\ell(x). \tag{3.22}$$

$$\overline{\partial}_t\frac{\bar{\ell}_t^p(x)}{p}\leq\frac{1}{1-t}\bar{\ell}_t^p(x),\qquad\forall t\in D_{\bar{\ell}}(x). \tag{3.23}$$

The first and the last points of the next Theorem can be compared with [**35**, Theorem 2.13] in the case $p=2$.

Theorem 3.19 (Time-derivatives of energies bound second time-derivatives of potentials). *Let $\varphi:X\to\mathbb{R}$ be a p-Kantorovich potential. Then the following holds true:*

1. *For all $x\in\mathrm{e}_t(G_\varphi)$ with $t\in(0,1)$, we have:*
$$\ell_t^+(x)=\ell_t^-(x)=\bar{\ell}_t^+(x)=\bar{\ell}_t^-(x)=\ell(\gamma).$$

(2) For all $x \in X$, $\mathring{G}_\varphi(x) \ni t \mapsto \ell_t(x) = \bar{\ell}_t(x)$ is locally Lipschitz and, provided $\ell(\gamma) > 0$, the following estimate holds true

$$\frac{1-s}{1-t} \le \frac{\ell_t(x)}{\ell_s(x)} \le \frac{s}{t}, \quad 0 < t \le s < 1.$$

(3) For all $(x,t) \in D(\mathring{G}_\varphi) \subset D_\ell \cap D_{\bar{\ell}}$ we have that the following estimate holds true for the upper and lower second derivatives $z \in \{\underline{\mathcal{P}}_2\bar{\varphi}_t(x), \overline{\mathcal{P}}_2\varphi_t(x)\}$ in time of (3.1):

$$-\frac{p-1}{t}\ell_t^p(x) \le \underline{\partial}_t \frac{(p-1)\ell_t^p(x)}{p} \le \underline{\mathcal{P}}_2\varphi_t(x) \le z \le \overline{\mathcal{P}}_2\bar{\varphi}_t(x) \le \overline{\partial}_t \frac{(p-1)\bar{\ell}_t^p(x)}{p} \le \frac{p-1}{1-t}\ell_t^p(x).$$

Proof. Let $(x,t) \in D(\mathring{G}_\varphi)$ (recall (3.20)). An application of Lemma 3.10 implies that there exist $y^\pm, z^\pm \in X$ such that

$$- \varphi_t(x) = \frac{\mathsf{d}(x, y_\pm)^p}{pt^{p-1}} - \varphi(y_\pm),$$
$$- \bar{\varphi}_t(x) = -\frac{\mathsf{d}(x, z_\pm)^p}{pt^{p-1}} + \varphi^c(z_\pm).$$

Since $\varphi_t(x) = \bar{\varphi}_t(x)$ by Lemma 3.15, we can equate the two expressions, obtaining that the assumption (3.16) in the Lemma 3.14 is satisfied. Hence x is the t-midpoint of a geodesic connecting $y_\pm$ and $z_\pm$ for all four possibilities. The same lemma guarantees that

$$\frac{\mathsf{d}(x, y^\pm)}{t} = \frac{\mathsf{d}(x, z^\pm)}{1-t}$$

and thus $\ell_t^\pm(x) = \bar{\ell}_t^\pm(x)$. Recall now that if $x = \gamma_t$ for some $\gamma \in G_\varphi$ then Corollary 3.13 implies that

$$Q_t(-\varphi)(x) = -\varphi_t(x) = \frac{\mathsf{d}(x, \gamma_0)^p}{pt^{p-1}} - \varphi(\gamma_0)$$

and thus the sequence $\{y_n\}$ with $y_n \equiv \gamma_0$ is in the class of admissible sequences for the infimum and supremum in the definition of $D^\pm_{-\varphi}(x,t)$. Hence

$$t\ell_t^-(x) = D^-_{-\varphi}(x,t) \le \mathsf{d}(x, \gamma_0) = t\ell(\gamma) \le D^+_{-\varphi}(x,t) = t\ell_t^+(x),$$

and *1.* follows.

In order to prove *2.*, let $\gamma^t, \gamma^s \in G_\varphi$ be such that $\gamma^t_t = \gamma^s_s = x$, for some $t, s \in (0,1)$. Then

$$\frac{\ell(\gamma^\alpha)^p}{p} - \varphi(\gamma^\alpha_0) = \varphi^c(\gamma^\alpha_1) \le \frac{\mathsf{d}(\gamma^\alpha_1, \gamma^\beta_0)^p}{p} - \varphi(\gamma^\beta_0)$$

for $(\alpha, \beta) = (s,t)$ and $(\alpha, \beta) = (t,s)$. Summing the two inequalities, we obtain that

$$\ell(\gamma^t)^p + \ell(\gamma^s)^p \le \mathsf{d}(\gamma^t_0, \gamma^s_1)^p + \mathsf{d}(\gamma^t_1, \gamma^s_0)^p,$$

hence the set $\{(\gamma^t_0, \gamma^t_1), (\gamma^s_0, \gamma^s_1)\}$ turns out to be d^p-cyclic monotone. For what concerns the right hand side we can observe that, since $\gamma^t_t = \gamma^s_s = x$, it holds

$$\mathsf{d}(\gamma^\alpha_0, \gamma^\beta_1) \le \mathsf{d}(\gamma^\alpha_0, x) + \mathsf{d}(x, \gamma^\beta_1) = \alpha\ell(\gamma^\alpha) + (1-\beta)\ell(\gamma^\beta).$$

Thus the inequality above rewrites in the following way:

$$\ell(\gamma^t)^p + \ell(\gamma^s)^p \le (t\ell(\gamma^t) + (1-s)\ell(\gamma^s))^p + (s\ell(\gamma^s) + (1-t)\ell(\gamma^t))^p.$$

More precisely, it is in the form

$$a^p + b^p \le c^p + d^p,$$

with a,b,c,d positive and such that $a+b=c+d$. Applying Karamata inequality, it turns out that an inequality of this type implies $a,b\in(\min\{c,d\},\max\{c,d\})$. In our case, if $s>t$, we get

$$\frac{1-s}{1-t}\le\frac{\ell_t(x)}{\ell_s(x)}\le\frac{s}{t},$$

thus $\ell_.(x)$ is locally lipschitz.

To obtain *3.*, as in (3.2) let us define $\tilde h = h,\bar h$ as

$$\tilde h(\varepsilon):=2(\tilde\varphi_{t_0+\varepsilon}(x)-\tilde\varphi_{t_0}(x)-\varepsilon\partial_t\tilde\varphi_{t_0}(x)).$$

Recall that, by Lemma 3.15, for all $t\in[0,1]$ it holds $\varphi_t\le\bar\varphi_t$ with the equality satisfied in the case $x\in \mathrm{e}_t(G_\varphi)$. Moreover, since $\mathring{G}_\varphi(x)\subset D_\ell(x)\cap D_{\bar\ell}(x)$, the maps $t\mapsto\tilde\varphi_t(x)$ are differentiable at $t_0\in\mathring{G}_\varphi(x)$ and $(p-1)\ell^p_{t_0}(x)/p=\partial_t|_{t=t_0}\varphi_t(x)=\partial_t|_{t=t_0}\bar\varphi_t(x)=(p-1)\bar\ell^p_{t_0}(x)/p$. These facts imply that $h\le\tilde h$ on $(-t_0,1-t_0)$. Dividing by ε^2 and taking subsequential limits, we obtain

$$\underline{\mathcal{P}}_2\varphi_t(x)\le\underline{\mathcal{P}}_2\bar\varphi_t(x),\quad \overline{\mathcal{P}}_2\varphi_t(x)\le\overline{\mathcal{P}}_2\bar\varphi_t(x).$$

Combining these inequalities with those of Lemma 3.5, (3.22) and (3.23) we get the claim. □

We conclude with the following result; for its proof we refer to [**35**, Corollary 3.13].

COROLLARY 3.20. *For all $x\in X$, for a.e. $t\in\mathring{G}_\varphi(x)$, $\partial_t\ell^p_t(x)$ and $\partial_t\bar\ell^p_t(x)$ exist, coincide, and satisfy:*

$$\begin{aligned}-\frac{\ell^p_t(x)}{t}\le\partial_t\frac{\ell^p_t(x)}{p}&=\partial_t\frac{\ell^p_t(x)\,|_{\mathring{G}_\varphi(x)}}{p}\\ &=\partial_t\frac{\bar\ell^p_t(x)\,|_{\mathring{G}_\varphi(x)}}{p}=\partial_t\frac{\bar\ell^p_t(x)}{p}\le\frac{\bar\ell^p_t(x)}{1-t}.\end{aligned}\tag{3.24}$$

REMARK 3.21. Recall that we already proved that $\partial^\pm_\tau|_{\tau=s}\varphi_\tau(x)=(p-1)\ell^\pm_s(x)^p/p$ and $\ell^\pm_s(\gamma_s)=\ell$ for all $s\in(0,1)$.

3.2.4. Third order inequality and consequences. Just as the solution to a Hamilton-Jacobi equation with Hamiltonian $H(w)=|w|^{p'}/p'$ behaves affinely in time on its characteristics, (3.14) similarly shows that the t interpolant φ_t of a Kantorovich potential becomes an affine function of time t along a (φ,p)-Kantorovich geodesic γ_t. The goal of this section and the next is to show that $\partial^2_t\varphi_t$ is non-decreasing along such curves and provide a positive lower bound (3.34)–(3.35) for the slope of $z(t):=[\partial^2_t\varphi_t](\gamma_t)$ — at least under certain regularity hypotheses which can be subsequently verified for a large enough family of (φ,p)-Kantorovich geodesics that serve our purposes. For $p=p'=2$, such estimates were proved in [**35**], but their proof does not generalize to our case. However, Cavalletti and Milman [**35**] also provided a heuristic argument in the smooth setting which can be adapted to $p\neq2$ as follows.

REMARK 3.22. Let us start from the Hamilton-Jacobi equation

$$\partial_t\varphi_t=H(\nabla\varphi_t)$$

satisfied by the time t interpolant φ_t of a p-Kantorovich potential φ on a Riemannian manifold. Differentiating in t gives

$$\partial^2_t\varphi_t=DH|_{\nabla\varphi_t}(\nabla\partial_t\varphi_t).\tag{3.25}$$

Setting $z(t)=\left[\partial^2_t\varphi_t\right](\gamma_t)$where γ_t is the time t evaluation of a φ-Kantorovich geodesic, we observe using $\gamma'(t)=-DH(\nabla\varphi_t)$ that

$$z'(t)=\partial^3_t\varphi_t(\gamma_t)-\left\langle\nabla\partial^2_t\varphi_t(\gamma_t),DH(\nabla\varphi_t(\gamma(t)))\right\rangle.$$

On the other hand

$$\partial_t^3 \varphi_t = D^2 H|_{\nabla\varphi_t}(\nabla\partial_t\varphi_t, \nabla\partial_t\varphi_t) + DH|_{\nabla\varphi_t}(\nabla\partial_t^2\varphi_t).$$

Inserting this into the previous equation yields

$$\begin{aligned} z'(t) &= D^2 H|_{\nabla\varphi_t}(\nabla\partial_t\varphi_t, \nabla\partial_t\varphi_t) \\ &= |\nabla\varphi_t(\gamma(t))|^{p'-2}|\nabla\partial_t\varphi_t(\gamma(t))|^2 + (p'-2)|\nabla\varphi_t(\gamma(t))|^{p'-4}\left\langle\nabla\varphi_t(\gamma(t)), \nabla\partial_t\varphi_t(\gamma(t))\right\rangle^2. \end{aligned}$$

Convexity of $H(w) = |w|^{p'}/p'$ shows that $z(t)$ is non-decreasing (hence confirming differentiability a.e.) and allows its derivative to be estimated from below in terms of $|\nabla\varphi_t(\gamma(t))|$ and $|\nabla\partial_t\varphi_t(\gamma(t))|$ — both of which exist a.e. since φ_t is locally semiconvex in the halfspace $t > 0$. The Cauchy-Schwarz inequality gives

$$\begin{aligned} z'(t) &\geq (p'-1)|\nabla\varphi_t(\gamma(t))|^{p'-4}\langle\nabla\varphi_t(\gamma(t)), \nabla\partial_t\varphi_t(\gamma(t))\rangle^2 \\ &= \frac{1}{p-1}\frac{z(t)^2}{\ell_t^p}, \end{aligned}$$

where and $\ell_t = |DH(\nabla\varphi_t)|$ and $(p-1)(p'-1) = 1$ have been recalled and (3.25) has been used to identify $z(t) = |\nabla\varphi_t(\gamma_t)|^{p'-2}\left\langle\nabla\varphi_t(\gamma_t), \nabla\partial_t\varphi_t(\gamma_t)\right\rangle$. At least heuristically, this establishes (3.35).

In order to obtain rigorous estimates on third order variations of Kantorovich potentials, we introduce the quantities $\tilde{r} \in \{r, \bar{r}\}$ which measure the time partial of energies along a fixed (φ, p)-Kantorovich geodesic (which plays the role of a characteristic in the nonsmooth setting); for every $s \in (0,1)$ set

$$\begin{aligned} \tilde{r}_+^\gamma(s) = \tilde{r}_+(s) &:= \overline{\partial}_\tau|_{\tau=s}\frac{(p-1)}{p}\tilde{\ell}_\tau^p(\gamma_s) = (p-1)\tilde{\ell}^{p-1}\overline{\partial}_\tau|_{\tau=s}\tilde{\ell}_\tau(\gamma_s), \\ \tilde{r}_-^\gamma(s) = \tilde{r}_-(s) &:= \underline{\partial}_\tau|_{\tau=s}\frac{(p-1)}{p}\tilde{\ell}_\tau^p(\gamma_s) = (p-1)\tilde{\ell}^{p-1}\underline{\partial}_\tau|_{\tau=s}\tilde{\ell}_\tau(\gamma_s). \end{aligned}$$

By definition, $\tilde{r}_-(s) \leq \tilde{r}_+(s)$; moreover, equality holds $\tilde{r}_-(s) = \tilde{r}_+(s) = \tilde{r}$ if and only if the map $\tau \mapsto (p-1)\tilde{\ell}_\tau^p(\gamma_s)/p$ is differentiable at $\tau = s$ with derivative $\tilde{r}$.
We also define upper and lower second order Peano derivatives in time (Definition 3.4) $\tilde{q}_\pm \in \{q_\pm, \bar{q}_\pm\}$ of the (forward and backward) interpolated Kantorovich potentials respectively, evaluated along the same characteristic, as follows:

$$\begin{aligned} \tilde{q}_+(s) &:= \overline{\mathcal{P}}_2\tilde{\varphi}_s(x)|_{x=\gamma_s} = \limsup_{\varepsilon\to 0}\frac{\tilde{h}(s,\varepsilon)}{\varepsilon^2}, \\ \tilde{q}_-(s) &:= \underline{\mathcal{P}}_2\tilde{\varphi}_s(x)|_{x=\gamma_s} = \liminf_{\varepsilon\to 0}\frac{\tilde{h}(s,\varepsilon)}{\varepsilon^2}, \end{aligned}$$

where $\tilde{h}(s,\varepsilon)$ is defined analogously to (3.2). By definition, $\tilde{q}_-(s) = \tilde{q}_+(s) = \tilde{q}$ hold if and only if the map $\tau \mapsto \tilde{\varphi}_\tau(\gamma_s)$ has second-order Peano derivative at $\tau = s$ given by $\tilde{q}$. We summarize the relation between $\tilde{q}_\pm$ and $\tilde{r}_\pm$ implied by Lemma 3.3 and Lemma 3.5 in the following

Corollary 3.23 (First differentiability of energy is equivalent to second differentiability of potential). *The following statements are equivalent for a given $s \in (0,1)$:*

(1) $\tilde{r}_-(s) = \tilde{r}_+(s) = \tilde{r} \in \mathbb{R}$, i.e. the map $D_{\tilde{\ell}}(\gamma_s) \ni \tau \mapsto (p-1)\tilde{\ell}_\tau^p(\gamma_s)/p$ is differentiable at $\tau = s$ with derivative $\tilde{r}$.
(2) $\tilde{q}_-(s) = \tilde{q}_+(s) = \tilde{q} \in \mathbb{R}$, i.e. the map $(0,1) \ni \tau \mapsto \tilde{\varphi}_\tau(\gamma_s)$ has second order Peano derivative at $\tau = s$ given by $\tilde{q}$.

If one of the two conditions above is satisfied, the map $(0,1) \ni \tau \mapsto \tilde{\varphi}_\tau(\gamma_s)$ *is twice differentiable at* $\tau = s$*, and we have :*

$$\partial^2_\tau|_{\tau=s}\tilde{\varphi}_\tau(\gamma_s) = \partial_\tau|_{\tau=s}\frac{(p-1)\tilde{\ell}^p_\tau(\gamma_s)}{p} = (p-1)\tilde{\ell}^{p-1}\cdot\partial_\tau|_{\tau=s}\ell_\tau(\gamma_s) = \tilde{r} = \tilde{q}.$$

We are now in a position to obtain lower bounds on the incremental ratio of $\tilde{q}$. This provides the required third-order information concerning φ_t even when the upper and lower derivatives in question do not agree. For the geometric interpretation of the following discretized differential inequalities, we refer to the discussion of the case $p = 2$ in [**35**, Section 5.1].

