An Integral-Type Lyapunov Function Approach for Control Synthesis and Disturbance Attenuation for a Class of Nonlinear Systems

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A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy in Engineering

September 2020

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To my grandparents Juventino and

Magdalena, my aunt Elvira and my Godmother Alma,

who unfortunately will not be able to see me fulfilling this dream.

概要

非線形システムに対する制御系設計と外乱抑制のための積分型
リアプノフ関数アプローチ

モレノ・サエンス　ハイロ

非線形システムには多様なシステム表現があり、すべての非線形システムに対して統一的に設計法を議論することは困難である。そこで、非線形制御では、扱う非線形システムのクラスを限定し、そのクラスに応じた理論構築が個別に行われてきた。本論文では、多項式ファジィシステムで表現可能な非線形システムのクラスに対して、積分型リアプノフ関数を用いることで、設計条件の保守性を軽減することを試みる。また、外乱抑制を目的とした制御系設計法において、多項式ファジィシステムに対するHamilton-Jacobi-Isaacs (HJI) 方程式の近似解を求めるために、sum-of-squaresに基づく新しい解法アルゴリズムを提案する。ベンチマーク設計問題を通して、従来手法との比較検討を行い、本設計手法の有効性を明らかにする。本論文は6章で構成され、概要は以下の通りである。

第1章では緒論を述べる。本研究の背景や目的を述べ、他の関連手法に対する本研究の位置付けを説明する。

第2章では、本研究の対象システムであるファジィシステム/多項式ファジィシステム、および、それらの非線形記述能力について述べるとともに、本論文で提案する設計条件の導出や解法において重要な役割を担うsum-of-squares、および、H∞制御問題について述べる。

第3章では、線積分型ファジィリアプノフ関数を用いた安定解析と制御系設計について新しい提案を行う。とくに、線積分型多項式ファジィリアプノフ関数を用いることで、従来から用いられてきたファジィリアプノフ関数のシステムの解軌道に沿った時間微分時に現れるメンバーシップ関数の時間微分の複雑な項を消去できることを明らかにし、これにより可解設計問題へ定式化できることを示す。

第4章では、第3章で提案した制御系設計手法を線積分型高次多項式ファジィリアプノフ関数へ拡張し、それに基づくsum-of-squares条件を導出する。　Sum-of-squaresの枠組みを用いることで、従来の線形行列不等式条件では扱えなかった多項式リアプノフ関数の高次次数化を可能とし、設計条件の保守性の軽減を成し遂げる。ベンチマーク設計問題を通して、提案手法の有効性を検証する。

第5章では、多項式ファジィシステムに対する外乱抑制制御を論じる。外乱抑制制御を実現するために、多項式ファジィシステムの$H\infty$制御問題に対するsum-of-squares設計条件を導出する。多項式ファジィシステムに対するHamilton-Jacobi-Isaacs (HJI) 方程式の近似解を求めるために、 sum-of-squaresに基づく新しい解法アルゴリズムを提案する。ベンチマーク設計問題を通して、従来手法との比較検討を行い、本設計手法の有効性を明らかにする。

第6章では、結論を述べる。本研究のまとめと問題点、および、今後の展望について述べる。

Abstract

In contrast to linear control, a general and systematic methodology to study stability and stabilization of nonlinear systems does not exist. The development of fuzzy logic by L. Zadeh in the mid-sixties led one decade later to the work of E. Mamdani who implemented a fuzzy algorithm scheme to control a laboratory-built steam engine, and it represented a watershed to consider fuzzy logic as an alternative to control nonlinear systems. However, this approach is based on heuristic rules and the lack of a mathematical model describing the system implies that some performance requirements such as optimality and robustness cannot be guaranteed. The pioneer work in 1985 of T. Takagi and M. Sugeno overcame this drawback with the introduction of a mathematical tool to construct a fuzzy representation of a system. The Takagi-Sugeno representation uses fuzzy IF-THEN rules with local linear state-space realizations as a consequence to describe a nonlinear system. In the late 2000s, this idea was extended to the polynomial case, reducing, in general, the number of fuzzy rules and extending the region of approximation of the fuzzy model.

Model-based fuzzy control schemes have drawn attention from control community around the globe, and have become a workaround to design controllers for complicated nonlinear systems. For this purpose, Lyapunov's second method plays a central role. Nevertheless, the search for a single quadratic Lyapunov function in common for a set of state equations brings conservative results. The introduction and utilization of multiple Lyapunov functions such as fuzzy Lyapunov functions, piecewise Lyapunov function, and integral-type Lyapunov functions have reduced this conservativeness.

This thesis addresses the problem of improving sum-of-squares-based stability and control synthesis conditions by using an integral-type Lyapunov function, also known as line integral fuzzy Lyapunov function, which is a more general case of the quadratic one. In contrast to the standard fuzzy Lyapunov functions, integral-type functions become independent on the time derivative of the membership functions. Moreover, this idea is generalized to an integral-type

polynomial form, bringing more relaxed results than the aforementioned proposal. Finally, the proposed Lyapunov function will work as an approximator of the value function of the Hamilton-Jacobi-Isaac's equation, which is the solution for the H infinity problem in the context of differential games.

The present thesis is structured as follows: Chapter 1 introduces an overview of the control problem (nonlinear and fuzzy control), the objective of this thesis and related works. Secondly, Chapter 2 presents the Takagi-Sugeno and polynomial fuzzy representations, the sum-of-squares decomposition, the H infinity control problem and differential games as well as the mathematical concepts that are used to relax the proposed conditions. In Chapter 3 the integral-type Lyapunov function presented by Rhee et al., is used to find SOS stability analysis conditions for model-based fuzzy control systems relaxed by using copositive-based idea. Moreover, the stabilization problem is relaxed via the Positivstellensatz. Then, the work introduced by Rhee et al., is generalized in Chapter 4 to the case that the integrand is a polynomial vector field, resulting in the polynomial form of the integral-type Lyapunov function. Iterative SOS conditions for control design are presented by means of the extended Lyapunov function proposed in the present thesis. Chapter 5 addresses the two-player zero-sum game to study the H infinity problem. Iterative SOS conditions are presented and the simultaneous policy update algorithm is employed to enhance the approximation of the solution of the Hamilton-Jacobi-Isaacs equation for the polynomial fuzzy system case. A summary of the outcome and discussion presented in previous chapters as well as future direction of the current research are presented in Chapter 6.

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List of Acronyms

HJB	Hamilton-Jacobi-Bellman
HJI	Hamilton-Jacobi-Isaacs
MFs	Membership functions
LMI	Linear matrix inequalities
PDC	Parallel distributed compensation
SOS	Sum of squares
SPUA	Simultaneous policy update algorithm

List of Symbols

$\mathbb{R}[\boldsymbol{x}]$	Polynomial ring with real coefficients
$\mathbb{S}[\boldsymbol{x}]$	Cone of sum of squares polynomials
$\mathbb{Z}^{\geq 0}$	Nonnegative integer
$\mathcal{L}_2[0,\infty)$	Space of square-integrable functions in $[0,\infty)$
$\boldsymbol{x}_0$	Initial condition
$\dot{\boldsymbol{x}}$	$\frac{d\boldsymbol{x}}{dt}$
$\dot{V}(\boldsymbol{x})$	$\frac{dV(\boldsymbol{x})}{dt} = \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}}$

Acknowledgements

Doing a PhD has been a dream that I have had since my last year of the bachelor's studies, and it is just around the corner. In this journey called 'pursuing a PhD in Japan' there are a lot of people to whom I want to say thank you in the lines coming next.

First of all, to my parents Rogelia and Jairo Cristino who have always worked hard to provide me with a lot of opportunities in my life. Thank you for your love and friendship.

Naturally, I am deeply indebted to my supervisor Prof. Kazuo Tanaka, who welcomed me into his laboratory the first time I came to Japan as an exchange student, and gave me the recommendation to come back as a PhD student. Thank you for the time to discuss my research, your emails and talks to encourage me, and for all the opportunities to be a reviewer of conference and journal papers that have helped me a lot to improve my skills.

People come and go every single day, but a few always stay as my friends. I am really happy to have made friends with Edgar, Edgarito, Julio, Gibran and his family, Soutarou, Natsuki, Yuki, Nobu, Seiko, James, Daniel, Amigo Cheng, AJ, Carlos and his family, William, Jean, Marzieh and Julie. Thanks a lot for your advice, talks, words of encouragement, for inviting me to have lunch, drink a beer, have a walk or smoke a cigarette. Those moments have made my stay in Japan much easier.

I want to dedicate these lines to Prof. Suwako Uehara, who has been more than my *baito* boss, she has been an excellent friend who has listened to me when I have faced happy or difficult times in Japan. Thank you so much for always trusting me and for opening your lovely home to me.

I am deeply grateful to Mr. Takeo Kanazawa for teaching me more about the life in Japan besides university life and for always encouraging me. He has given me the opportunity to visit companies and meet with their owners, meet the Ambassadors of some Latin American countries in social gathering, some unique experiences.

We live far from each other, but my friends in Mexico have also played an important role in this journey. Thanks Jonhs, Wero, Jimmy, Alexia, Jessica, Memo, Darich, Alan, Adiel, Ivan Gordo, my cousins Adriana Mares and Jatziry Moreno, Talia, Danna, and especially Mariana, who always has the right words, the right meme, the perfect timing to have wonderful conversations or just hum a song to make me laugh.

I would like to express my gratitude to people who are and have been members of Tanaka-Lab. Especially Dr. Chen, Dr. Lizhen and Dr. Alissa for your time to discuss and advice in my research. I am also very appreciative of all the support Yuriko Sakata has always given to me.

I want to say thanks to MEXT for the scholarship provided to support my first three year in Japan. I am also grateful to Prof. Atsuko Jeffreys and Prof. Choo for giving me some opportunities to work for them, and to Mr. Hideki Kinai for letting me be part of *Osekkai Japan.*

And last but not least, I want to thank my aunt Yaned, uncles Alfredo and Daniel, my Goddaughter, Godfather and all my relatives.

1

Introduction

"The way to get started is to quit talking and begin doing."

— Walt Disney

The history of automatic control systems dates back to 270 BC with the work developed by Ktesibios of Alexandria. He designed a water clock consisting of two feedback forms, one floating valve to guarantee a constant flow of water into a tank and a siphon to return to the lower level of the clock when the maximum level in the tank was reached. Later works, such as the automata described by Heron of Alexandria in *Pneumatica*, the control of the level of water in a steam engine boiler designed by Sutton Thomas Wood, and the construction of the first automatic windmill by Edmund Lee are part of the the early period of automatic control.

The first major contribution of automatic control systems to engineering came in the XVIII century with James Watt with the introduction of his velocity regulator (also known as Watt's governor) for a steam engine. His work improved the efficiency of steam engines and opened the way to the Industrial Revolution. Nevertheless, subsequent studies showed that Watt's governor had some troubles such as variations in the velocity instead of staying in a constant value, and in the worst case, an unlimited increase of the velocity, or in other words, instability.

Before James Clerk Maxwell presented his work titled *On governors* in 1868, the design of automatic control system was by trial and error. However, Maxwell demonstrated that the stability of a steam engine equipped with a Watt's governor depends on the coefficients of its differential equation, and gave a criterion for differential equations up to 4th order. This work was a watershed to consider the automatic control as a mathematical problem, giving the basis of the control theory. The following decades were a period of progress in the control theory field with the works of Edward Routh and Adolf Hurwitz generalizing Maxwell's criterion to higher order, Aleksandr Lyapunov and his stability method based on a generalized energy function, and Oliver Heaviside and his study of systems using the concept of transfer function.

During the first fifty years of last century, classical control flourished with the relevant works of Nicolas Minorsky in 1922 who introduced the idea of a proportional-integral-derivative (PID) controller for an automatic steering system. Years later, Harold Black, an inventor at Bell Laboratories, investigated the benefits of using a negative feedback to reduce the noise in amplifiers and in cooperation with Harry Nyquist, proposed a stability criterion based on the polar-plot of a complex function. Some time afterwards, Hendrik Bode introduced the phase and magnitude plots as well as the study of closed-loop stability by means of the concepts of gain and phase margins. Then, John Ziegler and Nathaniel Nichols gave tuning rules to determine the parameters of a PID controller and Walter Evans presented his root locus method to have a graphical representation of the location of the closed-loop poles in the complex s-plane.

Frequency domain methods from classical control faced their limitation in the study of multivariable and nonlinear systems. The description of a system via state-space models paved the way to the development of the modern control theory. In contrast to classical control, the time domain techniques from modern control are applicable to both linear and nonlinear control systems and it thrived during the Cold War with the works in dynamic programming of Richard Bellman, the development of the maximum principle by Lev Pontryagin and the filtering problem solved by Rudolf Kalman.

Computers started to play a central role in control engineering at the time when systems became more and more complex, and the intelligent control, whose methods comprises fuzzy control, neural network-based control and genetic algorithms, emerged as a prominent alternative to deal with them [1–5].

1.1 An Overview of Fuzzy Control

The idea of a multi-valued logic started in the Ancient Greece with Plato who thought that there were more logical values besides true and false. But it was not until the early 20th century when Jan Lukasiewicz introduced the three-valued logic, which includes ‘possible’ as a third value and it is an option other than the bi-valued Aristotelian logic [6]. The excellent work of Lotfi Zadeh in 1965 introduced the mathematics of fuzzy sets and fuzzy logic [7], which is a multi-valued logic whose truth values can be any number between 0 and 1. One decade later, the first control application of fuzzy logic was presented by Ebrahim Mamdani who controlled a laboratory-built steam engine [8].

The main feature of Mamdani's approach is the capability to capture human operators' experience on a process in a set of fuzzy IF-THEN rules, and these rules become the heart of the Mamdani-type fuzzy logic controller. Nonetheless, considering the fact that a mathematical model is not required to design this heuristic fuzzy controller, some basic requirements such as optimality, robustness and so on, cannot be guaranteed. This drawback was overcome with the introduction of the model-based fuzzy control, in which the Takagi-Sugeno fuzzy model [9] has been one of the most fruitful approaches. The difference resides in the consequent part of the fuzzy IF-THEN rules, which is a state-space representation of a linear system describing local dynamics, and all the consequent parts blended together exactly represent, locally or globally, the nonlinear system under study [10]. A strong advantage of representing a nonlinear system as a Takagi-Sugeno fuzzy model is that stability and stabilization conditions based on a quadratic Lyapunov function can be expressed in the form of linear matrix inequalities (LMI), and there already exists efficient numerical methods to solve them [11]. By the end of the 2000s decade, the work presented in [12] introduced a more general representation: the polynomial fuzzy model. Here, the consequence parts are not restricted to be linear state-space realization, but polynomial state-space forms. Unfortunately, LMI solvers cannot be directly used. In order to deal with this polynomial representation, the referred work made use of the sum of squares (SOS) optimization which had been effectively developed a few year earlier [13].

Both Takagi Sugeno and polynomial fuzzy model-based approaches leverage Lyapunov methods to study stability and synthesize stabilizing controllers, and quadratic Lyapunov function is the most commonly used doubtlessly (see [10, 14, 15] and references therein). Conditions via quadratic Lyapunov functions are generally simple, however, they tend to be conservative. For the sake of reducing the conservativeness, new forms of Lyapunov functions have been introduced in the literature, such as non-quadratic [16], piecewise function [17,18], polynomial [12, 19], fuzzy function [20, 21], to mention but a few. The latter form follows the same fuzzy IF-THEN rules structure, with the difference that the consequent parts are quadratic functions. In general, it brings better results. Nevertheless, since the inferred Lyapunov function includes membership functions (MFs), their time derivatives appear when applying Lyapunov method, complicating the conditions. The upper bound of the time derivatives is usually used instead, but it is not easy to determine due to the dependence on states and control input [20]. The work in [22] introduced the line integral fuzzy Lyapunov function, which is an alternative form of the fuzzy Lyapunov functions that makes stability conditions independent of the derivative with respect to time of the MFs, and it has been employed with success to lessen the conservativeness for nonlinear systems expressed as Takagi-Sugeno forms [23–26].

Without any doubt, stability is the most important attribute of a control system. However, the closed-loop system is also expected to accomplish desired performance objectives, for instance optimality and robustness. H_∞ control design framework [27, 28] is used to synthesize controllers that mitigates the effect of external disturbance in the state variables, showing its effectiveness in the model-based fuzzy control field with the works [29–32] and references therein. In the context of differential games, the H_∞ problem can be expressed as a two-player zero-sum game [33, 34]. The control law and external disturbance are the players which are at odds with each other, one of the players is attempting to minimize a cost functional, and the other to maximize it. The solution of this minimax optimization problem is analogous to find a solution for the Hamilton-Jacobi-Isaacs (HJI) equation, which is a first order nonlinear partial differential equation. In the context of linear systems, this problem reduces to solve an algebraic Ricatti equation, which is well defined and easily solved by numerical methods. Nevertheless, for general nonlinear systems, there might not be solution for the HJI equation. Policy iteration is an alternative method to approximate the solution of the two-player zero-sum game for a nonlinear system. This procedure assumes that a control input law is known a priori and consists of two steps [35]. The first step, known as policy evaluation, solves a more tractable HJI equation whose solution is used in the policy improvement step to make better the control input, doing again until the convergence of the solution is reached. It is worth to mention that the "more tractable" HJI equation is still hard to solve. Therefore, approximation techniques are used to express the value function and adaptive dynamic programming [35–38] has been an excellent method to deal with it. However, a drawback of using neural network-based adaptive dynamic programming methods is an inherent characteristic stated by the universal approximation theorem, which says that a neural network with at least one hidden layer can be close to a continuous function only on a compact set [39, 40]. As an alternative, Hamilton-Jacobi-Bellman (HJB) equation has been converted to a set of inequalities [41] that make possible the extension to higher degree polynomials, and as a result the works in [42] and [43] have successfully computed by means of SOS optimization an approximated solution for the HJB and HJI equations associated to polynomial nonlinear systems, respectively.

1.2 Outline and Contributions

This thesis presents the results of the study on stability and stabilization of a class of nonlinear systems. In spite of the model-based fuzzy control has become an workaround to represent, and consequently, study nonlinear systems, there are still open problems that draw attention from fuzzy control community. Quadratic Lyapunov function gives a simple and elegant characterization of Lyapunov's second method, however, this quadratic form has its limitations as well. One of those handicaps is the fact to find a single quadratic function in common for the set of state-space realization that defines the fuzzy model.

The summation structure of the fuzzy model brings multiple-summation form in stabilization and performance behaviour conditions that complicates the reduction of the Lyapunov inequalities to linear matrix inequalities (LMI) or sum of squares (SOS) conditions.

Throughout the present thesis, the research focus its attention in the following points to decrease the inherent conservatism of the model-based fuzzy control.

- By means of polynomial fuzzy model. A vast body of literature related to polynomial fuzzy system have shown the improvement on the results compared to the Takagi-Sugeno fuzzy system. Moreover, the use of SOS paves the way for increasing the degree of the Lyapunov function, and employing mathematical techniques such as Positivstellensatz and copositivity property to enhance the conditions.

- By means of more general Lyapunov functions. The novel work in [22] introduced an integral-type form, which is a variation of fuzzy Lyapunov functions avoiding its biggest drawback: handle with the time derivative of the MFs. Furthermore, this study proposes a more general setting of the aforementioned function that brings a relaxation on the results compared to other current methods.

- By means of including polynomial restrictions for the MFs. Conservative results emerge from the conditions to check positivity of multiple-summation. A large number of techniques have been proposed in the literature, but it remains as an open problem. This study considers two options, first the substitution of the MFs by quadratic variables, and secondly, the replacement by first-degree variables instead with the inclusion of the fact that the summation of the MFs is equal to 1 and adding polynomial restrictions via the S-procedure.

This thesis is structured as follows.

- The second chapter of this thesis introduces the mathematical background and essential definitions which are going to be employed in the sequel to obtain the main results and contributions of the present dissertation.

- The third chapter of the present thesis leverages the Lyapunov function introduced in [22]. In contrast to other current criteria that make use of this integral-type form, this research has employed it in the study of polynomial fuzzy systems. The combination of the integral-type Lyapunov function and polynomial fuzzy system have considerably enhanced the results as shown in the examples. Copositivity property and Positivstellensatz refutation have been applied as relaxation techniques in the stability and stabilization problems, respectively. These results are part of author's works [44] and [45].

- The fourth chapter of the present thesis generalizes the Lyapunov function discussed in previous section to a polynomial setting. Rather than considering gradients of quadratic forms in the integrand, this study focuses its attention on gradients of higher-even-degree-homogeneous polynomials. The contributions are the derivation of SOS-based stabilization conditions by using the proposed function, whose relaxation considers two ideas, improving the conditions by means of Positivstellensatz and S-procedure. These results are part of author's works [46] and [47].

- The fifth chapter of the present thesis confronts the disturbance attenuation problem. First of all, a solution via quadratic stabilization is proposed. Then, the research tackles this problem by means of differential games. The contribution here is to bring the policy iteration algorithms to the fuzzy control framework, presented as SOS conditions and the relaxation is performed by making use of integral-type Lyapunov function proposed in Chapter 3 and S-procedure idea from Chapter 4. These results are part of author's work [47].

- The sixth chapter of this thesis summarizes and discusses the results presented in previous chapters as well as introduces a general idea of future work that this research can lead to.

1.3 Related Works

Model-based fuzzy control is a fruitful research topic in the control community. The novel line integral fuzzy Lyapunov function idea in [22] has led the way to the works [23–26, 48], to name but a few. These works have focus their attention in the Takagi-Sugeno fuzzy model and have shown the improvement on the results.

While doing this research, the works in [49, 50] have also studied the generalization of the integral-type function presented by Rhee et al. in [22]. Former proposes a new path independent structure which covers a larger class of nonlinear systems expressed in Takagi-Sugeno form that can be tackled with the Lyapunov function under study, while latter introduces a general setting of the integral term to the polynomial case, and it only gives stability conditions

Novel relaxation techniques have taken into account the MFs in the conditions. The work in [51] have suggested that the lack of knowledge on the shape of the MFs in the conditions is a source of conservatism and the work in [52] has successfully given membership-function-dependent conditions for the guarantee cost control case. Other studies [53–55] consider bounds based on the time derivative of the MFs, multisimplex representation, and matrix operations derived from the fact that the result of adding fuzzy-MFs is equal to one, respectively

Regarding to the H_∞ problem, the studies in [29, 31, 56] have given LMI conditions for Takagi-Sugeno models, synchronization [57] and sliding mode controller [58] for polynomial fuzzy models, and filtering problem conditions [26] via integral-type Lyapunov functions. From the point of view of differential games, the H_∞ has been studied mainly by means of neural networks approaches [35–38], and to the best of author's knowledge, the work in [43] for polynomial (non-fuzzy) nonlinear systems is the only in the sum of squares context.

2

Preliminaries

"Study hard what interests you the most in the most undisciplined, irreverent and original manner possible."

— Richard Feynman

This second chapter includes basic definitions, necessary mathematical tools, and a brief explanation of stability theory, model-based fuzzy control, and the relation between disturbance attenuation and differential games that will be employed in the sequel. Throughout the present thesis, bold letters denote matrices and vectors; and scalars otherwise. For the ease of notation, initial condition $\boldsymbol{x}(t=0)$ will be written as $\boldsymbol{x}_0$ and variables depending on time such as state-space variables $\boldsymbol{x}(t)$, control input $u(t)$, external disturbance $w(t)$, and output $y(t)$ will be simply denoted as $\boldsymbol{x}$, u, w, and y, respectively.

2.1 Definitions, notations and mathematical tools

2.1.1 Positive Definiteness

A continuous multivariate function $V(x_1, x_2, \ldots, x_n) = V(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ is called positive definite if for all $\boldsymbol{x} \in \mathbb{R} - \{\mathbf{0}\}$, the function satisfies $V(\boldsymbol{x}) > 0$ and $V(\mathbf{0}) = 0$. If $V(\boldsymbol{x}) \geq 0$ at $\boldsymbol{x} \neq \mathbf{0}$ and $V(\mathbf{0}) = 0$, then it is said to be positive semidefinite. Moreover, a function satisfying that $-V(\boldsymbol{x}) > 0$ or $-V(\boldsymbol{x}) \geq 0$ at $\boldsymbol{x} \neq 0$ and $V(\mathbf{0}) = 0$ is called respectively negative definite or negative semidefinite [28]. Readers should not confuse the concept of positive definite function with a nonnegative function, simply denoted as $h(\boldsymbol{x}) \geq 0$.

Let $\boldsymbol{P}$ be a square matrix of order n. Similarly, $\boldsymbol{P}$ is named a positive define ($\boldsymbol{P} > \mathbf{0}$), positive semidefinite ($\boldsymbol{P} \geq \mathbf{0}$), negative definite ($\boldsymbol{P} < \mathbf{0}$) or negative semidefinite ($\boldsymbol{P} \leq \mathbf{0}$) matrix if the resulting function $V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x}$ satisfies any of the above definitions [11].

2.1.2 Sum of Squares Decomposition

Let $\mathbb{R}[\boldsymbol{x}]$ be the polynomial ring. Define the cone of sum of squares (SOS) polynomials as the set

$$\mathbb{S}[\boldsymbol{x}] := \left\{ \sum_{i=1}^{z} q_i(\boldsymbol{x})^2 \;\middle|\; q_i(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}],\ z \in \mathbb{N} \right\}. \tag{2.1}$$

As a consequence of the above definition, a form $p(\boldsymbol{x}) \in \mathbb{S}[\boldsymbol{x}]$ is a nonnegative function [13]. The satisfaction of $p(\boldsymbol{x}) - \phi(\boldsymbol{x}) \in \mathbb{S}[\boldsymbol{x}]$ for a given $\phi(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$ that is positive definite, guarantees that $p(\boldsymbol{x}) > 0, \forall \boldsymbol{x} \neq 0,\ p(\boldsymbol{0}) = 0$. Now, consider the case that $\boldsymbol{P}(\boldsymbol{x})$ is a square polynomial matrix of order m and define a vector column $\boldsymbol{y} = [y_1, y_2, \ldots, y_m]^T$ whose entries are independent on $\boldsymbol{x}$. If $\boldsymbol{y}^T\boldsymbol{P}(\boldsymbol{x})\boldsymbol{y} \in \mathbb{S}[\boldsymbol{x}, \boldsymbol{y}]$ then $\boldsymbol{P}(\boldsymbol{x}) \geq 0$ [59]. There are some third-party MATLAB toolboxes that solve SOS optimization problems and this research has made used of the toolbox SOSOPT. The author refers readers to the manual [60] for further explanation of the toolbox.

2.1.3 Copositivity

Consider the problem of determining if a matrix $\boldsymbol{M} \in \mathbb{R}^{n\times n}$ is positive for all vector $\boldsymbol{y} \in \mathbb{R}^{n\times 1}$ taking values in the nonnegative orthant, that is to say

$$\boldsymbol{y}^T\boldsymbol{M}\boldsymbol{y} \geq 0,\ \forall \boldsymbol{y} \geq 0. \tag{2.2}$$

Then, $\boldsymbol{M}$ will be copositive. The aforementioned verification problem is a well-known computational hard problem [13], a natural way to rewrite this problem is considering the change of variable $y_i = \hat{y}_i^2$, then latter condition becomes

$$\hat{\boldsymbol{y}}^T\boldsymbol{M}\hat{\boldsymbol{y}} = \sum_{i=1}^{n}\sum_{j=1}^{n} \hat{y}_i^2\hat{y}_j^2 m_{ij} \geq 0. \tag{2.3}$$

Here, m_{ij} denotes the entry of the matrix $\boldsymbol{M}$ being situated in the row 'i' and column 'j', and $\hat{\boldsymbol{y}} = [\hat{y}_1^2, \ldots, \hat{y}_n^2]^T$. By Polya theorem [13], a relaxed condition in terms of SOS is given by

$$\left(\sum_{k=1}^{n} \hat{y}_k^2\right)^s \sum_{i=1}^{n}\sum_{j=1}^{n} \hat{y}_i^2\hat{y}_j^2 m_{ij} \in \mathbb{S}[\hat{\boldsymbol{y}}], \tag{2.4}$$

where $s \in \mathbb{Z}^{\geq 0}$ is the Polya exponent.

2.1.4 Positivstellensatz

The Positivstellensatz is a powerful mathematical tool belonging to real algebraic geometry, which characterizes positive polynomials on a semialgebraic set [61]. Consider a finite sequence of polynomials inequalities $f_1(\boldsymbol{x}) \geq 0, \ldots, f_{z_f}(\boldsymbol{x}) \geq 0$ and polynomial equations $g_1(\boldsymbol{x}) = 0, \ldots, g_{z_g}(\boldsymbol{x}) = 0$. If there exist $\sigma_0(\boldsymbol{x}),\ \sigma_i(\boldsymbol{x}) \in \mathbb{S}[\boldsymbol{x}]$ and $\tau_i(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$ such that the Positivstellensatz refutation

$$\sigma_0(\boldsymbol{x}) + \sum_{i=1}^{z_f} \sigma_i(\boldsymbol{x}) f_i(\boldsymbol{x}) + \sum_{i=1}^{z_g} \tau_i(\boldsymbol{x}) g_i(\boldsymbol{x}) = -1, \tag{2.5}$$

holds true, then the semialgebraic set

$$\left\{ \boldsymbol{x} \in \mathbb{R}^n \left| \begin{matrix} f_1(\boldsymbol{x}) \geq 0, \ldots, f_{z_f}(\boldsymbol{x}) \geq 0, \\ g_1(\boldsymbol{x}) = 0, \ldots, g_{z_g}(\boldsymbol{x}) = 0. \end{matrix} \right. \right\} = \emptyset. \tag{2.6}$$

2.1.5 S-Procedure

As a consequence of the Positivstellensatz, the work presented in [62] generalized the well-known S-procedure for quadratic forms [11]. Given polynomials $f_0(\boldsymbol{x}), \ldots, f_{z_f}(\boldsymbol{x})$, the following condition

$$\bigcap_{i=1}^{z_f} \left\{ \boldsymbol{x} \in \mathbb{R}^n \middle| f_i(\boldsymbol{x}) \geq 0 \right\} \subseteq \left\{ \boldsymbol{x} \in \mathbb{R}^n \middle| f_0(\boldsymbol{x}) \geq 0 \right\}, \tag{2.7}$$

is verified if there exist multipliers $\sigma_i(\boldsymbol{x}) \in \mathbb{S}[\boldsymbol{x}]$ such that

$$f_0(\boldsymbol{x}) - \sum_{i=1}^{z_f} \sigma_i(\boldsymbol{x}) f_i(\boldsymbol{x}) \in \mathbb{S}[\boldsymbol{x}]. \tag{2.8}$$

2.1.6 Schur Complement

Suppose $\boldsymbol{M} \in \mathbb{R}^{p \times p}$, $\boldsymbol{N} \in \mathbb{R}^{q \times p}$, $\boldsymbol{L} \in \mathbb{R}^{q \times q}$ and $\boldsymbol{L} > 0$ is invertible. Consider the matrix inequality below.

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{N}^T \\ \boldsymbol{N} & \boldsymbol{L}^{-1} \end{bmatrix} > \boldsymbol{0}. \tag{2.9}$$

Then the Schur complement is expressed as $\boldsymbol{M} - \boldsymbol{N}^T \boldsymbol{L} \boldsymbol{N} > 0$. The importance of this result is the capability to transform a bilinear relationship into a higher-dimension linear condition [11].

2.2 Stability in the Sense of Lyapunov

Let

$$\dot{\boldsymbol{x}} = \boldsymbol{\mathcal{F}}(\boldsymbol{x}), \tag{2.10}$$

be a nonlinear system. Here, the column vector $\boldsymbol{x} = [x_1, x_2, \ldots, x_n]^T$ represents the state variables and $\boldsymbol{\mathcal{F}}(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous. Define the equilibrium point as the value of $\boldsymbol{x}$ that makes the time derivative equal to zero, that is to say $\boldsymbol{\mathcal{F}}(\boldsymbol{x}) = \mathbf{0}$. Without loss of generality, define $\boldsymbol{x} = \mathbf{0}$ as the equilibrium point. There are several definitions of stability (e.g. input-output stability), notwithstanding, Lyapunov theory addresses the stability of the zero equilibrium. The existence of $\delta > 0$ satisfying

$$||\boldsymbol{x}_0|| < \delta \Rightarrow ||\boldsymbol{x}(t)|| < \varepsilon, \ \forall t \geq 0, \tag{2.11}$$

for each $\varepsilon > 0$ verifies stability of the origin of (2.10), see Figure 2.1. Moreover, if the selection of δ fulfills

$$||\boldsymbol{x}_0|| < \delta \Rightarrow \lim_{t \to \infty} \boldsymbol{x}_0 = 0, \tag{2.12}$$

then, the origin is said to be asymptotically stable. Finally, it is unstable if it is not stable [28]. The beauty of Lyapunov stability theory is that generalizes the concept of energy for a conservative dynamic system to general systems. In a conservative system, the energy is a positive function decreasing to zero as the states approach to an stable equilibrium [63]. Aleksandr Lyapunov proved that other functions with the same properties as the energy functions can be used to determine stability of the equilibrium of general systems.

