8-1-2020

Graph edge coloring and a new approach to the overfull conjecture

Yan Cao

Georgia State University

GRAPH EDGE COLORING AND A NEW APPROACH TO THE OVERFULL CONJECTURE

by

YAN CAO

Under the Direction of Guantao Chen, PhD

ABSTRACT

The graph edge coloring problem is to color the edges of a graph such that adjacent edges receives different colors. Let G be a simple graph with maximum degree Δ. The minimum number of colors needed for such a coloring of G is called the chromatic index of G, written $\chi'(G)$. We say G is of class one if $\chi'(G) = \Delta$, otherwise it is of class 2. A majority of edge coloring papers is devoted to the Classification Problem for simple graphs. A graph G is said to be *overfull* if $|E(G)| > \Delta\lfloor |V(G)|/2 \rfloor$. Hilton in 1985 conjectured that every graph G of class two with $\Delta(G) > \frac{|V(G)|}{3}$ contains an overfull subgraph H with $\Delta(H) = \Delta(G)$. In this thesis, I will introduce some of my researches toward the Classification Problem of simple graphs, and a new approach to the overfull conjecture together with some new techniques and ideas.

INDEX WORDS: Critical graphs, Overfull graphs, Multifan, Lollipop. Elementary set.

GRAPH EDGE COLORING AND A NEW APPROACH TO THE OVERFULL CONJECTURE

by

YAN CAO

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

in the College of Arts and Sciences

Georgia State University

2020

GRAPH EDGE COLORING AND A NEW APPROACH TO THE OVERFULL CONJECTURE

by

YAN CAO

Committee Chair: Guantao Chen

Committee: Zhongshan Li

Songling Shan

Hendricus Van der Holst

Yi Zhao

Electronic Version Approved:

Office of Graduate Studies

College of Arts and Sciences

Georgia State University

August 2020

ACKNOWLEDGEMENTS

The completion of this research could not have been possible without the assistance of so many people whose names may not all be enumerated. Nonetheless, I would like to express my special appreciation to the following:

Dr. Guantao Chen, my advisor, for the support, guidance, valuable comments that benefited me much not only in the completion of this study but also in my life.

Drs. Zhongshan Li, Songling Shan, Hendricus Van der Holst and Yi Zhao, for serving on my dissertation committee and for their time and effort in checking this manuscript.

My parents, Mr. Qingguang Cao and Mrs. Fengli Li, for their love, caring, and support in every aspect throughout my graduate program.

Shushan He and Dr. Guangming Jing, for their support and caring as friends.

At last, I would like to thank everyone who helped me in Atlanta.

TABLE OF CONTENTS

LIST OF FIGURES

0 INTRODUCTION

Graph edge coloring is a well established subject in the field of graph theory, it is one of the basic combinatorial optimization problems. The edge coloring problem is to color the edges of a graph G such that each edge receive a color and adjacent edges, that is, different edges incident to a common vertex, receive different colors. The minimum number of colors needed for such a coloring of G is called the chromatic index of G, written $\chi'(G)$. By Vizing-Gupta's theorem, for a simple graph G, we have $\chi'(G) = \Delta$ or $\Delta + 1$. A simple graph G is of class 1 if $\chi'(G) = \Delta$, otherwise it is of class 2. By a result of Holyer [15], the determination of the chromatic index is an NP-hard optimization problem.

A graph G is said to be *critical* if any of its proper subgraph has chromatic index smaller than G. In 1968, Vizing [24] made a number of conjectures regarding structural properties of critical class two graphs, including Vizing's average degree conjecture, Vizing's independence conjecture, Vizing's 2-factor conjecture, etc. The main motivation for Vizing to study critical class two graphs is the Classification Problem for simple graphs and the fact that any class two graph contains a critical class two graph with the same maximum degree as a subgraph. Among all Vizing's conjectures on critical class two graphs, the average degree conjecture is without doubt the most important one. In the first part of this thesis, I will present a proof towards Vizing's average degree conjecture for critical class two graphs, showing that this conjecture is true asymptotically.

Let G be a simple graph. If G has an induced subgraph H satisfying $|E(H)| > \Delta\lfloor\frac{|V(H)|}{2}\rfloor$; such an induced subgraph of G is called an *overfull subgraph* of G. We say that G is an *overfull graph* if G is an overfull subgraph of itself. Note that any overfull subgraph of G has odd order and the same maximum degree as G. Hence if G has an overfull subgraph, then G is class two. However, the converse statement is not true since the existence of a counterexample P^* obtained from Peterson graph P by deleting one vertex. In 1986, Chetwynd and Hilton proposed the following well known overfull conjecture. For a simple graph G, if $\Delta(G) > \frac{1}{3}|V(G)|$, then G is Class 2 implies that G contains an overfull subgraph H

with $\Delta(H) = \Delta(G)$. Applying Edmonds' matching polytope theorem, Seymour [19] showed that whether a graph containing an overfull subgraph can be determined in polynomial time. A number of outstanding conjectures listed in *Twenty Pretty Edge Coloring Conjectures* in [21] lie in deciding when a critical class two graph is overfull. In the second part of this thesis, I will present a proof of vertex-splitting conjecture, which is a weakening conjecture of overfull conjecture.

There is only small progress towards overfull conjecture itself, and all of these known results are far away from solving it. Up to now, there have been mainly two ways to approach the conjecture. That is, adding minimum degree conditions or raising maximum degree conditions much higher than $|V(G)|/3$. In the last part of this thesis, I will present some of my attempts on attacking the overfull conjecture through a new approach. That is, using the properties of critical class two graphs and the connection between critical graphs and overfullness. Although the problem I proposed in the last section is not completed solved in this thesis, I believe that the new techniques and ideas developed in that section could be very useful on attacking the overfull conjecture.

1 VIZING'S AVERAGE DEGREE CONJECTURE

1.1 Introduction

In this thesis, we consider simple graphs (finite graphs without loops or parallel edges) only. Let G be a graph. As usual, we denote by $V(G)$ and $E(G)$ the *vertex set* and the *edge set* of G, respectively. For a vertex $v \in V(G)$, let $N(v) = \{w \in V(G) : vw \in E(G)\}$ be the *neighborhood* of v in G and let $d(v) = |N(v)|$ be the *degree* of v in G. Let $N[v] = N(v) \cup \{v\}$. For a vertex set U, let $N(U) = \bigcup_{u \in U} N(u)$. Denote by $\Delta(G)$ and $\delta(G)$ the maximum degree and minimum degree of G, respectively. An *edge coloring* of a graph G is a coloring of the edges of G such that each edge receives a color and adjacent edges, that is, distinct edges having a common end, receive different colors. The minimum number of colors in such an edge coloring of G is called the *chromatic index* of G, written $\chi'(G)$. For a positive integer k, let $\mathcal{C}^k(G)$ denote the set of all edge colorings of G with color set $[k]$, where $[k] = \{1, 2, \ldots, k\}$. Clearly $\chi'(G)$ is the smallest k such that $\mathcal{C}^k(G) \neq \emptyset$. For a vertex $v \in V(G)$ and a coloring $\varphi \in \mathcal{C}^k(G)$, let $\varphi(v) = \{\varphi(vw) : w \in N(v)\}$ and $\bar{\varphi}(v) = [k] \backslash \varphi(v)$. We call $\varphi(v)$ the set of *colors present* at v and $\bar{\varphi}(v)$ the set of *colors missing* at v with respect to φ. For a vertex set U, let $\bar{\varphi}(U) = \bigcup_{u \in U} \bar{\varphi}(u)$. A graph G is Δ-*critical* if $\Delta(G) = \Delta$, $\chi'(G) = \Delta(G) + 1$ and $\chi'(H) < \chi'(G)$ for every proper subgraph H of G. Let $\overline{d}(G)$ denote the average degree of G, i.e., $\overline{d}(G) = 2|E(G)|/|V(G)|$. In 1968, Vizing [24] made a number of conjectures regarding structural properties of Δ-critical graphs, including the following two.

Conjecture 1.1. *If G is a Δ-critical graph of order n, then*

- $\overline{d}(G) \geq \Delta - 1 + n/3$ (Vizing's Average Degree Conjecture)*, and*
- $\alpha(G) \leq n/2$*, where $\alpha(G)$ is the independence number of G* (Vizing's Independence Number Conjecture)*.*

For a brief survey, we refer to the paper [2] by Cao and Chen. In that paper, we prove that $\overline{d}(G) \geq \frac{3}{4}\Delta - 8$ for all Δ-critical graphs. Besides this exact result, we also prove the following approximate result:

Theorem 1.2. [[2], Theorem 2] *There exist two functions D_0 and D_1 from $(0, 1)$ to $\mathbb{R}$ such that for any positive real number $\epsilon \in (0, 1)$, if G is a Δ-critical graph with $\Delta \geq D_1(\epsilon)$ and $\delta(G) \geq D_0(\epsilon)$, then $\overline{d}(G) \geq (1 - \epsilon)\Delta$.*

In the study of graph edge coloring for simple graphs, we have Vizing's Adjacency Lemma (VAL, Lemma 1.10), the short Kierstead path (Lemma 1.11) and their refinements. VAL investigates the neighbors around a vertex, while the short Kierstead path investigates a path of length three. The largest distance between two vertices in their structures is up to three. In this paper, we prove the following technique theorem and enlarge the distance from three to four, which helps us eliminate the minimum degree condition in Theorem 1.2.

For any $v \in V(G)$, let $L(v) = N(N(v)) \backslash N[v]$, i.e. the set of vertices at distance 2 from v. For an edge $xy \in E(G)$, let φ be a coloring in $\mathcal{C}^\Delta(G - xy)$. We denote a vertex in $N(y)$ by y_α if the edge between y and this vertex is colored by α under φ.

Theorem 1.3. *Let $d \geq 2$ be an integer, G be a Δ-critical graph, $xy \in E(G)$ and $\varphi \in \mathcal{C}^{\Delta}(G-xy)$. If $\Delta \geq (d+1)(d+3)$ and $d(x) \leq d$, then for every $\alpha \in \bar{\varphi}(x)$, $d(y_\alpha) \geq \Delta - d + 2$ and there are at most $2d^2 - d - 2$ vertices in $N(y_\alpha) \cup L(y_\alpha)$ with degree at most d.*

The proof of Theorem 1.3 is divided into two parts, an extremal result and a structural result. The latter result is the core of the proof. Their statements and proofs will be given in subsection 1.5 after the proof of the main theorem below, which is given in subsection 1.4.

Theorem 1.4. *There exists a function D from $(0,1)$ to $\mathbb{R}$ such that for any positive real number $\epsilon \in (0,1)$, if G is a Δ-critical graph with $\Delta \geq D(\epsilon)$, then $\overline{d}(G) \geq (1-\epsilon)\Delta$. Moreover, we may assume $D(\epsilon) \leq (\frac{16}{\epsilon^3})^{8/\epsilon}$ when $\epsilon < \frac{2}{3}$.*

The "moreover" part will be shown in subsection 1.2 for a specific function $D(\epsilon)$. Theorem 1.4 shows that the Vizing's average degree conjecture is asymptotically true. As a consequence, we immediately obtain the following corollary which implies that the Vizing's independence conjecture is also asymptotically true.

Theorem 1.5. *There exists a function D' from $(0,1)$ to $\mathbb{R}$ such that for any positive real number $\epsilon \in (0,1)$, if G is a Δ-critical graph of order n with $\Delta \geq D'(\epsilon)$, then $\alpha(G) \leq (\frac{1}{2}+\epsilon)n$.*

Proof. Let $D'(\epsilon) = D(2\epsilon)$. Assume on the contrary that $\alpha(G) > (\frac{1}{2}+\epsilon)n$. Then let X be an independent vertex set of G with size greater than $(\frac{1}{2}+\epsilon)n$. Let $Y = V(G)\backslash X$. So $|Y| < (\frac{1}{2}-\epsilon)n$. Since $\sum_{x \in X} d(x) = |E(X,Y)|$ (where $E(X,Y)$ is the set of edges with one end in X and the other end in Y), and $|E(X,Y)| \leq \sum_{y \in Y} d(y) < (\frac{1}{2}-\epsilon)n\Delta$, we have $\sum_{x\in X} d(x) + \sum_{y \in Y} d(y) < (1-2\epsilon)n\Delta$. But by Theorem 1.4, $\sum_{x\in X} d(x) + \sum_{y\in Y} d(y) \geq (1-2\epsilon)n\Delta$ when $\Delta(G) \geq D'(\epsilon) = D(2\epsilon)$, giving a contradiction. □

1.2 Basics and calculations.

In [2], it was shown that Theorem 1.2 is true with the following two functions:

$$D_0(\epsilon) = \begin{cases} \left\lceil 2(\frac{1}{3\epsilon})^3 + 2\sqrt{(\frac{1}{3\epsilon})^6 - (\frac{1}{3\epsilon})^3}\right\rceil & \text{if } \epsilon < \frac{1}{3}, \\ 2 & \text{if } \epsilon \geq \frac{1}{3}. \end{cases}$$

$$D_1(\epsilon) = \max\left\{f(\epsilon), \frac{3c_0+1}{\rho^2}, \frac{N+c_0}{\epsilon^3}\right\}.$$

Here $c_0 = c_0(\epsilon) = \lceil \frac{1-\epsilon}{\epsilon} \rceil$, $\rho = \rho(\epsilon) = \frac{\epsilon^3}{\epsilon^2+1}$, $N = N(\epsilon) = (c_0+1)(\frac{1}{\rho}+1)^{3c_0+1}$, and $f(\epsilon) = \frac{31}{\epsilon}$ if $\epsilon > \frac{30}{31}$ and $f(\epsilon) = \frac{1}{\epsilon^2}(3c_0^4 + 12c_0^3 + 10c_0^2 + 4c_0 + 1)$ otherwise. In fact the definition of $D_0(\epsilon)$ in [2] does not include the ceiling brackets, but it makes no difference to Theorem 1.2 since $\delta(G)$ is an integer.

Theorem 1.2 was proved by a typical discharging method with some adjacency lemmas. The discharging rule is the following:

Let q be a positive real number less than $\Delta(G)$. We initially assign to each vertex x of G a charge $M(x) = d(x)$ and redistribute the charge according to the following discharging rule:

- **R1:** Each vertex y with degree larger than q distributes $d(y) - q$ equally among all neighbors of y with degree less than q.

Denote by $M'(x)$ the resulting charge on each vertex x.

In the proof of Theorem 1.2 in [2], we proved Claim 3.1 and Claim 3.2 which together imply the following stronger result.

Theorem 1.6. [2] *For any $\epsilon \in (0,1)$, let $D_1(\epsilon)$ and $D_0(\epsilon)$ be defined as the above. If G is a Δ-critial graph with $\Delta \geq D_1(\epsilon)$ and $q = (1-\epsilon)\Delta$, then $M'(x) \geq q$ for any $x \in V(G)$ with $d(x) \geq D_0(\epsilon)$.*

Theorem 1.6 implies Theorem 1.2 immediately. We will use this stronger result and Theorem 1.3 to prove Theorem 1.4 in subsection 1.4. More specifically, we will show that Theorem 1.4 is true for $D(\epsilon) = D_1(\frac{\epsilon}{2})$. We first prove the following result which confirms the "moreover" part of Theorem 1.4.

Lemma 1.7. *If $\epsilon < \frac{2}{3}$, then $D(\epsilon) = D_1(\frac{\epsilon}{2}) < (\frac{16}{\epsilon^3})^{8/\epsilon}$*

Proof. Let $\epsilon_1 = \epsilon/2$. In the proof of this lemma, let $c_0 = c_0(\epsilon_1)$, $\rho = \rho(\epsilon_1)$ and $N = N(\epsilon_1)$.

Since $\epsilon < \frac{2}{3}$, we have that $\epsilon_1 < \frac{1}{3}$ and the following inequalities.

$$3 \le \left\lceil \frac{1}{\epsilon_1} \right\rceil - 1 = c_0 = \left\lceil \frac{1-\epsilon_1}{\epsilon_1} \right\rceil < \frac{1}{\epsilon_1} \quad \text{and} \quad \frac{1}{\epsilon_1^3} < \frac{1}{\rho} = \frac{\epsilon_1^2+1}{\epsilon_1^3} < \frac{\frac{1}{9}+1}{\epsilon_1^3} < \frac{\sqrt{2}}{\epsilon_1^3}.$$

Thus $\frac{N+c_0}{\epsilon_1^3} > N = (c_0+1)(\frac{1}{\rho}+1)^{3c_0+1} > \frac{1}{\epsilon_1} \cdot (\frac{1}{\epsilon_1^3})^{10} > \frac{3^{24}}{\epsilon_1^7}$, while $f(\epsilon_1) \le \max\{\frac{30c_0^4}{\epsilon_1^2}, \frac{31}{\epsilon_1}\} < \frac{31}{\epsilon_1^7}$ and $\frac{3c_0+1}{\rho^2} < \frac{4c_0}{\rho^2} < \frac{8}{\epsilon_1^7}$. Hence, $D_1(\epsilon_1) = \max\left\{f(\epsilon_1), \frac{3c_0+1}{\rho^2}, \frac{N+c_0}{\epsilon_1^3}\right\} = \frac{N+c_0}{\epsilon_1^3}$.

Note that $3 \le \left\lceil \frac{1-\epsilon_1}{\epsilon_1} \right\rceil = c_0 < \frac{1}{\epsilon_1} \le c_0 + 1$ and $c_0 < \frac{1}{\rho} = \frac{1}{\epsilon_1} + \frac{1}{{\epsilon_1}^3} < \frac{2}{{\epsilon_1}^3} - 2$, since $2{\epsilon_1}^3 + {\epsilon_1}^2 < 1$. Thus

$$N + c_0 = (c_0+1)(\frac{1}{\rho}+1)^{3c_0+1} + c_0 < (c_0+1)(\frac{1}{\rho}+2)^{3c_0+1} < (\frac{2}{{\epsilon_1}^3})^{3c_0+2}$$

and so

$$\frac{N+c_0}{{\epsilon_1}^3} < (\frac{2}{{\epsilon_1}^3})^{3c_0+3} < (\frac{2}{{\epsilon_1}^3})^{\frac{4}{\epsilon_1}}.$$

Thus $D(\epsilon) = D_1(\epsilon_1) \le (\frac{16}{\epsilon^3})^{\frac{8}{\epsilon}}$. □

Let $C(\epsilon) = 2D_0(\epsilon)^2 - D_0(\epsilon) - 2$. The following result will be used in the proof of Theorem 1.4.

Lemma 1.8. *$D(\epsilon) > \epsilon D(\epsilon) > 2^{18} D_0(\frac{\epsilon}{2})^2$. In particular, $D(\epsilon) > \frac{2}{\epsilon} C(\frac{\epsilon}{2})$.*

Proof. Let $\epsilon_1 = \epsilon/2 < \frac{1}{2}$, so that $\frac{1}{\epsilon_1} > 2$. If $\epsilon_1 < \frac{1}{3}$ then $2 < D_0(\epsilon_1) \le \left\lceil 4(\frac{1}{3\epsilon_1})^3 \right\rceil < \frac{1}{\epsilon_1^3}$, else $D_0(\epsilon_1) = 2 < \frac{1}{\epsilon_1^3}$. Also with the conventions of the previous proof, $c_0 \ge 2$, $\frac{1}{\rho} = \frac{\epsilon_1^2+1}{\epsilon_1^3} > \frac{1}{\epsilon^3}$ and $D(\epsilon) = D_1(\epsilon_1) > \frac{N}{\epsilon_1^3} > \frac{1}{\epsilon_1^3}(\frac{1}{\rho}+1)^{3c_0+1} > \frac{1}{\epsilon_1^3}(\frac{1}{\rho})^7 > \frac{1}{\epsilon_1^{24}} > \frac{D_0(\epsilon_1)^2}{\epsilon_1^{18}} > 2^{17} \cdot \frac{D_0(\epsilon_1)^2}{\epsilon_1}$. The required inequalities easily follow. □

1.3 Coloring preliminaries and adjacency lemmas

In this subsection, we always assume that G is a Δ-critical graph, $xy \in E(G)$ and $\varphi \in \mathcal{C}^\Delta(G - xy)$. We will re-state these assumptions in each lemma for their completeness.

A set $X \subseteq V(G)$ is called *elementary* with respect to φ if the sets $\bar{\varphi}(v)$ with $v \in X$ are pairwise disjoint. For a color $\alpha \in [\Delta]$, let E_α denote the set of edges e of G with $\varphi(e) = \alpha$

and call it a *color class* with respect to φ. Clearly, E_α is a matching of G. So if α and β are two colors, then the spanning subgraph H of G with edge set $E_\alpha \cup E_\beta$ has maximum degree at most 2, so every component of H is either a path or an even cycle (whose edges are colored alternately with α and β) and we refer to such a component as an (α, β)*-chain* of G with respect to φ. For a vertex v of G, let $P_v(\alpha, \beta, \varphi)$ denote the unique component of H that contains the vertex v. Let $\varphi' = \varphi / P_v(\alpha, \beta, \varphi)$ denote the mapping obtained from φ by switching the colors α and β on the edges of $P_v(\alpha, \beta, \varphi)$. Then, clearly, $\varphi' \in \mathcal{C}^\Delta(G - xy)$ is an edge coloring of $G - xy$ with color set $[\Delta]$, too. This switching operation is called a *Kempe change.*

A *multi-fan* at x with respect to edge $e = xy \in E(G)$ and coloring $\varphi \in \mathcal{C}^\Delta(G - e)$ is a sequence $F = (x, e_1, y_1, \ldots, e_p, y_p)$ with $p \geq 1$ consisting of edges $e_1, e_2, \ldots, e_p$ and vertices $x, y_1, y_2, \ldots, y_p$ satisfying the following two conditions:

- The edges $e_1, e_2, \ldots, e_p$ are distinct, $e_1 = e$ and $e_i = xy_i$ for $i = 1, \ldots, p$.
- For every edge e_i with $2 \leq i \leq p$, there is a vertex y_j with $1 \leq j < i$ such that $\varphi(e_i) \in \bar{\varphi}(y_j)$.

Notice that multi-fan is slightly more general than Vizing-fan which requires $j = i - 1$ in the second condition. The following lemma shows that a multi-fan is elementary. The proof can be found in the book [21].

Lemma 1.9. [Stiebitz, Scheide, Toft and Favrholdt [21]] *Let $F = (x, e_1, y_1, \ldots, e_p, y_p)$ be a multi-fan at x with respect to e and φ. Then the following statements hold:*

(a) $\{x, y_1, y_2, \ldots, y_p\}$ is elementary.

(b) If $\alpha \in \bar{\varphi}(x)$ and $\beta \in \bar{\varphi}(y_i)$ for some i, then $P_x(\alpha, \beta, \varphi) = P_{y_i}(\alpha, \beta, \varphi)$.

For a vertex $x \in V(G)$ and a given positive number q, let $\sigma_q(x, y) = |\{z \in N(y) \setminus \{x\} : d(z) \geq q\}|$.

Lemma 1.10. [Vizing's Adjacency Lemma [22]] *If G is a Δ-critical graph, then $\sigma_\Delta(x, y) \geq \Delta - d(x) + 1$ for every $xy \in E(G)$.*

A *Kierstead path* with respect to $e = xy$ and $\varphi \in \mathcal{C}^\Delta(G - e)$ is a sequence $K = (y_0, e_1, y_1, \dots, e_p, y_p)$ with $p \geq 1$ consisting of edges $e_1, e_2, \dots, e_p$ and vertices $y_0, y_1, \dots, y_p$ satisfying the following two conditions:

- The vertices $y_0, y_1, \dots, y_p$ are distinct, $e_1 = e$ and $e_i = y_i y_{i-1}$ for $1 \leq i \leq p$.
- For every edge e_i with $2 \leq i \leq p$, there is a vertex y_j with $0 \leq j < i$ such that $\varphi(e_i) \in \bar{\varphi}(y_j)$.

Clearly a Kierstead path with 3 vertices is a multi-fan with center y_1. For a Kierstead path with 4 vertices, some elementary properties were shown below.

Lemma 1.11. [Kostochka and Stiebitz [21], Luo and Zhao [17]] *If $K = (y_0, e_1, y_1, e_2, y_2, e_3, y_3)$ is a Kierstead path with respect to e_1 and φ, then $V(K)$ is elementary unless $d(y_1) = d(y_2) = \Delta(G)$, in which case, all colors in $\bar{\varphi}(y_0), \bar{\varphi}(y_1), \bar{\varphi}(y_2)$ and $\bar{\varphi}(y_3)$ are distinct except one possible common missing color in $\bar{\varphi}(y_3) \cap (\bar{\varphi}(y_0) \cup \bar{\varphi}(y_1))$.*

A *short broom* with respect to xy and $\varphi \in \mathcal{C}^\Delta(G - xy)$ is a sequence $B = (y_0, e_1, y_1, \dots, e_p, y_p)$ with $p \geq 3$ such that for all $i \geq 3$, $e_i = y_2 y_i$ and $(y_0, e_1, y_1, e_2, y_2, e_i, y_i)$ is a Kierstead path with respect to xy and φ.

Lemma 1.12. [Chen, Chen and Zhao [6]] *Let $B = \{y_0, e_1, y_1, e_2, y_2, \dots, e_p, y_p\}$ be a short broom with respect to xy and φ. If $|\bar{\varphi}(y_0) \cup \bar{\varphi}(y_1)| \geq 4$ and $\min\{d(y_1), d(y_2)\} < \Delta$, then $V(B)$ is elementary under φ.*

Recently, the lemma above was extended to general brooms by Cao, Chen, Jing, Stiebitz and Toft. Since the stronger result is not used in this paper, we refer the reader to the survey paper [4].

1.4 Proof of Theorem 1.4

In this subsection, we prove our main result, Theorem 1.4, by assuming the truth of Theorem 1.3. Since the "moreover" part was proved by Lemma 1.7, we only need to prove the following:

Theorem 1.4*. *Let $D(\epsilon) = D_1(\frac{\epsilon}{2})$. If G is a Δ-critical graph with $\Delta \geq D(\epsilon)$, then $\overline{d}(G) \geq (1-\epsilon)\Delta$.*

Proof. Recall that $D(\epsilon) = D_1(\frac{\epsilon}{2}) > \frac{2}{\epsilon}C(\frac{\epsilon}{2})$ where $C(\epsilon) = 2D_0(\epsilon)^2 - D_0(\epsilon) - 2$ and $D_1(\epsilon), D_0(\epsilon)$ are defined in subsection 1.2.

Let $q = (1 - \frac{\epsilon}{2})\Delta$. We initially assign to each vertex x of G a charge $M(x) = d(x)$ and redistribute the charge according to the following steps:

- **R1:** Each vertex y with degree larger than q distributes $d(y) - q$ equally among all neighbors of y with degree less than q. Denote by $M'(x)$ the resulting charge on each vertex x.

- **R2:** After **R1**, for a vertex v with $d(v) \geq D_0(\frac{\epsilon}{2})$, if $N(v) \cup L(v)$ contains at most $C(\frac{\epsilon}{2})$ vertices with degree less than $D_0(\frac{\epsilon}{2})$, then v distributes 1 to each vertex in $L(v)$ with degree less than $D_0(\frac{\epsilon}{2})$. Denote by $M''(x)$ the resulting charge on each vertex x.

We will show that $M''(x) \geq (1-\epsilon)\Delta$ for each $x \in V(G)$, which implies the theorem.

By Theorem 1.6, we have $M'(x) \geq q$ for each vertex x with degree at least $D_0(\frac{\epsilon}{2})$. Hence after **R1**, only the vertices with degree less than $D_0(\frac{\epsilon}{2})$ may have charge less than $q = (1 - \frac{\epsilon}{2})\Delta$.

Claim 1.1. *For every $x \in V(G)$ with degree at least $D_0(\frac{\epsilon}{2})$, $M''(x) \geq (1-\epsilon)\Delta$.*

Proof. If $N(v) \cup L(v)$ contains at most $C(\frac{\epsilon}{2})$ vertices with degree less than $D_0(\frac{\epsilon}{2})$, then since $\Delta \geq D(\epsilon) \geq \frac{2}{\epsilon}C(\frac{\epsilon}{2})$, after **R2**, x still has charge at least $q - C(\frac{\epsilon}{2}) \geq (1-\epsilon)\Delta + \frac{\epsilon}{2}\Delta - C(\frac{\epsilon}{2}) \geq (1-\epsilon)\Delta$. If $N(v) \cup L(v)$ contains more than $C(\frac{\epsilon}{2})$ may vertices with degree less than $D_0(\frac{\epsilon}{2})$, then it does not send any charge to other vertices in **R2**, $M''(x) \geq q > (1-\epsilon)\Delta$. □

Claim 1.2. *For every $x \in V(G)$ with degree less than $D_0(\frac{\epsilon}{2})$, $M''(x) \geq (1-\epsilon)\Delta$.*

Proof. Pick a neighbor y of x, let $\varphi \in \mathcal{C}^\Delta(G - xy)$. By Lemma 1.8, we have $\Delta \geq D(\epsilon) \geq (D_0(\frac{\epsilon}{2}) + 1)(D_0(\frac{\epsilon}{2}) + 3)$. Then by Theorem 1.3, for any missing color α of x, y_α has degree at least $\Delta - D_0(\frac{\epsilon}{2}) + 2 > D_0(\frac{\epsilon}{2})$ and $N(y_\alpha) \cup L(y_\alpha)$ contains at most $C(\frac{\epsilon}{2})$ vertices with degree less than $D_0(\frac{\epsilon}{2})$. (y_α exists by Lemma 1.9.) So y_α distributes 1 to x. Since there are at least $\Delta - D_0(\frac{\epsilon}{2}) + 1$ colors missing at x, we have $M''(x) \geq M'(x) + (\Delta - D_0(\frac{\epsilon}{2}) + 1) > \Delta - D_0(\frac{\epsilon}{2}) + 1 > (1-\epsilon)\Delta$. □

Combining **Claim 1.1** and **Claim 1.2**, we proved Theorem 1.4*. □

1.5 Proof of Theorem 1.3

1.5.1 General setting

Let $d \geq 2$ be an integer, G be a Δ-critical graph, $xy \in E(G)$ with $d(x) \leq d$, $\varphi \in \mathcal{C}^\Delta(G - xy)$ and $\alpha \in \bar{\varphi}(x)$. Notice that $\{x, y, y_\alpha\}$ is a multi-fan at y with respect to xy and φ. Thus by Lemma 1.9, we have $d(y_\alpha) \geq |\bar{\varphi}(x)| + |\bar{\varphi}(y)| \geq \Delta - d + 2$. This proves the first conclusion of Theorem 1.3.

Let $A(\varphi) = \bar{\varphi}(x) \cup \bar{\varphi}(y)$ and $B(\varphi) = [\Delta] \backslash A(\varphi) = \varphi(x) \backslash \bar{\varphi}(y)$. We will use A and B instead of $A(\varphi)$ and $B(\varphi)$ when the coloring φ is clear. Easy to see that $|A| + |B| = \Delta$ and $|B| \leq d(x) - 2 \leq d - 2$. Let $N_A := N_A(y_\alpha) = \{v \in N(y_\alpha) \backslash \{y\} \ : \ \varphi(y_\alpha v) \in A\}$ and $N_B := N_B(y_\alpha) = N(y_\alpha) \backslash N_A = \{v \in N(y_\alpha) \ : \ \varphi(y_\alpha v) \in B\}$. Let $L_A = N(N_A) \backslash (N[y_\alpha] \cup \{x\})$ and $L_B = N(N_B) \backslash (N[y_\alpha] \cup \{x\})$. We say a vertex is *d-small* if its degree is at most d. Let T be the set of all *d-small* vertices in $V(G)$. Now in order to prove Theorem 1.3, we need to show that if $\Delta \geq (d+1)(d+3)$ then $|(N(y_\alpha) \cup L(y_\alpha)) \cap T| \leq 2d^2 - d - 2 = (d-1)^2 + (d+1)(d-1) + (d-2)$.

Definition 1. *Let G be a Δ-critical graph, $xy \in E(G)$, $\varphi \in \mathcal{C}^\Delta(G - xy)$ and $\alpha \in \bar{\varphi}(x)$. We call a seven vertex set $\{x, y, y_\alpha, z_1, z_2, t_1, t_2\}$ a φ-**fork** on xyy_α if it satisfies* (1) *for each $i \in \{1, 2\}$, $z_i \in N_A$, $t_i \in N(z_i)$; and* (2) *$\varphi(z_i t_i) \in A \cap \bar{\varphi}(t_{3-i})$. We call t_1, t_2 the two* ends *of the fork.*

To complete the proof of Theorem 1.3, we will prove the following two propositions instead. They contradict each other if we suppose on the contrary of Theorem 1.3.