Theorem 3.24 (Third-order difference quotient bounds on potential along its characteristics). *For all* $0 < s < t < 1$ *and both possibilities for* $\pm$*, we have*

$$\frac{q_+(t) - q_-(s)}{t-s} \geq \frac{s}{t}\frac{r_\pm(s)^2}{(p-1)\ell^p}, \tag{3.26}$$

$$\frac{\bar{q}_+(t) - \bar{q}_-(s)}{t-s} \geq \frac{1-t}{1-s}\frac{\bar{r}_\pm(t)^2}{(p-1)\ell^p}. \tag{3.27}$$

The proof of the analogous estimate for $p = 2$ ([**35**, Theorem 5.2]) does not work in the general case $p > 1$.

Proof. By definition of the Hopf-Lax transform and by Lemma 3.10, we have that

$$-\varphi_{s+\varepsilon}(\gamma_s) = Q_{s+\varepsilon}(-\varphi)(\gamma_s) = \frac{\mathsf{d}(y^\pm_\varepsilon,\gamma_s)^p}{p(s+\varepsilon)^{p-1}} - \varphi(y^\pm_\varepsilon),$$

with $\mathsf{d}(y^\pm_\varepsilon,\gamma_s) = D^\pm_{-\varphi}(\gamma_s, s+\varepsilon) = (s+\varepsilon)\ell^\pm_{s+\varepsilon}(\gamma_s) =: D^\pm_{s+\varepsilon}$. Moreover, the following inequality trivially holds:

$$-\varphi_{t+\varepsilon}(\gamma_t) \leq \frac{\mathsf{d}(y^\pm_\varepsilon,\gamma_t)^p}{p(t+\varepsilon)^{p-1}} - \varphi(y^\pm_\varepsilon).$$

Subtracting the two expressions above, we obtain:

$$\varphi_{t+\varepsilon}(\gamma_t) - \varphi_{s+\varepsilon}(\gamma_s) \geq -\frac{\mathsf{d}(y^\pm_\varepsilon,\gamma_t)^p}{p(t+\varepsilon)^{p-1}} + \frac{\mathsf{d}(y^\pm_\varepsilon,\gamma_s)^p}{p(s+\varepsilon)^{p-1}},$$

hence recalling (3.2)

$$\begin{aligned}\frac{1}{2}(h(t,\varepsilon) - h(s,\varepsilon)) &\geq -\varphi_t(\gamma_t) + \varphi_s(\gamma_s) - \frac{\mathsf{d}(y^\pm_\varepsilon,\gamma_t)^p}{p(t+\varepsilon)^{p-1}} + \frac{\mathsf{d}(y^\pm_\varepsilon,\gamma_s)^p}{p(s+\varepsilon)^{p-1}},\\ &= (t-s)\frac{\ell^p}{p} - \frac{\mathsf{d}(y^\pm_\varepsilon,\gamma_t)^p}{p(t+\varepsilon)^{p-1}} + \frac{\mathsf{d}(y^\pm_\varepsilon,\gamma_s)^p}{p(s+\varepsilon)^{p-1}},\\ &= (t-s)\frac{\ell^p}{p} - \frac{\mathsf{d}(y^\pm_\varepsilon,\gamma_t)^p}{p(t+\varepsilon)^{p-1}} + \frac{(s+\varepsilon)(\ell^\pm_{s+\varepsilon}(\gamma_s))^p}{p}. \end{aligned} \tag{3.28}$$

We need now an estimate from below of the second term. In order to do that, let us observe that

$$\mathsf{d}(y^\pm_\varepsilon,\gamma_t) \leq \mathsf{d}(y^\pm_\varepsilon,\gamma_s) + \mathsf{d}(\gamma_s,\gamma_t) = D^\pm_{s+\varepsilon} + D_t - D_s,$$

where we put $D_r = r\ell = \mathsf{d}(\gamma_r,\gamma_0)$, for $r = s,t$. In particular,

$$\begin{aligned}D^\pm_{s+\varepsilon} + D_t - D_s &= (s+\varepsilon)(\ell^\pm_{s+\varepsilon}(\gamma_s) - \ell_s(\gamma_s)) + (t+\varepsilon)\ell\\ &= (t+\varepsilon)\left[\frac{s+\varepsilon}{t+\varepsilon}\ell^\pm_{s+\varepsilon}(\gamma_s) + \left(1 - \frac{s+\varepsilon}{t+\varepsilon}\right)\ell\right].\end{aligned}$$

Thus, substituting this expression in (3.28), we get

$$\frac{h(t,\varepsilon)-h(s,\varepsilon)}{2\varepsilon^2} \geq \frac{t+\varepsilon}{p\varepsilon^2}\left[\frac{t-s}{t+\varepsilon}\ell^p + \frac{s+\varepsilon}{t+\varepsilon}(\ell^{\pm}_{s+\varepsilon}(\gamma_s))^p - \left(\frac{s+\varepsilon}{t+\varepsilon}\ell^{\pm}_{s+\varepsilon}(\gamma_s) + \frac{t-s}{t+\varepsilon}\ell\right)^p\right]. \tag{3.29}$$

In other words, denoting with $f(x) := x^p$ and defining for every $\lambda \in [0,1]$ the functions

$$s_{x,y}(\lambda) = \lambda f(x) + (1-\lambda) f(y), \qquad g_{x,y}(\lambda) = f(\lambda x + (1-\lambda) y),$$

we want to estimate from below the quantity $s_{x,y}(\lambda) - g_{x,y}(\lambda)$ for the following choices of λ, x, y:

$$\lambda = \frac{s+\varepsilon}{t+\varepsilon}, \qquad x = \ell^{\pm}_{s+\varepsilon}(\gamma_s), \qquad y = \ell = \ell_s(\gamma_s). \tag{3.30}$$

Appling the following inequality $s_{x,y}(\lambda) - g_{x,y}(\lambda) \geq \min_{[y,x]} f'' \cdot \frac{\lambda(1-\lambda)}{2}(x-y)^2$, for all $\lambda \in [0,1]$, with the choices of x, y, λ given by (3.30), we get

$$\frac{h(t,\varepsilon)-h(s,\varepsilon)}{2\varepsilon^2} \geq \frac{t+\varepsilon}{p\varepsilon^2}\left[\min_{z\in[\ell_s(\gamma_s),\ell^{\pm}_{s+\varepsilon}(\gamma_s)]} z^{p-2} \cdot \frac{p(p-1)}{2} \cdot \frac{t-s}{t+\varepsilon} \cdot \frac{s+\varepsilon}{t+\varepsilon} \cdot (\ell^{\pm}_{s+\varepsilon}(\gamma_s) - \ell_s(\gamma_s))^2\right]. \tag{3.31}$$

Taking appropriate subsequential limits as $\varepsilon \to 0$, we obtain

$$\frac{q^+(t) - q^-(s)}{2(t-s)} \geq \frac{s}{t}\frac{(p-1)}{2}\ell^{p-2}(\partial_\tau|_{\tau=s}\ell^{\pm}_\tau(\gamma_s))^2.$$

In particular, it turns out that

$$\frac{q^+(t) - q^-(s)}{t-s} \geq \frac{s}{t}\frac{r_\pm(s)^2}{(p-1)\ell^p}.$$

Next, we will deduce inequality (3.27) from (3.26) by simply using the duality between φ and φ^c. Indeed, since by definition it holds that $\bar{\varphi}_t = -\varphi^c_{1-t}$, we deduce that :

$$\bar{h}^{\varphi}_{\gamma}(r,\varepsilon) = -h^{\varphi^c}_{\gamma^c}(1-r,-\varepsilon).$$

Moreover, it holds

$$\frac{(p-1)(\ell^{\varphi^c,\pm}_{1-r-\varepsilon}(\gamma^c_{1-r}))^p}{p} = -\partial^{\mp}_r \varphi^c_{1-r-\varepsilon}(\gamma^c_{1-r}) = \partial^{\mp}_r \varphi_{r+\varepsilon}(\gamma_r) = \frac{(p-1)(\bar{\ell}^{\varphi,\pm}_{r+\varepsilon}(\gamma_r))^p}{p};$$

hence, choosing as φ, γ, ε, s, t respectively φ^c, γ_c, $-\varepsilon$,$1-t$, $1-s$ we get the second claim.

□

We start by noticing an immediate consequence of Theorem 3.24:

COROLLARY 3.25. *For both $\tilde{q} = q, \bar{q}$, the functions $t \mapsto \tilde{q}_\pm(t)$ are monotone non-decreasing on $(0,1)$.*

We now combine previous regularity results on time behaviour of Kantorovich potential with Theorem 3.24 in order to have a clear statement on the third order variation of Kantorovich potentials.

THEOREM 3.26 (A priori third-order bounds for potential along its characteristics). *Assume that for **a.e.** $t \in (0,1)$:*

$$(0,1) \ni \tau \mapsto \tilde{\varphi}_\tau(\gamma_t) \quad \textit{is twice differentiable at } \tau = t \quad \textit{for both} \quad \tilde{\varphi} = \varphi, \bar{\varphi}, \tag{3.32}$$

in any of the equivalent senses of Corollary 3.23 and that moreover:

$$\partial^2_\tau|_{\tau=t}\varphi_\tau(\gamma_t) = \partial^2_\tau|_{\tau=t}\bar{\varphi}_\tau(\gamma_t) \quad \textit{for a.e. } t \in (0,1).$$

If there exists a continuous function z for which

$$\partial_\tau^2|_{\tau=t}\varphi_\tau(\gamma_t) = \partial_\tau^2|_{\tau=t}\bar\varphi_\tau(\gamma_t) = z(t) \quad \textit{for a.e. } t \in (0,1),$$

then (3.32) holds for ***all*** $t \in (0,1)$ *and for all* $t \in (0,1)$

$$\partial_\tau^2|_{\tau=t}\varphi_\tau(\gamma_t) = \partial_\tau^2|_{\tau=t}\bar\varphi_\tau(\gamma_t) = \partial_\tau|_{\tau=t}\frac{(p-1)\ell_\tau^p(\gamma_t)}{p} = \partial_\tau|_{\tau=t}\frac{(p-1)\bar\ell_\tau^p(\gamma_t)}{p} = z(t). \tag{3.33}$$

Finally, the following third order information on $\varphi_t(x)$ at $x = \gamma_t$ holds true:

$$\frac{z(t)-z(s)}{t-s} \geq \sqrt{\frac{s}{t}\frac{1-t}{1-s}}\frac{|z(s)||z(t)|}{(p-1)\ell^p}, \quad \forall\ 0<s<t<1. \tag{3.34}$$

In particular, for any point $t \in (0,1)$ where $z(t)$ is differentiable we have

$$z'(t) \geq \frac{z(t)^2}{(p-1)\ell^p}. \tag{3.35}$$

Proof. By Corollary 3.23, it follows that $\tilde q_-(t) = \tilde q_+(t) = z(t)$ for a.e. $t \in (0,1)$. More precisely, the same holds true for every $t \in (0,1)$ by the monotonicity of $\tilde q_\pm$ and the continuity of z; thus, (3.33) is satisfied. Moreover, Corollary 3.23 also implies that $\tilde r_-(t) = \tilde r_+(t) = z(t)$ for both $\tilde r = r, \bar r$ and for all $t \in (0,1)$. Taking the geometric mean of (3.26) and (3.27), we get (3.34). Finally, passing to the limit as $s \to t$ in (3.34), we obtain (3.35). □

The assumptions of Theorem 3.26 will hold true for a.e. $t \in (0,1)$ only for a certain family of Kantorovich geodesics; nonetheless, this will be enough for our purposes.

Finally, inequality (3.35) will be crucial to deduce concavity of certain one-dimensional factors. We include here a result that will be used later. For its proof we refer to [**35**, Lemma 5.7].

Lemma 3.27 (Concavity restatement). *Assume that for some locally absolutely continuous function z on $(0,1)$ we have:*

$$\partial_\tau|_{\tau=t}\frac{(p-1)\ell_\tau^p(\gamma_t)}{p} = z(t) \quad \textit{for a.e. } t \in (0,1).$$

Then for any fixed $r_0 \in (0,1)$, the function:

$$L(r) = \exp\left(-\frac{1}{\ell^p(p-1)}\int_{r_0}^r \partial_\tau|_{\tau=t}\frac{(p-1)\ell_\tau^p(\gamma_t)}{p}\,dt\right) = \exp\left(-\frac{1}{\ell^p(p-1)}\int_{r_0}^r z(t)\,dt\right)$$

is concave on $(0,1)$.

3.2.5. Time propagation of Intermediate Kantorovich potentials. Finally we recall the definition of time-propagated intermediate Kantorovich potentials as introduced in [**35**].

Definition 3.28. Given a Kantorovich potential $\varphi : X \to \mathbb{R}$ and $s,t \in (0,1)$, define the t-propagated s-Kantorovich potential Φ_s^t on the domain $D_\ell(t)$ where forward speed is well-defined and its time-reversed version $\bar\Phi_s^t$ on the domain $D_{\bar\ell}(t)$ from (3.21), by:

$$\Phi_s^t := \varphi_t + (t-s)\frac{\ell_t^p}{p} \text{ on } D_\ell(t), \qquad \bar\Phi_s^t := \bar\varphi_t + (t-s)\frac{\bar\ell_t^p}{p} \text{ on } D_{\bar\ell}(t).$$

Using Theorem 3.19, it follows that for all $s,t \in (0,1)$:

$$\Phi_s^t = \bar\Phi_s^t = \varphi_s \circ e_s \circ (\mathrm{e}_t|_{G_\varphi}^{-1}), \qquad \text{on } \mathrm{e}_t(G_\varphi). \tag{3.36}$$

Indeed, for any $\gamma \in G_\varphi$ it holds

$$\Phi_s^t(\gamma_t) = \varphi_t(\gamma_t) + (t-s)\frac{\ell_t(\gamma_t)^p}{p} = \varphi_t(\gamma_t) + (t-s)\frac{\ell(\gamma)^p}{p} = \varphi_s(\gamma_s).$$

Consequently, on $\mathrm{e}_t(G_\varphi)$, $\Phi_s^t = \bar\Phi_s^t$ is identified as the push-forward of φ_s via $\mathrm{e}_t \circ \mathrm{e}_s^{-1}$, i.e. its propagation along G_φ from time s to time t.

Proposition 3.29 (Linear expansion of energy in time generates propagation of potential). *For any $s \in (0,1)$, the following properties hold:*

(1) The maps $(x,t) \mapsto \Phi_s^t(x)$ and $(x,t) \mapsto \bar\Phi_s^t(x)$ are continuous on D_ℓ and on $D_{\bar\ell}$ respectively;

(2) For each $x \in X$, denoting $\tilde\Phi \in \{\Phi, \bar\Phi\}$ and the corresponding $\tilde\ell \in \{\ell, \bar\ell\}$, the map $D_{\tilde\ell}(x) \ni t \mapsto \tilde\Phi_s^t(x)$ is differentiable at t if and only if $D_{\tilde\ell}(x) \ni t \mapsto \tilde\ell_t^p(x)$ is differentiable at t or if $t = s \in D_{\tilde\ell}(x)$. In particular, $t \mapsto \tilde\Phi_s^t(x)$ is a.e. differentiable. At any point of differentiability:

$$\partial_t \tilde\Phi_s^t(x) = \tilde\ell_t^p(x) + (t-s)\frac{\partial_t \tilde\ell_t^p(x)}{p}$$

In particular, if $s \in D_{\tilde\ell}(x)$ then $\partial_t|_{t=s}\tilde\Phi_s^t(x)$ exists and is given by $\tilde\ell_t^p(x)$.

(3) For each $x \in X$, the map $G_\varphi \ni t \mapsto \Phi_s^t(x) = \bar\Phi_s^t(x)$ is locally Lipschitz;

(4) For all $t \in (0,1)$:

$$\begin{cases} \underline{\partial}_t \Phi_s^t(x) \geq \dfrac{s}{t}\ell_t^p(x), & t \geq s \\ \overline{\partial}_t \Phi_s^t(x) \leq \dfrac{s}{t}\ell_t^p(x), & t \leq s \end{cases} \quad \forall x \in D_\ell(t); \qquad \begin{cases} \overline{\partial}_t \bar\Phi_s^t(x) \leq \dfrac{1-s}{1-t}\bar\ell_t^p(x), & t \geq s \\ \underline{\partial}_t \bar\Phi_s^t(x) \geq \dfrac{1-s}{1-t}\bar\ell_t^p(x), & t \leq s \end{cases} \quad \forall x \in D_{\bar\ell}(t). \tag{3.37}$$

Proof. By lower semi-continuity and Corollary 3.13, 1) and 2) follow trivially. By Corollary 3.17 and Theorem 3.19, 3) holds true. To see 4), observe that for every $x \in D_{\tilde\ell}(t)$,

$$\underline{\partial}_t \tilde\Phi_s^t(x) = \tilde\ell_t^p(x) + (t-s)\underline{\partial}_t \frac{\tilde\ell_t^p(x)}{p}, \quad t \geq s$$
$$\underline{\partial}_t \tilde\Phi_s^t(x) = \tilde\ell_t^p(x) + (t-s)\overline{\partial}_t \frac{\tilde\ell_t^p(x)}{p}, \quad t \leq s$$

with analogous identities holding for $\overline{\partial}_t \tilde\Phi_s^t(x)$. Using estimates (3.22) and (3.23) of Corollary 3.18, the claim follows. ☐

3.3. Curvature-Dimension conditions: from $\mathsf{CD}_p(K,N)$ to $\mathsf{CD}^1(K,N)$

We will now focus on the main goal of this chapter: to show for essentially non-branching spaces that the synthetic ($p=2$) curvature-dimension condition can be equivalently formulated in terms of entropic convexity conditions along p-Wasserstein geodesics for any any other $p > 1$. Our approach will be to show for such spaces that $\mathsf{CD}_p(K,N)$ for $p > 1$ is equivalent to $\mathsf{CD}^1(K,N)$, which is an appropriate concavity statement about the factor measures which arise whenever $\mathfrak{m}$ is disintegrated along the needles of the signed distance to the zero level-set of an arbitrary continuous function.