2.2.1 Second Method of Lyapunov

Lemma 1. Let $V(\boldsymbol{x}) : D \to \mathbb{R}^n$ be a C^1 function, where $D \subseteq \mathbb{R}^n$ is containing the origin. The zero equilibrium of (2.10) is stable if a function $V(\boldsymbol{x})$, whose trajectories monotonically decrease and is radially unbounded i.e., $\lim_{\boldsymbol{x} \to \infty} V(\boldsymbol{x}) \to \infty$, exists and fulfills

$$V(\boldsymbol{x}) > 0 \text{ and } \dot{V}(\boldsymbol{x}) = \frac{dV(\boldsymbol{x})}{dt} \leq 0, \ \forall \boldsymbol{x} \in \mathbb{R}^n - \{\mathbf{0}\} \text{ and } V(\mathbf{0}) = 0. \tag{2.13}$$

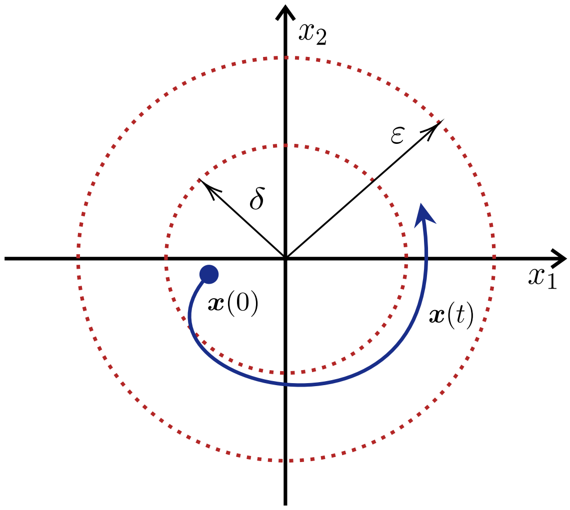

Figure 2.1: Stability according to the theory of Lyapunov.

Furthermore, if $\dot{V}(\boldsymbol{x}) < 0$, the equilibrium is asymptotically stable [28]. This function was given the name of Lyapunov function.

2.3 Model-Based Fuzzy Control

Consider a nonlinear system whose dynamics are modeled as the state equations below.

$$\dot{\boldsymbol{x}} = \boldsymbol{\mathcal{F}}(\boldsymbol{x}) + \boldsymbol{\mathcal{G}}(\boldsymbol{x})u. \tag{2.14}$$

Here, $\boldsymbol{\mathcal{F}}(\boldsymbol{x})$ and $\boldsymbol{\mathcal{G}}(\boldsymbol{x})$ are matrices of appropriate dimensions whose entries are Lipschitz continuous nonlinear functions with the assumption that $\boldsymbol{\mathcal{F}}(\mathbf{0}) = \mathbf{0}$ and u is the input control variable [28]. Equation (2.14) becomes $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u$, which is the general form of the state-space realization of a linear system, when $\boldsymbol{\mathcal{F}}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ and $\boldsymbol{\mathcal{G}}(\boldsymbol{x}) = \boldsymbol{B}$, with $\boldsymbol{A}$ and $\boldsymbol{B}$ being constant matrices.

2.3.1 Takagi-Sugeno Form

Mamdani-type fuzzy controller emerged as an alternative to control complicated plants since a mathematical model is not required [8]. Instead, the designer synthesizes the fuzzy controller based on the expertise of human operators on the plant via a set of IF-THEN rules and using fuzzy inference [7]. However, the lack of a mathematical model became also a drawback since desired behaviour such as optimal and robust performance cannot be ensured. The introduction of the Takagi-Sugeno fuzzy model in [9] overcame this issue and became an alternative method to represent, verify stability and design controllers for nonlinear systems since then. Fuzzy rules in Takagi-Sugeno form are structured as

$$i^{th} \text{ model rule: IF } z_1 \text{ is } M_{i1} \text{ and } \cdots z_m \text{ is } M_{im} \text{ THEN: } \dot{\boldsymbol{x}} = \boldsymbol{A}_i \boldsymbol{x} + \boldsymbol{B}_i u, \tag{2.15}$$

for all $i = 1, 2, \ldots, r$. Here, $z_1, z_2, \ldots, z_m$ are premise variables, $M_{i1}, M_{i2}, \ldots, M_{im}$ are fuzzy sets, r is the number of rules, $\boldsymbol{A}_i \in \mathbb{R}^{n \times n}$ and $\boldsymbol{B}_i \in \mathbb{R}^{n \times 1}$. It is worthwhile to mention that the consequence parts of the above fuzzy rules are linear state-space realizations. The defuzzified system is given as

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{r} h_i(\boldsymbol{z}) \big\{ \boldsymbol{A}_i \boldsymbol{x} + \boldsymbol{B}_i u \big\}, \tag{2.16}$$

with

$$h_i(\boldsymbol{z}) = \prod_{j=1}^{m} M_{ij}(z_j). \tag{2.17}$$

In above equation, $M_{ij}(z_j)$ are the membership function (MF) associated with the fuzzy set M_{ij}. Therefore

$$\sum_{i=1}^{r} h_i(\boldsymbol{z}) = 1, \quad 0 \leq h_i(\boldsymbol{z}) \leq 1 \ \forall i. \tag{2.18}$$

There are several approaches to obtain a Takagi-Sugeno fuzzy model, such as sector nonlinearity [10]. Here, a brief explanation of the sector nonlinearity will be addressed. The dynamics of a simple pendulum with friction are given by the state equations

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -10 \sin x_1 - x_2. \end{aligned} \tag{2.19}$$

Define the premise variable $z = \frac{\sin x_1}{x_1}$ and

$$\max_{x_1} z = 1, \quad \min_{x_1} z = -0.2172 \tag{2.20}$$

The sector nonlinearity idea [10] allows expressing the premise variable as

$$z = \max_{x_1} z \cdot M_{11}(z) + \min_{x_1} z \cdot M_{21}(z), \tag{2.21}$$

where $M_{11}(z)$ and $M_{21}(z)$ are the MFs related the the fuzzy sets M_{11} and M_{21}, respectively.

Therefore, $M_{11}(z) + M_{21}(z) = 1$ and consequently

$$M_{11}(z) = \begin{cases} \frac{\sin x_1 + 0.2172 x_1}{1.2172 x_1}, & x_1 \neq 0 \\ 1, & x_1 = 0 \end{cases}, \quad M_{21}(z) = \begin{cases} \frac{x_1 - \sin x_1}{1.2172 x_1}, & x_1 \neq 0 \\ 0, & x_1 = 0 \end{cases}. \tag{2.22}$$

Thus

$$\begin{aligned} 1^{st} \text{ model rule: IF } z \text{ is } M_{11} \text{ THEN: } \dot{\boldsymbol{x}} = \boldsymbol{A}_1 \boldsymbol{x}, \\ 2^{nd} \text{ model rule: IF } z \text{ is } M_{21} \text{ THEN: } \dot{\boldsymbol{x}} = \boldsymbol{A}_2 \boldsymbol{x}. \end{aligned} \tag{2.23}$$

Then, the inferred fuzzy model is

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{2} h_i(z) \boldsymbol{A}_i \boldsymbol{x}. \tag{2.24}$$

For this fuzzy system in Takagi-Sugeno form, the state and input matrices are

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \; \boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 \\ 2.172 & -1 \end{bmatrix}, \tag{2.25}$$

with MFs $h_1(z) = M_{11}(z)$ and $h_2(z) = M_{21}(z)$.

2.3.2 Polynomial Form

Fuzzy systems in polynomial form were introduced in [12] and extends the well-known Takagi-Sugeno fuzzy model to a more general setting. The structure of the polynomial fuzzy model resembles the structure of the Takagi-Sugeno fuzzy model with the difference that the consequence parts admit nonlinear (polynomial) state-space realization as seen in equation below.

$$i^{th} \text{ model rule: IF } z_1 \text{ is } M_{i1} \text{ and } \cdots z_m \text{ is } M_{im} \text{ THEN: } \dot{\boldsymbol{x}} = \boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{x})u, \tag{2.26}$$

where $\boldsymbol{A}_i(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]^{n \times n}$ and $\boldsymbol{B}_i(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]^{n \times 1}$ and the fuzzy inferred model expressed as

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{r} h_i(\boldsymbol{z}) \big\{ \boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{x})u \big\}. \tag{2.27}$$

The sector nonlinearity method briefly explained in previous section can be also used to obtain a polynomial fuzzy model. However, a more accurate representation can be obtained by using a Taylor series-based approximation [51]. For the nonlinear system (2.19) representing a simple pendulum with friction, the premise variable $\hat{z} = \sin x_1$ can be written as

$$\hat{z} = f_q(x) + R_q(x)x^q, \tag{2.28}$$

where

$$f_q(x) := \sum_{k=1}^{q-1} \frac{f^{[k]}(0)}{k!} x^k. \tag{2.29}$$

is the $(q-1)$th-order Taylor series expansion and $R_q(x)$ is the Taylor remainder. Considering a second-order expansion of the sinusoidal function, the Taylor remainder of $\hat{z} = \sin x_1$ becomes

$$R_3(x_1) = \frac{\hat{z} - f_3(x_1)}{x_1^3} = \frac{\sin x_1 - x_1}{x_1^3}. \tag{2.30}$$

Rewriting the Taylor remainder as

$$R_3(x_1) = \max_{x_1} R_3(x_1) \cdot M_{11}(\hat{z}) + \min_{x_1} R_3(x_1) \cdot M_{21}(\hat{z}), \tag{2.31}$$

with $M_{11}(\hat{z}) + M_{21}(\hat{z}) = 1$ and

$$\max_{x_1} R_3(x_1) = 0, \quad \min_{x_1} R_3(x_1) = -\frac{1}{6}. \tag{2.32}$$

Hence

$$\begin{aligned} \hat{z} &= x_1 - \frac{1}{6}x_1^3 M_{21}(\hat{z}) = x_1\big(M_{11}(\hat{z}) + M_{21}(\hat{z})\big) - \frac{1}{6}x_1^3 M_{21}(\hat{z}) \\ &= x_1 M_{11}(\hat{z}) + \Big(x_1 - \frac{1}{6}x_1^3\Big) M_{21}(\hat{z}). \end{aligned} \tag{2.33}$$

Finally, the polynomial fuzzy model of the simple pendulum with friction (2.19) is

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{2} h_i(\hat{z}) \boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x}, \tag{2.34}$$

with

$$\boldsymbol{A}_1(\boldsymbol{x}) = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \ \boldsymbol{A}_2(\boldsymbol{x}) = \begin{bmatrix} 0 & 1 \\ -10 + \frac{10}{6}x_1^2 & -1 \end{bmatrix}, \tag{2.35}$$

and

$$h_1(\hat{z}) = M_{11}(\hat{z}) = \begin{cases} \frac{6(\sin x_1 - x_1)}{x_1^3} + 1, & x_1 \neq 0 \\ 0, & x_1 = 0 \end{cases}, \quad h_2(\hat{z}) = M_{21}(\hat{z}) = 1 - h_1(\hat{z}). \tag{2.36}$$

2.3.3 Stability Analysis

The study of stability of model-based fuzzy systems makes use of Lyapunov's theory explained in section 2.2. The asymptotic stability conditions for fuzzy systems have the general form written below.

$$\begin{aligned} V(\boldsymbol{x}) > 0, \\ \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} < 0,\ \forall i = 1, 2, \ldots, r, \end{aligned} \tag{2.37}$$

for all $\boldsymbol{x} \neq \boldsymbol{0}$ and $V(\boldsymbol{0}) = 0$. Here, the Lyapunov function candidate $V(\boldsymbol{x})$ can assume the form of a quadratic function [10], polynomial function [12], non-quadratic function [64], multiple function [17, 20, 22], and so on. For a quadratic Lyapunov function $V(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}$ and a Takagi-Sugeno fuzzy model, previous equation converts to an LMI condition.

$$\boldsymbol{P} > \boldsymbol{0}, \quad -\boldsymbol{A}_i^T\boldsymbol{P} - \boldsymbol{P}\boldsymbol{A}_i > \boldsymbol{0},\ \forall i. \tag{2.38}$$

2.3.4 Stabilization

Analogous to the fuzzy systems aforementioned, parallel distributed compensation (PDC) has the structure

$$i^{th} \text{ model rule: IF } z_1 \text{ is } M_{i1} \text{ and } \cdots z_m \text{ is } M_{im} \text{ THEN: } u = -\boldsymbol{F}_i(\boldsymbol{x})\boldsymbol{x}. \tag{2.39}$$

Here, $\boldsymbol{F}_i(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]^{1\times n}$ are the feedback gain vectors. The defuzzification process of the PDC controller is calculated as

$$u = -\sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{F}_i(\boldsymbol{x})\boldsymbol{x}. \tag{2.40}$$

Inserting the PDC control law above in the open-loop system (2.27) leads to the feedback system below.

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\{\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x})\}\boldsymbol{x}. \tag{2.41}$$

Control synthesis conditions based on Lyapunov method have the following general setting

$$\begin{aligned} V(\boldsymbol{x}) &> 0, \\ \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\left\{\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\Big(\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x})\Big)\boldsymbol{x}\right\} &< 0, \end{aligned} \tag{2.42}$$

at $\boldsymbol{x} \neq \boldsymbol{0}$ with $V(\boldsymbol{0}) = 0$. Note that latter condition has two challenges. First, it involves a double-fuzzy summation and checking its positiveness is still an open problem. Researchers around the globe have proposed some method to deal with this issue, see [10, 14, 29, 65–67] and references therein. Second, the condition includes two decision variables in a single term, therefore they are bilinear conditions. By using a quadratic Lyapunov function as in (2.38) with a Takagi-Sugeno fuzzy system, LMI conditions are

$$\begin{aligned} \boldsymbol{X} &> \boldsymbol{0}, \\ -\boldsymbol{X}\boldsymbol{A}_i^T - \boldsymbol{A}_i\boldsymbol{X} + \boldsymbol{M}_i^T\boldsymbol{B}_i^T + \boldsymbol{B}_i\boldsymbol{M}_i &> \boldsymbol{0}, \\ -\boldsymbol{X}\boldsymbol{A}_i^T - \boldsymbol{A}_i\boldsymbol{X} - \boldsymbol{X}\boldsymbol{A}_j^T - \boldsymbol{A}_j\boldsymbol{X}& \\ +\boldsymbol{M}_j^T\boldsymbol{B}_i^T + \boldsymbol{B}_i\boldsymbol{M}_j + \boldsymbol{M}_i^T\boldsymbol{B}_j^T + \boldsymbol{B}_j\boldsymbol{M}_i &> \boldsymbol{0}, \quad i < j, \end{aligned} \tag{2.43}$$

where $\boldsymbol{X} = \boldsymbol{P}^{-1}$ and $\boldsymbol{M}_i = \boldsymbol{F}_i\boldsymbol{X}$. As seen, using the quadratic Lyapunov function leads to simple and linear conditions, yet conservative results.

2.4 Integral-type Lyapunov Function

The present thesis deals with the line integral below introduced in [22].

$$V(\boldsymbol{x}) = 2\int_{\mathcal{C}} \zeta(\boldsymbol{\psi}) \cdot \boldsymbol{d\psi}. \tag{2.44}$$

Here, $\mathcal{C}$ is any curve that connects the origin state $\boldsymbol{0}$ with the current state $\boldsymbol{x}$, $\boldsymbol{\psi}$ denotes the integration variable and $(\cdot)$ is the inner product. Define the integrand by setting

$$\zeta(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{P}(\boldsymbol{x}). \tag{2.45}$$

Line integral (2.44) along C can be seen as a Lyapunov function under the strict assumption that it is independent of the path [22]. Let $\boldsymbol{P}_0 \in \mathbb{R}^{n\times n}$ be a matrix whose entries inside the main diagonal are set to be zero and $\boldsymbol{D}_i = diag(d^i_{11}, \cdots, d^i_{nn})$. Consider the following fuzzy rules.

$$i^{th} \text{ rule: IF } x_1 \text{ is } M_{i1} \text{ and } \cdots x_n \text{ is } M_{in} \text{ THEN: } \boldsymbol{P}(\boldsymbol{x}) = \boldsymbol{P}_0 + \boldsymbol{D}_i. \tag{2.46}$$

Note that this integral-type Lyapunov function is applicable to model-based fuzzy systems whose jth fuzzy set in the ith fuzzy rule depends exclusively on the x_j state variable, in other words $M_{ij}(x_j)$. The defuzzification process of (2.46) leads to the expression

$$\boldsymbol{P}(\boldsymbol{x}) = \boldsymbol{P}_0 + \sum_{i=1}^{r} h_i(\boldsymbol{x})\boldsymbol{D}_i. \tag{2.47}$$

Following the selection criteria of the main diagonal entries of $\boldsymbol{D}_i$ stated in [22] ensures that the line integral (2.44) is path independent. When the premise variable x_l belongs to the same fuzzy set in different rules (e.g. $M_{pl} = M_{ql}$), the lth entries of the matrices $\boldsymbol{D}_p$ and $\boldsymbol{D}_q$ have to be the same (i.e. $d^p_{ll} = d^q_{ll}$). Under the assumption that (2.44) is independent of the path, the substitution of $\boldsymbol{\psi} = \tau\boldsymbol{x}$ brings the condition

$$V(\boldsymbol{x}) = 2\int_C \boldsymbol{\zeta}(\boldsymbol{\psi})\boldsymbol{d\psi} = 2\int_0^1 \boldsymbol{\zeta}(\tau\boldsymbol{\psi})\boldsymbol{x}d\tau = 2\int_0^1 \tau\boldsymbol{x}^T\left(\boldsymbol{P}_0 + \sum_{i=1}^{r} h_i(\tau\boldsymbol{x})\boldsymbol{D}_i\right)\boldsymbol{x}d\tau. \tag{2.48}$$

Therefore, if $\boldsymbol{P}_0 + \boldsymbol{D}_i > 0,\ \forall i \in \{1,\ldots,r\}$ implies that $\boldsymbol{P}(\boldsymbol{x}) > 0 \Rightarrow V(\boldsymbol{x}) > 0$. For more details on the stability and stabilization conditions, please refer to [22–25].

2.5 Disturbance Attenuation and Differential Games

Let $\chi(t) : [0,\infty) \to \mathbb{R}$ be a piecewise continuous function. The set containing all the continuous signals represented by $\chi(t)$ with finite energy, in other words

$$\int_0^\infty ||\chi(t)||^2 dt < \infty \tag{2.49}$$

with $||\cdot||$ being the Euclidean norm, receives the name of $\mathcal{L}_2[0,\infty)$ space. In addition, the nonnegative value

$$||\chi(t)||_2 = \sqrt{\int_0^\infty ||\chi(t)||^2 dt}, \tag{2.50}$$

defines the $\mathcal{L}_2$ gain of the signal. The state-space equation (2.51) describes a nonlinear system with an external disturbance.

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{\mathcal{F}}(\boldsymbol{x}) + \boldsymbol{\mathcal{G}}(\boldsymbol{x})u + \boldsymbol{\mathcal{K}}(\boldsymbol{x})w, \\ \boldsymbol{y} &= \boldsymbol{\mathcal{M}}(\boldsymbol{x}). \end{aligned} \tag{2.51}$$

Above equation is a more general representation of a nonlinear system than (2.14). Here, $\boldsymbol{\mathcal{K}}(\boldsymbol{x})$ and $\boldsymbol{\mathcal{M}}(\boldsymbol{x})$ are vectors of nonlinear functions, $w \in \mathcal{L}_2[0,\infty)$ is the exogenous disturbance signal and $\boldsymbol{y}$ is the measured output. Let $\boldsymbol{z}$ be the performance output and define the $\mathcal{L}_2$ gain of the system as

$$\sup_{||w||_2 \neq 0} \frac{||\boldsymbol{z}||_2}{||w||_2} \leq \gamma. \tag{2.52}$$

For an input-output system, the $\mathcal{L}_2$ gain is a measure of the maximal gain from input w to output $\boldsymbol{z}$. Now, consider that $\boldsymbol{z} = [\boldsymbol{y}, \sqrt{R}u]^T$. The existence of a positive definite function $V(\boldsymbol{x})$ satisfying

$$\dot{V}(\boldsymbol{x}) + \boldsymbol{y}^T\boldsymbol{y} + u^T R u - \gamma^2 w^T w \leq 0, \tag{2.53}$$

guarantees that (2.52) hold true. This is clear to see when integrating with respect to t from 0 to T equation (2.53). The assumption that $\boldsymbol{x}_0 = \mathbf{0}$ leads to

$$V(\boldsymbol{x}(T)) + \int_0^T (\boldsymbol{y}^T\boldsymbol{y} + u^T R u - \gamma^2 w^T w)dt \leq 0. \tag{2.54}$$

The quantity $V(\boldsymbol{x}(T))$ is nonnegative. Thus

$$\sqrt{\frac{\int_0^T (\boldsymbol{y}^T\boldsymbol{y} + u^T R u)dt}{\int_0^T w^T w dt}} \leq \gamma. \tag{2.55}$$

2.5.1 Two-Player Zero-Sum Game

Differential games belong to the branch of mathematics known as game theory and studies the modeling of cooperation and conflict of decision-makers in the context of dynamical systems [33, 34]. Consider the cost functional below.

$$\mathcal{J}(\boldsymbol{x}_0, u, w) = \int_0^\infty \left(y^T y + u^T \boldsymbol{R} u - \gamma^2 w^T w\right) dt. \tag{2.56}$$

Here, $\boldsymbol{x}_0$ denotes the initial condition. The players u and w are at odds with each other and the victory of one implies the defeat of the other. This is equivalent to $\mathcal{J}^1(\boldsymbol{x}_0, u, w) = \mathcal{J}(\boldsymbol{x}_0, u, w) = -\mathcal{J}^2(\boldsymbol{x}_0, u, w)$ and u and w have the goal to minimize $\mathcal{J}^1$ and $\mathcal{J}^2$, respectively. Define

$$V^*(\boldsymbol{x}_0) = \inf_u \sup_w \mathcal{J}(\boldsymbol{x}, u, w), \tag{2.57}$$

as the two-player zero-sum game where $V^*(\boldsymbol{x}_0)$ is the value if optimal strategies are employed. The Nash equilibrium of the game (2.57) is the saddle point (u^*, w^*), which exists if the Nash equilibrium condition $\mathcal{J}(\boldsymbol{x}_0, u^*, w) \leq \mathcal{J}(\boldsymbol{x}_0, u^*, w^*) \leq \mathcal{J}(\boldsymbol{x}_0, u, w^*)$ holds true. Let

$$V(\boldsymbol{x}) = \int_t^\infty \left(y^T y + u^T \boldsymbol{R} u - \gamma^2 w^T w\right) d\tau \tag{2.58}$$

be the value function for a fixed policy pair (u, w). The differential form obtained by using Leibniz's formula is

$$\mathcal{H}(\boldsymbol{x}, V(\boldsymbol{x}), u, w) := \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \left\{\boldsymbol{\mathcal{F}}(\boldsymbol{x}) + \boldsymbol{\mathcal{G}}(\boldsymbol{x}) u + \boldsymbol{\mathcal{K}}(\boldsymbol{x}) w\right\} + y^T y + u^T \boldsymbol{R} u - \gamma^2 w^T w = 0, \tag{2.59}$$

with $\mathcal{H}(\boldsymbol{x}, V(\boldsymbol{x}), u, w)$ being the Hamiltonian. Isaacs' condition requires that

$$\inf_u \sup_w \mathcal{H}(\boldsymbol{x}, V(\boldsymbol{x}), u, w) = \sup_w \inf_u \mathcal{H}(\boldsymbol{x}, V(\boldsymbol{x}), u, w), \tag{2.60}$$

holds for all control and disturbance policies (u, w). Previous conditions is necessary for the existence of the saddle point. The stationary points are calculated by

$$\frac{\partial \mathcal{H}(\boldsymbol{x}, V(\boldsymbol{x}), u, w)}{\partial u} = 0, \quad \frac{\partial \mathcal{H}(\boldsymbol{x}, V(\boldsymbol{x}), u, w)}{\partial w} = 0, \tag{2.61}$$

and lead to the expressions of the policies u and w stated below.

$$u = -\frac{1}{2} \boldsymbol{R}^{-1} \boldsymbol{\mathcal{G}}^T(\boldsymbol{x}) \left(\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\right)^T, \tag{2.62}$$

$$w = \frac{1}{2\gamma^2} \boldsymbol{\mathcal{K}}^T(\boldsymbol{x}) \left(\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\right)^T. \tag{2.63}$$

The replacement of the policy pair given by (2.62), (2.63) and the output y from (2.51) in (2.59) brings the Hamilton-Jacobi-Isaacs (HJI) equation

$$\frac{\partial V^*(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{\mathcal{F}}(\boldsymbol{x}) - \frac{1}{4}\frac{\partial V^*(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{\mathcal{G}}(\boldsymbol{x})\boldsymbol{R}^{-1}\boldsymbol{\mathcal{G}}^T(\boldsymbol{x})\left(\frac{\partial V^*(\boldsymbol{x})}{\partial \boldsymbol{x}}\right)^T + \frac{1}{4\gamma^2}\frac{\partial V^*(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{\mathcal{K}}(\boldsymbol{x})\boldsymbol{\mathcal{K}}^T(\boldsymbol{x})\left(\frac{\partial V^*(\boldsymbol{x})}{\partial \boldsymbol{x}}\right)^T + \boldsymbol{\mathcal{M}}(\boldsymbol{x})^T\boldsymbol{\mathcal{M}}(\boldsymbol{x}) = 0. \tag{2.64}$$

The minimum solution denoted as $V^*(\boldsymbol{x}) \geq 0$ satisfies $V(\boldsymbol{x}) \geq V^*(\boldsymbol{x}) \geq 0$ for any other function $V(\boldsymbol{x})$ solving the HJI equation. The Nash equilibrium (u^*, w^*) is given by (2.62) and (2.63) considering the partial derivative of $V^*(\boldsymbol{x})$.

2.5.2 Policy Iteration

Finding the solution of the HJI equation is a requirement to design an H_∞ controller. Unfortunately, it is a partial differential equation that is hard to solve for general nonlinear systems. Policy iteration methods are algorithms that allow approximating the value function assuming that an initial admissible stabilizing control law u is known. In general, they consists in two steps: 1) in the policy evaluation a solution for the simplified HJI equation including the admissible control policy is found, and 2) the policy improvement updates the admissible control policy by means of using the solution computed in the previous step, doing this steps again until the solution converges [68]. Updating the disturbance policy during the second step was suggested in [69] and the convergence of the solution and stability of this modified policy iteration method were studied in [35]. This modification of the policy iteration algorithm is called simultaneous policy update algorithm (SPUA), and it is presented below (see flowchart in Figure 2.2).

Algorithm 1. SPUA for nonlinear systems.

Step 1: Assuming that an initial admissible control law u_0 for the nonlinear system (2.51) at $w_0 = 0$ is known, set $i = 0$ for a given $\gamma > 0$.

Step 2: Find the solution $V_i(\boldsymbol{x})$ of the equation below, satisfying that $V_i(\boldsymbol{x}) \geq 0$ and $V_i(\boldsymbol{0}) = 0$

.

$$\frac{\partial V_i(\boldsymbol{x})}{\partial \boldsymbol{x}}\big(\boldsymbol{\mathcal{F}}(\boldsymbol{x}) + \boldsymbol{\mathcal{G}}(\boldsymbol{x})u_i + \boldsymbol{\mathcal{K}}(\boldsymbol{x})w_i\big) + y^Ty + u_i^T\boldsymbol{R}u_i - \gamma^2 w_i^T w_i = 0. \tag{2.65}$$

Step 3: With the solution $V_i(\boldsymbol{x})$ previously found, update the policy pair by means of

$$\begin{aligned} u_{i+1} &= -\frac{1}{2}\boldsymbol{R}^{-1}\boldsymbol{\mathcal{G}}^T(\boldsymbol{x})\left(\frac{\partial V_i(\boldsymbol{x})}{\partial \boldsymbol{x}}\right)^T, \\ w_{i+1} &= \frac{1}{2\gamma^2}\boldsymbol{\mathcal{K}}^T(\boldsymbol{x})\left(\frac{\partial V_i(\boldsymbol{x})}{\partial \boldsymbol{x}}\right)^T. \end{aligned} \tag{2.66}$$

Step 4: Increase $i = i + 1$ and return to Step 2, repeat these steps until convergence of $V_i(\boldsymbol{x})$ is reached.

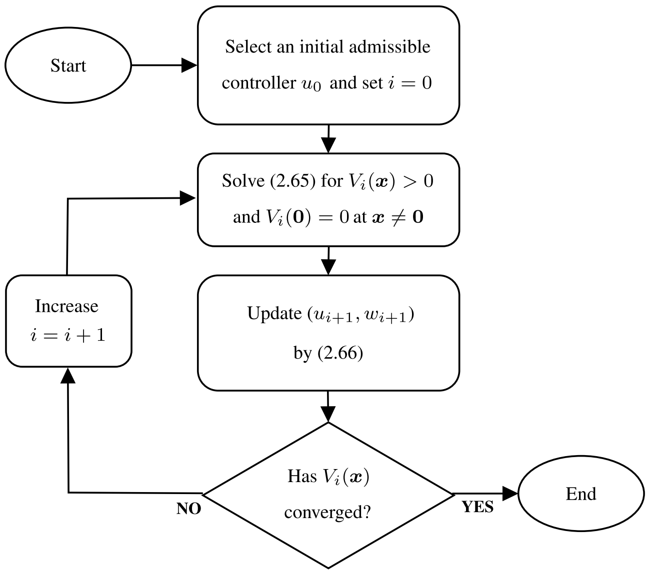

Figure 2.2: Flowchart of the conventional SPUA.

2.5.3 Relaxed SPUA

The conventional SPUA requires to solve a more tractable differential equation in the evaluation policy step. However, it is still hard to find a solution, or in the worst of the cases, the solution cannot be written as elementary functions. The works [41–43] have opted for the relaxation of the dynamic programming problem to an optimization problem, that is to say, find a solution $V(\boldsymbol{x}) > 0,\ V(\boldsymbol{0}) = 0$ at $\boldsymbol{x} \neq \boldsymbol{0}$ of the following minimizing linear programming problem

$$\min_{V(\boldsymbol{x})} \int \cdots \int_{\Omega} V(\boldsymbol{x}) dx_1 \cdots dx_n \;\; \text{subject to} \\ \mathcal{H}(\boldsymbol{x}, V(\boldsymbol{x}), u, w) \leq 0, \tag{2.67}$$

where $\Omega \subseteq \mathbb{R}^n$ and the origin is an element of this subset. A solution $V(\boldsymbol{x})$ satisfying (2.67) is not an strict solution of the HJI equation, yet a lower bound [70] or a upper bound [42,43] of the real cost.

2.5.4 Converse Optimal Problem

The converse problem to the optimal control problem formulated in [71] consists in finding a class of nonlinear systems for which a given performance and a given storage function, the latter is the solution of the optimal control problem. The converse problem is also described by the HJI equation. However, since the value function and performance are given, the HJI reduces to an algebraic equation in the unknowns $\boldsymbol{\mathcal{F}}(\boldsymbol{x})$, $\boldsymbol{\mathcal{G}}(\boldsymbol{x})$ and $\boldsymbol{\mathcal{K}}(\boldsymbol{x})$, instead of solving a first-order nonlinear partial differential equation in unknown $V(\boldsymbol{x})$ when the vectors $\boldsymbol{\mathcal{F}}(\boldsymbol{x})$, $\boldsymbol{\mathcal{G}}(\boldsymbol{x})$ and $\boldsymbol{\mathcal{K}}(\boldsymbol{x})$ are given.

Consider the nonlinear system below.

$$\begin{aligned} \dot{x}_1 &= -\frac{19}{6}x_1 + \frac{3}{2}x_1x_2^2 - \frac{7}{3}x_2 - \frac{x_2^2}{6x_2^2+6} - \frac{1}{3}x_2 \arctan(x_2) + x_2u + w, \\ \dot{x}_2 &= x_1, \\ y &= x_1. \end{aligned} \tag{2.68}$$

For the performance index $\int_0^\infty (y^T y + u^T u - \gamma_0^2 w^T w)dt$ and a minimum attenuation factor $\gamma_0 = \frac{1}{\sqrt{2}}$, the value function and optimal controller are

$$V(\boldsymbol{x}) = 3x_1^2 + 7x_2^2 + x_2^2 \arctan(x_2), \tag{2.69}$$

$$u = -3x_1x_2. \tag{2.70}$$

Proof. By choosing the value function as (2.58), its gradient is

$$\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} = [D_{x_1}V, D_{x_2}V] = \left[6x_1, 14x_2 + \frac{x_2^2}{x_2^2+1} + 2x_2 \arctan(x_2)\right]^T. \tag{2.71}$$

Considering the unknowns $\boldsymbol{\mathcal{F}}(\boldsymbol{x}) = [f_1(\boldsymbol{x}), f_2(\boldsymbol{x})]^T$, $\boldsymbol{\mathcal{G}}(\boldsymbol{x}) = [g_1(\boldsymbol{x}), 0]$, $\boldsymbol{\mathcal{K}}(\boldsymbol{x}) = [k_1(\boldsymbol{x}), 0]$ and define the output as $y = x_1$. The substitution in the HJI equation leads to

$$\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \begin{bmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \end{bmatrix} - \frac{1}{4} \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \begin{bmatrix} g_1(\boldsymbol{x}) \\ 0 \end{bmatrix} \begin{bmatrix} g_1(\boldsymbol{x}) \\ 0 \end{bmatrix}^T \left(\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \right)^T + \frac{1}{4\gamma_0^2} \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \begin{bmatrix} k_1(\boldsymbol{x}) \\ 0 \end{bmatrix} \begin{bmatrix} k_1(\boldsymbol{x}) \\ 0 \end{bmatrix}^T \left(\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \right)^T + x_1^2 = 0. \quad (2.72)$$

Reducing the algebraic expression, it is obtained

$$6x_1 f_1(\boldsymbol{x}) + \left(14x_2 + \frac{x_2^2}{x_2^2+1} + 2x_2 \arctan(x_2) \right) f_2(\boldsymbol{x}) - \frac{1}{4}(6x_1)^2 g_1^2(\boldsymbol{x}) + \frac{1}{4\gamma_0^2}(6x_1)^2 k_1^2(\boldsymbol{x}) + x_1^2 = 0. \quad (2.73)$$

Isolating $f_1(\boldsymbol{x})$

$$f_1(\boldsymbol{x}) = -\frac{\left(14x_2 + \frac{x_2^2}{x_2^2+1} + 2x_2 \arctan(x_2) \right) f_2(\boldsymbol{x})}{6x_1} + \frac{3x_1 g_1^2(\boldsymbol{x})}{2} - \frac{3x_1 k_1^2(\boldsymbol{x})}{2\gamma_0^2} - \frac{x_1}{6}. \quad (2.74)$$

By choosing $f_2(\boldsymbol{x}) = x_1$, we get

$$f_1(\boldsymbol{x}) = -\frac{7}{3}x_2 - \frac{x_2^2}{6(x_2^2+1)} - \frac{x_2}{3}\arctan(x_2) + \frac{3x_1 g_1^2(\boldsymbol{x})}{2} - \frac{3x_1 k_1^2(\boldsymbol{x})}{2\gamma_0^2} - \frac{x_1}{6}. \quad (2.75)$$

At this point, one can freely choose $g_1(\boldsymbol{x})$ and $k_1(\boldsymbol{x})$ to make it as complicated as desired, for simplicity, $g_1(\boldsymbol{x}) = x_2$, $k_1(\boldsymbol{x}) = 1$ and $\gamma_0 = \frac{1}{\sqrt{2}}$ have been chosen to obtain

$$f_1(\boldsymbol{x}) = -\frac{7}{3}x_2 - \frac{x_2^2}{6(x_2^2+1)} - \frac{x_2}{3}\arctan(x_2) + \frac{3}{2}x_1 x_2^2 - \frac{19}{6}x_1. \quad (2.76)$$

□

3

Stability Study and Synthesis of Controllers

"Nothing is more powerful than an idea whose time has come."