Proposition A. *Let $d \geq 2$ be an integer, G be a Δ-critical graph, $xy \in E(G)$, $\varphi \in \mathcal{C}^\Delta(G - xy)$ and $\alpha \in \bar{\varphi}(x)$. If $x \in T$, $\Delta \geq (d+1)(d+3)$ and $|(N(y_\alpha) \cup L(y_\alpha)) \cap T| > (d-1)^2 + (d+1)(d-$*

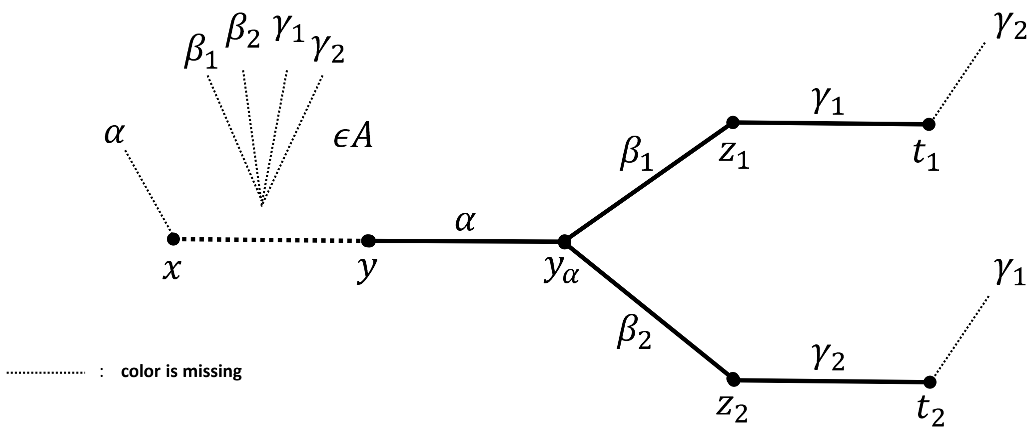

Figure 1.1. φ-fork on xyy_α.

$1) + (d-2)$, then there exist $\varphi' \in \mathcal{C}^\Delta(G-xy)$ and a φ'-fork on xyy_α such that $t_1, t_2 \in T$, where y_α is defined under φ and t_1, t_2 are two ends of the fork.

Proposition B. *Let G be a Δ-critical graph and $xy \in E(G)$. For any $\varphi \in \mathcal{C}^\Delta(G-xy)$ and $\alpha \in \bar\varphi(x)$, there does not exist a φ-fork on xyy_α such that $\Delta \geq d(x)+d(t_1)+d(t_2)+1$ where t_1, t_2 are the two ends of the fork.*

Remark 1.3. *For any $\varphi \in \mathcal{C}^\Delta(G-xy)$, since $\bar\varphi(x) \cap \bar\varphi(y) = \emptyset$, we have $|(\bar\varphi(x) \cup \bar\varphi(y)) \cap \bar\varphi(t_1) \cap \bar\varphi(t_2)| \geq |\bar\varphi(x) \cup \bar\varphi(y)| - |\varphi(t_1)| - |\varphi(t_2)| \geq \Delta - d(x) + 1 + 1 - d(t_1) - d(t_2)$. Thus $\Delta \geq d(x) + d(t_1) + d(t_2) + 1$ implies that $|(\bar\varphi(x) \cup \bar\varphi(y)) \cap \bar\varphi(t_1) \cap \bar\varphi(t_2)| \geq 3$. In other words, it implies that there always exist 3 colors in $A(\varphi) \cap \bar\varphi(t_1) \cap \bar\varphi(t_2)$ for any coloring $\varphi \in \mathcal{C}^\Delta(G-xy)$.*

Clearly Theorem 1.3 follows from the two propositions above since $(d+1)(d+3) > 3d+1$ for $d \geq 2$. We now give the proofs of the two propositions.

1.5.2 Proof of Proposition A

Proof. Let $d \geq 2$ be an integer, G be a Δ-critical graph, $xy \in E(G)$ with $x \in T$, $\varphi \in \mathcal{C}^\Delta(G-xy)$ and $\alpha \in \bar\varphi(x)$. We first verify a fact: $(*)$ For any $v \in V(G)$, $N[v] = \{v\} \cup N(v)$ contains at most $d-1$ *d-small* vertices.

Suppose v is *d-small*. For any $w \in N(v)$, we have $d(w) \geq \Delta - d(v) + 1 + 1$ by Lemma 1.10. Thus $d(w) \geq \Delta - d + 2 > d$. Hence in this case $|N[v] \cap T| = 1$. Suppose v is not *d-small*. Then by Lemma 1.10, $N(v)$ contains at least $\Delta - d + 1$ vertices with degree Δ if $N(v) \cap T \neq \emptyset$. Thus in this case, $|N[v] \cap T| \leq d - 1$. The fact is verified.

Notice that the coloring φ satisfies the following two conditions:

(1) $\varphi \in \mathcal{C}^{\Delta}(G - xy)$;

(2) $\varphi(yy_\alpha) \in \bar{\varphi}(x)$ for the fixed vertex y_α.

We use the word "fixed" since some operations in later proofs may change the color on yy_α. In this subsection, y_α only denotes the vertex y_α under the original coloring φ.

Suppose that the number of *d-small* vertices in $N(y_\alpha) \cup L(y_\alpha)$ is at least $(d-1)^2 + (d+1)(d-1) + (d-2) + 1$.

Claim 1.4. *For any coloring φ satisfying* (1) *and* (2), $|L_{A(\varphi)} \cap T| \geq (d+1)(d-1) + (d-2) + 1$.

Proof. Let φ be a coloring satisfying (1) and (2) (where y_α is already fixed). Notice that $N(y_\alpha) \cup L(y_\alpha)$ is the union of $\{y\}, N(y) \backslash \{y_\alpha\}, N_A, N_B, L_A$ and L_B (some of these sets may overlap). We will give upper bounds of the numbers of *d-small* vertices in $\{y\}, N(y) \backslash \{y_\alpha\}, N_A, N_B$ and L_B respectively.

By the fact $(*)$, for any vertex v, $N[v] = \{v\} \cup N(v)$ contains at most $d - 1$ *d-small* vertices. Recall $|B| \leq d - 2$, thus the total number of *d-small* vertices in $N_B \cup L_B$ is at most $(d-2)(d-1)$, and $N(y) \backslash \{y_\alpha\}$ contains at most $d - 1$ *d-small* vertices.

Notice that $\{x, y, y_\alpha\} \cup N_A$ is the vertex set of a simple broom. By Lemma 1.9 and 1.11, each vertex in this set except x has degree at least $|\bar{\varphi}(x)| - 1 \geq \Delta - d + 1 - 1 > d$. Thus $\{y\}$ and N_A do not contain any *d-small* vertex. So in total, there are at most $(d-1)^2$ *d-small* vertices in $\{y\}, N(y) \backslash \{y_\alpha\}, N_A, N_B$ and L_B. Hence there are at least $(d+1)(d-1) + (d-2) + 1$ *d-small* vertices in L_A. □

Claim 1.5. *There is a coloring φ' satisfying* (1), (2) *and the following: There exist $d + 2$ distinct vertices $z_1, z_2, \ldots, z_{d+2}$ in $N_{A(\varphi')}$ and $d + 2$ d-small vertices $t_1, t_2, \ldots, t_{d+2}$ in $L_{A(\varphi')}$ such that $z_i t_i = e_i \in E(G)$ and $\varphi'(e_i) \in A(\varphi')$.*

Proof. For any coloring φ' satisfying (1) and (2), by Claim 1.4, we have $|L_{A(\varphi')} \cap T| \geq (d+1)(d-1)+(d-2)+1$. By the definition of $L_{A(\varphi')}$, for each $t \in L_{A(\varphi')} \cap T$, we can pick a vertex $z_t \in N_{A(\varphi')}$ such that $z_t t \in E(G)$. (Different t may correspond to the same z_t.) Let $E_{zt} = \{z_t t : t \in L_{A(\varphi')} \cap T\}$, then $|E_{zt}| \geq (d+1)(d-1)+(d-2)+1$. Denote by $a(\varphi')$ the number of edges in E_{zt} with colors in A under coloring φ', i.e. $a(\varphi') = |\{e \in E_{zt} : \varphi'(e) \in A(\varphi')\}|$. Let φ' be a coloring satisfying (1) and (2) such that $a(\varphi')$ is maximum over all such colorings.

Suppose the claim is not true, then we have the size of the set (not multi-set) $\{z_t : \varphi'(z_t t) \in A(\varphi')\}$ is at most $d+1$, i.e. $|\{y_\alpha z_t : \varphi'(z_t t) \in A(\varphi')\}| \leq d+1$. Recall that each vertex of G has at most $d-1$ *d-small* neighbors by $(*)$, thus $a(\varphi') \leq |\{z_t : \varphi'(z_t t) \in A(\varphi')\} \cdot |(d-1) \leq (d+1)(d-1)$. Since $|E_{zt}| \geq (d+1)(d-1)+d-1$, at least $d-1$ edges in E_{zt} are colored by colors in B. Recall $|B| \leq d-2$, by the Pigeonhole Principle, there are at least two edges in E_{zt} sharing the same color in B. Without loss of generality, let us say $\varphi'(z_1 t_1) = \varphi'(z_2 t_2) = \gamma \in B$ where $t_1, t_2 \in L_{A(\varphi')} \cap T$ and $z_1 = z_{t_1}$, $z_2 = z_{t_2}$. Notice that these two edges share the same color, so $z_1 \neq z_2$.

Since t_1, t_2 and x are *d-small* vertices, and $\Delta \geq (d+1)(d+3)$, they have at least $\Delta - d - d - (d-1) \geq d^2 + d + 4$ common missing colors (which are in $\bar{\varphi}'(x) \subseteq A$). We choose a color δ from these common missing colors such that $\delta \notin \{\varphi'(yy_\alpha), \varphi'(y_\alpha z_1), \varphi'(y_\alpha z_2)\} \cup \{\varphi'(e) : e \in E_{zt},\ \varphi'(e) \in A(\varphi')\} \cup \{y_\alpha z_t : \varphi'(z_t t) \in A(\varphi')\}$. This is possible since we only need to avoid at most $3 + a(\varphi') + (d+1) \leq 3 + (d+1)(d-1) + (d+1) < d^2 + d + 4$ colors.

Now consider $P_x(\delta, \gamma, \varphi')$. This path has two endpoints, thus at least one of t_1, t_2 is not on this path, say t_1. Then we may let $\varphi'' = \varphi'/P_{t_1}(\delta, \gamma, \varphi')$. By the choice of δ, we have φ'' satisfies (1) and (2) and $a(\varphi'') > a(\varphi')$ (with additional edge $z_1 t_1$ counted), which contradict the fact that $a(\varphi')$ is maximum. □

For convenience, we still denote the coloring satisfying the claim above by φ.

Claim 1.6. *Under the coloring φ above, we may further assume the following: There exist i, j such that $\varphi(e_i) \in \bar{\varphi}(t_j)$ and $\varphi(e_j) \in \bar{\varphi}(t_i)$. In other words, $\{x, y, y_\alpha, z_i, z_j, t_i, t_j\}$ is a φ-fork on xyy_α and* **Proposition A** *is true.*

Proof. Let $\varphi(e_1) = \gamma_1$. Since $\Delta \geq (d+1)(d+3)$, vertices $x, t_1, t_2, \dots, t_{d+2}$ have a common missing color, say δ. Notice that $\delta \neq \gamma_1$ since $\delta \in \bar{\varphi}(t_1)$. Then at most one of $t_2, t_3, \dots, t_{d+2}$ is on $P_{t_1}(\gamma_1, \delta, \varphi)$. We switch γ_1 and δ on all other (γ_1, δ)-chains to get φ_1. In this new coloring, all but one of $t_2, t_3, \dots, t_{d+2}$ miss the color γ_1. Now we claim that φ_1 satisfies the conditions in **Claim 5.2**.

Clearly condition (1) holds for φ_1. Note that $\gamma_1 \in A(\varphi) = \bar{\varphi}(x) \cup \bar{\varphi}(y)$ and $\delta \in \bar{\varphi}(x)$. If $\gamma_1 \in \bar{\varphi}(y)$ then x and y are the endpoints of a (γ_1, δ)-chain not passing through t_1, and so $\gamma_1 \in \bar{\varphi}_1(x)$ and $\delta \in \bar{\varphi}_1(y)$. Otherwise, if $\gamma_1 \in \bar{\varphi}(x)$, then $\gamma_1, \delta \in \bar{\varphi}_1(x)$. In both cases, $A(\varphi_1) = A(\varphi)$, and $\varphi_1(yy_\alpha) \in \bar{\varphi}_1(x)$ regardless of whether ot not $\varphi(yy_\alpha) \in \{\gamma_1, \delta\}$; thus condition (2) holds for φ_1. Finally, because $\gamma_1, \delta \in A(\varphi) = A(\varphi_1)$, the remaining conditions in **Claim 5.2** hold.

We assume without loss of generality that γ_1 is missing at $t_2, t_3, \dots, t_{d+1}$ under φ_1. Let $\varphi_1(e_i) = \gamma_i$. Now if one of $\gamma_2, \gamma_3, \dots, \gamma_{d+1}$ is missing at t_1, we are done. Suppose not, then all these colors are in $\varphi_1(t_1) \backslash \{\gamma_1\}$. Since $|\varphi_1(t_1) \backslash \{\gamma_1\}| = d(t_1) - 1 \leq d - 1$, there are two colors of them, say γ_2, γ_3, that are the same. Now we use the same idea as before. Since $\Delta \geq (d+1)(d+3) > 4d$, so x, t_1, t_2, t_3 have a common missing color, say δ'. Notice that at most one of t_2, t_3 is on $P_{t_1}(\gamma_2, \delta', \varphi_1)$; assume t_2 is not on this path. Then we switch γ_2 and δ' on $P_{t_2}(\gamma_2, \delta', \varphi_1)$ to get φ_2. Similarly we have φ_2 still satisfies our previous assumptions. Moreover, e_2 is colored by $\delta' \in \bar{\varphi}_2(t_1)$. Since $\varphi_2(e_1) = \gamma_1 \in \bar{\varphi}_2(t_2)$, the claim holds for $(i, j) = (1, 2)$ under φ_2. □

1.5.3 Proof of Proposition B

Assume the existence of a φ-fork F as in **Proposition B**. We will obtain a contradiction in three stages. We first show that φ can be assumed to be F-*rainbow*, meaning that all colors on the edges of F are distinct. We then show that we may further assume that all these colors are missing at the two ends of F except γ_1, γ_2, which are missing at one of the ends of F. Finally, we reach a contradiction by some Kempe changes.

Ler $F = \{x, y, y_\alpha, z_1, z_2, t_1, t_2\}$ be a φ-fork on xyy_α where $\alpha \in \bar{\varphi}(x)$ and $\Delta \geq d(x) +$

$d(t_1)+d(t_2)+1$. For $i=1,2$, denote the colors of $y_\alpha z_i$ and $z_i t_i$ by β_i and γ_i, respectively. From the definition of a fork, note that $\{\alpha,\beta_1,\beta_2,\gamma_1,\gamma_2\} \subseteq A(\varphi) = \bar{\varphi}(x) \cup \bar{\varphi}(y)$.

If θ, η are colors such that $\theta \in \bar{\varphi}(x)$ and $\eta \in \bar{\varphi}(y)$, then by Lemma 1.9, $P_x(\theta,\eta,\varphi) = P_y(\theta,\eta,\varphi)$; we call $P_x(\theta,\eta,\varphi)$ an *xy-chain* (with respect to φ). It is *F-avoiding* if it does not pass along any edge of F. We will make frequent use of the fact that if $P_x(\theta,\eta,\varphi)$ is an F-avoiding xy-chain and $\varphi' = \varphi/P_x(\theta,\eta,\varphi)$, then $A(\varphi') = A(\varphi)$ and F is a φ'-fork whose edges are colored the same in φ' as in φ; the sole point of this change is that now $\theta \in \bar{\varphi}'(y)$ and $\eta \in \bar{\varphi}'(x)$. (It does not matter whether $P_x(\theta,\eta,\varphi)$ passes through t_1 and/or t_2, although, if it does not, then that often helps in seeing that it is F-avoiding.)

A *delta-color* for F,φ is a color $\delta \in A(\varphi) \cap \bar{\varphi}(t_1) \cap \bar{\varphi}(t_2)$ such that $\delta \notin \{\alpha,\beta_1,\beta_2,\gamma_1,\gamma_2\}$.

Remark 1.7. *If $\{\alpha,\beta_1,\beta_2\} \not\subseteq \bar{\varphi}(t_1)\cap\bar{\varphi}(t_2)$, then at most two of $\alpha,\beta_1,\beta_2,\gamma_1,\gamma_2$ are in $\bar{\varphi}(t_1)\cap\bar{\varphi}(t_2)$, since clearly $\gamma_1,\gamma_2 \notin \bar{\varphi}(t_1)\cap\bar{\varphi}(t_2)$. It therefore follows from* **Remark 1.3** *that, in this case, there exists a delta-color for F,φ.*

For any coloring $\varphi \in \mathcal{C}^\Delta(G-xy)$ such that F is a φ-fork, we denote by $\delta(\varphi)$ an arbitrary delta-color for F,φ (assuming one exists). We will often define $\delta = \delta(\varphi)$ and consider the pair (φ,δ). If we subsequently transform φ into another coloring φ' such that F is a φ'-fork, we will assume unless stated otherwise that δ is necessarily a delta-color for F,φ'; we will write $(\varphi',\delta(\varphi'))$ if we need to choose a new delta-color for F,φ'.

Claim 1.8. *We may assume that φ is an F-rainbow coloring.*

Proof. Suppose on the contrary that there is no F-rainbow coloring φ such that F is a φ-fork. We will show that this leads to a contradiction.

Assume F is a φ-fork with its edges colored as above. Since φ is proper, it follows that α,β_1,β_2 are distinct, $\gamma_1 \neq \beta_1$ and $\gamma_2 \neq \beta_2$. We also have $\gamma_1 \neq \gamma_2$, since $\gamma_1 \in \bar{\varphi}(t_2)$ and $\gamma_2 \in \bar{\varphi}(t_1)$ by the definition of a fork. Using symmetry (interchanging subscripts 1 and 2), we thus have four cases to consider:

Case 1: $\gamma_1 = \alpha$, $\gamma_2 \notin \{\alpha,\beta_1,\beta_2\}$;

Case 2: $\gamma_1 = \beta_2$, $\gamma_2 \notin \{\alpha,\beta_1,\beta_2\}$;

Case 3: $\gamma_1 = \alpha$, $\gamma_2 = \beta_1$;

Case 4: $\gamma_1 = \beta_2$, $\gamma_2 = \beta_1$.

Since in each case either α or β_2 equals γ_1 and so is not in $\bar{\varphi}(t_1) \cap \bar{\varphi}(t_2)$, it follows from **Remark 1.7** that there exists a delta-color $\delta = \delta(\varphi)$ for F, φ. We consider the pair (φ, δ). In each case, we will concentrate on the colors in $\bar{\varphi}(y)$; notice that $\bar{\varphi}(y) \neq \emptyset$ since edge xy is uncolored.

In proving the result in each case, we will assume that it has already been proved in all previous cases. Logically, Case 3 comes after Case 2. But the arguments for Cases 1 and 3 are so similar that we write them out together.

Cases 1 and 3: $\gamma_1 = \alpha \in \bar{\varphi}(t_2)$, and $\gamma_2 \notin \{\alpha, \beta_1, \beta_2\}$ (Case 1) or $\gamma_2 = \beta_1$ (Case 3).

(a1) In each case, $\delta \in \bar{\varphi}(x)$. For, if not, then $\delta \in \bar{\varphi}(y)$. By Lemma 1.9, $P_x(\delta, \alpha, \varphi) = P_y(\delta, \alpha, \varphi)$, and this path passes along yy_α but not through t_1 or t_2 or along any other edge of F. Let $\varphi' = \varphi / P_{t_1}(\delta, \alpha, \varphi)$, so that $\varphi'(z_1 t_1) = \delta \in \bar{\varphi}'(t_2)$, since $\alpha, \delta \in \bar{\varphi}(t_2)$. Then F is a φ'-fork. In Case 1, φ' is now F-rainbow, a contradiction. In Case 3, $(\varphi', \delta(\varphi'))$ is in Case 2, with subscripts 1 and 2 interchanged, which also leads to a contradiction.

(a2) Suppose there is a color $\eta \in \bar{\varphi}(y)$ such that $P_x(\delta, \eta, \varphi)$ does not pass along $y_\alpha z_1$ or $y_\alpha z_2$ (which is automatic if $\eta \notin \{\beta_1, \beta_2\}$). Then $P_x(\delta, \eta, \varphi)$ is an F-avoiding xy-chain. Let $\varphi' = \varphi / P_x(\delta, \eta, \varphi)$. Then $\delta \in \bar{\varphi}'(y)$, which contradicts (a1).

(a3) Suppose $\beta_1 \in \bar{\varphi}(y) \cap \bar{\varphi}(t_1)$. Then $P_y(\alpha, \beta_1, \varphi) = yy_\alpha z_1 t_1$. But $P_y(\alpha, \beta_1, \varphi)$ must end at x by Lemma 1.9, giving a contradiction. (Noe that this argument does not use δ.)

(a4) Suppose $\beta_1 \in \bar{\varphi}(y) \setminus \bar{\varphi}(t_1)$ and the path $P_x(\delta, \beta_1, \varphi) = P_y(\delta, \beta_1, \varphi)$ passes along $y_\alpha z_1$. Note that this path misses t_1 and t_2. Let $\varphi' = \varphi / P_{t_1}(\delta, \beta_1, \varphi)$; then $\beta_1 \in \bar{\varphi}'(y) \cap \bar{\varphi}'(t_1)$ and we get a contradiction by (a3). (Note that (a4) cannot arise in Case 3, when $\beta_1 = \gamma_2 \in \bar{\varphi}(t_1)$.)

(a5) The only remaining possibility is that $\bar{\varphi}(y) = \{\beta_2\}$ (which implies $\delta, \gamma_2 \in \bar{\varphi}(x)$) and the path $P_x(\delta, \beta_2, \varphi) = P_y(\delta, \beta_2, \varphi)$ passes along $y_\alpha z_2$. Note that this path misses t_1 and t_2. Switch δ and β_2 on all other (δ, β_2)-chains to get φ'. Then $\beta_2 \in \bar{\varphi}'(t_1) \cap \bar{\varphi}'(t_2)$, and so $P_x(\gamma_2, \beta_2, \varphi')$ is an F-avoiding xy-chain. Let $\varphi'' = \varphi' / P_x(\gamma_2, \beta_2, \varphi')$; then $\gamma_2 \in \bar{\varphi}''(y)$. If $\gamma_2 \neq \beta_1$ (Case 1) then $(\varphi'', \delta(\varphi''))$ gives a contradiction by (a2) with $\eta = \gamma_2$; otherwise (Case

3) it gives a contradiction by (a3), since $\beta_1 = \gamma_2 \in \bar{\varphi}''(t_1)$.

Case 2 and 4: $\gamma_1 = \beta_2 \in \bar{\varphi}(t_2)$, and $\gamma_2 \notin \{\alpha, \beta_1, \beta_2\}$ (Case 2) or $\gamma_2 = \beta_1$ (Case 4).

(b1) We claim that the path $P_{t_1}(\delta, \gamma_1, \varphi) = P_{t_1}(\delta, \beta_2, \varphi)$ passes along $y_\alpha z_2$. Otherwise, let $\varphi' = \varphi/P_{t_1}(\delta, \gamma_1, \varphi)$, so that $\varphi'(z_1 t_1) = \delta \in \bar{\varphi}'(t_2)$, since $\delta, \gamma_1 \in \bar{\varphi}(t_2)$. Then F is a φ'-fork. In Case 2, φ' is now F-rainbow, a contradiction. In Case 4, $(\varphi', \delta(\varphi'))$ is in Case 2, with subscripts 1 and 2 interchanged, which also leads to a contradiction.

(b2) Suppose $\beta_2 \in \bar{\varphi}(y)$ and $\alpha \in \bar{\varphi}(t_1) \cap \bar{\varphi}(t_2)$. Then the path $P_x(\alpha, \beta_2, \varphi) = P_y(\alpha, \beta_2, \varphi)$ passes along yy_α and $y_\alpha z_2$ but not through t_1 or t_2. Let $\varphi' = \varphi/P_{t_1}(\alpha, \beta_2, \varphi)$, so that $\varphi'(z_1 t_1) = \alpha \in \bar{\varphi}'(t_2)$, since $\alpha, \beta_2 \in \bar{\varphi}(t_2)$. Now (φ', δ) is in Case 1 or Case 3 depending on whether (φ, δ) was in Case 2 or Case 4, which leads to a contradiction.

(b3) Suppose $\delta \in \bar{\varphi}(y)$ and $\alpha \in \bar{\varphi}(t_1) \cap \bar{\varphi}(t_2)$. We may assume by (b2) that $\beta_2 \in \bar{\varphi}(x)$, and by (b1) that $P_{t_1}(\delta, \beta_2, \varphi)$ passes along $y_\alpha z_2$. Thus $P_x(\delta, \beta_2, \varphi)$ is an F-avoiding xy-chain. Let $\varphi' = \varphi/P_x(\delta, \beta_2, \varphi)$. Then $\beta_2 \in \bar{\varphi}'(y)$, and so (φ', δ) gives a contradiction by (b2).

(b4) Suppose $\delta \in \bar{\varphi}(y)$. Then the path $P_x(\delta, \alpha, \varphi) = P_x(\delta, \alpha, \varphi)$ passes along yy_α, but not through t_1 or t_2. Switch colors δ and α on all other (δ, α)-chains to give φ', so that $\alpha \in \bar{\varphi}'(t_1) \cap \bar{\varphi}'(t_2)$. Note that $\delta \in \bar{\varphi}'(y)$. Let $\delta' = \delta(\varphi')$. If $\delta' \in \bar{\varphi}'(y)$ then (φ', δ') gives a contradiction by (b3). So assume $\delta' \in \bar{\varphi}'(x)$. Then $P_x(\delta, \delta', \varphi')$ is an F-avoiding xy-chain. Let $\varphi'' = \varphi'/P_x(\delta, \delta', \varphi')$. Then $\delta' \in \bar{\varphi}''(y)$, and (φ'', δ') gives a contradiction by (b3).

(b5) Suppose $\beta_2 \in \bar{\varphi}(y)$. We may assume by (b4) that $\delta \in \bar{\varphi}(x)$. By (b1), $P_{t_1}(\delta, \beta_2, \varphi)$ passes along $y_\alpha z_2$, and so $P_x(\delta, \beta_2, \varphi)$ is an F-avoiding xy-chain. Let $\varphi' = \varphi/P_x(\delta, \beta_2, \varphi)$, so that $\delta \in \bar{\varphi}'(y)$. Then (φ', δ) gives a contradiction by (b4).

(b6) Suppose $\delta, \beta_2 \in \bar{\varphi}(x)$ and there is a color $\eta \in \bar{\varphi}(y)$ such that $P_x(\delta, \eta, \varphi)$ does not pass along $y_\alpha z_1$ (which is automatic if $\eta \neq \beta_1$). Then $P_x(\delta, \eta, \varphi)$ is an F-avoiding xy-chain. Let $\varphi' = \varphi/P_x(\delta, \eta, \varphi)$, so that $\delta \in \bar{\varphi}'(y)$. Then (φ', δ) gives a contradiction by (b4).

(b7) The only remaining possiblility is that $\bar{\varphi}(y) = \{\beta_1\}$ (which implies $\delta, \beta_2 \in \bar{\varphi}(x)$) and the path $P_x(\delta, \beta_1, \varphi) = P_y(\delta, \beta_1, \varphi)$ passes along $y_\alpha z_1$. Note that this path misses t_1 and t_2. Switch δ and β_1 on all other (δ, β_1)-chains to get φ'. We now consider the two cases separately.

In Case 2, we note that $\beta_1 \in \bar{\varphi}'(t_1) \cap \bar{\varphi}'(t_2)$, and so $P_x(\beta_2, \beta_1, \varphi')$ is an F-avoiding xy-chain (since it avoids the (β_1, β_2)-path $t_1 z_1 y_\alpha z_2$). Let $\varphi'' = \varphi'/P_x(\beta_2, \beta_1, \varphi')$; then $\beta_2 \in \bar{\varphi}''(y)$, and $(\varphi'', \delta(\varphi''))$ gives a contradiction by (b5).

In Case 4, note that $\varphi'(z_2 t_2) = \delta \in \bar{\varphi}'(t_1)$, since $\varphi(z_2 t_2) = \beta_1 \in \bar{\varphi}(t_1)$. So $(\varphi', \delta(\varphi'))$ is in Case 2, which also leads to a contradiction.

This completes the proof of Claim 1.8. □

By Claim 1.8, we now assume that $\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2$ are all distinct.

Claim 1.9. *We may further assume that* $\{\alpha, \beta_1, \beta_2\} \subseteq \bar{\varphi}(t_1) \cap \bar{\varphi}(t_2)$.

Proof. Assume this is false. Let $T(\varphi) = \{\alpha, \beta_1, \beta_2\} \cap \bar{\varphi}(t_1) \cap \bar{\varphi}(t_2)$ and $t(\varphi) = |T(\varphi)|$. Assume φ is chosen so that $t(\varphi)$ is as large as possible, and $t(\varphi) < 3$. Let $\delta = \delta(\varphi)$ be a delta-color for F, φ, which exists by **Remark 1.7**. We will get a contradiction in two cases.

Case 1: $\alpha \notin T(\varphi)$.

(a1) We claim that $\delta \in \bar{\varphi}(x)$. For, if not, then $\delta \in \bar{\varphi}(y)$. By Lemma 1.9, $P_x(\alpha, \delta, \varphi) = P_y(\alpha, \delta, \varphi)$, and this path passes along $y y_\alpha$ but not through t_1 or t_2. Switch α and δ on all other (α, δ)-chains to get φ'. Then $\alpha \in T(\varphi')$. So $t(\varphi') = t(\varphi) + 1$, which contradicts the choice of φ.

(a2) Suppose there is a color $\eta \in \bar{\varphi}(y)$ such that $P_x(\delta, \eta, \varphi)$ does not pass along $y_\alpha z_1$ or $y_\alpha z_2$ (which is automatic if $\eta \notin \{\beta_1, \beta_2\}$). Then $P_x(\delta, \eta, \varphi)$ is an F-avoiding xy-chain. Let $\varphi' = \varphi/P_x(\delta, \eta, \varphi)$. Then $\delta \in \bar{\varphi}'(y)$, and (φ', δ) gives a contradiction by (a1).

(a3) The only remaining possibility is that $\bar{\varphi}(y) \subseteq \{\beta_1, \beta_2\}$, which implies $\{\gamma_1, \gamma_2\} \subseteq \bar{\varphi}(x)$. Assume without loss of generality that $\beta_1 \in \bar{\varphi}(y)$, then the path $P_x(\delta, \beta_1, \varphi) = P_y(\delta, \beta_1, \varphi)$ passes along $y_\alpha z_1$ by (a2). Clearly this path misses t_1 and t_2. Switch colors δ and β_1 on all other (δ, β_1)-chains to get φ'. Then $\beta_1 \in T(\varphi') \subseteq \bar{\varphi}'(t_1) \cap \bar{\varphi}'(t_2)$, which means that $t(\varphi') \geq t(\varphi)$ and so $t(\varphi') = t(\varphi)$ by the maximality of $t(\varphi)$. Now $P_x(\gamma_1, \beta_1, \varphi')$ is an F-avoiding xy-chain. So let $\varphi'' = \varphi'/P_x(\gamma_1, \beta_1, \varphi')$ so that $\gamma_1 \in \bar{\varphi}''(y)$. Then $(\varphi'', \delta(\varphi''))$ gives a contradiction by (a2) with $\eta = \gamma_1$.

Case 2: $\{\beta_1, \beta_2\} \nsubseteq T(\varphi)$.

Assume without loss of generality that $\beta_1 \notin T(\varphi)$. By Case 1, we may assume that $\alpha \in T(\varphi)$.

(b1) We claim that $\beta_1 \in \bar{\varphi}(x)$. For otherwise, suppose $\beta_1 \in \bar{\varphi}(y)$. Then the path $P_x(\alpha, \beta_1, \varphi) = P_y(\alpha, \beta_1, \varphi)$ passes along yy_α and $y_\alpha z_1$ but not through t_1 or t_2. Switch colors α and β_1 on all other (α, β_1)-chains to get φ'. Then $\beta_1 \in T(\varphi')$, and so $T(\varphi') = (T(\varphi) \cup \{\beta_1\}) \setminus \{\alpha\}$ and $t(\varphi') = t(\varphi)$. So (φ', δ) is in Case 1, which leads to a contradiction.

(b2) We further claim that $\delta \in \bar{\varphi}(x)$. For otherwise, suppose $\delta \in \bar{\varphi}(y)$. The path $P_x(\delta, \beta_1, \varphi) = P_y(\delta, \beta_1, \varphi)$ misses t_1 and t_2. If this path does not pass along $y_\alpha z_1$ then it is an F-avoiding xy-chain, so let $\varphi' = \varphi / P_x(\delta, \beta_1, \varphi)$; then $\beta_1 \in \bar{\varphi}'(y)$ and we get a contradiction by (b1). So assume that this path passes along $y_\alpha z_1$. Switch δ and β_1 on all other (δ, β_1)-chains to get φ'. Then $\beta_1 \in T(\varphi')$ and $t(\varphi) > t(\varphi)$, which contradicts the choice of φ.

(b3) Suppose there is a color $\eta \in \bar{\varphi}(y)$ such that $\eta \neq \beta_2$. Then $P_x(\delta, \eta, \varphi)$ is an F-avoiding xy-chain. Let $\varphi' = \varphi / P_x(\delta, \eta, \varphi)$. Then $\delta \in \bar{\varphi}'(y)$, which contradicts (b2).