The first implication that we will address is the following one: if $(X,\mathsf{d},\mathfrak{m})$ is a p-essentially non-branching metric measure space verifying $\mathsf{CD}_p(K,N)$ then it satisfies $\mathsf{CD}^1(K,N)$ (actually the stronger $\mathsf{CD}^1_{Lip}(K,N)$).

3.3.1. Curvature estimates. Recalling Definition 1.50, one will observe that to prove $(X,\mathsf{d},\mathfrak{m})$ verifies $\mathsf{CD}^1_u(K,N)$ it suffices to show that, for $\mathfrak{q}$-a.e. $\alpha \in Q$, the one dimensional metric measure space $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ is a $\mathsf{CD}(K,N)$ space, i.e. if X_α is isometric to $[0, L_\alpha]$ where L_α is the length of X_α then,

$$\mathfrak{m}_\alpha = h_\alpha \mathcal{L}^1 \llcorner_{[0,L_\alpha]}, \qquad \left(h_\alpha^{\frac{1}{N-1}}\right)'' + \frac{K}{N-1} h_\alpha^{\frac{1}{N-1}} \leq 0,$$

where the inequality has to be understood in the distributional sense. Notice indeed that, by construction, the transport rays X_α are the maximal totally-ordered subsets of $\mathcal{T}_u^b \subset X$ under the partial-ordering $\leq_u$ given by Γ_u.

First we recall a result relating d^p-cyclically monotone sets to d-cyclically monotone set, presented in [**31**] for $p = 2$.

Lemma 3.30 (Certain d-cyclically monotone sets are also d^p-cyclical monotone). *Let $p > 1$ be any real number and let $\Delta \subset \Gamma_u$ be any set such that*

$$(x_0, y_0), (x_1, y_1) \in \Delta \implies (u(y_1) - u(y_0)) \cdot (u(x_1) - u(x_0)) \geq 0.$$

Then Δ is d^p-cyclically monotone.

Proof. By hypothesis the set

$$\Lambda := \{(u(x), u(y)) : (x, y) \in \Delta\} \subset \mathbb{R}^2$$

is monotone in the Euclidean sense. Since this corresponds to a one-dimensional transportation problem, it is well-known ([**98**], p.75) that Λ is also c-cyclically monotone, for any cost $c(x,y) = \vartheta(|x - y|)$ with $\vartheta : [0, +\infty) \to [0, +\infty)$ convex and such that $\vartheta(0) = 0$. Hence, in particular, Λ is $|\cdot|^p$-cyclically monotone.

Fix now $\{(x_i, y_i)\}_{i=1}^n \subset \Delta$. Using that u is 1-Lipschitz and $\Delta \subset \Gamma$, it turns out that

$$\begin{aligned}\sum_{i=1}^n \mathsf{d}^p(x_i, y_i) &= \sum_{i=1}^n |u(x_i) - u(y_i)|^p \\ &\leq \sum_{i=1}^n |u(x_i) - u(y_{i+1})|^p \leq \sum_{i=1}^n \mathsf{d}^p(x_i, y_{i+1}).\end{aligned}$$

Hence the claim. □

Example 1 (Dimensional count in the smooth case). If d is the geodesic distance on an n-dimensional Riemannian manifold X (or Euclidean space), then — away from the cut locus — any d-cyclically monotone subset Δ is contained in a $n+1$ dimensional subset of X^2, the extra dimension being due to the degeneracy of d along the direction of transport [**81**]. On the other hand, if the left projection $P_1(\Delta) \subset \{\tilde{u} = 0\}$ for some C^1 function $\tilde{u}$ whose derivative is non-vanishing on its zero set, we expect the dimension of Δ to be reduced to n, which coincides with the dimensional bound on a d^p-cyclically monotone set for $p > 1$. This example helps motivate both the previous lemma and the construction to follow.

Similarly, in the nonsmooth setting, fixing $\delta \in \mathbb{R}$ and considering pairs $\Delta \subset \Gamma_u$ of partners $(x, y) \in \Delta$ whose lower endpoint lies on a fixed level set $u(y) = \delta$, it follows that Δ is d^p-cyclically monotone for all $p > 1$. Equivalently, for each $C \subset \mathcal{T}_u^b$ and $\delta \in \mathbb{R}$, the set $\Delta := (C \times \{u = \delta\}) \cap \Gamma_u$ is d^p-cyclically monotone. Setting

$$C_\delta = P_1((C \times \{u = \delta\}) \cap \Gamma_u),$$

we see that if $\mathfrak{m}(C_\delta) > 0$, then by Theorem 1.30, there exists a unique $\nu \in \mathrm{OptGeo}_p(\mu_0, \mu_1)$ such that

$$(\mathrm{e}_0)_\sharp \nu = \mathfrak{m}(C_\delta)^{-1} \mathfrak{m} \llcorner C_\delta, \qquad (\mathrm{e}_0, \mathrm{e}_1)_\sharp \nu (C \times \{u = \delta\} \cap \Gamma_u) = 1,$$

and whose push-forwards by e_t verify the entropic concavity statement 1.35 for all $t \in [0,1]$. Letting C and δ vary, it is a standard procedure, see for example [**30**], to deduce that:

- for $\mathfrak{q}$-a.e. $\alpha \in Q$, the conditional probabilities $\mathfrak{m}_\alpha$ are absolutely continuous w.r.t. $\mathcal{L}^1\llcorner_{X_\alpha}$;
- if $\mathfrak{m}_\alpha = h_\alpha \mathcal{L}^1\llcorner_{X_\alpha}$, then $h_\alpha > 0$ in the relative interior of X_α and is locally Lipschitz.

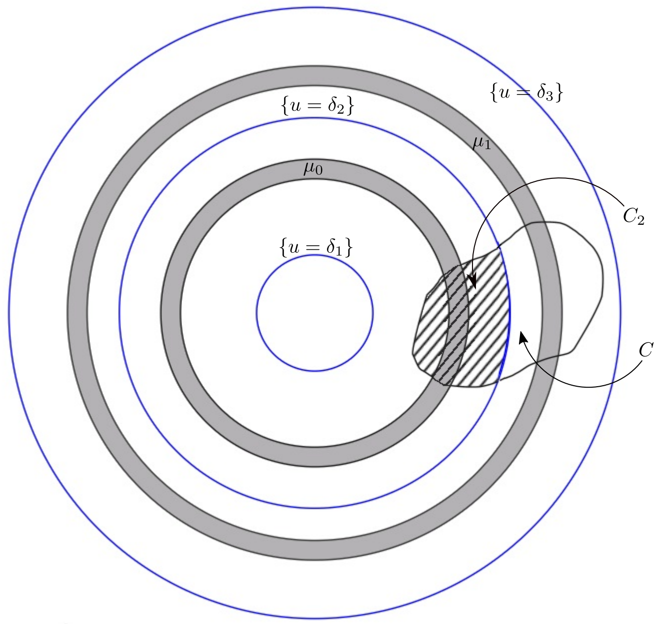

FIGURE 1. (The sets C_δ) Transporting μ_0 to μ_1 along radial transport geodesics determined by a radial 1-Lipschitz function u associated to the radial Kantorovich potential φ. If we assume that u behaves like the euclidean norm, it holds that $C_{\delta_1} = \varnothing, C_{\delta_2} = C_2, C_{\delta_3} = C$.

The next step is to prove the $\mathsf{CD}(K,N)$ inequality for $\mathfrak{q}$-a.e. one-dimensional density h_α. This follows repeating verbatim the proof of [**36**, Theorem 4.2] where the same implication was proved assuming $\mathsf{CD}_{2,loc}(K,N)$ and 2-essentially non-branching. The main ingredient being Lemma 3.30 for $p = 2$, the argument carries over for any $p > 1$.

Combining what has been discussed so far, we obtain the following:

THEOREM 3.31 (Non-branching $\mathsf{CD}_{p,loc}$ spaces are CD^1_{Lip}). *Let $(X, \mathsf{d}, \mathfrak{m})$ be a p-essentially non-branching m.m.s. satisfying the $\mathsf{CD}_{p,loc}(K,N)$ condition for some $p \in (1,+\infty)$, $K \in \mathbb{R}$, and $N \in [1,\infty)$ and $\mathfrak{m}(X) = 1$.*

Then, for any fixed 1-Lipschitz function $u : X \to \mathbb{R}$, the transport relation R^b_u induces on the transport set a disintegration of $\mathfrak{m}\llcorner_{\mathcal{T}_u}$ into conditional measures, $\mathfrak{m}_\alpha$, that for $\mathfrak{q}$-a.e. α satisfy $\mathfrak{m}_\alpha = h_\alpha \mathcal{L}^1\llcorner_{X_\alpha}$ and:

$$h_\alpha((1-s)t_0 + st_1)^{1/(N-1)} \geq \sigma^{(1-s)}_{K,N-1}(t_1 - t_0) h_\alpha(t_0)^{1/(N-1)} + \sigma^{(s)}_{K,N-1}(t_1 - t_0) h_\alpha(t_1)^{1/(N-1)},$$

for all $s \in [0,1]$ and for $t_0, t_1 \in [0, L_\alpha]$ with $t_0 < t_1$, where we have identified the transport ray X_α with the real interval $[0, L_\alpha]$ having the same length.

Notice that the $\mathfrak{q}$-measurability of the disintegration, ensured by the Disintegration Theorem, implies joint measurability of the map $(\alpha, t) \to h_\alpha(t)$.

REMARK 3.32 (Enhancing CD^1_{Lip}). It is worth underlining that the conclusion of Theorem 3.31 is actually stronger than claiming that $(X, \mathsf{d}, \mathfrak{m})$ verifies $\mathsf{CD}^1_{Lip}(K,N)$. Notice, indeed, that while $\mathsf{CD}^1_{Lip}(K,N)$ asks for *a* disintegration of $\mathfrak{m}\llcorner_{\mathcal{T}_u}$ (no partition required, see Definition

1.50) where each conditional measure is concentrated along a maximal transport ray and verifies $\mathsf{CD}(K,N)$, Theorem 3.31 shows that we have a *partition* of the transport set made of maximal transport rays and the associated *essentially unique* disintegration verifies $\mathsf{CD}(K,N)$ (recall Remark 1.46). In what follows we will show that this property is enough to prove that $(X,\mathsf{d},\mathfrak{m})$ also verifies $\mathsf{CD}_q(K,N)$ for any $q>1$, provided it is also q-essentially non-branching.

To complete the picture we mention that in [**35**, Proposition 8.13] it is shown that $\mathsf{CD}^1_{Lip}(K,N)$ coupled with essentially non-branching (hence $p=2$) implies that the disintegration of $\mathfrak{m}\llcorner_{\mathcal{T}^b_u}$ coming from the partition induced by the transport relation R^b_u indeed verifies all the conditions required by $\mathsf{CD}^1_{Lip}(K,N)$. We refer to [**35**, Proposition 8.13] for additional details.

3.4. Curvature-Dimension conditions: from $\mathsf{CD}^1(K,N)$ to $\mathsf{CD}_q(K,N)$

Before tackling Theorem 3.1 and Corollary 3.1 we explore an example which illustrates some of the strategies and notations used.

EXAMPLE 2 (Radial transport). Let $X=\mathbb{R}^n$, d be Euclidean distance, and set $\mathfrak{m}=\mathcal{L}^n$. Let $\mu_0(dx)=\frac{1}{\omega_n|x|^{n-1}}\mathcal{L}^n\llcorner_{A_{1,2}}(dx)$ and $\mu_1(dx)=\frac{1}{\omega_n|x|^{n-1}}\mathcal{L}^n\llcorner_{A_{3,4}}(dx)$ where for $0<s<r<\infty$, $A_{s,r}$ is defined as the spherical shell

$$A_{s,r}=B_r(0)\setminus B_s(0).$$

We use the cost $c(x,y)=\frac{\mathsf{d}(x,y)^q}{q}$ where $1<q<\infty$. For this transport problem, the optimal map is $T(x)=(|x|+2)\frac{x}{|x|}$, the Kantorovich potential is $\varphi(x)=-2^{q-1}|x|$, and its interpolated potentials are

$$\varphi_t(x)=\begin{cases}\frac{-|x|^q}{qt^{q-1}}, & \text{if } |x|\le 2t,\\ -2^{q-1}\left[|x|-\frac{2t}{q'}\right], & \text{if } 2t<|x|,\end{cases}$$

where q' is the Hölder dual to q. It is possible to show that the set G_φ of (φ,q)-Kantorovich geodesics (3.12) consist of all segments of length two pointed away from the origin. Notice that not all such geodesics are involved in the transport of μ_0 to μ_1: indeed only those starting in the source $A_{1,2}$ (and therefore ending in the target $A_{3,4}$) are. In particular, only the subset of geodesics starting at a point in $A_{1,2}$ will have mass passing along them at all times $t\in(0,1)$. This restriction should be compared to condition 3 from Definition 3.35. In particular, we use G to denote a good subset of G_φ of full measure which meet the stipulations of Definition 3.35. Since we wish to apply the Disintegration Theorem, we have to associate the geodesics of G_φ with the transport rays of a 1-Lipschitz function. We do so by choosing our 1-Lipschitz function to be the signed distance to a level set of φ. In our example we can use the norm since φ is a monotone radial function. However, in the general case, we must use the signed distance $d_{a,s}:=d_{\varphi_s-a}$ with respect to the a level set of φ. Note that the ordinary distance function was not used so that we could refer to the level sets of $d_{a,s}$ uniquely. This idea is the basis of the discussion in subsection 3.4.1. In both cases we see that we are working with a subset, G, of the transport set according to the 1-Lipschitz function we chose. This should be compared to Lemma 3.36.

Finally, we demonstrate how the change of variables formula from Theorem 3.41 applies to our example. For $0<t<1$ and $\gamma\in G$, the interpolating maps, measures, and densities are given by:

$$T_t(x) = (|x| + 2t)\,\frac{x}{|x|},$$
$$\mu_t(dx) = \frac{1}{\omega_n |x|^{n-1}} \mathcal{L}^n \llcorner_{A_{1+2t,2+2t}}(dx),$$
$$\text{and } \rho_t(\gamma_t) = \frac{1}{\omega_n\,(|\gamma_0| + 2t)^{n-1}}.$$

Hence, for $s,t \in (0,1)$ we have

$$\frac{\rho_t(\gamma_t)}{\rho_s(\gamma_s)} = \left(\frac{1+\ell+2s}{1+\ell+2t}\right)^{n-1} \tag{3.38}$$

if $|\gamma_0| = 1+\ell$. For fixed $s \in (0,1)$, we note that if $a = -2^{q-1}\left[1+\ell+\frac{2s}{q}\right]$ for $0 \le \ell \le 1$ and $\gamma \in G$ is a geodesic such that $|\gamma_0| = 1+\ell$ then

$$\varphi_s(\gamma_s) = a.$$

In particular, using this notation we can write $G_{a,s} = \{\gamma \in G : \varphi_s(\gamma_s) = a\}$. Hence,

$$\mathrm{e}_s(G_{a,s}) = \partial B_{1+\ell+2s}(0)$$
$$\mathrm{e}_{[0,1]}(G_{a,s}) = A_{1+\ell,3+\ell}$$

where $-\ell = 2^{1-q}a + 1 + \frac{2s}{q}$. Using the Disintegration Theorem, as in (**??**), for any $1 \le \ell \le 1$, we obtain

$$\mathcal{L}^n \llcorner_{A_{1+\ell,3+\ell}}(dx) = \int_{\partial B_1(0)} |x|^{n-1} \mathcal{H}^1 \llcorner_{\{r\alpha | 1+\ell \le r \le 3+\ell\}} \mathcal{H}^{n-1}(d\alpha)$$

where $\mathcal{H}^k$ denotes k-dimensional Hausdorff measure. Notice that we can rewrite this as

$$\mathcal{L}^n \llcorner_{A_{1+2\ell,3+2\ell}} = \int_{\partial B_{1+\ell+2s}(0)} g^{a,s}(\alpha,\cdot)_\sharp \left(2\left(\frac{1+\ell+2t}{1+\ell+2s}\right)^{n-1} \chi_{[0,1]}(t)dt\right) \mathcal{H}^{n-1}(d\alpha)$$
$$= \int_0^1 g^{a,s}(\cdot,t)_\sharp \left(2\left(\frac{1+\ell+2t}{1+\ell+2s}\right)^{n-1} \chi_{[0,1]}(t) d\mathcal{H}^{n-1}\right) \mathcal{L}^1(dt) \tag{3.39}$$

where $g^{a,s} : \mathrm{e}_s(G_{a,s}) \times [0,1] \to X$ and $g^{a,s}(\alpha,\cdot) = \mathrm{e}_s \llcorner_{G_{a,s}}^{-1}(\alpha)$. Hence, $h^{a,s}_\alpha(t) = \left(\frac{1+\ell+2t}{1+\ell+2s}\right)^{n-1}$ where we have normalized this function so that $h^a_\alpha(s) = 1$. Next notice that

$$\Phi^t_s(x) = -2^{q-1}\left[|x| - 2t + \frac{2s}{q}\right]. \tag{3.40}$$

We may also compute that

$$\partial_\tau \Big|_{\tau=t} \Phi^\tau_s(x) = 2^q.$$

This allows us to show that

$$\frac{\partial_\tau \Big|_{\tau=t} \Phi^\tau_s(\gamma_t)}{\ell^q(\gamma)} \cdot \frac{1}{h^{\varphi_s(\gamma_s),s}_{\gamma_s}(t)} = \left(\frac{1+\ell+2s}{1+\ell+2t}\right)^{n-1}$$

if $|\gamma_0| = 1+\ell$ which, of course, matches (3.38) and verifies Theorem 3.41. Note that in general one will not have such explicit information. As such, an expression like (3.39) will be not at disposal; hence, it is necessary to deduce information by comparing the disintegration described in (3.39) with another one. Observe that the measure being pushed forward in (3.39) lives on $\mathrm{e}_t(G_{a,s})$ and was obtained from a disintegration with respect to a time varying partition of $\mathrm{e}_t(G_{a,s})$. For the second disintegration we instead focus on varying the level set values a to form a partition of $\mathrm{e}_t(G)$. This description should be compared with subsection 3.4.2 and the comparison done in subsection 3.4.3.