— Victor Hugo

This chapter addresses the stability and stabilization problems by means of the integral-type Lyapunov function presented in [22]. The first study considers the polynomial fuzzy system (2.27) under zero input condition. The time derivative of $V(\boldsymbol{x})$ involves a double-fuzzy summation that is relaxed by using the copositive idea. The stabilization problem makes use of the Positivstellensatz refutation to characterize the polynomials conditions on the semialgebraic set of interest.

3.1 Stability Analysis

Theorem 3.1. Let (2.27) at $u = 0$ be a fuzzy system in polynomial form describing the dynamic behaviour of a nonlinear system with the zero equilibrium state. If there exist matrices $\boldsymbol{P}_0,\ \boldsymbol{D}_i \in \mathbb{R}^{n\times n}$ and $s \in \mathbb{Z}^{\geq 0}$ such that, for given $\epsilon > 0$, polynomials $\epsilon_{ij}(\boldsymbol{x}) > 0$, the conditions

$$\boldsymbol{x}^T\{\boldsymbol{P}_0 + \boldsymbol{D}_i - \epsilon \boldsymbol{I}\}\boldsymbol{x} \in \mathbb{S}[\boldsymbol{x}]\ \forall i, \tag{3.1}$$

$$\left(\sum_{k=1}^{r} \hat{h}_k^2\right)^s \sum_{i=1}^{r}\sum_{j=1}^{r} \hat{h}_i^2\hat{h}_j^2 \boldsymbol{\Lambda}_{ij}(\boldsymbol{x}) \in \mathbb{S}[\hat{\boldsymbol{h}}, \boldsymbol{x}]\ \forall i, j, \tag{3.2}$$

with $\boldsymbol{\Lambda}_{ij}(\boldsymbol{x}) = -\boldsymbol{x}^T\Big(\boldsymbol{A}_i^T(\boldsymbol{x})(\boldsymbol{P}_0 + \boldsymbol{D}_j) + (\boldsymbol{P}_0 + \boldsymbol{D}_j)\boldsymbol{A}_i(\boldsymbol{x}) + \epsilon_{ij}(\boldsymbol{x})\boldsymbol{I}\Big)\boldsymbol{x}$ and $\hat{\boldsymbol{h}} = [\hat{h}_1^2\ \hat{h}_2^2 \cdots \hat{h}_r^2]$ hold true, then the origin is asymptotically stable.

Proof. This demonstration leverages the integral-type Lyapunov function candidate (2.44). Keep in mind that the square matrices $\boldsymbol{P}_0$ and $\boldsymbol{D}_i$ have to be constructed under the guidelines to ensure independence of the path. The vector $\frac{dV(\boldsymbol{x})}{dt}$ is expressed as

$$\begin{aligned}\dot{V}(\boldsymbol{x}) &= 2\boldsymbol{x}^T\boldsymbol{P}(\boldsymbol{x})\dot{\boldsymbol{x}} \\ &= \dot{\boldsymbol{x}}^T\boldsymbol{P}(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{P}(\boldsymbol{x})\dot{\boldsymbol{x}}.\end{aligned} \tag{3.3}$$

The replacement of (2.27) assuming that $u = 0$ in (3.3) brings the following condition

$$= \sum_{i=1}^{r} h_i(\boldsymbol{x})\{\boldsymbol{x}^T\boldsymbol{A}_i^T(\boldsymbol{x})\boldsymbol{P}(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{P}(\boldsymbol{x})\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x}\}. \tag{3.4}$$

Substituting (2.47) and factorizing one obtains

$$= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{x})h_j(\boldsymbol{x})\boldsymbol{x}^T\{\boldsymbol{A}_i^T(\boldsymbol{x})(\boldsymbol{P}_0 + \boldsymbol{D}_j) + (\boldsymbol{P}_0 + \boldsymbol{D}_j)\boldsymbol{A}_i(\boldsymbol{x})\}\boldsymbol{x}. \tag{3.5}$$

The nonnegativity property of the MFs $h_1(\boldsymbol{x}), \cdots, h_r(\boldsymbol{x})$ permits the substitutionn $\hat{h}_i^2 = h_i(\boldsymbol{x})$, $\hat{h}_j^2 = h_j(\boldsymbol{x})$ to consider them as quadratic polynomial variables and become part of the conditions. Thus

$$\dot{V}(\boldsymbol{x}) = \sum_{i=1}^{r}\sum_{j=1}^{r} \hat{h}_i^2\hat{h}_j^2\boldsymbol{x}^T\{\boldsymbol{A}_i^T(\boldsymbol{x})(\boldsymbol{P}_0 + \boldsymbol{D}_j) + (\boldsymbol{P}_0 + \boldsymbol{D}_j)\boldsymbol{A}_i(\boldsymbol{x})\}\boldsymbol{x}. \tag{3.6}$$

The conditions $\dot{V}(\boldsymbol{x}) < 0$ is verified if $-\dot{V}(\boldsymbol{x}) - \epsilon(\boldsymbol{x}) \in \mathbb{S}[\boldsymbol{x}]$. Finally, copositivity property for the double-fuzzy summation brings more relaxed results. □

3.1.1 Stability Analysis Examples

Example 1. Consider the 4-rule fuzzy model in Takagi-Sugeno form below.

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{4} h_i(\boldsymbol{x})\boldsymbol{A}_i\boldsymbol{x}, \tag{3.7}$$

where the state matrices are

$$\boldsymbol{A}_1 = \begin{bmatrix} -5 & -4 \\ -1 & a \end{bmatrix}, \qquad \boldsymbol{A}_2 = \begin{bmatrix} -4 & -4 \\ \frac{1}{5}(3b-2) & \frac{1}{5}(3a-4) \end{bmatrix},$$

$$\boldsymbol{A}_3 = \begin{bmatrix} -3 & -4 \\ \frac{1}{5}(2b-3) & \frac{1}{5}(2a-6) \end{bmatrix}, \ \boldsymbol{A}_4 = \begin{bmatrix} -2 & -4 \\ b & -2 \end{bmatrix}.$$

The varying parameters are in the range $a \in [-13, 0]$ and $b \in [0, 390]$ and the normalized MFs are

$$h_1(\boldsymbol{x}) = M_1^1(x_1)M_2^1(x_2),\ h_2(\boldsymbol{x}) = M_1^1(x_1)M_2^2(x_2),$$
$$h_3(\boldsymbol{x}) = M_1^2(x_1)M_2^1(x_2),\ h_4(\boldsymbol{x}) = M_1^2(x_1)M_2^2(x_2),$$

with

$$M_\lambda^1(x_\lambda) = \begin{cases} 0.5(1-\sin(x_\lambda)) & \text{if } |x_\lambda| \le \frac{\pi}{2} \\ 0 & \text{if } x_\lambda > \frac{\pi}{2} \\ 1 & \text{if } x_\lambda < -\frac{\pi}{2} \end{cases},$$
$$M_\lambda^2(x_\lambda) = 1 - M_\lambda^1(x_\lambda),\quad \lambda \in \{1, 2\}.$$

Therefore, the appropriate diagonal matrices $\boldsymbol{D}_i$ that ensure the independence of the path are given by

$$\begin{aligned} \boldsymbol{D}_1 &= \begin{bmatrix} d_{11}^1 & 0 \\ 0 & d_{22}^1 \end{bmatrix},\ \boldsymbol{D}_2 = \begin{bmatrix} d_{11}^1 & 0 \\ 0 & d_{22}^2 \end{bmatrix}, \\ \boldsymbol{D}_3 &= \begin{bmatrix} d_{11}^3 & 0 \\ 0 & d_{22}^1 \end{bmatrix},\ \boldsymbol{D}_4 = \begin{bmatrix} d_{11}^3 & 0 \\ 0 & d_{22}^2 \end{bmatrix}, \end{aligned} \tag{3.8}$$

and

$$\boldsymbol{P}_0 = \begin{bmatrix} 0 & p_{12} \\ p_{12} & 0 \end{bmatrix}. \tag{3.9}$$

Above fuzzy model has been used in [22, 23] as a benchmark example. The purpose is to find the largest feasible region where a Lyapunov function can be found to check stability of the equilibrium of the system (3.7) when the parameters a and b vary in discrete steps. First of all, the standard polynomial Lyapunov function approach [12] to determine stability of the system is used. Figure 3.1 depicts the feasible areas of the system in Example 1 for quadratic, fourth-degree, sixth-degree and eighth-degree Lyapunov functions. A symbol in the coordinate (a, b) marks when a feasible solution for conditions in [12] was found, proving stability of the zero equilibrium.

The next step is to compare the proposed SOS conditions with other criteria based on

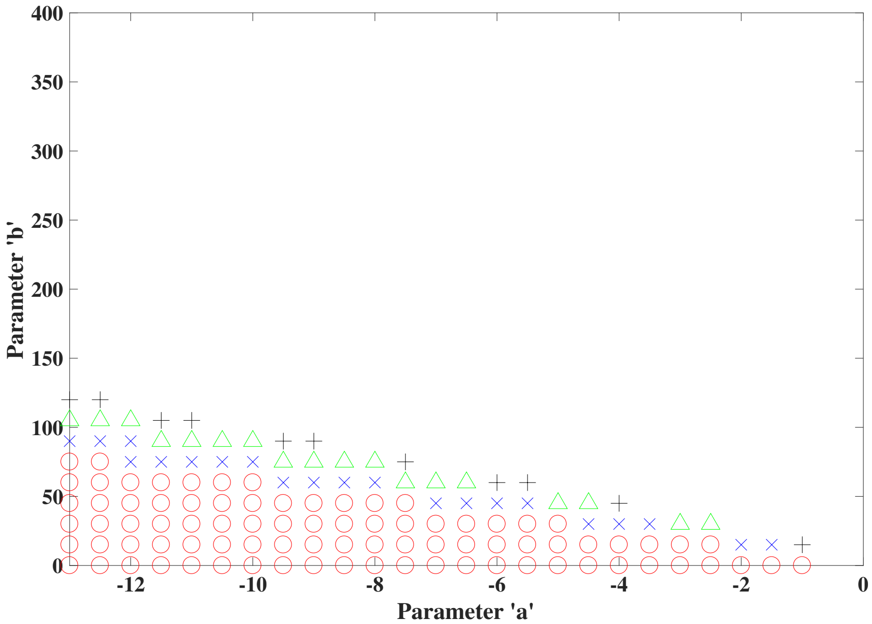

Figure 3.1: Feasible region from conditions in [12] in Example 1 using quadratic Lyapunov function (○) and higher-degree polynomial Lyapunov functions: quartic (×), hexic (△) and octic (+).

the Lyapunov function introduced by [22]. The results are shown in Figure 3.2.

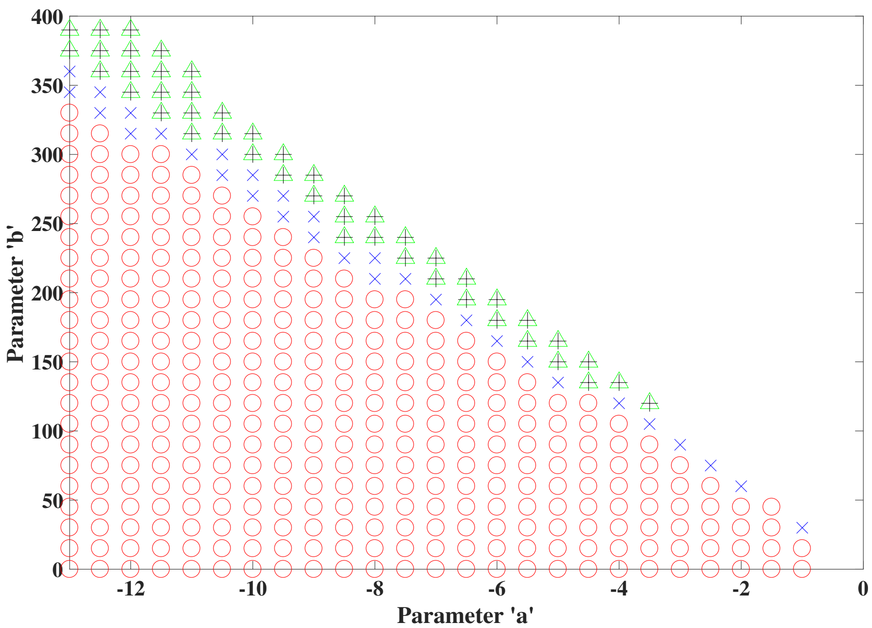

Figure 3.2: Feasible region from conditions in [22] (○), conditions in [23] (×), conditions in [25] (△) and SOS conditions in Theorem 3.1 with Polya exponent $s = 2$ (+).

As seen in Figure 3.2, this proposal verifies that $\boldsymbol{x} = \mathbf{0}$ of the system in Example 1 is asymptotically stable at $a = -12$ and $b = 390$, the solutions are

$$\begin{aligned}
\boldsymbol{P}_0 + \boldsymbol{D}_1 &= \begin{bmatrix} 2.1379 & -0.0101 \\ -0.0101 & 0.1455 \end{bmatrix}, \\
\boldsymbol{P}_0 + \boldsymbol{D}_2 &= \begin{bmatrix} 2.1379 & -0.0101 \\ -0.0101 & 0.0523 \end{bmatrix}, \\
\boldsymbol{P}_0 + \boldsymbol{D}_3 &= \begin{bmatrix} 5.5892 & -0.0101 \\ -0.0101 & 0.1455 \end{bmatrix}, \\
\boldsymbol{P}_0 + \boldsymbol{D}_4 &= \begin{bmatrix} 5.5892 & -0.0101 \\ -0.0101 & 0.0523 \end{bmatrix}.
\end{aligned} \tag{3.10}$$

The trajectories in the phase plane at $\boldsymbol{x}_0 = [-0.8, 3]^T$, $\boldsymbol{x}_0 = [-0.8, -2.1]^T$, $\boldsymbol{x}_0 = [-0.1, -3]^T$, $\boldsymbol{x}_0 = [0.5, -1]^T$, $\boldsymbol{x}_0 = [0.2, 1]^T$, and $\boldsymbol{x}_0 = [-0.5, 0.4]^T$ are exhibited in Figure 3.3, and Figure 3.4 shows the states response for $\boldsymbol{x}_0 = [-0.5, 0.4]^T$.

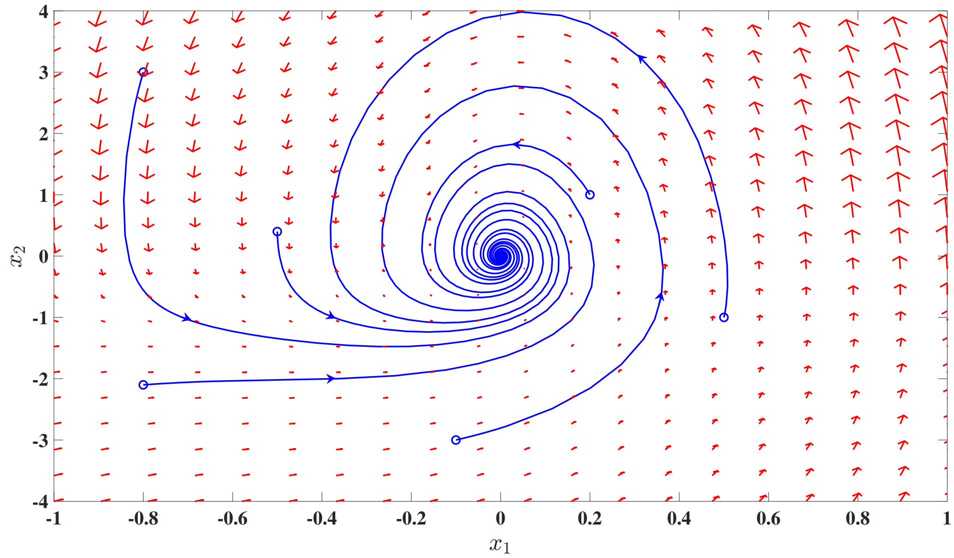

Figure 3.3: Phase trajectories in the plane $x_1 - x_2$ of the system in Takagi-Sugeno form in Example 1, setting the parameter at $a = -12$ and $b = 390$.

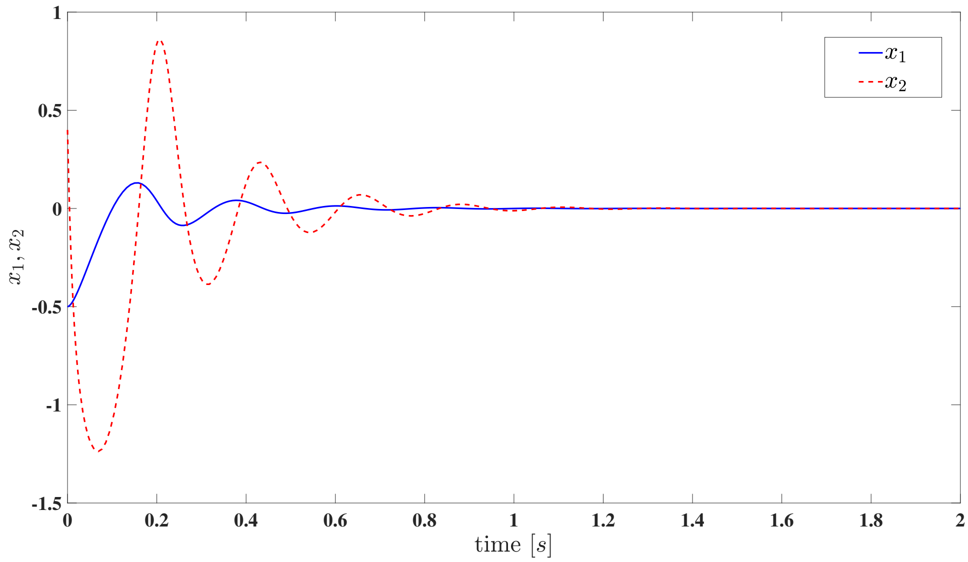

Figure 3.4: State-variable response for $a = -12$ and $b = 390$ at $\boldsymbol{x}_0 = [-0.5, 0.4]^T$.

Example 2. Consider the state equations below.

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{2} h_i(\boldsymbol{x})\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x},$$

where

$$\boldsymbol{A}_1(\boldsymbol{x}) = \begin{bmatrix} -1.1098x_1^2 + 0.17975x_1x_2 - x_2^2 + x_1 - 1 & 1 \\ -1 & -1 \end{bmatrix},$$
$$\boldsymbol{A}_2(\boldsymbol{x}) = \begin{bmatrix} -1.1807x_1^2 + 0.18751x_1x_2 - x_2^2 + x_1 - 1 & 1 \\ 0.2172 & -1 \end{bmatrix}.$$

and MFs

$$h_1(x_1) = \frac{1 + \tanh x_1}{2}, \; h_2(x_1) = \frac{1 - \tanh x_1}{2}.$$

Different from Example 1, the aforementioned state equations are in a polynomial fuzzy form. Therefore, LMI conditions presented in [22,23,25] do not work to study stability of the fuzzy system in polynomial form. On the other hand, our SOS conditions are feasible with the following solutions

$$\boldsymbol{P}_1 = \begin{bmatrix} 0.2367 & 0 \\ 0 & 0.3087 \end{bmatrix}, \; \boldsymbol{P}_2 = \begin{bmatrix} 0.2112 & 0 \\ 0 & 0.3087 \end{bmatrix}. \tag{3.11}$$

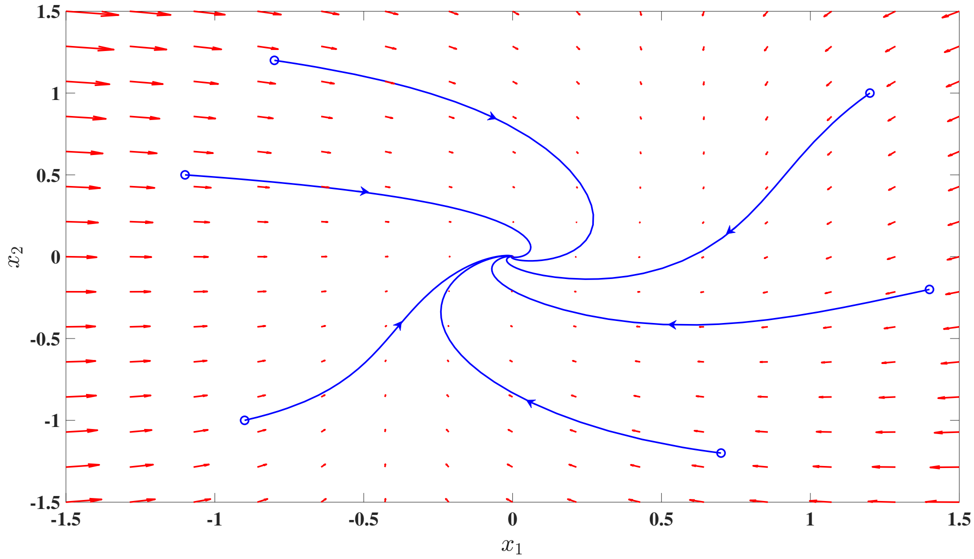

Figure 3.5: Phase trajectories in the plane $x_1 - x_2$ of the system in Example 2.

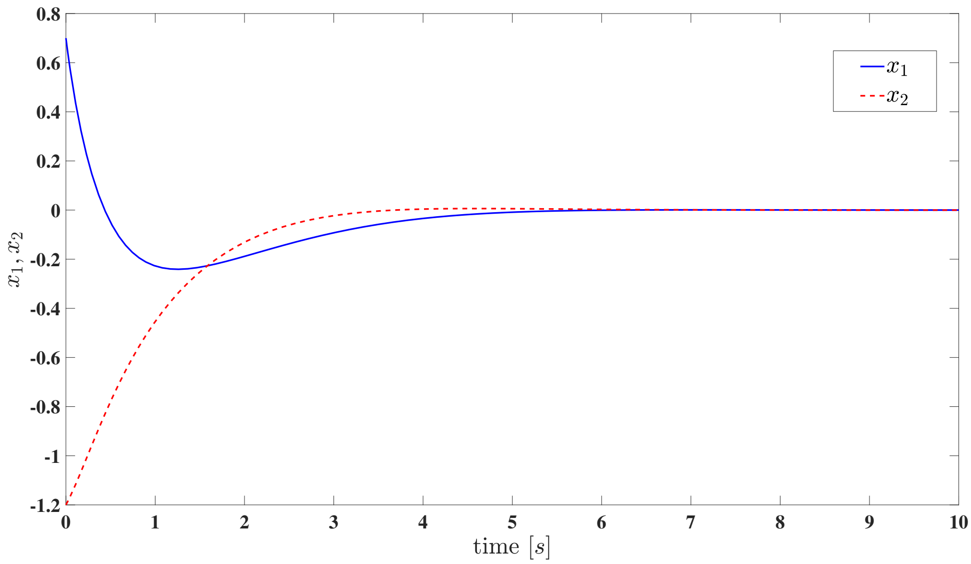

Figure 3.6: State-variable response at $\boldsymbol{x}_0 = [0.7, -1.2]^T$.

For initial conditions $\boldsymbol{x}_0 = [-1.1, 0.5]^T$, $\boldsymbol{x}_0 = [-0.8, 1.2]^T$, $\boldsymbol{x}_0 = [1.2, 1]^T$, $\boldsymbol{x}_0 = [-0.9, -1]^T$, $\boldsymbol{x}_0 = [1.4, -0.2]^T$, and $\boldsymbol{x}_0 = [0.7, -1.2]^T$, the trajectories in the phase plane are depicted in Figure 3.5, and Figure 3.6 illustrates the time response of the states variables at $\boldsymbol{x}_0 = [0.7, -1.2]^T$.

3.2 Control Synthesis

Theorem 3.2. The zero solution $\boldsymbol{x} = \boldsymbol{0}$ of the polynomial fuzzy system on the form (2.41) is feedback stabilizable with a PDC control law (2.40) if there exists square matrices $\boldsymbol{P}_0 + \boldsymbol{D}_i$, polynomial feedback gain vectors $\boldsymbol{F}_j(\boldsymbol{x})$ and Positivstellensatz multipliers $\tau_l(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$, $\sigma_l(\boldsymbol{x}),\ \rho_{ij}(\boldsymbol{x}) \in \mathbb{S}[\boldsymbol{x}]$ such that

$$\boldsymbol{x}^T\{\boldsymbol{P}_0 + \boldsymbol{D}_i - \epsilon \boldsymbol{I}\}\boldsymbol{x} \in \mathbb{S}[\boldsymbol{x}]\ \forall i, \tag{3.12}$$

$$\begin{aligned} -\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{l=1}^{r} \hat{h}_i^2\hat{h}_j^2\hat{h}_k^2\hat{h}_l^2 \Big\{\sigma_l(\boldsymbol{x})\Big[2\boldsymbol{x}^T\Big((\boldsymbol{P}_0+\boldsymbol{D}_k)\boldsymbol{A}_i(\boldsymbol{x}) - (\boldsymbol{P}_0+\boldsymbol{D}_k)\boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x}) \\ - \alpha(\boldsymbol{P}_0+\boldsymbol{D}_k)\Big)\boldsymbol{x}\Big] + \tau_l(\boldsymbol{x}) + \rho_{ij}(\boldsymbol{x})\Big\} + \sum_{l=1}^{r}\hat{h}_l^2\tau_l(\boldsymbol{x}) \in \mathbb{S}[\hat{\boldsymbol{h}}, \boldsymbol{x}], \end{aligned} \tag{3.13}$$

where $\epsilon > 0$ is a given small real number, $\alpha < 0$ and $\hat{\boldsymbol{h}} = [\hat{h}_1^2, \cdots, \hat{h}_r^2]$.

Proof. The time derivative of the integral-type (2.44) is

$$\dot{V}(\boldsymbol{x}) = 2\boldsymbol{x}^T\boldsymbol{P}(\boldsymbol{x})\dot{\boldsymbol{x}}. \tag{3.14}$$

Substituting (2.41) and factorizing, it becomes

$$= 2\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{x})h_j(\boldsymbol{x})\boldsymbol{x}^T\Big\{\boldsymbol{P}(\boldsymbol{x})\big(\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x})\big)\Big\}\boldsymbol{x}. \tag{3.15}$$

Now, replacing (2.47), one can rewrite latter equation as

$$2\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} h_i(\boldsymbol{x})h_j(\boldsymbol{x})h_k(\boldsymbol{x})\boldsymbol{x}^T\Big\{(\boldsymbol{P}_0+\boldsymbol{D}_k)\boldsymbol{A}_i(\boldsymbol{x}) - (\boldsymbol{P}_0+\boldsymbol{D}_k)\boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x})\Big\}\boldsymbol{x} < 0. \tag{3.16}$$

Recalling the property of the MFs $\sum_{i=1}^{r} h_i(\boldsymbol{x}) = 1$. Moreover, since MFs are nonnegative, one can replace them as $\hat{h}_i^2 = h_i(\boldsymbol{x})$ and consider the following set conditions

$$\left\{ \hat{\boldsymbol{h}}, \boldsymbol{x} \in \mathbb{R}^{n+r} \;\middle|\; \begin{array}{c} \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\hat{h}_i^2\hat{h}_j^2\hat{h}_k^2\Big\{2\boldsymbol{x}^T\Big[(\boldsymbol{P}_0+\boldsymbol{D}_k)\boldsymbol{A}_i(\boldsymbol{x}) \\ -(\boldsymbol{P}_0+\boldsymbol{D}_k)\boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x}) - \alpha(\boldsymbol{P}_0+\boldsymbol{D}_k)\Big]\boldsymbol{x}\Big\} \geq 0, \\ \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\hat{h}_i^2\hat{h}_j^2\hat{h}_k^2 - 1 = 0 \end{array} \right\} = \emptyset. \tag{3.17}$$

By making use of the Positivstellensatz, the semialgebraic set is empty if the equality below is satisfied.

$$s_0(\hat{\boldsymbol{h}},\boldsymbol{x}) + s_1(\hat{\boldsymbol{h}},\boldsymbol{x})\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\hat{h}_i^2\hat{h}_j^2\hat{h}_k^2\Big\{2\boldsymbol{x}^T\Big[(\boldsymbol{P}_0+\boldsymbol{D}_k)\boldsymbol{A}_i(\boldsymbol{x}) - (\boldsymbol{P}_0+\boldsymbol{D}_k)\boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x}) - \alpha(\boldsymbol{P}_0+\boldsymbol{D}_k)\Big]\boldsymbol{x}\Big\} + t(\hat{\boldsymbol{h}},\boldsymbol{x})\left[\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\hat{h}_i^2\hat{h}_j^2\hat{h}_k^2 - 1\right] = -1. \quad (3.18)$$

The multiplication of previous equation by an SOS polynomial denoted as $q(\hat{\boldsymbol{h}},\boldsymbol{x})$ brings the expression

$$q(\hat{\boldsymbol{h}},\boldsymbol{x})s_1(\hat{\boldsymbol{h}},\boldsymbol{x})\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\hat{h}_i^2\hat{h}_j^2\hat{h}_k^2\Big\{2\boldsymbol{x}^T\Big[(\boldsymbol{P}_0+\boldsymbol{D}_k)\boldsymbol{A}_i(\boldsymbol{x}) - (\boldsymbol{P}_0+\boldsymbol{D}_k)\boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x}) - \alpha(\boldsymbol{P}_0+\boldsymbol{D}_k)\Big]\boldsymbol{x}\Big\} + q(\hat{\boldsymbol{h}},\boldsymbol{x})t(\hat{\boldsymbol{h}},\boldsymbol{x})\left[\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\hat{h}_i^2\hat{h}_j^2\hat{h}_k^2 - 1\right] + q(\hat{\boldsymbol{h}},\boldsymbol{x}) = -q(\hat{\boldsymbol{h}},\boldsymbol{x})s_0(\hat{\boldsymbol{h}},\boldsymbol{x}). \quad (3.19)$$

Note that the resulting polynomial $q(\hat{\boldsymbol{h}},\boldsymbol{x})s_1(\hat{\boldsymbol{h}},\boldsymbol{x})$ is SOS and $q(\hat{\boldsymbol{h}},\boldsymbol{x})t(\hat{\boldsymbol{h}},\boldsymbol{x})$ is an element of the polynomial ring (not SOS). Therefore, for simplicity $q(\hat{\boldsymbol{h}},\boldsymbol{x})s_1(\hat{\boldsymbol{h}},\boldsymbol{x}) = \sum_{l=1}^{r}\hat{h}_l^2\sigma_l(\boldsymbol{x})$, $q(\hat{\boldsymbol{h}},\boldsymbol{x})t(\hat{\boldsymbol{h}},\boldsymbol{x}) = \sum_{l=1}^{r}\hat{h}_l^2\tau_l(\boldsymbol{x})$ where $\sigma_l(\boldsymbol{x}) \in \mathbb{S}[\hat{\boldsymbol{h}},\boldsymbol{x}]$ and $\tau_l(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$. Moreover, the isolated term $q(\hat{\boldsymbol{h}},\boldsymbol{x})$ is rewritten as $\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{l=1}^{r}\hat{h}_i^2\hat{h}_j^2\hat{h}_k^2\hat{h}_l^2\rho_{ij}(\boldsymbol{x})$ for $\rho_{ij}(\boldsymbol{x}) \in \mathbb{S}[\hat{\boldsymbol{h}},\boldsymbol{x}]$. Then, equation (3.19) becomes condition (3.13), finishing the proof. □

3.2.1 Path Following Algorithm

Conditions stated in Theorem 3.2 represent a non-convex optimization problem, therefore, conventional SOS solvers cannot find a solution. We leverage an iterative SOS method (also called path-following) to solve this bilinear formulation.

Algorithm 2. Path following for the stabilization problem

Step 1. Assume that each consequent part of the fuzzy system in polynomial form is an independent nonlinear systems and solve the following conditions

$$\begin{aligned} \boldsymbol{x}^T\{\boldsymbol{X}_j - \epsilon \boldsymbol{I}\}\boldsymbol{x} \in \mathbb{S}[\boldsymbol{x}], \\ -\boldsymbol{v}^T\Big(\boldsymbol{A}_j(\boldsymbol{x})\boldsymbol{X}_j - \boldsymbol{B}_j(\boldsymbol{x})\boldsymbol{M}_j(\boldsymbol{x}) + \boldsymbol{X}_j\boldsymbol{A}_j^T(\boldsymbol{x}) - \boldsymbol{M}_j^T(\boldsymbol{x})\boldsymbol{B}_j^T(\boldsymbol{x})\Big)\boldsymbol{v} \in \mathbb{S}[\boldsymbol{v},\boldsymbol{x}], \end{aligned} \tag{3.20}$$

where $\epsilon > 0$ is a small real number, vector $\boldsymbol{v}$ do not depend on $\boldsymbol{x}$ and $\boldsymbol{F}_j(\boldsymbol{x}) = \boldsymbol{M}_j(\boldsymbol{x})\boldsymbol{X}^{-1}$.