(b4) The only other possibility is that $\bar{\varphi}(y) = \{\beta_2\}$; then $\gamma_2 \in \bar{\varphi}(x)$. We may assume $\beta_2 \in T(\varphi) \subseteq \bar{\varphi}(t_1) \cap \bar{\varphi}(t_2)$, since otherwise $\beta_2 \in \bar{\varphi}(x)$ by (b1) applied to β_2. Thus $P_x(\gamma_2, \beta_2, \varphi)$ is an F-avoiding xy-chain. Let $\varphi' = \varphi / P_x(\gamma_2, \beta_2, \varphi)$, so that $\gamma_2 \in \bar{\varphi}'(y)$. Then (φ', δ) gives a contradiction by (b3) with $\eta = \gamma_2$. □

By Claim 1.9, we now assume that $\{\alpha, \beta_1, \beta_2\} \subseteq \bar{\varphi}(t_1) \cap \bar{\varphi}(t_2)$.

Claim 1.10. *We may further assume that one of γ_1, γ_2 is missing at y.*

Proof. Assume this is false, then $\gamma_1, \gamma_2 \in \bar{\varphi}(x)$. Choose $\delta \in A(\varphi) \cap \bar{\varphi}(t_1)$ such that $\delta \notin \{\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2\}$. This is possible because the inequality in **Remark 1.3** shows that $|A(\varphi) \cap \bar{\varphi}(t_1)| \geq 3 + d(t_2) \geq 5$, and $\gamma_1 \notin \bar{\varphi}(t_1)$.

(a1) We claim that $\beta_1, \beta_2 \in \bar{\varphi}(x)$. Suppose otherwise. Since we do not use δ here, we may assume without loss of generality that $\beta_1 \in \bar{\varphi}(y)$. Then $P_x(\gamma_1, \beta_1, \varphi)$ is an F avoiding xy-chain. Let $\varphi' = \varphi / P_x(\gamma_1, \beta_1, \varphi)$, so that $\gamma_1 \in \bar{\varphi}'(y)$. This contradicts our assumption that the result is false.

(a2) We further claim that $\delta \in \bar{\varphi}(x)$. For suppose $\delta \in \bar{\varphi}(y)$. Note that $\delta \in \bar{\varphi}(t_1)$ and $\gamma_1 \in \bar{\varphi}(t_2)$. So $P_x(\delta, \gamma_1, \varphi)$ is an F-avoiding xy-chain. Let $\varphi' = \varphi/P_x(\delta, \gamma_1, \varphi)$. Then $\gamma_1 \in \bar{\varphi}'(y)$, which is again contrary to assumption.

(a3) Let $\eta \in \bar{\varphi}(y)$. By assumption and (a1), $\eta \notin \{\beta_1, \beta_2, \gamma_1, \gamma_2\}$, and so $P_x(\delta, \eta, \varphi)$ is an F-avoiding xy-chain. Let $\varphi' = \varphi/P_x(\delta, \eta, \varphi)$, so that $\delta \in \bar{\varphi}'(y)$; then (φ', δ) contradicts (a2). This completes the proof. □

We further assume without loss of generality that $\gamma_1 \in \bar{\varphi}(y)$. Now starting from this specific coloring φ, we are going to find a contradiction. The following process contains many Kempe changes. The purpose for doing such a long and complicated process is to adjust the position of the uncolored edge and some other details so that after this process, we can find a way to color the entire graph G (including the uncolored edge) with Δ colors, which gives a contradiction.

Claim 1.11. *There exist colorings $\varphi_1, \varphi_2 \in \mathcal{C}^{\Delta}(G - xy)$ such that F is both a φ_1-fork and a φ_2-fork with edges colored as in φ except for edge $e_1 = z_1t_1$, and the colors $\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2$ have the same properties as in φ except for the modifications below:*

(a) *$\varphi_1(e_1) = \alpha \notin \bar{\varphi}_1(t_1)$, $\gamma_1 \in \bar{\varphi}_1(t_1) \cap A(\varphi_1)$ and $\beta_2 \in \bar{\varphi}_1(y)$.*

(b) *$\varphi_2(e_1) = \beta_2 \notin \bar{\varphi}_2(t_1)$, $\gamma_1 \in \bar{\varphi}_2(t_1) \cap A(\varphi_2)$ and $\beta_1 \in \bar{\varphi}_2(y)$.*

Proof. (a) Note that the path $P_x(\alpha, \gamma_1, \varphi) = P_y(\alpha, \gamma_1, \varphi)$ passes along yy_α but not through t_1 or t_2. Let $\varphi' = \varphi/P_{t_1}(\alpha, \gamma_1, \varphi)$; then $\varphi'(e_1) = \alpha \notin \bar{\varphi}'(t_1)$ and $\gamma_1 \in \bar{\varphi}'(y) \cap \bar{\varphi}'(t_1) \cap \bar{\varphi}'(t_2)$. If $\beta_2 \in \bar{\varphi}'(y)$, we let $\varphi_1 = \varphi'$.

So assume $\beta_2 \in \bar{\varphi}'(x)$. If $\gamma_2 \in \bar{\varphi}'(y)$, let $\varphi'' = \varphi'$; otherwise, if $\gamma_2 \in \bar{\varphi}'(x)$, then $P_x(\gamma_1, \gamma_2, \varphi')$ is an F-avoiding xy-chain and we let $\varphi'' = \varphi'/P_x(\gamma_1, \gamma_2, \varphi')$. Either way, $\gamma_2 \in \bar{\varphi}''(y)$. Now $P_x(\beta_2, \gamma_2, \varphi'')$ is an F-avoiding xy-chain, and we let $\varphi_1 = \varphi''/P_x(\beta_2, \gamma_2, \varphi'')$. Then $\beta_2 \in \bar{\varphi}_1(y)$.

(b) The path $P_x(\alpha, \beta_2, \varphi_1) = P_y(\alpha, \beta_2, \varphi_1)$ passes along yy_α and $y_\alpha z_2$ but not through t_1 or t_2. Let $\varphi' = \varphi_1/P_{t_1}(\alpha, \beta_2, \varphi_1)$; then $\varphi'(e_1) = \beta_2 \notin \bar{\varphi}'(t_1)$ and $\alpha \in \bar{\varphi}'(t_1) \cap \bar{\varphi}'(t_2)$. If $\beta_1 \in \bar{\varphi}'(y)$, we let $\varphi_2 = \varphi'$.

So assume $\beta_1 \in \bar{\varphi}'(x)$, and note that we still have $\beta_1 \in \bar{\varphi}'(t_1) \cap \bar{\varphi}'(t_2)$. Then $P_x(\beta_1, \beta_2, \varphi')$ is an F-avoiding xy-chain, and we let $\varphi_2 = \varphi'/P_x(\beta_1, \beta_2, \varphi')$. Then $\beta_1 \in \bar{\varphi}_2(y)$. □

Under the coloring φ_2 obtained by Claim 1.11 (b), we recolor xy by α, yy_α by β_1 and leave $y_\alpha z_1$ uncolored to obtain ψ. We list all properties we need in the following proof about coloring ψ for reference. We have $\psi \in \mathcal{C}^\Delta(G - y_\alpha z_1)$, $\psi(xy) = \alpha$, $\psi(yy_\alpha) = \beta_1$, $\psi(e_1) = \beta_2$, $\psi(y_\alpha z_2) = \beta_2$, $\psi(e_2) = \gamma_2$. We also have $\beta_2, \gamma_1, \gamma_2 \in \bar{\psi}(x) \cup \bar{\psi}(y)$, $\alpha \in \bar{\psi}(y_\alpha)$, $\beta_1 \in \bar{\psi}(z_1)$, $\gamma_1, \gamma_2, \beta_1, \alpha \in \bar{\psi}(t_1)$ and $\gamma_1, \beta_1, \beta_2, \alpha \in \bar{\psi}(t_2)$.

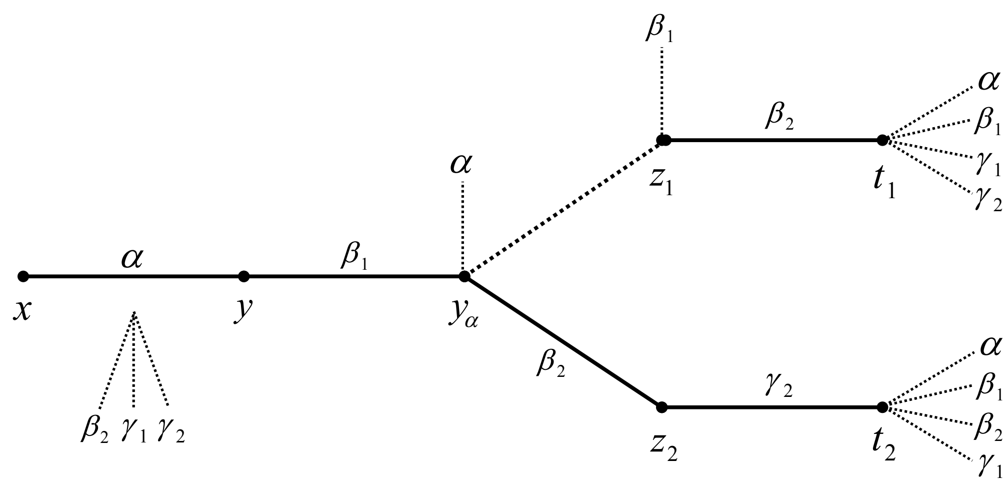

Figure 1.2. ψ on the fork.

Notice that β_1 is missing at both ends of e_1 under ψ, we can actually recolor e_1 by β_1 whenever we want. This little technique lead us to the following claim:

Claim 1.12. *The other endpoint of* $P_{y_\alpha}(\alpha, \beta_2, \psi)$ *is* t_1.

Proof. Recolor e_1 by β_1 to get ψ', $\psi' \in \mathcal{C}^\Delta(G - y_\alpha z_1)$ since $\beta_1 \in \bar{\psi}(z_1) \cap \bar{\psi}(t_1)$. Then $\beta_2 \in \bar{\psi}'(z_1)$, $P_{y_\alpha}(\alpha, \beta_2, \psi') = P_{z_1}(\alpha, \beta_2, \psi')$ by Lemma 1.9. Color e_1 by β_2 to return to ψ, we have the other endpoint of $P_{y_\alpha}(\alpha, \beta_2, \psi)$ is t_1. □

By $\beta_2 \in \bar{\psi}(x) \cup \bar{\psi}(y)$ and the claim above, we know that the (α, β_2)-chain containing xy does not pass through t_1, z_1, y_α, z_2. We switch α, β_2 on this chain to obtain ψ_1. Since $\beta_2 \in \bar{\psi}(x) \cup \bar{\psi}(y)$, we have $\alpha \in \bar{\psi}_1(x) \cup \bar{\psi}_1(y)$. In fact α is not missing at y since otherwise, recolor yy_α by α and color $y_\alpha z_1$ by β_1, we colored the whole graph with Δ colors, which gives a contradiction. Thus $\alpha \in \bar{\psi}_1(x)$. In ψ_1 we have $P_{y_\alpha}(\alpha, \beta_1, \psi_1) = P_{z_1}(\alpha, \beta_1, \psi_1)$ passes along yy_α, let $\psi_2 = \psi_1 / P_x(\alpha, \beta_1, \psi_1)$, then $\beta_1 \in \bar{\psi}_2(x)$.

With the adjusted coloring ψ_2, we finally reach the point that is very close to a contradiction. Now consider $P_{y_\alpha}(\alpha, \gamma_2, \psi_2)$, we split the proof into two cases:

Case 1: The other endpoint of $P_{y_\alpha}(\alpha, \gamma_2, \psi_2)$ is not x or y.

In this case we ignore vertices z_2 and t_2 and use only the path $xyy_\alpha z_1 t_1$. Let $\psi' = \psi_2 / P_{y_\alpha}(\alpha, \gamma_2, \psi_2)$. Under ψ', we have $\gamma_2 \in \bar{\psi}'(y_\alpha)$. Since $\gamma_2 \in \bar{\psi}_2(x) \cup \bar{\psi}_2(y) = \bar{\psi}'(x) \cup \bar{\psi}'(y)$, in fact we have $\gamma_2 \in \bar{\psi}'(x)$. Otherwise recolor yy_α by γ_2 and color $y_\alpha z_1$ by β_1, we colored the whole graph with Δ colors, a contradiction. By the same proof as Claim 1.12 with γ_2 in place of α, we have the other endpoint of $P_{y_\alpha}(\gamma_2, \beta_2, \psi')$ is t_1, so let $\psi'' = \psi' / P_x(\gamma_2, \beta_2, \psi')$, then $\psi''(xy) = \gamma_2$. Notice that $\beta_1 \in \bar{\psi}''(x)$, thus switch γ_2, β_1 on the path xyy_α and color $y_\alpha z_1$ by β_1, we colored the whole graph with Δ colors, which gives a contradiction.

Case 2: The other endpoint of $P_{y_\alpha}(\alpha, \gamma_2, \psi_2)$ is x or y.

In this case we ignore vertices x and y and use only the path $t_1 z_1 y_\alpha z_2 t_2$. Note that $P_{t_2}(\alpha, \gamma_2, \psi_2)$ is disjoint with $P_{y_\alpha}(\alpha, \gamma_2, \psi_2)$ in this case. Let $\psi' = \psi_2 / P_{t_2}(\alpha, \gamma_2, \psi_2)$, then $\psi'(e_2) = \alpha$. Note that $\beta_2 \in \bar{\psi}'(t_2)$, thus switch α, β_2 on the path $t_2 z_2 y_\alpha$, recolor e_1 by β_1 and color $y_\alpha z_1$ by β_2, we colored the whole graph with Δ colors, which gives a contradiction.

Now **Proposition B** is proved and Theorem 1.3 follows immediately. □

2 VERTEX-SPLITTING CONJECTURE

2.1 Introduction

For two integer p, q with $q \geq p$, we use $[p, q]$ to denote the set of all integer between p and q, inclusively. We consider only simple graphs. Let G be a graph with maximum degree $\Delta(G) =$

Δ. Let $v \in V(G)$ with $d_G(v) = t \geq 2$ and $N_G(v) = \{u_1, \ldots, u_t\}$. A *vertex-splitting* in G at v into two vertices v_1 and v_2 gives a new graph G' such that $V(G') = (V(G) \setminus \{v\}) \cup \{v_1, v_2\}$ and $E(G') = E(G-v) \cup \{v_1v_2\} \cup \{v_1u_i : i \in [1,s]\} \cup \{v_2u_i : i \in [s+1,t]\}$, where $s \in [1, t-1]$ is any integer. We say G' is obtained from G by a vertex-splitting.

If a graph G has too many edges, i.e., $|E(G)| > \Delta\lfloor |V(G)|/2 \rfloor$, then we have to color $E(G)$ using exactly $(\Delta+1)$ colors. Such graphs are called *overfull*. Clearly, overfull graphs have an odd order, all regular graphs of an odd order are overfull, and all graphs obtained from a regular Class 1 graph by a vertex-splitting is overfull.

Being overfull is definitely a cause for a graph to be Class 2, but is that the only cause? Hilton and Zhao [12] in 1997 conjectured the following: Let G be an n-vertex Class 1 Δ-regular graph with $\Delta > \frac{n}{3}$. If G^* is obtained from G by a vertex-splitting, then G^* is Δ-critical (vertex-splitting conjecture). Clearly, G^* is overfull. The conjecture asserts that for every $e \in E(G^*)$, $G^* - e$ is no longer Class 2. In other words, being overfull is the only cause for G^* to be Class 2. This conjecture was verified when $\Delta \geq \frac{n}{2}(\sqrt{7}-1) \approx 0.82n$ by Hilton and Zhao [12] in 1997. Song [20] in 2002 showed that the conjecture holds for a particular class of n-vertex Class 1 Δ-regular graphs with $\Delta \geq \frac{n}{2}$. No other progress on this conjecture has been achieved since then. We support this conjecture as below.

Theorem 2.1. *Let n and Δ be positive integers such that $\Delta \geq \frac{3(n-1)}{4}$. If G is obtained from an $(n-1)$-vertex Δ-regular Class 1 graph by a vertex-splitting, then G is Δ-critical.*

If G is an n-vertex overfull graph, then $|E(G)| \geq \Delta(n-1)/2 + 1$. Thus $\sum_{v \in V(G)}(\Delta - d_G(v)) \leq \Delta - 2$. Therefor, if G has a vertex of degree 2, then all other vertices of G are of maximum degree. Is the converse of this statement true? That is, when will be a Class 2 graph with a degree 2 vertex overfull? We investigate this question and show that this happens when Δ is large. In general, for two adjacent vertices $u, v \in V(G)$, we call (u, v) a *full-deficiency pair* of G if $d(u) + d(v) = \Delta(G) + 2$. In particular, if v is of degree 2 in a Δ-critical graph G, then each neighbor u of v has degree Δ by Vizing's Adjacency Lemma (this lemma will be introduced in subsection 2.2). Therefore, (u, v) is a full-deficiency pair of G. We obtain the following result.

Theorem 2.2. *Let n and Δ be positive integers such that $\Delta \geq \frac{3(n-1)}{4}$, and G be an n-vertex Δ-critical graph. If G has a full-deficiency pair, then G is overfull. Consequently, G is obtained from an $(n-1)$-vertex Δ-regular Class 1 multigraph by a vertex-splitting.*

Theorem 2.2 partially supports a conjecture of Chetwynd and Hilton from 1986 [7, 8]. The conjecture states the following: Let G be a simple graph with $\Delta(G) > \frac{1}{3}|V(G)|$. Then G is Class 2 implies that G contains an overfull subgraph H with $\Delta(H) = \Delta(G)$ (overfull conjecture). The overfull conjecture was confirmed only for some special classes of graphs. Chetwynd and Hilton [8] in 1989 verified the conjecture for n-vertex graphs with $\Delta \geq n-3$. Hoffman and Rodger [14] in 1992 confirmed the conjecture for complete multipartite graphs. Plantholt [18] in 2004 showed that the overfull conjecture is affirmative for graphs G with an even order n, maximum degree Δ and minimum degree δ satisfying $(3\delta - \Delta)/2 \geq cn$ for any $c \geq \frac{3}{4}$. The overfull conjecture was also confirmed for large regular graphs in 2013 [9, 10].

Both the overfull conjecture and the vertex-splitting conjecture are best possible in terms of the condition on the maximum degree, by considering the critical Class 2 graph P^*, which is obtained from the Petersen graph by deleting a vertex. Hilton and Zhao [12] proved that the overfull conjecture implies the vertex-splitting conjecture.

The results in Theorem 2.1 and Theorem 2.2 together imply that all n-vertex Δ-critical graphs with a vertex of degree 2 can be obtained from an $(n-1)$-vertex Δ-regular Class 1 multigraph by a vertex splitting, when $\Delta \geq \frac{3(n-1)}{4}$. Thereby, these results provide a way of constructing dense Δ-critical graphs, which are known to be hard. The reminder of this paper is organized as follows. We introduce some definitions and preliminary results in subsection 2.2. In subsection 2.3, we prove Theorem 2.1 and Theorem 2.2 by assuming the truth of several lemmas. These lemmas will be proved in subsection 2.4.

2.2 Definitions and Preliminary Results

Let G be a graph. For $e \in E(G)$, $G - e$ denotes the graph obtained from G by deleting the edge e. The symbol Δ is reserved for $\Delta(G)$, the maximum degree of G throughout this thesis. A *k-vertex* in G is a vertex of degree exactly k in G, and a *k-neighbor* of a vertex v

is a neighbor of v that is a k-vertex in G. For $u, v \in V(G)$, we use $\text{dist}_G(u, v)$ to denote the distance between u and v, which is the length of a shortest path connecting u and v in G. For $S \subseteq V(G)$, define $\text{dist}_G(u, S) = \min_{v \in S} \text{dist}_G(u, v)$.

Let G be a graph and $\varphi \in \mathcal{C}^k(G - e)$ for some edge $e \in E(G)$ and some integer $k \geq 0$. For a color α, a sequence of *Kempe* $(\alpha, *)$-*changes* is a sequence of Kempe changes that each involves the exchanging of the color α and another color from $[1, k]$. Let $x, y \in V(G)$, and $\alpha, \beta, \gamma \in [1, k]$ be three colors. If x and y are contained in a same (α, β)-chain of G with respect to φ, we say x and y are (α, β)-*linked* with respect to φ. Otherwise, x and y are (α, β)-*unlinked* with respect to φ. Without specifying φ, when we just say x and y are (α, β)-linked or x and y are (α, β)-unlinked, we mean they are linked or unlinked with respect to the current edge coloring. Let P be an (α, β)-chain of G with respect to φ that contains both x and y. If P is a path, denote by $P_{[x,y]}(\alpha, \beta, \varphi)$ the subchain of P that has endvertices x and y. By *swapping colors* along $P_{[x,y]}(\alpha, \beta, \varphi)$, we mean exchanging the two colors α and β on the path $P_{[x,y]}(\alpha, \beta, \varphi)$. The notion $P_{[x,y]}(\alpha, \beta)$ always represents the (α, β)-chain with respect to the current edge coloring. Define $P_x(\alpha, \beta, \varphi)$ to be an (α, β)-chain or an (α, β)-subchain of G with respect to φ that starts at x and ends at a different vertex missing exactly one of α and β. (If x is an endvertex of the (α, β)-chain that contains x, then $P_x(\alpha, \beta, \varphi)$ is unique. Otherwise, we take one segment of the whole chain to be $P_x(\alpha, \beta, \varphi)$. We will specify the segment when it is used.) If u is a vertrex on $P_x(\alpha, \beta, \varphi)$, we write $u \in P_x(\alpha, \beta, \varphi)$; and if uv is an edge on $P_x(\alpha, \beta, \varphi)$, we write $uv \in P_x(\alpha, \beta, \varphi)$. Similarly, the notion $P_x(\alpha, \beta)$ always represents the (α, β)-chain with respect to the current edge coloring. If $u, v \in P_x(\alpha, \beta)$ such that u lies between x and v, then we say that $P_x(\alpha, \beta)$ *meets* u *before* v. Suppose that $\alpha \in \overline{\varphi}(x)$ and $\beta, \gamma \in \varphi(x)$. An $(\alpha, \beta) - (\beta, \gamma)$ *swap at* x consists of two operations: first swaps colors on $P_x(\alpha, \beta, \varphi)$ to get an edge k-coloring φ', and then swaps colors on $P_x(\beta, \gamma, \varphi')$. By convention, an (α, α)-swap at x does nothing at x. Suppose the current color of an edge uv of G is α, the notation $uv : \alpha \to \beta$ means to recolor the edge uv using the color β. Recall that $\overline{\varphi}(x)$ is the set of colors not present at x. If $|\overline{\varphi}(x)| = 1$, we will also use $\overline{\varphi}(x)$ to denote the color that is missing at x.

Let $\alpha, \beta, \gamma, \tau, \eta \in [1, k]$. We will use a matrix with two rows to denote a sequence of operations taken on φ. Each entry in the first row represents a path or a sequence of vertices. Each entry in the second row, indicates the action taken on the object above this entry. We require the operations to be taken to follow the "left to right" order as they appear in the matrix. For example, the matrix below indicates three sequential operations taken on the graph based on the coloring from the previous step:

$$\begin{bmatrix} P_{[a,b]}(\alpha, \beta) & rs & ab \\ \alpha/\beta & \gamma \to \tau & \eta \end{bmatrix}.$$

Step 1 Swap colors on the (α, β)-subchain $P_{[a,b]}(\alpha, \beta, \varphi)$.

Step 2 Do $rs : \gamma \to \tau$.

Step 3 Color the edge ab using color η.

Let T be a sequence of vertices and edges of G. We denote by $V(T)$ the set of vertices from $V(G)$ that are contained in T, and by $E(T)$ the set of edges from $E(G)$ that are contained in T.

2.2.1 Multifan

Let G be a graph, $e = rs_1 \in E(G)$ and $\varphi \in \mathcal{C}^k(G - e)$ for some integer $k \geq 0$. A *multifan* centered at r with respect to e and φ is a sequence $F_\varphi(r, s_1 : s_p) := (r, rs_1, s_1, rs_2, s_2, \ldots, rs_p, s_p)$ with $p \geq 1$ consisting of distinct vertices $r, s_1, s_2, \ldots, s_p$ and distinct edges $rs_1, rs_2, \ldots, rs_p$ satisfying the following condition:

(F1) For every edge rs_i with $i \in [2, p]$, there exists $j \in [1, i-1]$ such that $\varphi(rs_i) \in \overline{\varphi}(s_j)$.

We will simply denote a multifan $F_\varphi(r, s_1 : s_p)$ by F if φ and the vertices and edges in $F_\varphi(r, s_1 : s_p)$ are clear. Let $F_\varphi(r, s_1 : s_p)$ be a multifan. By its definition, for any $p^* \in [1, p]$, $F_\varphi(r, s_1 : s_{p^*})$ is a multifan. The following result comes directly from Lemma 1.9.

Lemma 2.3. *Let G be a Class 2 graph and $F_\varphi(r, s_1 : s_p)$ be a multifan with respect to a critical edge $e = rs_1$ and a coloring $\varphi \in \mathcal{C}^\Delta(G-e)$. Then the following statements hold.*

(a) $V(F)$ is φ-elementary.

(b) Let $\alpha \in \overline{\varphi}(r)$. Then for every $i \in [1,p]$ and $\beta \in \overline{\varphi}(s_i)$, r and s_i are (α,β)-linked with respect to φ.

Let $F_\varphi(r, s_1 : s_p)$ be a multifan. We call $s_{\ell_1}, s_{\ell_2}, \dots, s_{\ell_k}$, a subsequence of $s_1 : s_p$, an α-sequence with respect to φ and F if the following holds:

$$\varphi(rs_{\ell_1}) = \alpha \in \overline{\varphi}(s_1), \quad \varphi(rs_{\ell_i}) \in \overline{\varphi}(s_{\ell_{i-1}}), \quad i \in [2,k].$$

A vertex in an α-sequence is called an *α-inducing vertex* with respect to φ and F, and a missing color at an α-inducing vertex is called an *α-inducing color*. For convenience, α itself is also an α-inducing color. We say β is *induced by* α if β is α-inducing. By Lemma 1.9 and the definition of multifan, each color in $\overline{\varphi}(V(F))$ is induced by a unique color in $\overline{\varphi}(s_1)$. Also if α_1, α_2 are two distinct colors in $\overline{\varphi}(s_1)$, then an α_1-sequence is disjoint with an α_2-sequence. For two distinct α-inducing colors β and δ, we write $\delta \prec \beta$ if there exists an α-sequence $s_{\ell_1}, s_{\ell_2}, \dots, s_{\ell_k}$ such that $\delta \in \overline{\varphi}(s_{\ell_i})$, $\beta \in \overline{\varphi}(s_{\ell_j})$ and $i < j$. For convenience, $\alpha \prec \beta$ for any α-inducing color $\beta \neq \alpha$. As a consequence of Lemma 1.9, we have the following properties for a multifan. A proof of the result can be found in [3, Lemma 3.2].

Lemma 2.4. *Let G be a Class 2 graph and $F_\varphi(r, s_1 : s_p)$ be a multifan with respect to a critical edge $e = rs_1$ and a coloring $\varphi \in \mathcal{C}^\Delta(G-e)$. For two colors $\delta \in \overline{\varphi}(s_i)$ and $\lambda \in \overline{\varphi}(s_j)$ with $i, j \in [1,p]$ and $i \neq j$, the following statements hold.*

(a) If δ and λ are induced by different colors, then s_i and s_j are (δ,λ)-linked with respect to φ.

(b) If δ and λ are induced by the same color, $\delta \prec \lambda$, and s_i and s_j are (δ,λ)-unlinked with respect to φ, then $r \in P_{s_j}(\lambda, \delta, \varphi)$.

2.2.2 Kierstead path

Let G be a graph, $e = v_0v_1 \in E(G)$, and $\varphi \in \mathcal{C}^k(G-e)$ for some integer $k \geq 0$. A *Kierstead path* with respect to e and φ is a sequence $K = (v_0, v_0v_1, v_1, v_1v_2, v_2, \ldots, v_{p-1}, v_{p-1}v_p, v_p)$ with $p \geq 1$ consisting of distinct vertices $v_0, v_1, \ldots, v_p$ and distinct edges $v_0v_1, v_1v_2, \ldots, v_{p-1}v_p$ satisfying the following condition:

(K1) For every edge $v_{i-1}v_i$ with $i \in [2,p]$, there exists $j \in [1, i-1]$ such that $\varphi(v_{i-1}v_i) \in \overline{\varphi}(v_j)$.

Clearly a Kierstead path with at most 3 vertices is a multifan. We consider Kierstead paths with 4 vertices. The result below comes directly from Lemma 1.11.

Lemma 2.5. *Let G be a Class 2 graph, $e = v_0v_1 \in E(G)$ be a critical edge, and $\varphi \in \mathcal{C}^\Delta(G-e)$. If $K = (v_0, v_0v_1, v_1, v_1v_2, v_2, v_2v_3, v_3)$ is a Kierstead path with respect to e and φ, then the following statements hold.*

(a) If $\min\{d_G(v_2), d_G(v_3)\} < \Delta$, then $V(K)$ is φ-elementary.

(b) $|\overline{\varphi}(v_3) \cap (\overline{\varphi}(v_0) \cup \overline{\varphi}(v_1))| \leq 1$.

2.3 Proof of Theorems 2.1 and 2.2

We will prove Theorems 2.1 and 2.2 based on the following lemmas, whose proof will be presented in the subsection 2.4.

General properties on Kierstead paths with 5 vertices was proved by the first author of this paper [1]. Here we stress only one of the cases.

Lemma 2.6. *Let G be a Class 2 graph, $ab \in E(G)$ be a critical edge, $\varphi \in \mathcal{C}^\Delta(G-ab)$, and $K = (a, ab, b, bu, u, us, s, st, t)$ be a Kierstead path with respect to ab and φ. If $|\overline{\varphi}(t) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))| \geq 3$, then the following hold:*

(a) There exists $\varphi^* \in \mathcal{C}^\Delta(G-ab)$ satisfies the following properties:

(i) $\varphi^*(bu) \in \overline{\varphi}^*(a) \cap \overline{\varphi}^*(t)$,

(ii) $\varphi^*(us) \in \overline{\varphi}^*(b) \cap \overline{\varphi}^*(t)$, *and*

(iii) $\varphi^*(st) \in \overline{\varphi}^*(a)$.

(b) $d_G(b) = d_G(u) = \Delta$.

Figure 2.1 shows a Kierstead path with the properties described in (a).

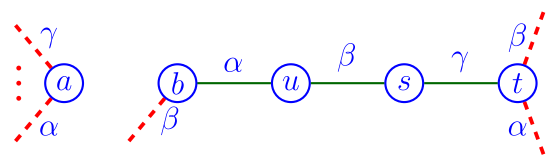

Figure 2.1. Colors on a Kierstead path of 5 vertices

Lemma 2.7. *Let G be a Class 2 graph, $ab \in E(G)$ be a critical edge, $\varphi \in \mathcal{C}^\Delta(G - ab)$, and $K = (a, ab, b, bu, u, us, s, st, t)$ and $K^* = (a, ab, b, bu, u, ux, x)$ be two Kierstead paths with respect to ab and φ, where $x \notin V(K)$. If $|\overline{\varphi}(t) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))| \geq 4$ and $\overline{\varphi}(x) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$, then $d_G(x) = \Delta$.*

A *short-kite* H is a graph with

$$V(H) = \{a, b, c, u, x, y\} \quad \text{and} \quad E(H) = \{ab, ac, bu, cu, ux, uy\}.$$

The lemma below reveals some properties of a short-kite with specified colors on its edges.

Lemma 2.8. *Let G be a Class 2 graph, $H \subseteq G$ be a short-kite with $V(H) = \{a, b, c, u, x, y\}$, and let $\varphi \in \mathcal{C}^\Delta(G - ab)$. Suppose*

$$K = (a, ab, b, bu, u, ux, x) \quad \textit{and} \quad K^* = (b, ab, a, ac, c, cu, u, uy)$$

are two Kierstead path with respect to ab and φ. If $\overline{\varphi}(x) \cup \overline{\varphi}(y) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$, then $\max\{d_G(x), d_G(y)\} = \Delta$.

A *kite* H is a graph with

$$V(H) = \{a, b, c, u, s_1, s_2, t_1, t_2\} \quad \text{and} \quad E(H) = \{ab, ac, bu, cu, us_1, us_2, s_1t_1, s_2t_2\}.$$

The lemma below reveals some properties of a kite with specified colors on its edges.

Lemma 2.9. *Let G be a Class 2 graph, $H \subseteq G$ be a kite with $V(H) = \{a, b, c, u, s_1, s_2, t_1, t_2\}$, and let $\varphi \in \mathcal{C}^\Delta(G - ab)$. Suppose*

$$K = (a, ab, b, bu, u, us_1, s_1, s_1t_1, t_1) \quad \textit{and} \quad K^* = (b, ab, a, ac, c, cu, u, us_2, s_2, s_2t_2, t_2)$$

are two Kierstead paths with respect to ab and φ. If $\varphi(s_1t_1) = \varphi(s_2t_2)$, then $|\overline{\varphi}(t_1) \cap \overline{\varphi}(t_2) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))| \le 4$.

The following result comes directly from Proposition B in Section 1.

Lemma 2.10. *Let G be a Δ-critical graph, $ab \in E(G)$, and $\{u, s_1, s_2, t_1, t_2\} \subseteq V(G)$. If $\Delta \ge d_G(a) + d_G(t_1) + d_G(t_2) + 1$, then for any $\varphi \in \mathcal{C}^\Delta(G - ab)$, G does not contain a fork on $\{a, b, u, s_1, s_2, t_1, t_2\}$ with respect to φ.*

We need the following two additional results to prove Theorem 2.2. Since all vertices not missing a given color α are saturated by the matching that consists of all edges colored by α in G, we have the following result.