Let $(X,\mathsf{d},\mathfrak{m})$ be a p-essentially non-branching metric measure space satisfying $\mathsf{CD}_p(K,N)$ and, consequently from Theorem 3.31, also the strengthening of $\mathsf{CD}^1_{Lip}(K,N)$ described in Remark 3.32. This will be needed to close the argument: the strengthening of $\mathsf{CD}^1_{Lip}(K,N)$ will give a "canonical" way of disintegrating the measure $\mathfrak{m}$ that will be crucial to implement our strategy.

Given any $q>1$, we will prove that $(X,\mathsf{d},\mathfrak{m})$ also verifies $\mathsf{CD}_q(K,N)$, provided the space is q-essentially non-branching as well. Recall that without loss of generality we can assume $\mathrm{supp}(\mathfrak{m})=X$ and we have the standing assumption that $\mathfrak{m}(X)=1$.

Fix $\mu_0,\mu_1\in\mathcal{P}_q(X,\mathsf{d},\mathfrak{m})$. From the curvature assumption it follows that (X,d) is a geodesic space, hence, from Section 1.2, $(\mathcal{P}_q(X),W_q)$ is a geodesic space as well; therefore the set of q-optimal dynamical plan $\mathrm{OptGeo}_q(\mu_0,\mu_1)$ is not empty.

Recall moreover that $\mathsf{CD}_p(K,N)$ implies qualitative non-degeneracy (1.4) by [**61**], hence Theorem 1.30 yields a unique $\nu\in\mathrm{OptGeo}_q(\mu_0,\mu_1)$ and

$$[0,1]\ni t\mapsto \mu_t := (\mathrm{e}_t)_\sharp\nu = \rho_t\mathfrak{m}.$$

Finally, let $\varphi: X\to\mathbb{R}$ be a q-Kantorovich potential for the Optimal transport problem from μ_0 to μ_1, with cost $c:=\mathsf{d}^q/q$. Recall that $G_\varphi\subset\mathrm{Geo}(X)$ denotes the set of (φ,q)-Kantorovich geodesics, i.e. all the geodesics γ for which

$$\varphi(\gamma_0)+\varphi^c(\gamma_1)=\frac{\mathsf{d}^q(\gamma_0,\gamma_1)}{q}.$$

We will also denote with G^0_φ the set of null (φ,q)-Kantorovich geodesic defined as follows:

$$G^0_\varphi := \{\gamma\in G_\varphi : \ell(\gamma)=0\},$$

and its complement in G_φ by G^+_φ.

Using [**35**, Proposition 9.1], the $\mathsf{MCP}(K,N)$ implies some non-trivial regularity properties on the time behaviour of the density ρ_t: indeed the implication (1) $\Rightarrow$ (4) of [**35**, Proposition 9.1] gives a Lipschitz-type bound whenever μ_1 reduces to a Dirac mass δ_o for some $o\in X$ (notice that from [**35**, Remark 9.4] this implication does not require any type of essential non-branching property). Then the case of a general μ_1 can be obtained via approximation: using the q-essential non-branching property in its equivalent formulation given by Theorem 1.30, one can repeat the arguments of [**35**, Proposition 9.1] in the implications (4) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) where the main points were uniqueness of optimal dynamical plans and upper semi-continuity of entropies, both still valid in our framework. We summarize this discussion in the next statement:

Corollary 3.33 (Logarithmic finite difference bounds for interpolating densities along characteristics). *Let $(X,\mathsf{d},\mathfrak{m})$ be a q-essentially non-branching m.m.s. verifying $\mathsf{MCP}(K,N)$. Then for all $\mu_0,\mu_1\in\mathcal{P}_q(X)$ with $\mu_0\ll\mathfrak{m}$ there exists a unique $\nu\in\mathrm{OptGeo}_q(\mu_0,\mu_1)$ and a map $S:X\to\mathrm{Geo}(X)$ such that $\nu=S_\sharp\mu_0$.*

Moreover $\mu_t=(\mathrm{e}_t)_\sharp\nu\ll\mathfrak{m}$ for $t\in[0,1)$ and there exist versions of the densities $\rho_t=\frac{d\mu_t}{d\mathfrak{m}}$, such that for ν-a.e. $\gamma\in\mathrm{Geo}(X)$, for all $0\le s\le t<1$, it holds

$$\rho_s(\gamma_s)>0,\quad \left(\tau_{K,N}^{(\frac{s}{t})}(\mathsf{d}(\gamma_0,\gamma_t))\right)^N \le \frac{\rho_t(\gamma_t)}{\rho_s(\gamma_s)} \le \left(\tau_{K,N}^{(\frac{1-t}{1-s})}(\mathsf{d}(\gamma_s,\gamma_1))\right)^{-N}. \tag{3.41}$$

In particular, for ν-a.e. γ, the map $t\mapsto\rho_t(\gamma_t)$ is locally Lipschitz on $(0,1)$ and upper semi-continuous at $t=0$.

A further consequence of Corollary 3.33 can be obtained considering the regularity property of the map $t \mapsto \mathfrak{m}(\mathrm{e}_t(G))$, for some compact subset G of φ-Kantorovich geodesics (see for instance [**35**, Proposition 9.6]).

Proposition 3.34 (Near continuity of the evolution of $\mathrm{spt}\mu_t$). *Let $(X, \mathsf{d}, \mathfrak{m})$ be a q-essentially non-branching m.m.s. verifying $\mathsf{MCP}(K,N)$. For $\mu_0, \mu_1 \in \mathcal{P}_q(X)$ with $\mu_0 \ll \mathfrak{m}$, let ν denote the unique element of $\mathrm{OptGeo}_q(\mu_0, \mu_1)$.*

Then for any compact set $G \subset \mathrm{Geo}(X)$ with $\nu(G) > 0$, such that (3.41) holds true for all $\gamma \in G$ and $0 \leq s \leq t < 1$, it holds for any $t \in (0,1)$:

$$\lim_{\varepsilon \to 0+} \frac{\mathcal{L}^1\big(G(x) \cap (t-\varepsilon, t+\varepsilon)\big)}{2\varepsilon} = 1 \quad \textit{in } L^1(\mathrm{e}_t(G), \mathfrak{m}),$$

where $G(x) = \bigcup_{\gamma \in G} \gamma^{-1}(x)$.

Finally, we conclude this first part by recalling the definition of a special class of Kantorovich geodesics.

Definition 3.35 (Good collections of geodesics). Given $\mu_0, \mu_1 \in \mathcal{P}_q(X)$ with $\mu_0 \ll \mathfrak{m}$, we say that $G \subset G_\varphi^+$ is a *good* subset of geodesics if the following properties hold true:

(1) G is compact;
(2) there exists a constant $c > 0$ such that for every $\gamma \in G$: $c \leq \ell(\gamma) \leq 1/c$;
(3) for every $\gamma \in G$, $\rho_t(\gamma_t) > 0$ for all $t \in [0,1]$ and the map $(0,1) \ni t \mapsto \rho_t(\gamma_t)$ is continuous;
(4) the claim of Proposition 3.34 holds true for G;
(5) The map $\mathrm{e}_t|_G : G \to X$ is injective.

From now on we will assume $G \subset G_\varphi^+$ to be a good subset. In particular all the results contained in Sections 3.4.1, 3.4.2, and 3.4.3 will be obtained tacitly assuming any optimal dynamical plan to be concentrated on a good subset of geodesics.

We will dispose of this assumption in Section 3.4.4 via an approximation argument. Notice indeed that under q-essentially non-branching and $\mathsf{MCP}(K,N)$ for any $\nu \in \mathrm{OptGeo}_q(\mu_0, \mu_1)$ with $\mu_0 \ll \mathfrak{m}$, and any $\varepsilon > 0$ there exists a *good* compact subset $G^\varepsilon \subset G_\varphi^+$ such that $\nu(G^\varepsilon) \geq \nu(G_\varphi^+) - \varepsilon$ for any $\varepsilon > 0$. Without loss of generality, we can also assume that G^ε increases along any given sequence of ε decreasing to 0.

In what follows we will use a suitable collection of L^1-optimal transport problems to decompose the Jacobian of the evolution of the W_q-geodesic $t \to \mu_t$ and to obtain key estimates on both components: our interest will be focused on finding a codimension-1 Jacobian orthogonal to the evolution and a one-dimensional counterpart. For both of these factors, curvature estimates will be obtained via L^1-optimal transport techniques, in particular Theorem 3.31, by comparing two families of conditional measures: one coming from the aforementioned L^1-optimal transport problem and the other one from the q-Kantorovich potential.

The decomposition technique will be very similar to the one developed in [**35**]; we will not repeat all the proofs but just list the main differences and include additional details where needed.

3.4.1. L^1 Partition. For $s \in [0,1]$ and $a \in \mathbb{R}$, we define the set of geodesics $G_{a,s} \subset G_\varphi$ as follows:

$$G_{a,s} = \{\gamma \in G : \varphi_s(\gamma_s) = a\}.$$

Let us observe that since G is compact and $\mathrm{e}_s : G \to X$ is continuous, $\mathrm{e}_s(G)$ is still compact. Moreover, for $s \in (0,1)$, $\varphi_s : X \to \mathbb{R}$ is continuous and hence $G_{a,s}$ is compact as well.
Let us fix $a \in \varphi_s(\mathrm{e}_s(G))$. The aim of the next subsection will be to analyze the structure of the evolution of the set $G_{a,s}$, i.e. $\mathrm{e}_{[0,1]}(G_{a,s})$.

From now on we will denote the signed-distance function from a level set a of φ_s with $d_{a,s} := d_{\varphi_s - a}$ (recall the notation of (1.19)). Since $d_{a,s}$ is a 1-Lipschitz function, we can associate to it all the sets introduced in Section 1.4.3, including the transport ordering $\Gamma_{d_{a,s}} = \leq_{d_{a,s}}$,the relation $R_{d_{a,s}} = \Gamma_{d_{a,s}} \cup \Gamma^{-1}_{d_{a,s}}$ and set $\mathcal{T}_{d_{a,s}} \subset P_1(R_{d_{a,s}})$.

Lemma 3.36. *Let (X, d) be a geodesic space. Once $s \in [0,1]$ and $a \in \varphi_s(\mathrm{e}_s(G))$ are fixed, then for each $\gamma \in G_{a,s}$ and for every $0 \leq r \leq t \leq 1$, $(\gamma_r, \gamma_t) \in \Gamma_{d_{a,s}}$. In particular,*

$$\mathrm{e}_{[0,1]}(G_{a,s}) \subset \mathcal{T}_{\mathsf{d}_{a,s}}.$$

The proof goes along the same lines of [**35**, Lemma 10.3]; we have included it for the reader's convenience.

Proof. Let us fix $\gamma \in G_{a,s}$. By Corollary 3.13 and Lemma 3.15 (2), we have that if $s \in [0,1)$ then for any $x \in \{\varphi_s = a\}$, it holds

$$\frac{\mathsf{d}^p(\gamma_s,\gamma_1)}{p(1-s)^{p-1}} = \varphi_s(\gamma_s) + \varphi^c(\gamma_1) = \varphi_s(x) + \varphi^c(\gamma_1) \leq \bar{\varphi}_s(x) + \varphi^c(\gamma_1) \leq \frac{\mathsf{d}^p(x,\gamma_1)}{p(1-s)^{p-1}}.$$

Hence $\mathsf{d}(\gamma_s,\gamma_1) \leq \mathsf{d}(x,\gamma_1)$. In the same way, if $s \in (0,1]$, then for any $y \in \{\varphi_s = a\}$ we have that

$$\frac{\mathsf{d}^p(\gamma_s,\gamma_0)}{ps^{p-1}} = \varphi(\gamma_0) - \varphi_s(\gamma_s) = \varphi(\gamma_0) - \varphi_s(y) \leq \frac{\mathsf{d}^p(y,\gamma_0)}{ps^{p-1}}.$$

So $\mathsf{d}(\gamma_s,\gamma_0) \leq \mathsf{d}(y,\gamma_0)$, which is also trivially satisfied in the case $s = 0$. Thus, for any $x, y \in \{\varphi_s = a\}$ we have

$$\mathsf{d}(\gamma_0,\gamma_1) \leq \mathsf{d}(\gamma_0, x) + \mathsf{d}(y,\gamma_1).$$

Taking the infimum over x and y we get that

$$\mathsf{d}(\gamma_0,\gamma_1) \leq d_{a,s}(\gamma_0) - d_{a,s}(\gamma_1),$$

where the sign of $\mathsf{d}_{a,s}$ was determined by the fact that $s \mapsto \varphi_s(\gamma_s)$ is decreasing. More precisely, the latter relation turns out to hold as an equality by 1-Lipschitz regularity of $d_{a,s}$, thus $(\gamma_0,\gamma_1) \in \Gamma_{\mathsf{d}_{a,s}}$. This implies that for every $0 \leq r \leq t \leq 1$, $(\gamma_r,\gamma_t) \in \Gamma_{d_{a,s}}$. □

By Theorem 3.31 we have that, choosing $u = d_{a,s}$, the following disintegration formula holds

$$\mathfrak{m}\llcorner_{\mathcal{T}_{\mathsf{d}_{a,s}}} = \int_Q \hat{\mathfrak{m}}^{a,s}_\alpha \hat{\mathfrak{q}}^{a,s}(d\alpha), \tag{3.42}$$

where Q is a section of the partition of $\mathcal{T}^b_{d_{a,s}}$ given by the equivalence classes $\{R^b_{d_{a,s}}(\alpha)\}_{\alpha \in Q}$, and for $\hat{q}^{a,s}$-a.e. $\alpha \in Q$, $\hat{\mathfrak{m}}^{a,s}_\alpha$ is a probability measure supported on the transport ray $X_\alpha = R_{d_{a,s}}(\alpha)$ and $(X_\alpha, \mathsf{d}, \hat{\mathfrak{m}}^{a,s}_\alpha)$ verifies $\mathsf{CD}(K,N)$. By Lemma 3.36, it follows that:

$$\mathfrak{m}\llcorner_{\mathrm{e}_{[0,1]}(G_{a,s})} = \int_Q \hat{\mathfrak{m}}^{a,s}_\alpha \llcorner_{\mathrm{e}_{[0,1]}(G_{a,s})} \hat{\mathfrak{q}}^{a,s}(d\alpha).$$

From the very definition of $G_{a,s}$ and the p-essentially non-branching property, in the previous disintegration formula the quotient set Q can be naturally identified with $\mathrm{e}_s(G_{a,s})$; moreover, we can consider the Borel parametrization

$$g^{a,s} : \mathrm{e}_s(G_{a,s}) \times [0,1] \to X, \quad g^{a,s}(\alpha,\cdot) = (\mathrm{e}_s\llcorner_{G_{a,s}})^{-1}(\alpha),$$

yielding the following disintegration formula:

$$\mathfrak{m}\llcorner_{\mathrm{e}_{[0,1]}(G_{a,s})} = \int_{\mathrm{e}_s(G_{a,s})} g^{a,s}(\alpha,\cdot)_\sharp \left(h^{a,s}_\alpha \cdot \mathcal{L}^1\llcorner_{[0,1]}\right) \mathfrak{q}^{a,s}(d\alpha), \tag{3.43}$$

where $\mathfrak{q}^{a,s}$ is a Borel measure concentrated on $\mathrm{e}_s(G_{a,s})$, and for $\mathfrak{q}^{a,s}$-a.e. $\alpha \in \mathrm{e}_s(G_{a,s})$, $h^{a,s}_\alpha$ is a $\mathsf{CD}(\ell_s(\alpha)^2 K, N)$ density on $[0,1]$. Notice that the factor $\ell_s(\alpha)^2 = \mathcal{H}^1(X_\alpha)^2$ is due to the reparametrization of the transport ray on $[0,1]$. This permits, invoking Fubini's theorem, to reverse the order of integration so to have:

$$\mathfrak{m}\llcorner_{\mathrm{e}_{[0,1]}(G_{a,s})} = \int_{[0,1]} g^{a,s}(\cdot,t)_\sharp(h^{a,s}_\cdot(t)\cdot\mathfrak{q}^{a,s})\mathcal{L}^1(dt) = \int_{[0,1]} \mathfrak{m}^{a,s}_t\,\mathcal{L}^1(dt), \tag{3.44}$$

where we defined

$$\mathfrak{m}^{a,s}_t := g^{a,s}(\cdot,t)_\sharp(h^{a,s}_\cdot(t)\cdot\mathfrak{q}^{a,s}).$$

Finally, the previous disintegration formula does not change if we multiply and divide conditional measures by $h^{a,s}_\alpha(s)$; therefore, changing $\mathfrak{q}^{a,s}$, we can assume $h^{a,s}_\alpha(s)=1$, yielding $\mathfrak{m}^{a,s}_s=\mathfrak{q}^{a,s}$ and

$$\mathfrak{m}^{a,s}_t := g^{a,s}(\cdot,t)_\sharp(h^{a,s}_\cdot(t)\cdot\mathfrak{m}^{a,s}_s). \tag{3.45}$$

Moreover (see [**35**, Proposition 10.7]), for any $s\in(0,1)$ and $a\in\varphi_s(\mathrm{e}_s(G))$, the map

$$(0,1)\ni t\mapsto\mathfrak{m}^{a,s}_t$$

is continuous in the weak topology and if $\mathfrak{m}(\mathrm{e}_{[0,1]}(G_{a,s}))>0$, then $\mathfrak{m}^{a,s}_t(\mathrm{e}_t(G_{a,s}))>0$, for all $t\in(0,1)$. Finally,

$$\forall t\in[0,1]\quad \mathfrak{m}^{a,s}_t(\mathrm{e}_t(G_{a,s}))=\|\mathfrak{m}^{a,s}_t\|\le C\,\mathfrak{m}(\mathrm{e}_{[0,1]}(G_{a,s})),$$

for some $C>0$ depending only on K, N and $\{\ell(\gamma):\gamma\in G_{a,s}\}$.