Step 2. Set $\boldsymbol{P}_i = \boldsymbol{D}_i$ and employ the vectors $\boldsymbol{F}_j(\boldsymbol{x})$ calculated in Step 1 to find a solution for the following minimizing problem

$$\min_{\boldsymbol{P}_i} \alpha \text{ subject to (3.22) and (3.23)}, \tag{3.21}$$

with

$$\boldsymbol{x}^T\{\boldsymbol{P}_i - \epsilon \boldsymbol{I}\}\boldsymbol{x} \in \mathbb{S}[\boldsymbol{x}] \ \forall i, \tag{3.22}$$

$$-\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} \hat{h}_i^2\hat{h}_j^2\hat{h}_k^2\Lambda_{ijk}(\boldsymbol{x}) \in \mathbb{S}[\hat{\boldsymbol{h}},\boldsymbol{x}], \tag{3.23}$$

where $\Lambda_{ijk}(\boldsymbol{x}) = 2\boldsymbol{x}^T\big\{\boldsymbol{P}_k\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{P}_k\boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x}) - \alpha\boldsymbol{P}_k\big\}\boldsymbol{x}$. Set $\eta = 0$ and $\boldsymbol{P}_i^\eta = \boldsymbol{P}_i$.

Step 3. Set $\boldsymbol{P}_i = \boldsymbol{P}_i^\eta$ and $\sigma_l(\boldsymbol{x}) = 1$. Solve the minimizing problem below.

$$\min_{\boldsymbol{F}_j(\boldsymbol{x}),\tau_l(\boldsymbol{x}),\rho_{ij}(\boldsymbol{x})} \alpha \text{ subject to (3.12)-(3.13)} \tag{3.24}$$

If any feasible solutions are found with $\alpha < 0$, then they satisfy Theorem (3.2). Otherwise, go to step 4.

Step 4. By using $\boldsymbol{P}_i$ and the previous computed values of $\boldsymbol{F}_j(\boldsymbol{x})$, $\sigma_l(\boldsymbol{x})$, $\tau_l(\boldsymbol{x})$ and $\rho_{ij}(\boldsymbol{x})$ solve the following SOS minimizing problem

$$\min_{\delta\boldsymbol{F}_j(\boldsymbol{x}),\delta\boldsymbol{P}_i,\delta\sigma_l(\boldsymbol{x}),\delta\tau_l(\boldsymbol{x}),\delta\rho_{ij}(\boldsymbol{x})} \alpha \text{ subject to (3.26)-(3.34)} \tag{3.25}$$

The SOS conditions are

$$\boldsymbol{x}^T\{\boldsymbol{P}_i + \delta\boldsymbol{P}_i - \epsilon\boldsymbol{I}\}\boldsymbol{x} \in \mathbb{S}[\boldsymbol{x}], \ \forall i, \tag{3.26}$$

where ϵ is a given small positive real number and $\delta\boldsymbol{P}_i = \delta\boldsymbol{P}_0 + \delta\boldsymbol{D}_i$.

$$-\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{l=1}^{r}\hat{h}_i^2\hat{h}_j^2\hat{h}_k^2\hat{h}_l^2\Big\{\sigma_l(\boldsymbol{x})\Big[\Lambda_{ijk}(\boldsymbol{x})+\delta\Lambda_{ijk}(\boldsymbol{x})\Big]+\delta\sigma_l(\boldsymbol{x})\Big[\Lambda_{ijk}(\boldsymbol{x})\Big]$$
$$+\tau_l(\boldsymbol{x})+\delta\tau_l(\boldsymbol{x})+\rho_{ij}(\boldsymbol{x})+\delta\rho_{ij}(\boldsymbol{x})\Big\}+\sum_{l=1}^{r}\hat{h}_l^2\big\{\tau_l(\boldsymbol{x})+\delta\tau_l(\boldsymbol{x})\big\}\in\mathbb{S}[\hat{\boldsymbol{h}},\boldsymbol{x}], \quad (3.27)$$

where, $\delta\Lambda_{ijk}(\boldsymbol{x}) = 2\boldsymbol{x}^T\big\{\delta\boldsymbol{P}_k\boldsymbol{A}_i(\boldsymbol{x}) - \delta\boldsymbol{P}_k\boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x}) - \boldsymbol{P}_k\boldsymbol{B}_i(\boldsymbol{x})\delta\boldsymbol{F}_j(\boldsymbol{x}) - \alpha\delta\boldsymbol{P}_k\big\}\boldsymbol{x}$.

$$\sigma_l(\boldsymbol{x})+\delta\sigma_l(\boldsymbol{x})\in\mathbb{S}[\boldsymbol{x}],\ \forall l, \quad (3.28)$$

$$\rho_{ij}(\boldsymbol{x})+\delta\rho_{ij}(\boldsymbol{x})\in\mathbb{S}[\boldsymbol{x}],\forall i,j \quad (3.29)$$

$$\boldsymbol{v}_1^T\begin{bmatrix}\epsilon_P\boldsymbol{P}_i^2 & \delta\boldsymbol{P}_i\\ \delta\boldsymbol{P}_i & \boldsymbol{I}\end{bmatrix}\boldsymbol{v}_1\in\mathbb{S}[\boldsymbol{v}_1],\ \forall i, \quad (3.30)$$

$$\boldsymbol{v}_2^T\begin{bmatrix}\epsilon_F\boldsymbol{F}_j(\boldsymbol{x})\boldsymbol{F}_j^T & \delta\boldsymbol{F}_j\\ \delta\boldsymbol{F}_j^T & \boldsymbol{I}\end{bmatrix}\boldsymbol{v}_2\in\mathbb{S}[\boldsymbol{v}_2,\boldsymbol{x}],\ \forall j, \quad (3.31)$$

$$\boldsymbol{v}_3^T\begin{bmatrix}\epsilon_\sigma\sigma_l(\boldsymbol{x})^2 & \delta\sigma_l(\boldsymbol{x})\\ \delta\sigma_l(\boldsymbol{x}) & 1\end{bmatrix}\boldsymbol{v}_3\ \in\mathbb{S}[\boldsymbol{v}_3,\boldsymbol{x}],\ \forall l, \quad (3.32)$$

$$\boldsymbol{v}_4^T\begin{bmatrix}\epsilon_\tau\tau_l(\boldsymbol{x})^2 & \delta\tau_l(\boldsymbol{x})\\ \delta\tau_l(\boldsymbol{x}) & 1\end{bmatrix}\boldsymbol{v}_4\in\mathbb{S}[\boldsymbol{v}_4,\boldsymbol{x}],\ \forall l, \quad (3.33)$$

$$\boldsymbol{v}_5^T\begin{bmatrix}\epsilon_\rho\rho_{ij}(\boldsymbol{x})^2 & \delta\rho_{ij}(\boldsymbol{x})\\ \delta\rho_{ij}(\boldsymbol{x}) & 1\end{bmatrix}\boldsymbol{v}_5\in\mathbb{S}[\boldsymbol{v}_5,\boldsymbol{x}],\ \forall i,j, \quad (3.34)$$

where $\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \boldsymbol{v}_3,\ \boldsymbol{v}_4,\ \boldsymbol{v}_5$ are vectors that do not depend on $\boldsymbol{x}$ and ϵ_P, ϵ_F, ϵ_σ, ϵ_τ, ϵ_ρ, are small positive real numbers. For $\delta\boldsymbol{P}_i$ obtained from (3.25), set $\boldsymbol{P}_i^{\eta+1} = \boldsymbol{P}_i + \delta\boldsymbol{P}_i$. Then, increase $\eta = \eta + 1$ and return to step 3.

3.2.2 Design Examples

Example 3. The following is a nonlinear system in a three-rule Takagi-Sugeno form

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{3} h_i(\boldsymbol{x})\{\boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{B}_i\boldsymbol{u}\}, \tag{3.35}$$

with state matrices and input vectors given below

$$\boldsymbol{A}_1 = \begin{bmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{bmatrix}, \ \boldsymbol{A}_2 = \begin{bmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{bmatrix}, \ \boldsymbol{A}_3 = \begin{bmatrix} -a & -4.33 \\ 0 & 0.05 \end{bmatrix},$$

$$\boldsymbol{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \boldsymbol{B}_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix}, \ \boldsymbol{B}_3 = \begin{bmatrix} -b+6 \\ -1 \end{bmatrix},$$

and MFs

$$\begin{aligned} h_1(x_1) &= \frac{\cos(10x_1)+1}{4}, \\ h_2(x_1) &= \frac{\sin(10x_1)+1}{4}, \\ h_3(x_1) &= \frac{-\cos(10x_1)-\sin(10x_1)+2}{4}. \end{aligned}$$

The present benchmark example has been thoroughly studied in the literature (see [14,15, 72] and references therein). This Takagi-Sugeno fuzzy model includes two varying parameters (a and b). Setting $a = 2$, the purpose is to determine the maximum value of the parameter b such that a stabilizing control can be designed. Table 3.1 summarizes the results obtained by using proposed conditions and other existing results. It is worth mentioning that the criterion in this thesis admits polynomial feedback gain vectors $\boldsymbol{F}_j(\boldsymbol{x})$, however constant feedback gain vectors $\boldsymbol{F}_j$ are used instead to fairly make a comparison with the LMI-based approaches. As

Table 3.1: Comparative results on the maximum value of parameter b in Example 3

Method	b_{max}
Theorem 3.2	6.9
Method in [72]	6.5
Theorem 5 in [14]	6.5
Theorem 5 in [15]	6

seen in Table 3.1, the b_{max} obtained using our proposal is higher than other b_{max} obtained via other existing approaches. For $a = 2$ and $b = 6.9$, the solutions of the SOS conditions in Theorem 3.2, with $\alpha = -0.0022$ are

$$\boldsymbol{F}_1 = \begin{bmatrix} 4.0241 & 0.7055 \end{bmatrix}, \boldsymbol{F}_2 = \begin{bmatrix} 1.0221 & 1.0051 \end{bmatrix}, \boldsymbol{F}_3 = \begin{bmatrix} -0.2238, -3.7678 \end{bmatrix},$$

$$\boldsymbol{P}_0 + \boldsymbol{D}_1 = \begin{bmatrix} 0.0476 & 0.0957 \\ 0.0957 & 2.3981 \end{bmatrix},$$

$$\boldsymbol{P}_0 + \boldsymbol{D}_2 = \begin{bmatrix} 0.0050 & 0.0957 \\ 0.0957 & 2.3981 \end{bmatrix},$$

$$\boldsymbol{P}_0 + \boldsymbol{D}_3 = \begin{bmatrix} 0.7473 & 0.0957 \\ 0.0957 & 2.3981 \end{bmatrix}.$$

Figure 3.7 illustrates the phase trajectories of the uncontrolled systems while Figure 3.8 shows trajectories of the feedback system in Takagi-Sugeno form in Example 3, setting $a = 2$ and $b = 6.9$ at $\boldsymbol{x}_0 = [-0.3, 1.1]^T$, $\boldsymbol{x}_0 = [-1.2, 0.9]^T$, $\boldsymbol{x}_0 = [1.1, 0.7]^T$, $\boldsymbol{x}_0 = [0.4, -1.1]^T$, $\boldsymbol{x}_0 = [1.3, -0.4]^T$, and $\boldsymbol{x}_0 = [-1.2, -0.8]^T$.

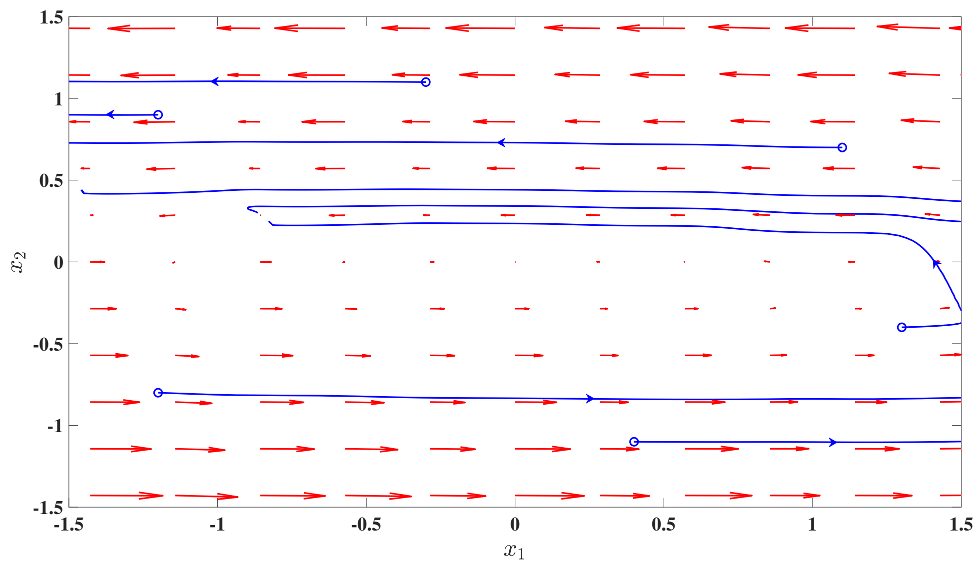

Figure 3.7: Phase trajectories in the plane $x_1 - x_2$ of the uncontrolled system in Example 3

The time plot of state-variables and $u(t)$ of the feedback fuzzy system in Example 3 at $\boldsymbol{x}_0 = [-1.2, -0.8]^T$ are illustrated in Figure 3.9.

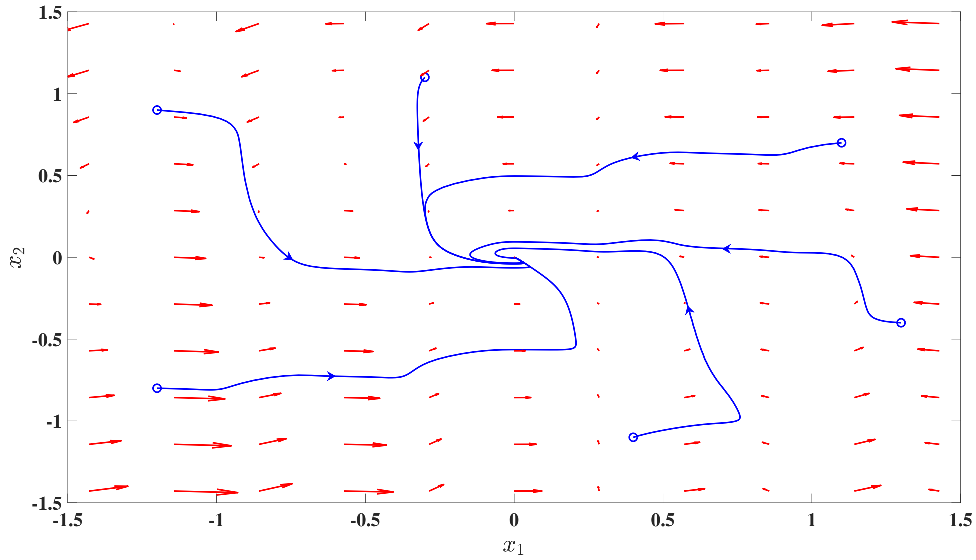

Figure 3.8: Phase trajectories in the plane $x_1 - x_2$ of the feedback system in Example 3

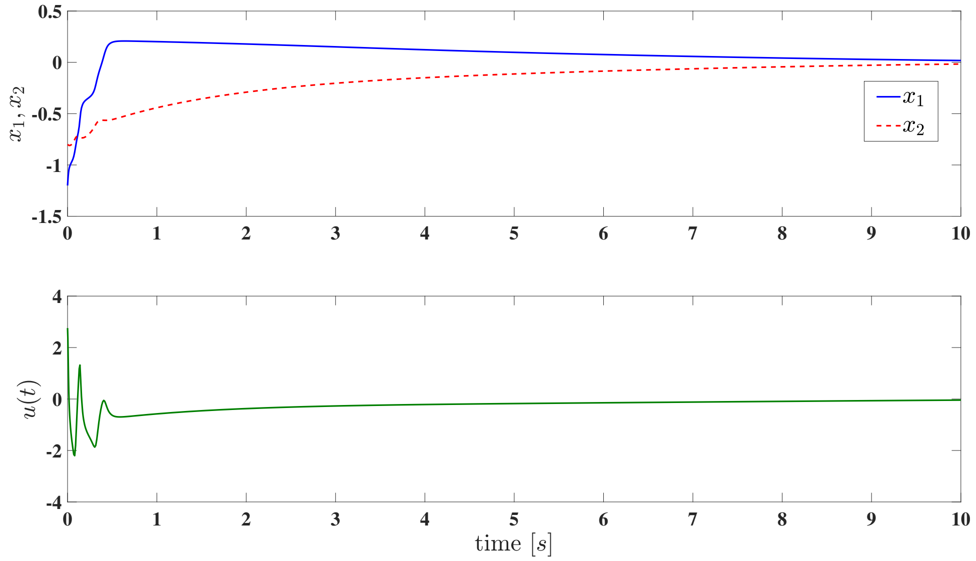

Figure 3.9: State-variable response (top) and control input response (bottom) of the feedback system in Example 3 at $\boldsymbol{x}_0 = [-1.2, -0.8]^T$

Finally, the Positivstellensatz multipliers are

$$\sigma_1(\boldsymbol{x}) = 0.9647, \ \sigma_2(\boldsymbol{x}) = 0.0226, \ \sigma_3(\boldsymbol{x}) = 0.0036,$$

$$\tau_1(\boldsymbol{x}) = 0.0002242x_1^2 + 0.00051458x_1x_2 + 0.0031725x_2^2,$$

$$\tau_2(\boldsymbol{x}) = 0.0022178x_1^2 + 0.00079401x_1x_2 + 7.1601 \times 10^{-05}x_2^2,$$

$$\tau_3(\boldsymbol{x}) = 0.00028751x_1^2 + 0.00010249x_1x_2 + 1.2752 \times 10^{-05}x_2^2,$$

$$\rho_{11}(\boldsymbol{x}) = 0.00022953x_1^2 + 0.0017554x_1x_2 + 0.006447x_2^2,$$

$$\rho_{12}(\boldsymbol{x}) = 0.015167x_1^2 + 0.0066612x_1x_2 + 0.0015829x_2^2,$$

$$\rho_{13}(\boldsymbol{x}) = 0.00022028x_1^2 + 0.00028119x_1x_2 + 0.003818x_2^2,$$

$$\rho_{21}(\boldsymbol{x}) = 0.015167x_1^2 + 0.0066612x_1x_2 + 0.0015829x_2^2,$$

$$\rho_{22}(\boldsymbol{x}) = 0.00053782x_1^2 + 0.00019231x_1x_2 + 1.7528 \times 10^{-05}x_2^2,$$

$$\rho_{23}(\boldsymbol{x}) = 0.0053062x_1^2 + 0.0018981x_1x_2 + 0.00017115x_2^2,$$

$$\rho_{31}(\boldsymbol{x}) = 0.00022028x_1^2 + 0.00028119x_1x_2 + 0.003818x_2^2,$$

$$\rho_{32}(\boldsymbol{x}) = 0.0053062x_1^2 + 0.0018981x_1x_2 + 0.00017115x_2^2,$$

$$\rho_{33}(\boldsymbol{x}) = 0.00013471x_1^2 + 4.9438 \times 10^{-05}x_1x_2 + 1.0557 \times 10^{-05}x_2^2.$$

Example 4. Consider the state equation below.

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{3} h_i(\boldsymbol{x})\big\{\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{u})\big\}, \tag{3.36}$$

where

$$\begin{aligned}
\boldsymbol{A}_1(\boldsymbol{x}) &= \begin{bmatrix} 1.59 + x_1^2 - 2x_2^2 - x_1x_2 & -7.29 + 2x_1x_2 \\ 0.01 & -x_1^2 - x_2^2 \end{bmatrix}, \\
\boldsymbol{A}_2(\boldsymbol{x}) &= \begin{bmatrix} 0.02 + x_1^2 - 2x_2^2 - x_1x_2 & -4.64 + 2x_1x_2 \\ 0.35 & 0.21 - x_1^2 - x_2^2 \end{bmatrix}, \\
\boldsymbol{A}_3(\boldsymbol{x}) &= \begin{bmatrix} -a + x_1^2 - 2x_2^2 - x_1x_2 & -4.33 + 2x_1x_2 \\ 0 & 0.05 - x_1^2 - x_2^2 \end{bmatrix},
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{B}_1(\boldsymbol{x}) &= \begin{bmatrix} 1 + x_1 + x_1^2 \\ 0 \end{bmatrix}, \\
\boldsymbol{B}_2(\boldsymbol{x}) &= \begin{bmatrix} 8 + x_1 + x_1^2 \\ 0 \end{bmatrix},
\end{aligned}$$

$$\boldsymbol{B}_3(\boldsymbol{x}) = \begin{bmatrix} -b+6+x_1+x_1^2 \\ -1 \end{bmatrix}.$$

For this fuzzy system in polynomial form, the MFs are defined as

$$\begin{aligned} h_1(x_1) &= \frac{1}{1+e^{62.5x_1+6}}, \; h_2(x_1) = \frac{1}{1+e^{-62.5x_1+6}}, \\ h_3(x_1) &= 1-h_1(x_1)-h_2(x_1). \end{aligned}$$

This example shows that the SOS conditions presented in this chapter can be applied to fuzzy systems in polynomial form, in contrast to other existing criteria based on the integral-type Lyapunov function introduced by [22] limited to study fuzzy systems in Takagi-Sugeno form. Setting the parameters $a = 2$ and $b = 6$, Figure 3.10 shows the solutions of the feedback polynomial fuzzy model at $\boldsymbol{x}_0 = [0.4, 0.7]^T$, $\boldsymbol{x}_0 = [1, 1.3]^T$, $\boldsymbol{x}_0 = [-1.3, -0.6]^T$, $\boldsymbol{x}_0 = [1.1, -0.9]^T$, $\boldsymbol{x}_0 = [1.3, 0.2]^T$, and $\boldsymbol{x}_0 = [-0.1, -1.1]^T$.

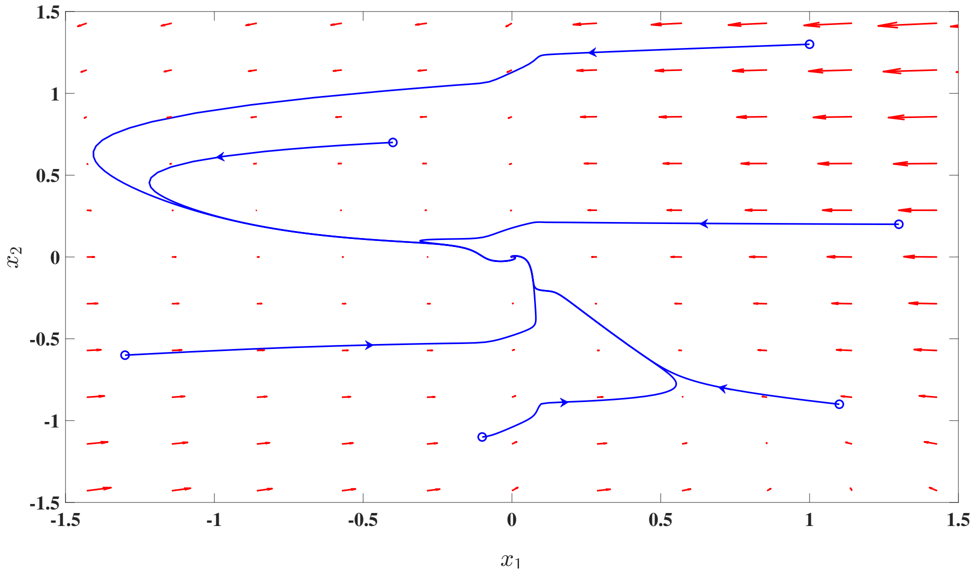

Figure 3.10: Phase trajectories in the plane $x_1 - x_2$ of the feedback system in Example 4.

Figure 3.11 depicts the time plot of state-variables and control input $u(t)$ for the feedback system in polynomial fuzzy form at $\boldsymbol{x}_0 = [-0.1, -1.1]^T$.

The feedback gain vectors and matrices $\boldsymbol{P}_0 + \boldsymbol{D}_i$ computed by using Theorem 3.2, with

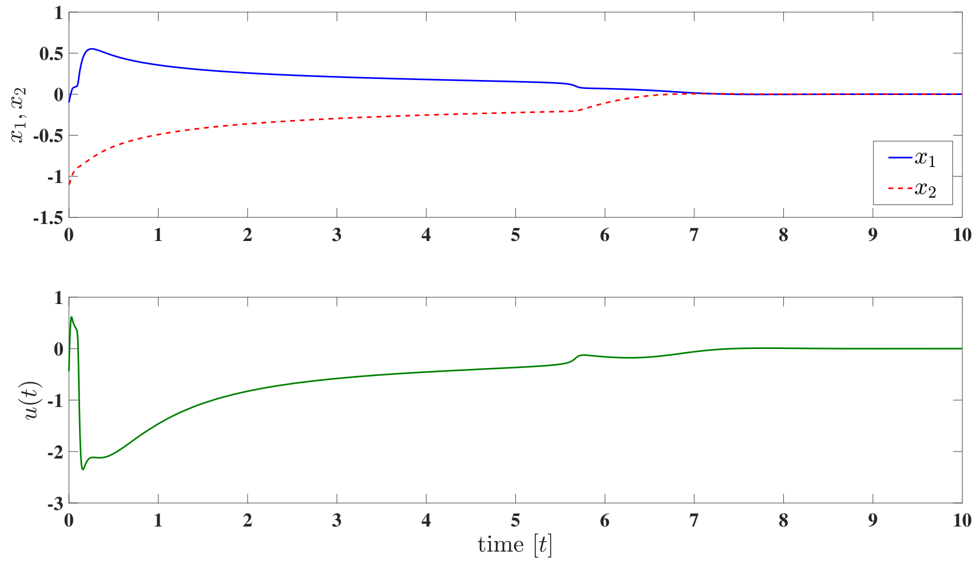

Figure 3.11: State-variable response (top) and control input response (bottom) of the feedback system in Example 4 at $\boldsymbol{x}_0 = [-0.1, -1.1]^T$

$\alpha = -0.0089$, are

$$\boldsymbol{F}_1 = \begin{bmatrix} 5.6713 & 1.6926 \end{bmatrix}, \boldsymbol{F}_2 = \begin{bmatrix} 2.2789 & 1.0365 \end{bmatrix}, \boldsymbol{F}_3 = \begin{bmatrix} 1.4221 & -2.3872 \end{bmatrix}.$$

$$\boldsymbol{P}_0 + \boldsymbol{D}_1 = \begin{bmatrix} 0.0880 & 0.1626 \\ 0.1626 & 1.8927 \end{bmatrix},$$

$$\boldsymbol{P}_0 + \boldsymbol{D}_2 = \begin{bmatrix} 0.2488 & 0.1626 \\ 0.1626 & 1.8927 \end{bmatrix},$$

$$\boldsymbol{P}_0 + \boldsymbol{D}_3 = \begin{bmatrix} 0.8007 & 0.1626 \\ 0.1626 & 1.8927 \end{bmatrix}.$$

For this example in polynomial fuzzy form, the Positivstellensatz multipliers are

$$\sigma_1(\boldsymbol{x}) = 0.8622, \ \sigma_2(\boldsymbol{x}) = 0.8591, \ \sigma_3(\boldsymbol{x}) = 0.7692,$$

$$\tau_1(\boldsymbol{x}) = 0.02351x_1^4 + 0.0059104x_1^3x_2 + 0.28998x_1^2x_2^2 + 0.5049x_1x_2^3 + 1.1711x_2^4,$$

$$\tau_2(\boldsymbol{x}) = 0.25775x_1^4 + 0.68616x_1^3x_2 + 1.4982x_1^2x_2^2 - 0.017196x_1x_2^3 + 1.2894x_2^4,$$

$$\tau_3(\boldsymbol{x}) = 0.02689x_1^4 - 0.16946x_1^3x_2 + 0.94501x_1^2x_2^2 - 0.071207x_1x_2^3 + 1.1448x_2^4,$$

$$\rho_{11}(\boldsymbol{x}) = 0.013241x_1^4 + 0.028462x_1^3x_2 + 0.14023x_1^2x_2^2 + 0.14045x_1x_2^3 + 0.34022x_2^4,$$

$$\rho_{12}(\boldsymbol{x}) = 0.2437x_1^4 + 0.20146x_1^3x_2 + 0.54625x_1^2x_2^2 - 0.013846x_1x_2^3 + 0.58157x_2^4,$$

$$\rho_{13}(\boldsymbol{x}) = 0.017915x_1^4 - 0.071747x_1^3x_2 + 0.53138x_1^2x_2^2 + 0.0075253x_1x_2^3 + 0.5778x_2^4,$$

$$\rho_{21}(\boldsymbol{x}) = 0.2437x_1^4 + 0.20146x_1^3x_2 + 0.54625x_1^2x_2^2 - 0.013847x_1x_2^3 + 0.58157x_2^4,$$

$$\rho_{22}(\boldsymbol{x}) = 0.059357x_1^4 + 0.14703x_1^3x_2 + 0.38265x_1^2x_2^2 + 0.019832x_1x_2^3 + 0.35359x_2^4,$$

$$\rho_{23}(\boldsymbol{x}) = 0.22341x_1^4 + 0.12976x_1^3x_2 + 0.48914x_1^2x_2^2 + 0.016644x_1x_2^3 + 0.57108x_2^4,$$

$$\rho_{31}(\boldsymbol{x}) = 0.017915x_1^4 - 0.071747x_1^3x_2 + 0.53138x_1^2x_2^2 + 0.007525x_1x_2^3 + 0.57779x_2^4,$$

$$\rho_{32}(\boldsymbol{x}) = 0.22341x_1^4 + 0.12976x_1^3x_2 + 0.48914x_1^2x_2^2 + 0.016643x_1x_2^3 + 0.57107x_2^4,$$

$$\rho_{33}(\boldsymbol{x}) = 0.011697x_1^4 - 0.031294x_1^3x_2 + 0.27328x_1^2x_2^2 + 0.0021582x_1x_2^3 + 0.31911x_2^4.$$

3.3 Discussion and Conclusions of the Chapter

This chapter has provided stability and stabilization SOS-based conditions for polynomial model-based fuzzy systems derived through integral-type functions (2.44). Examples 1 and 3 have shown that proposed criteria have improved the results, or obtained the same for the second-order stability case, compared to some current LMI methods.

In Example 1, the conditions were firstly compared to the criterion based on polynomial Lyapunov functions; for this case, the use of multiple Lyapunov functions have been proved to bring more relaxed results rather than the search for a single common Lyapunov function (see [17, 18, 20, 22]). Compared to other existing integral-type-based conditions, the use of the copositivity property instead of the use of well-known double-fuzzy summation relaxation techniques [10, 14, 29] has significantly decreased the inherent conservativeness by including the MFs and their properties in the conditions. Proposed conditions and the criteria introduced in [25] have the same results for this example, however, the latter proposed a specific structure that is only valid for second-order Takagi-Sugeno fuzzy systems, this restriction is not presented in the proposal of this thesis.

In Example 3, the stabilization proposal was tested in a benchmark Takagi-Sugeno fuzzy model. Thanks to the use of SOS optimization, the Positivstellensatz [13] was applied to provide a certificate for the positive definiteness of the Lyapunov's second method for stability conditions in a semialgebraic set defined by the MFs. The same as in stability conditions, the substitution of MFs by quadratic polynomial variables allows expressing the conditions as multiple-fuzzy summation rather than using parameterized techniques, such as [14], which brings more relaxed conditions.

Finally, compared to other existing integral-type-based works in terms of LMIs, SOS-based conditions given in the present chapter are useful to study stability and control synthesis for polynomial fuzzy systems as shown in Examples 2 and 4.

4

Generalized Integral-type Lyapunov Function

"Innovation distinguishes between a leader and a follower."

— Steve Jobs

This chapter covers the generalization of the integral-type Lyapunov function presented by [22]. The proposal considers a more general setting of the integrand, extending from quadratic forms to higher even degree polynomial forms. Then, the study focuses on the derivation of control synthesis conditions, and the Positivstellensatz and a MFs knowledge-based approach are used to relax the conditions.

4.1 Polynomial setting of the integral-type Lyapunov function

From now on, the present thesis deals with a Lyapunov function with the structure

$$V(\boldsymbol{x}) = \int_{\mathcal{C}} \boldsymbol{\zeta}(\boldsymbol{\psi}) \cdot \boldsymbol{d\psi}. \tag{4.1}$$

In the same manner of the integral-type Lyapunov function presented by [22], consider a curve $\mathcal{C}$ in the state-space extending from the zero state to the current state $\boldsymbol{x}$, the vector $\boldsymbol{\psi}$ is a dummy-variable and

$$\boldsymbol{\zeta}(\boldsymbol{x}) = \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} = \sum_{i=1}^{r} h_i(\boldsymbol{x}) \frac{\partial v_i^{[\lambda]}(\boldsymbol{x})}{\partial \boldsymbol{x}}, \tag{4.2}$$

is the function to be integrated. Define $v_i^{[\lambda]}(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$ and $v_i^{[\lambda]}(\boldsymbol{0}) = 0$ as homogeneous polynomials of even degree λ. Writing the polynomials as

$$v_i^{[\lambda]}(\boldsymbol{x}) = \sum_{j=1}^{n} a_{i,j} x_j^{\lambda} + \sum_{\delta \in \mathbb{D}} b_\delta \boldsymbol{x}^\delta, \tag{4.3}$$

Here, $a_{i,j}$ and b_δ are the coefficients. Now, define $\mathbb{D}$ to be the set

$$\mathbb{D} = \Big\{(\delta_1, \delta_2, \cdots, \delta_n) \in \mathbb{N}^n \big| \sum_{\rho=1}^{n} \delta_\rho = \lambda, \delta_\rho \neq \lambda\Big\}. \tag{4.4}$$

The multi-index notation denotes the monomials $\boldsymbol{x}^\delta = x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n}$ and their coefficients $b_\delta = b_{\delta_1,\delta_2 \cdots \delta_n}$ for all $\delta \in \mathbb{D}$. Observe that second term after the equal sign in equation (4.3) represents the addition of multivariate monomials in $\boldsymbol{x}$ and their coefficients b_δ have to be set equal in $v_i^{[\lambda]}(\boldsymbol{x})$ for all $i = 1, 2 \ldots, r$. The method to select the coefficients $a_{i,j}$ of the univariate monomials, i.e. first summation after the equal sign in equation (4.3), is like the criteria explained in [22] to choose the elements of the diagonal matrices. In summary, rules with the same x_j-based fuzzy sets share the same coefficients $a_{i,j}$ in their polynomials $v_i^{[\lambda]}(\boldsymbol{x})$, and the coefficients are independent otherwise.