Lemma 2.11 (Parity Lemma)**.** *Let G be an n-vertex graph and $\varphi \in \mathcal{C}^\Delta(G)$. Then for any color $\alpha \in [1, \Delta]$, $|\{v \in V(G) : \alpha \in \overline{\varphi}(v)\}| \equiv n \pmod 2$.*

Lemma 2.12. *If G is an n-vertex Class 2 graph with a full-deficiency pair (a, b) such that ab is a critical edge of G, then G satisfies the following properties.*

(i) For every $x \in (N_G(a) \cup N_G(b)) \setminus \{a, b\}$, $d_G(x) = \Delta$;

(ii) For every $x \in V(G) \setminus \{a, b\}$, if $\mathrm{dist}_G(x, \{a, b\}) = 2$, then $d_G(x) \ge \Delta - 1$. Furthermore, if $d_G(a) < \Delta$ and $d_G(b) < \Delta$, then $d_G(x) = \Delta$;

(iii) *For every* $x \in V(G) \setminus \{a,b\}$, *if* $d_G(x) \geq n - |N_G(b) \cup N_G(a)|$, *then* $d_G(x) \geq \Delta - 1$. *Furthermore, if* $d_G(a) < \Delta$ *and* $d_G(b) < \Delta$, *then* $d_G(x) = \Delta$;

(iv) *If there exists* $x \in V(G) \setminus \{a,b\}$ *such that* $d_G(x) < \Delta$, *then there exists* $y \in V(G) \setminus \{a,b,x\}$ *such that* $d_G(y) < \Delta$.

Proof. We let $\varphi \in \mathcal{C}^\Delta(G - ab)$ and

$$F = (b, ba, a)$$

be the multifan with respect to ab and φ. By Lemma 1.9,

$$|\overline{\varphi}(F)| = 2\Delta + 2 - (d_G(a) + d_G(b)) = 2\Delta + 2 - (\Delta + 2) = \Delta. \tag{1}$$

By Lemma 1.9, for every $\varphi' \in \mathcal{C}^\Delta(G - ab)$, $\{a,b\}$ is φ'-elementary and for every $i \in \overline{\varphi}'(a)$ and $j \in \overline{\varphi}'(b)$, a and b are (i,j)-linked with respect to φ'. We will use this fact very often.

Since all the Δ colors appear in $\overline{\varphi}(F)$, each of $N_G(a) \cup \{b\}$ and $N_G(b) \cup \{a\}$ is the vertex set of a multifan with respect to ab and φ. By Lemma 1.9 and (**1**), we know that for every $x \in (N_G(a) \cup N_G(b)) \setminus \{a,b\}$, $d_G(x) = \Delta$. This proves (i).

For (ii), let $x \in V(G) \setminus \{a,b\}$ such that $\text{dist}_G(x, \{a,b\}) = 2$. We assume that $\text{dist}_G(x,b) = 2$ and let $u \in (N_G(b)) \setminus \{a\}) \cap N_G(x)$. Then by (**1**), $K = (a, ab, b, bu, u, ux, x)$ is a Kierstead path with respect to ab and φ. By (**1**) and Lemma 1.11, it follows that $d_G(x) \geq \Delta - 1$. If $d_G(a) < \Delta$ and $d_G(b) < \Delta$, by (**1**) and Lemma 1.11, we get $d_G(x) = \Delta$.

For (iii), let $x \in V(G) \setminus \{a,b\}$ such that $d_G(x) \geq n - |N_G(b) \cup N_G(a)|$. By (i), we may assume that $x \notin (N_G(a) \cup N_G(b)) \setminus \{a,b\}$. Thus $d_G(x) \geq n - |N_G(b) \cup N_G(a)|$ implies that there exists $u \in ((N_G(a) \cup N_G(b))) \cap N_G(x)$. Therefore, $\text{dist}_G(x, \{a,b\}) = 2$. Now Statement (ii) yields the conclusion. Statement (iv) is a consequence of (**1**) and the Parity Lemma. $\square$

Corollary 2.13. *Let* G *be an* n*-vertex Class 2 graph with a full-deficiency pair* (a,b) *such that* ab *is a critical edge of* G. *If* $\Delta \geq \frac{3(n-1)}{4}$, *then there exists at most one vertex* $x \in V(G) \setminus \{a,b\}$ *such that* $d_G(x) = \Delta - 1$.

Proof. Assume to the contrary that there exist distinct $x, y \in V(G) \setminus \{a, b\}$ such that $d_G(x) = d_G(y) = \Delta - 1$. By Lemma 2.12 (i), $x, y \notin (N_G(a) \cup N_G(b)) \setminus \{a, b\}$. By Lemma 2.12 (iii), we may assume that $d_G(b) = \Delta$. Thus $d_G(a) = 2$ as $d_G(a) + d_G(b) = \Delta + 2$. Let c be the other neighbor of a in G. Since (a, c) is a full-deficiency pair of G as well, we may assume $x, y \notin N_G(c)$.

Since $d_G(b) = d_G(c) = \Delta$ and $d_G(x) = d_G(y) = \Delta - 1$, we get $|N_G(b) \cap N_G(c)| \geq \frac{n}{2} - 1$ and $|N_G(x) \cap N_G(y)| \geq \frac{n}{2} - 2$. Since $b, c, x, y \notin N_G(b) \cap N_G(c)$ and $b, c, x, y \notin N_G(x) \cap N_G(y)$, we get $|N_G(b) \cap N_G(c) \cap N_G(x) \cap N_G(y)| \geq 1$. Let $u \in N_G(b) \cap N_G(c) \cap N_G(x) \cap N_G(y)$, H be the short-kite with $V(H) = \{a, b, c, u, x, y\}$, and let $\varphi \in \mathcal{C}^\Delta(G - ab)$. As $\{a, b\}$ is φ-elementary, $|\overline{\varphi}(a) \cup \overline{\varphi}(b)| = 2\Delta + 2 - (d_G(a) + d_G(b)) = \Delta$ and so $\overline{\varphi}(a) \cup \overline{\varphi}(b) = [1, \Delta]$. Thus $K = (a, ab, b, bu, u, ux, x)$ and $K^* = (b, ab, a, ac, c, cu, u, uy)$ are two Kierstead paths with respect to ab and φ, and $\overline{\varphi}(x) \cup \overline{\varphi}(y) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$. However, $d_G(x) = d_G(y) = \Delta - 1$, contradicting Lemma 2.8. □

Proof of Theorem 2.1. Since G is overfull, G is Class 2. We show that every edge of G is critical. Suppose to the contrary that there exists $xy \in E(G)$ such that xy is not a critical edge of G. Let

$$G^* = G - xy.$$

Then $\chi'(G^*) = \Delta + 1$.

Since ab is a critical edge of G, $ab \neq xy$. Also, since ab is a critical edge of G, and any Δ-coloring of $G - ab$ gives a Δ-coloring of $G^* - ab$, ab is also a critical edge of G^*. Since $d_{G^*}(x) = d_{G^*}(y) = \Delta - 1$, we reach a contradiction to Corollary 2.13. □

Proof of Theorem 2.2. Let (a, b) be a full-deficiency pair of G. It suffices to only show that for every $v \in V(G) \setminus \{a, b\}$, $d_G(v) = \Delta$. To see this, let G be a Δ-critical graph with a full-deficiency pair (a, b) and for every $v \in V(G) \setminus \{a, b\}$, $d_G(v) = \Delta$. Let $\varphi \in \mathcal{C}^\Delta(G - ab)$. Since $\overline{\varphi}(a) \cap \overline{\varphi}(b) = \emptyset$ and $d_G(a) + d_G(b) = \Delta + 2$, $\varphi(a) \cap \varphi(b) = \emptyset$. Thus, identifying a and b in G gives a Δ-coloring of a Δ-regular multigraph G^*. This implies that $|V(G^*)| = n - 1$ is even. So n is odd. Consequently, G is overfull.

Thus, for the sake of contradiction, we assume that there exists $x \in V(G) \setminus \{a, b\}$ such that $d_G(x) < \Delta$. By Lemma 2.12 (iv), there exits $y \in V(G) \setminus \{a, b, x\}$ such that $d_G(y) < \Delta$. Furthermore, by Lemma 2.12 (iii) and Corollary 2.13, there exists at most one vertex $x \in V(G)\setminus\{a, b\}$ such that $d_G(x) = \Delta - 1$, and for all other vertex $y \in V(G)\setminus\{a, b, x\}$, if $d_G(y) < \Delta$, then $d_G(y) < n - |N_G(b) \cup N_G(a)|$. This gives a vertex $t \in V(G) \setminus \{a, b\}$ such that $d_G(t) < n - |N_G(b) \cup N_G(a)|$. Let $\varphi \in \mathcal{C}^\Delta(G - ab)$ and

$$F = (b, ba, a)$$

be the multifan with respect to ab and φ. By Lemma 1.9,

$$|\overline{\varphi}(F)| = 2\Delta + 2 - (d_G(a) + d_G(b)) = \Delta. \quad (\mathbf{2})$$

Assume, without loss of generality, that $d_G(b) \geq d_G(a)$. Then $d_G(b) \geq \frac{3(n-1)}{8} + 1$ as $d_G(a) + d_G(b) = \Delta + 2 \geq \frac{3(n-1)}{4} + 2$. By Lemma 2.12 (i) and (ii), we assume that $\text{dist}_G(t, \{a, b\}) \geq 3$. Since $\Delta \geq \frac{3(n-1)}{4}$, for any $s \in N_G(t)$ with $d_G(s) \geq \Delta - 1$ (such s exists as t is adjacent to at least two Δ-neighbors by VAL), we conclude that there exists $u \in N_G(b) \cap N_G(s)$. Now by (**2**), $K = (a, ab, b, bu, u, us, s, st)$ is a Kierstaed path with respect to ab and φ. This implies that $d_G(b) = d_G(u) = \Delta$ by Lemma 2.6 (b). Thus $d_G(a) = 2$. We let c be the other Δ-neighbor of a.

As $d_G(a) = 2$ and $ab \in E(G)$, $|N_G(b) \cup N_G(a)| \geq \Delta + 1 > \frac{3n}{4}$. Since G is Δ-critical, VAL implies that for every $s \in N_G(t)$, $d_G(s) \geq \Delta + 2 - d_G(t) \geq \Delta + 2 + |N_G(b) \cup N_G(a)| - n \geq n - |N_G(b) \cup N_G(a)|$. Thus, by Lemma 2.12 (iii) and Corollary 2.13, there exists at most one vertex $s \in N_G(t)$ such that $d_G(s) < \Delta$. In this case, $d_G(s) = \Delta - 1$.

Next, we claim that

$$\text{for any } x \in V(G) \setminus \{a, b\},\ d_G(x) < \Delta \Rightarrow d_G(x) < n - |N_G(b) \cup N_G(a)| \leq n - \Delta - 1. \quad (\mathbf{3})$$

Assume to the contrary that $d_G(x) \geq n - |N_G(b) \cup N_G(a)|$. By Lemma 2.12 (iii), we have

$d_G(x) = \Delta - 1$. Again, as $\Delta \geq \frac{3(n-1)}{4}$ and every vertex in $(N_G(a) \cup N_G(b) \cup N_G(t)) \setminus \{a, b\}$ has degree at least $\Delta - 1$, for any $s \in N_G(t)$, we conclude that there exists $u \in N_G(b) \cap N_G(s) \cap N_G(x)$. Now by (**2**), $K = (a, ab, b, bu, u, us, s, st)$ and $K^* = (a, ab, b, bu, u, ux, x)$ are two Kierstaed paths with respect to ab and φ. Clearly, $\overline{\varphi}(t) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$ and $\overline{\varphi}(x) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$, and $|\overline{\varphi}(t)| \geq \Delta - (n - \Delta - 1) = 2\Delta - n + 1 \geq \frac{n-1}{2}$. Since $n \geq |V(K) \cup \{x\}| = 6$, $\Delta \geq \frac{3(n-1)}{4}$ implies that $\Delta \geq 4$. As $\{b, s, t, x\} \cap N_G(b) = \emptyset$, we see that $n \geq 4 + 4 = 8$. Hence, $|\overline{\varphi}(t)| \geq \lceil \frac{n-1}{2} \rceil \geq 4$, achieving a contradiction to Lemma 2.7.

By Lemma 2.12 (iii) and (iv) and the conclusion in (**3**), we let $t_1, t_2 \in V(G) \setminus \{a, b\}$ such that both of them have degree less than $n - \Delta$. Let $s_1 \in N_G(t_1)$ and $s_2 \in N_G(t_2)$ be any two distinct vertices. Since G is Δ-critical, VAL implies that for every $s_i \in N_G(t_i)$, $d_G(s_i) \geq 2\Delta - n > n - \Delta$. Thus, By Lemma 2.12 (iii) and (iv) and the conclusion in (**3**), $d_G(s_1) = d_G(s_2) = \Delta$. Thus, since $b, c, s_1, s_2 \notin N_G(b) \cap N_G(c)$ and $b, c, s_1, s_2 \notin N_G(s_1) \cap N_G(s_2)$,

$$
\begin{aligned}
|N_G(b) \cap N_G(c) \cap N_G(s_1) \cap N_G(s_2)| &\geq |N_G(s_1) \cap N_G(s_2)| - (n - |N_G(b) \cap N_G(c)| - 4), \\
&\geq |N_G(s_1) \cap N_G(s_2)| + |N_G(b) \cap N_G(c)| + 4 - n \\
&\geq 2\Delta - n + 2\Delta - n + 4 - n \geq 1,
\end{aligned}
$$

as $\Delta \geq \frac{3(n-1)}{4}$. Let $u \in N_G(b) \cap N_G(c) \cap N_G(s_1) \cap N_G(s_2)$. Then H with $V(H) = \{a, b, c, u, s_1, s_2, t_1, t_2\}$ is kite. By (**2**), both

$$
K = (a, ab, b, bu, u, us_1, s_1, s_1t_1, t_1) \quad \text{and} \quad K^* = (b, ab, a, ac, c, cu, u, us_2, s_2, s_2t_2, t_2)
$$

are Kierstead paths with respect to ab and φ. Let

$$
A = \{t \in V(G) \setminus \{a, b\} : d_G(t) < n - \Delta\}.
$$

We consider two cases below.

Case 1: there exist two distinct $t_1, t_2 \in A$ such that $\varphi(t_1) \cap \varphi(t_2) \neq \emptyset$.

In this case, we choose $s_1 \in N_G(t_1)$ and $s_2 \in N_G(t_2)$ such that $\varphi(s_1t_1) = \varphi(s_2t_2)$. Let $\Gamma = \overline{\varphi}(t_1) \cap \overline{\varphi}(t_2)$. Since $\overline{\varphi}(a) \cup \overline{\varphi}(b) = [1, \Delta]$, $\Gamma \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$. By (**2**) and the assumption of this case, $|\Gamma| \geq \Delta - 2(n - \Delta - 2) = 3\Delta - 2n + 4 \geq \frac{n-1}{4} + 2$. Since $\Delta \geq \frac{3(n-1)}{4} \geq \frac{3}{4}|(V(H)| - 1)$, $\Delta \geq 6$. Since also $s_1, s_2, t_1, t_2 \notin N_G(b)$, we have $n \geq |V(H)| + 3 \geq 11$. Thus, $|\Gamma| \geq \lceil \frac{n-1}{4} \rceil + 2 \geq 5$, contradicting Lemma 2.9.

Case 2: for each two distinct $t_1, t_2 \in A$, it holds that $\varphi(t_1) \cap \varphi(t_2) = \emptyset$.

By (**2**) and the assumption of this case, we see that H^* with $V(H^*) = \{a, b, u, s_1, s_2, t_1, t_2\}$ is a fork. However, by (**3**), $d_G(a) + d_G(t_1) + d_G(t_2) \leq 2 + 2(n - \Delta - 2) = 2n - 2\Delta - 2 < \Delta$, as $\Delta \geq \frac{3(n-1)}{4}$, contradicting Lemma 2.10. The proof is now completed.

□

2.4 Proof of Lemmas 2.6 to 2.9

Lemma 2.6. *Let G be a Class 2 graph, $ab \in E(G)$ be a critical edge, $\varphi \in \mathcal{C}^\Delta(G - ab)$, and $K = (a, ab, b, bu, u, us, s, st, t)$ be a Kierstead path with respect to ab and φ. If $|\overline{\varphi}(t) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))| \geq 3$, then the following hold:*

(a) There exists $\varphi^* \in \mathcal{C}^\Delta(G - ab)$ satisfies the following properties:

- *(i) $\varphi^*(bu) \in \overline{\varphi}^*(a) \cap \overline{\varphi}^*(t)$,*
- *(ii) $\varphi^*(us) \in \overline{\varphi}^*(b) \cap \overline{\varphi}^*(t)$, and*
- *(iii) $\varphi^*(st) \in \overline{\varphi}^*(a)$.*

(b) $d_G(b) = d_G(u) = \Delta$.

Proof. By Lemma 1.9, for every $\varphi' \in \mathcal{C}^\Delta(G - ab)$, $\{a, b\}$ is φ'-elementary and for every $i \in \overline{\varphi}'(a)$ and $j \in \overline{\varphi}'(b)$, a and b are (i, j)-linked with respect to φ'.

Let $\Gamma = \overline{\varphi}(t) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))$, and $\alpha, \beta \in \Gamma$. If $\alpha, \beta \in \overline{\varphi}(a)$, then we let $\lambda \in \overline{\varphi}(b)$, and do a (β, λ)-swap at b. If $\alpha, \beta \in \overline{\varphi}(b)$, then we let $\lambda \in \overline{\varphi}(a)$, and do a (β, λ)-swap at a. Therefore, we may assume that

$$\alpha \in \overline{\varphi}(a) \quad \text{and} \quad \beta \in \overline{\varphi}(b).$$

If $\varphi(bu) = \delta \neq \alpha$, then we do an (α, δ)-swap at t, and rename the color δ as α and vice versa. Thus we may assume

$$\varphi(bu) = \alpha.$$

Assume first that $\varphi(us) \in \overline{\varphi}(b)$. We do a $(\beta, \varphi(us))$-swap at t and still call the resulting coloring by φ, we see that $\varphi(us) \in \overline{\varphi}(b) \cap \overline{\varphi}(t)$. By permuting the name of the colors, we let $\varphi(us) = \beta$. Let $\varphi(st) = \gamma$. Since $\alpha, \beta \in \overline{\varphi}(t)$, $\gamma \neq \alpha, \beta$. If $\gamma \in \overline{\varphi}(a)$, we are done. So we assume $\gamma \in \overline{\varphi}(b) \cup \overline{\varphi}(u)$. We color ab by α and uncolor bu. Denote this resulting coloring by φ'. Then $K' = (b, bu, u, us, s, st, t)$ is a Kierstead path with respect to bu and φ'. However, $\alpha, \beta \in \overline{\varphi}'(t) \cap (\overline{\varphi}'(b) \cup \overline{\varphi}'(u))$, showing a contradiction to Lemma 1.11.

Thus we let $\varphi(us) = \delta \in \overline{\varphi}(a)$. Then $\delta \neq \beta$ by Lemma 1.9. Let $\varphi(st) = \gamma$. Clearly, $\gamma \neq \alpha, \beta, \delta$. We have either $\gamma \in \overline{\varphi}(a)$ or $\gamma \in \overline{\varphi}(b) \cup \overline{\varphi}(u)$. We consider three cases below.

Case 1. $\gamma \in \overline{\varphi}(b)$**.**

If $u \in P_a(\beta, \delta) = P_b(\beta, \delta)$, we do a (β, δ)-swap at t. Since a and b are (δ, γ)-linked and $u \in P_t(\delta, \gamma)$, we do a (δ, γ)-swap at a. This gives a desired coloring φ^*.

If $u \notin P_a(\beta, \delta) = P_b(\beta, \delta)$, we first do a (β, δ)-swap at a and then a (β, γ)-swap at a. Again this gives a desired coloring φ^*.

Case 2. $\gamma \in \overline{\varphi}(u)$**.**

If $\delta \in \Gamma$, since b and u are (β, γ)-linked by Lemma 1.9, we do (β, γ)-swap at t. Now $u \in P_t(\delta, \beta)$, we do a (β, δ)-swap at a. This gives a desired coloring φ^*. Thus we assume $\delta \notin \Gamma$. Since b and u are (β, γ)-linked by Lemma 1.9, and a and u are (δ, γ)-linked by Lemma 2.4 (a), we do $(\beta, \gamma) - (\gamma, \delta)$-swaps at t. Finally, since $u \in P_t(\beta, \delta)$, we do a (β, δ)-swap at a. This gives a desired coloring φ^*.

Case 3. $\gamma \in \overline{\varphi}(a)$**.**

If $\delta \in \Gamma$, we do a (β, γ)-swap at t and then a (β, δ)-swap at a to get a desired coloring φ^*. Thus we assume $\delta \notin \Gamma$. Let $\tau \in \Gamma \setminus \{\alpha, \beta\}$. If $\tau \in \overline{\varphi}(u)$, since a and u are (δ, γ)-linked by Lemma 2.4 (a), we do a (τ, δ)-swap at t. This gives back to the previous case that $\delta \in \Gamma$. Next we assume $\tau \in \overline{\varphi}(b)$. It is clear that $u \in P_a(\tau, \delta) = P_b(\tau, \delta)$, as otherwise, a (τ, δ)-swap at a gives a desired coloring. Thus we do a (τ,δ)-swap at t, giving back to the previous case that $\delta \in \Gamma$.

Now we assume $\tau \in \overline{\varphi}(a)$. If $u \notin P_a(\beta,\delta)$, we do a (β,δ)-swap at a. Since a and b are (α,δ)-linked and $u \in P_a(\alpha,\delta)$, we do an (α,δ) swap at t. Now since $u \in P_t(\gamma,\delta)$, we do a (γ,δ)-swap at a, and do $(\beta,\gamma)-(\gamma,\alpha)$-swaps at t. Since a and b are (τ,γ)-linked, we do a (τ,γ)-swap at t, and then a (β,γ) -swap at a. Now since $u \in P_t(\beta,\delta)$, we do a (β,δ)-swap at a. This gives a desired coloring. Thus, we assume $u \in P_a(\beta,\delta)$. We do a (β,δ)-swap at t, and then a (τ,β)-swap at t. Next we do a (β,γ)-swap at a and then a (γ,δ)-swap at a. This gives a desired coloring for (b).

For statement (b), let $\varphi^* \in \mathcal{C}^\Delta(G-ab)$ satisfying (i)–(iii). Let $\alpha,\gamma \in \overline{\varphi}^*(a)$, $\beta \in \overline{\varphi}^*(b)$ with $\alpha,\beta \in \overline{\varphi}(t)$ such that

$$\varphi^*(bu)=\alpha, \quad \varphi^*(us)=\beta, \quad \text{and} \quad \varphi^*(st)=\gamma.$$

Let $\tau \in \overline{\varphi}^*(t)\setminus\{\alpha,\beta\}$. Suppose to the contrary first that $d_G(b) \le \Delta-1$. Let $\lambda \in \overline{\varphi}^*(b)\setminus\{\beta\}$. We do $(\tau,\lambda)-(\lambda,\gamma)$-swaps at t. Now we color ab by α and uncolor bu to get a coloring φ'. Then $K'=(b,bu,u,us,s,st,t)$ is a Kierstead path with respect to bu and φ'. However, $\alpha,\beta \in \overline{\varphi}'(t)\cap(\overline{\varphi}'(b)\cup\overline{\varphi}'(u))$, contradicting Lemma 1.11.

Assume then that $d_G(b)=\Delta$ and $d_G(u)\le \Delta-1$. Let $\lambda \in \overline{\varphi}^*(u)$. Since (a,ab,b,bu,u) is a multifan, $\lambda \notin \{\alpha,\beta,\gamma\}$. Since u and b are (β,λ)-linked and u and a are (γ,λ)-linked by Lemma 2.4 (b), we do $(\beta,\lambda)-(\lambda,\gamma)$-swap(s) at t. Now we color ab by α and uncolor bu to get a coloring φ'. Then $K'=(b,bu,u,us,s,st,t)$ is a Kierstead path with respect to bu and φ'. However, $\alpha \in \overline{\varphi}'(t)\cap\overline{\varphi}'(u)$, contradicting Lemma 1.11, since $d_G(u)<\Delta$. □

Lemma 2.7. *Let G be a Class 2 graph, $ab \in E(G)$ be a critical edge, $\varphi \in \mathcal{C}^\Delta(G-ab)$, and $K=(a,ab,b,bu,u,us,s,st,t)$ and $K^*=(a,ab,b,bu,u,ux,x)$ be two Kierstead paths with respect to ab and φ, where $x \notin V(K)$. If $|\overline{\varphi}(t)\cap(\overline{\varphi}(a)\cup\overline{\varphi}(b))|\ge 4$ and $\overline{\varphi}(x)\subseteq\overline{\varphi}(a)\cup\overline{\varphi}(b)$, then $d_G(x)=\Delta$.*

Proof. Assume to the contrary that $d_G(x) \le \Delta-1$. Since $\overline{\varphi}(x) \subseteq \overline{\varphi}(a)\cup\overline{\varphi}(b)$, Lemma 1.11 gives that $d_G(x) = \Delta - 1$. By Lemma 2.6, $d_G(b) = d_G(u) = \Delta$ and we assume that $\overline{\varphi}(b) = \beta$, $\varphi(bu) = \alpha$, $\varphi(us) = \beta$, $\varphi(st) = \gamma$, $\alpha, \gamma \in \overline{\varphi}(a)$, and $\alpha, \beta \in \overline{\varphi}(t)$. In the following, when we swap colors, we always make sure that the colors on the edges bu and us are unchanged. The color on the edge st might be changed, but the new color will still be a color from $\overline{\varphi}(a)$. This is guaranteed by using the elementary fact that for every coloring $\varphi' \in \mathcal{C}^{\Delta}(G - ab)$, a and b are (i, j)-linked for every $i \in \overline{\varphi}'(a)$ and every $j \in \overline{\varphi}'(b)$. We use this fact every often without even mentioning it.

Let $\varphi(ux) = \delta$ and $\overline{\varphi}(x) = \tau$. We first claim that if $\delta \neq \gamma$, then we may assume $\delta \in \overline{\varphi}(t)$. Clearly, $\delta \neq \alpha, \beta$, and since K^* is a Kierstead path and $\overline{\varphi}(b) = \beta$, we have $\varphi(ux) = \delta \in \overline{\varphi}(a)$. Let $\Gamma = \overline{\varphi}(t) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))$, and let $\{\alpha, \beta, \eta, \lambda\} \subseteq \Gamma$. Suppose that $\delta \notin \overline{\varphi}(t)$. We do $(\beta, \gamma) - (\gamma, \eta)$-swaps at b. Denote the new coloring by φ'.

If $ux \notin P_b(\eta, \delta)$, based on φ', we do an (η, δ) -swap at b and and then do an (α, δ)-swap at t. If $\tau = \gamma$, we do $(\delta, \gamma) - (\gamma, \eta)$-swaps at b and then do an (α, η)-swap at t. Finally we do $(\eta, \lambda) - (\lambda, \gamma) - (\gamma, \beta)$-swaps at b. Thus we assume $\tau \neq \gamma$. Clearly, $\tau \neq \alpha$. If $\tau = \beta$, we simply do a (β, δ)-swap at b, then $(\beta, \gamma) - (\gamma, \eta)$-swaps at b, an (α, η)-swap at t, and finally $(\eta, \lambda) - (\lambda, \gamma) - (\gamma, \beta)$-swaps at b. Thus, $\tau \neq \alpha, \beta$. We do $(\delta, \tau) - (\tau, \eta)$-swaps at b, an (α, η)-swap at t, and finally do $(\eta, \lambda) - (\lambda, \gamma) - (\gamma, \beta)$-swaps at b.

Thus, we assume that $ux \notin P_b(\eta, \delta)$. Based on φ', we do an (η, δ)-swap at t and then $(\eta, \lambda) - (\lambda, \gamma) - (\gamma, \beta)$-swaps at b.

After the operations above, we have $\varphi(bu) = \alpha$, $\varphi(us) = \beta$, $\varphi(st) = \gamma$, $\varphi(ux) = \delta$ and $\overline{\varphi}(x) = \tau$, and $\alpha, \beta, \delta, \lambda \in \Gamma$.

Case 1: $\overline{\varphi}(x) = \gamma$.

Recall that $\alpha, \beta, \delta \in \Gamma$. We color ab by α, recolor bu by β, and uncolor us. Note that u and t are (α, γ)-linked, as otherwise an (α, γ)-swap at u and a (β, γ)-swap at s gives a coloring φ' such that $\gamma \in \overline{\varphi}'(u) \cap \overline{\varphi}'(s)$. Thus we do an (α, γ)-swap at both a and x, recolor ux by α, and then a (β, δ)-swap at both x and a. It is clear that $ux \in P_t(\alpha, \gamma)$ and $P_t(\alpha, \gamma)$

meets u before x. We now do the following operations:

$$\begin{bmatrix} P_{[t,u]}(\alpha,\gamma) & ux & bu & ab \\ \alpha/\gamma & \alpha\to\beta & \beta\to\gamma & \gamma\to\beta \end{bmatrix}.$$

Based on the coloring above, we do $(\alpha,\beta)-(\beta,\delta)$-swaps at both x and a, and then an (α,δ)-swap at x. Denote the new coloring by φ'. Now we do an (α,β)-swap at t and color us by α, giving a Δ-coloring of G.

Case 2: $\overline{\varphi}(x)\neq\gamma$ **and** $\varphi(ux)\neq\varphi(st)=\gamma$.

Recall that $\alpha,\beta,\delta\in\Gamma$ and $|\Gamma|\geq 4$. Let $\{\alpha,\beta,\delta,\lambda\}\subseteq\Gamma$. We show that there is a coloring φ' such that $\{\alpha,\delta\}\subseteq\overline{\varphi}'(t)$ and $\overline{\varphi}'(x)=\beta$. Since we already have $\{\alpha,\delta\}\subseteq\overline{\varphi}(t)$, we assume that $\overline{\varphi}(x)=\tau\neq\beta$. Thus $\tau\in\overline{\varphi}(a)$. If $\tau=\alpha$, we simply do an (α,β)-swap at x. Therefore, $\tau\neq\alpha$. It is possible that $\tau=\lambda$, but we deal with this together with the case that $\tau\neq\lambda$. We first do $(\beta,\gamma)-(\gamma,\lambda)-(\lambda,\tau)$-swaps at b, then we do an (α,τ)-swap at both x and t. Now we do $(\tau,\gamma)-(\gamma,\beta)$-swaps at b, and an (α,β)-swap at both x and t. We now derive a contradiction based on the coloring of $E(K)\cup E(K^*)$, as shown in Figure 2.2.

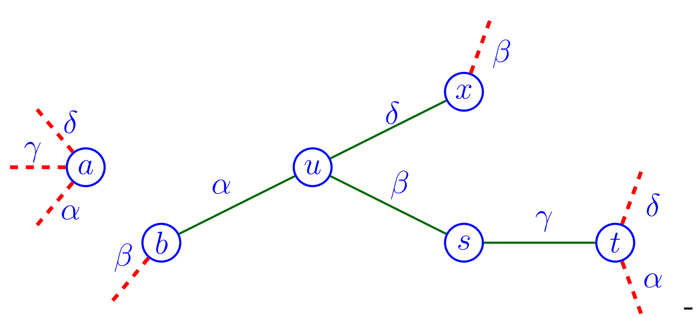

Figure 2.2. Colors on the edges of K and K^*

We color ab by α, recolor bu by β, and uncolor us. We then do an (α,β)-swap at both x and t, and an (α,δ)-swap at x. Now since u and s are (β,δ)-linked, we do a (β,δ)-swap at both x and a. Since u and t are (δ,γ)-linked, we do a (δ,γ)-swap at a. Finally, we do $(\beta,\delta)-(\delta,\alpha)$-swaps at x. Now $P_u(\alpha,\beta)=uba$, so u and s are (α,β)-unlined. We do an (α,β)-swap at u and color us by β. This gives a Δ-coloring of G, showing a contradiction.

Case 3: $\overline{\varphi}(x) = \tau \neq \gamma$ **and** $\varphi(ux) = \varphi(st) = \gamma$.

We again let $\Gamma = \overline{\varphi}(t) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))$, and let $\{\alpha, \beta, \lambda\} \subseteq \Gamma$. We show that this case can be converted to Case 2. We first claim that $\overline{\varphi}(x) = \tau \neq \beta$. As otherwise, we first do an (α, β)-swap at x, and then a (β, γ)-swap at b. Now, $P_b(\gamma, \alpha) = bux$, showing a contradiction to the fact that a and b are (α, γ)-linked. Next, we claim that $\tau \neq \alpha$. As otherwise, we simply do a (β, γ)-swap at b and achieve a same contradiction as above. Thus, $\tau \neq \alpha, \beta$.