3.4.2. L^q partition. We will now consider a decomposition of $\mathfrak{m}$ into conditional measures induced by Kantorovich potentials.

Hence for any $s,t\in(0,1)$, let us consider $a\in\Phi^t_s(\mathrm{e}_t(G))=\varphi_s(\mathrm{e}_s(G))$. With such a choice of a, the compact set $\mathrm{e}_t(G)$ admits a partition given by $\mathrm{e}_t(G)\cap\{\Phi^t_s=a\}_{a\in\mathbb{R}}$.

Continuity of Φ^t_s makes it possible to apply the Disintegration Theorem. Since $\mathfrak{m}[\mathrm{e}_t(G)]<\infty$, there exists an essentially unique disintegration of $\mathfrak{m}\llcorner_{\mathrm{e}_t(G)}$ strongly consistent with respect to the quotient map Φ^t_s:

$$\mathfrak{m}\llcorner_{\mathrm{e}_t(G)} = \int_{\varphi_s(\mathrm{e}_s(G))}\hat{\mathfrak{m}}^t_{a,s}\mathfrak{q}^t_s(da) \tag{3.46}$$

where $\mathfrak{q}^t_s=(\Phi^t_s)_\sharp\mathfrak{m}\llcorner_{\mathrm{e}_t(G)}$ and $\hat{\mathfrak{m}}^t_{a,s}$ is a probability measure concentrated on the set $\mathrm{e}_t(G)\cap\{\Phi^t_s=a\}=\mathrm{e}_t(G_{a,s})$.

Notice that, as one would expect, being the image of a time propagation of an intermediate Kantorovich potential, the quotient set $\varphi_s(\mathrm{e}_s(G))$ does not depend on t.

The next follows with no modification from [**35**, Proposition 10.8].

Proposition 3.37. *The following properties hold true:*

1. *For any $s,t,\tau\in(0,1)$, the quotient measures $\mathfrak{q}^t_s$ and $\mathfrak{q}^\tau_s$ are mutually absolutely continuous;*
2. *For any $s,t\in(0,1)$, the quotient measure $\mathfrak{q}^t_s$ is absolutely continuous with respect to Lebesgue measure $\mathcal{L}^1$ on $\mathbb{R}$.*

Employing what we obtained so far, we can rewrite (3.46) in the following way:

$$\mathfrak{m}\llcorner_{\mathrm{e}_t(G)} = \int_{\varphi_s(\mathrm{e}_s(G))}\mathfrak{m}^t_{a,s}\mathcal{L}^1(da), \tag{3.47}$$

where $\mathfrak{m}^t_{a,s}:=(d\mathfrak{q}^t_s/d\mathcal{L}^1)\cdot\hat{\mathfrak{m}}^t_{a,s}$ is concentrated on $\mathrm{e}_t(G_{a,s})$ for $\mathcal{L}^1$-a.e. $a\in\varphi_s(e_s(G))$.

Over the set $\mathrm{e}_t(G)$ we also have the measure μ_t; as it can be lifted to the set $\mathrm{Geo}(X)$, it makes sense to notice that the family of sets $\{G_{a,s}\}_{a\in\mathbb{R}}$ provides a partition of G. Hence an application of the Disintegration Theorem guarantees the existence of an essentially unique disintegration of ν strongly consistent with respect $\varphi_s \circ \mathrm{e}_s$:

$$\nu = \int_{\varphi_s(\mathrm{e}_s(G))} \nu_{a,s} \mathfrak{q}_s^{\nu}(da) \tag{3.48}$$

where the probability measure $\nu_{a,s}$ is concentrated on $G_{a,s}$ for $\mathfrak{q}_s^{\nu}$-a.e. $a \in \varphi_s(\mathrm{e}_s(G))$. In particular, $q_s^{\nu}(\varphi_s(\mathrm{e}_s(G))) = ||\nu|| = 1$.

Multiplying (3.47) by ρ_t and applying $(\mathrm{e}_t)_\sharp$ to (3.48) produces the same measure μ_t: this permits to deduce what follows. For all the missing details we refer to [**35**, Corollary 10.10].

Corollary 3.38. *We have the following*

(1) *For any $s \in (0,1)$, the quotient measure $\mathfrak{q}_s^{\nu}$ is mutually absolutely continuous with respect to $\mathfrak{q}_s^s$. In particular, it is absolutely continuous with respect to $\mathcal{L}^1$.*
(2) *For any $s,t \in (0,1)$ and $\mathcal{L}^1$-a.e. $a \in \varphi_s(\mathrm{e}_s(G))$:*

$$\rho_t \cdot \mathfrak{m}_{a,s}^t = q_s^{\nu}(a) \cdot (\mathrm{e}_t)_\sharp \nu_{a,s},$$

where $q_s^{\nu} := d\mathfrak{q}_s^{\nu}/d\mathcal{L}^1$. In particular, $\mathfrak{m}_{a,s}^t$ and $(\mathrm{e}_t)_\sharp \nu_{a,s}$ are mutually absolutely continuous for $\mathfrak{q}_s^{\nu}$-a.e. $a \in \varphi_s(\mathrm{e}_s(G))$.
(3) *For any $s \in (0,1)$ and $\mathfrak{q}_s^{\nu}$-a.e. $a \in \varphi_s(e_s(G))$, the maps*

$$[0,1] \ni t \mapsto \rho_t \cdot \mathfrak{m}_{a,s}^t, \quad [0,1] \ni t \mapsto (\mathrm{e}_t)_\sharp \nu_{a,s}$$

coincide for $\mathcal{L}^1$-a.e. $t \in [0,1]$ up to a positive multiplicative constant $C_{a,s}$ depending only on a,s.

3.4.3. Comparison between conditional measures. We will now link the seemingly unrelated disintegrations (3.44) and (3.47). Observe that $\mathfrak{m}_{a,s}^t$ and $\mathfrak{m}_t^{a,s}$ are concentrated on $\mathrm{e}_t(G_{a,s})$, for each $t \in (0,1)$ for $\mathcal{L}^1$-a.e. $a \in \varphi_s(e_s(G))$ and for each $a \in \varphi_s(e_s(G))$ and all $t \in (0,1)$, respectively. The common feature of the two families of conditional measures $\mathfrak{m}_{a,s}^t$ and $\mathfrak{m}_t^{a,s}$ is that they are both coming from a disintegration formula with quotient measure the Lebesgue measure. We can exploit this property in the next lemma.

Lemma 3.39. *For every $s,t \in (0,1)$ and $a \in \varphi_s(e_s(G))$, the limit*

$$\mathfrak{m}_t^{a,s} = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \mathfrak{m} \llcorner_{\mathrm{e}_{[t-\varepsilon,t+\varepsilon]}(G_{a,s})}$$

holds true in the weak topology.

Proof. Since $(0,1) \ni t \mapsto \mathfrak{m}_t^{a,s}$ is continuous in the weak topology, and so together with (3.44), we see that for any $f \in C_b(X)$:

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_X f(z) \mathfrak{m} \llcorner_{\mathrm{e}_{[t-\varepsilon,t+\varepsilon]}(G_{a,s})}(dz) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \left(\int_X f(z) \mathfrak{m}_\tau^{a,s}(dz) \right) \mathcal{L}^1(d\tau) = \int_X f(z) \mathfrak{m}_t^{a,s}(dz),$$

thereby concluding the proof. □

We are now in position to compare $\mathfrak{m}_{a,s}^t$ and $\mathfrak{m}_t^{a,s}$ by comparing $\mathfrak{m}$ in a neighborhood of $\mathrm{e}_t(G_{a,s})$ obtained first varying t and then varying a. We refer to [**35**, Theorem 11.3] for all the details in the case $q = 2$ and simply note that the argument works for any $q > 1$; (the main ingredients needed for the proof are the disintegration formulas (3.44), (3.47) and temporal regularity of Φ_s^t obtained in Section 3.2).

Theorem 3.40 (Relating factorization by potential values and by φ-Kantorovich geodesics via Fubini). *For any $s \in (0,1)$,*

$$\mathfrak{m}_s^{a,s} = \ell_s^p \cdot \mathfrak{m}_{a,s}^s, \quad \textit{for} \quad \mathcal{L}^1\textit{-a.e. } a \in \varphi_s(\mathrm{e}_s(G)).$$

Moreover, for any $s \in (0,1)$ *and* $\mathcal{L}^1$*-a.e.* $t \in (0,1)$ *including at* $t = s$, $\partial_t \Phi_s^t(x)$ *exists and is positive, and for* $\mathcal{L}^1$*-a.e.* $a \in \varphi_s(\mathrm{e}_s(G))$ *and* $\mathfrak{m}_{a,s}^t$*-a.e.* x *we have:*

$$\mathfrak{m}_t^{a,s} = \partial_t \Phi_s^t \cdot \mathfrak{m}_{a,s}^t, \quad \text{for} \quad \mathcal{L}^1\text{-a.e.} \quad a \in \varphi_s(\mathrm{e}_s(G)). \tag{3.49}$$

3.4.4. Change of variable formula. Building on Theorem 3.40, we are now in position to write the Jacobian associated to the evolution of μ_t as the product of two factors. All the results obtained until now will be used to prove the following:

THEOREM 3.41 (Change of variables formula). *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a* p*-essentially non branching m.m.s. satisfying* $\mathsf{CD}_p(K,N)$ *and assume it is also* q*-essentially non branching. Let us consider* $\mu_0, \mu_1 \in \mathcal{P}_q(X, \mathsf{d}, \mathfrak{m})$ *and let* ν *denote the unique element of* $\mathrm{OptGeo}_q(\mu_0, \mu_1)$*. Setting* $\mu_t = (\mathrm{e}_t)_\sharp \nu \ll \mathfrak{m}$*, we will consider the densities* $\rho_t := d\mu_t/d\mathfrak{m}$, $t \in [0,1]$*, given by Corollary 3.33. Then for any* $s \in (0,1)$*, for* $\mathcal{L}^1$*-a.e.* $t \in (0,1)$ *and* ν*-a.e.* $\gamma \in G_\varphi^+$, $\partial_\tau|_{\tau=t} \Phi_s^\tau(\gamma_t)$ *exists and the following formula holds:*

$$\frac{\rho_t(\gamma_t)}{\rho_s(\gamma_s)} = \frac{\partial_\tau|_{\tau=t} \Phi_s^\tau(\gamma_t)}{\ell^p(\gamma)} \cdot \frac{1}{h_{\gamma_s}^{\varphi_s(\gamma_s),s}(t)}. \tag{3.50}$$

Here $h_{\gamma_s}^{\varphi_s(\gamma_s),s}$ *is the* $\mathsf{CD}(\ell(\gamma)^2 K, N)$ *density on* $[0,1]$ *from* (3.43)*, renormalized in such a way* $h_{\gamma_s}^{\varphi_s(\gamma_s),s}(s) = 1$*. Finally, for all* $\gamma \in G_\varphi^0$*, it holds:*

$$\rho_t(\gamma_t) = \rho_s(\gamma_s), \qquad \forall t, s \in [0,1]. \tag{3.51}$$

PROOF. By [**35**, Lemma 6.11] and the discussion below Definition 1.37, $(X, \mathsf{d}, \mathfrak{m})$ verifies $\mathsf{MCP}(K,N)$ and Corollary 3.33 guarantees the existence of versions of the densities satisfying (3.41). For any $\varepsilon > 0$, there exists a *good* compact subset $G^\varepsilon \subset G_\varphi^+$ such that $\nu(G^\varepsilon) \geq \nu(G_\varphi^+) - \varepsilon$ and such that G_ε increases along a sequence of ε decreasing to 0.

Fixing $\varepsilon > 0$ on this sequence and the good subset G^ε, let us set

$$\nu^\varepsilon = \frac{1}{\nu(G^\varepsilon)} \nu \llcorner_{G^\varepsilon}, \quad \mu_t^\varepsilon := (\mathrm{e}_t)_\sharp \nu^\varepsilon \ll \mathfrak{m}.$$

In particular we have that $\mu_t^\varepsilon = \frac{1}{\nu(G^\varepsilon)} \mu \llcorner_{\mathrm{e}_t(G_\varepsilon)}$, for all $t \in [0,1]$ and therefore:

$$\mu_t^\varepsilon = \rho_t^\varepsilon \mathfrak{m}, \quad \rho_t^\varepsilon := \frac{1}{\nu(G^\varepsilon)} \rho_t|_{\mathrm{e}_t(G^\varepsilon)}, \ \forall t \in [0,1].$$

As we proved in Corollary 3.38, for each $s \in (0,1)$ and $\mathfrak{q}_s^{\varepsilon,s}$-a.e. $a \in \varphi_s(\mathrm{e}_s(G^\varepsilon))$, the map $[0,1] \ni t \mapsto \rho_t \cdot \mathfrak{m}_{a,s}^{\varepsilon,t}$ coincides for $\mathcal{L}^1$-a.e. $t \in [0,1]$ with the geodesic $t \mapsto (\mathrm{e}_t)_\sharp \nu_{a,s}^\varepsilon$ up to a constant $C_{a,s}^\varepsilon > 0$. Hence, for such s and a, for $\mathcal{L}^1$ a.e $t \in [0,1]$, we have that for any Borel set $H \subset G^\varepsilon$ the quantity

$$\int_{\mathrm{e}_t(H)} \rho_t^\varepsilon(x) \mathfrak{m}_{a,s}^{\varepsilon,t}(dx) = C_{a,s}^\varepsilon \int_{\mathrm{e}_t(H)} (\mathrm{e}_t)_\sharp \nu_{a,s}^\varepsilon(dx) = C_{a,s}^\varepsilon \nu_{a,s}^\varepsilon(H) \tag{3.52}$$

is constant in t, where in the last equality we used the injectivity of the map $\mathrm{e}_t : G^\varepsilon \to X$. By Theorem 3.40, for $\mathcal{L}^1$-a.e. $t \in (0,1)$ and $\mathcal{L}^1$-a.e. $a \in \varphi_s(G_s^\varepsilon)$, $\partial_t \Phi_s^t(x)$ exists and is positive for $\mathfrak{m}_{a,s}^{\varepsilon,t}$-a.e. x; moreover (3.49) holds. Thus, for all a, s and t for which the previous condition and (3.52) hold, we have

$$\begin{aligned} C_{a,s}^\varepsilon \nu_{a,s}^\varepsilon(H) &= \int_{\mathrm{e}_t(H)} \rho_t^\varepsilon(x) \mathfrak{m}_{a,s}^{\varepsilon,t}(dx) = \int_{\mathrm{e}_t(H)} \rho_t^\varepsilon(x) (\partial_t \Phi_s^t(x))^{-1} \mathfrak{m}_t^{\varepsilon,a,s}(dx) \\ &= \int_{\mathrm{e}_s(H)} \rho_t^\varepsilon(g^{a,s}(\alpha,t)) (\partial_\tau|_{\tau=t} \Phi_s^\tau(g^{a,s}(\alpha,t)))^{-1} h_\alpha^{a,s}(t) \mathfrak{m}_s^{\varepsilon,a,s}(d\alpha) \\ &= \int_{\mathrm{e}_s(H)} \rho_t^\varepsilon(g^{a,s}(\alpha,t)) (\partial_\tau|_{\tau=t} \Phi_s^\tau(g^{a,s}(\alpha,t)))^{-1} h_\alpha^{a,s}(t) \ell_s^p(\alpha) \mathfrak{m}_{a,s}^{\varepsilon,s}(d\alpha) \end{aligned} \tag{3.53}$$

where the two last equalities follow from (3.45) and Theorem 3.40, respectively.