Lemma 2. The integral (4.1) depends only on the starting and finishing states, i.e. it is path independent, if coefficients of the polynomials (4.3) are chosen under the aforementioned criterion.

Proof. The vector $\boldsymbol{\xi}(\boldsymbol{x}) = [\xi_1(\boldsymbol{x}), \xi_2(\boldsymbol{x}), \cdots, \xi_n(\boldsymbol{x})]$ is a path independent vector field, also called a conservative vector field [73], if the condition

$$\frac{\partial \xi_p(\boldsymbol{x})}{\partial x_q} = \frac{\partial \xi_q(\boldsymbol{x})}{\partial x_p}, \tag{4.5}$$

holds for any pair $(p, q) \in \{1, 2, \cdots, n\}^2$ excluding $p \neq q$. The vector gradient of the homogeneous polynomials are

$$\begin{aligned} \frac{\partial v_i^{[\lambda]}(\boldsymbol{x})}{\partial \boldsymbol{x}} &= \left[\lambda a_{i,1} x_1^{\lambda-1} + \frac{\partial}{\partial x_1} \sum_{\delta \in \mathbb{D}} b_\delta \boldsymbol{x}^\delta, \cdots, \lambda a_{i,n} x_n^{\lambda-1} + \frac{\partial}{\partial x_n} \sum_{\delta \in \mathbb{D}} b_\delta \boldsymbol{x}^\delta\right] \\ &= \left[\lambda a_{i,1} x_1^{\lambda-1}, \cdots, \lambda a_{i,n} x_n^{\lambda-1}\right] + \frac{\partial}{\partial \boldsymbol{x}} \sum_{\delta \in \mathbb{D}} b_\delta \boldsymbol{x}^\delta. \end{aligned} \tag{4.6}$$

The fuzzy blending of the gradients brings equation (4.2) into the form

$$\sum_{i=1}^{r} h_i(\boldsymbol{x}) \frac{\partial v_i^{[\lambda]}(\boldsymbol{x})}{\partial \boldsymbol{x}} = \sum_{i=1}^{r} h_i(\boldsymbol{x}) \left[\lambda a_{i,1} x_1^{\lambda-1}, \cdots, \lambda a_{i,n} x_n^{\lambda-1} \right] + \frac{\partial}{\partial \boldsymbol{x}} \sum_{\delta \in \mathbb{D}} b_\delta \boldsymbol{x}^\delta. \tag{4.7}$$

Observe that last element on the right-hand side in equation (4.7) is a gradient of a function and implies that is a conservative vector field.

Notation in equation (2.17) may be confusing to prove that the fuzzy vector is independent of the path, since subscript i in $M_{ij}(x_j)$ represents the rule in which the fuzzy set belongs to, and j is associated with the premise variable related to the fuzzy set under study. For the avoidance of doubt, let $\boldsymbol{\Delta}_{x_\rho}$ be the standard simplex

$$\boldsymbol{\Delta}_{x_\rho} = \left\{ \left(M_\rho^1(x_\rho), M_\rho^2(x_\rho) \right) \middle| \sum_{\kappa=1}^{2} M_\rho^\kappa(x_\rho) = 1,\ 0 \leq M_\rho^\kappa(x_\rho) \leq 1 \right\}. \tag{4.8}$$

Note that MFs $h_i(\boldsymbol{x})$ are in fact members of the resulting set

$$\left\{ h_1(\boldsymbol{x}), \ldots, h_r(\boldsymbol{x}) \right\} = \boldsymbol{\Delta}_{x_1} \times \left(\boldsymbol{\Delta}_{x_2} \times \cdots \times \left(\boldsymbol{\Delta}_{x_{m-1}} \times \boldsymbol{\Delta}_{x_m} \right) \cdots \right), \tag{4.9}$$

where $M_\rho^1(x_\rho)$, $M_\rho^2(x_\rho)$ are MFs related to the corresponding fuzzy set given by sector nonlinearity [10,51], the number of premise variables is denoted as m and $\times$ represents the Cartesian product. Note and keep in mind that $M_{ij}(x_j) \in \boldsymbol{\Delta}_{x_j}$. For the sake of simplicity and with no loss of generality, pay attention on the first entry on the first vector in the right-hand side of (4.7), which is

$$\sum_{i=1}^{r} h_i(\boldsymbol{x}) \lambda a_{i,1} x_1^{\lambda-1} = \sum_{i=1}^{r} \prod_{j=1}^{m} M_{ij}(x_j) \lambda a_{i,1} x_1^{\lambda-1}. \tag{4.10}$$

As aforementioned, when two fuzzy rules include the same fuzzy set, coefficients $a_{i,j}$ are identical in both polynomials related to those rules, and it is clear that both MFs $h_i(\boldsymbol{x})$ include the same $M_\rho^\kappa(x_\rho)$. Rename all those coefficients as a_ρ^κ and factorize them to obtain

$$\sum_{i=1}^{r} \prod_{j=1}^{m} M_{ij}(x_j) \lambda a_{i,1} x_1^{\lambda-1} = \sum_{\kappa=1}^{2} \lambda a_1^\kappa M_1^\kappa(x_1) \sum_{i=1}^{r/2} \prod_{j=2}^{m} M_{ij}(x_j) x_1^{\lambda-1}. \tag{4.11}$$

Observe that

$$\sum_{i=1}^{r/2} \prod_{j=2}^{m} M_{ij}(x_j) = \sum_{\kappa_2=1}^{2} M_2^{\kappa_2}(x_2) \sum_{\kappa_3=1}^{2} M_2^{\kappa_3}(x_3) \cdots \sum_{\kappa_m=1}^{2} M_2^{\kappa_m}(x_m) = 1. \tag{4.12}$$

Equation (4.11) reduces to

$$\sum_{i=1}^{r}\prod_{j=1}^{m}M_{ij}(x_j)\lambda a_{i,1}x_1^{\lambda-1}=\sum_{\kappa=1}^{2}\lambda a_1^{\kappa}M_1^{\kappa}(x_1)x_1^{\lambda-1}. \tag{4.13}$$

Therefore, first entry depends on x_1. In general, the p-th entry exclusively depends on x_p and condition (4.5) holds true due to $\frac{\partial \xi_p(x_p)}{\partial x_q}=0$ for $p\neq q$. □

The demonstration that $V(\boldsymbol{x})$ in (4.1) is positive definite and radially unbounded function comes next. Before going any further, the following is an useful relation concerning homogeneous functions [74].

Lemma 3. Let $\theta(\boldsymbol{x})$ be a homogeneous function with degree of homogeneity κ, said otherwise $\theta(\tau\boldsymbol{x})=\tau^{\kappa}\theta(\boldsymbol{x})$. The function satisfies the condition

$$\frac{\partial\theta(\boldsymbol{x})}{\partial\boldsymbol{x}}\cdot\boldsymbol{x}=\kappa\theta(\boldsymbol{x}). \tag{4.14}$$

Moreover, all entries of the vector gradient, denoted as $\frac{\partial\theta(\boldsymbol{x})}{\partial x_i}$, are also homogeneous of degree $\kappa-1$.

Lemma 4. The fulfillment of $v_i^{[\lambda]}(\boldsymbol{x})>0,\ \forall\boldsymbol{x}\neq 0,\ v_i^{[\lambda]}(\boldsymbol{0})=0$ and $\lim_{|\boldsymbol{x}|\to\infty}v_i^{[\lambda]}(\boldsymbol{x})=+\infty$ implies that $V(\boldsymbol{x})$ is a candidate Lyapunov function.

Proof. The substitution of equation (4.2) in (4.1) brings the following equation

$$V(\boldsymbol{x})=\int_{\mathcal{C}}\sum_{i=1}^{r}h_i(\boldsymbol{\psi})\frac{\partial v_i^{[\lambda]}(\boldsymbol{\psi})}{\partial\boldsymbol{\psi}}\cdot d\boldsymbol{\psi}. \tag{4.15}$$

As stated by Lemma 2, line integral (4.15) is independent of the path by following the coefficients selection criteria. Denote $\boldsymbol{\psi}=\eta\boldsymbol{x},\ 0\leq\eta\leq 1$ as the parametric form of line segment extending from $\boldsymbol{0}$ to $\boldsymbol{x}$, then

$$V(\boldsymbol{x})=\int_0^1\sum_{i=1}^{r}h_i(\eta\boldsymbol{x})\frac{\partial v_i^{[\lambda]}(\eta\boldsymbol{x})}{\eta\partial\boldsymbol{x}}\cdot\boldsymbol{x}d\eta. \tag{4.16}$$

Recalling that $v_i^{[\lambda]}(\boldsymbol{x})$ are homogeneous polynomials. By means of Lemma 3 the equation below is obtained

$$\begin{aligned} V(\boldsymbol{x}) &= \int_0^1 \eta^{\lambda-1} \sum_{i=1}^{r} h_i(\eta \boldsymbol{x}) \frac{\partial v_i^{[\lambda]}(\boldsymbol{x})}{\partial \boldsymbol{x}} \cdot \boldsymbol{x} d\eta \\ &= \sum_{i=1}^{r} \left(\int_0^1 \lambda \eta^{\lambda-1} h_i(\eta \boldsymbol{x}) d\eta \right) v_i^{[\lambda]}(\boldsymbol{x}). \end{aligned} \tag{4.17}$$

The nonnegativity property of the MFs entails that $\lambda\eta^{\lambda-1}h_i(\eta\boldsymbol{x}) \geq 0$ in $\eta \in [0,1]$ for $i \in \{1, 2, \cdots, r\}$ and $\int_0^1 \lambda\eta^{\lambda-1}h_i(\eta\boldsymbol{x})d\eta \geq 0$ verified with the monotony of the integral. Observe that $\int_0^1 \lambda\eta^{\lambda-1}\sum_{i=1}^{r} h_i(\eta\boldsymbol{x})d\eta = \int_0^1 \lambda\eta^{\lambda-1}d\eta = \eta^\lambda|_0^1 = 1$. Consequently, $V(\boldsymbol{x}) > 0$ at $x \neq 0$ and $v_i^{[\lambda]}(\mathbf{0}) = 0$ if $v_i^{[\lambda]}(\boldsymbol{x})$ are positive definite. Lastly, $V(\boldsymbol{x})$ is radially unbounded due to $v_i^{[\lambda]}(\boldsymbol{x}) \to \infty$ as $||\boldsymbol{x}|| \to \infty$, thereby concluding the proof. □

Remark 1. *Integral-type Lyapunov function introduced in [22] is in fact a special case of the proposed Lyapunov function assuming that the polynomials $v_i^{[\lambda]}(\boldsymbol{x})$ are quadratic forms.*

Remark 2. *Assuming that all the polynomial $v_i^{[\lambda]}(\boldsymbol{x})$ are fixed to be equal, the Lyapunov function (4.1) reduces to a homogeneous polynomial Lyapunov function [12].*

Remark 3. *A linear combination $v_i(\boldsymbol{x}) = v_i^{[\lambda]}(\boldsymbol{x}) + v_i^{[\lambda-2]}(\boldsymbol{x}) + \cdots + v_i^{[2]}(\boldsymbol{x})$ of polynomials satisfying path independence structure, entails*

$$\begin{aligned} V(\boldsymbol{x}) &= \sum_{i=1}^{r} \left(\int_0^1 \lambda\eta^{\lambda-1}h_i(\eta\boldsymbol{x})d\eta \right) v_i^{[\lambda]}(\boldsymbol{x}) + \sum_{i=1}^{r} \left(\int_0^1 (\lambda-2)\eta^{\lambda-3}h_i(\eta\boldsymbol{x})d\eta \right) v_i^{[\lambda-2]}(\boldsymbol{x}) + \ldots \\ &+ \sum_{i=1}^{r} \left(\int_0^1 2\eta h_i(\eta\boldsymbol{x})d\eta \right) v_i^{[2]}(\boldsymbol{x}). \end{aligned} \tag{4.18}$$

Thus, $V(\boldsymbol{x})$ is positive definite if conditions $v_i^{[\lambda]}(\boldsymbol{x}) > 0$, $v_i^{[\lambda-2]}(\boldsymbol{x}) > 0$, $\ldots$, $v_i^{[2]}(\boldsymbol{x}) > 0$, $v_i^{[\cdot]}(\mathbf{0}) = 0$ at $\boldsymbol{x} \neq 0$ hold true for all $i \in 1, 2, \ldots, r$.

4.2 Stability and Stabilization Analysis

Corollary 1. Fuzzy system (2.27) at $u = 0$ is asymptotically stable if there exist $v_i^{[\lambda]}(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$ and a nonnegative integer s such that, for given polynomials $\epsilon_i(\boldsymbol{x}) > 0$, $\epsilon_{ij}(\boldsymbol{x}) > 0$, the conditions

$$v_i^{[\lambda]}(\boldsymbol{x}) - \epsilon_i(\boldsymbol{x}) \in \mathbb{S}[\boldsymbol{x}] \; \forall i, \tag{4.19}$$

$$-\left(\sum_{k=1}^{r}\hat{h}_k^2\right)^s\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{h}_i^2\hat{h}_j^2\left\{\frac{\partial v_j^{[\lambda]}(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x}+\epsilon_{ij}(\boldsymbol{x})\right\}\in\mathbb{S}[\hat{\boldsymbol{h}},\boldsymbol{x}]\ \forall i,j, \tag{4.20}$$

hold true with $\hat{\boldsymbol{h}}=[\hat{h}_1^2\ \hat{h}_2^2\cdots\hat{h}_r^2]$.

Proof. This result is an immediate corollary from Theorem 3.1 with the difference that these conditions have employed the proposed Lyapunov function, rather than the integral-type form introduced by [22]. □

Corollary 2. Consider a polynomial fuzzy system (2.41) with the origin as equilibrium. The existence of polynomials $v_i^{[\lambda]}(\boldsymbol{x})$, Positivistellensatz multipliers $\sigma_l(\boldsymbol{x})$, $\rho_{ij}(\boldsymbol{x})\in\mathbb{S}[\boldsymbol{x}]$, $\tau_l(\boldsymbol{x})\in\mathbb{R}[\boldsymbol{x}]$, polynomial gain vectors $\boldsymbol{F}_j(\boldsymbol{x})$ of suitable dimensions and $\alpha<0$ satisfying (4.21) and (4.22), proves that the zero equilibrium of the system is feedback asymptotically stabilizable.

$$v_i^{[\lambda]}(\boldsymbol{x})-\epsilon(\boldsymbol{x})\in\mathbb{S}[\boldsymbol{x}]\ \forall i, \tag{4.21}$$

$$\begin{aligned}-\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{l=1}^{r}\hat{h}_i^2\hat{h}_j^2\hat{h}_k^2\hat{h}_l^2\Big\{\sigma_l(\boldsymbol{x})\Big[\frac{\partial v_k^{[\lambda]}(\boldsymbol{x})}{\partial \boldsymbol{x}}\{\boldsymbol{A}_i(\boldsymbol{x})-\boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x})\boldsymbol{x}\}-\alpha V_k(\boldsymbol{x})\Big]\\+\tau_l(\boldsymbol{x})+\rho_{ij}(\boldsymbol{x})\Big\}+\sum_{l=1}^{r}\hat{h}_l^2\tau_l(\boldsymbol{x})\in\mathbb{S}[\hat{\boldsymbol{h}},\boldsymbol{x}],\end{aligned} \tag{4.22}$$

where $\hat{\boldsymbol{h}}=[\hat{h}_1^2,\cdots,\hat{h}_r^2]$ and a given $\epsilon(\boldsymbol{x})$ positive definite polynomial.

Proof. This result is an immediate corollary from Theorem 3.2 with the difference that these conditions have employed the proposed Lyapunov function, rather than the integral-type form introduced by [22]. □

4.2.1 Examples

Example 5. Consider the stable polynomial fuzzy model below.

$$\dot{\boldsymbol{x}}=\sum_{i=1}^{4}h_i(\boldsymbol{x})\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x}, \tag{4.23}$$

with matrices $\boldsymbol{A}_i(\boldsymbol{x})$ defined as

$$\boldsymbol{A}_1(\boldsymbol{x}) = \begin{bmatrix} -4-2x_1^4 & -4 \\ -1 & -2 \end{bmatrix},$$
$$\boldsymbol{A}_2(\boldsymbol{x}) = \begin{bmatrix} -2 & -4 \\ 20 & -2-x_1^2-5x_2^2 \end{bmatrix},$$
$$\boldsymbol{A}_3(\boldsymbol{x}) = \begin{bmatrix} -3.8-x_1^4 & -4 \\ -1 & -2 \end{bmatrix},$$
$$\boldsymbol{A}_4(\boldsymbol{x}) = \begin{bmatrix} -2 & -4 \\ 20 & -2-5x_1^2-x_2^2 \end{bmatrix},$$

and MFs written as $h_1(\boldsymbol{x}) = M_1^1(x_1)M_2^1(x_2)$, $h_2(\boldsymbol{x}) = M_1^1(x_1)M_2^2(x_2)$, $h_3(\boldsymbol{x}) = M_1^2(x_1)M_2^1(x_2)$, $h_4(\boldsymbol{x}) = M_1^2(x_1)M_2^2(x_2)$ and $M_1^1(x_1) = 0.5(1+\sin x_1)$, $M_1^2(x_1) = 0.5(1-\sin x_1)$, $M_2^1(x_2) = 0.5(1+\sin x_2)$, $M_2^2(x_2) = 0.5(1-\sin x_2)$. The fourth-degree polynomials coming next are used to verify stability by means of the generalized integral-type Lyapunov function.

$$\begin{aligned} v_1^{[4]}(\boldsymbol{x}) &= 0.0728x_1^4 + 0.0354x_2^4 + 0.0537x_1^2x_2^2, \\ v_2^{[4]}(\boldsymbol{x}) &= 0.0728x_1^4 + 0.0041x_2^4 + 0.0537x_1^2x_2^2, \\ v_3^{[4]}(\boldsymbol{x}) &= 0.0713x_1^4 + 0.0354x_2^4 + 0.0537x_1^2x_2^2, \\ v_4^{[4]}(\boldsymbol{x}) &= \underbrace{0.0713x_1^4 + 0.0041x_2^4}_{\sum_{j=1}^{2} a_{i,j}x_j^4} + \underbrace{0.0537x_1^2x_2^2}_{b_{\delta=(2,2)}\boldsymbol{x}^{\delta}} \end{aligned} \tag{4.24}$$

This examples will demonstrate that the proposed structure (4.1) is actually a Lyapunov function. The gradient vectors of above polynomials are

$$\begin{aligned} \frac{\partial v_1^{[4]}(\boldsymbol{x})}{\partial \boldsymbol{x}} &= \left[0.2912x_1^3 + 0.1074x_1x_2^2\ ,\ 0.1074x_1^2x_2 + 0.1416x_2^3\right], \\ \frac{\partial v_2^{[4]}(\boldsymbol{x})}{\partial \boldsymbol{x}} &= \left[0.2912x_1^3 + 0.1074x_1x_2^2\ ,\ 0.1074x_1^2x_2 + 0.0164x_2^3\right], \\ \frac{\partial v_3^{[4]}(\boldsymbol{x})}{\partial \boldsymbol{x}} &= \left[0.2852x_1^3 + 0.1074x_1x_2^2\ ,\ 0.1074x_1^2x_2 + 0.1416x_2^3\right], \\ \frac{\partial v_4^{[4]}(\boldsymbol{x})}{\partial \boldsymbol{x}} &= \left[0.2852x_1^3 + 0.1074x_1x_2^2\ ,\ 0.1074x_1^2x_2 + 0.0164x_2^3\right]. \end{aligned}$$

Fuzzy blending brings the expression

$$
\begin{aligned}
\sum_{i=1}^{4} h_i(\boldsymbol{x}) \frac{\partial v_i^{[4]}(\boldsymbol{x})}{\partial \boldsymbol{x}} =& \big[0.2912 h_1(\boldsymbol{x}) x_1^3 + 0.2912 h_2(\boldsymbol{x}) x_1^3 + 0.2852 h_3(\boldsymbol{x}) x_1^3 + 0.2852 h_4(\boldsymbol{x}) x_1^3 \ , \\
& 0.1416 h_1(\boldsymbol{x}) x_2^3 + 0.0164 h_2(\boldsymbol{x}) x_2^3 + 0.1416 h_3(\boldsymbol{x}) x_2^3 + 0.0164 h_4(\boldsymbol{x}) x_2^3\big] \\
& + \big[0.1074 x_1 x_2^2 \ , \ 0.1074 x_1^2 x_2\big].
\end{aligned}
$$

Factorizing

$$
\begin{aligned}
&= \big[0.2912 M_1^1(x_1) \cancelto{1}{\big(M_2^1(x_2) + M_2^2(x_2)\big)} x_1^3 + 0.2852 M_1^2(x_1) \cancelto{1}{\big(M_2^1(x_2) + M_2^2(x_2)\big)} x_1^3 \ , \\
&\quad 0.1416 M_2^1(x_2) \cancelto{1}{\big(M_1^1(x_1) + M_1^2(x_1)\big)} x_2^3 + 0.0164 M_2^2(x_2) \cancelto{1}{\big(M_1^1(x_1) + M_1^2(x_1)\big)} x_2^3\big] \\
&\quad + \big[0.1074 x_1 x_2^2 \ , \ 0.1074 x_1^2 x_2\big] \\
&= \big[0.2912 M_1^1(x_1) x_1^3 + 0.2852 M_1^2(x_1) x_1^3 \ , \ 0.1416 M_2^1(x_2) x_2^3 + 0.0164 M_2^2(x_2) x_2^3\big] \\
&\quad + \big[0.1074 x_1 x_2^2 \ , \ 0.1074 x_1^2 x_2\big].
\end{aligned}
$$

Define

$$
\begin{aligned}
[\xi_1(x_1), \xi_2(x_2)] = \big[&0.2912 M_1^1(x_1) x_1^3 + 0.2852 M_1^2(x_1) x_1^3 + 0.1074 x_1 x_2^2 \ , \\
&0.1416 M_2^1(x_2) x_2^3 + 0.0164 M_2^2(x_2) x_2^3 + 0.1074 x_1^2 x_2\big].
\end{aligned}
$$

Then, by conditions (4.5) it leads to

$$
\frac{\partial \xi_1(x_1)}{\partial x_2} = 0.2148 x_1 x_2 = \frac{\partial \xi_2(x_2)}{\partial x_1}. \tag{4.25}
$$

Consequently, it is path independent and line integral (4.1) is a Lyapunov function for the Takagi-Sugeno fuzzy system in this example. For the initial conditions $\boldsymbol{x}_0 = [-0.9, -1]$, $\boldsymbol{x}_0 = [-1.3, 1.2]$, $\boldsymbol{x}_0 = [1.2, -0.7]$, $\boldsymbol{x}_0 = [0.6, -1.3]$, $\boldsymbol{x}_0 = [1.25, 1]$ and $\boldsymbol{x}_0 = [-0.7, 0.9]$ the phase trajectories in the $x_1 - x_2$ plane are depicted in the figure below.

Example 6. The 3-rule Takagi-Sugeno fuzzy model in Example 3 presented in subsection 3.2.2 will be tackled again.

As in Example 3, this Takagi-Sugeno fuzzy benchmark model is used to compare the performance of the proposed SOS conditions presented in this chapter. Setting $a = 2$, the

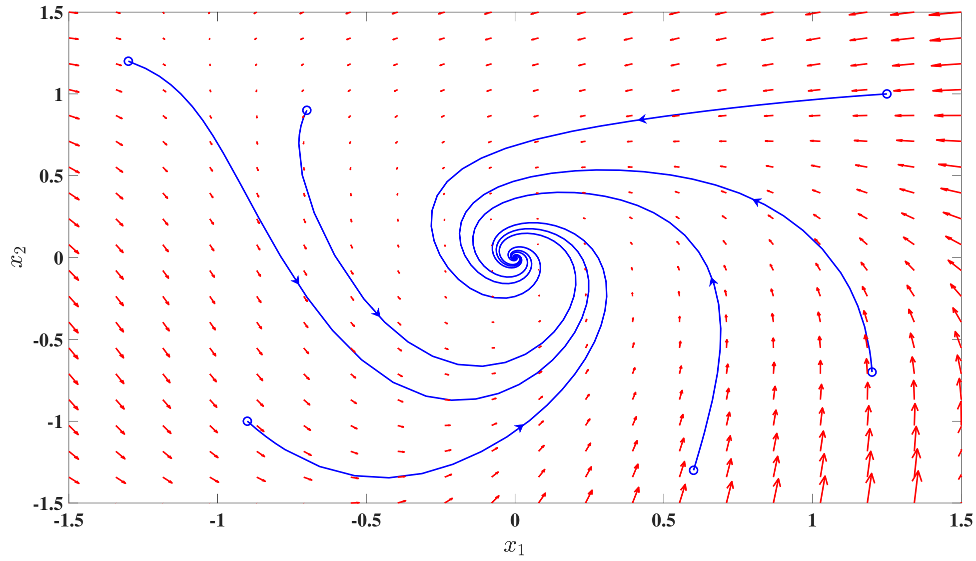

Figure 4.1: Phase trajectories in the plane $x_1 - x_2$ of the system in Example 5.

purpose is to find the maximum value of the parameter b for which the design conditions are feasible. The results obtained by using some existing criteria in the literature and the proposal in this chapter are listed in Table 4.1.

Table 4.1: Comparative results on the maximum value of parameter b in Example 6

Method	b_{max}
Corollary 2 (quartic)	7.8
Corollary 2 (quadratic)	6.9
Theorem 5 in [75]	6.6
Theorem 5 in [14]	6.5
Theorem 5 in [15]	6
Theorem 1 in [48]	2.5

According to Table 4.1, SOS conditions are feasible for $b = 7.8$ with $\alpha = -0.0065$ and considering quartic polynomials, which are

$$
\begin{aligned}
V_1^{[4]}(\boldsymbol{x}) &= 0.01207x_1^4 - 0.018838x_1^3x_2 + 0.14625x_1^2x_2^2 + 0.26761x_1x_2^3 + 1.4055x_2^4, \\
V_2^{[4]}(\boldsymbol{x}) &= 0.01863x_1^4 - 0.018838x_1^3x_2 + 0.14625x_1^2x_2^2 + 0.26761x_1x_2^3 + 1.4055x_2^4, \\
V_3^{[4]}(\boldsymbol{x}) &= 0.00415x_1^4 - 0.018838x_1^3x_2 + 0.14625x_1^2x_2^2 + 0.26761x_1x_2^3 + 1.4055x_2^4.
\end{aligned}
\tag{4.26}
$$

The stabilizing vectors $\boldsymbol{F}_j$ are

$$\boldsymbol{F}_1 = \begin{bmatrix} 13.1627 & 2.8852 \end{bmatrix}, \ \boldsymbol{F}_2 = \begin{bmatrix} 48.9872 & 38.3044 \end{bmatrix}, \ \boldsymbol{F}_3 = \begin{bmatrix} -13.5236 & -123.8248 \end{bmatrix}. \quad (4.27)$$

In order to demonstrate that the feedback system is stable, Figure 4.2 shows the trajectories in the phase plane at $\boldsymbol{x}_0 = [-0.1, 0.5]$, $\boldsymbol{x}_0 = [-0.9, -1.1]$, $\boldsymbol{x}_0 = [1.2, 0.9]$, $\boldsymbol{x}_0 = [0.5, -0.5]$, $\boldsymbol{x}_0 = [1.3, -0.5]$ and $\boldsymbol{x}_0 = [-1.3, 0.6]$.

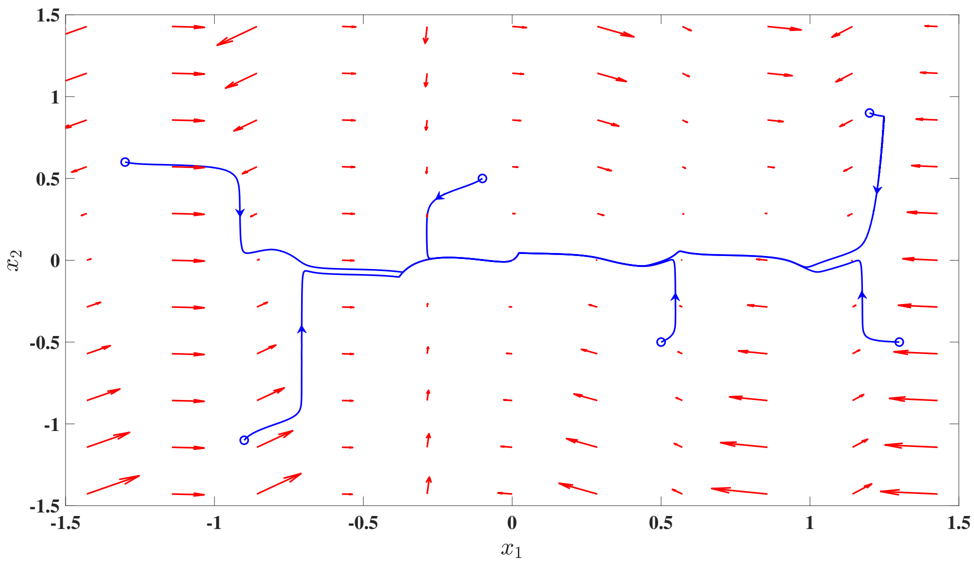

Figure 4.2: Phase trajectories in the plane $x_1 - x_2$ of the feedback system in Example 6.