We do a (β, γ)-swap at b and an (α, γ)-swap at t. Now do a (τ, γ)-swap at both x and t, a (γ, λ)-swap at b, an (λ, α)-swap at t, and finally a (β, λ)-swap at b. Let the new coloring be φ'. We see that $\varphi'(st) = \lambda \neq \varphi'(ux) = \tau$. We verify that it still holds that $|\overline{\varphi}'(t) \cap (\overline{\varphi}'(a) \cup \overline{\varphi}'(b))| \geq 4$. If $\tau \in \Gamma$, we now have $\alpha, \beta, \gamma, \tau \in \overline{\varphi}'(t) \cap (\overline{\varphi}'(a) \cup \overline{\varphi}'(b))$. If $\tau \notin \Gamma$, then $(\Gamma \setminus \{\lambda\}) \cup \{\tau\} \subseteq \overline{\varphi}'(t) \cap (\overline{\varphi}'(a) \cup \overline{\varphi}'(b))$. □

Lemma 2.8. *Let G be a Class 2 graph, $H \subseteq G$ be a short-kite with $V(H) = \{a, b, c, u, x, y\}$, and let $\varphi \in \mathcal{C}^{\Delta}(G - ab)$. Suppose*

$$K = (a, ab, b, bu, u, ux, x) \quad \text{and} \quad K^* = (b, ab, a, ac, c, cu, u, uy)$$

are two Kierstead path with respect to ab and φ. If $\overline{\varphi}(x) \cup \overline{\varphi}(y) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$, then $\max\{d_G(x), d_G(y)\} = \Delta$.

Proof. Assume to the contrary that $\max\{d_G(x), d_G(y)\} \leq \Delta - 1$. Since both K and K^* are Kierstead paths and $\overline{\varphi}(x) \cup \overline{\varphi}(y) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$, Lemma 1.11 and (b) implies that $d_G(b) = d_G(u) = \Delta$ and $d_G(x) = d_G(y) = \Delta - 1$.

Let $\overline{\varphi}(b) = \{1\}$. Then $\varphi(ac) = 1$. We may assume $\varphi(uy) = 1$. The reasoning is below. Since a and b are $(1, \alpha)$-linked for every $\alpha \in \overline{\varphi}(y) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$, we may assume $\overline{\varphi}(y) = 1$. Then a $(1, \varphi(uy))$-swap at y gives a coloring, call it still φ, such that $\varphi(uy) = 1$. We consider now two cases.

Case 1: $\overline{\varphi}(x) = \overline{\varphi}(y)$.

Let $\varphi(ux) = \gamma$, and $\overline{\varphi}(x) = \overline{\varphi}(y) = \eta$. As $\varphi(uy) = \overline{\varphi}(b) = 1$, $1 \notin \{\gamma, \eta\}$. As both K and K^* are Kierstead paths and $\overline{\varphi}(x) \cup \overline{\varphi}(y) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$, $\gamma, \eta \in \overline{\varphi}(a)$. Denote by $P_u(1, \gamma)$ the

$(1,\gamma)$-subchain starting at u that does not include the edge ux.

Claim 2.1. *We may assume that $P_u(1,\gamma)$ ends at x, some vertex $z \in V(G)\setminus\{a,b,c,u,x,y\}$, or passing c ends at a.*

Proof. Note that $P_a(1,\gamma) = P_b(1,\gamma)$. If $u \notin P_a(1,\gamma)$, then the $(1,\gamma)$-chain containing u is a cycle or a path with endvertices contained in $V(G)\setminus\{a,b,c,u,x,y\}$. Thus $P_u(1,\gamma)$ ends at x or some $z \in V(G)\setminus\{a,b,c,u,x,y\}$. Hence we assume $u \in P_a(1,\gamma)$. As a consequence, $P_u(1,\gamma)$ ends at either b or a. If $P_x(1,\gamma)$ ends at b, we color ab by 1, uncolor ac, and exchange the vertex labels b and c. This gives an edge Δ-coloring of $G-ab$ such that $P_u(1,\gamma)$ ends at a. Thus, if $u \in P_a(1,\gamma)$, we may always assume that $P_u(1,\gamma)$ ends at a. $\square$

Let $\varphi(bu) = \delta$. Again, $\delta \in \overline{\varphi}(a)$. Figure 2.3 depicts the colors and missing colors on these specified edges and vertices, respectively. Clearly, $\delta \neq 1,\gamma$. Since a and b are $(1,\delta)$-linked with respect to φ, $\eta \neq \delta$. Thus, γ,δ and η are pairwise distinct.

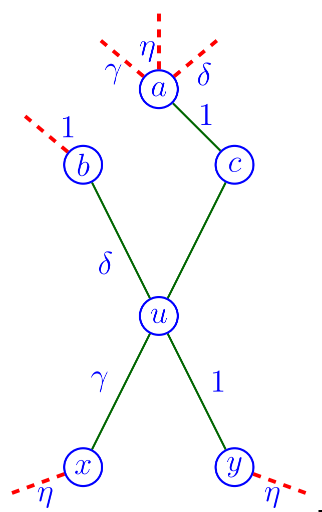

Figure 2.3. Colors on the edges connecting x and y to b

Claim 2.2. *It holds that $ub \in P_y(\eta,\delta)$ and $P_y(\eta,\delta)$ meets u before b.*

Proof. Let φ' be obtained from φ by coloring ab by δ and uncoloring bu. Note that $\overline{\varphi}'(b) = 1, \overline{\varphi}'(u) = \delta$ and $\varphi'(uy) = 1$. Thus $F^* = (u, ub, b, uy, y)$ is a multifan and so u and y are (η,δ)-linked by Lemma 1.9. By uncoloring ab and coloring bu by δ, we get back the original coloring φ. Therefore, under the coloring φ, $u \in P_y(\eta,\delta)$ and $P_y(\eta,\delta)$ meets u before b. $\square$

We apply the following operations based on φ:

$$\begin{bmatrix} ux & P_{[u,y]}(\eta,\delta) & ub & P_u(1,\gamma) & ab \\ \gamma \to \eta & \delta/\eta & \delta \to 1 & 1/\gamma & \delta \end{bmatrix}.$$

By Claim 2.1, $P_u(1,\gamma)$ does not end at b. In any case, the above operations give an edge Δ-coloring of G. This contradicts the earlier assumption that $\chi'(G) = \Delta + 1$.

Case 2: $\overline{\varphi}(x) \neq \overline{\varphi}(y)$.

Let

$$\varphi(bu) = \alpha, \quad \varphi(ux) = \beta, \quad \overline{\varphi}(x) = \tau, \quad \text{and} \quad \overline{\varphi}(y) = \gamma.$$

As $\varphi(uy) = \overline{\varphi}(b) = 1$, $1 \notin \{\alpha,\beta,\gamma\}$. Also, since a and b are $(1,\alpha)$-linked, $\gamma \neq \alpha$. Since both K and K^* are Kierstead paths and $\overline{\varphi}(x) \cup \overline{\varphi}(y) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$, we have $\alpha,\beta,\tau,\gamma \in \overline{\varphi}(a)$.

Claim 2.3. *We may assume $\overline{\varphi}(x) = \tau = 1$.*

Proof. If $uy \notin P_x(1,\tau)$, we simply do a $(1,\tau)$-swap at x. Thus, we assume that $u \in P_x(1,\tau)$. We first do a $(1,\tau)$-swap at b, then an (α,τ)-swap at x. Then we do a (γ,τ)-swap at b. Finally, a $(1,\gamma)$-swap at b and a $(1,\alpha)$-swap at x give the desired coloring. $\square$

Since $ux \in P_x(1,\beta)$, and a and b are $(1,\beta)$-linked, we do a $(1,\beta)$-swap at b. Now we color ab by α, recolor bu by β and uncolor ux, see Figure 2.4 for a depiction.

Note that

$$F^* = (u, ux, x, uy, y), \quad K^* = (x, xu, u, ub, b, ba, a)$$

are, respectively, a multifan and a Kierstead path. By Lemma 1.9, u and y are (α,γ)-linked, and u and x are (α,β)-linked and $(1,\alpha)$-linked. Thus, we do an (α,γ)-swap at a, an (α,β)-swap at a, a $(1,\alpha)$-swap at a, and then an (α,γ)-swap at a. Now $P_u(\alpha,\beta) = uba$, contradicting Lemma 1.9 that u and x are (α,β)-linked. The proof is now completed. $\square$

Lemma 2.9. *Let G be a Class 2 graph, $H \subseteq G$ be a kite with $V(H) = \{a,b,c,u,s_1,s_2,t_1,t_2\}$,*

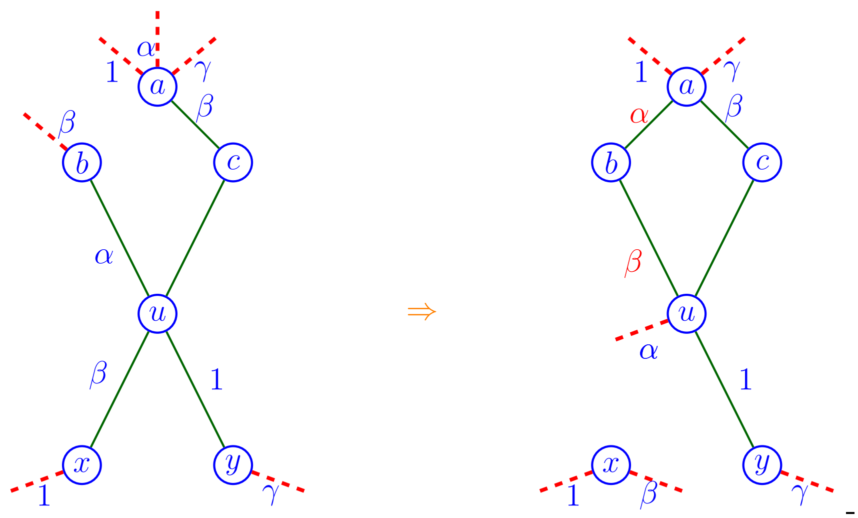

Figure 2.4. Colors on the edges connecting x and y to b

and let $\varphi \in \mathcal{C}^{\Delta}(G-ab)$. *Suppose*

$$K = (a, ab, b, bu, u, us_1, s_1, s_1t_1, t_1) \quad \textit{and} \quad K^* = (b, ab, a, ac, c, cu, u, us_2, s_2, s_2t_2, t_2)$$

are two Kierstead paths with respect to ab *and* φ. *If* $\varphi(s_1t_1) = \varphi(s_2t_2)$, *then* $|\overline{\varphi}(t_1) \cap \overline{\varphi}(t_2) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))| \leq 4$.

Proof. Let $\Gamma = \overline{\varphi}(t_1) \cap \overline{\varphi}(t_2) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))$. Assume to the contrary that $|\Gamma| \geq 5$. By considering K and applying Lemma 2.6, we conclude that $d_G(b) = d_G(u) = \Delta$. We show that there exists $\varphi^* \in \mathcal{C}^{\Delta}(G-ab)$ satisfying the following properties:

(i) $\varphi^*(bu), \varphi^*(cu), \varphi^*(us_2) \in \overline{\varphi}^*(a) \cap \overline{\varphi}^*(t_1) \cap \varphi^*(t_2)$,

(ii) $\varphi^*(us_1) \in \overline{\varphi}^*(b) \cap \overline{\varphi}^*(t_1) \cap \varphi^*(t_2)$, and

(iii) $\varphi^*(s_1t_1) = \varphi^*(s_2t_2) \in \overline{\varphi}^*(a)$.

See Figure 2.5 for a depiction of the colors described above.

Let $\alpha, \beta, \tau, \delta \in \Gamma$, and let $\varphi(s_1t_1) = \varphi(s_2t_2) = \gamma$. We may assume that $\alpha \in \overline{\varphi}(a)$ and $\beta \in \overline{\varphi}(b)$. Otherwise, since $d_G(b) = \Delta$, we have $\alpha, \beta \in \overline{\varphi}(a)$. Let $\lambda \in \overline{\varphi}(b)$. As a and b are (β, λ)-linked, we do a (β, λ)-swap at b. Note that this operation may change some colors of

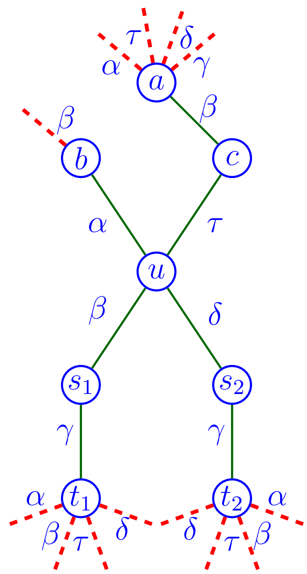

Figure 2.5. Colors on the edges of a kite

the edges of K and K^*, but they are still Kierstead paths with respect to ab and the current coloring.

Since $d_G(b) = d_G(u) = \Delta$, and $\beta \in \overline{\varphi}(b) \cap \overline{\varphi}(t_1)$, we know that $\gamma \in \overline{\varphi}(a)$, as K_1 is a Kierstead path. Next, we may assume that $\varphi(bu) = \alpha$. If not, let $\varphi(bu) = \alpha'$. Since a and b are (α, β)-linked, we do an (α, β)-swap at b. Now a and b are (α, α')-linked, we do an (α, α')-swap at b. Finally, we do an (α', β)-swap at b. All these swaps do not change the colors in Γ, so now we get the color on bu to be α.

We may now assume that $\varphi(cu) = \tau$. If not, let $\varphi(cu) = \tau'$. Since a and b are (β, τ)-linked, we do a (β, τ)-swap at b. Then do $(\tau, \tau') - (\tau', \beta)$-swaps at b.

Finally, we show that we can modify φ to get φ' such that $\varphi'(us_1) = \beta$ and $\varphi'(us_2) = \delta$. Assume firstly that $\varphi(us_1) = \beta' \neq \beta$. If $\beta' \in \Gamma$, we do $(\beta, \gamma) - (\gamma, \beta')$-swaps at b. Thus, we assume $\beta' \notin \Gamma$. Let $\lambda \in \Gamma \setminus \{\alpha, \beta, \tau, \delta\}$. If $u \notin P_a(\beta, \beta') = P_b(\beta, \beta')$, we simply do a (β, β')-swap at b. Thus, we assume $u \in P_a(\beta, \beta') = P_b(\beta, \beta')$. We do a (β, β')-swap at both t_1 and t_2. Since a and b are (β, λ)-linked, we do a (β, λ)-swap at both t_1 and t_2. Now we do $(\beta, \gamma) - (\gamma, \beta')$-swaps at b. By switching the role of β and β', we have $\varphi(us_1) = \beta$. Lastly, we show that $\varphi(us_2) = \delta$.

Note that $bu \in P_{t_1}(\alpha, \gamma)$. Otherwise, let $\varphi' = \varphi/P_{t_1}(\alpha, \gamma)$. Then $P_b(\alpha, \beta) = bus_1t_1$, showing a contradiction to the fact that a and b are (α, β)-linked with respect to φ'. Thus, $bu \in P_{t_1}(\alpha, \gamma)$. Next, we claim that $P_{t_1}(\alpha, \gamma)$ meets u before b. As otherwise, we do the following operations to get a Δ-coloring of G:

$$\begin{bmatrix} s_1t_1 & P_{[s_1,b]}(\alpha, \gamma) & us_1 & bu & ab \\ \gamma \to \alpha & \alpha/\gamma & \beta \to \alpha & \alpha \to \beta & \gamma \end{bmatrix}.$$

This gives a contradiction to the assumption that G is Δ-critical. Thus, we have that $P_{t_1}(\alpha, \gamma)$ meets u before b. This implies that it is not the case that $P_{t_2}(\alpha, \gamma)$ meets u before b. In turn, this implies that $u \in P_a(\beta, \delta') = P_b(\beta, \delta')$. As otherwise, we get a Δ-coloring of G by doing a (β, δ)-swap along the (β, δ)-chain containing u, and then doing the same operation as above with t_2 playing the role of t_1.

Since $u \in P_a(\beta, \delta') = P_b(\beta, \delta')$, we do a (β, δ')-swap at both t_1 and t_2. As $u \in P_a(\beta, \tau) = P_b(\beta, \tau)$, we do a (β, τ)-swap at both t_1 and t_2. Since $us_1 \in P_{t_1}(\beta, \gamma)$, we do a (β, γ)-swap at b, then a (γ, λ)-swap at b. Since a and b are (τ, λ)-linked, we do a (τ, λ)-swap at both t_1 and t_2. Now $(\lambda, \delta) - (\delta, \gamma) - (\gamma, \beta)$-swaps at b give a desired coloring.

Still, by the same arguments as above, we have that $P_{t_1}(\alpha, \gamma)$ meets u before b, and $u \in P_a(\beta, \delta) = P_b(\beta, \delta)$. Let $P_u(\beta, \delta)$ be the (β, δ)-chain starting at u not including the edge us_2. It is clear that $P_u(\beta, \delta)$ ends at either a or b. We may assume that $P_u(\beta, \delta)$ ends at a. Otherwise, we color ab by β, uncolor ac, and let τ play the role of α. Let $P_u(\alpha, \gamma)$ be the (α, γ)-chain starting at u not including the edge bu, which ends at t_1 by our earlier argument. We do the following operations to get a Δ-coloring of G:

$$\begin{bmatrix} P_u(\alpha, \gamma) & bu & P_u(\beta, \delta) & us_2t_2 & ab \\ \alpha/\gamma & \alpha \to \beta & \beta/\delta & \delta/\gamma & \alpha \end{bmatrix}.$$

This gives a contradiction to the assumption that G is Δ-critical. The proof is now finished.

□

3 A NEW APPROACH TO THE OVERFULL CONJECTURE

3.1 Introduction

Let G be a graph with maximum degree $\Delta(G) = \Delta$. The *core* of G, denoted G_Δ, is the subgraph of G induced by its vertices of degree Δ.

For a graph or a multigraph G with $|V(G)| \geq 3$, define its density

$$\omega(G) = \max_{X \subseteq V(G), |X| \geq 3} \frac{|E(G[X])|}{\lfloor |X|/2 \rfloor}, \tag{4}$$

or zero by convention if $|V(G)| \leq 2$. Clearly $\omega(G)$ is always achieved by some $X \subseteq V(G)$ with an odd cardinality. Note that $\omega(G)$ is a lower bound on $\chi'(G)$, since every matching of G contains at most $\lfloor |X|/2 \rfloor$ edges with both endpoints in X for every $X \subseteq V(G)$. We call G *overfull* if $|E(G)| > \Delta \lfloor |V(G)|/2 \rfloor$. Thus, if G is overfull, $\omega(G) \geq \frac{|E(G)|}{\lfloor |V(G)|/2 \rfloor} > \Delta$. Consequently, $|V(G)|$ is odd and $\chi'(G) = \Delta + 1$.

Although it is NP-complete to compute the chromatic index of a graph G, it is still worth to investigate the properties of Class 2 graphs. As we mentioned above, an overfull graph G is of Class 2. So it is natural to ask the following question: Does every Class 2 graph G contains an overfull subgraph H with $\Delta(H) = \Delta(G)$? The answer is no. A well know counter-example is the graph P^*, which is obtained from the Peterson graph by deleting a vertex. But with this observation, Hilton in 1985 proposed the following conjecture.

Conjecture 3.1 (Hilton's overfull conjecture [7, 8])**.** *Every class two graph G with $\Delta(G) > \frac{1}{3}|V(G)|$ contains an overfull subgraph H with $\Delta(H) = \Delta(G)$.*

There is only small progress towards overfull conjecture, and all of these known results are far away from solving it. Up to now, there have been mainly two ways to approach the conjecture. That is, adding minimum degree conditions or raising maximum degree conditions much higher than $|V(G)|/3$.

Theorem 3.2 (Plantholt [?])**.** *Let G be a graph with even order n, maximum degree Δ, and minimum degree δ. If $\delta \geq (\sqrt{7}/3)n - 1 \approx 0.88n$, then G is of Class 2 if and only if G*

contains an overfull subgraph H with $\Delta(H) = \Delta(G)$.

Theorem 3.3 (Chetwynd and Hilton [8])**.** *Let G be a graph with $\Delta(G) = |V(G)| - 3$. Then G is of Class 2 if and only if G contains an overfull subgraph H with $\Delta(H) = \Delta(G)$.*

Classifying a graph as Class 1 or Class 2 is a very difficult problem in general even when restricted to the class of graphs with maximum degree three, see [16]. Therefore, this problem is usually studied on particular classes of graphs. One possibility is to consider graphs whose core has a simple structure (see [21, Sect. 4.2]). Vizing [23] proved that if G_Δ has at most two vertices then G is Class 1. Fournier [11] generalized Vizing's result by showing that if G_Δ contains no cycles then G is Class 1. Thus a necessary condition for a graph to be Class 2 is to have a core that contains cycles. Hilton and Zhao [13] considered the problem of classifying graphs whose core is the disjoint union of cycles. Only a few such graphs are known to be Class 2. These include the overfull graphs and the graph P^*. In 1996, Hilton and Zhao [13] proposed the following conjecture.

Conjecture 3.4 (Core Conjecture)**.** *Let G be a connected simple graph with $\Delta \geq 3$ and $\Delta(G_\Delta) \leq 2$. Then G is Class 2 if and only if G is overfull or $G = P^*$.*

We call a connected Class 2 graph G with $\Delta(G_\Delta) \leq 2$ a *Hilton-Zhao graph (HZ-graph).* Note that Vizing's Adjacency Lemma implies that every vertex in a Class 2 graph has at least two neighbors of degree $\Delta(G)$, and so the core of an HZ-graph is in fact 2-regular. Clearly, P^* is an HZ-graph with $\chi'(P^*) = 4$ and $\Delta(P^*) = 3$. Hence the Core Conjecture is equivalent to the claim that every HZ-graph $G \neq P^*$ with $\Delta(G) \geq 3$ is overfull. A first breakthrough of this conjecture was achieved in 2003, when Cariolaro and Cariolaro [5] settled the base case $\Delta = 3$. In the summer of 2019, Chen, Jing, Song and I confirmed the Core Conjecture for all HZ-graphs G with $\Delta \geq 4$ as below.

Theorem 3.5. *Let G be a connected graph with $\Delta(G) = \Delta$. If $\Delta \geq 4$ and $\Delta(G_\Delta) \leq 2$, then G is Class 2 if and only if G is overfull.*

In the proof of Theorem 3.5, we noticed an important fact that every HZ-graph is Δ-

critical. This fact leads to a new approach to the overfull conjecture. A graph G with maximum degree Δ is called (*edge*)-Δ-*critical* if $\chi'(G) = \Delta + 1$ and $\chi'(H) \leq \Delta$ for any proper subgraph H of G. Note that to prove the overfull conjecture, it is sufficient to prove it for all Δ-critical graphs since *every simple graph G contains a Δ-critical subgraph H with $\Delta(H) = \Delta(G)$.*

Under a coloring φ of G, a vertex set X is called an *elementary set* with respect to φ if u and v do not share any common missing color for any two vertices $u, v \in X$. Many sets in Δ-critical graphs have been proved to be elementary under certain conditions, such as the vertex sets of Vizing fans, Kierstead paths, Tashkinov trees and short brooms. In fact, there is another strong connection between critical graphs and the overfull conjecture.

Proposition 3.6. *If G is a critical graph and $xy \in E(G)$, then G is overfull if and only if $V(G)$ is elementary under a coloring $\varphi \in \mathcal{C}^{\Delta}(G - xy)$.*

A proof of the above proposition can be found in the book [21] of Stiebitz et al. (See Theorem 1.4 and Proposition 4.13 in the book.) This result implies that the overfull conjecture can be proved by finding a large elementary set. Though this time we need to show that the entire vertex set of the graph is elementary. Inspired by the properties above, I proposed the following conjecture, which is closely related to the core conjecture and the overfull conjecture.

Conjecture 3.7. *Let G be a Δ-critical graph with $\Delta > \frac{|V(G)|}{2} + 1$. If there exists a vertex r with degree Δ such that exactly two neighbors of r have degree Δ, then G is overfull.*

Obviously this conjecture implies the core conjecture for graphs with maximum degree larger than $|V(G)|/2 + 1$. Our next step is to remove the condition 'there exists a vertex v with degree Δ such that exactly two neighbors of v have degree Δ'. According to the partial proof I have obtained on this conjecture, this condition seems necessary. But If we can find a way to remove it, the strengthened result will be a great improvement towards overfull conjecture. In fact, after removing the condition about r, conjecture 3.7 is very close to the just overfull conjecture, which is slightly weaker than the overfull conjecture.

Conjecture 3.8 (Just Overfull Conjecture)**.** *Let G be a graph with $\Delta(G) = \Delta \geq |V(G)|/2$, then G is Δ-critical if and only if G is just overfull, i.e.,* $|E(G)| = \Delta(G) \left\lfloor \frac{|V(G)|}{2} \right\rfloor + 1$.

Unfortunately, I do not have a complete proof of conjecture 3.7. In the following subsections, I will present my current work on this conjecture and some new structural results, which I think are very useful on attacking the overfull conjecture.

3.2 Preliminary and Lemmas

Let G be a graph. The closed neighborhood of v in G, denoted $N_G[v]$, is defined by $N_G(v) \cup \{v\}$. We simply write $N(v), N[v]$, and $d(v)$ if G is clear. We write $u \not\sim v$ if u is nonadjacent to v in G. Let $i \geq 1$ be an integer and $v \in V(G)$. Define

$$V_i = \{w \in V(G) \,:\, d_G(w) = i\}, \qquad N_i(v) = N_G(v) \cap V_i, \quad \text{and} \quad N_i[v] = N_i(v) \cup \{v\}.$$

For $X \subseteq V(G)$, we define $N_G(X) = \bigcup_{w \in X} N_G(w)$ and $N_i(X) = N_G(X) \cap V_i$. For $H \subseteq G$, we simple write $N_G(H)$ for $N_G(V(H))$.

Let $x, y \in V(G)$, φ be a coloring of G and $\alpha, \beta, \gamma \in [1, k]$ be three colors. If $P_x(\alpha, \beta, \varphi)$ is a path, define $P_x^+(\alpha, \beta, \varphi)$ to be an (α, β)-chain or an (α, β)-subchain of G with respect to φ that starts at x and ends at a different vertex missing exactly one of α and β. If x is an endvertex of the (α, β)-chain that contains x, then $P_x^+(\alpha, \beta, \varphi) = P_x(\alpha, \beta, \varphi)$ is unique. Otherwise, $P_x(\alpha, \beta, \varphi)$ is split by x into two subpaths, and we take one subpath to be $P_x^+(\alpha, \beta, \varphi)$. We will specify the subpath when it is used. If u is a vertrex on $P_x(\alpha, \beta, \varphi)$, we write $u \in P_x(\alpha, \beta, \varphi)$; and if uv is an edge on $P_x(\alpha, \beta, \varphi)$, we write $uv \in P_x(\alpha, \beta, \varphi)$. We also define these notions for $P_x^+(\alpha, \beta, \varphi)$. Similarly, the notion $P_x(\alpha, \beta)$ (or $P_x^+(\alpha, \beta)$) always represents the (α, β)-chain (or subchain) with respect to the current edge coloring. If $u, v \in P_x(\alpha, \beta)$ (or $P_x^+(\alpha, \beta)$ respectively) such that u lies between x and v, then we say that $P_x(\alpha, \beta)$ (or $P_x^+(\alpha, \beta)$ respectively) *meets u before v*. Suppose that $\alpha \in \overline{\varphi}(x)$ and $\beta, \gamma \in \varphi(x)$. An $(\alpha, \beta) - (\beta, \gamma)$ *swap at x* consists of two operations: first swaps colors on $P_x(\alpha, \beta, \varphi)$ to get an edge k-coloring φ', and then swaps colors on $P_x(\beta, \gamma, \varphi')$. By convention, an (α, α)-

swap at x does nothing. Suppose the current color of an edge uv of G is α, the notation $uv : \alpha \to \beta$ means to recolor the edge uv using the color β. Recall that $\overline{\varphi}(x)$ is the set of colors not present at x. If $|\overline{\varphi}(x)| = 1$, we will also use $\overline{\varphi}(x)$ to denote the color that is missing at x.

Let T be a sequence of vertices and edges of G. We denote by $V(T)$ the set of vertices from $V(G)$ that are contained in T, and by $E(T)$ the set of edges from $E(G)$ that are contained in T. For a coloring $\varphi' \in \mathcal{C}^{\Delta}(G-e)$, φ' is called *T-stable* with respect to φ if $\overline{\varphi}'(x) = \overline{\varphi}(x)$ for every $x \in V(T)$, and $\varphi'(f) = \varphi(f)$ for every $f \in E(T)$. Clearly, T-stable is an equivalent relationship, and so φ is T-stable with respect to itself. Let P be an (α,β)-path. If P does not contain any edge in $E(T)$, then P is called *E(T)-avoiding*; if additionally P does not contain any vertex in $V(T)$, then P is called *T-avoiding*. A Kempe change is called T-avoiding if the Kempe chain in this change is T-avoiding.

Let $w \in V(G)$ and $p \geq 1$. A *star* centered at w with p leafs is a subgraph of G that is isomorphic to the complete bipartite graph $K_{1,p}$ such that w has degree p in the subgraph. If $v_1, \ldots, v_p \in N_G(w)$ are the leafs, we denote the star by $S(w; v_1, \ldots, v_p)$.

Let a, b be two positive integers. If $b \geq a$, we abbreviate a vertex sequence $(s_a, s_{a+1}, \ldots, s_b)$ as $s_a : s_b$. If $b < a$, then $s_a : s_b$ denotes an empty sequence. The notation $[a, b]$ stands for the set $\{a, \ldots, b\}$ if $b \geq a$, and $\emptyset$ otherwise. If $F = (a_1, \ldots, a_t)$ is a sequence, then for a new entry b, (F, b) denotes the sequence $(a_1, \ldots, a_t, b)$.

3.2.1 Multifan at r in Conjecture 3.7

In this subsection, we always assume that G is a Δ-critical graph, and there exists a vertex r with degree Δ such that exactly two neighbors of r have degree Δ. Under these assumptions, for any multifan at r, we add a further requirement in its definition as follows: all vertices of the fan except the center have degree $\Delta - 1$. In the remainder of this paper, we use this new definition for all multifans at r under the assumption of conjecture 3.7. The following fact shows that the above definition is well defined.

Lemma 3.9. *Let G be a Δ-critical graph and r be a vertex with degree Δ such that $|N_{\Delta}(r)| =$*

2. *If $s \in N(r) \setminus N_\Delta(r)$, then $d(s) = \Delta - 1$.*

Proof. By Lemma 1.10, we have $|N_\Delta(r)| = 2 \geq \Delta - d(r) + 1$. Hence $d(r) = \Delta - 1$. □

Let $F_\varphi(r, s_1 : s_p)$ be a multifan. By its definition, except s_1, every other s_i misses exactly one color with respect to φ in F. Note that $|\overline{\varphi}(s_1)| = 2$, and so every color in $\overline{\varphi}(V(F))$ is induced by one of the two colors in $\overline{\varphi}(s_1)$. So $s_1, s_2, \ldots, s_p$ can be divided into two sequences. Therefore, we can equip F with additional properties.

We call a multifan F with respect to rs_1 and $\varphi \in \mathcal{C}^\Delta(G - rs_1)$ a *typical multifan*, denoted by $F_\varphi(r, s_1 : s_\alpha : s_\beta)$, if $\overline{\varphi}(r) = \{1\}$, $\overline{\varphi}(s_1) = \{2, \Delta\}$ and the following hold.

(1) Either $|V(F)| = 2$, or $|V(F)| \geq 3$ and there exist $\alpha \in [2, \beta]$ such that $s_2, \ldots, s_\alpha$ is a 2-inducing sequence and $s_{\alpha+1}, \ldots, s_\beta$ is a Δ-inducing sequence of F where $\beta = |V(F)| - 1$.

(2) If $|V(F)| \geq 3$, then for each $i \in [2, \beta]$, $\varphi(rs_i) = i$ and $\overline{\varphi}(s_i) = i + 1$ except for $i = \alpha + 1$. If $\alpha + 1 \in [2, \beta]$, then $\varphi(rs_{\alpha+1}) = \Delta$ and $\overline{\varphi}(s_{\alpha+1}) = \alpha + 2$.

Clearly by relabelling vertices and colors if necessary, any multifan at r can be assumed to be a typical multifan. If $\alpha \geq 2$ and $\beta > \alpha$, we say F has *two sequences*. Otherwise we say F has *single sequence*. For a typical multifan $F = F_\varphi(r, s_1 : s_\alpha : s_\beta)$, if $\alpha = \beta$, then we say $F = F_\varphi(r, s_1 : s_\alpha)$ is a *typical 2-inducing multifan*. The graph given in Figure 3.1 depicts a typical multifan within the neighborhood of a Δ-vertex r.

The following Lemma indicates that under the assumptions of conjecture 3.7, any multifan at r can be assumed to be a typical multifan that has single sequence.