Since the left-hand side of (3.53) does not depend on t, it follows that for all $s \in (0,1)$ and for $q_s^{\varepsilon,s}$-a.e. $a \in \varphi_s(e_s(G^\varepsilon))$, there exists a subset $T \subset (0,1)$ of full $\mathcal{L}^1$ measure such that for all $H \subset G^\varepsilon_{a,s}$ the map

$$T \ni t \mapsto \int_{\mathrm{e}_s(H)} \rho_t^\varepsilon(g^{a,s}(\alpha,t))(\partial_\tau|_{\tau=t}\Phi_s^\tau(g^{a,s}(\alpha,t)))^{-1}h_\alpha^{a,s}(t)\ell_s^p(\alpha)\mathfrak{m}^{\varepsilon,s}_{a,s}(d\alpha),$$

is constant. In particular, since any Borel subset of $\mathrm{e}_s(G_{a,s})$ can be written in the form $\mathrm{e}_s(H)$, we have that for $t,t' \in T$

$$\rho_{t'}^\varepsilon(\gamma_{t'})(\partial_\tau|_{\tau=t'}\Phi_s^\tau(\gamma_{t'}))^{-1}h^{a,s}_{\gamma_s}(t') = \rho_t^\varepsilon(\gamma_t)(\partial_\tau|_{\tau=t}\Phi_s^\tau(\gamma_t))^{-1}h^{a,s}_{\gamma_s}(t), \tag{3.54}$$

for $\mathfrak{m}^{\varepsilon,s}_{a,s}$-a.e. $\alpha \in \mathrm{e}_s(G^\varepsilon_{a,s})$ where $\gamma = \mathrm{e}_s^{-1}(\alpha) = g^{a,s}(\alpha,\cdot) \in G^\varepsilon_{a,s}$, with the exceptional set depending on t,t'. Recall that, by Corollary 3.38, given $t' \in T$, $\partial_\tau|_{\tau=t'}\Phi_s^\tau(\gamma^\alpha_{t'})$ exists for $\mathfrak{m}^{\varepsilon,s}_{a,s}$-a.e. $\alpha \in \mathrm{e}_s(G^\varepsilon_{a,s})$. Thus, in particular, the equality (3.54) holds for a countable sequence of $\{t'\} \subset T$ dense in $(0,1)$. Using the normalization $h^{a,s}_{\gamma_s}(s) = 1$, the continuity of $h^{a,s}_{\gamma_s}(\cdot)$, $\rho^\varepsilon_\cdot(\gamma_\cdot)$ and the fact that

$$\lim_{T \ni t' \to s} \partial_\tau|_{\tau=t'}\Phi_s^\tau(\gamma^\alpha_{t'}) = \ell_s(\gamma^\alpha_s)^p = \ell(\gamma^\alpha)^p,$$

it is possible to pass to the limit for $t' \to s$ in (3.54)

$$\rho_s^\varepsilon(\gamma_s)\ell(\gamma)^{-p} = \rho_t^\varepsilon(\gamma_t)(\partial_\tau|_{\tau=t}\Phi_s^\tau(\gamma_t))^{-1}h^{a,s}_{\gamma_s}(t), \tag{3.55}$$

for $\mathfrak{m}^{\varepsilon,s}_{a,s}$-a.e. $\alpha \in \mathrm{e}_s(G^\varepsilon_{a,s})$, with $\gamma = \mathrm{e}_s^{-1}(\alpha) \in G^\varepsilon_{a,s}$.

By corollary 3.38, the measures $\mathfrak{m}^{\varepsilon,s}_{a,s}$ and $(\mathrm{e}_s)_\sharp\nu^\varepsilon_{a,s}$ are mutually absolutely continuous for $\mathfrak{q}_s^{\varepsilon,s}$-a.e. $a \in \varphi_s(\mathrm{e}_s(G^\varepsilon))$. In particular, this implies that for all $s \in (0,1)$, for $\mathfrak{q}_s^{\varepsilon,s}$-a.e. $a \in \varphi_s(\mathrm{e}_s(G^\varepsilon))$ and $\mathcal{L}^1$-a.e. $t \in (0,1)$, the equality (3.54) holds for $\nu_{a,s}$-a.e. γ. By Corollary 3.38, it follows that the measures $\mathfrak{q}_s^{\varepsilon,s}$ and $\mathfrak{q}_s^{\varepsilon,\nu}$ are mutually absolutely continuous; thus, by the disintegration formula (3.48), it follows that for all $s \in (0,1)$ and $\mathcal{L}^1$-a.e. $t \in (0,1)$:

$$\rho_s^\varepsilon(\gamma_s)\ell(\gamma)^{-p} = \rho_t^\varepsilon(\gamma_t)(\partial_\tau|_{\tau=t}\Phi_s^\tau(\gamma_t))^{-1}h^{\varphi_s(\gamma_s),s}_{\gamma_s}(t),$$

for ν-a.e. $\gamma \in G^\varepsilon$. Passing to the limit as $\varepsilon \to 0$ along the chosen sequence, it turns out that all $s \in (0,1)$, $\mathcal{L}^1$-a.e. $t \in (0,1)$ and ν-a.e. $\gamma \in G^+_\varphi$ satisfy

$$\rho_s(\gamma_s)\ell(\gamma)^{-p} = \rho_t(\gamma_t)(\partial_\tau|_{\tau=t}\Phi_s^\tau(\gamma_t))^{-1}h^{\varphi_s(\gamma_s),s}_{\gamma_s}(t).$$

By Fubini 's Theorem, for ν-a.e. $\gamma \in G^+_\varphi$, we have that (3.50) holding for $\mathcal{L}^1$-a.e. $s,t \in (0,1)$. □

Remark 3.42. All of the results of this section also hold for $\bar\Phi^t_s$ in place of Φ^t_s. Indeed, recall that for all $x \in X$, $\Phi^t_s(x) = \bar\Phi^t_s(x)$ for $t \in \mathring{G}_\varphi(x)$, and that by Proposition 3.29, $\partial_t\Phi^t_s(x) = \partial_t\bar\Phi^t_s(x)$ for a.e. $t \in \mathring{G}_\varphi(x)$. As these were the only two properties used in the above derivation the assertion follows.

By Proposition 3.29, we know that the differentiability points of $\tau \mapsto \tilde\Phi^\tau_s(x)$ and $\tau \mapsto \tilde\ell^p_\tau(x)$ coincide for all $\tau \neq s$ and at these points

$$\partial_\tau\tilde\Phi^\tau_s(x) = \tilde\ell^p_\tau(x) + (\tau - s)\partial_\tau\frac{\tilde\ell^p_\tau(x)}{p}.$$

Hence by Remark 3.42, we deduce that for ν-a.e. geodesic $\gamma \in G^+_\varphi$ and for a.e. $t \in (0,1)$ both quantities

$$\partial_\tau|_{\tau=t}\ell^p_\tau(\gamma_t) = \partial_\tau|_{\tau=t}\bar\ell^p_\tau(\gamma_t)$$

exist and coincide. We can therefore rewrite the change of variable formula in the following way: for ν-a.e. geodesic $\gamma \in G_\varphi^+$

$$\frac{\rho_s(\gamma_s)}{\rho_t(\gamma_t)} = \frac{h_{\gamma_s}^{\varphi_s(\gamma_s),s}(t)}{1+(t-s)\frac{\partial_\tau|_{\tau=t}\ell_\tau^p(\gamma_t)}{p\ell(\gamma)^p}} = \frac{h_{\gamma_s}^{\varphi_s(\gamma_s),s}(t)}{1+(t-s)\partial_\tau|_{\tau=t}\log\bar{\ell}_\tau(\gamma_t)}, \quad \text{for a.e. } t,s\in(0,1). \tag{3.56}$$

For sake of brevity, once the geodesic γ is fixed, we will use the following notation: $\rho(t) = \rho_t(\gamma_t)$, $h_s(t) := h_{\gamma_s}^{\varphi_s(\gamma_s)}(t)$ and $K_0 = K\cdot\ell(\gamma)^2$. We recall that, by Corollary 3.33 and (3.43), given by Theorem 3.31, the following properties hold true for ν-a.e $\gamma \in G_\varphi^+$:

(A) $(0,1)\ni t\mapsto \rho(t)$ is locally Lipschitz and strictly positive.
(B) For all $s\in(0,1)$, h_s is a $\mathsf{CD}(K_0,N)$ density on $[0,1]$ satisfying $h_s(s)=1$.

Fix now a geodesic $\gamma\in G_\varphi^+$ satisfying the change of variable formula (3.58), (A), (B) above.

The formula (3.56) implies that there exists a set $I\subset(0,1)$ of full measure such that for all $s\in I$ the functions

$$t\mapsto \partial_\tau|_{\tau=t}\frac{\tilde{\ell}_\tau^p/p(\gamma_t)}{\tilde{\ell}(\gamma)^p}, \quad t\mapsto z_s(t) := \frac{\frac{\rho(t)}{\rho(s)}h_s(t)-1}{t-s}$$

coincide a.e. on $(0,1)$ for both $\tilde{\ell}\in\{\ell,\bar{\ell}\}$, with z_s defined on $(0,1)\setminus\{s\}$. Hence, by continuity, the functions $\{z_s\}_{s\in I}$ must all coincide, where defined, with a unique function $t\mapsto z(t)$ defined on $(0,1)$ such that

$$z(t) = \frac{\partial}{\partial\tau}\bigg|_{\tau=t}\log\ell_\tau(\gamma_t) = \frac{\partial}{\partial\tau}\bigg|_{\tau=t}\log\bar{\ell}_\tau(\gamma_t), \text{ for a.e. } t\in(0,1). \tag{3.57}$$

Since $\mathsf{CD}(K,N)$ densities are locally Lipschitz in the interior of the domain where they are defined, we see that z is locally Lipschitz in $(0,1)$ from (3.58). Combining (3.57) with the third order information provided by Theorem 3.26 (up to constant factors) yields:

(C) $(0,1)\ni t\mapsto z(t)$ is locally Lipschitz. Moreover, for any $\delta\in(0,1/2)$ there exists $C_\delta>0$ so that:

$$\frac{z(t)-z(s)}{t-s} \geq (1-C_\delta(t-s))|z(s)||z(t)|, \ \forall\, 0<\delta\leq s<t\leq 1-\delta<1.$$

In particular, $z'(t)\geq z^2(t)$ for a.e. $t\in(0,1)$.

To summarize, the change of variable formula can be rewritten in the following form:

$$\frac{\rho(s)}{\rho(t)} = \frac{h_s(t)}{1+(t-s)z(t)}, \qquad \text{for all } t,s\in(0,1), \tag{3.58}$$

where $z(t)$ coincides for all $t\in(0,1)$ with the second Peano derivative of $\tau\mapsto\varphi_\tau(\gamma_t)$ and of $\tau\mapsto\bar{\varphi}_\tau(\gamma_t)$ at $\tau=t$. These second Peano derivatives exist for all $t\in(0,1)$ and are a continuous function.

We are therefore in position to obtain the aforementioned factorization of the "Jacobian". It has been already proved in [**35**] (see Theorem 12.3) that properties (A), (B), (C) together with the change of variable formula (3.58) are enough to obtain a factorization of the real function $1/\rho(t)$ into a product $L(t)Y(t)$, in which the first factor $L(t)$ is concave due to dilational and dimensional effects (analogous to the Brunn-Minkowski inequality on $(\mathbb{R}^n,|\cdot|,\mathcal{L}^n)$), while the latter term $Y(t)$ captures the effects of the curvature of $(X,\mathsf{d},\mathfrak{m})$. In the smooth case $\rho(t)^{-1/n}$ would be interpreted as the mean-free path between particles during transport.

Theorem 3.43 (Isolating curvature effects in the volume distortion along the direction transported [**35**, Theorem 12.3]). *If the change of variable formula* (3.58) *holds and the properties (A), (B), (C) are satisfied, then*

$$\frac{1}{\rho_t(\gamma_t)} = L(t)Y(t) \quad \forall t \in (0,1),$$

where L is concave and Y is a $\mathsf{CD}(K_0,N)$ density on $(0,1)$.

3.4.5. Main Theorems. Finally, combining the results proved so far in Section 3.3 and Section 3.4 we close the circle proving:

Theorem 3.44 (Non-branching CD_p spaces are CD^1_{Lip} hence CD_q). *Let $(X,\mathsf{d},\mathfrak{m})$ be a p-essentially non-branching m.m.s. verifying $\mathsf{CD}_p(K,N)$ for some $p>1$. If $(X,\mathsf{d},\mathfrak{m})$ is also q-essentially non-branching for some $q>1$, then it verifies $\mathsf{CD}_q(K,N)$.*

Proof. Consider $\mu_0,\mu_1 \in \mathcal{P}_q(X,\mathsf{d},\mathfrak{m})$. Recall that $\mathsf{CD}_p(K,N)$ implies (X,d) to be a geodesic space, hence the same is true for $(\mathcal{P}_q(X),W_q)$. Moreover, it implies (X,d) is $\mathsf{MCP}(K,N)$, hence qualitatively non-degenerate. Since $(X,\mathsf{d},\mathfrak{m})$ is assumed to be q-essentially non-branching, Theorem 1.30 yields a unique $\nu \in \mathrm{OptGeo}_q(\mu_0,\mu_1)$ and

$$[0,1] \ni t \mapsto \mu_t := (\mathrm{e}_t)_\sharp \nu \ll \mathfrak{m}.$$

Let $\rho_t := d\mu_t/d\mathfrak{m}$ be the versions of the densities guaranteed by Corollary 3.33. Finally let $\varphi : X \to \mathbb{R}$ be a Kantorovich potential for the optimal transport problem from μ_0 to μ_1, with cost $c := \mathsf{d}^q/q$. Recall that $G_\varphi \subset \mathrm{Geo}(X)$ denote the set of (φ,q)-Kantorovich geodesics, i.e. all the geodesics γ for which

$$\varphi(\gamma_0) + \varphi^c(\gamma_1) = \frac{\mathsf{d}^q(\gamma_0,\gamma_1)}{q}.$$

As already observed, ν will be concentrated on $G_\varphi = G_\varphi^+ \cup G_\varphi^0$, where G_φ^+ and G_φ^0 denote the subsets of positive and zero length (φ,q)-Kantorovich geodesics respectively.
By the change of variables formula obtained in Theorem 3.41 (which relies on the $\mathsf{CD}^1_{Lip}(K,N)$ conclusion of Theorem 3.31), for ν-a.e. geodesic $\gamma \in G_\varphi^+$:

$$\frac{\rho_s(\gamma_s)}{\rho_t(\gamma_t)} = \frac{h_{\gamma_s}^{\varphi_s(\gamma_s),s}(t)}{1+(t-s)\frac{\partial_\tau|_{\tau=t}\ell^p_\tau(\gamma_t)}{p\ell(\gamma)^p}} = \frac{h_{\gamma_s}^{\varphi_s(\gamma_s),s}(t)}{1+(t-s)\frac{\partial_\tau|_{\tau=t}\bar{\ell}^p_\tau(\gamma_t)}{p\ell(\gamma)^p}}, \quad \text{for a.e. } t,s \in (0,1) \tag{3.59}$$

where for all $s \in (0,1), h_s = h_{\gamma_s}^{\varphi_s(\gamma_s),s}$ is a $\mathsf{CD}(K_0,N)$ density, with $K_0 = \ell(\gamma)^2 K$ and $h_s(s)=1$. Since Corollary 3.33 implies the Lipschitz regularity of $t \mapsto \rho_t(\gamma_t)$, assumptions (A) and (B) of the Theorem 3.43 are satisfied. Moreover, the third order information on the Kantorovich potential φ guarantees also the validity of the assumption (C) of the Theorem 3.43. Hence for ν-a.e. $\gamma \in G_\varphi^+$, it holds

$$\frac{1}{\rho_t(\gamma_t)} = L(t)Y(t), \ \forall t \in (0,1)$$

where L is a concave function and Y is a $\mathsf{CD}(K_0,N)$ density on $(0,1)$.

It is now a standard application of Hölder's inequality that gives us the validity of the $\mathsf{CD}_q(K,N)$ inequality along the W_q-geodesic μ_t: fix $t_0,t_1 \in (0,1)$ and set $t_\alpha = \alpha t_1 + (1-\alpha)t_0$, where $\alpha \in [0,1]$. Using that $\sigma^{(\alpha)}_{K_0,N}(\theta) = \sigma^{(\alpha)}_{K,N}(\theta\ell(\gamma))$, it holds true:

$$
\begin{aligned}
\rho_{t_\alpha}^{-\frac{1}{N}}(\gamma_{t_\alpha}) &= L^{\frac{1}{N}}(t_\alpha)Y^{\frac{1}{N}}(t_\alpha)\\
&\geq \big(\alpha L(t_1) + (1-\alpha)L(t_0)\big)^{\frac{1}{N}} \cdot \big(\sigma^{(\alpha)}_{K_0,N-1}(|t_1-t_0|)Y^{\frac{1}{N-1}}(t_1) + \sigma^{(1-\alpha)}_{K_0,N-1}(|t_1-t_0|)Y^{\frac{1}{N-1}}(t_0)\big)^{\frac{N-1}{N}}\\
&\geq \alpha^{\frac{1}{N}}\sigma^{(\alpha)}_{K_0,N-1}(|t_1-t_0|)^{\frac{N-1}{N}}Y^{\frac{1}{N}}(t_1)L^{\frac{1}{N}}(t_1) + (1-\alpha)^{\frac{1}{N}}\sigma^{(1-\alpha)}_{K_0,N-1}(|t_1-t_0|)^{\frac{N-1}{N}}Y^{\frac{1}{N}}(t_0)L^{\frac{1}{N}}(t_0)\\
&= \alpha^{\frac{1}{N}}\sigma^{(\alpha)}_{K,N-1}(|t_1-t_0|\ell(\gamma))^{\frac{N-1}{N}}\rho_{t_1}^{-\frac{1}{N}}(\gamma_{t_1}) + (1-\alpha)^{\frac{1}{N}}\sigma^{(1-\alpha)}_{K,N-1}(|t_1-t_0|\ell(\gamma))^{\frac{N-1}{N}}\rho_{t_0}^{-\frac{1}{N}}(\gamma_{t_0})\\
&= \tau^{(\alpha)}_{K,N}(\mathsf{d}(\gamma_{t_0},\gamma_{t_1}))\rho_{t_1}^{-\frac{1}{N}}(\gamma_{t_1}) + \tau^{(1-\alpha)}_{K,N}(\mathsf{d}(\gamma_{t_0},\gamma_{t_1}))\rho_{t_0}^{-\frac{1}{N}}(\gamma_{t_0}).
\end{aligned}
\tag{3.60}
$$

