APPLICATIONS OF SYMMETRY ANALYSIS TO PHYSICALLY RELEVANT DIFFERENTIAL EQUATIONS

by

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Abstract

We investigate the role of Lie symmetries in generating solutions to differential equations that arise in particular physical systems. We first provide an overview of the Lie analysis and review the relevant symmetry analysis of differential equations in general. The Lie symmetries of some simple ordinary differential equations are found to illustrate the general method. Then we study the properties of particular ordinary differential equations that arise in astrophysics and cosmology using the Lie analysis of differential equations. Firstly, a system of differential equations arising in the model of a relativistic star is generated and a governing nonlinear equation is obtained for a linear equation of state. A comprehensive symmetry analysis is performed on this equation. Secondly, a second order nonlinear ordinary differential equation arising in the model of the early universe is described and a detailed symmetry analysis of this equation is undertaken. Our objective in each case is to find explicit Lie symmetry generators that may help in analysing the model.

Dedication

To my wife Nelly,

my daughter Zama

and my son Lungelo.

Declaration

This study was carried out from February 2003 to February 2005 under the supervision of Dr K.S. Govinder and Prof S.D. Maharaj in the School of Mathematical Sciences, University of KwaZulu-Natal, Howard College Campus, Durban.

I declare that this dissertation represents the original work of the author. Where the work of others has been used references have been made. It has not been submitted in any form for any degree to any other institution nor has it been published before.

M. C. Kweyama

February 2005

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Dr Govinder introduced me to the exciting field of symmetries and the Lie method of solving differential equations, and also clarified many details intrinsic for this approach. To do some tedious calculations he introduced me to the symbolic manipulation packages Program LIE and Mathematica which have been used extensively in this dissertation.

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Contents

Chapter 1

Introduction

Most physical situations involving the rates of change can be modelled with differential equations. It is therefore crucial that we analyse differential equations in a systematic manner. The solutions to these equations may be found using either numerical or analytical methods. While great advances have been made in numerical methods, analytic solutions tend to be more desirable than the numeric ones.

One of the most important methods of finding analytical solutions of nonlinear problems is through symmetry analysis. In this dissertation we undertake a Lie symmetry analysis of some important differential equations arising in Mathematical Physics.

1.1 Outline

Lie was a Norwegian mathematician who, through his dedication, commitment and contributions to the development of mathematics, is considered to be one of the greatest mathematicians of the nineteenth century. As a result we begin our study by reviewing his inspirational and motivating life story.

In Chapter 2 we outline the basic ideas behind the Lie analysis. We illustrate, through examples, the method of finding Lie point symmetries. As some equations do not have the required number of point symmetries various extensions of the classical Lie approach have been considered. One such extension involves finding so-called hidden symmetries (Abraham-Shrauner 1993, Abraham-Shrauner and Guo 1992 and Abraham-Shrauner and Guo 1994) which have been shown to lead to solutions of a number of equations that do not possess Lie point symmetries with suitable Lie algebras. This is done through the reduction or the increase of the order of a differential equation. Again, using examples, we illustrate how the reduction and the increase of order are undertaken. In this dissertation we apply the reduction and the increase of order of ordinary differential equations extensively in an attempt to find solutions to some important differential equations.

In Chapter 3 we consider the following system of field equations

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} \tag{1.1}$$

$$4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} = \frac{p}{C} \tag{1.2}$$

$$4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + \left(\dot{Z}x - Z + 1\right)y = 0 \tag{1.3}$$

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arising in astrophysics and we briefly explain how it is generated. We generate a new master equation by eliminating Z in (1.1) − (1.3) to obtain a third order nonlinear ordinary differential equation in y only. We then perform a comprehensive symmetry analysis of this resulting equation.

In Chapter 4 we consider a second order nonlinear ordinary differential equation, viz.

$$2H\ddot{H} + 6H^2\dot{H} - \dot{H}^2 + aH^2 = b \tag{1.4}$$

which arises in cosmology. Again we perform a symmetry analysis of this equation. In addition we subject the equation to the Painlevé integrability test. Ultimately we apply nonlocal symmetries to reduce the equation to quadratures.

We conclude with a brief discussion of our results in Chapter 5.