Lemma 3.10. *Let G be a Δ-critical graph, r be a vertex with degree Δ such that $|N_\Delta(r)| = 2$, $s_1 \in N_{\Delta-1}(r)$ and $\varphi \in \mathcal{C}^\Delta(G - rs_1)$. Then for every multifan $F = F_\varphi(r, s_1, s_p)$ centered at r, there exists a coloring $\varphi' \in \mathcal{C}^\Delta(G - rs_p)$ and a typical multifan F^* centered at r with respect to rs_p and φ' such that $V(F^*) = V(F)$ and F^* has single sequence.*

Proof. By the definition of multifan, s_p is the last η-inducing color for some $\eta \in \overline{\varphi}(s_1)$. Thus we may assume, without loss of generality, that $F = F_\varphi(r, s_1 : s_\alpha : s_\beta)$ is a typical multifan and $s_p = s_\beta$. Clearly if F has only single sequence then we are done. Thus

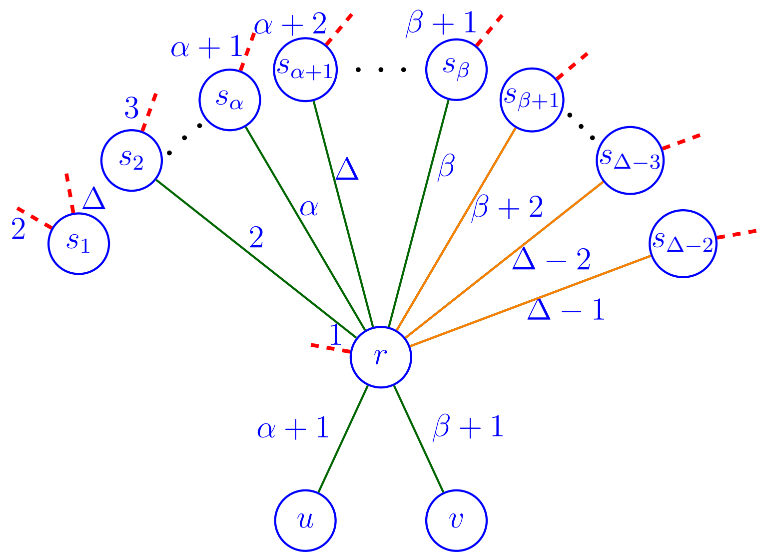

Figure 3.1. A typical multifan $F_\varphi(r, s_1 : s_\alpha : s_\beta)$ in the neighborhood of r, where a dashed line at a vertex indicates a color missing at the vertex.

we assume that $\beta \geq \alpha+1 \geq 3$. Let φ' be obtained from φ by uncoloring rs_β, doing $rs_i : i \to i+1$ for each $i \in [\alpha+1, \beta-1]$ and coloring rs_1 by Δ. Now $\overline{\varphi}'(s_\beta) = \{\beta, \beta+1\}$, $F^* = (r, rs_\beta, s_\beta, rs_{\beta-1}, s_{\beta-1}, \ldots, rs_{\alpha+1}, s_{\alpha+1}, rs_1, s_1, rs_2, s_2, \ldots, rs_\alpha, s_\alpha)$ is a β-inducing multifan with respect to rs_β and φ'. We obtain the desired coloring and multifan. □

A multifan $F_\varphi(r, s_1 : s_t)$ is called **maximum** at r if $|V(F)|$ is maximum among all multifans with respect to rs_i for some $s_i \in N_{\Delta-1}(r)$ and $\varphi' \in \mathcal{C}^k(G - rs_i)$.

A sequence of distinct vertices $s_{h_1}, s_{h_t}, \ldots, s_{h_t} \in N_{\Delta-1}(r)$ form a *rotation* if

(1) $\{s_{h_1}, s_{h_t}, \ldots, s_{h_t}\}$ is φ-elementary, and

(2) for each ℓ with $\ell \in [1, t]$, it holds $\varphi(rs_{h_\ell}) = \overline{\varphi}(s_{h_{\ell-1}})$ where $h_0 = h_t$.

Assume $N_{\Delta-1}(r) = \{s_1, s_2, \ldots, s_{\Delta-2}\}$. Let i, j be integers with $2 \leq i \leq j \leq \Delta-2$. Then the *shifting from* s_i *to* s_j is an operation that, for each ℓ with $\ell \in [i, j]$, replaces the current color of rs_ℓ by the color in $\overline{\varphi}(s_\ell)$. For a rotation $s_{h_1}, s_{h_2}, \ldots, s_{h_t}$, the *shifting from* s_{h_1} *to* s_{h_t} is an operation that, for each ℓ with $\ell \in [1, t]$, replaces the current color of rs_{h_ℓ} by the color in $\overline{\varphi}(s_{h_\ell})$. We will apply shifting either on a sequence of vertices from a multifan

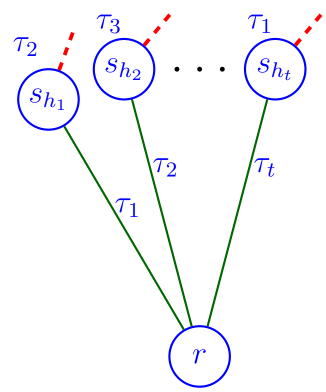

Figure 3.2. A rotation in the neighborhood of r.

at r or on a rotation. Note that we sometimes have $i > j$ when applying a shifting. In that case the shifting does not change any color.

3.2.2 Lollipop at r in Conjecture 3.7

In this subsection, we still assume that G is a Δ-critical graph, and there exists a vertex r with degree Δ such that exactly two neighbors of r have degree Δ. Let $e = rs_1 \in E(G)$ with $s_1 \in N_{\Delta-1}(r)$, and let $\varphi \in \mathcal{C}^\Delta(G-e)$. Then a *lollipop* centered at r (depicted in Figure 3.3) is a sequence $L = (F, ru, u, ux, x)$ of distinct vertices and edges such that $F = F_\varphi(r, s_1 : s_\alpha : s_\beta)$ is a typical multifan, $u \in N_\Delta(r)$ and $x \in N(u) \setminus N_\Delta(u)$ with $x \notin \{s_1, \dots, s_\beta\}$.

The following results about lollipop in subsections 2.4.1 and 2.4.2 are proved in [3]. All these results are proved under the following assumptions: G is a Δ-critical graph, r is a vertex with degree Δ such that $|N_\Delta(r)| = 2$, $s_1 \in N_{\Delta-1}(r)$, $\varphi \in \mathcal{C}^\Delta(G - rs_1)$ and $L = (F_\varphi(r, s_1, s_\alpha, s_\beta), ru, u, ux, x)$ is a lollipop centered at r.

3.2.3 Fundamental properties of a lollipop

Lemma 3.11 (Claim 4.1 and 4.5 in [3]). *If $F = F_\varphi(r, s_1, s_\alpha)$ is a typical 2-inducing multifan at r, then the following hold.*

- *We may assume that $\varphi(ru) = \alpha + 1$, which is the last 2-inducing color of F.*

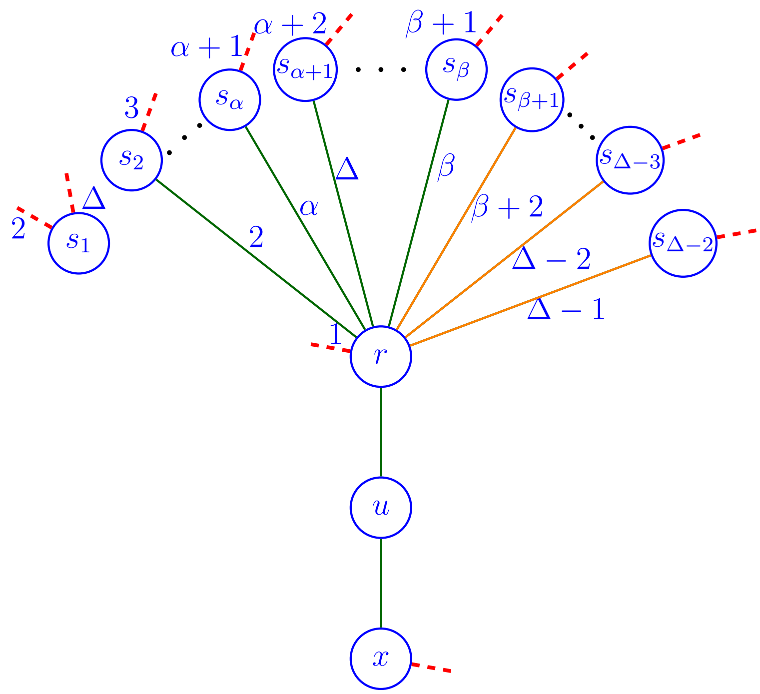

Figure 3.3. A lollipop centered at r, where x can be the same as some s_i for $i \in [\beta+1, \Delta-2]$.

- *If $\varphi(ru) = \alpha+1$, then for any $z \in N(u) \setminus (N_\Delta(u) \cup N(r))$, there is an F-stable coloring $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$ with respect to φ such that $\varphi'(ru) = \alpha+1$ and $\alpha+1 \in \overline{\varphi}'(z)$.*

In particular, if $x \notin N(r)$ (x is defined in L), then we may assume that $\varphi(ru) = \alpha+1 \in \bar{\varphi}(x)$.

Lemma 3.12 (Claim 4.2 in [3])**.** *Suppose $F = F_\varphi(r, s_1, s_\alpha)$ is a maximum typical 2-inducing multifan at r. For any $z \in N(u) \setminus (N_\Delta(u) \cup V(F))$ and any F-stable $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$, if $\varphi'(ru) = \alpha+1 \in \overline{\varphi}'(z)$, then $\varphi'(uz) \in \overline{\varphi}'(V(F)) \setminus \{1\}$.*

Lemma 3.13 (Claim 4.7 in [3])**.** *Suppose $F = F_\varphi(r, s_1, s_\alpha)$ is a maximum typical 2-inducing multifan at r. Let $x, y \in N(u) \setminus (N_\Delta(u) \cup V(F))$ be distinct, and $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$ be any F-stable coloring with $\varphi'(ru) = \alpha+1$. If $\overline{\varphi}'(x) \cap (\overline{\varphi}'(V(F)) \setminus \{1\}) \neq \emptyset$, then $\overline{\varphi}'(y) \cap \overline{\varphi}'(V(F)) = \emptyset$ and y and r are $(\tau, 1)$-linked for any color $\tau \in \bar{\varphi}'(y)$.*

Lemma 3.14. *If $\varphi(ru) = \alpha+1 \in \overline{\varphi}(x)$ and $\varphi(ux) = \tau$ is a 2-inducing color with respect to φ and F, then the following statements hold.*

(a) $ux \in P_r(\tau, 1)$,

(b) furthermore, let $P_x^+(\tau, 1)$ *be the* $(\tau, 1)$*-path starting at* x *not containing* ux*, we have* $P_x^+(\tau, 1)$ *ends at* r.

(c) For any 2-inducing color δ *with* $\tau \prec \delta$, $r \in P_{s_1}(\Delta, \delta) = P_{s_{\delta-1}}(\Delta, \delta)$.

(d) For any Δ*-inducing color* δ, $r \in P_{s_{\delta-1}}(\delta, \alpha+1) = P_{s_\alpha}(\delta, \alpha+1)$*, where* $s_{\Delta-1} = s_1$ *if* $\delta = \Delta$*. Moreover,* $P_{s_1}(\Delta, \alpha+1)$ *meets* r *before* u*, and so it meets* u *before* s_α.

(e) For any 2-inducing color δ *with* $\delta \prec \tau$, $r \in P_{s_\alpha}(\delta, \alpha+1) = P_{s_{\delta-1}}(\delta, \alpha+1)$*. Moreover* $P_{s_{\delta-1}}(\delta, \alpha+1)$ *meets* r *before* u*, and so it meets* u *before* s_α.

The "moreover" parts of Lemma 3.14 (d) and (e) are not stated in [3] explicitly. But they can be easily verified by their proofs in [3].

For a color α, a sequence of *Kempe* $(\alpha, *)$*-changes* is a sequence of Kempe changes that each involves the exchanging of the color α and another color from $[1, k]$.

Lemma 3.15. *If* $\varphi(ru) = \alpha + 1$ *and there is a vertex* $s_{h_1} \in \{s_{\beta+1}, \dots, s_{\Delta-2}\}$ *with* $\varphi(rs_{h_1}) = \tau_1 \in \{\beta+2, \cdots, \Delta-1\}$*, then the following statements hold.*

(1) If exists a vertex $w \in V(G) \setminus (V(F) \cup \{s_{h_1}\})$ *such that* $w \in P_r(\tau_1, 1, \varphi')$ *for any* F**-stable** $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$ *obtained from* φ *through a sequence of* F*-avoiding Kempe* $(1, *)$*-changes, then there exists a sequence of distinct vertices* $s_{h_1}, s_{h_2}, \dots, s_{h_t} \in \{s_{\beta+1}, \dots, s_{\Delta-2}\}$ *satisfying the following conditions:*

(a) $\varphi(rs_{h_{i+1}}) = \overline{\varphi}(rs_{h_i}) \in \{\beta+2, \cdots, \Delta-1\}$ *for each* $i \in [1, t-1]$.

(b) s_{h_i} *and* r *are* $(\overline{\varphi}(s_{h_i}), 1)$*-linked with respect to* φ *for each* $i \in [1, t]$.

(c) $\overline{\varphi}(s_{h_t}) = \tau_1$.

(2) If $\overline{\varphi}(x) = \alpha + 1$ *and there exists a vertex* $w \in V(G) \setminus (V(F) \cup \{s_{h_1}\})$ *such that* $w \in P_r(\tau_1, 1, \varphi')$ *for any* L**-stable** $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$ *obtained from* φ *through a sequence of* L*-avoiding Kempe* $(1, *)$*-changes, then there exists a sequence of distinct vertices* $s_{h_1}, s_{h_2}, \dots, s_{h_t} \in \{s_{\beta+1}, \dots, s_{\Delta-2}\}$ *satisfying the following conditions:*

(a) $\varphi(rs_{h_{i+1}}) = \overline{\varphi}(rs_{h_i}) \in \{\beta+2, \cdots, \Delta-1\}$ *for each* $i \in [1, t-1]$.

(b) s_{h_i} *and* r *are* $(\overline{\varphi}(s_{h_i}), 1)$*-linked with respect to* φ *for each* $i \in [1, t-1]$.

(c) $\overline{\varphi}(s_{h_t}) = \tau_1$ *or* $\overline{\varphi}(s_{h_t}) = \alpha+1$. *If* $\overline{\varphi}(s_{h_t}) = \tau_1$, *then* s_{h_t} *and* r *are* $(\tau_1, 1)$*-linked with respect to* φ.

The next result is hidden in the proof of Lemma 3.12. Since it is not stated explicitly in [3], we give its proof here for completeness. Note that this result does not require F to be maximum.

Lemma 3.16. *Suppose* $F = F_\varphi(r, s_1, s_\alpha)$ *is a typical* 2*-inducing multifan at* r, $\varphi(ru) = \alpha+1 \in \bar{\varphi}(x)$ *and* $\varphi(ux) = \tau_1 \notin \bar{\varphi}(V(F))$. *Let* s_{h_1} *be the vertex in* $N(r)$ *such that* $\varphi(rs_{h_1}) = \tau_1$. *If* $u \neq s_{h_1}$, *then* $u \in P_r(\tau_1, 1, \varphi')$ *for any* L**-stable** $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$ *obtained from* φ *through a sequence of* L*-avoiding Kempe* $(1, *)$*-changes. Moreover, if* $s_{h_1}, s_{h_2}, \ldots, s_{h_t}$ *is the sequence obtained by applying Lemma 3.15 (2) with* $w = u$, *then* $\bar{\varphi}(s_{h_t}) = \alpha+1$, *i.e.,* $s_{h_1}, s_{h_2}, \ldots, s_{h_t}$ *is not a rotation.*

Proof. Let φ' be a L-stable coloring obtained from φ through a sequence of L-avoiding Kempe $(1, *)$-changes. Then $\varphi'(ux) = \varphi(ux) = \tau_1$. We first show $u \in P_r(\tau_1, 1, \varphi')$. Suppose not, we may let $\varphi^* = \varphi'/P_u(1, \tau_1, \varphi')$. Then $\varphi^*(ux) = 1$, and so $P_r(1, \alpha+1, \varphi^*)$ ends at x but not s_α, contradicting Lemma 1.9.

For the "moreover" part, by Lemma 3.15 (2) (c) we have $P_r(1, \tau_1) = P_{s_{h_t}}(1, \tau_1)$. Since it has been proved that $u \in P_r(\tau_1, 1, \varphi)$, we have $u \in P_r(1, \tau_1) = P_{s_{h_t}}(1, \tau_1)$. Let $P_u^+(1, \tau_1)$ be the $(1, \tau_1)$-path starting at u not containing ux. Now by shifting from s_{h_1} to s_{h_t} if necessary, we may assume (by symmetry) that $P_u^+(1, \tau_1)$ ends at s_{h_t}. Then we do the following to obtain φ'.

$$\begin{bmatrix} ur & P_u^+(1, \tau_1) & ux \\ (\alpha+1)/1 & 1/\tau_1 & \tau_1/(\alpha+1) \end{bmatrix}$$

Then $(r, rs_1, s_1, \ldots, s_\alpha)$ is still a multifan under φ'. But $\alpha+1 \in \bar{\varphi}'(r) \cap \bar{\varphi}'(s_\alpha)$, giving a contradiction to Lemma 1.9. $\square$

By applying Lemma 3.16, we obtain the following result.

Lemma 3.17. *Suppose $F = F_\varphi(r, s_1, s_\alpha)$ is a typical 2-inducing multifan at r, $\varphi(ru) = \alpha + 1 \in \bar{\varphi}(x)$ and $\varphi(ux) = i \in \{2, 3, \ldots, \alpha\}$. Then $r \in P_x(\alpha + 1, i) = P_{s_\alpha}(\alpha + 1, i)$ and $r \notin P_{s_{i-1}}(\alpha + 1, i)$.*

Proof. By Lemma 2.4 (b), we only need to show $P_{s_{i-1}}(\alpha + 1, i) \neq P_{s_\alpha}(\alpha + 1, i)$. Suppose on the contrary that $P_{s_{i-1}}(\alpha + 1, i) = P_{s_\alpha}(\alpha + 1, i)$. Then $x \notin P_{s_\alpha}(\alpha + 1, i)$, and so as r. Let $\varphi' = \varphi/P_{s_{i-1}}(\alpha+1, i)$. Then $(r, rs_1, s_1, \ldots, s_{i-1})$ is a 2-inducing multifan. Note that $\alpha + 1$ is the last 2-inducing color, $\varphi'(ux) = i \notin \{\Delta, \alpha + 1, 1, 2, \ldots, i - 1\}$, $s_i, s_{i+1}, \ldots, s_\alpha$ is a rotation and $\varphi'(rs_i) = i$. Thus we get a contradiction by Lemma 3.16. □

3.2.4 Adjacency in a lollipop

Lemma 3.18. *Suppose $F = F_\varphi(r, s_1 : s_\alpha : s_\beta)$ is a typical multifan and $N_{\Delta-1}(r) = \{s_1, s_2, \ldots, s_{\Delta-2}\}$. If $\varphi(ru) = \alpha+1 \in \overline{\varphi}(x)$ and $\varphi(ux) = \Delta$, then the following two statements hold.*

- *If $u \sim s_1$, then $\varphi(us_1)$ is a Δ-inducing color.*
- *If $u \sim s_\alpha$, then $\varphi(us_\alpha)$ is a Δ-inducing color.*

In particular, if additionally $F = F_\varphi(r, s_1 : s_\alpha)$ is a typical 2-inducing multifan, then $u \not\sim s_1$ and $u \not\sim s_\alpha$.

Lemma 3.19. *Suppose $F = F_\varphi(r, s_1 : s_\alpha)$ is a typical 2-inducing multifan and $N_{\Delta-1}(r) = \{s_1, s_2, \ldots, s_{\Delta-2}\}$. If $\varphi(ru) = \alpha + 1 \in \overline{\varphi}(x)$ and $\varphi(ux) = \mu \in \overline{\varphi}(V(F))$ is a 2-inducing color, then $u \not\sim s_{\mu-1}$ and $u \not\sim s_\mu$.*

3.3 Approach to Conjecture 3.7

Conjecture 3.7. Let G be a Δ-critical graph with $\Delta > \frac{|V(G)|}{2} + 1$. If there exists a vertex r with degree Δ such that exactly two neighbors of r have degree Δ, then G is overfull.

In order to show that G is overfull, we need to show that $|E(G)| \geq \Delta(\frac{|V(G)|-1}{2}) + 1$ and $n = |V(G)|$ is odd. Since $\sum_{v \in V(G)} d(v) = 2|E(G)|$, the above inequality is equivalent to

$\sum_{v\in V(G)}(\Delta - d(v)) \le \Delta - 2$. Note that by Lemma 3.9, we have $\sum_{v\in N[r]}(\Delta - d(v)) = \Delta - 2$. Therefore, we need to prove that n is odd and the following claim.

Claim 3.1. *For any $v \in V(G) \setminus N[r]$, $d(v) = \Delta$.*

Suppose Claim 3.1 is false. Let x be a vertex in $V(G) \setminus N[r]$ with degree less than Δ. By Lemma 1.10, we have $N_\Delta(x) \ne \emptyset$. Thus we have two cases: either $N_\Delta(r) \cap N_\Delta(x) \ne \emptyset$ or $N_\Delta(r) \cap N_\Delta(x) = \emptyset$. If we can find a contradiction in each of these two cases, then we are done. Unfortunately, we did not completely solve them. In this thesis, we only show that there is a contradiction in the first case $N_\Delta(r) \cap N_\Delta(x) \ne \emptyset$.

Let $u \in N_\Delta(r) \cap N_\Delta(x)$. We will show the following result in the next subsection.

Claim 3.2. $|N(u) \cap N_{\Delta-1}(r)| \le 1$.

Note that if Claim 3.2 is true, then we can achieve a contradiction as follows. By Claim 3.2, we have $|N(r) \cap N(u)| \le 2$. Thus $|N(r) \cup N(u)| \ge \Delta + \Delta - 2 = 2\Delta - 2 > n$, giving a contradiction.

3.4 Proof of Claim 3.2

Let G be a Δ-critical graph, r be a vertex with degree Δ such that $|N_\Delta(r)| = 2$, x be a vertex in $V(G) \setminus N[r]$ with degree less than Δ and u be a vertex in $N_\Delta(r) \cap N_\Delta(x)$.

Claim 3.2. $|N(u) \cap N_{\Delta-1}(r)| \le 1$.

Proof. Let $N_{\Delta-1}(r) = \{s_1, \ldots, s_{\Delta-2}\}$. We choose a vertex in $N_{\Delta-1}(r)$, say s_1, a coloring $\varphi \in \mathcal{C}^\Delta(G - rs_1)$ and a multifan F with respect to rs_1 and φ such that F is maximum at r. (Maximum multifan is defined in subsection 3.2.1.) Assume that $\overline{\varphi}(r) = 1$ and $\overline{\varphi}(s_1) = \{2, \Delta\}$, and $F = F_\varphi(r, s_1 : s_p)$ is such a multifan. By Lemma 3.10, we may assume that $F_\varphi(r, s_1 : s_p) = F_\varphi(r, s_1 : s_\alpha)$ is a typical 2-inducing multifan, where $\alpha = p$. Since $x \notin N(r)$, by Lemma 3.11, we may further assume that $\varphi(ru) = \alpha + 1 \in \bar{\varphi}(x)$. Then by Lemma 3.12, we have that $\varphi(ux)$ is either Δ or a 2-inducing color. Let $L = (F, ru, u, ux, x)$, then L is a lollipop.

In order to show Claim 3.2, we will prove the following four claims.

Claim 3.3. *If $\varphi(ux) = \Delta$, then $N(u) \cap \{s_1, s_2, \ldots, s_\alpha\} = \emptyset$.*

Claim 3.4. *If $\varphi(ux) = \Delta$, then $N(u) \cap \{s_{\alpha+1}, s_{\alpha+2}, \ldots, s_{\Delta-2}\} = \emptyset$.*

Claim 3.5. *If $\varphi(ux)$ is a 2-inducing color, then $|N(u) \cap \{s_1, s_2, \ldots, s_\alpha\}| \le 1$.*

Claim 3.6. *If $\varphi(ux)$ is a 2-inducing color, then $N(u) \cap \{s_{\alpha+1}, s_{\alpha+2}, \ldots, s_{\Delta-2}\} = \emptyset$.*

Clearly the above four claims together imply Claim 3.2. We now prove them one by one.

Claim 3.3. If $\varphi(ux) = \Delta$, then $N(u) \cap \{s_1, s_2, \ldots, s_\alpha\} = \emptyset$.

To show this claim, similar to Lemma 3.18, we will prove the following stronger result.

Claim 3.3*. Without assuming the typical multifan $F = F_\varphi(r, s_1 : s_\alpha : s_\beta)$ is 2-inducing, if $\varphi(ux) = \Delta$ and $u \sim s_i$ for some $i \in [1.\alpha]$, then $\varphi(us_i) \in [\alpha + 2, \beta + 1]$, i.e., $\varphi(us_i)$ is a Δ-inducing color.

Proof. Suppose on the contrary that $u \sim s_i$ for some $i \in [1, \alpha]$ and $\varphi(us_i) \notin [\alpha+2, \beta+1]$. By Lemma 3.18, we have that $i \in [2, \alpha-1]$. Let $\varphi(us_i) = \tau$. Clearly $\tau \notin \{i, i+1, \alpha+1, \Delta\}$. Let $P_u^+(\Delta, i+1)$ be the $(\Delta, i+1)$-path starting from u not containing ux. We first prove three subclaims without using the assumption $\varphi(us_i) \notin [\alpha + 2, \beta + 1]$. Some of these subclaims will also be used in the proof of Claim 3.5.

Subclaim 4.1.1. $u, r \in P_{s_1}(\Delta, i+1) = P_{s_i}(\Delta, i+1)$.

Proof. By Lemma 2.4, we have $P_{s_1}(\Delta, i+1) = P_{s_i}(\Delta, i+1)$. So we only need to show that both u and r are contained in this path. Suppose on the contrary, we have the following cases.

Suppose that $u \notin P_{s_1}(\Delta, i+1)$ and $r \in P_{s_1}(\Delta, i+1)$. We interchange Δ and $i+1$ on $P_u(\Delta, i+1)$ to obtain φ'. In this case φ' is F stable and $\varphi'(ux) = i+1 = \bar{\varphi}'(s_i)$. Thus we get a contradiction by Lemma 3.19.

Suppose that $u \in P_{s_1}(\Delta, i+1)$ and $r \notin P_{s_1}(\Delta, i+1)$. Then $P_u^+(\Delta, i+1)$ ends at either

s_1 or s_i. In both cases, we do the following to obtain φ'.

$$\begin{bmatrix} s_{i+1} : s_\alpha & ur & P_u^+(\Delta, i+1) & ux \\ \text{shift} & (\alpha+1)/(i+1) & (i+1)/\Delta & \Delta/(\alpha+1) \end{bmatrix}$$

Then $(r, rs_1, s_1, \dots, s_i)$ is a multifan under φ'. But $\bar{\varphi}'(s_1) \cap \bar{\varphi}'(s_i) \neq \emptyset$ in both cases, giving a contradiction to Lemma 1.9.

Suppose that $u \notin P_{s_1}(\Delta, i+1)$ and $r \notin P_{s_1}(\Delta, i+1)$. Let $\varphi' = \varphi/P_{s_1}(\Delta, i+1)$. Then $\bar{\varphi}'(s_i) = \Delta$ and $\bar{\varphi}'(s_1) = \{2, i+1\}$. Uncolor rs_i, color rs_1 by 2 and shift from s_2 to s_{i-1} to obtain φ''. Then $(r, rs_i, s_i, rs_{i-1}, s_{i-1}, \dots, s_1, rs_{i+1}, s_{i+1}, \dots, s_\alpha)$ is an i-inducing multifan under φ''. Also we have that $\alpha+1$ is the last i-inducing color, $u \sim s_i$ and $\varphi''(ux) = \Delta = \bar{\varphi}''(s_i)$. Thus we obtain a contradiction by Lemma 3.19. This completes the proof of the subclaim. □

Subclaim 4.1.2. There is a $(\Delta, i+1)$-path between r and u which does not contain ux or rs_{i+1}, i.e., $P_u^+(\Delta, i+1)$ contains r, and it meets r before s_{i+1}.

Proof. Suppose on the contrary, then we have two cases. Either $r \notin P_u^+(\Delta, i+1)$ or $r \in P_u^+(\Delta, i+1)$ and $P_u^+(\Delta, i+1)$ meets s_{i+1} before r. We will show that there is a contradiction in each of the above two cases.

Suppose $r \notin P_u^+(\Delta, i+1)$. In this case, $P_u^+(\Delta, i+1)$ is F-edge-avoiding, and it ends at either s_1 or s_i by subclaim 4.1.1. In both cases, We do the following operation to obtain φ'.

$$\begin{bmatrix} s_{i+1} : s_\alpha & ur & P_u^+(\Delta, i+1) & ux \\ \text{shift} & (\alpha+1)/(i+1) & (i+1)/\Delta & \Delta/(\alpha+1) \end{bmatrix}$$

Then $(r, rs_1, s_1, \dots, s_i)$ is a multifan under φ'. But $\bar{\varphi}'(s_1) \cap \bar{\varphi}'(s_i) \neq \emptyset$ in both cases, giving a contradiction to Lemma 1.9.

Suppose $r \in P_u^+(\Delta, i+1)$ and $P_u^+(\Delta, i+1)$ meets s_{i+1} before r. In this case let $P_r^+(\Delta, i+1)$ be the $(\Delta, i+1)$-path starting at r not containing rs_{i+1}. Similarly, the other end of $P_r^+(\Delta, i+1)$ is either s_1 or s_i. We do the following operation to obtain φ' (where rux

is the path with vertices r, u, x and edges ru and ux).

$$\begin{bmatrix} s_{i+1} : s_\alpha & P_r^+(\Delta, i+1) & rux \\ \text{shift} & (i+1)/\Delta & \Delta/(\alpha+1) \end{bmatrix}$$

Then $(r, rs_1, s_1, \ldots, s_i)$ is a multifan under φ'. But $\bar{\varphi}'(s_1) \cap \bar{\varphi}'(s_i) \neq \emptyset$ in both cases, giving a contradiction to Lemma 1.9. This completes the proof of the subclaim. □

Denote by $P_{[u,r]}(\Delta, i+1)$ the $(\Delta, i+1)$-path established by the above subclaim.

Subclaim 4.1.3. For every L-stable $\varphi^* \in \mathcal{C}^\Delta(G - rs_1)$, it holds that $\varphi^*(us_i) \neq 1$. Furthermore, if $\varphi^*(us_i) = \varphi(us_i) = \tau$, then $us_i \in P_r(\tau, 1, \varphi^*)$. Consequently, if φ^* is obtained form φ through a sequence of L-avoiding Kempe $(1, *)$-changes, then $\varphi^*(us_i) = \varphi(us_i) = \tau$ and $us_i \in P_r(\tau, 1, \varphi^*)$.

Proof. We first show that $\varphi^*(us_i) \neq 1$. Suppose not, we do the following to obtain φ'.

$$\begin{bmatrix} s_{i+1} : s_\alpha & P_{[u,r]}(\Delta, i+1) & ux & ur & us_i \\ \text{shift} & (i+1)/\Delta & \Delta/(\alpha+1) & (\alpha+1)/1 & 1/(i+1) \end{bmatrix}$$

Then $\Delta \in \bar{\varphi}'(r) \cap \bar{\varphi}'(s_1)$, giving a contradiction.

We now prove the "furthermore" part. Suppose on the contrary that $us_i \notin P_r(\tau, 1, \varphi^*)$. Then clearly $P_u(\tau, 1, \varphi^*)$ is F-avoiding. Hence $\varphi' = \varphi^*/P_u(\tau, 1, \varphi^*)$ is L-stable. But $\varphi'(us_i) = 1$, giving a contradiction to the first part. □

Now by the above subclaim and the assumption $\varphi(us_i) = \tau \notin [\alpha+2, \beta+1]$, we have the following three cases. τ is a 2-inducing color with $\tau \prec i$; τ is a 2-inducing color with $i+1 \prec \tau$; and $\tau \notin \bar{\varphi}(V(F))$.

Case 1. τ is a 2-inducing color with $\tau \prec i$.

Let $P_u^+(1, \tau)$ be the $(1, \tau)$-path starting from u not containing us_i. We consider the other end of it. Since $us_i \in P_r(1, \tau)$, the other end of $P_u^+(1, \tau)$ is either s_τ or r by Lemma 1.9.

Suppose $P_u^+(1,\tau)$ ends at $s_{\tau-1}$. We do the following operation to obtain φ'.

$$\begin{bmatrix} s_{i+1}:s_\alpha & P_u^+(1,\tau) & us_i & P_{[u,r]}(\Delta,i+1) & ux & ur \\ \text{shift} & 1/\tau & \tau/(i+1) & (i+1)/\Delta & \Delta/(\alpha+1) & (\alpha+1)/1 \end{bmatrix}$$

Then $\Delta \in \bar{\varphi}'(r) \cap \bar{\varphi}'(s_1)$, giving a contradiction.

Suppose $P_u^+(1,\tau)$ ends at r. Then since $us_i \in P_r(1,\tau) = P_{s_{\tau-1}}(1,\tau)$, we have that $P_{s_{\tau-1}}(1,\tau)$ meets s_i before u. We do the following operation to obtain φ'.

$$\begin{bmatrix} P_{[s_{\tau-1},s_i]}(1,\tau) & s_\tau:s_{i-1} & rs_i & s_{i+1}:s_\alpha & us_i & P_{[u,r]}(\Delta,i+1) & ux & ur \\ 1/\tau & \text{shift} & i\to 1 & \text{shift} & \tau/(i+1) & (i+1)/\Delta & \Delta/(\alpha+1) & (\alpha+1)/\tau \end{bmatrix}$$

Then $\Delta \in \bar{\varphi}'(r) \cap \bar{\varphi}'(s_1)$, giving a contradiction.

Case 2. τ is a 2-inducing color with $i+1 \prec \tau$.