The state-variables and control input $u(t)$ at $\boldsymbol{x}_0 = [-1.3, 0.6]$ are depicted in Figure 4.3. For this example in polynomial fuzzy form, the Positivstellensatz multipliers are

$$\sigma_1(\boldsymbol{x}) = 0.5260, \ \sigma_2(\boldsymbol{x}) = 0.6694, \ \sigma_3(\boldsymbol{x}) = 1.6914,$$

$$\tau_1(\boldsymbol{x}) = 3.5505 \times 10^{-5}x_1^4 - 0.0002x_1^3x_2 + 0.0023x_1^2x_2^2 + 0.0039x_1x_2^3 + 0.0016x_2^4,$$

$$\tau_2(\boldsymbol{x}) = 0.0020x_1^4 - 0.0010x_1^3x_2 - 0.0008x_1^2x_2^2 + 0.0020x_1x_2^3 + 0.0012x_2^4,$$

$$\tau_3(\boldsymbol{x}) = 5.9602 \times 10^{-5}x_1^4 - 0.0002x_1^3x_2 + 0.0061x_1^2x_2^2 - 0.0058x_1x_2^3 + 0.0050x_2^4,$$

$$\rho_{11}(\boldsymbol{x}) = 2.5681 \times 10^{-5}x_1^4 - 9.4485 \times 10^{-5}x_1^3x_2 + 0.0007x_1^2x_2^2 + 0.0013x_1x_2^3 + 0.0006x_2^4,$$

$$\rho_{12}(\boldsymbol{x}) = 0.0433x_1^4 - 0.0829x_1^3x_2 + 0.0432x_1^2x_2^2 + 0.2295x_1x_2^3 + 0.10094x_2^4,$$

$$\rho_{13}(\boldsymbol{x}) = 3.8386 \times 10^{-5}x_1^4 - 0.0002x_1^3x_2 + 0.0236x_1^2x_2^2 - 0.1405x_1x_2^3 + 0.27963x_2^4,$$

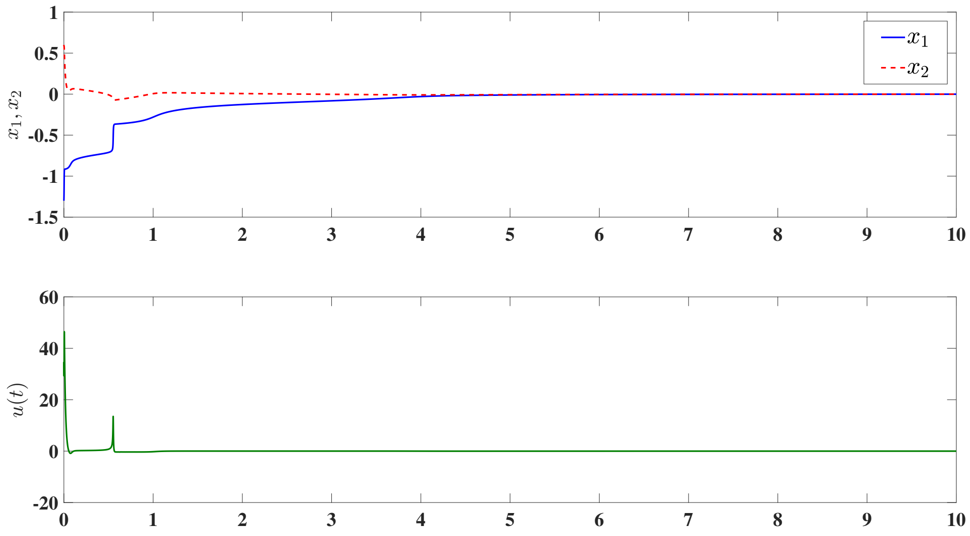

Figure 4.3: State-variable response (top) and control input response (bottom) of the feedback system in Example 6 at $\boldsymbol{x}_0 = [-1.3, 0.6]$.

$$\rho_{21}(\boldsymbol{x}) = 0.0433x_1^4 - 0.0829x_1^3x_2 + 0.0432x_1^2x_2^2 + 0.2295x_1x_2^3 + 0.1009x_2^4,$$

$$\rho_{22}(\boldsymbol{x}) = 0.0008x_1^4 - 0.0003x_1^3x_2 - 0.0002x_1^2x_2^2 + 0.0009x_1x_2^3 + 0.0006x_2^4,$$

$$\rho_{23}(\boldsymbol{x}) = 0.0009x_1^4 - 0.0024x_1^3x_2 + 0.0040x_1^2x_2^2 - 0.0004x_1x_2^3 + 0.0015x_2^4,$$

$$\rho_{31}(\boldsymbol{x}) = 3.8386 \times 10^{-5}x_1^4 - 0.0002x_1^3x_2 + 0.02364x_1^2x_2^2 - 0.1405x_1x_2^3 + 0.2796x_2^4,$$

$$\rho_{32}(\boldsymbol{x}) = 0.0009x_1^4 - 0.0024x_1^3x_2 + 0.0040x_1^2x_2^2 - 0.0004x_1x_2^3 + 0.0014x_2^4,$$

$$\rho_{33}(\boldsymbol{x}) = 4.2473 \times 10^{-5}x_1^4 - 0.0002x_1^3x_2 + 0.0052x_1^2x_2^2 - 0.0055x_1x_2^3 + 0.0051x_2^4.$$

4.3 Control Synthesis (S-procedure relaxation)

Theorem 4.1. The polynomial fuzzy system (2.27) is stabilizable to the origin by a PDC controller (2.39) on the condition that there exist $v_k^{[\lambda]}(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$, feedback gain vectors $\boldsymbol{F}_j(\boldsymbol{x})$ of suitable dimensions and S-procedure multipliers $\sigma_\iota(\boldsymbol{x})$, $\tau_{\lambda\iota}(\boldsymbol{x}) \in \mathbb{S}[\boldsymbol{x}]$ such that

$$\sum_{i=1}^{r} \hat{h}_i v_i^{[\lambda]}(\boldsymbol{x}) - \epsilon_1(\boldsymbol{x}) - \sum_{\iota=1}^{z} \tau_{\lambda\iota}(\boldsymbol{x}) p_\iota(\boldsymbol{h}) \in \mathbb{S}[\boldsymbol{h}, \boldsymbol{x}], \tag{4.28}$$

$$-\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} \hat{h}_i\hat{h}_j\hat{h}_k \frac{\partial v_k^{[\lambda]}(\boldsymbol{x})}{\partial \boldsymbol{x}} \Big\{\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x})\Big\}\boldsymbol{x} - \epsilon_2(\boldsymbol{x}) - \sum_{\iota=1}^{z} \sigma_\iota(\boldsymbol{x}) p_\iota(\boldsymbol{h}) \in \mathbb{S}[\boldsymbol{h}, \boldsymbol{x}]. \tag{4.29}$$

Here, $\epsilon_1(\boldsymbol{x}) > 0$ and $\epsilon_2(\boldsymbol{x}) > 0$ are positive definite polynomials, $\boldsymbol{h} = [\hat{\mu}_{11}, \hat{\mu}_{12}, \dots, \hat{\mu}_{1q}]$. Polynomial restrictions $p_\iota(\boldsymbol{h})$ are written based on the domain of the variables $\boldsymbol{h}$, and $\hat{h}_i$ represent the $i-th$ element of the set resulting from the Cartesian product of the simplexes.

$$\{\hat{h}_1, \dots, \hat{h}_i, \dots, \hat{h}_r\} = \{\hat{\mu}_{11}, 1-\hat{\mu}_{11}\} \times \Big(\{\hat{\mu}_{12}, 1-\hat{\mu}_{12}\} \times \Big(\cdots \times \{\hat{\mu}_{1q}, 1-\hat{\mu}_{1q}\}\Big)\cdots\Big). \tag{4.30}$$

Proof. This theorem considers the property $M_j^1(x_j) + M_j^2(x_j) = 1$. Recall the definition of standard simplex in equation (4.8) and the description of MFs $h_i(\boldsymbol{x})$ as elements of the Cartesian product (4.9) Substituting $M_j^1(x_j) = \hat{\mu}_{1j}$, $M_j^2(x_j) = 1 - \hat{\mu}_{1j}$, where $\hat{\mu}_{1j}$ are polynomial variables. As a result, sets $\{M_j^1(x_j), M_j^2(x_j)\}$ become $\{\hat{\mu}_{1j}, 1-\hat{\mu}_{1j}\}$ and satisfy $\sum_{i=1}^{r} \hat{h}_i = 1$. The derivative of (4.1) with respect to time is

$$\dot{V}(\boldsymbol{h}, \boldsymbol{x}) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} \hat{h}_i\hat{h}_j\hat{h}_k \frac{\partial v_k^{[\lambda]}(\boldsymbol{x})}{\partial \boldsymbol{x}} \Big\{\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x})\Big\}\boldsymbol{x}. \tag{4.31}$$

Observe that previous equation is a polynomial form in both variables $\boldsymbol{h}$ and $\boldsymbol{x}$. The next step is to define a set of constraints $\mathcal{S} = \{p_1(\boldsymbol{h}) \geq 0, \dots, p_z(\boldsymbol{h}) \geq 0\}$ based on MFs, inserted by means of the S-procedure. Therefore, $-\dot{V}(\boldsymbol{h}, \boldsymbol{x})$ will be positive in $\mathcal{S}$ whether (4.29) is satisfied with multipliers $\sigma_\iota(\boldsymbol{x}) \in \mathbb{S}[\boldsymbol{x}]$. Analogously, a less conservative verification of positiveness of $V(\boldsymbol{x})$ is to make use of constraints on the variables that replace the MFs. Equation (4.17) and the monotony property validate that the positive definiteness of the integrand entails that $V(\boldsymbol{x}) > 0$, $V(\boldsymbol{0}) > 0$ at $\boldsymbol{x} \neq 0$.

Because λ is an even positive integer and $\eta \in [0, 1]$, one just need to verify that the inequality $\sum_{i=1}^{r} h_i(\eta\boldsymbol{x})v_i^{[\lambda]}(\boldsymbol{x}) > 0$ holds true at $\boldsymbol{x} \neq 0$. Replacing $h_i(\eta\boldsymbol{x}) = \hat{h}_i$ brings the expression $\sum_{i=1}^{r} \hat{h}_i v_i^{[\lambda]}(\boldsymbol{x})$ which in terms of $\boldsymbol{x}$ and $\boldsymbol{h}$, and the S-procedure is applied to check positivity in $\mathcal{S}$. □

Remark 4. *For a linear combination as stated in Remark 3, $V(\boldsymbol{x})$ is positive definite if condition (4.28) are satisfied for $\lambda, \lambda - 2, \dots, 2$.*

4.3.1 Design Examples

Example 7. Consider again the benchmark problem in Example 3 in subsection 3.2.2.

Finding the maximum value of the varying parameter b when fixing $a = 2$ has been the object of study of many literature in the model-based fuzzy control field. In contrast to Example 3 and Example 6, the study of this example considers knowledge of the membership function (see Figure 4.4) via polynomial constraints.

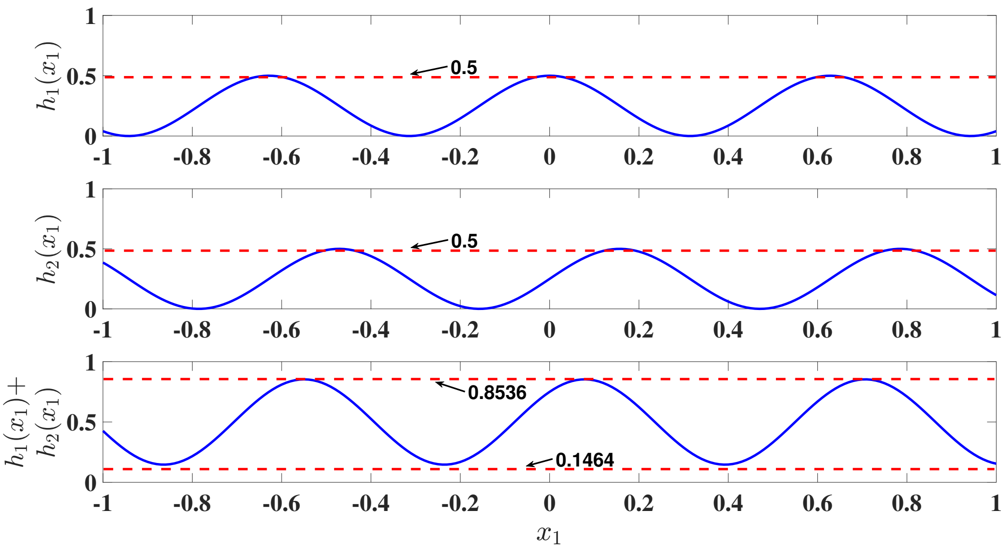

Figure 4.4: MFs in Example 7

It became clear that $0 \leq h_1(x_1) \leq 0.5$, $0 \leq h_2(x_1) \leq 0.5$ and $0.1464 \leq h_1(x_1) + h_2(x_1) \leq 0.8536$. Replacing the MFs by linear polynomial variables as $h_1(\boldsymbol{x}) = \hat{\mu}_1$, $h_2(\boldsymbol{x}) = \hat{\mu}_2$, $h_3(\boldsymbol{x}) = 1 - \hat{\mu}_1 - \hat{\mu}_2$ and making use of above inequalities to define the following set of polynomials restrictions in the variables $\hat{\mu_1}$ and $\hat{\mu_2}$

$$\mathcal{S} = \{\hat{\mu}_1(0.5 - \hat{\mu}_1) \geq 0,\ \hat{\mu}_2(0.5 - \hat{\mu}_2) \geq 0,\ (\hat{\mu}_1 + \hat{\mu}_2 - 0.1464)(0.8536 - \hat{\mu}_1 - \hat{\mu}_2) \geq 0\}. \quad (4.32)$$

Table 4.2: Comparative results on the maximum value of parameter b in Example 7

Method	b_{max}
Theorem 4.1 ($\lambda = 4$)	9
Theorem 4.1 ($\lambda = 2$)	8.4
Corollary 1 in [55]	8
Theorem 1 (quartic) in [76]	8
Theorem 4.1 (Common quadratic Lyapunov function)	7.4
Theorem 1 (quadratic) in [76]	7
Theorem 5 in [75]	6.6
Theorem 5 in [14]	6.5
Theorem 5 in [15]	6

As stated in Table 4.2, the proposed SOS conditions in Theorem 4.1 are feasible with $\lambda = 4$ at $a = 2$ and $b = 9$. The solutions are

$$\begin{aligned} \boldsymbol{F}_1 &= [0.7099, -4.298], \ \boldsymbol{F}_2 = [1.6225, 3.6693], \\ \boldsymbol{F}_3 &= [-0.3806, -7.765], \end{aligned} \tag{4.33}$$

and

$$V(\boldsymbol{x}) = 1.3318 \times 10^{-5} x_1^4 - 3.0747 \times 10^{-5} x_1^3 x_2 + 0.0014 x_1^2 x_2^2 + 0.0017 x_1 x_2^3 + 0.0069 x_2^4, \tag{4.34}$$

and S-Procedure multipliers are

$$\begin{aligned} \sigma_1(\boldsymbol{x}) &= 0.0003x_1^4 - 0.0019x_1^3x_2 + 0.0113x_1^2x_2^2 + 0.0024x_1x_2^3 + 0.0449x_2^4, \\ \sigma_2(\boldsymbol{x}) &= 0.0001x_1^4 - 0.0033x_1^3x2 + 0.0276x_1^2x_2^2 - 0.0725x1x_2^3 + 0.0772x_2^4, \\ \sigma_3(\boldsymbol{x}) &= 1.4032 \times 10^{-6}x_1^4 - 3.2644 \times 10^{-6}x_1^3x_2 - 0.0002x_1^2x_2^2 - 0.0017x_1x_2^3 + 0.0347x_2^4. \end{aligned}$$

Note that all polynomials $v_i(\boldsymbol{x})$ resulted to be the identical, reducing to the standard quartic Lyapunov function. The trajectories in the phase plane at $\boldsymbol{x}_0 = [-0.5, 0.9]$, $\boldsymbol{x}_0 = [-0.9, 1.2]$, $\boldsymbol{x}_0 = [1.1, 0.9]$, $\boldsymbol{x}_0 = [0.4, -1.1]$, $\boldsymbol{x}_0 = [1.3, -0.4]$, $\boldsymbol{x}_0 = [-1.4, -1]$ are shown in Figure 4.5 and Figure 4.6 illustrates the time plot of state-variables, $u(t)$ and Lyapunov function at $\boldsymbol{x}_0 = [-1.4, -1]$.

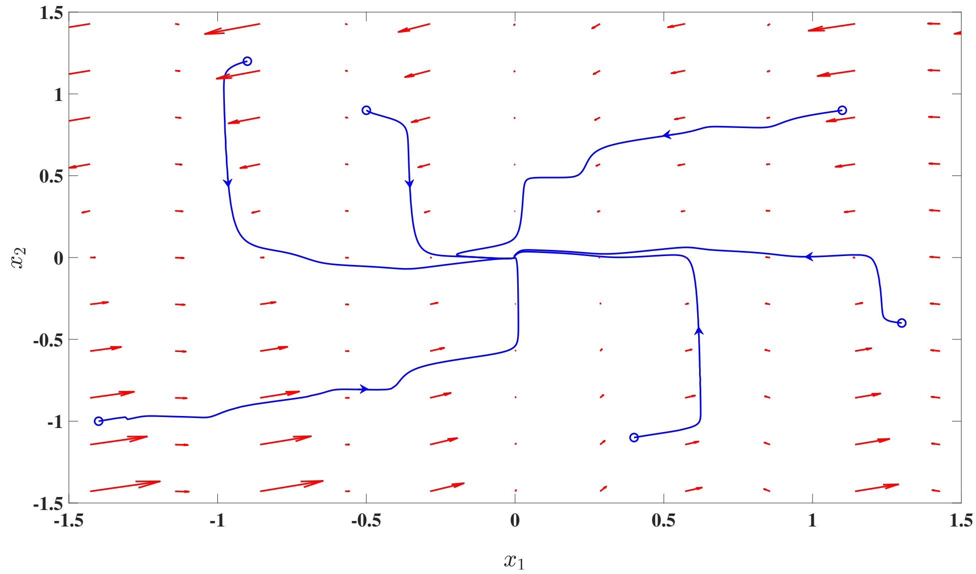

Figure 4.5: Phase trajectories in the plane $x_1 - x_2$ of the feedback system in Example 7.

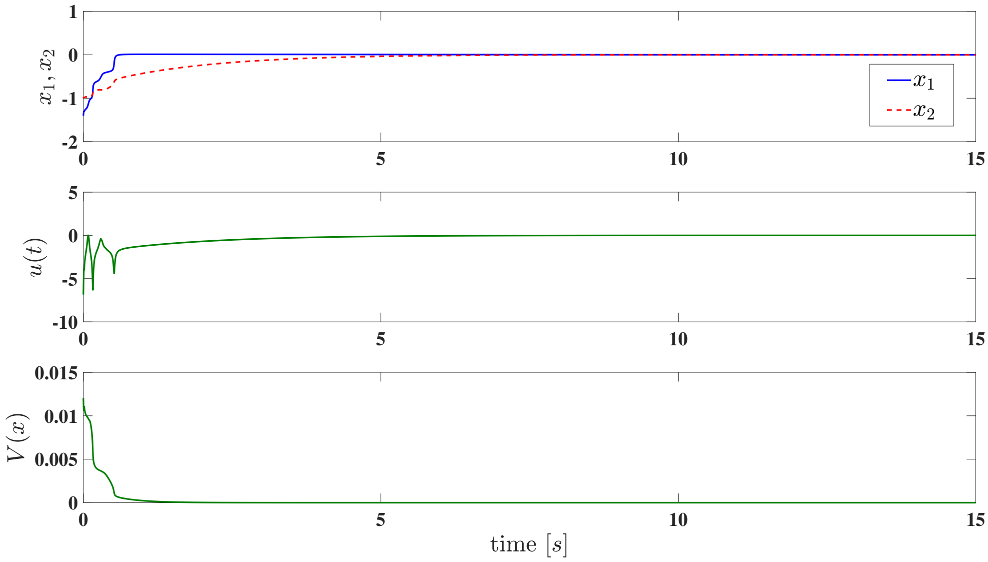

Figure 4.6: From top to bottom, state-variable response, time plot of u and time response of Lyapunov function of the feedback system in Example 7

Example 8. This example considers a slightly modification of the well-known benchmark Example 3 in subsection 3.2.2, consisting of the same state matrices $\boldsymbol{A}_i$ and input vectors $\boldsymbol{B}_i$ with MFs stated below.

$$\begin{aligned} h_1(x_1) &= \frac{\cos(x_1)+2}{10}, \quad h_2(x_1) = \frac{\sin(x_1)+3}{10}, \\ h_3(x_1) &= 1 - h_1(x_1) - h_2(x_1). \end{aligned} \tag{4.35}$$

This modification of the well-known benchmark example was borrowed from [53] and was employed to demonstrate the significance of adding knowledge on the MFs in the design conditions. In the cited work, authors leveraged the fuzzy quadratic Lyapunov function and reached a maximum value of parameter $b = 9.5$ at $a = 2$. To fairly compare the proposed SOS conditions in Theorem 4.1, set $\lambda = 2$ and define the following set of polynomial restrictions (see Figure 4.9).

$$\mathcal{S} = \{(\hat{\mu}_1 - 0.1)(0.3 - \hat{\mu}_1) \geq 0,\ (\hat{\mu}_2 - 0.2)(0.4 - \hat{\mu}_2) \geq 0,\ (\hat{\mu}_1 + \hat{\mu}_2 - 0.3586)(0.6414 - \hat{\mu}_1 - \hat{\mu}_2) \geq 0\}. \tag{4.36}$$

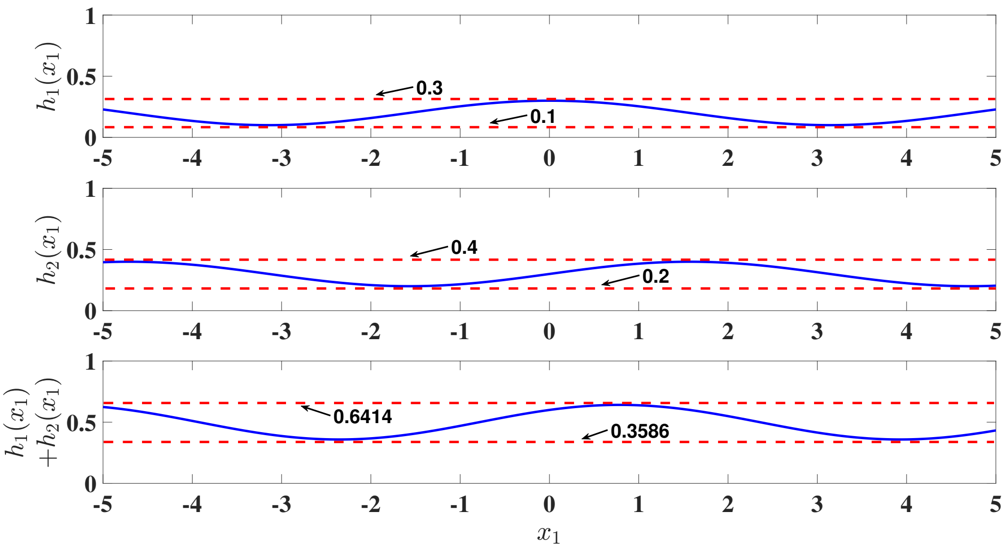

Figure 4.7: MFs in Example 8.

Using this information, SOS conditions are feasible for $a = 2$ and $b = 11$ with the solutions below.

$$v_1^{[2]}(\boldsymbol{x}) = 0.372x_1^2 - 1.56x_1x_2 + 14.3172x_2^2,$$

$$v_2^{[2]}(\boldsymbol{x}) = 2.103x_1^2 - 1.56x_1x_2 + 14.3172x_2^2,$$

$$v_3^{[2]}(\boldsymbol{x}) = 0.6x_1^2 - 1.56x_1x_2 + 14.3172x_2^2.$$

$$\boldsymbol{F}_1 = \boldsymbol{F}_2 = \boldsymbol{F}_3 = [0.48985, -1.4883],$$

and the S-procedure multipliers are

$$\tau_1(\boldsymbol{x}) = 11.7457x_1^2 - 0.1189x_1x_2 + 12.9667x_2^2,$$

$$\tau_2(\boldsymbol{x}) = 11.9096x_1^2 - 0.0943x_1x_2 + 12.8531x_2^2,$$

$$\tau_3(\boldsymbol{x}) = 8.3449x_1^2 - 0.1908x_1x_2 + 10.2065x_2^2,$$

$$\sigma_1(\boldsymbol{x}) = 14.3476x_1^2 - 5.2459x_1x_2 + 19.4738x_2^2,$$

$$\sigma_2(\boldsymbol{x}) = 20.6315x_1^2 + 10.8021x_1x_2 + 19.7758x_2^2,$$

$$\sigma_3(\boldsymbol{x}) = 12.7527x_1^2 + 8.8439x_1x_2 + 24.31x_2^2.$$

Figure 4.8 shows the phase trajectories in the plane $x_1 - x_2$ at $\boldsymbol{x}_0 = [-1.3, -0.6]^T$, $\boldsymbol{x}_0 = [-0.5, 0.7]^T$, $\boldsymbol{x}_0 = [1.2, 0.6]^T$, $\boldsymbol{x}_0 = [0.6, -1.1]^T$, $\boldsymbol{x}_0 = [1.4, 0.1]^T$, and $\boldsymbol{x}_0 = [-0.8, -1]^T$. On the other hand, Figure 4.9 represents graphically the time plot of the states, control input and Lyapunov function at $\boldsymbol{x}_0 = [-0.8, -1]^T$.

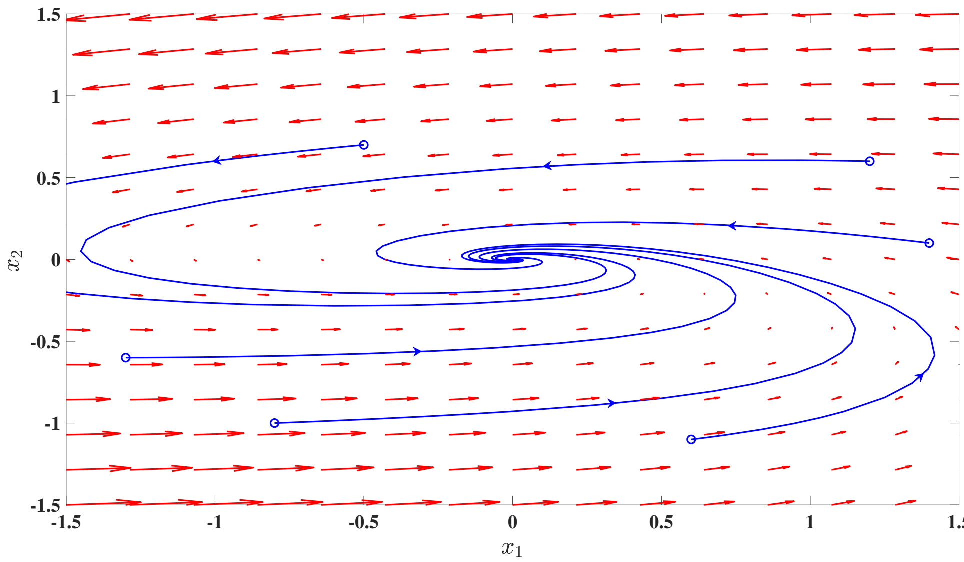

Figure 4.8: Phase trajectories in the plane $x_1 - x_2$ of the feedback system in Example 8

4.4 Discussion and Conclusions of the Chapter

This chapter has presented and given the proof of a novel integral-type Lyapunov function, which is a higher-degree polynomial setting of the form employed in previous chapter. The purpose of Example 5 has been to demonstrate that the proposal is in fact a Lyapunov

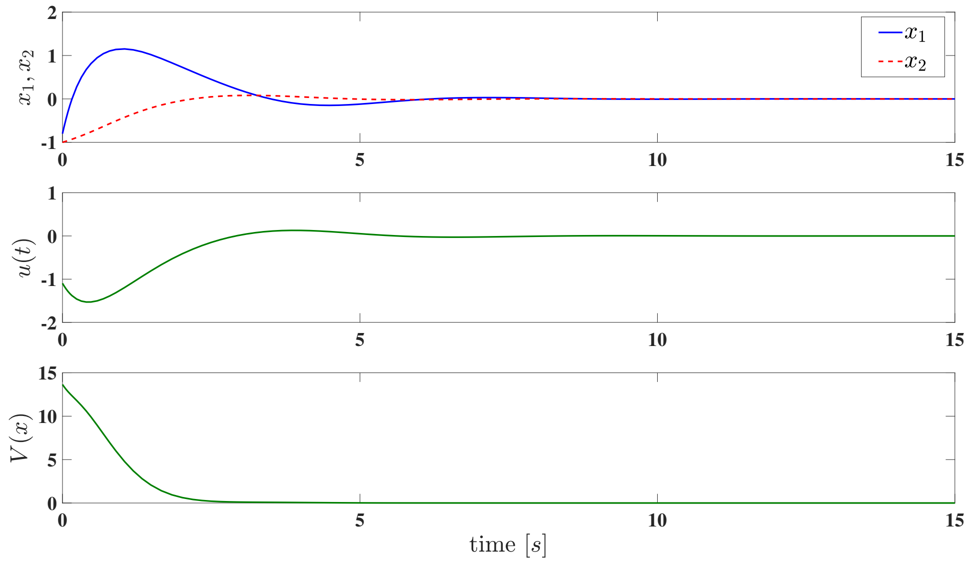

Figure 4.9: From top to bottom, state-variable response, time plot of u and time response of Lyapunov function of the feedback system in Example 8.

function for the system under study whereas Example 6 has brought more relaxed results when increasing the degree of the function. Latter example uses in fact the same Takagi-Sugeno benchmark fuzzy system as in Example 3, and it shows that the novel proposal brings more relaxed results than conditions in Theorem 3.2, this is mainly due to the fact that the higher the degree of the Lyapunov function is, the more the results are relaxed. Furthermore, as in previous section, the use of multiple Lyapunov functions, the Positivstellensatz certification and the substitution of the MFs by quadratic polynomial variables contribute to improve the stabilization conditions.

Last two examples (7 and 8) in the present chapter used the novel relaxation technique. Rather than including the basic properties of MFs as semialgebraic set conditions via the Positivstellensatz, the novel relaxation technique defines a set of polynomial restrictions which depend on the knowledge on the MFs and are included via the S-procedure. Both examples use the same benchmark model as in Example 6, and the inclusion of knowledge on the MFs in the conditions have reflected an improvement on the results, getting a higher maximum value for the parameter b. In spite of the matrices $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ are the same in Examples 7 and 8, the MFs differ from each other. By employing conditions which exclude knowledge of the MFs, the results must be the same. Therefore, the proposal has demonstrated the significance of using conditions that contain polynomial restrictions representing knowledge on the MFs as a part of the control synthesis conditions.

5

Disturbance Attenuation Control

> *"He who seeks for methods without having a definite problem in mind seeks for the most part in vain."*
>
> — David Hilbert

Previous two chapters have delimited the study of open-loop and feedback stability of nonlinear systems in polynomial fuzzy form in the absence of external disturbance signals. The present chapter covers the disturbance attenuation problem in the model-based fuzzy control framework. To this end, disturbed nonlinear system (2.51) is expressed as the next state-space realization

$$\begin{aligned}\dot{\boldsymbol{x}} &= \sum_{i=1}^{r} h_i(\boldsymbol{x})\Big\{\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{x})u + \boldsymbol{E}_i(\boldsymbol{x})w\Big\},\\ y &= \sum_{i=1}^{r} h_i(\boldsymbol{x})\boldsymbol{C}_i(\boldsymbol{x})\boldsymbol{x}.\end{aligned} \tag{5.1}$$

The disturbance attenuation problem will be firstly tackled via quadratic stabilization as follows.

5.1 Disturbance attenuation control via quadratic stabilization

Theorem 5.1. The control law $u = -\sum_{i=1}^{r} \boldsymbol{F}_i(\boldsymbol{x})\boldsymbol{x}$ quadratically stabilizes the zero equilibrium of (2.51) in the polynomial fuzzy form (5.1) at $w = 0$, and renders the $\mathcal{L}_2$ gain of the feedback system less or equal than γ at $w \neq 0$ if there exist a symmetric matrix $\boldsymbol{P}$ and polynomial vectors $\boldsymbol{M}_i(\boldsymbol{x})$ of appropriate dimensions satisfying the SOS conditions below.

$$\begin{aligned} \min_{\boldsymbol{P},\boldsymbol{M}_1(\boldsymbol{x}),\ldots,\boldsymbol{M}_r(\boldsymbol{x})} &\gamma \text{ subject to} \\ \boldsymbol{\vartheta}_1^T(\boldsymbol{P}-\epsilon\boldsymbol{I})\boldsymbol{\vartheta}_1 &\in \mathbb{S}[\boldsymbol{\vartheta}_1], \\ -\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{h}_i^2\hat{h}_j^2\boldsymbol{\vartheta}_2^T\boldsymbol{\Gamma}_{ij}(\boldsymbol{x})\boldsymbol{\vartheta}_2 &\in \mathbb{S}[\hat{\boldsymbol{h}},\boldsymbol{\vartheta}_2,\boldsymbol{x}], \end{aligned} \tag{5.2}$$

Here, ϵ is a small enough positive number selected in advance, $\boldsymbol{\vartheta}_1$ and $\boldsymbol{\vartheta}_2$ are vectors depending neither $\boldsymbol{x}$ nor w and

$$\boldsymbol{\Gamma}_{ij}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{\Lambda}_{ij}(\boldsymbol{x}) & \boldsymbol{E}_i(\boldsymbol{x}) & \boldsymbol{P}\boldsymbol{C}_i^T(\boldsymbol{x}) & \boldsymbol{M}_j^T(\boldsymbol{x}) \\ \boldsymbol{E}_i^T(\boldsymbol{x}) & -\gamma^2\boldsymbol{I} & \boldsymbol{0} & 0 \\ \boldsymbol{C}_i(\boldsymbol{x})\boldsymbol{P} & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{M}_j(\boldsymbol{x}) & 0 & \boldsymbol{0} & -\boldsymbol{R}^{-1} \end{bmatrix}, \tag{5.3}$$

with $\boldsymbol{M}_j(\boldsymbol{x}) = \boldsymbol{F}_j(\boldsymbol{x})\boldsymbol{P}$ and

$$\boldsymbol{\Lambda}_{ij}(\boldsymbol{x}) = \boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{P} - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{M}_j(\boldsymbol{x}) + \boldsymbol{P}\boldsymbol{A}_i^T(\boldsymbol{x}) - \boldsymbol{M}_j^T(\boldsymbol{x})\boldsymbol{B}_i^T(\boldsymbol{x}).$$

Proof. This proof leverages a quadratic form $V(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{P}^{-1}\boldsymbol{x}$ for $\boldsymbol{P}^{-1} > 0$ as Lyapunov function, whose time derivative is

$$\dot{V}(\boldsymbol{x}) = \dot{\boldsymbol{x}}^T\boldsymbol{P}^{-1}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{P}^{-1}\dot{\boldsymbol{x}}. \tag{5.4}$$

Using above equation, polynomial fuzzy model (5.1) and PDC controller, inequality (2.53) becomes

$$\begin{aligned} \sum_{i=1}^{r}\sum_{j=1}^{r}h_i(\boldsymbol{x})h_j(\boldsymbol{x})\Big\{\boldsymbol{x}^T\big(\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x})\big)^T\boldsymbol{P}^{-1}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{P}^{-1}\big(\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x})\big)\boldsymbol{x} \\ + \boldsymbol{x}^T\boldsymbol{P}^{-1}\boldsymbol{E}_i(\boldsymbol{x})w + w^T\boldsymbol{E}_i(\boldsymbol{x})^T\boldsymbol{P}^{-1}\boldsymbol{x} - \gamma^2 w^T w\Big\} \\ + \left(\sum_{i=1}^{r}h_i(\boldsymbol{x})\boldsymbol{C}_i(\boldsymbol{x})\boldsymbol{x}\right)^T\left(\sum_{i=1}^{r}h_i(\boldsymbol{x})\boldsymbol{C}_i(\boldsymbol{x})\boldsymbol{x}\right) \\ + \left(\sum_{i=1}^{r}h_i(\boldsymbol{x})\boldsymbol{F}_i(\boldsymbol{x})\boldsymbol{x}\right)^T R\left(\sum_{i=1}^{r}h_i(\boldsymbol{x})\boldsymbol{F}_i(\boldsymbol{x})\boldsymbol{x}\right) \le 0, \end{aligned} \tag{5.5}$$

which is equivalent to the following matrix form

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{x})h_j(\boldsymbol{x})\left\{\begin{bmatrix}\boldsymbol{x}\\ w\end{bmatrix}^T\begin{bmatrix}\tilde{\boldsymbol{\Lambda}}_{ij}(\boldsymbol{x}) & \boldsymbol{P}^{-1}\boldsymbol{E}_i(\boldsymbol{x})\\ \boldsymbol{E}_i(\boldsymbol{x})^T\boldsymbol{P}^{-1} & -\gamma^2\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ w\end{bmatrix}\right.$$
$$\left. + \begin{bmatrix}\boldsymbol{x}\\ w\end{bmatrix}^T\begin{bmatrix}\boldsymbol{C}_i^T(\boldsymbol{x}) & \boldsymbol{F}_i^T(\boldsymbol{x})\\ 0 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{I} & 0\\ 0 & R\end{bmatrix}\begin{bmatrix}\boldsymbol{C}_i(\boldsymbol{x}) & 0\\ \boldsymbol{F}_i(\boldsymbol{x}) & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ w\end{bmatrix}\right\} \leq 0, \tag{5.6}$$

where $\tilde{\boldsymbol{\Lambda}}_{ij}(\boldsymbol{x}) = \big(\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x})\big)^T\boldsymbol{P}^{-1} + \boldsymbol{P}^{-1}\big(\boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x})\big)$. Applying Schur complement, it becomes

$$\begin{bmatrix}\tilde{\boldsymbol{\Lambda}}_{ij}(\boldsymbol{x}) & \boldsymbol{P}^{-1}\boldsymbol{E}_i(\boldsymbol{x}) & \boldsymbol{C}_i^T(\boldsymbol{x}) & \boldsymbol{F}_j^T(\boldsymbol{x})\\ \boldsymbol{E}_i^T(\boldsymbol{x})\boldsymbol{P}^{-1} & -\gamma^2\boldsymbol{I} & 0 & 0\\ \boldsymbol{C}_i(\boldsymbol{x}) & 0 & -\boldsymbol{I} & 0\\ \boldsymbol{F}_j(\boldsymbol{x}) & 0 & 0 & -R^{-1}\end{bmatrix} \leq 0. \tag{5.7}$$

In order to avoid nonconvex terms, we multiply both sides by $diag(\boldsymbol{P}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I})$ and equation (5.6) transforms into (5.3). The substitution of the MFs by quadratic polynomial variables $h_i(\boldsymbol{x}) = \hat{h}_i^2$ concludes the proof of this Theorem. □

5.2 Disturbance Attenuation Control via Differential Games

In contrast to previous section, here the disturbance attenuation problem is formulated as a differential game and the integral-type Lyapunov function is used as an approximator of the solution of the HJI equation. Inserting the polynomial fuzzy model (5.1) in equation (2.59) then becomes

$$\mathcal{H}(\boldsymbol{x}, V(\boldsymbol{x}), u, w) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{x})h_j(\boldsymbol{x})\left\{\frac{\partial v_j^{[\lambda]}(\boldsymbol{x})}{\partial \boldsymbol{x}}\{\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{x})u + \boldsymbol{E}_i(\boldsymbol{x})w\}\right.$$
$$\left. + \boldsymbol{x}^T\boldsymbol{C}_i^T(\boldsymbol{x})\boldsymbol{C}_j(\boldsymbol{x})\boldsymbol{x}\right\} + u^T\boldsymbol{R}u - \gamma^2 w^T w, \tag{5.8}$$

and the associated control and disturbance policies result to be

$$u = -\frac{1}{2}\boldsymbol{R}^{-1}\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(\boldsymbol{x})h_j(\boldsymbol{x})\boldsymbol{B}_i^T(\boldsymbol{x})\left(\frac{\partial v_j^{[\lambda]}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right)^T, \tag{5.9}$$

$$w = \frac{1}{2\gamma^2}\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(\boldsymbol{x})h_j(\boldsymbol{x})\boldsymbol{E}_i^T(\boldsymbol{x})\left(\frac{\partial v_j^{[\lambda]}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right)^T. \tag{5.10}$$

The disturbance attenuation problem addressed in this section consists in two steps: 1) find an initial admissible control policy or policy pair, and 2) enhance the controller by means of the SPUA. An admissible control policy (or policy pair) guarantee that the feedback system is stable at $\boldsymbol{x} \neq \boldsymbol{0}$ and the performance index (2.56) is finite.