Recall that, by Corollary 3.33, the function $t \mapsto \rho_t(\gamma_t)$ is upper semi-continuous at the end-points; so, it follows that for ν-a.e. $\gamma \in G^+_\varphi$ the inequality (3.60) holds true for all $t_0, t_1 \in [0,1]$. In particular, setting $t_0 = 0, t_1 = 1$, we have that for all $\alpha \in [0,1]$:

$$
\rho_\alpha^{-\frac{1}{N}}(\gamma_\alpha) \geq \tau^{(\alpha)}_{K,N}(\mathsf{d}(\gamma_0,\gamma_1))\rho_1^{-\frac{1}{N}}(\gamma_1) + \tau^{(1-\alpha)}_{K,N}(\mathsf{d}(\gamma_0,\gamma_1))\rho_0^{-\frac{1}{N}}(\gamma_0); \tag{3.61}
$$

the latter inequality being satisfied for ν-a.e.$\gamma \in G^+_\varphi$. We now claim that (3.61) is also satisfied for every $\gamma \in G^0_\varphi$, confirming in this way the validity of the $\mathsf{CD}(K,N)$ condition. Indeed, in this case the map $\alpha \mapsto \rho_\alpha(\gamma_\alpha)$ turns out to be constant by the Theorem 3.41 and then (3.61) is trivially satisfied as an equality, since $\tau^{(\alpha)}_{K,N}(0) = \alpha$, for every $\alpha \in [0,1]$. Thus, the claim. ☐

COROLLARY 3.45 (Local-to-Global). *Fix any $p > 1$ and $K, N \in \mathbb{R}$ with $N > 1$. Let $(X,\mathsf{d},\mathfrak{m})$ be a p-essentially non-branching metric measure space verifying $\mathsf{CD}_{p,loc}(K,N)$ and such that (X,d) is a length space with* $\mathrm{supp}(\mathfrak{m}) = X$. *Then $(X,\mathsf{d},\mathfrak{m})$ verifies $\mathsf{CD}_p(K,N)$.*

CHAPTER 4

Displacement convexity of the Entropy and the distance cost Optimal Transportation

As we have seen so far, Lott-Sturm-Villani theory is based on the characterisation of Ricci curvature lower bounds in terms of displacement convexity of certain entropy functionals along W_2-geodesics. The theory of m.m.s.'s verifying $\mathsf{CD}(K,N)$ has then extensively developed, leading to a rich and fruitful approach to the geometry of m.m.s.'s by means of Optimal-Transport. Nevertheless, substantial recent advancements in the theory (localization paradigm and local-to-global property) have been obtained considering the different point of view of L^1-Optimal transport problems. This has led to a different curvature dimension condition, called $\mathsf{CD}^1(K,N)$ (introduced for the first time in [**35**]), formulated in terms of one-dimensional curvature properties of integral curves of Lipschitz maps. In this chapter we show that the two approaches produce the same curvature-dimension condition reconciling the two definitions. In particular we show that the $\mathsf{CD}^1(K,N)$ condition can be formulated in terms of displacement convexity along W_1-geodesics. In order to state the main result of this chapter, we need first to introduce some preliminary notions.

As we have seen in the Section 1.3, one can also prescribe the convexity inequality (1.5) to hold along a W_p-geodesic, getting to the more general definition of $\mathsf{CD}_p(K,N)$ (see definition 1.35). In this chapter we will deal with the case $p=1$, that, due to the lack of strict convexity of the exponent, needs a more refined definition.

Definition 4.1. Given $K,N\in\mathbb{R}$ with $N\geq 1$ we say that a metric measure space $(X,\mathsf{d},\mathfrak{m})$ satisfies the $\mathsf{CD}_1(K,N)$ condition if and only if for each pair of $\mu_0,\mu_1\in\mathcal{P}_1(X,\mathsf{d},\mathfrak{m})$ there exists a Borel probability measure $\boldsymbol{\pi}\in\mathcal{P}(\mathrm{Geo}(X))$ concentrated on constant speed geodesics, such that $(\mathrm{e}_0,\mathrm{e}_1)_\sharp\boldsymbol{\pi}\in\mathrm{Opt}_1(\mu_0,\mu_1)$, $\mu_t:=(\mathrm{e}_t)_\sharp\boldsymbol{\pi}\ll\mathfrak{m}$ and for which the inequality

$$S_{N'}(\mu_t|\mathfrak{m})\leq-\int_{X\times X}\left[\tau_{K,N'}^{(1-t)}(\mathsf{d}(\gamma_0,\gamma_1))\rho_0^{-1/N'}(\gamma_0)+\tau_{K,N'}^{(t)}(\mathsf{d}(\gamma_0,\gamma_1))\rho_1^{-1/N'}(\gamma_1)\right]\boldsymbol{\pi}(d\gamma)\tag{4.1}$$

holds for all $t\in[0,1]$ and all $N'\geq N$.

Remark 4.2. Notice that since we are dealing with the 1-transportation distance, there are dynamic transport plans which are not concentrated on constant speed geodesics. Insisting on this property in the definition above seems the natural choice to make a connection with the analogous definitions for $p>1$, see e.g. Lemma 4.4.

Recall that, to avoid pathologies, we assume $\mathrm{supp}(\mathfrak{m})=X$. Now we state the main result of this chapter:

Theorem 4.3. *Let $(X,\mathsf{d},\mathfrak{m})$ be an essentially non-branching metric measure space and further assume $\mathfrak{m}(X)=1$. Then $(X,\mathsf{d},\mathfrak{m})$ satisfies the $\mathsf{CD}^1_{Lip}(K,N)$ condition if and only if it satisfies the $\mathsf{CD}_1(K,N)$ condition.*

We will present separately the two implications needed for the proof of Theorem 4.3. In order to prove that $\mathsf{CD}_1(K,N)$ implies $\mathsf{CD}^1_{Lip}(K,N)$, in Section 4.1 we adapt the approach used in Section 3.3 to prove that $\mathsf{CD}_p(K,N)\implies\mathsf{CD}^1_{Lip}(K,N)$. In Section 4.2 we will prove that, under $\mathsf{CD}^1_{Lip}(K,N)$, given any two measures $\mu_0,\mu_1\in\mathcal{P}_1(X,\mathsf{d},\mathfrak{m})$, there exists a W_1-geodesic $\{\mu_t\}$ interpolating them and verifying the Entropy inequality (4.1). For this purpose, we will consider the Kantorovich potential $u : X \to \mathbb{R}$ associated to this L^1-optimal transport problem and, being the latter 1-Lipschitz, we deduce the validity of the $\mathsf{CD}^1_u(K,N)$ condition. Disintegrating μ_0, μ_1 along the transport rays of u, the geodesic μ_t will be obtained as superposition of the one-dimensional geodesics interpolating their marginals.

4.1. $\mathsf{CD}_1(K,N) \implies \mathsf{CD}^1_{Lip}(K,N)$

Fix $u : X \to \mathbb{R}$ a 1-Lipschitz function and let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching m.m.s. verifying $\mathsf{CD}_1(K,N)$ with $\mathfrak{m}(X) = 1$.

Step 1. Disintegration formula.
First notice that $\mathsf{CD}_1(K,N)$ implies, reasoning for instance like [**94**] in the case $p = 2$, that the space is proper. Moreover $\mathsf{CD}_1(K,N)$ implies the following variant of $\mathsf{MCP}(K,N)$ (see definition 1.37 and refer to [**75**], [**94**] for further insights):

Lemma 4.4. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. with $\mathfrak{m}(X) = 1$ and satisfying $\mathsf{CD}_1(K,N)$. Then $(X, \mathsf{d}, \mathfrak{m})$ satisfies the following version of $\mathsf{MCP}(K,N)$: for any $\mu_0 \in \mathcal{P}(X)$ with $\mu_0 \ll \mathfrak{m}$ and $x_0 \in X$, there exists a curve (μ_t) which is a W_p-geodesic for any $p \in [1, \infty)$ such that $\mu_t = \rho_t \mathfrak{m} + \mu_t^s$ for all $t \in [0,1)$ and*

$$\int_X \rho_t^{-1/N'} \mu_t \geq \int_X \tau_{K,N'}^{(1-t)}(\mathsf{d}(x,x_0)) \rho_0^{-1/N'}(x) \mu_0(dx), \tag{4.2}$$

holds true for all $t \in [0,1)$ and $N' \geq N$.

Proof. Let $\mu_0 \in \mathcal{P}(X, \mathsf{d}, \mathfrak{m})$ and $x_0 \in X$ be given. Since $\mathrm{supp}(\mathfrak{m}) = X$, we can consider $\mu_{1,\varepsilon} := c_\varepsilon \mathfrak{m}\llcorner_{B_\varepsilon(x_0)}$, with $c_\varepsilon > 0$ normalisation constant. Let $\boldsymbol{\pi}_\varepsilon$ be given by Definition 4.1 and put $\mu_{t,\varepsilon} := (\mathrm{e}_t)_* \boldsymbol{\pi}_\varepsilon$. It is classical to check that properness of X implies that $\boldsymbol{\pi}_\varepsilon$ is precompact and therefore we can obtain a limit dynamical plan $\boldsymbol{\pi}$ inducing a geodesic from μ_0 to δ_0. Validity of (4.2) simply follows by lower semicontinuity of entropy and the claim follows. □

The version of $\mathsf{MCP}(K,N)$ obtained in Lemma 4.4 is actually equivalent to the classical one, provided the space is essentially non-branching: we refer for its proof to [**35**, Lemma 6.13] (see also [**83**, Section 5]). Hence in our framework we can directly use the classical $\mathsf{MCP}(K,N)$. It was shown in [**75**, Lemma 2.5] that for $\mathsf{MCP}(K,N)$ spaces a doubling condition holds true; the latter implies that every bounded closed ball in X is totally bounded. Therefore, since X is complete, it is proper (i.e., all bounded closed sets are compact) and thus geodesic. Hence, as discussed in Theorem 1.49, the following disintegration formula is valid:

$$\mathfrak{m}\llcorner_{\mathcal{T}_u} = \mathfrak{m}\llcorner_{\mathcal{T}_u^b} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha), \tag{4.3}$$

where for $\mathfrak{q}$-a.e. $\alpha \in Q$ we have $\mathfrak{m}_\alpha \in \mathcal{P}(X)$, with $\mathfrak{m}_\alpha(X \setminus X_\alpha) = 0$. Recall that the notation X_α is used to denote the equivalence class of the element $\alpha \in Q$ that is, in particular, a transport ray. Notice that the first identity follows from the essentially non-branching assumption and the discussion after Theorem 1.45.

Hence it is only left to show that $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ satisfy $\mathsf{CD}(K,N)$.

Step 2. Intermediate regularity of conditional measures.
It is already present in the literature how to improve the validity of (4.2) to any $\mu_1 \in \mathcal{P}(X)$, provided the space is essentially non-branching and the geodesic $(\mu_t)_{t\in[0,1]}$ is a W_2-geodesic (see [**38**, Theorem 1.1]).

This will be enough to deduce a first result on the regularity of $\mathfrak{m}_\alpha$. Indeed, by Theorem 1.49, localization for $\mathsf{MCP}(K,N)$ holds true; in particular, for $\mathfrak{q}$-a.e. α, $\mathfrak{m}_\alpha = h_\alpha \mathcal{H}^1\llcorner_{X_\alpha}$ and the one-dimensional metric measure space $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ verifies $\mathsf{MCP}(K,N)$. Moreover, h_α is strictly positive in the relative interior of X_α and locally Lipschitz.

Step 3. $\mathsf{CD}(K,N)$ estimates for one-dimensional spaces.
In order to conclude, it remains to show that for $\mathfrak{q}$-a.e. $\alpha \in Q$, the one-dimensional metric measure space $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ satisfies $\mathsf{CD}(K,N)$. Consider the ray map g defined in (1.17); via g we are able to identify the set of definition of the densities h_α with real intervals. We start with the following preliminary result.

LEMMA 4.5. *For any $\bar{Q} \subseteq Q$ Borel set with positive $\mathfrak{q}$-measure and for $R_0, R_1, L_0, L_1 \in \mathbb{R}$ such that $R_0 < R_1$, $L_0, L_1 > 0$ and $[R_0, R_1 + L_1]$ belongs to the domain of $\mathfrak{q}$-a.e. h_α, it holds:*

$$
\begin{aligned}
&(L_t)^{\frac{1}{N}} \sup_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_t) \\
&\geq (L_0)^{\frac{1}{N}} \tau_{K,N}^{(1-t)}(\mathsf{d}(R_0,R_1)) \inf_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_0) + (L_1)^{\frac{1}{N}} \tau_{K,N}^{(t)}(\mathsf{d}(R_0,R_1)) \inf_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_1),
\end{aligned}
\tag{4.4}
$$

for every $t \in [0,1]$, where $R_t = (1-t)R_0 + tR_1$ (the same holds for L_t).

PROOF. **Step 1.**
Fix $\bar{Q} \subseteq Q$ Borel set with positive $\mathfrak{q}$-measure and consider $R_0, R_1, L_0, L_1 \in \mathbb{R}$ such that $R_0 < R_1$ and $L_0, L_1 > 0$. Define for $i = 1, 2$ the probability measures:

$$
\mu_i = \frac{1}{\mathfrak{q}(\bar{Q})} \int_{\bar{Q}} g(\alpha, \cdot)_\sharp \left(\frac{1}{\varepsilon L_i} \mathcal{L}^1 \llcorner_{[R_i, R_i + \varepsilon L_i]} \right) \mathfrak{q}(d\alpha).
$$

First of all observe that, for such measures, the transport has to be performed along the rays $\{X_\alpha\}_{\alpha \in \bar{Q}}$. For sure an optimal plan with this property exists, since the plan π rearranging the mass monotonically along each ray is optimal; hence $\operatorname{supp} \pi \subset \Gamma$, so it is d-cyclically monotone and therefore W_1-optimal. The aim is to prove that all the other optimal plans enjoy the same property.

Indeed, if not, there would exist at least one optimal plan $\bar{\pi}$ such that, for some $\bar{Q}_1 \subset \bar{Q}$ of positive $\mathfrak{q}$-measure and for some $S \subset \mathbb{R}$, it holds

$$
\bar{\pi}\{(g(\alpha, s), g(\alpha', s')) : \alpha, \alpha' \in \bar{Q}_1, s, s' \in S \,\text{with}\, \alpha \neq \alpha'\} > 0,
$$

with $\bar{Q}_1 \times S \subset \mathrm{Dom}\,(g)$. Let us consider the plan

$$
\pi^* = \frac{\pi + \bar{\pi}}{2};
$$

trivially, it is still optimal for the couple μ_0, μ_1. By construction this plan splits some points, generating in this way a set of branching points with positive measure. This will lead to a contradiction. Consider indeed the Kantorovich potential v associated to the W_1-optimal transport problem between μ_0 and μ_1, possibly different from the 1-Lipschitz function u we fixed above. Theorem 1.45 applied to v implies that necessarily $\mathfrak{m}(A_{\pm,v}) = 0$. Since $A_{\pm,v}$ will contain $P_1(\{(g(\alpha,s), g(\alpha',s')) : \alpha, \alpha' \in \bar{Q}_1, s, s' \in S \,\text{with}\, \alpha \neq \alpha'\})$ considered above, and $\mu_0 \ll \mathfrak{m}$, the contradiction with $\bar{\pi}(\{(g(\alpha,s), g(\alpha',s')) : \alpha, \alpha' \in \bar{Q}_1, s, s' \in S \,\text{with}\, \alpha \neq \alpha'\}) > 0$ follows. Hence, every optimal plan will have support contained in the set

$$
A^\varepsilon_{\bar{Q}} := \cup_{\alpha \in \bar{Q}} g(\alpha, [R_0, R_0 + \varepsilon L_0]) \times g(\alpha, [R_1, R_1 + \varepsilon L_1]).
$$

Step 2.
Since by definition $\mu_0, \mu_1 \ll \mathfrak{m}$, there exists a dynamic transport plan $\boldsymbol{\pi}$ as in Definition 4.1 such that for $\mu_t := (\mathrm{e}_t)_* \boldsymbol{\pi} = \rho_t \mathfrak{m}$ the inequality (4.1) holds true. Step 1 above and the fact that $\boldsymbol{\pi}$ is concentrated on constant speed geodesics completely characterize $\boldsymbol{\pi}$; in particular we have that for q-a.e. $\alpha \in Q$ the function ρ_t is 0 $\mathfrak{m}_\alpha$-a.e. outside the 'interval' $g(\alpha, [R_t, R_t + \varepsilon L_t])$. Hence, using the Disintegration Theorem and Jensen inequality, we can estimate the left-hand

side of (4.1) by:

$$\begin{aligned}\int_X \rho_t^{1-\frac{1}{N}}\, d\mathfrak{m} &= \int_{\bar{Q}}\int_{X_\alpha} \rho_t(x)^{1-\frac{1}{N}} m_\alpha(dx)\mathfrak{q}(d\alpha) = \int_{\bar{Q}}\int_{R_t}^{R_t+\varepsilon L_t} \rho_t(g(\alpha,s))^{1-\frac{1}{N}} h_\alpha(s) ds\mathfrak{q}(d\alpha)\\
&\leq (\varepsilon L_t)\int_{\bar{Q}} \sup_{[R_t,R_t+\varepsilon L_t]} h_\alpha^{\frac{1}{N}} \fint_{R_t}^{R_t+\varepsilon L_t} \big(\rho_t(g(\alpha,s))h_\alpha(s)\big)^{1-\frac{1}{N}} ds\mathfrak{q}(d\alpha)\\
&\leq (\varepsilon L_t)^{\frac{1}{N}}\int_{\bar{Q}} \sup_{[R_t,R_t+\varepsilon L_t]} h_\alpha^{\frac{1}{N}} \left(\int_{R_t}^{R_t+\varepsilon L_t} \rho_t(g(\alpha,s))h_\alpha(s)ds\right)^{1-\frac{1}{N}} \mathfrak{q}(d\alpha)\\
&\leq (\varepsilon L_t\mathfrak{q}(\bar{Q}))^{\frac{1}{N}} \sup_{\bar{Q}}\left(\sup_{[R_t,R_t+\varepsilon L_t]} h_\alpha^{\frac{1}{N}}\right).\end{aligned}$$