To show this case, we will introduce a concept called the *symmetric coloring of* φ. Note that by Lemma 2.4, we have $P_{s_1}(\Delta,\alpha+1) = P_{s_\alpha}(\Delta,\alpha+1)$. Hence $P_x(\alpha+1,\Delta)$ does not ends at s_1 or s_α. Clearly $P_x(\alpha+1,\Delta)$ is F-edge-avoiding. We let φ' be the coloring obtained from φ by the following operation if $\alpha > 1$.

$$\begin{bmatrix} P_x(\alpha+1,\Delta) & rs_\alpha & rs_1 & s_2:s_{\alpha-1} \\ (\alpha+1)/\Delta & \text{uncolor} & \text{color by 2} & \text{shift} \end{bmatrix}$$

If $\alpha = 1$, let $\varphi' = \varphi/P_x(\alpha+1,\Delta)$. We call coloring φ' the symmetric coloring of φ. Note that under φ', $(r, rs_\alpha, s_\alpha, rs_{\alpha-1}, s_{\alpha-1}, \ldots, s_1, rs_{\alpha+1}, s_{\alpha+1}, \ldots, s_\beta)$ is a multifan. $s_\alpha, s_{\alpha-1}, \ldots, s_1$ is an α-inducing sequence and $s_{\alpha+1}, \ldots, s_\beta$ is an $\alpha+1$-inducing sequence. Also we have $\varphi'(ru) = \Delta \in \bar{\varphi}'(x)$, Δ is the last α-inducing color, $\varphi'(ux) = \alpha+1 \in \bar{\varphi}'(s_\alpha)$, $\varphi'(rs_i) = i+1$ and $\varphi'(us_i) = \varphi(us_i) = \tau$. Now under φ', we have $\tau \prec i+1$ since previously $i+1 \prec \tau$ under φ. Thus by considering this symmetric coloring of φ, we are back to **Case 1**, which gives a contradiction.

Case 3. $\tau \notin \bar{\varphi}(V(F))$.

In this case, by subclaim 4.1.3, we can apply Lemma 3.15 (2) with $\tau = \tau_1$ and $w = u$ to obtain a sequence $s_{h_1}, s_{h_2}, \ldots, s_{h_t}$ satisfying (a), (b) and (c) in Lemma 3.15 (2). Let $\varphi(rs_{h_i}) = \tau_i$ for $i \in [1,t]$ and $\bar{\varphi}(s_{h_i}) = \tau_{i+1}$ for $i \in [1, t-1]$. We first show the following property: $(*)$ $s_{h_1}, s_{h_2}, \ldots, s_{h_t}$ can not form a rotation, i.e., $\bar{\varphi}(s_{h_t}) = \alpha + 1$.

Suppose on the contrary that $\bar{\varphi}(s_{h_t}) = \tau_1 = \tau$. By Lemma 3.15 (2) (b), we have $P_r(1, \tau_1) = P_{s_{h_t}}(1, \tau_1)$. Also $u \in P_r(1, \tau_1)$ by subclaim 4.1.3. Let $P_u^+(1, \tau_1)$ be the $(1, \tau_1)$-path starting at u not containing us_i. Then by shifting from s_{h_1} to s_{h_t} if necessary (and by symmetry), we may assume that $P_u^+(1, \tau_1)$ ends at s_{h_t}, and so it does not contain rs_{h_1}. Now we do the following operation to obtain φ'.

$$\begin{bmatrix} s_{i+1} : s_\alpha & P_u^+(1, \tau_1) & us_i & P_{[u,r]}(\Delta, i+1) & ux & ur \\ \text{shift} & 1/\tau_1 & \tau_1/(i+1) & (i+1)/\Delta & \Delta/(\alpha+1) & (\alpha+1)/1 \end{bmatrix}$$

Then $\Delta \in \bar{\varphi}'(r) \cap \bar{\varphi}'(s_1)$, giving a contradiction. This proves $(*)$.

We then show the following property: $(**)$ we can get a contradiction if the missing color $\alpha + 1$ at x is replaced by τ_1.

Suppose now the missing color $\alpha + 1$ at x is replaced by τ_1. We still call this coloring φ for convenience. Then F is still a multifan at r, $(s_{h_1}, s_{h_2}, \ldots, s_{h_t})$ is still a sequence satisfying Lemma 3.15 (2) (a) and $\bar{\varphi}(s_{h_t}) = \alpha + 1$. Notice that $P_{s_{h_t}}(\alpha + 1, 1)$ does not pass through r or s_α by Lemma 1.9. We do the following to obtain φ'.

$$\begin{bmatrix} P_{s_{h_t}}(\alpha+1, 1) & rs_{h_t} & rs_{h_{t-1}} & \cdots & rs_1 \\ (\alpha+1)/1 & \tau_t \to 1 & \tau_{t-1} \to \tau_t & \cdots & \tau_1 \to \tau_2 \end{bmatrix}$$

Then $\bar{\varphi}'(r) = \varphi'(us_i) = \tau_1 \in \bar{\varphi}'(x)$. Interchange τ_1 and 1 on every $(\tau_1, 1)$-chain (or say relabel τ_1 as 1) to get φ''. Then $\bar{\varphi}''(r) = \varphi''(us_i) = 1 \in \bar{\varphi}''(x)$. Let $\varphi''' = \varphi''/P_x(1, \alpha+1, \varphi'')$. Note that $us_i \notin P_x(1, \alpha+1, \varphi'')$ since $P_x(1, \alpha+1, \varphi'')$ is disjoint from $P_r(1, \alpha+1, \varphi'')$ (which contains us_i) by Lemma 1.9. Hence $\varphi'''(us_i) = 1$, and so φ''' gives a contradiction to subclaim 4.1.3. This proves $(**)$.

Now we are ready to find a contradiction in case 3. Consider the $(\alpha + 1, \tau_1)$-path

$P_{s_\alpha}(\alpha+1,\tau_1)$. Firstly, $P_{s_\alpha}(\alpha+1,\tau_1)$ contains r. Since otherwise, by interchanging $\alpha+1$ and τ_1 on this path, we get a larger multifan $(r, rs_1, s_1, \dots, s_\alpha, rs_{h_1}, s_{h_1}, \dots, s_{h_t})$, contradicting the maximality of F. Thus we have $r \in P_{s_\alpha}(\alpha+1,\tau_1)$. Let z be the other end of $P_{s_\alpha}(\alpha+1,\tau_1)$.

Suppose $z = x$. Then $s_\alpha, x \notin P_{s_{h_t}}(\alpha+1,\tau_1)$. Let $\varphi' = \varphi/P_{s_{h_t}}(\alpha+1,\tau_1)$. We get a contradiction by $(*)$. Suppose $z = s_{h_t}$. Let $\varphi' = \varphi/P_x(\alpha+1,\tau_1)$. Then we get a contradiction by $(**)$. Notice that this also implies that $t \geq 2$.

The only remaining case is $z \notin \{x, s_{h_t}\}$. We also have $P_x(\alpha+1,\tau_1) = P_{s_{h_t}}(\alpha+1,\tau_1)$ by the above proof. Now by $r \in P_{s_\alpha}(\alpha+1,\tau_1)$, we have the following two cases.

Suppose $P_{s_\alpha}(\alpha+1,\tau_1)$ meets s_{h_1} before r. Let φ' be obtained from φ by interchanging $\alpha+1$ and τ_1 along $P_x(\alpha+1,\tau_1) = P_{s_{h_t}}(\alpha+1,\tau_1)$ and then shifting from s_{h_1} to s_{h_t}. Since $P_{s_\alpha}(\alpha+1,\tau_1,\varphi)$ meets s_{h_1} before r under φ, $P_{s_\alpha}(\alpha+1,\tau_1,\varphi')$ ends at s_{h_1}, and it does not contain r. Let $\varphi'' = \varphi'/P_{s_\alpha}(\alpha+1,\tau_1,\varphi')$. Then $(r, rs_1, s_1, \dots, s_\alpha, rs_{h_t}, s_{h_t}, rs_{h_{t-1}}, s_{h_{t-1}}, \dots, s_{h_1})$ is a multifan at r under φ'', contradicting the maximality of F.

Suppose $P_{s_\alpha}(\alpha+1,\tau_1)$ meets r before s_{h_1}, i.e., $P_{s_\alpha}(\alpha+1,\tau_1)$ meets u before r. Then $P_{[s_{h_1},z]}(\alpha+1,\tau_1)$ does not contain r or u or s_α. Similar to the case above, we obtain φ' from φ by interchanging $\alpha+1$ and τ_1 along $P_x(\alpha+1,\tau_1) = P_{s_{h_t}}(\alpha+1,\tau_1)$ and then shifting from s_{h_1} to s_{h_t}. Then let $\varphi'' = \varphi'/P_{[s_{h_1},z]}(\alpha+1,\tau_1,\varphi')$. Now $s_{h_t}, s_{h_{t-1}}, \dots, s_{h_1}$ is a sequence satisfying Lemma 3.15 (2) (a) and $\tau_1 \in \bar{\varphi}''(x)$. We get a contradiction by $(**)$. This completes the proof of Claim 3.3*. □

Claim 4.2. If $\varphi(ux) = \Delta$, then $N(u) \cap \{s_{\alpha+1}, s_{\alpha+2}, \dots, s_{\Delta-2}\} = \emptyset$.

We first introduce some concepts and tools. Let $B = N(u) \cap \{s_{\alpha+1}, s_{\alpha+2}, \dots, s_{\Delta-2}\}$. For any $y \in B$, by Lemma 3.13, we have $\bar{\varphi}(y) \notin \bar{\varphi}(V(F))$ and y and r are $(1, \bar{\varphi}(y))$-linked. Consequently, for any L-stable coloring φ' obtained from φ by a sequence of $(1,*)$-Kempe changes, we have $\bar{\varphi}'(y) = \bar{\varphi}(y)$ and y and r are $(1, \bar{\varphi}(y))$-linked under φ'. Thus we can apply Lemma 3.15 with $u = y$ and $\tau_1 = \bar{\varphi}(y)$ to obtain a sequence. Denote this sequence by *the y-sequence.*

For a vertex $y \in B$ with the y-sequence $s_{h_1}, s_{h_2}, \dots, s_{h_t}$. We say y is of type I if $y \neq s_{h_t}$. Otherwise we say y is of type II. Clearly if y is type II, then $\bar{\varphi}(s_{h_t}) = \bar{\varphi}(y)$, and so $t \geq 2$ and

$s_{h_1}, s_{h_2}, \ldots, s_{h_t}$ form a rotation. If y is type I, then by Lemma 3.13 and Lemma 3.15 (2) (c), we have $\bar{\varphi}(s_{h_t}) = \alpha + 1$. Since otherwise, $P_r(1, \bar{\varphi}(y))$ will have three ends r, y and s_{h_t}.

We first show the following: $(*)$ any vertex $y \in B$ can be assumed to be type I.

Let $y \in B$ be a type II vertex and denote $\bar{\varphi}(y)$ by τ_1. Let the y-sequence be $s_{h_1}, s_{h_2}, \ldots, s_{h_t}$ which satisfies the three conclusions of Lemma 3.15 (2). Let $\varphi(rs_{h_i}) = \tau_i$ for $i \in [1, t]$ and $\bar{\varphi}(s_{h_i}) = \tau_{i+1}$ for $i \in [1, t-1]$. Note that $y = s_{h_t}$ and $t \geq 2$. Consider the path $P_{s_\alpha}(\alpha + 1, \tau_1)$. Firstly, $P_{s_\alpha}(\alpha + 1, \tau_1)$ contains r. Since otherwise, by interchanging $\alpha + 1$ and τ_1 on this path, we get a larger multifan $(r, rs_1, s_1, \ldots, s_\alpha, rs_{h_1}, s_{h_1}, \ldots, s_{h_t})$, contradicting the maximality of F. Thus we have $r \in P_{s_\alpha}(\alpha + 1, \tau_1)$. Moreover, we claim that $P_{s_\alpha}(\alpha + 1, \tau_1)$ meets u before r. Since otherwise, by shifting from s_{h_1} to s_{h_t} and then interchanging $\alpha + 1$ and τ_1 on $P_{[s_\alpha, s_{h_1}]}(\alpha + 1, \tau_1)$, we get a larger multifan $(r, rs_1, s_1, \ldots, s_\alpha, rs_{h_t}, s_{h_t}, rs_{h_{t-1}}, s_{h_{t-1}}, \ldots, s_{h_1})$, contradicting the maximality of F. Let z be the other end of $P_{s_\alpha}(\alpha + 1, \tau_1)$. We will either find a contradiction or change y to a type I vertex in the following three cases.

Suppose $z = x$. Then $r, s_\alpha, x \notin P_y(\alpha{+}1, \tau_1)$. Let $\varphi' = \varphi / P_y(\alpha{+}1, \tau_1)$, then $\bar{\varphi}'(y) = \alpha{+}1$. We get a contradiction by Lemma 3.13.

Suppose $z \notin \{x, y\}$. Let φ_1 be obtained from φ by interchanging $\alpha + 1$ and τ_1 on both $P_x(\alpha + 1, \tau_1)$ and $P_y(\alpha + 1, \tau_1)$. Then $\tau_1 \in \bar{\varphi}_1(x)$ and $\bar{\varphi}_1(y) = \bar{\varphi}_1(s_{h_t}) = \alpha + 1$. Note that $r \notin P_y(\alpha + 1, 1, \varphi_1)$ by Lemma 1.9. Let $\varphi_2 = \varphi_1 / P_y(\alpha + 1, 1, \varphi_1)$, we have $\bar{\varphi}_2(y) = 1$. Now we do the following operation to obtain φ'.

$$\begin{bmatrix} rs_{h_t} & rs_{h_{t-1}} & \cdots & rs_1 \\ \tau_t \to 1 & \tau_{t-1} \to \tau_t & \cdots & \tau_1 \to \tau_2 \end{bmatrix}$$

Then $\bar{\varphi}'(r) = \bar{\varphi}'(s_{h_1}) = \tau_1 \in \bar{\varphi}'(x)$. Interchange τ_1 and 1 on every $(\tau_1, 1)$-chain (or say relabel τ_1 as 1) to get φ''. Then $\bar{\varphi}''(r) = \bar{\varphi}''(s_{h_1}) = 1 \in \bar{\varphi}'(x)$. Note that both $P_x(1, \alpha{+}1, \varphi')$ and $P_{s_{h_1}}(1, \alpha + 1, \varphi')$ do not contain r by Lemma 1.9. Interchange 1 and $\alpha + 1$ on both $P_x(1, \alpha + 1, \varphi')$ and $P_{s_{h_1}}(1, \alpha + 1, \varphi')$ to get φ'''. Now $\bar{\varphi}'''(y) = \tau_t$; y is a type I vertex with the y-sequence $s_{h_{t-1}}, s_{h_{t-2}}, \ldots, s_{h_1}$, as desired.

The only remaining case is $z = y$. In this case $r, s_\alpha, y \notin P_x(\alpha+1, \tau_1)$. Let $\varphi_1 = \varphi/P_x(\alpha+1,\tau_1)$, then $\bar{\varphi}_1(x) = \tau_1$. Consider the path $P_{s_1}(\Delta, \tau_1, \varphi_1)$. Firstly, $P_{s_1}(\Delta, \tau_1, \varphi_1)$ contains r. Since otherwise, by interchanging Δ and τ_1 on this path, we get a larger multifan $(r, rs_1, s_1, \ldots, s_\alpha, rs_{h_1}, s_{h_1}, \ldots, s_{h_t})$, contradicting the maximality of F. Thus we have $r \in P_{s_1}(\Delta, \tau_1, \varphi_1)$. Moreover, the other end of $P_{s_1}(\Delta, \tau_1, \varphi_1)$ is x. Since otherwise, we may obtain a coloring φ^* by interchanging τ_1 and Δ on $P_x(\Delta, \tau_1, \varphi_1)$ and then a $(\Delta, 1) - (1, \alpha+1)$ swap at x. Then $\bar{\varphi}^*(x) = \alpha+1$ and $\varphi^*(ux) = \tau_1 \notin \bar{\varphi}^*(V(F))$, giving a contradiction to Lemma 3.12. Therefore we have $r \in P_{s_1}(\Delta, \tau_1, \varphi_1) = P_x(\Delta, \tau_1, \varphi_1)$, and so $r \notin P_y(\Delta, \tau_1, \varphi_1)$. Let $\varphi_2 = \varphi_1/P_y(\Delta, \tau_1, \varphi_1)$. Then $\bar{\varphi}_2(y) = \varphi_2(ux) = \Delta$. Let $\varphi_3 = \varphi_2/P_y(\Delta, 1, \varphi_2)$. Note that if $ux \notin P_y(\Delta, 1, \varphi_2)$, then we may continue with the proof in the case $z \notin \{x, y\}$ with φ_3 in place of the coloring φ_2 in that case. Thus we assume that $ux \in P_y(\Delta, 1, \varphi_2)$, and so $\bar{\varphi}_3(y) = \varphi_3(ux) = 1$. Now we do the following to obtain φ'.

$$\begin{bmatrix} rs_{h_t} & rs_{h_{t-1}} & \cdots & rs_1 \\ \tau_t \to 1 & \tau_{t-1} \to \tau_t & \cdots & \tau_1 \to \tau_2 \end{bmatrix}$$

Then $\bar{\varphi}'(r) = \tau_1 \in \bar{\varphi}'(x)$. Note that $r \notin P_x(\tau_1, \alpha+1, \varphi')$ by Lemma 1.9. Let $\varphi'' = \varphi'/P_x(\tau_1, \alpha+1, \varphi')$. Then $\alpha + 1 \in \bar{\varphi}''(x)$, $\varphi''(ux) = 1 \notin \bar{\varphi}''(V(F))$, giving a contradiction to Lemma 3.12. This completes the proof of $(*)$.

Now we ready to show Claim 3.4, that is, $B = \emptyset$. Suppose on the contrary that there exists a vertex $y \in B$. By $(*)$, we assume that y is a type I vertex. Let the y sequence be $s_{h_1}, s_{h_2}, \ldots, s_{h_t}$ with $\bar{\varphi}(y) = \tau_1$, $\varphi(rs_{h_i}) = \tau_i$ for $i \in [1, t]$ and $\bar{\varphi}(s_{h_i}) = \tau_{i+1}$ for $i \in [1, t-1]$. By the property of type I vertices (proved in the second paragraph of the proof of this claim), we have $\bar{\varphi}(s_{h_t}) = \alpha + 1$. We will prove the following stronger result: $(**)$ there is a contradiction even when the assumption $x \in V(G) \setminus N[r]$ is replaced by $x \in V(G) \setminus (V(F) \cup \{s_{h_1}, s_{h_2}, \ldots, s_{h_t}\})$. Denote $\varphi(uy)$ by δ.

Subclaim 4.2.1. $\varphi(uy) = \delta \in \bar{\varphi}(V(F)) \setminus \{1\}$.

Proof. Suppose on the contrary that $\delta \notin \bar{\varphi}(V(F)) \setminus \{1\}$. Consider $P_{s_\alpha}(\alpha+1, \tau_1)$. Firstly,

$P_{s_\alpha}(\alpha+1,\tau_1)$ contains r. Since otherwise, by interchanging $\alpha+1$ and τ_1 on this path, we get a larger multifan $(r, rs_1, s_1, \dots, s_\alpha, rs_{h_1}, s_{h_1}, \dots, s_{h_t})$, contradicting the maximality of F. If y is not the other end of $P_{s_\alpha}(\alpha+1,\tau_1)$, then we may let $\varphi' = \varphi/P_y(\alpha+1,\tau_1)$. Clearly φ' is F-stable, hence $\delta \in \bar{\varphi}'(V(F)) \setminus \{1\} = \bar{\varphi}(V(F)) \setminus \{1\}$ by Lemma 3.12, as desired.

Thus we assume that $y \in P_{s_\alpha}(\alpha+1,\tau_1)$. Then $x, s_{h_t} \notin P_{s_\alpha}(\alpha+1,\tau_1)$. Let φ_1 be obtained from φ by interchanging $\alpha+1$ and τ_1 on both $P_x(\alpha+1,\tau_1)$ and $P_{s_{h_t}}(\alpha+1,\tau_1)$. Then $\bar{\varphi}_1(s_{h_t}) = \tau_1 \in \bar{\varphi}_1(x)$. Consider $P_{s_1}(\Delta,\tau_1,\varphi_1)$. By the same proof three paragraphs above, we have $r \in P_{s_1}(\Delta,\tau_1,\varphi_1) = P_x(\Delta,\tau_1,\varphi_1)$, and so r is not contained by $P_y(\Delta,\tau_1,\varphi_1)$ or $P_{s_{h_t}}(\Delta,\tau_1,\varphi_1)$. We do $(\tau_1,\Delta)-(\Delta,1)$ swap at both y and s_{h_t} to obtain φ_2. Note that r and s_1 are $(\Delta,1)$-linked by Lemma 1.9, and so φ_2 is F-stable. Now we consider the following two cases.

Suppose $\delta = 1$. In this case $\varphi_2(uy) = \Delta \neq \delta$ and $\varphi_2(ux) = 1$. We do the following to obtain φ'.

$$\begin{bmatrix} rs_{h_t} & rs_{h_{t-1}} & \cdots & rs_1 \\ \tau_t \to 1 & \tau_{t-1} \to \tau_t & \cdots & \tau_1 \to \tau_2 \end{bmatrix}$$

Then $\bar{\varphi}'(r) = \tau_1 \in \bar{\varphi}'(x)$. Note that $r \notin P_x(\tau_1,\alpha+1,\varphi')$ by Lemma 1.9. Let $\varphi'' = \varphi'/P_x(\tau_1,\alpha+1,\varphi')$. Then $\alpha+1 \in \bar{\varphi}''(x)$, $\varphi''(ux) = 1 \notin \bar{\varphi}''(V(F))$, giving a contradiction to Lemma 3.12.

Suppose $\delta \neq 1$. Then $\varphi_2(uy) = \delta = \varphi(uy)$. Let $\varphi_3 = \varphi_2/P_y(1,\alpha+1,\varphi_2)$. Then φ_3 is F-stable by Lemma 1.9. Thus by Lemma 3.12 with y in place of x, we have $\delta \in \bar{\varphi}_3(V(F)) \setminus \{1\} = \bar{\varphi}(V(F)) \setminus \{1\}$, as desired. □

Let $P_u^+(1,\delta)$ be the $(1,\delta)$-path starting at u not containing uy. Note that if $u \notin P_r(1,\delta) = P_{s_{\delta-1}}(1,\delta)$, then we may interchange δ and 1 on $P_u(1,\delta)$ to obtain φ^*. Since φ^* is L-stable and $\varphi^*(uy) = 1$, we get a contradiction by subclaim 4.2.1. Thus we have $u \in P_r(1,\delta) = P_{s_{\delta-1}}(1,\delta)$, and therefore $P_u^+(1,\delta)$ either ends at $s_{\delta-1}$ or r. By considering the symmetric coloring of φ (introduced in the proof of Claim 3.3* Case 2), we may assume without loss of generality that $P_u^+(1,\delta)$ ends at $s_{\delta-1}$.

Subclaim 4.2.2. $r, u \in P_{s_1}(\Delta,\tau_1) = P_y(\Delta,\tau_1)$. Moreover, there exits a (Δ,τ_1)-path

$P_{[u,r]}(\Delta,\tau_1)$ between u and r which does not contain rs_{h_1} or ux.

Proof. Firstly, $P_{s_1}(\Delta,\tau_1,\varphi_1)$ contains r. Since otherwise, by interchanging Δ and τ_1 on this path, we get a larger multifan $(r, rs_1, s_1, \ldots, s_\alpha, rs_{h_1}, s_{h_1}, \ldots, s_{h_t})$, contradicting the maximality of F. Thus we have $r \in P_{s_1}(\Delta,\tau_1,\varphi_1)$. Secondly, $u \in P_{s_1}(\Delta,\tau_1)$. Since otherwise, by interchanging Δ and τ_1 on $P_u(\Delta,\tau_1)$, we obtain a coloring φ^* with $\varphi^*(ux) = \tau_1 \notin \bar{\varphi}^*(V(F))$, giving a contradiction to Lemma 3.12. Thus $u \in P_{s_1}(\Delta,\tau_1)$. Finally we have $y \in P_{s_1}(\Delta,\tau_1)$. Since otherwise, let $\varphi^* * = \varphi/P_y(\Delta,\tau_1)$, we get a contradiction by Lemma 3.13. This proves the first part of the subclaim.

Let $P_u^+(\tau_1,\Delta)$ be the (τ_1,Δ)-path starting at u not containing ux. Suppose that the "moreover" part fails. Then we have the following two cases: either $r \notin P_u^+(\tau_1,\Delta)$ or $r \in P_u^+(\tau_1,\Delta)$ and $P_u^+(\tau_1,\Delta)$ meets s_{h_1} before r.

Suppose $r \notin P_u^+(\tau_1,\Delta)$. Then $P_u^+(\tau_1,\Delta)$ ends at either s_1 or y. We do the following to obtain φ'.

$$\begin{bmatrix} s_{h_1} : s_{h_t} & ur & P_u^+(\tau_1,\Delta) & ux \\ \text{shift} & (\alpha+1)/\tau_1 & \tau_1/\Delta & \Delta/(\alpha+1) \end{bmatrix}$$

Then in both cases, $(r, rs_1, s_1, \ldots, s_\alpha, rs_{h_t}, s_{h_t}, rs_{h_{t-1}}, s_{h_{t-1}}, \ldots, s_{h_1})$ is a multifan at r under φ', contradicting the maximality of F.

Suppose $r \in P_u^+(\tau_1,\Delta)$ and $P_u^+(\tau_1,\Delta)$ meets s_{h_1} before r. Let $P_r^+(\tau_1,\Delta)$ be the (τ_1,Δ)-path starting at r not containing rs_{h_1}. Then we have $u \notin P_r^+(\tau_1,\Delta)$ and $P_r^+(\tau_1,\Delta)$ ends at either s_1 or y. We do the following to obtain φ'.

$$\begin{bmatrix} s_{h_1} : s_{h_t} & P_r^+(\tau_1,\Delta) & rux \\ \text{shift} & \tau_1/\Delta & \Delta/(\alpha+1) \end{bmatrix}$$

Then in both cases, $(r, rs_1, s_1, \ldots, s_\alpha, rs_{h_t}, s_{h_t}, rs_{h_{t-1}}, s_{h_{t-1}}, \ldots, s_{h_1})$ is a multifan at r under φ', contradicting the maximality of F. This completes the proof of subclaim 4.2.2. □

Recall that $P_u^+(1,\delta)$ ends at $s_{\delta-1}$. Now by subclaim 4.2.2, we may do the following to obtain φ'.

$$\begin{bmatrix} s_{h_1} : s_{h_t} & P_{[u,r]}(\Delta, \tau_1) & ux & ur & P_u^+(1,\delta) & uy \\ \text{shift} & \tau_1/\Delta & \Delta/(\alpha+1) & (\alpha+1)/1 & 1/\delta & \delta/\tau_1 \end{bmatrix}$$

Then $\Delta \in \bar{\varphi}'(r) \cap \bar{\varphi}'(s_1)$, giving a contradiction. This completes the proof of $(**)$, and Claim 3.4 follows immediately. □

Claim 4.3. If $\varphi(ux)$ is a 2-inducing color, then $|N(u) \cap \{s_1, s_2, \ldots, s_\alpha\}| \leq 1$.

To show Claim 3.5, we will prove the following stronger result.

Claim 3.5*. If $\varphi(ux) = i$ is a 2-inducing color, then $N(u) \cap \{s_1, s_2, \ldots, s_\alpha\} \subseteq \{s_1, s_2, \ldots, s_{i-2}\}$ and $\varphi(us) = \Delta$ for any $s \in N(u) \cap \{s_1, s_2, \ldots, s_\alpha\}$.

Proof. Suppose on the contrary. Then by Lemma 3.19, either there exists $j \geq i+1$ such that $u \sim s_j$, or there exists $j \leq i-2$ such that $u \sim s_j$ and $\varphi(us_j) \neq \Delta$.

Case 1. There exists $j \geq i+1$ such that $u \sim s_j$.

Let φ^* be obtained from φ by the following operation. (Note that $\alpha \geq 2$ since there exist two distinct 2-inducing colors $\alpha+1$ and i.)

$$\begin{bmatrix} rs_i & rs_1 & s_2 : s_{i-1} \\ \text{uncolor} & \text{color by 2} & \text{shift} \end{bmatrix}$$

Then by applying Claim 3.3* on φ^*, we have $\varphi(us_j) = \varphi^*(us_j) \in \{\Delta, 2, 3, \ldots, i-1\}$. Denote $\varphi(us_j)$ by δ. Note that subclaim 4.1.2 is proved without supposing on the contrary of Claim 3.3*, so we may also apply it here. Hence there exists an $(i, j+1)$-path between u and r which does not contain ux or rs_{j+1}. Therefore, this path contains s_{i-1}. So under the original coloring φ, there exists an $(i, j+1)$-path between u and s_{i-1} which does not contain ux or rs_{j+1} or rs_i. Denote this path by $P_{[u,s_{i-1}]}(i, j+1)$. By applying Lemma 3.14 (d) and (e) on φ, we have that there exists a $(\delta, \alpha+1)$-path $P_{[s_{\delta-1},u]}(\delta, \alpha+1)$ which does not contain us_j (where $s_{\Delta-1} = s_1$). Note that $r \in P_{[s_{\delta-1},u]}(\delta, \alpha+1)$, and if $\delta \neq \Delta$, then

$rs_\delta \in P_{[s_{\delta-1},r]}(\delta,\alpha+1)$. Now let φ' be obtained from φ by the following operation.

$$\begin{bmatrix} us_j & P_{[u,s_{i-1}]}(i,j+1) & ux & P_{[s_{\delta-1},u]}(\delta,\alpha+1) \\ \delta/(j+1) & (j+1)/i & i/(\alpha+1) & (\alpha+1)/\delta \end{bmatrix}$$

if $\delta = \Delta$, then $(r, rs_1, s_1, \ldots, s_{i-1}, rs_{j+1}, s_{j+1}, \ldots, s_\alpha)$ is a multifan under φ'. But $\alpha+1 \in \bar{\varphi}'(s_1) \cap \bar{\varphi}'(s_\alpha)$, giving a contradiction to Lemma 1.9.

If $\delta \neq \Delta$, note that $\bar{\varphi}'(s_{\delta-1}) = \varphi'(rs_\delta) = \alpha+1$, then $(r, rs_1, s_1, \ldots, s_{i-1}, rs_{j+1}, s_{j+1}, \ldots, s_\alpha)$ is still a multifan under φ'. But $\alpha+1 \in \bar{\varphi}'(s_{\delta-1}) \cap \bar{\varphi}'(s_\alpha)$, giving a contradiction. This completes the proof of Case 1.

Case 2. There exists $j \le i-2$ such that $u \sim s_j$ and $\varphi(us_j) \neq \Delta$.

Denote $\varphi(us_j)$ by δ. Clearly $\delta \notin \{i, j, j+1, \alpha+1\}$. So we have the following five cases: $\delta = 1$; $\delta \in \{2, 3, \ldots, j-1\}$; $\delta \in \{j+2, j+3, \ldots, i-1\}$; $\delta \in \{i+1, i+2, \ldots, \alpha\}$ and $\delta \notin \bar{\varphi}(V(F))$.

Case 2.1. $\delta = 1$.

Let $P_u^+(1,i)$ be the $(1,i)$-path starting at u not containing ux. Note that $ux \in P_r(1,i) = P_{s_{i-1}}(1,i)$ by Lemma 1.9 and Lemma 3.14 (a). Thus By Lemma 3.14 (b), we have that $P_u^+(1,i)$ ends at s_{i-1}. Also by Lemma 3.14 (e), there exists a $(j+1,\alpha+1)$-path $P_{[u,s_\alpha]}(j+1,\alpha+1)$ not containing ru. By Lemma 3.14 (e) again, we have $P_x(j+1,\alpha+1)$ is disjoint from $P_{[u,s_\alpha]}(j+1,\alpha+1)$, and $r, s_j \notin P_x(j+1,\alpha+1)$. We obtain φ' by the the following operation.

$$\begin{bmatrix} P_x(j+1,\alpha+1)) & P_u^+(1,i) & ux & P_{[u,s_\alpha]}(j+1,\alpha+1) & ur \\ (\alpha+1)/(j+1) & 1/i & i/(j+1) & (j+1)/(\alpha+1) & (\alpha+1)/1 \end{bmatrix}$$

Then under φ', the path consists by ux and us_j is a $(i,j+1)$-chain. Interchange i and $j+1$ on this chain to obtain φ''. Now $\bar{\varphi}''(s_j) = i$, $\bar{\varphi}''(s_\alpha) = j+1$ and $\bar{\varphi}''(s_{i-1}) = \varphi''(ru) = 1$. So $(r, rs_1, s_1, \ldots, s_j, rs_i, s_i, rs_{i+1}, s_{i+1}, \ldots, s_\alpha, rs_{j+1}, s_{j+1}, rs_{j+2}, s_{j+2}, \ldots, s_{i-1})$ is a 2-inducing multifan under φ''. The last 2-inducing color is 1, $\bar{\varphi}''(r) = \alpha+1$, $\bar{\varphi}''(x) = j+1$ and

$\varphi''(ux) = \bar{\varphi}''(s_j) = i$. Thus we can do a $(j+1, \alpha+1) - (\alpha+1, 1)$ swap at x to obtain φ'''. By Lemma 1.9, φ''' is F-stable. Since $\varphi'''(ux) = \varphi''(ux) = \bar{\varphi}''(s_j) = \bar{\varphi}'''(s_j)$, we get a contradiction by Lemma 3.19.