5.2.1 Path Following Algorithm

The challenge of finding a solution for the bilinear disturbance attenuation conditions is that the optimization problem includes two variables to be minimized. To avoid this difficulty, below algorithm considers γ as a decision variable restricted to be positive and only minimizes a single variable. Recall that this algorithm is searching for an initial admissible control policy (or policy pair), and the attenuation level γ is going to be minimized via policy iteration afterwards. Just for the sake of simplicity, this algorithm considers a polynomial Lyapunov function instead of the integral-type form.

Algorithm 3. Path following for the disturbance attenuation problem

Step 1: Set $k = 0$ and define the policy pair $(\hat{u}_k, \omega)$ with $\hat{u}_0 = \sum_{i=1}^{r}\hat{h}_i\rho_i(\boldsymbol{x})$ for $\rho_i(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$, and ω a polynomial as $\boldsymbol{x}$ to be included in the SOS conditions.

Step 2: Find a feasible solution for the optimization problem

$$\min_{\mathcal{V}_k(\boldsymbol{x}),\hat{\gamma},W_{ij}(\boldsymbol{x}),\sigma_\iota(\boldsymbol{x})} \alpha \text{ subject to}$$

$$\mathcal{V}_k(\boldsymbol{x}) - \varepsilon(\boldsymbol{x}) \in \mathbb{S}[\boldsymbol{x}], \tag{5.11}$$

$$-\sum_{i=1}^{r}\hat{h}_i\frac{\partial \mathcal{V}_k(\boldsymbol{x})}{\partial \boldsymbol{x}}\{\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{x})\hat{u}_k + \boldsymbol{E}_i(\boldsymbol{x})\omega\} + \cdots$$

$$-\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{h}_i\hat{h}_j\{\boldsymbol{x}^T\boldsymbol{C}_i^T(\boldsymbol{x})\boldsymbol{C}_j(\boldsymbol{x})\boldsymbol{x}-\alpha W_{ij}(\boldsymbol{x})\}$$

$$-\hat{u}_k^T\boldsymbol{R}\hat{u}_k+\hat{\gamma}\omega^T\omega-\sum_{\iota=1}^{q}\sigma_\iota(\boldsymbol{x})p_\iota(\boldsymbol{h})\in\mathbb{S}[\boldsymbol{x},\boldsymbol{h},\omega], \tag{5.12}$$

$$W_{ij}(\boldsymbol{x})\in\mathbb{S}[\boldsymbol{x}],\ \forall i,j, \tag{5.13}$$

$$\sigma_\iota(\boldsymbol{x})\in\mathbb{S}[\boldsymbol{x}],\forall\iota, \tag{5.14}$$

$$\hat{\gamma}>0. \tag{5.15}$$

Step 3: A feasible solution with $\alpha<0$ means that $\hat{u}_k$ is an admissible control policy. Otherwise, define $\mathcal{V}(\boldsymbol{x})=\mathcal{V}_k(\boldsymbol{x})$ and consider the next minimizing problem

$$\min_{\delta\mathcal{V}(\boldsymbol{x}),\hat{\gamma},W_{ij}(\boldsymbol{x}),\sigma_\iota(\boldsymbol{x})}\alpha \text{ subject to}$$

$$-\sum_{i=1}^{r}\hat{h}_i\frac{\partial\big(\mathcal{V}(\boldsymbol{x})+\delta\mathcal{V}(\boldsymbol{x})\big)}{\partial\boldsymbol{x}}\Big\{\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x}+\boldsymbol{E}_i(\boldsymbol{x})\omega\Big\}+\hat{\gamma}\omega^T\omega+...$$

$$-\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{h}_i\hat{h}_j\Bigg\{\boldsymbol{x}^T\boldsymbol{C}_i^T(\boldsymbol{x})\boldsymbol{C}_j(\boldsymbol{x})\boldsymbol{x}-\frac{1}{4}\boldsymbol{R}^{-1}\Bigg(\frac{\partial\mathcal{V}(\boldsymbol{x})}{\partial\boldsymbol{x}}\boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{B}_j^T(\boldsymbol{x})\Big(\frac{\partial\mathcal{V}(\boldsymbol{x})}{\partial\boldsymbol{x}}\Big)^T$$

$$+\frac{\partial\mathcal{V}(\boldsymbol{x})}{\partial\boldsymbol{x}}\boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{B}_j^T(\boldsymbol{x})\Big(\frac{\partial\delta\mathcal{V}(\boldsymbol{x})}{\partial\boldsymbol{x}}\Big)^T+\frac{\partial\delta\mathcal{V}(\boldsymbol{x})}{\partial\boldsymbol{x}}\boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{B}_j^T(\boldsymbol{x})\Big(\frac{\partial\mathcal{V}(\boldsymbol{x})}{\partial\boldsymbol{x}}\Big)^T\Bigg)-\alpha W_{ij}(\boldsymbol{x})\Bigg\}$$

$$-\sum_{\iota=1}^{q}\sigma_\iota(\boldsymbol{x})p_\iota(\boldsymbol{h})\in\mathbb{S}[\boldsymbol{x},\boldsymbol{h},\omega], \tag{5.16}$$

$$\mathcal{V}(\boldsymbol{x})+\delta\mathcal{V}(\boldsymbol{x})-\varepsilon(\boldsymbol{x})\in\mathbb{S}[\boldsymbol{x}], \tag{5.17}$$

$$\boldsymbol{v}^T\begin{bmatrix}\epsilon_{\mathcal{V}}\mathcal{V}^2(\boldsymbol{x}) & \delta\mathcal{V}(\boldsymbol{x})\\ \delta\mathcal{V}(\boldsymbol{x}) & 1\end{bmatrix}\boldsymbol{v}\in\mathbb{S}[\boldsymbol{x},\boldsymbol{v}], \tag{5.18}$$

$$W_{ij}(\boldsymbol{x})\in\mathbb{S}[\boldsymbol{x}],\ \forall i,j, \tag{5.19}$$

$$\sigma_\iota(\boldsymbol{x})\in\mathbb{S}[\boldsymbol{x}],\forall\iota, \tag{5.20}$$

$$\hat{\gamma}>0. \tag{5.21}$$

The entries of vector $\boldsymbol{v}$ do not depend on $\boldsymbol{x}$ and is employed to verify positivity of the polynomial matrix as stated in subsection 2.1.2, and $\epsilon_{\mathcal{V}}>0$ is a small real number. Using the solution $\delta\mathcal{V}(\boldsymbol{x})$ update the control policy as

$$u_{k+1} = -\frac{1}{2}\boldsymbol{R}^{-1}\sum_{i=1}^{r}\hat{h}_i\boldsymbol{B}_i^T(\boldsymbol{x})\left(\frac{\partial\big(\mathcal{V}(\boldsymbol{x})+\delta\mathcal{V}(\boldsymbol{x})\big)}{\partial\boldsymbol{x}}\right)^T, \tag{5.22}$$

and increase $k = k+1$. Return to Step 2.

5.2.2 SPUA for Model-Based Fuzzy Control Systems

As stated in subsection 2.5.2, policy iteration methods are useful algorithms to approximate the value function of the Hamilton-Jacobi equations as long as the designer knows an initial admissible control policy (or policy pair). Algorithm 3 gives a methodology to find an initial initial setting, but any other method can be used instead, for instance a solution from Theorem 5.1. Once an initial setting is known, one can write the Hamiltonian as follows.

$$\begin{aligned} H_\mu(\boldsymbol{h},\boldsymbol{x},v_j(\boldsymbol{x}),u,w,\gamma) := -\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{h}_i\hat{h}_j\Bigg\{\frac{\partial v_j^{[\lambda]}(\boldsymbol{x})}{\partial\boldsymbol{x}}\Big\{\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x}+\boldsymbol{B}_i(\boldsymbol{x})u+\boldsymbol{E}_i(\boldsymbol{x})w\Big\} \\ +\boldsymbol{x}^T\boldsymbol{C}_i^T(\boldsymbol{x})\boldsymbol{C}_j(\boldsymbol{x})\boldsymbol{x}\Bigg\} - u^T\boldsymbol{R}u+\gamma^2w^Tw-\sum_{\iota=1}^{q}\sigma_\iota(\boldsymbol{x})p_\iota(\boldsymbol{h}) \in \mathbb{S}[\boldsymbol{h},\boldsymbol{x}]. \end{aligned} \tag{5.23}$$

Define $\Omega \subset \mathbb{R}^n$ as the region where the attenuation is expected the most. The SOS-based SPUA method is explained below (see flowchart in Figure 5.1).

Algorithm 4. SOS-based SPUA for model-based fuzzy control systems

Step 1: Define (u_0, w_0) as the initial admissible policy pair and select an arbitrary $\gamma > 0$ and set $\theta = 0$.

Step 2: Find a feasible solution for the SOS conditions coming next

$$\sum_{i=1}^{r}\hat{h}_i v_{\theta,i}^{[\lambda]}(\boldsymbol{x}) - \epsilon(\boldsymbol{x}) - \sum_{\iota=1}^{z}\tau_{\lambda\iota}(\boldsymbol{x})p_\iota(\boldsymbol{h}) \quad \in \quad \mathbb{S}[\boldsymbol{h},\boldsymbol{x}], \tag{5.24}$$

$$\sum_{i=1}^{r}\hat{h}_i\Big(v_{\theta-1,i}^{[\lambda]}(\boldsymbol{x}) - v_{\theta,i}^{[\lambda]}(\boldsymbol{x})\Big) - \sum_{\iota=1}^{z}\varsigma_{\lambda\iota}(\boldsymbol{x})p_\iota(\boldsymbol{h}) \quad \in \quad \mathbb{S}[\boldsymbol{h},\boldsymbol{x}], \tag{5.25}$$

$$H_\mu(\boldsymbol{x}, v_{\theta,i}^{[\lambda]}(\boldsymbol{x}), u_\theta, w_\theta) \quad \in \quad \mathbb{S}[\boldsymbol{h},\boldsymbol{x}], \tag{5.26}$$

with $\tau_{\lambda_\iota}(\boldsymbol{x}),\ \varsigma_{\lambda_\iota}(\boldsymbol{x}) \in \mathbb{S}[\boldsymbol{x}]$.

Step 3: The new policy pair is given as

$$u_{\theta+1} = -\frac{1}{2}\boldsymbol{R}^{-1}\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{h}_i\hat{h}_j\boldsymbol{B}_i^T(\boldsymbol{x})\left(\frac{\partial v_{\theta,j}^{[\lambda]}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right)^T, \tag{5.27}$$

$$w_{\theta+1} = \frac{1}{2\gamma^2}\sum_{i=1}^{r}\sum_{j=1}^{r}\hat{h}_i\hat{h}_j\boldsymbol{E}_i^T(\boldsymbol{x})\left(\frac{\partial v_{\theta,j}^{[\lambda]}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right)^T. \tag{5.28}$$

Step 4: If polynomials satisfy

$$\sqrt{\int\cdots\int_{\Omega}(v_{\theta-1,i}^{[\lambda]}(\boldsymbol{x}) - v_{\theta,i}^{[\lambda]}(\boldsymbol{x}))^2 dx_1\cdots dx_n} \le \epsilon, \tag{5.29}$$

for a small given $\epsilon > 0$, convergence has been reached and bring the algorithm to an end. Or else increase $\theta = \theta + 1$ and return to Step 2.

Remark 5. *This algorithm finds an upper bound [42, 43] on the solution of the HJI equation. Therefore, $V_\theta(\boldsymbol{x}) \ge V_{\theta+1}(\boldsymbol{x}) \ge 0$. Using this fact and equation (4.17) lead to (5.25).*

Remark 6. *Since $V_\theta(\boldsymbol{x}) \ge V_{\theta+1}(\boldsymbol{x})$, it entails that $\int_\Omega V_\theta(\boldsymbol{x}) \ge \int_\Omega V_{\theta+1}(\boldsymbol{x})$. The Lyapunov function used in this proposed algorithm already has a line integral structure, which may have a nonelementary antiderivative. Therefore, this study considers the integral of $v_\theta^{[\lambda]}(\boldsymbol{x})$ instead.*

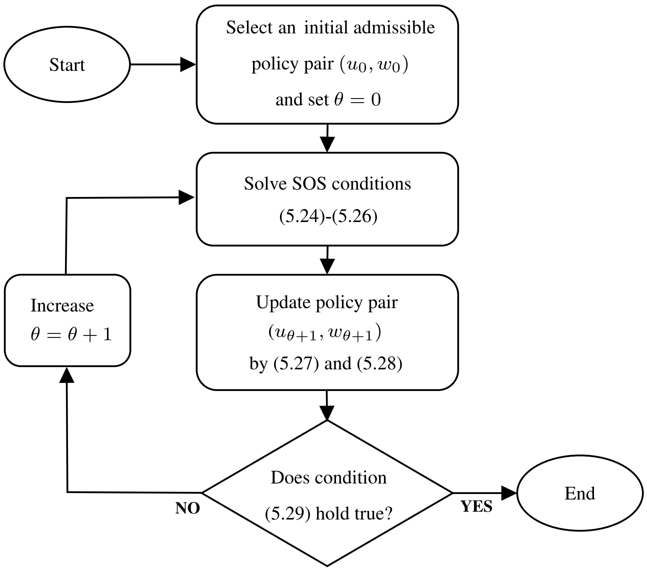

Figure 5.1: Flowchart of the SOS-based SPUA

Algorithm 4 finds a solution for the suboptimal disturbance attenuation problem. The solution of the near-optimal problem is given below.

$$\min_{v_j^{[\lambda]}(\boldsymbol{x}),\sigma_\iota(\boldsymbol{x}),\tau_{\lambda\iota}(\boldsymbol{x})} \gamma_\xi \text{ subject to} \tag{5.30}$$

$$\sum_{i=1}^{r} \hat{h}_i v_{\theta,i}^{[\lambda]}(\boldsymbol{x}) - \epsilon(\boldsymbol{x}) - \sum_{\iota=1}^{z} \tau_{\lambda\iota}(\boldsymbol{x}) p_\iota(\boldsymbol{h}) \in \mathbb{S}[\boldsymbol{h},\boldsymbol{x}], \tag{5.31}$$

$$\mathcal{H}_\mu(\boldsymbol{x}, v_{\theta,i}^{[\lambda]}(\boldsymbol{x}), u_\theta, w_\theta) \in \mathbb{S}[\boldsymbol{h},\boldsymbol{x}]. \tag{5.32}$$

Here, the attenuation level γ_ξ is the minimum value for the minimizing problem above for a given policy pair (u_ξ, w_ξ). Setting $\gamma = \gamma_\xi$ and $(u_0, w_0) = (u_\xi, w_\xi)$, Algorithm 4 computes an upper bound on the value function for this γ. Defining the policy pair $(u_{\xi+1}, w_{\xi+1})$ as the suboptimal solution found in Algorithm 4 and increase $\xi = \xi + 1$, optimization problem (5.30)-(5.32) is solved again until $|\gamma_{\xi-1} - \gamma_\xi| \leq \epsilon_\xi$ for a small $\epsilon_\xi > 0$. The convergence of the attenuation level entails that a near-optimal solution has been found (see flowchart in Figure 5.2).

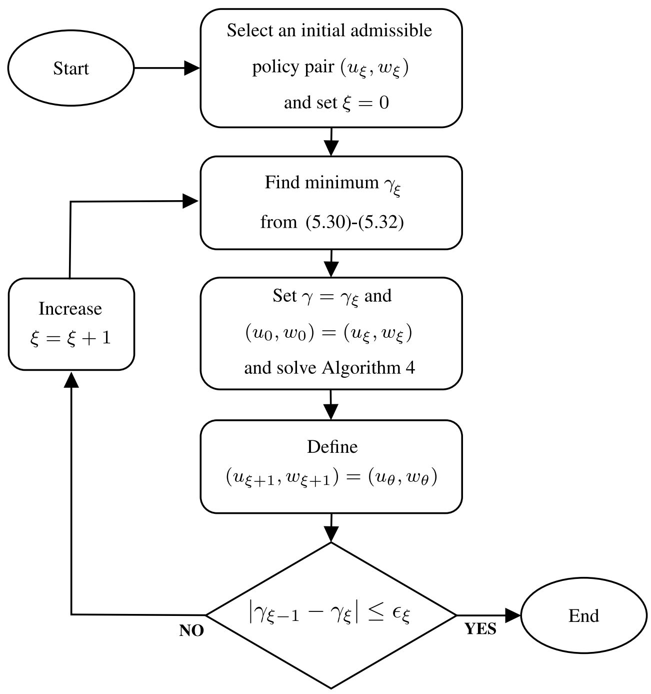

Figure 5.2: Flowchart of the near-optimal searching method

5.2.3 Disturbance Attenuation Control Examples

Example 9. Consider the following four-rule fuzzy system in Takagi-Sugeno form

$$\begin{aligned}\dot{\boldsymbol{x}} &= \sum_{i=1}^{4} h_i(\boldsymbol{x})\{\boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{B}_i u + \boldsymbol{E}_i w\},\\ y &= \sum_{i=1}^{4} h_i(\boldsymbol{x})\boldsymbol{C}_i\boldsymbol{x}.\end{aligned} \tag{5.33}$$

Here,

$$\begin{aligned}
\boldsymbol{A}_1 &= \begin{bmatrix} -0.8 & 2.8 \\ 2 & -3 \end{bmatrix},\ \boldsymbol{A}_2 = \begin{bmatrix} -0.8 & 2.2 \\ 0.8 & -1 \end{bmatrix},\\
\boldsymbol{A}_3 &= \begin{bmatrix} -3.4 & 1 \\ -1.4 & -2.4 \end{bmatrix},\ \boldsymbol{A}_4 = \begin{bmatrix} -3.4 & 0.4 \\ -2.6 & -3.6 \end{bmatrix},\\
\boldsymbol{B}_i &= \begin{bmatrix} 0.3 \\ 1.3 \end{bmatrix},\ \boldsymbol{C}_i = \begin{bmatrix} 0.9 & 1.7 \end{bmatrix},\ \forall i \in \{1,2,3,4\},\\
\boldsymbol{E}_1 &= \begin{bmatrix} 0.4 \\ -1.4 \end{bmatrix},\ \boldsymbol{E}_2 = \begin{bmatrix} -1 \\ 0.4 \end{bmatrix},\ \boldsymbol{E}_3 = \begin{bmatrix} 1 \\ -2.2 \end{bmatrix},\ \boldsymbol{E}_4 = \begin{bmatrix} -0.4 \\ -0.4 \end{bmatrix}.
\end{aligned}$$

The MFs are defined as follows

$$\begin{aligned}
h_1(\boldsymbol{x}) &= \mu_1(x_1)\mu_1(x_2),\ h_2(\boldsymbol{x}) = \mu_1(x_1)\mu_2(x_2),\\
h_3(\boldsymbol{x}) &= \mu_2(x_1)\mu_1(x_2), h_4(\boldsymbol{x}) = \mu_2(x_1)\mu_2(x_2),
\end{aligned}$$

where

$$\begin{aligned}
\mu_1(x_1) &= 1 - \cos^2(x_1),\ \mu_2(x_1) = 1 - \mu_1(x_1),\\
\mu_1(x_2) &= \big(1 - \sin^2(x_2)\big)e^{\cos(x_2)}/(1 + e^{\cos(x_2)}),\ \mu_2(x_2) = 1 - \mu_1(x_2).
\end{aligned}$$

Table below compares the minimum value of the attenuation levels γ reached with both the feasible solution computed via the SOS convex conditions in Theorem 5.1 and the SOS-based SPUA for polynomial fuzzy system when $\boldsymbol{R} = 10$.

Table 5.1: Comparative results on the minimum value of γ in Example 9

Method	Minimum value of γ	Reduction rate of γ
Theorem 5.1	2.7925	-
SOS-based SPUA*	2.4609	11.875%
SOS-based SPUA	2.2591	19.101%

As indicated in Table 5.1, convex conditions from Theorem 5.1 are feasible with a minimum $\gamma = 2.7925$, whose resulting matrix $\boldsymbol{P}$ and vectors $\boldsymbol{M}_i$ are

$$\boldsymbol{P} = \begin{bmatrix} 0.1128 & -0.0914 \\ -0.0914 & 0.1666 \end{bmatrix}, \tag{5.34}$$

$$\begin{aligned} \boldsymbol{M}_1 &= \begin{bmatrix} 0.0837 & 0.0570 \end{bmatrix}, \ \boldsymbol{M}_2 = \begin{bmatrix} 0.0643 & 0.0912 \end{bmatrix}, \\ \boldsymbol{M}_3 &= \begin{bmatrix} 0.0157 & 0.1334 \end{bmatrix}, \ \boldsymbol{M}_4 = \begin{bmatrix} 0.0449 & 0.0779 \end{bmatrix}. \end{aligned} \tag{5.35}$$

As aforesaid, the SOS-based SPUA requires an initial control policy u_0. Define the initial control policy as

$$u_0 = -\sum_{i=1}^{4} h_i(\boldsymbol{x})\boldsymbol{M}_i\boldsymbol{P}^{-1}\boldsymbol{x}, \tag{5.36}$$

and the subset of the state-space where the effect of the disturbance is expected to be mitigated the most is

$$\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 : |x_i| < 1, \forall i \in \{1, 2\}\}. \tag{5.37}$$

Consider the substitution $h_i(\boldsymbol{x}) = \hat{h}_i^2$ and the initial control policy u_0 in (5.36). Three iterations later the SOS-based SPUA converged to $\gamma = 2.4609$ and the resulting controller is defined as

$$\begin{aligned} u = -2.116\hat{h}_1^2x_1 - 1.6392\hat{h}_1^2x_2 - 2.116\hat{h}_2^2x_1 - 2.7531\hat{h}_2^2x_2 - 2.6617\hat{h}_3^2x_1 - 1.6392\hat{h}_3^2x_2 \\ -2.6617\hat{h}_4^2x_1 - 2.7531\hat{h}_4^2x_2. \end{aligned} \tag{5.38}$$

Next step is to introduce knowledge on the MFs by replacing $\hat{h}_1 = \hat{\mu}_{11}\hat{\mu}_{12}$, $\hat{h}_2 = \hat{\mu}_{11}(1 - \hat{\mu}_{12})$, $\hat{h}_3 = (1 - \hat{\mu}_{11})\hat{\mu}_{12}$, $\hat{h}_4 = (1 - \hat{\mu}_{11})(1 - \hat{\mu}_{12})$ and introducing the following polynomial in the variables $\hat{\mu}_{11}$, $\hat{\mu}_{12}$ via the S-procedure

$$\mathcal{S} = \{\hat{\mu}_{11}(1 - \hat{\mu}_{11}) \geq 0, \ \hat{\mu}_{12}(0.7311 - \hat{\mu}_{12}) \geq 0\}. \tag{5.39}$$

The SOS-based SPUA converged to $\gamma = 2.2591$ after 4 iterations and the solution is

$$V(\boldsymbol{x}) = 20.9496x_1^2 + 24.9114x_1x_2 + 13.9061x_2^2, \tag{5.40}$$

with S-procedure multipliers are

$$\sigma_1(\boldsymbol{x}) = 35.8684x_1^2 - 69.7384x_1x_2 + 48.4823x_2^2,$$

$$\sigma_2(\boldsymbol{x}) = 186.5475x_1^2 + 221.9611x_1x_2 + 104.8123x_2^2.$$

For this specific example, the variable representing the external disturbance w was replaced by a linear variable ω due to the substitution on the disturbance policy (5.10) brings an infeasible solution. Figure 5.3 shows that the controller (5.9) at $w = 0$ stabilizes the origin since the trajectories of the initial states $\boldsymbol{x}_0 = [0.8, 1.2]^T$, $\boldsymbol{x}_0 = [0.9, 0.1]^T$, $\boldsymbol{x}_0 = [1.2, -1.3]^T$, $\boldsymbol{x}_0 = [-1.2, 0]^T$, $\boldsymbol{x}_0 = [-0.8, -1.1]^T$ and $\boldsymbol{x}_0 = [-0.1, 1.4]^T$ reach the equilibrium state. On the other hand, Figure 5.4 depicts the time plot of the variables $\boldsymbol{x}$, u and y for a null initial condition and the exogenous disturbance

$$w = \begin{cases} 8te^{-(t-10)}\cos(t-10), & \text{if} \quad t \geq 10 \\ 0, & \text{otherwise} \end{cases}, \tag{5.41}$$

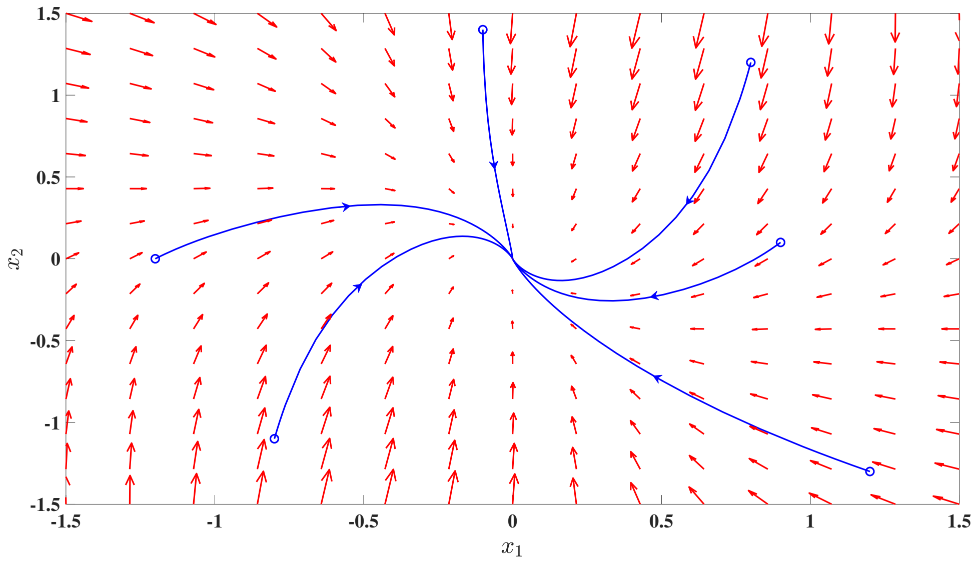

Figure 5.3: Phase trajectories in the plane $x_1 - x_2$ of the feedback system in Example 9

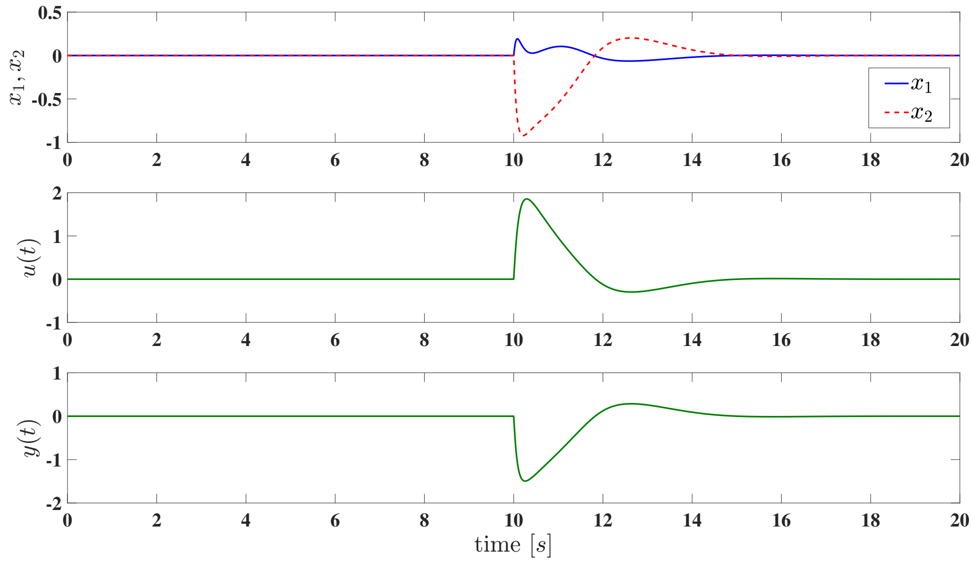

Figure 5.4: From top to bottom, state-variable response, time plot of u and output y of the feedback system in Example 9.

The cost function (2.56) includes a term $u^T \boldsymbol{R} u$, where $\boldsymbol{R} > 0$. This design parameter is useful to penalize the control input. The larger the value of $\boldsymbol{R}$ is, the more the control input is penalized. Figure 5.5 and 5.6 depict the time plot of the states variables, output and control input when the penalization parameter is $\boldsymbol{R} \in \{0.1, 1, 10\}$ at $\gamma = 2.2591$ for an initial condition $\boldsymbol{x}_0 = [0.9, 1.25]$ and external disturbance signal w given by (5.41), demonstrating the benefit of including this term in the design conditions.

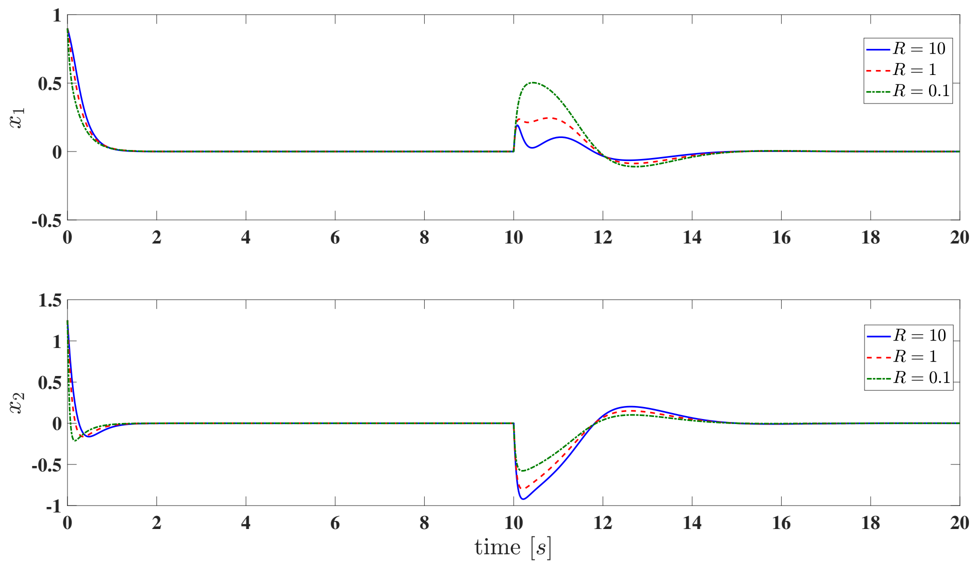

Figure 5.5: Time plot of the states of the feedback system in Example 9 when changing the penalization parameter $\boldsymbol{R}$.