1.2 Sophus Marius Lie

Sophus Marius Lie was born in 1842 in Norway. He mastered all his school subjects equally well and consequently did not find it easy to choose his career when he finished school. Ultimately, he pursued the study of mathematics and natural sciences. When he read the works of the geometers Poncelet and Plucker, he was so inspired that he began to publish an uninterrupted stream of research papers for many years (Yaglom 1988).

In 1870 Lie visited Paris with the aim of meeting Jordan and Darboux. He was highly influenced by Darboux's profound works on differential geometry, particularly the combination of differential geometry and the theory of differential equations. When the

Franco-Prussian war broke out in 1871 Lie, while wandering the French countryside, was suspected to be a German spy because of his poor French and strange appearance and was therefore arrested and imprisoned. While in prison he spent considerable time making various mathematical jottings on some aspects of Plucker's line geometry. He worked on his doctoral thesis which, after leaving prison, he submitted to the University of Kristiana. Darboux used his influence to have Lie released from prison (Cantwell 2002 and Yaglom 1988).

While in Paris, Lie met Jordan whose research was centred around the theory of groups. He was greatly impressed with Jordan's conviction on the importance of this theory in the future development of mathematics. In fact it led to Lie being one of those who introduced group theoretic concepts into all branches of mathematics. Lie devoted his entire life to the study of the theory of continuous groups and the notion of symmetry. (Continuous groups are now known as Lie groups). It was during his time that the concept of symmetry, which had long been the concern of artists, evolved into a fundamental idea in mathematics and science. Lie was one of the scientists that contributed most notably to this evolution. He also used the concept of invariance, which is integrally intertwined with the concept of symmetry, to make a significant impact to the field of differential equations in his theory of transformation groups (Yaglom 1988). Today symmetry analysis constitutes the most important widely applicable method for finding analytical solutions of non-linear problems (Cantwell 2002). Lie's theory rested on his discovery of the intimate connection between continuous groups and specific algebraic systems now known as Lie algebras.

Lie never ran out of ideas and his whole life was filled with intense creative work. When writing, Lie carefully set down details and provided many examples. Hence most of his papers and books were very long. He believed quite sensibly that any natural mathematical theory should be transparent, and that difficulties in mathematics usually arise not from the essence of the problem but from badly conceived definitions. Lie valued his students and any up-and-coming mathematicians. He gave ideas generously to the potential mathematicians he met on the way (Yaglom 1988).

Lie was one of the last great mathematicians of the nineteenth century. He died in Oslo on February 18, 1899 (Yaglom 1988).

1.3 Definitions

Here we provide various definitions of the concepts pertinent to our approach so that the subsequent analysis can easily be followed. For further information, we refer the reader to Bluman and Anco (2002), Bluman and Kumei (1989), Cantwell (2002), Dresner (1999), Hydon (2000) and Olver (1993).

1.3.1 Lie Groups

Group

A group G is a set of elements with a law of composition ϕ between elements satisfying the following conditions (Bluman and Anco 2002, Bluman and Kumei 1989 and Olver 1993):

CLOSURE: If a and b are elements of G, so is $\phi(a, b)$.

ASSOCIATIVITY: For any elements a, b and c of G

$$\phi(a, \phi(b, c)) = \phi(\phi(a, b), c). \tag{1.5}$$

IDENTITY ELEMENT: G contains a unique element I, called an identity element, such that for any element a of G

$$\phi(a, I) = a = \phi(I, a). \tag{1.6}$$

INVERSE ELEMENT: For any element a of G there exists a unique element in G denoted by a^{-1} such that

$$\phi(a, a^{-1}) = I = \phi(a^{-1}, a). \tag{1.7}$$

The element a^{-1} is called the inverse of a.

If two elements a and b of a group satisfy the condition

$$\phi(a, b) = \phi(b, a) \tag{1.8}$$

they are said to commute. If all the elements of a group commute with each other, the group is called an **Abelian** group.

Group of transformations

A set of transformations

$$\bar{x} = X(x, \varepsilon)$$

defined for each x in $D \subset R$, depending on the parameter ε lying in the set $S \subset R$, with $\phi(\varepsilon, \delta)$ defining a composition of parameters ε and δ in S, forms a **group of transformations** on D if (Bluman and Anco 2002):

(i) for each parameter ε in S the transformations are one-to-one onto D.