Arguing similarly, the right-hand side of (4.1) can be estimated in the following way where $\pi = (\mathrm{e}_0,\mathrm{e}_1)_\sharp\boldsymbol{\pi}$:

$$\begin{aligned}&\int_{X\times X} \rho_0^{-\frac{1}{N}}(x)\tau_{K,N}^{(1-t)}(\mathsf{d}(x,y)) + \rho_1^{-\frac{1}{N}}(y)\tau_{K,N}^{(t)}(\mathsf{d}(x,y))\pi(dx,dy)\\
&\geq \inf_{A^\varepsilon_{\bar{Q}}} \tau_{K,N}^{(1-t)}(\mathsf{d}(x,y)) \int_X \rho_0^{1-\frac{1}{N}}(x)\mathfrak{m}(dx) + \inf_{A^\varepsilon_{\bar{Q}}} \tau_{K,N}^{(t)}(\mathsf{d}(x,y)) \int_X \rho_1^{1-\frac{1}{N}}(y)\mathfrak{m}(dy)\\
&\geq (\varepsilon\mathfrak{q}(\bar{Q}))^{\frac{1}{N}}\Bigg[\inf_{A^\varepsilon_{\bar{Q}}} \tau_{K,N}^{(1-t)}(\mathsf{d}(x,y)) \inf_{\bar{Q}}\left(\inf_{[R_0,R_0+\varepsilon L_0]} h_\alpha^{\frac{1}{N}}\right)(L_0)^{\frac{1}{N}}\\
&\qquad + \inf_{A^\varepsilon_{\bar{Q}}} \tau_{K,N}^{(t)}(\mathsf{d}(x,y)) \inf_{\bar{Q}}\left(\inf_{[R_1,R_1+\varepsilon L_1]} h_\alpha^{\frac{1}{N}}\right)(L_1)^{\frac{1}{N}}\Bigg].\end{aligned}$$

Hence, considering both the estimates obtained so far, we get

$$\begin{aligned}(L_t)^{\frac{1}{N}} \sup_{\bar{Q}}\left(\sup_{[R_t,R_t+\varepsilon L_t]} h_\alpha^{\frac{1}{N}}\right) \geq\ & \inf_{A^\varepsilon_{\bar{Q}}} \tau_{K,N}^{(1-t)}(\mathsf{d}(x,y)) \inf_{\bar{Q}}\left(\inf_{[R_0,R_0+\varepsilon L_0]} h_\alpha^{\frac{1}{N}}\right)(L_0)^{\frac{1}{N}}\\
&+ \inf_{A^\varepsilon_{\bar{Q}}} \tau_{K,N}^{(t)}(\mathsf{d}(x,y)) \inf_{\bar{Q}}\left(\inf_{[R_1,R_1+\varepsilon L_1]} h_\alpha^{\frac{1}{N}}\right)(L_1)^{\frac{1}{N}}.\end{aligned}$$

Sending $\varepsilon \to 0$, we obtain

$$(L_t)^{\frac{1}{N}} \sup_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_t) \geq (L_0)^{\frac{1}{N}} \inf_{A_{\bar{Q}}} \tau_{K,N}^{(1-t)}(\mathsf{d}(x,y)) \inf_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_0) + (L_1)^{\frac{1}{N}} \inf_{A_{\bar{Q}}} \tau_{K,N}^{(t)}(\mathsf{d}(x,y)) \inf_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_1),$$

where $A_{\bar{Q}} := \cup_{\alpha\in\bar{Q}}\{(g(\alpha,R_0), g(\alpha,R_1))\}$. Since $g(\alpha,\cdot)$ is an isometry, (4.4) is proved. □

We are now ready to prove the following:

Proposition 4.6. *For $\mathfrak{q}$-a.e. $\alpha \in Q$, the metric measure space $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ satisfies $\mathsf{CD}(K,N)$.*

Proof. By remark 1.34, to prove the claim is sufficient to show that:

$$h_\alpha((1-t)R_0 + tR_1)^{\frac{1}{N-1}} \geq \sigma_{K,N-1}^{(1-t)}(R_1-R_0)h_\alpha(R_0)^{\frac{1}{N-1}} + \sigma_{K,N-1}^{(t)}(R_1-R_0)h_\alpha(R_1)^{\frac{1}{N-1}}, \tag{4.5}$$

for all $t\in[0,1]$ and for $R_0, R_1 \in [0, L_\alpha]$ with $R_0 < R_1$, where we have identified the transport ray X_α with the real interval $[0, L_\alpha]$ having the same length.
As already done in [**36**], it is sufficient to show that for every $R_0, R_1 \in [0, L_\alpha]$ with $R_0 < R_1$ and $L_0, L_1 > 0$, we have that for $\mathfrak{q}$-a.e. $\alpha \in Q$

$$(L_t)^{\frac{1}{N}} h_\alpha^{\frac{1}{N}}(R_t) \geq (L_0)^{\frac{1}{N}} \tau_{K,N}^{(1-t)}(R_1-R_0) h_\alpha^{\frac{1}{N}}(R_0) + (L_1)^{\frac{1}{N}} \tau_{K,N}^{(t)}(R_1-R_0) h_\alpha^{\frac{1}{N}}(R_1), \tag{4.6}$$

for all $t \in [0,1]$, where $L_t = (1-t)L_0 + tL_1$ (the same for R_t). Indeed, if this is the case taking also into account the already established continuity of h_α, one can make the choice

$$L_0 = \frac{\sigma^{(1-t)}_{K,N-1}(R_1 - R_0)h(R_0)^{\frac{1}{N-1}}}{1-t}, \quad L_1 = \frac{\sigma^{(t)}_{K,N-1}(R_1 - R_0)h(R_1)^{\frac{1}{N-1}}}{t},$$

obtaining exactly (4.5). Thus, our aim will be proving (4.6). Arguing by contraddiction, let us assume that there exist $R_0, R_1 \in [0, L_\alpha]$, $L_0, L_1 > 0$ with $R_0 + L_0, R_1 + L_1 < L_\alpha$ and a Borel set $Q_1 \subseteq Q$ with positive $\mathfrak{q}$-measure such that for every $\alpha \in Q_1$ it holds:

$$(L_t)^{\frac{1}{N}} h_\alpha^{\frac{1}{N}}(R_t) < (L_0)^{\frac{1}{N}} \tau^{(1-t)}_{K,N}(R_1 - R_0) h_\alpha^{\frac{1}{N}}(R_0) + (L_1)^{\frac{1}{N}} \tau^{(t)}_{K,N}(R_1 - R_0) h_\alpha^{\frac{1}{N}}(R_1). \tag{4.7}$$

By Lusin Theorem, there exists a Borel set $Q_2 \subset Q_1$ with positive $\mathfrak{q}$-measure on which the maps $\alpha \mapsto h_\alpha(R_i)$, for $i = 0, t, 1$ are continuous. Hence, fixed $\delta > 0$, there exists $Q_3 \subset Q_2$ with positive $\mathfrak{q}$-measure such that

$$(L_t)^{\frac{1}{N}} h_\alpha^{\frac{1}{N}}(R_t) < (L_0)^{\frac{1}{N}} \tau^{(1-t)}_{K,N}(R_1 - R_0) h_\alpha^{\frac{1}{N}}(R_0) + (L_1)^{\frac{1}{N}} \tau^{(t)}_{K,N}(R_1 - R_0) h_\alpha^{\frac{1}{N}}(R_1) - \delta, \quad \forall \alpha \in Q_3.$$

In particular, for every $\bar{Q} \subset Q_3$ compact set with positive $\mathfrak{q}$-measure:

$$(L_t)^{\frac{1}{N}} \sup_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_t) < (L_0)^{\frac{1}{N}} \tau^{(1-t)}_{K,N}(R_1 - R_0) \sup_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_0) + (L_1)^{\frac{1}{N}} \tau^{(t)}_{K,N}(R_1 - R_0) \sup_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_1) - \delta.$$

Combining the latter inequality with (4.4), we deduce that for any $\bar{Q} \subset Q_3$ Borel set with positive $\mathfrak{q}$-measure

$$\begin{aligned}
&(L_0)^{\frac{1}{N}} \tau^{(1-t)}_{K,N}(R_1 - R_0) \inf_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_0) + (L_1)^{\frac{1}{N}} \tau^{(t)}_{K,N}(R_1 - R_0) \inf_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_1) < \\
&(L_0)^{\frac{1}{N}} \tau^{(1-t)}_{K,N}(R_1 - R_0) \sup_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_0) + (L_1)^{\frac{1}{N}} \tau^{(1-t)}_{K,N}(R_1 - R_0) \sup_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_1) - \delta.
\end{aligned}$$

Since the parameter δ does not depend on $\bar{Q}$, we obtain a contradiction. ☐

This concludes the proof of the implication: from $\mathsf{CD}_1(K,N)$ to $\mathsf{CD}^1_{Lip}(K,N)$. We will next move to the opposite implication.

4.2. $\mathsf{CD}^1_{Lip}(K,N) \implies \mathsf{CD}_1(K,N)$

Notice that $\mathsf{CD}^1_{Lip}(K,N)$ implies that $(X, \mathsf{d}, \mathfrak{m})$ is a proper geodesic space and verifies $\mathsf{MCP}(K,N)$ (see for all the details [**35**]).

Let $\mu_0, \mu_1 \in \mathcal{P}_1(X, \mathsf{d}, \mathfrak{m})$ be given. We will construct a W_1-geodesic verifying the Entropy inequality. Consider therefore $u : X \to \mathbb{R}$ a Kantorovich potential associated to the transport problem between μ_0, μ_1 with cost d. Considering the associated Γ_u, it holds that any optimal transport plan π has to be concentrated over Γ_u, i.e. $\pi(\Gamma_u) = 1$. Moreover, with no loss in generality we can assume that μ_0 is concentrated over the transport set $\mathcal{T}^b_u$: indeed the part of μ_0 outside of $\mathcal{T}^b_u$ is left in place by π; in particular, it will not give any contribution in the Entropy inequality as $\tau^{(1-t)}_{K,N}(0) = 0$.

Since u is 1-Lipschitz, by the $\mathsf{CD}^1_u(K,N)$ condition there exist a family of rays $\{X_\alpha\}_{\alpha \in Q} \subset X$ and a disintegration of $\mathfrak{m}\llcorner_{\mathcal{T}_u}$ on $\{X_\alpha\}_{\alpha \in Q}$ such that:

$$\mathfrak{m}\llcorner_{\mathcal{T}_u} = \mathfrak{m}\llcorner_{\mathcal{T}^b_u} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha), \quad \text{with } \mathfrak{m}_\alpha(X_\alpha) = 1, \text{ for } \mathfrak{q}\text{-a.e. } \alpha \in Q, \tag{4.8}$$

where the first identity is given by Theorem 1.45 and $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha) \in \mathsf{CD}(K,N)$. It follows that

$$\mu_0 = \rho_0 \mathfrak{m} = \int_Q \rho_0 \, \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha) = \int_Q \mu_{0,\alpha} \mathfrak{q}_0(d\alpha) \tag{4.9}$$

where $\mu_{0,\alpha} = \rho_0 \mathfrak{m}_\alpha \cdot (\int \rho_0 \mathfrak{m}_\alpha)^{-1}$ and $\mathfrak{q}_0 = \mathfrak{Q}_\sharp(\mu_0)$ with $\mathfrak{Q}$ the quotient map. Then we claim that for any Borel set $C \subseteq Q$ it holds:

$$(\mathfrak{Q}^{-1}(C) \times X) \cap (\Gamma_u \setminus \{x = y\}) \cap (\mathcal{T}_u^b \times \mathcal{T}_u^b) = (X \times \mathfrak{Q}^{-1}(C)) \cap (\Gamma_u \setminus \{x = y\}) \cap (\mathcal{T}_u^b \times \mathcal{T}_u^b).$$

Indeed, since $\mu_0(\mathcal{T}_u^b) = \mu_1(\mathcal{T}_u^b) = 1$, then $\pi((\Gamma_u \setminus \{x = y\}) \cap \mathcal{T}_u^b \times \mathcal{T}_u^b) = 1$; hence if $x, y \in \mathcal{T}_u^b$ with $(x,y) \in \Gamma_u$, then it must be $\mathfrak{Q}(x) = \mathfrak{Q}(y)$ since $\mathcal{T}_u^b$ does not admit forward or backward branching points. This implies that

$$\begin{aligned}
\mu_0(\mathfrak{Q}^{-1}(C)) &= \pi((\mathfrak{Q}^{-1}(C) \times X) \cap (\Gamma_u \setminus \{x = y\})) \\
&= \pi(X \times \mathfrak{Q}^{-1}(C)) \cap (\Gamma_u \setminus \{x = y\}) \\
&= \mu_1(\mathfrak{Q}^{-1}(C));
\end{aligned}$$

in particular $\mathfrak{q}_0 = \mathfrak{q}_1 := \mathfrak{Q}_\sharp(\mu_1)$. Hence, we can write the following disintegration: $\mu_1 = \rho_1 \mathfrak{m} = \int_Q \rho_1 \, \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha) = \int_Q \mu_{1,\alpha} \mathfrak{q}_0(d\alpha)$, where $\mu_{1,\alpha} = \rho_1 \mathfrak{m}_\alpha \cdot (\int \rho_0 \mathfrak{m}_\alpha)^{-1}$ and, $\mathfrak{q}_0$-a.e., $\mu_{0,\alpha}, \mu_{1,\alpha}$ are probability measures on X_α. Furthermore, by construction they are absolutely continuous with respect to $\mathfrak{m}_\alpha$. By the $\mathsf{CD}^1_u(K,N)$ condition the metric measure space $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ satisfies $\mathsf{CD}(K,N)$; hence there exists an optimal dynamical plan ν_α such that $\rho_{t,\alpha} \mathfrak{m}_\alpha = \mu_{t,\alpha} = (\mathrm{e}_t)_\sharp \nu_\alpha$ is a W_1-geodesic interpolating $\mu_{0,\alpha}$ and $\mu_{1,\alpha}$ and

$$\rho_{t,\alpha}^{-\frac{1}{N'}}(\gamma_t) \geq \tau_{K,N'}^{(1-t)}(\mathsf{d}(\gamma_0,\gamma_1)) \rho_{0,\alpha}^{-\frac{1}{N'}}(\gamma_0) + \tau_{K,N}^{(t)}(\mathsf{d}(\gamma_0,\gamma_1)) \rho_{1,\alpha}^{-\frac{1}{N'}}(\gamma_1), \quad \text{for } \nu_\alpha \text{ a.e. } \gamma. \tag{4.10}$$

It is then natural to proceed gluing 1-dimensional geodesics: define $\nu = \int_Q \nu_\alpha \mathfrak{q}_0(d\alpha)$ and set $\mu_t = (\mathrm{e}_t)_\sharp \nu$. Observe that, it holds $\mu_t = \int_Q \mu_{t,\alpha} \mathfrak{q}_0(d\alpha)$ and we claim that $\{\mu_t\}$ is a W_1-geodesic interpolating μ_0 and μ_1. Indeed:

$$\begin{aligned}
W_1(\mu_t, \mu_s) &\leq \int_{X \times X} \mathsf{d}(x,y) (\mathrm{e}_t, \mathrm{e}_s)_\sharp \nu(dxdy) \\
&= \int_Q \int_{X_\alpha \times X_\alpha} \mathsf{d}(x,y) (\mathrm{e}_t, \mathrm{e}_s)_\sharp \nu_\alpha(dxdy) \mathfrak{q}_0(d\alpha) \\
&= |t-s| \int_Q \int_{X_\alpha \times X_\alpha} \mathsf{d}(x,y) (\mathrm{e}_0, \mathrm{e}_1)_\sharp \nu_\alpha(dxdy) \mathfrak{q}_0(d\alpha) \\
&= |t-s| \int_{X \times X} \mathsf{d}(x,y) (\mathrm{e}_0, \mathrm{e}_1)_\sharp \nu(dxdy) \\
&= |t-s| W_1(\mu_0, \mu_1).
\end{aligned}$$

The last equality follows from the optimality of the plan: indeed $(\mathrm{e}_0, \mathrm{e}_1)_\sharp \nu$ is concentrated on a d-cyclically monotone with marginals μ_0 and μ_1. To conclude, we show the convexity inequality (4.1) along the geodesic μ_t.

If $\mu_t = \rho_t \mathfrak{m}$, it follows from (4.8) that for each $t \in [0,1]$ it holds $\rho_{t,\alpha} = \frac{\rho_t}{\int \rho_0 \mathfrak{m}_\alpha}$. Hence the inequality (4.10) can be rewritten in the following way:

$$\rho_t^{-\frac{1}{N'}}(\gamma_t) \geq \tau_{K,N'}^{(1-t)}(\mathsf{d}(\gamma_0,\gamma_1)) \rho_0^{-\frac{1}{N'}}(\gamma_0) + \tau_{K,N}^{(t)}(\mathsf{d}(\gamma_0,\gamma_1)) \rho_1^{-\frac{1}{N'}}(\gamma_1), \quad \text{for } \nu_\alpha\text{-a.e. } \gamma. \tag{4.11}$$

Since for $\mathfrak{q}_0$-a.e. α the inequality (4.11) holds for ν_α−a.e. γ, a fortiori it holds true for ν−a.e. γ; hence, the claim is proved.

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