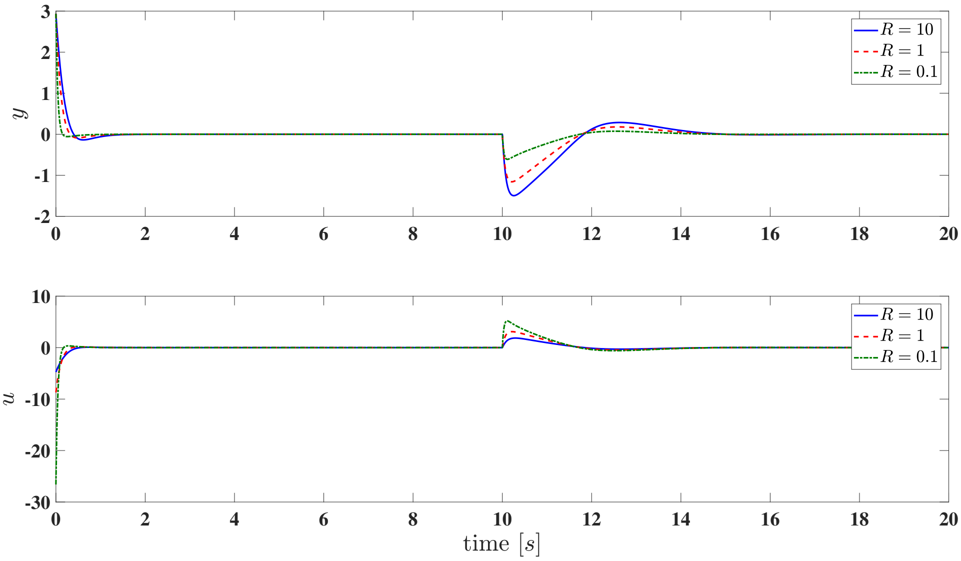

Figure 5.6: Time plot of the output (top) and control input (bottom) of the feedback system in Example 9 when changing the penalization parameter $\boldsymbol{R}$.

Now, consider the design conditions introduced in [31] for Takagi-Sugeno fuzzy systems, whose cost function excludes the input penalization term, which is equivalent to

$$\int_0^\infty \boldsymbol{y}^T \boldsymbol{y} dt \leq \gamma^2 \int_0^\infty w^T w dt. \tag{5.42}$$

The comparison of the time plot of the input pf the feedback system with controllers designed by using SOS-based SPUA proposal and conditions in [31] is shown in Figure 5.7. As expected, the transient response of the control input given by the SOS-based SPUA control feedback system is better since the amplitude of signal and overshot is less than those given by the control input of the feedback system with controller constructed by means of [31].

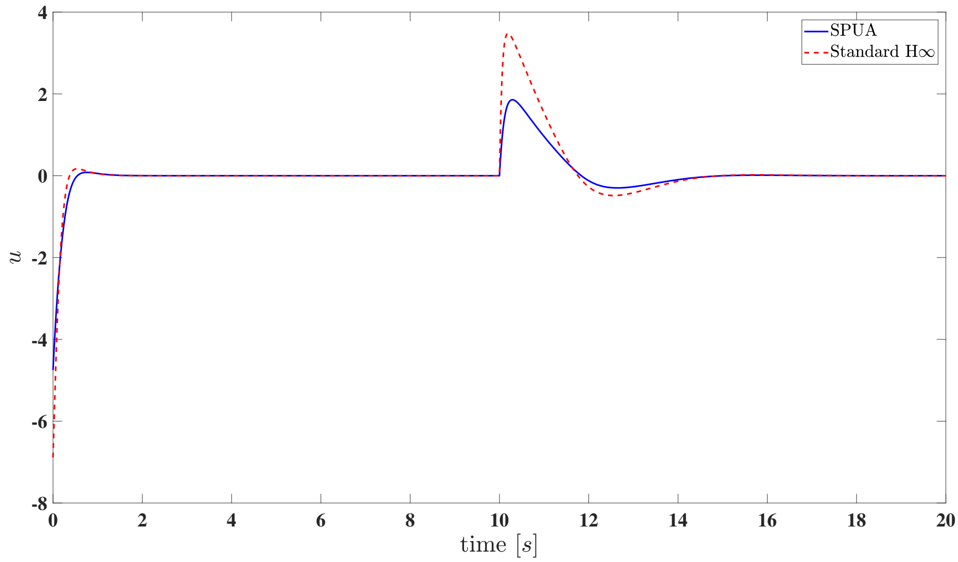

Figure 5.7: Comparison of the time plot of the control input of the feedback system in Example 9 with controllers design via the proposed SOS-based SPUA method and conditions in [31].

Example 10. The state equations below represent a second-order nonlinear system constructed via the converse HJB method (see subsection 2.5.4).

$$\begin{aligned}\dot{x}_1 &= -\frac{19}{6}x_1 + \frac{3}{2}x_1x_2^2 - \frac{7}{3}x_2 - \frac{x_2^2}{6(x_2^2+1)} - \frac{1}{3}x_2\arctan(x_2) + x_2u + w,\\ \dot{x}_2 &= x_1,\\ y &= x_1.\end{aligned} \tag{5.43}$$

Which is represented by the fuzzy system in polynomial form

$$\begin{aligned}\dot{\boldsymbol{x}} &= \sum_{i=1}^{3} h_i(x_2)\{\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{x})u + \boldsymbol{E}_i(\boldsymbol{x})w\},\\ y &= \sum_{i=1}^{3} h_i(x_2)\boldsymbol{C}_i(\boldsymbol{x})\boldsymbol{x},\end{aligned} \tag{5.44}$$

where

$$\boldsymbol{A}_1(\boldsymbol{x}) = \begin{bmatrix} -\frac{19}{6} + \frac{3}{2}x_2^2 & -2.7264 \\ 1 & 0 \end{bmatrix},$$

$$\boldsymbol{A}_2(\boldsymbol{x}) = \begin{bmatrix} -\frac{19}{6} + \frac{3}{2}x_2^2 & -4.7264 \\ 1 & 0 \end{bmatrix},$$
$$\boldsymbol{A}_3(\boldsymbol{x}) = \begin{bmatrix} -\frac{19}{6} + \frac{3}{2}x_2^2 & -1.7264 \\ 1 & 0 \end{bmatrix},$$

$$\boldsymbol{B}_1(\boldsymbol{x}) = \boldsymbol{B}_2(\boldsymbol{x}) = \boldsymbol{B}_3(\boldsymbol{x}) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix},$$

$$\boldsymbol{C}_1(\boldsymbol{x}) = \boldsymbol{C}_2(\boldsymbol{x}) = \boldsymbol{C}_3(\boldsymbol{x}) = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$\boldsymbol{E}_1(\boldsymbol{x}) = \boldsymbol{E}_2(\boldsymbol{x}) = \boldsymbol{E}_3(\boldsymbol{x}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and

$$h_1(x_2) = \frac{x_2}{6x_2^2+6} + \frac{1}{12}, \; h_2(x_2) = \frac{\arctan(x_2)}{9} + \frac{\pi}{18},$$
$$h_3(x_2) = 1 - h_1(x_2) - h_2(x_2).$$

It is important to note that convex conditions in Theorem 5.1 are infeasible for the fuzzy system in polynomial form described in this example. Nevertheless, the SOS-based SPUA method can find a disturbance attenuation controller. As demonstrated in subsection 2.5.4, both value function and optimal control policy are known, given by equations (2.69) and (2.70), respectively. The region in the state-space where the H_∞ performance is expected the most is defined as

$$\Omega = \left\{\boldsymbol{x} \in \mathbb{R}^2 : |x_i| < 1, \forall i \in \{1, 2\}\right\}. \tag{5.45}$$

Substituting $h_1(x_2) = \hat{\mu}_1$, $h_2(x_2) = \hat{\mu}_2$, $h_3(x_2) = 1 - \hat{\mu}_1 - \hat{\mu}_2$ and define the set of inequalities restrictions in terms of polynomial in the variables $\hat{\mu}_1$, $\hat{\mu}_1$.

$$\mathcal{S} = \left\{\hat{\mu}_1\left(\frac{1}{6} - \hat{\mu}_1\right) \leq 0, \; \hat{\mu}_2\left(\frac{\pi}{9} - \hat{\mu}_2\right) \leq 0\right\}. \tag{5.46}$$

For the sake of comparison, consider the initial control policy $u_0 = -10x_1x_2$ with the following three cases:

Case I: Polynomial Lyapunov function. SOS-based SPUA conditions are feasible with a fourth-degree polynomial Lyapunov function as a solution. The attenuation level tended to $\gamma = 0.7684$ and algorithm reached the solution

$$V(\boldsymbol{x}) = 0.42878x_2^4 + 3.0002x_1^2 + 1.1557x_1x_2 + 5.5858x_2^2, \tag{5.47}$$

after 4 iterations.

Case II: Integral-type Lyapunov function. SOS-based SPUA conditions are feasible with $v_i(\boldsymbol{x}) = v_i^{[4]}(\boldsymbol{x}) + v_i^{[2]}(\boldsymbol{x})$ for all $i \in \{1, 2, 3\}$. The attenuation level tended to $\gamma = 0.7071$ with the functions

$$\begin{aligned} v_1(\boldsymbol{x}) &= 0.000744x_2^4 + 3x_1^2 + 0.00198x_1x_2 + 8.1764x_2^2, \\ v_2(\boldsymbol{x}) &= 0.000744x_2^4 + 3x_1^2 + 0.00198x_1x_2 + 14.1764x_2^2, \\ v_3(\boldsymbol{x}) &= 0.000744x_2^4 + 3x_1^2 + 0.00198x_1x_2 + 5.1764x_2^2, \end{aligned} \tag{5.48}$$

and S-Procedure multipliers

$$\begin{aligned} \tau_{(\lambda=4),1}(\boldsymbol{x}) &= 0.017x_2^4 + 2.295x_1^2 + 0.0002x_1x_2 + 2.2887x_2^2, \\ \tau_{(\lambda=4),2}(\boldsymbol{x}) &= 0.0029x_2^4 + 2.2852x_1^2 + 0.0009x_1x_2 + 2.2581x_2^2, \\ \tau_{(\lambda=2),1}(\boldsymbol{x}) &= 0.017x_2^4 + 2.2948x_1^2 - 0.0009x_1x_2 + 2.745x_2^2, \\ \tau_{(\lambda=2),2}(\boldsymbol{x}) &= 0.0028639x_2^4 + 2.2845x_1^2 - 0.0038204x_1x_2 + 4.5634x_2^2, \\ \sigma_1(\boldsymbol{x}) &= 0.01247x_1^2x_2^2 + 0.024929x_1^2 - 0.0001254x_1x_2 + 0.15506x_2^2, \\ \sigma_2(\boldsymbol{x}) &= 0.002295x_1^2x_2^2 + 0.0053582x_1^2 - 0.0001687x_1x_2 + 0.043464x_2^2. \end{aligned}$$

Figure 5.8 depicts the time plot of y, u and disturbance attenuation (2.55) at $\boldsymbol{x}_0 = \boldsymbol{0}$ when the external disturbance signal is

$$w = \frac{15te^{-t/3}\cos(0.2t)}{t+1} \tag{5.49}$$

One can see that the control policy (5.9) renders the disturbance attenuation (2.55) of the stabilized polynomial fuzzy system when $t \to \infty$ less than γ^2.

Case III: Recasted nonlinear system. This final case addresses the attenuation control synthesis of the nonlinear system (5.43) using a non-fuzzy nonlinear technique presented in [43].

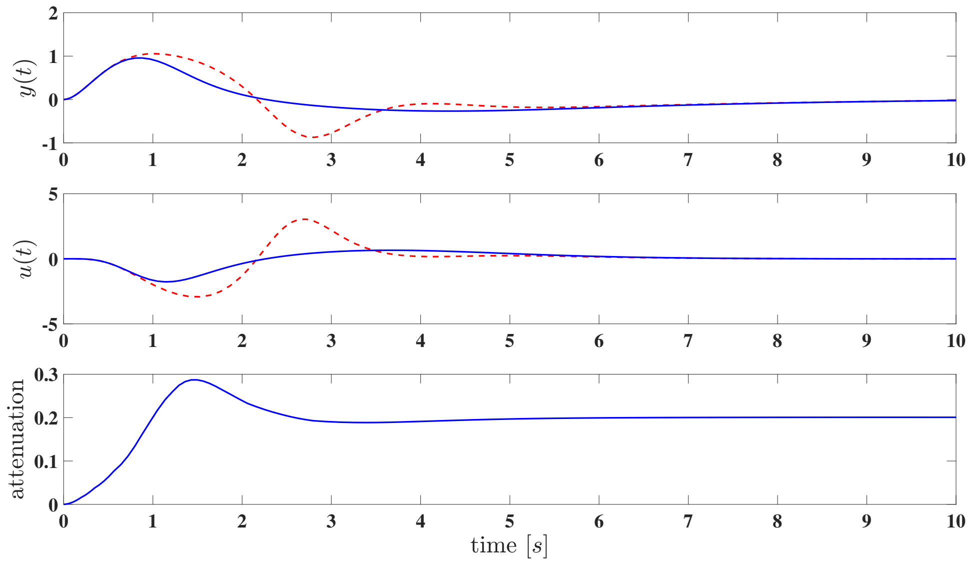

Figure 5.8: From top to bottom, time plot of y, u and disturbance attenuation. Feedback system (solid line) and system at $u = 0$ (dashed line) in Example 10.

The referred work introduces conditions for polynomial nonlinear systems. State equations (5.43) can be recasted [77] by introducing a new state variable $x_3 = \arctan(x_2)$ and it is rewritten as

$$\begin{aligned}
\dot{x}_1 &= -\frac{19}{6}x_1 + \frac{3}{2}x_1x_2^2 - \frac{7}{3}x_2 - \frac{x_2^2}{6(x_2^2+1)} - \frac{1}{3}x_2x_3 + x_2u + w, \\
\dot{x}_2 &= x_1, \\
\dot{x}_3 &= \frac{x_1}{x_2^2+1}, \\
y &= x_1.
\end{aligned} \tag{5.50}$$

The range of the function $\arctan(x_2)$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and it is introduced to the HJI inequality via S-procedure

$$-(x_2^2+1)\left(\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}\{\boldsymbol{\mathcal{F}}(\boldsymbol{x}) + \boldsymbol{\mathcal{G}}(\boldsymbol{x})u + \boldsymbol{\mathcal{K}}(\boldsymbol{x})w\} + y^Ty + u^T\boldsymbol{R}u - \gamma^2 w^Tw\right) \\ -\eta(\boldsymbol{x})\left(\frac{\pi}{4} - x_3^2\right) \in \mathbb{S}[\boldsymbol{x}]. \tag{5.51}$$

Here, $\eta(\boldsymbol{x}) \in \mathbb{S}[\boldsymbol{x}]$ is the S-procedure multiplier. SOS-based policy iteration conditions proposed in [43] tended to $\gamma = 0.7667$ after 4 iterations and the solution is

$$\begin{aligned} V(\boldsymbol{x}) &= 0.4866x_2^4 + 0.0401x_2^3x_3 - 0.000673x_2^2x_3^2 + 0.1282x_2x_3^3 - 0.03205x_3^4 \\ &+ 3.0002x_1^2 + 1.3107x_1x_2 + 5.6267x_2^2 - 0.99728x_2x_3 + 0.49864x_3^2. \end{aligned} \tag{5.52}$$

The comparison of the evolution of the attenuation level γ during the policy iterations algorithms for the above three cases are shown in Figure 5.9.

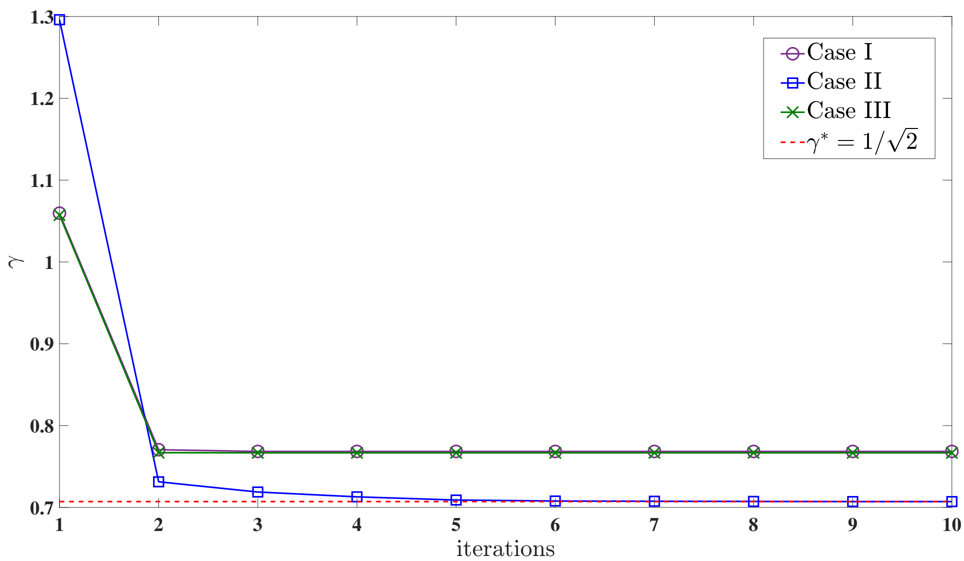

Figure 5.9: Iterations versus value of γ during the SOS-based SPUA for fuzzy system in Example 10.

Example 11. The dynamic behaviour of a second-order nonlinear system are represented by the following polynomial fuzzy system

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{2} h_i(x_1)\{\boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{B}_i(\boldsymbol{x})u + \boldsymbol{E}_i(\boldsymbol{x})w\}, \tag{5.53}$$

with $\boldsymbol{y} = \boldsymbol{x}$, and

$$\boldsymbol{A}_1(\boldsymbol{x}) = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ -1 + \frac{1}{6}x_1^2 - 0.0083x_1^4 & -1 \end{bmatrix}, \quad \boldsymbol{A}_2(\boldsymbol{x}) = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ -1 + \frac{1}{6}x_1^2 & -1 \end{bmatrix},$$

$$\boldsymbol{B}_1(\boldsymbol{x}) = \boldsymbol{B}_2(\boldsymbol{x}) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \quad \boldsymbol{E}_1(\boldsymbol{x}) = \boldsymbol{E}_2(\boldsymbol{x}) = \begin{bmatrix} 0.7 \\ 0 \end{bmatrix}.$$

MFs take the form

$$h_1(x_1) = \begin{cases} \frac{\sin(x_1)-x_1+\frac{1}{6}x_1^3}{0.0083x_1^5}, & \text{if} \quad x_1 \neq 0 \\ 1, & \text{if} \quad x_1 = 0 \end{cases},$$
$$h_2(x_1) = 1 - h_1(x_1).$$

In contrast to previous examples, here an initial controller is unknown and quadratic conditions in Theorem 5.1 are infeasible. Therefore, path following iterative method is applied with $\rho_i(\boldsymbol{x}) = x_1 + x_2$. Solutions

$$\mathcal{V}_k(\boldsymbol{x}) = 706.5819x_1^4 + 1250.6708x_1^2 + 1581.8412x_2^2,$$
$$\hat{u}_k = -1396.2548x_1^4 - 12.7893x_1^2,$$
$$\hat{\gamma} = 1920.7789$$

were found at $\alpha = -0.0992$. With this values as initial setting for the SOS-based SPUA, table below summarizes the iterations required to reached the attenuation level for quartic and hexic polynomials.

Table 5.2: Iterations required to converge to the attenuation level γ for the fuzzy system in Example 11

Degree	Iterations	Minimum value of γ
Quadratic	–	Infeasible
4th degree	12	1.2523
6th degree	6	0.9488

SOS-based SPUA gave the results below for quartic polynomials

$$v_1(\boldsymbol{x}) = 2.0384x_1^4 + 2.9304x_1^2 + 5.1714x_2^2,$$
$$v_2(\boldsymbol{x}) = 2.0385x_1^4 + 2.9307x_1^2 + 5.1714x_2^2,$$

and S-Procedure multiplier

$$\sigma_1(\boldsymbol{x}) = 5.2981 \times 10^{-5}x_1^4 - 0.014793x_1^3x_2 + 2.8857x_1^2x_2^2$$
$$+ 2.7005 \times 10^{-5}x_1^2 - 0.014675x_1x_2 + 2.7381x_2^2,$$

and for hexic polynomials

$$v_1(\boldsymbol{x}) = 0.0012038x_1^6 + 0.40137x_1^4 + 1.3991x_1^3x_2 + 1.2177x_1^2x_2^2 + 0.0047x_2^4 + 1.9188x_1^2 + 2.276x_1x_2 + 2.3755x_2^2,$$

$$v_2(\boldsymbol{x}) = 0.0012047x_1^6 + 0.41197x_1^4 + 1.3991x_1^3x2 + 1.2177x_1^2x_2^2 + 0.0047x_2^4 + 1.9206x_1^2 + 2.276x_1x_2 + 2.3755x_2^2,$$

the S-Procedure multipliers is

$$\sigma_1(\boldsymbol{x}) = 0.3049x_1^6 + 0.4367x_1^5x_2 + 0.4642x_1^4x_2^2 + 0.6122x_1^3x_2^3 + 0.6031x_1^2x_2^4 - 0.5672x_1^4 - 0.6512x_1^3x_2 + 0.4311x_1^2x_2^2 + 0.0786x_1x_2^3 + 0.0432x_2^4 + 0.282x_1^2 + 0.2747x_1x_2 + 0.0693x_2^2.$$

Figure 5.10 depicts the time plot of the states and control input for a null initial condition at $w = 5e^{-t}\sin t$.

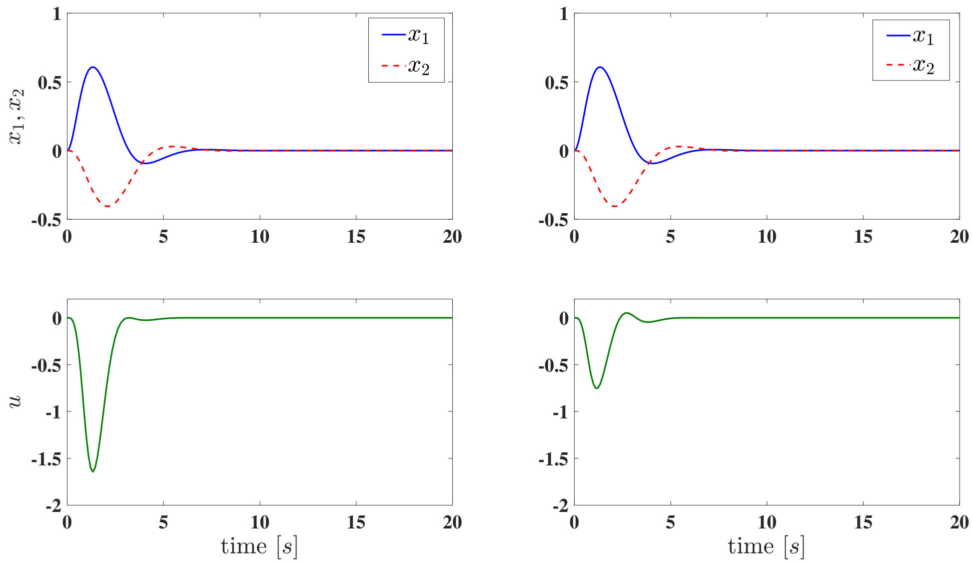

Figure 5.10: Time plot of the states for quartic polynomials (left-top) and hexic polynomials (right-top). Time plot of the control input for quartic polynomials (left-bottom) and hexic polynomials (right-bottom).

5.3 Discussion and Conclusions of the Chapter

This chapter has studied the disturbance attenuation problem in the context of quadratic stabilization and differential games. Example 9 proved that SOS-based SPUA method brings better results than quadratic H_∞ conditions. Convex conditions used the standard quadratic Lyapunov function, that is to say, a single common quadratic form to check stability of the closed-loop polynomial fuzzy system. These conditions are simple yet conservative. The feasible solution from proposed convex criteria has been used as an initial control policy for the SOS-based SPUA method which used the integral-type Lyapunov function as an approximator of the value function.

Furthermore, Examples 10 and 11 have demonstrated that nonconvex conditions and SPUA method via proposed integral-type Lyapunov function can find a feasible solution in cases when quadratic-based conditions have failed. These results concurred with discussion from previous chapters, in which it was concluded that a higher-degree polynomial form can bring more relaxed results than quadratic functions. Last but not least, the employment of a multiple form and the novel relaxation method which includes knowledge on the membership function have contributed to enhance the policy iteration algorithm, which is already known to be an efficient method to obtain a disturbance attenuation control law.

6

Conclusions and Future Work

"There is only one thing that makes a dream impossible to achieve: the fear of failure."

— Paulo Coelho

This dissertation has made used of the integral-type Lyapunov function to study stability, synthesize stabilizing controllers and design disturbance attenuation controllers for a class of nonlinear systems described by model-based fuzzy control systems.

Chapter 3 has introduced a relaxation in the stability condition via the copositivity property. Example 1 has illustrated that by using proposed SOS conditions one can get better or same results as other current integral-type-based conditions [22–25] without the need for adding slack variables that complicate the conditions. In contrast to previous works using the integral-type form, Example 2 has shown that the criterion in this thesis can be used to determine stability of fuzzy systems in polynomial form. For the stabilization problem, the Positivstellensatz refutation was applied to transform a semialgebraic set describing Lyapunov's second method for fuzzy systems into an SOS condition. Example 3 has dealt with a well-known benchmark Takagi-Sugeno fuzzy model whose maximum parameter b is larger than other current quadratic stabilization methods at $a = 2$. Analogously, Example 4 has studied a polynomial fuzzy system and found an stabilizing control at $a = 2$ and $b = 6$, values on which other current criteria [76, 78, 79] had failed to synthesize a control law.

Chapter 4 has presented a general setting of the integral-type Lyapunov function. Rather than leveraging gradients of quadratic functions, the proposal considers gradients of higher even degree polynomial forms. Example 5 studied stability of a four-rule fuzzy system in polynomial form with the generalized Lyapunov function, and it demonstrated the path independence property. Two relaxation techniques are considered in that chapter to improve the stabilization conditions. First of all, Positivstellensatz is employed along the same lines as in Chapter 3. Example 6 has illustrated that increasing the degree of the functions $v_i^{[\lambda]}(\boldsymbol{x})$

is possible to reach a larger value of the parameter b at $a = 2$ for the well-known benchmark example. Secondly, S-procedure has been applied to include polynomial restrictions on the variables replacing the MFs. Both Example 7 and 8 has made the most of the polynomial restrictions to illustrate that better results can be led for the Takagi-Sugeno benchmark problem if knowledge on MFs is added in the stabilization conditions.

Finally, Chapter 5 has faced the disturbance attenuation problem. For the sake of comparison, H_∞ quadratic conditions has been firstly obtained. Then, the study has proposed differential-games-based conditions solved via policy iteration methods. Example 9 has shown that a smaller value of the attenuation level can be reached by means of policy iteration than by means of quadratic stabilization. In Example 10 the results have illustrated that the generalized integral-type Lyapunov function proposed in Chapter 4 has led to the better results in comparison to the standard polynomial Lyapunov function for fuzzy systems and nonlinear (non-fuzzy) techniques. It should be noted that quadratic conditions and those based on the integral-type function introduced in [22] were infeasible. Last example has demonstrated the effectiveness of the proposed path following method and its results have evinced that the higher degree of the polynomials $v_i^{[\lambda]}(\boldsymbol{x})$, the smaller attenuation level can be reached.

According to the results, one can conclude that the proposed integral-type Lyapunov function and relaxation ideas provide an effective alternative to decrease the inherent conservatism of the model-based fuzzy control systems.

6.1 Future Work

To continue in the same line in differential games, the next step will be to consider the local stability and robust control for the H_∞ problem. On occasion, it is not possible to find a global stabilizing control, but at least one wants to design a control law that stabilizes the system in a certain region of the state space [62]. On the other hand, a mathematical model is just an approximation of the real dynamics of a system, and this model is not exempt of uncertainties that can lead to the closed-loop system to become unstable [33].

As an alternative of the H_∞ problem addressed in Chapter 5, one can consider the performance below.

$$\gamma^{-1} \int_0^\infty \boldsymbol{y}^T \boldsymbol{y} dt \leq \gamma \int_0^\infty w^T w dt.$$

This performance is achieved if the inequality

$$\frac{dV(\boldsymbol{x})}{dt} + \gamma^{-1}y^T y - \gamma w^T w \leq 0,$$

holds true for $V(\boldsymbol{x}) > 0$ at $\boldsymbol{x} \neq \mathbf{0}$ and $V(\mathbf{0}) = 0$. For nonlinear system (2.51) one has the following implication

$$\max_{w} \left(L_f V + L_g V u + L_k V w + \gamma^{-1}(\boldsymbol{x})\boldsymbol{\mathcal{M}}^T \boldsymbol{\mathcal{M}}(\boldsymbol{x}) - \gamma w^T w \right) \leq 0.$$

Here $L_\chi V$ denotes the Lie derivative of $V(\boldsymbol{x})$ along the vector field χ. The worst-case disturbance is then given as $w = 0.5\gamma^{-1}[L_k V]^T$. Latter inequality then becomes

$$L_f V + L_g V + \frac{1}{4\gamma} L_k V [L_k V]^T + \gamma^{-1} \boldsymbol{\mathcal{M}}(\boldsymbol{x})^T \boldsymbol{\mathcal{M}}(\boldsymbol{x}) \leq 0.$$

A necessary condition for the existance of a control law solving the problem of disturbance attenuation is if there exist a positive definite function $V(\boldsymbol{x})$ fulfilling the property

$$L_g V = 0 \ \Rightarrow \ L_f V + \frac{1}{4\gamma}[L_k V]^2 + \frac{1}{\gamma} \boldsymbol{\mathcal{M}}(\boldsymbol{x})^T \boldsymbol{\mathcal{M}}(\boldsymbol{x}) \leq 0,$$

Such function receives the name control Lyapunov function (CLF). A feedback law can be constructed in terms of the Lie derivatives of $V(\boldsymbol{x})$ as

$$u(t) = \begin{cases} -\frac{\vartheta(\boldsymbol{x}) + \sqrt{[\vartheta(\boldsymbol{x})]^2 + [L_g V]^4}}{L_g V} & \text{if } L_g V \neq 0, \\ 0 & \text{if } L_g V = 0. \end{cases}$$

with $\vartheta(\boldsymbol{x}) = L_f V + \frac{1}{4\gamma}[L_k V]^2 + \frac{1}{\gamma} h(\boldsymbol{x})^2$. Above equation is a modification of Sontag's formula for the disturbance attenuation problem [27]. In general, Sontag's formula possesses optimal and gain margin properties that are desired in the design of nonlinear control systems. In the context of model-based fuzzy control, this formula has been successfully applied for the stabilization problem [79] and for the finite-time problem [80]. However, to the best of author's knowledge, this formula has not been employed to tackle the disturbance attenuation problem yet.

Finally, the work presented in this thesis has given a solution for fuzzy systems of the disturbance attenuation problem via two-player zero-sum game and policy iteration. With regard to the optimal control problem via the solution of the Hamilton-Jacobi-Bellman equation, policy iteration algorithm has been implemented with success as an SOS optimization problem. Consequently, the logical next step will be to study the multiplayer non-zero-sum game, which has been only addressed in the framework of neural networks and machine learning [81, 82].

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List of Publications

Journal paper:

1. **Jairo Moreno Saenz**, Motoyasu Tanaka, and Kazuo Tanaka, "Relaxed Stabilization and Disturbance Attenuation Control Synthesis Conditions for Polynomial Fuzzy Systems," in *IEEE Transactions on Cybernetics*, **DOI**: 10.1109/TCYB.2019.2957154. Accepted, *Early Access*, (Impact Factor: 10.387). (Related to the contents of Chapters 4 and 5). Available at IEEE Xplore https://ieeexplore.ieee.org/document/8941079

International conference papers:

1. **Jairo Moreno Saenz**, Motoyasu Tanaka, and Kazuo Tanaka. "Stability analysis for polynomial fuzzy systems based on line-integral fuzzy Lyapunov function: A copositive relaxation approach," in *2017 Joint 17th World Congress of International Fuzzy Systems Association and 9th International Conference on Soft Computing and Intelligent Systems , SS-04-1-3*, Otsu, Japan, 2017, Jun. 27-30. (Related to the contents of Chapter 3) **DOI**: 10.1109/IFSA-SCIS.2017.8023262. Available at IEEE Xplore https://ieeexplore.ieee.org/document/8023262

2. **Jairo Moreno Saenz**, Motoyasu Tanaka, and Kazuo Tanaka. "Control synthesis for polynomial fuzzy systems using line-integral polynomial fuzzy Lyapunov function," in *2018 IEEE International Conference on Systems, Man, and Cybernetics, pp. 2925-2930*, Miyazaki, Japan, 2018, Oct. 7-10. (Related to the contents of Chapter 4) **DOI**: 10.1109/SMC.2018.00498. Available at IEEE Xplore https://ieeexplore.ieee.org/document/8616494

Other publication:

1. **Jairo Moreno Saenz,** Motoyasu Tanaka, and Kazuo Tanaka. "Stabilization conditions for a class of nonlinear systems via line-integral fuzzy Lyapunov function," in 第60回自動制御連合講演会, *SaB1-5*, Tokyo, Japan, 2017, Nov. 10-12. (Related to the contents of Chapter 3). **DOI**: 10.11511/jacc.60.0_376