(ii) S, with the law of composition ϕ, forms a group.

(iii) $\bar{x} = x$ when $\varepsilon = I$, i.e.

$$X(x, I) = x.$$

(iv) If $\bar{x} = X(x, \varepsilon)$, $\bar{\bar{x}} = X(x, \phi(\varepsilon, \delta))$.

Lie group of transformations

A one-parameter **Lie group of transformations** is a group of transformations which satisfies the following conditions in addition to the above:

(i) ϵ is a continuous parameter, i.e. S is an interval in R.

(ii) X is infinitely differentiable with respect to x in D and an analytic function of ϵ in S.

(iii) $\phi(\epsilon, \delta)$ is an analytic function of ϵ and δ and $\epsilon \in S$, $\delta \in S$.

1.3.2 Lie Algebras

Vector field

A vector field V on a set M assigns a tangent vector $V|_x$ to each point $x \in M$, with $V|_x$ varying smoothly from point to point. In local coordinates $(x^1, \cdots, x^m)$ a vector field has the form

$$V|_x = \xi^1(x)\frac{\partial}{\partial x^1} + \xi^2(x)\frac{\partial}{\partial x^2} + \cdots + \xi^m(x)\frac{\partial}{\partial x^m}, \tag{1.9}$$

where each $\xi^i(x)$ is a smooth function of x (Olver 1993).

Commutator

If G_1 and G_2 are vector fields then their commutator (also known as a **Lie bracket**) is defined as follows (Cantwell 2002, Hydon 2000 and Olver 1993):

$$[G_1, G_2] = G_1 G_2 - G_2 G_1. \tag{1.10}$$

As an example we consider the following two vector fields (Hydon 2000):

$$G_1 = \frac{\partial}{\partial x} \tag{1.11}$$

$$G_2 = x\frac{\partial}{\partial x} + \frac{3}{4}y\frac{\partial}{\partial y}. \tag{1.12}$$

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The commutator for the two vector fields is given by

$$
\begin{aligned}
[G_1, G_2] &= \left(\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x} + \frac{3}{4}y\frac{\partial}{\partial y}\right) - \left(x\frac{\partial}{\partial x} + \frac{3}{4}y\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}\right) \\
&= \frac{\partial}{\partial x} \\
&= G_1.
\end{aligned}
$$

Lie algebra

A **Lie algebra** is a vector space, L, on which commutation is defined and satisfies the following conditions (Hydon 2000 and Olver 1993):

CLOSURE: $G_1, G_2 \in L \Rightarrow [G_1, G_2] \in L$.

ANTISYMMETRY: $[G_1, G_2] = -[G_2, G_1]$

BILINEARITY:

$$
\begin{aligned}
{[k_1 G_1 + k_2 G_2, G_3]} &= k_1[G_1, G_2] + k_2[G_2, G_3] \\
{[G_1, k_1 G_2 + k_2 G_3]} &= k_1[G_1, G_2] + k_2[G_1, G_3]
\end{aligned}
$$

where k_1 and k_2 are constants.

JACOBI IDENTITY: $[G_1, [G_2, G_3]] + [G_2, [G_3, G_1]] + [G_3, [G_1, G_2]] = 0$

for all G_1, G_2 and G_3 in L.

If $[G_1, G_2] = 0$ then we say G_1 and G_2 commute. If all the elements of L commute L is called an **Abelian Lie algebra**.

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Solvable Lie algebra

A **solvable Lie algebra** L is a Lie algebra with the derived series

$$
\begin{aligned}
L \quad \supseteq \quad & L' = [L, L] \\
\supseteq \quad & L'' = [L', L'] \\
\supseteq \quad & \cdots \\
\supseteq \quad & L^{(k)} = \left[L^{(k-1)}, L^{(k-1)} \right]
\end{aligned}
$$

such that $L^{(k)} = 0$ for some $k > 0$ (Jacobson 1979).