Case 2.2. $\delta \in \{2, 3, \ldots, j-1\}$.

Firstly, by the same proof above, we have that there still exists a $(j+1, \alpha+1)$-path $P_{[u,s_\alpha]}(j+1, \alpha+1)$ not containing ru. We also have that $u \in P_r(1, \delta)$ since otherwise, by interchanging 1 and δ on $P_u(1, \delta)$, we are back to Case 2.1. Let $P_u^+(1, \delta)$ be the $(1, \delta)$-path staring from u not containing us_j. Then $P_u^+(1, \delta)$ ends at either $s_{\delta-1}$ or r. Notice that in the latter case, $P_u^+(1, \delta)$ meets s_δ before r, and so $P_{[u,s_\delta]}(1, \delta) = P_u^+(1, \delta) - r$.

If $P_u^+(1, \delta)$ ends at $s_{\delta-1}$, then we do the following to obtain φ'.

$$\begin{bmatrix} us_j & P_{[u,s_\alpha]}(j+1, \alpha+1) & ur & P_u^+(1, \delta) \\ \delta/(j+1) & (j+1)/(\alpha+1) & (\alpha+1)/1 & 1/\delta \end{bmatrix}$$

If $P_u^+(1, \delta)$ ends at r, then we do the following to obtain φ'.

$$\begin{bmatrix} us_j & s_\delta : s_j & P_{[u,s_\alpha]}(j+1, \alpha+1) & ur & P_{[u,s_\delta]}(1, \delta) \\ \delta/(j+1) & \text{shift} & (j+1)/(\alpha+1) & (\alpha+1)/1 & 1/\delta \end{bmatrix}$$

In the former case, $(r, rs_1, s_1, \ldots, s_{\delta-1})$ is a multifan; in the latter case, $(r, rs_1, s_1, \ldots, s_{\delta-1}, rs_j, s_j, rs_{j-1}, s_{j-1}, \ldots, s_\delta)$ is a multifan. In both cases, the last 2-inducing color of the multifan is $1 = \varphi'(ru)$. Note that $\bar{\varphi}'(r) = \alpha+1$, thus $r \notin P_x(\alpha = 1, 1, \varphi')$ by Lemma 1.9. Let $\varphi'' = \varphi/P_x(\alpha = 1, 1, \varphi')$. Then under φ'', $s_{j+1}, s_{j+2}, \ldots, s_\alpha$ form a rotation, $j+2 \le i \le \alpha$ and $\varphi''(ux) = \bar{\varphi}''(s_{i-1}) = i$. Thus we get a contradiction by Lemma 3.16.

Case 2.3. $\delta \in \{j+2, j+3, \ldots, i-1\}$.

By Lemma 3.14 (d) we have that $r \in P_{s_1}(\Delta, \alpha+1) = P_{s_\alpha}(\Delta, \alpha+1)$, and so $r \notin P_x(\Delta, \alpha+1)$. Let $\varphi_1 = \varphi/P_x(\Delta, \alpha+1)$. Note that $P_{s_1}(\Delta, i, \varphi_1) = P_{s_{i-1}}(\Delta, i)$. If $r \notin P_x(\Delta, i, \varphi_1)$, then by a $(\Delta, i) - (i, 1) - (1, \alpha+1)$ swap at x, we get a contradiction by Claim 3.3*. Thus we assume that $r \in P_x(\Delta, i, \varphi)$, and so $r \notin P_{s_1}(\Delta, i, \varphi_1) = P_{s_{i-1}}(\Delta, i)$.

Let $\varphi_2 = \varphi_1/P_{s_1}(\Delta, i, \varphi_1)$. Then do a $(\Delta,1)-(1,\alpha+1)$ swap at x to obtain φ_3. Finally we obtain φ' from φ_3 by uncoloring rs_{i-1}, shifting from s_2 to s_{i-2} and recoloring rs_1 by 2. Now under φ', $(r, rs_{i-1}, s_{i-1}, rs_{i-2}, s_{i-2}, \dots, s_1, rs_i, s_i, rs_{i+1}, s_{i+1}, \dots, s_\alpha)$ is an $i-1$-inducing multifan. Since $\delta \in \{j+2, j+3, \dots, i-1\}$, we are back to Case 2.2.

Case 2.4. $\delta \in \{i+1, i+2, \dots, \alpha\}$.

By the same proofs in Case 2.2, there still exists $P_{[u,s_\alpha]}(j+1,\alpha+1)$ and $P_u^+(1,\delta)$. Also, $P_u^+(1,\delta)$ ends at either $s_{\delta-1}$ or r. We first claim that in Case 2.4, we may assume that $P_u^+(1,\delta)$ ends at $s_{\delta-1}$.

In order to show the above claim, we introduce a concept called *i^+-symmetric coloring.* Note that by Lemma 3.17, we have $r \in P_x(\alpha+1,i) = P_{s_\alpha}(\alpha+1,i)$ and $r \notin P_{s_{i-1}}(\alpha+1,i)$. Clearly $P_{s_{i-1}}(\alpha+1,i)$ is F-avoiding. We let φ' be the coloring obtained from φ by the following operation.

$$\begin{bmatrix} P_{s_{i-1}}(\alpha+1,i) & s_i : s_\alpha & rux \\ i/(\alpha+1) & \text{shift} & (\alpha+1)/i \end{bmatrix}$$

We call coloring φ' the i^+-symmetric coloring of φ. Note that under the coloring φ', $(r, rs_1, s_1, \dots, s_{i-1}, rs_\alpha, s_\alpha, rs_{\alpha-1}, s_{\alpha-1}, \dots, s_i)$ is a 2-inducing multifan. The last 2-inducing color is $i = \varphi'(ru)$ and $i \in \bar{\varphi}'(x)$. Also we have $\varphi'(ux) = \alpha+1 = \bar{\varphi}'(s_{i-1})$ and $\varphi'(us_j) = \varphi(us_j) = \delta$. In Case 2.4, it is easy verify that the i^+-symmetric coloring of φ is still in Case 2,4. Thus by considering this coloring, we may assume without loss of generality that $P_u^+(1,\delta)$ ends at $s_{\delta-1}$. Now we do the following to obtain φ'.

$$\begin{bmatrix} us_j & P_{[u,s_\alpha]}(j+1,\alpha+1) & ur & P_u^+(1,\delta) \\ \delta/(j+1) & (j+1)/(\alpha+1) & (\alpha+1)/1 & 1/\delta \end{bmatrix}$$

Then under φ', $(r, rs_1, s_1, \dots, s_j, rs_\delta, s_\delta, rs_{\delta+1}, s_{\delta+1}, \dots, s_\alpha, rs_{j+1}, s_{j+1}, rs_{j+2}, s_{j+2}, \dots, s_{\delta-1})$ is a 2-inducing multifan. The last 2-inducing color is $1 = \varphi'(ru)$. We also have $\bar{\varphi}'(r) = \alpha+1$, $\bar{\varphi}'(s_j) = \delta$, $\varphi'(ux) = i$ and $\varphi'(us_j) = j+1$. Thus on the new multifan, we have $\bar{\varphi}'(s_j) \prec \varphi'(us_j) \prec \varphi'(ux)$. Note that $r \notin P_x(\alpha+1, 1, \varphi')$ by Lemma 1.9. Thus let $\varphi'' = \varphi'/P_x(\alpha+1,1,\varphi')$, we are back to Case 2.3.

Case 2.5. $\delta \notin \bar{\varphi}(V(F))$.

In this case, we denote $\delta = \varphi(us_j)$ by τ_1 for consistence. Note that $u \in P_r(1, \tau_1)$ since otherwise by interchanging 1 and τ_1 on $P_u(1, \tau_1)$, we are back to Case 2.1. In fact, the above statement also implies that $u \in P_r(1, \tau_1, \varphi')$ for any L-stable coloring φ' obtained from φ through a sequence of L-avoiding Kempe $(1, *)$-changes. Thus we may apply Lemma 3.15 (2) with $w = u$ to obtain a sequence $s_{h_1}, s_{h_2}, \ldots, s_{h_t}$ satisfying the three conclusions of Lemma 3.15 (2). Let $\varphi(rs_{h_i}) = \tau_i$ for $i \in [1, t]$ and $\bar{\varphi}(s_{h_i}) = \tau_{i+1}$ for $i \in [1, t-1]$. Then $\bar{\varphi}(s_{h_t})$ is either τ_1 or $\alpha + 1$.

Subcase 2.5.1 $\bar{\varphi}(s_{h_t}) = \tau_1$.

By the same proof in Case 2.2, there still exists $P_{[u,s_\alpha]}(j+1, \alpha+1)$. Let $P_u^+(1, \tau_1)$ be the $(1, \tau_1)$-path starting at u not containing us_j. Since $u \in P_r(1, \tau_1)$ and $P_r(1, \tau_1) = P_{s_{h_t}}(1, \tau_1)$ by Lemma 3.15 (2) (c), we have $P_u^+(1, \tau_1)$ ends at either r or s_{h_t}. By shifting from s_{h_1} to s_{h_t} if necessary, we may assume that $P_u^+(1, \tau_1)$ ends at s_{h_t}. Now we do the following to obtain φ'.

$$\begin{bmatrix} us_j & P_{[u,s_\alpha]}(j+1, \alpha+1) & ur & P_u^+(1, \tau_1) \\ \tau_1/(j+1) & (j+1)/(\alpha+1) & (\alpha+1)/1 & 1/\tau_1 \end{bmatrix}$$

Then $(r, rs_1, s_1, \ldots, s_j, rs_{h_1}, s_{h_1}, \ldots, s_{h_t})$ is a 2-inducing multifan. The last 2-inducing color is $1 = \varphi'(ru)$, $\bar{\varphi}'(r) = \alpha + 1$, $\varphi'(ux) = i \in [j+2, \alpha]$ and $s_{j+1}, s_{j+2}, \ldots, s_\alpha$ form a rotation. Note that $r \notin P_x(\alpha+1, 1)$ by Lemma 1.9, thus by interchanging $\alpha+1$ and 1 on $P_x(\alpha+1, 1)$, we obtain a contradiction by Lemma 3.16.

Subcase 2.5.2 $\bar{\varphi}(s_{h_t}) = \alpha + 1$.

Similar there still exists $P_{[u,s_\alpha]}(j+1, \alpha+1)$. Let $P_u^+(1, \tau_1)$ be the $(1, \tau_1)$-path starting at u not containing us_j. Since $u \in P_r(1, \tau_1)$, we have $P_u^+(1, \tau_1)$ ends at either r or z where $z \in V(G) \setminus (V(F) \cup \{s_{h_1}, s_{h_2}, \ldots, s_{h_t}, u, x\})$. Notice that in the former case, $P_u^+(1, \tau_1)$ meets s_{h_1} before r, and so $P_{[u,s_{h_1}]}(1, \tau_1) = P_u^+(1, \tau_1) - r$.

If $P_u^+(1,\tau_1)$ ends at r, we do the following to obtain φ'.

$$\begin{bmatrix} s_{h_1}:s_{h_t} & P_{[u,s_{h_1}]}(1,\tau_1) & us_j & P_{[u,s_\alpha]}(j+1,\alpha+1) & ur \\ \text{shift} & 1/\tau_1 & \tau_1/(j+1) & (j+1)/(\alpha+1) & (\alpha+1)/1 \end{bmatrix}$$

If $P_u^+(1,\tau_1)$ ends at z, we do the following to obtain φ'.

$$\begin{bmatrix} s_{h_1}:s_{h_t} & P_u^+(1,\tau_1) & us_j & P_{[u,s_\alpha]}(j+1,\alpha+1) & ur \\ \text{shift} & 1/\tau_1 & \tau_1/(j+1) & (j+1)/(\alpha+1) & (\alpha+1)/1 \end{bmatrix}$$

Then in both cases $(r, rs_1, s_1, \ldots, s_j)$ is a multifan. But $\bar{\varphi}'(r) = \bar{\varphi}'(s_j) = \tau_1$, giving a contradiction to Lemma 1.9. This completes the proof of Claim 3.5*. □

Claim 4.4. If $\varphi(ux)$ is a 2-inducing color, then $N(u) \cap \{s_{\alpha+1}, s_{\alpha+2}, \ldots, s_{\Delta-2}\} = \emptyset$.

The proof of this claim is very close to the proof of Claim 3.4. Let $\varphi(ux) = i \in [2,\alpha]$ and $B = N(u) \cap \{s_{\alpha+1}, s_{\alpha+2}, \ldots, s_{\Delta-2}\}$. Same as Claim 3.4, for any $y \in B$, we define y-sequence, type I vertex and type II vertex as before. Recall that for any $y \in B$ with the y-sequence $s_{h_1}, s_{h_2}, \ldots, s_{h_t}$, if y is of type I, then $\bar{\varphi}(s_{h_t}) = \alpha+1$; if y is of type II, then $y = s_{h_t}$ and $t \geq 2$.

We first show the following: $(*)$ any vertex $y \in B$ can be assumed to be type I.

Let $y \in B$ be a type II vertex and denote $\bar{\varphi}(y)$ by τ_1. Let the y-sequence be $s_{h_1}, s_{h_2}, \ldots, s_{h_t}$ which satisfies the three conclusions of Lemma 3.15 (2). Let $\varphi(rs_{h_i}) = \tau_i$ for $i \in [1,t]$ and $\bar{\varphi}(s_{h_i}) = \tau_{i+1}$ for $i \in [1, t-1]$. Note that $y = s_{h_t}$ and $t \geq 2$. Consider the path $P_{s_\alpha}(\alpha+1, \tau_1)$. Firstly, $P_{s_\alpha}(\alpha+1,\tau_1)$ contains r. Since otherwise, by interchanging $\alpha+1$ and τ_1 on this path, we get a larger multifan $(r, rs_1, s_1, \ldots, s_\alpha, rs_{h_1}, s_{h_1}, \ldots, s_{h_t})$, contradicting the maximality of F. Thus we have $r \in P_{s_\alpha}(\alpha+1,\tau_1)$. Moreover, we claim that $P_{s_\alpha}(\alpha+1,\tau_1)$ meets u before r. Since otherwise, by shifting from s_{h_1} to s_{h_t} and then interchanging $\alpha+1$ and τ_1 on $P_{[s_\alpha, s_{h_1}]}(\alpha+1,\tau_1)$, we get a larger multifan $(r, rs_1, s_1, \ldots, s_\alpha, rs_{h_t}, s_{h_t}, rs_{h_{t-1}}, s_{h_{t-1}}, \ldots, s_{h_1})$, contradicting the maximality of F. Let z be the other end of $P_{s_\alpha}(\alpha+1,\tau_1)$. We will either find a contradiction or change y to a type I vertex in the following three cases.

Suppose $z = x$. Then $r, s_\alpha, x \notin P_y(\alpha+1, \tau_1)$. Let $\varphi' = \varphi/P_y(\alpha+1, \tau_1)$, then $\bar{\varphi}'(y) = \alpha+1$. We get a contradiction by Lemma 3.13.

Suppose $z \notin \{x, y\}$. Let φ_1 be obtained from φ by interchanging $\alpha + 1$ and τ_1 on both $P_x(\alpha + 1, \tau_1)$ and $P_y(\alpha + 1, \tau_1)$. Then $\tau_1 \in \bar{\varphi}_1(x)$ and $\bar{\varphi}_1(y) = \bar{\varphi}_1(s_{h_t}) = \alpha + 1$. Note that $r \notin P_y(\alpha + 1, 1, \varphi_1)$ by Lemma 1.9. Let $\varphi_2 = \varphi_1/P_y(\alpha + 1, 1, \varphi_1)$, we have $\bar{\varphi}_2(y) = 1$. Now we do the following operation to obtain φ'.

$$\begin{bmatrix} rs_{h_t} & rs_{h_{t-1}} & \cdots & rs_1 \\ \tau_t \to 1 & \tau_{t-1} \to \tau_t & \cdots & \tau_1 \to \tau_2 \end{bmatrix}$$

Then $\bar{\varphi}'(r) = \bar{\varphi}'(s_{h_1}) = \tau_1 \in \bar{\varphi}'(x)$. Interchange τ_1 and 1 on every $(\tau_1, 1)$-chain (or say relabel τ_1 as 1) to get φ''. Then $\bar{\varphi}''(r) = \bar{\varphi}''(s_{h_1}) = 1 \in \bar{\varphi}'(x)$. Note that both $P_x(1, \alpha+1, \varphi')$ and $P_{s_{h_1}}(1, \alpha + 1, \varphi')$ do not contain r by Lemma 1.9. Interchange 1 and $\alpha + 1$ on both $P_x(1, \alpha + 1, \varphi')$ and $P_{s_{h_1}}(1, \alpha + 1, \varphi')$ to get φ'''. Now $\bar{\varphi}'''(y) = \tau_t$; y is a type I vertex with the y-sequence $s_{h_{t-1}}, s_{h_{t-2}}, \ldots, s_{h_1}$, as desired.

The only remaining case is $z = y$. In this case $r, s_\alpha, y \notin P_x(\alpha + 1, \tau_1)$. Let $\varphi_1 = \varphi/P_x(\alpha + 1, \tau_1)$, then $\bar{\varphi}_1(x) = \tau_1$. Consider the path $P_{s_{i-1}}(i, \tau_1, \varphi_1)$. Firstly, $P_{s_{i-1}}(i, \tau_1, \varphi_1)$ contains r. Since otherwise, by interchanging i and τ_1 on this path, we either get a larger multifan $(r, rs_1, s_1, \ldots, s_{i-1}, rs_{h_1}, s_{h_1}, \ldots, s_{h_t}, rs_i, s_i, rs_{i+1}, s_{i+1}, \ldots, s_\alpha)$ (when $P_{s_{i-1}}(i, \tau_1, \varphi_1)$ ends at s_{h_t}), contradicting the maximality of F; or we get a multifan $(r, rs_1, s_1, \ldots, s_{i-1}, rs_{h_1}, s_{h_1}, \ldots, s_{h_t})$ such that τ_1 is missing at both s_{i-1} and s_{h_t} (when $s_{h_t} \notin P_{s_{i-1}}(i, \tau_1, \varphi_1)$), contradicting Lemma 1.9. Thus we have $r \in P_{s_{i-1}}(i, \tau_1, \varphi_1)$. Moreover, the other end of $P_{s_{i-1}}(i, \tau_1, \varphi_1)$ is x. Since otherwise, we may obtain a coloring φ^* by interchanging τ_1 and i on $P_x(i, \tau_1, \varphi_1)$ and then an $(i, 1) - (1, \alpha + 1)$ swap at x. Then $\bar{\varphi}^*(x) = \alpha+1$ and $\varphi^*(ux) = \tau_1 \notin \bar{\varphi}^*(V(F))$, giving a contradiction to Lemma 3.12. Therefore we have $r \in P_{s_{i-1}}(i, \tau_1, \varphi_1) = P_x(i, \tau_1, \varphi_1)$, and so $r \notin P_y(i, \tau_1, \varphi_1)$. Let $\varphi_2 = \varphi_1/P_y(i, \tau_1, \varphi_1)$. Then $\bar{\varphi}_2(y) = \varphi_2(ux) = i$. Let $\varphi_3 = \varphi_2/P_y(i, 1, \varphi_2)$. Note that if $ux \notin P_y(i, 1, \varphi_2)$, then we may continue with the proof in the case $z \notin \{x, y\}$ with φ_3 in place of the coloring φ_2 in that case. Thus we assume that $ux \in P_y(i, 1, \varphi_2)$, and so $\bar{\varphi}_3(y) = \varphi_3(ux) = 1$. Now we do the following to obtain φ'.

$$\begin{bmatrix} rs_{h_t} & rs_{h_{t-1}} & \cdots & rs_1 \\ \tau_t \to 1 & \tau_{t-1} \to \tau_t & \cdots & \tau_1 \to \tau_2 \end{bmatrix}$$

Then $\bar{\varphi}'(r) = \tau_1 \in \bar{\varphi}'(x)$. Note that $r \notin P_x(\tau_1, \alpha+1, \varphi')$ by Lemma 1.9. Let $\varphi'' = \varphi'/P_x(\tau_1, \alpha+1, \varphi')$. Then $\alpha + 1 \in \bar{\varphi}''(x)$, $\varphi''(ux) = 1 \notin \bar{\varphi}''(V(F))$, giving a contradiction to Lemma 3.12. This completes the proof of $(*)$.

Now we ready to show Claim 3.6, that is, $B = \emptyset$. Suppose on the contrary that there exists a vertex $y \in B$. By $(*)$, we assume that y is a type I vertex. Let the y sequence be $s_{h_1}, s_{h_2}, \ldots, s_{h_t}$ with $\bar{\varphi}(y) = \tau_1$, $\varphi(rs_{h_i}) = \tau_i$ for $i \in [1, t]$ and $\bar{\varphi}(s_{h_i}) = \tau_{i+1}$ for $i \in [1, t-1]$. Recall that $\bar{\varphi}(s_{h_t}) = \alpha + 1$. Denote $\varphi(uy)$ by δ. We will prove the following stronger result: $(**)$ there is a contradiction even when the assumption $x \in V(G) \setminus N[r]$ is replaced by $x \in V(G) \setminus (V(F) \cup \{s_{h_1}, s_{h_2}, \ldots, s_{h_t}\})$.

Subclaim 4.4.1. $\varphi(uy) = \delta \in \bar{\varphi}(V(F)) \setminus \{1, \Delta\}$. Moreover, we may assume that $i \prec \delta$.

Proof. Consider $P_{s_\alpha}(\alpha+1, \tau_1)$. Firstly, $P_{s_\alpha}(\alpha+1, \tau_1)$ contains r. Since otherwise, after interchanging $\alpha + 1$ and τ_1 on this path, $(r, rs_1, s_1, \ldots, s_\alpha, rs_{h_1}, s_{h_1}, \ldots, s_{h_t})$ is a larger multifan, contradicting the maximality of F. Suppose y is not the other end of $P_{s_\alpha}(\alpha+1, \tau_1)$, then we may let $\varphi' = \varphi/P_y(\alpha+1, \tau_1)$. Clearly φ' is F-stable. If the other end of $P_y(\alpha+1, \tau_1)$ is not x, then we get a contradiction by Lemma 3.13. Thus $\bar{\varphi}'(x) = \tau_1$. Hence $\delta \in \bar{\varphi}'(V(F)) \setminus \{1\} = \bar{\varphi}(V(F)) \setminus \{1\}$ by Lemma 3.12 with y in place of x; and $\delta \neq \Delta$ by $(**)$ in the proof of Claim 3.4 with x and y switched. Note that if $\delta \prec i$, then we may use φ' instead of φ. Thus the "moreover" part also holds.

Thus we assume that $y \in P_{s_\alpha}(\alpha+1, \tau_1)$. Then $x, s_{h_t} \notin P_{s_\alpha}(\alpha+1, \tau_1)$. Let φ_1 be obtained from φ by interchanging $\alpha + 1$ and τ_1 on both $P_x(\alpha+1, \tau_1)$ and $P_{s_{h_t}}(\alpha+1, \tau_1)$. Then $\bar{\varphi}_1(s_{h_t}) = \tau_1 \in \bar{\varphi}_1(x)$. Consider $P_{s_{i-1}}(i, \tau_1, \varphi_1)$. By the same proof three paragraphs above, we have $r \in P_{s_{i-1}}(i, \tau_1, \varphi_1) = P_x(i, \tau_1, \varphi_1)$, and so r is not contained by $P_y(i, \tau_1, \varphi_1)$ or $P_{s_{h_t}}(i, \tau_1, \varphi_1)$. We do $(\tau_1, i) - (i, 1)$ swap at both y and s_{h_t} to obtain φ_2. Note that r and

s_{i-1} are $(i,1)$-linked by Lemma 1.9, and so φ_2 is F-stable. Now we consider the following two cases.

Suppose $\varphi_2(ux) = 1$. We do the following to obtain φ'.

$$\begin{bmatrix} rs_{h_t} & rs_{h_{t-1}} & \cdots & rs_1 \\ \tau_t \to 1 & \tau_{t-1} \to \tau_t & \cdots & \tau_1 \to \tau_2 \end{bmatrix}$$

Then $\bar{\varphi}'(r) = \tau_1 \in \bar{\varphi}'(x)$. Note that $r \notin P_x(\tau_1, \alpha+1, \varphi')$ by Lemma 1.9. Let $\varphi'' = \varphi'/P_x(\tau_1, \alpha+1, \varphi')$. Then $\alpha+1 \in \bar{\varphi}''(x)$, $\varphi''(ux) = 1 \notin \bar{\varphi}''(V(F))$, giving a contradiction to Lemma 3.12.

Suppose $\varphi_2(ux) = \varphi(ux) = i$. Then $\delta \neq 1$ and $\varphi_2(uy) = \delta = \varphi(uy)$. Let $\varphi_3 = \varphi_2/P_y(1, \alpha+1, \varphi_2)$. Then φ_3 is F-stable by Lemma 1.9. Thus by Lemma 3.12 with y in place of x, we have $\delta \in \bar{\varphi}_3(V(F)) \setminus \{1\} = \bar{\varphi}(V(F)) \setminus \{1\}$; and $\delta \neq \Delta$ by $(**)$ in the proof of Claim 3.4 with x and y switched. if $\delta \prec i$, then we may use φ' instead of φ. Thus the "moreover" part also holds. This completes the proof of subclaim 4.4.1. □

Let $P_u^+(1,\delta)$ be the $(1,\delta)$-path starting at u not containing uy. Note that if $u \notin P_r(1,\delta) = P_{s_{\delta-1}}(1,\delta)$, then we may interchange δ and 1 on $P_u(1,\delta)$ to obtain φ^*. Since φ^* is L-stable and $\varphi^*(uy) = 1$, we get a contradiction by subclaim 4.4.1. Thus we have $u \in P_r(1,\delta) = P_{s_{\delta-1}}(1,\delta)$, and therefore $P_u^+(1,\delta)$ either ends at $s_{\delta-1}$ or r. By considering the i^+-symmetric coloring of φ (introduced in the proof of Claim 3.5* Case 2.4), we may assume without loss of generality that $P_u^+(1,\delta)$ ends at $s_{\delta-1}$.

Let $P_u^+(\tau_1, i)$ be the (τ_1, i)-path starting at u not containing ux. We let $v \in X = \{s_{i-1}, s_i, r, s_{h_1}, y\}$ be the vertex such that $P_u^+(\tau_1, i)$ meets v before all other vertices in $X \cap V(P_u^+(\tau_1, i))$ if such a vertex v exists. Note that $v \neq r$. We consider the following three cases.

Suppose $v = s_{i-1}$. Then $P_u^+(\tau_1, i) = P_{[u, s_{i-1}]}(\tau_1, i)$. We do the following to obtain φ'.

$$\begin{bmatrix} P_u^+(\tau_1, i) & ux & ur & P_u^+(1,\delta) & uy \\ \tau_1/i & i/(\alpha+1) & (\alpha+1)/1 & 1/\delta & \delta/\tau_1 \end{bmatrix}$$

Then $(r, rs_1, s_1, \ldots, s_{i-1}, rs_{h_1}, s_{h_1}, rs_{h_2}, s_{h_2}, \ldots, s_{h_t})$ is a multifan. But $\bar{\varphi}'(r) = \bar{\varphi}'(s_{h_t}) = \alpha + 1$, giving a contradiction to Lemma 1.9.

Suppose $v = s_i$. Then $P_u^+(\tau_1, i)$ meets s_i before r. We do the following to obtain φ'.

$$\begin{bmatrix} s_{h_1} : s_{h_t} & P_{[u,r]}(\tau_1, i) & ux & ur & P_u^+(1,\delta) & uy \\ \text{shift} & \tau_1/i & i/(\alpha+1) & (\alpha+1)/1 & 1/\delta & \delta/\tau_1 \end{bmatrix}$$

Then $(r, rs_1, s_1, \ldots, s_{i-1})$ is a multifan. But $\bar{\varphi}'(r) = \bar{\varphi}'(s_{i-1}) = i$, giving a contradiction to Lemma 1.9.

Suppose $v \notin \{s_{i-1}, s_i\}$ or v does not exist. Let $z = v$ if v exists, otherwise let z be the other end of $P_u^+(\tau_1, i)$. Then in this case, $P_{[u,z]}(\tau_1, i)$ is F-avoiding. We do the following to obtain φ'.

$$\begin{bmatrix} s_{h_1} : s_{h_t} & P_{[u,z]}(\tau_1, i) & ux & ur \\ \text{shift} & \tau_1/i & i/(\alpha+1) & (\alpha+1)/\tau_1 \end{bmatrix}$$

Then $(r, rs_1, s_1, \ldots, s_\alpha, rs_{h_t}, s_{h_t}, rs_{h_{t-1}}, s_{h_{t-1}}, \ldots, s_{h_1})$ is a multifan, giving a contradiction to the maximality of F. This completes the proof of Claim 3.6 and Claim 3.2. □

REFERENCES

[1] Y. Cao. Kierstead path of 5 vertices. *A manuscript*, 2017.

[2] Y. Cao and G. Chen. On the average degree of edge chromatic critcal graphs. *Manuscript submitted*, 2018.

[3] Y. Cao, G. Chen, G. Jing, and S. Shan. Proof of the Core Conjecture of Hilton and Zhao. *arxiv.2004.00734*.

[4] Y. Cao, G. Chen, G. Jing, M. Stiebitz, and B. Toft. Graph edge coloring: A survey. *Graphs and Combinatorics, accepted*.

[5] David Cariolaro and Gianfranco Cariolaro. Colouring the petals of a graph. *Electron. J. Combin.*, 10:Research Paper 6, 11, 2003.

[6] G. Chen, X. Chen, and Y. Zhao. Hamiltonianicity of edge chromatic critical graph. *Discrete Math., accepted*.

[7] A. G. Chetwynd and A. J. W. Hilton. Star multigraphs with three vertices of maximum degree. *Math. Proc. Cambridge Philos. Soc.*, 100(2):303–317, 1986.

[8] A. G. Chetwynd and A. J. W. Hilton. The edge-chromatic class of graphs with maximum degree at least $|V| - 3$. In *Graph theory in memory of G. A. Dirac (Sandbjerg, 1985)*, volume 41 of *Ann. Discrete Math.*, pages 91–110. North-Holland, Amsterdam, 1989.

[9] Béla Csaba, Daniela Kühn, Allan Lo, Deryk Osthus, and Andrew Treglown. Proof of the 1-factorization and Hamilton decomposition conjectures. In *The Seventh European Conference on Combinatorics, Graph Theory and Applications*, volume 16 of *CRM Series*, pages 477–482. Ed. Norm., Pisa, 2013.

[10] Béla Csaba, Daniela Kühn, Allan Lo, Deryk Osthus, and Andrew Treglown. Proof of

the 1-factorization and Hamilton decomposition conjectures. *Mem. Amer. Math. Soc.*, 244(1154):v+164, 2016.

[11] Jean-Claude Fournier. Colorations des arêtes d'un graphe. *Cahiers Centre Études Recherche Opér.*, 15:311–314, 1973. Colloque sur la Théorie des Graphes (Brussels, 1973).

[12] A. J. W. Hilton and C. Zhao. Vertex-splitting and chromatic index critical graphs. volume 76, pages 205–211. 1997. Second International Colloquium on Graphs and Optimization (Leukerbad, 1994).

[13] A. J. W. Hilton and Cheng Zhao. On the edge-colouring of graphs whose core has maximum degree two. *J. Combin. Math. Combin. Comput.*, 21:97–108, 1996.

[14] D. G. Hoffman and C. A. Rodger. The chromatic index of complete multipartite graphs. *J. Graph Theory*, 16(2):159–163, 1992.

[15] I. Holyer. The NP-completeness of edge-coloring. *SIAM J. Comput.*, 10(4):718–720, 1981.

[16] I. Holyer. The NP-completeness of edge-coloring. *SIAM J. Comput.*, 10(4):718–720, 1981.

[17] R. Luo and Y. Zhao. Finding $\Delta(\Sigma)$ for a surface Σ of characteristic $\chi(\Sigma) = -5$. *J. Graph Theory*, 68(2):148–168, 2011.

[18] Michael Plantholt. Overfull conjecture for graphs with high minimum degree. *J. Graph Theory*, 47(2):73–80, 2004.

[19] P. D. Seymour. On multicolourings of cubic graphs, and conjectures of Fulkerson and Tutte. *Proc. London Math. Soc. (3)*, 38(3):423–460, 1979.

[20] Zi-Xia Song. A further extension of Yap's construction for Δ-critical graphs. *Discrete Math.*, 243(1-3):283–290, 2002.

[21] M. Stiebitz, D. Scheide, B. Toft, and L. M. Favrholdt. *Graph edge coloring.* Wiley Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., Hoboken, NJ, 2012. Vizing's theorem and Goldberg's conjecture, With a preface by Stiebitz and Toft.

[22] V. G. Vizing. Critical graphs with a given chromatic class (in Russian). *Diskret. Analiz No.*, 5:9–17, 1965.

[23] V. G. Vizing. Critical graphs with given chromatic class. *Diskret. Analiz No.*, 5:9–17, 1965.

[24] V. G. Vizing. Some unsolved problems in graph theory (in Russian). *Uspekhi Mat. Nauk*, 23:117–134, 1968.