1.3.3 Infinitesimal transformation

We consider a one-parameter transformation

$$
\begin{aligned}
\bar{x} &= X(x, y, \lambda) \\
\bar{y} &= Y(x, y, \lambda)
\end{aligned}
\tag{1.13}
$$

where λ is a continuous parameter. By taking the Taylor series expansion of this transformation about the point $\lambda = \lambda_0$, we generate

$$
\bar{x} = x + \left(\frac{\partial X}{\partial \lambda} \right)_{\lambda = \lambda_0} (\lambda - \lambda_0) + \cdots
\tag{1.14}
$$

$$
\bar{y} = y + \left(\frac{\partial Y}{\partial \lambda} \right)_{\lambda = \lambda_0} (\lambda - \lambda_0 + \cdots
\tag{1.15}
$$

The partial derivatives with respect to the group parameter λ evaluated at $\lambda = \lambda_0$ are referred to as the **infinitesimals** (Cantwell 2002) and are functions of x and y. We

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denote them by

$$\left(\frac{\partial X}{\partial \lambda}\right)_{\lambda=\lambda_0} = \xi(x, y) \tag{1.16}$$

$$\left(\frac{\partial Y}{\partial \lambda}\right)_{\lambda=\lambda_0} = \eta(x, y). \tag{1.17}$$

Considering the values of λ sufficiently close to λ_0, the coordinates of the transformation can be expressed as follows

$$\bar{x} = x + \xi(x, y)(\lambda - \lambda_0) \tag{1.18}$$

$$\bar{y} = y + \eta(x, y)(\lambda - \lambda_0) \tag{1.19}$$

where terms of second and higher degree in $(\lambda - \lambda_0)$ have been neglected. This transformation is known as an **infinitesimal transformation** (Dresner 1999).

1.3.4   Invariance under transformation

An **invariant** is that which remains unchanged when its constituents change. The concept of **invariance** has a physical basis in the conservation laws of mechanics.

A function f is said to be invariant under a Lie group if and only if

$$f(\bar{x}, \bar{y}) = f\left(X(x, y, \lambda), Y(x, y, \lambda)\right) = f(x, y) \tag{1.20}$$

i.e. the function must read the same when expressed in the new variables (Cantwell 2002).

A simple example of invariance under a continuous transformation is the rotation of a circle about an axis that is normal to its centre.

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1.3.5 Symmetry

A **symmetry** is an operation which leaves invariant that upon which it operates. A symmetry of a geometrical object is a transformation which leaves the object apparently unchanged.

Consider the transformation of infinitesimal form

$$\bar{x}_i = x_i + \varepsilon \xi_i \qquad i = 1, ..., n \tag{1.21}$$

where ε is a parameter of smallness. The equation (1.21) can be written as

$$\bar{x}_i = (1 + \varepsilon G)x_i \tag{1.22}$$

where

$$G = \xi_i \frac{\partial}{\partial x_i} \tag{1.23}$$

is a differential operator called the generator of the transformation (1.21).

We consider a particular case where

$$G = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}. \tag{1.24}$$

Under the action of the infinitesimal transformation generated by G, a function $f(x, y)$ becomes

$$\begin{aligned}
\bar{f}(\bar{x}, \bar{y}) &= (1 + \varepsilon G)f(x, y) \\
&= f + \varepsilon \left(\xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} \right).
\end{aligned} \tag{1.25}$$

If the form of f is unchanged, i.e.

$$\bar{f}(\bar{x}, \bar{y}) = f(x, y), \tag{1.26}$$

or

$$\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} = 0, \tag{1.27}$$

then G is called a **symmetry** of f. Mathematically all symmetries represent invariance under transformations. Some of these symmetries may be translations, reflections and rotations. Such symmetries are generally referred to as geometric. However, there are symmetries that may not have such a simple geometrical interpretation.

Chapter 2

Lie Theory of Differential Equations

2.1 Lie point symmetries of ordinary differential equations

A point symmetry is a symmetry in which the infinitesimals depend only on coordinates (Cantwell 2002). We describe a Lie point symmetry as a point symmetry that depends continuously on at least one parameter, i.e. the parameter(s) can vary continuously over a set of scalar nonzero measure.

Lie point symmetries of ordinary differential equations are of the form

$$G = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} \tag{2.1}$$

where the coefficients ξ and η are functions of x and y only.

To be able to apply a point transformation to an nth order ordinary differential equation

$$E\left(x, y, y', y'', ..., y^{(n)}\right) = 0 \tag{2.2}$$

where

$$y' \equiv \frac{dy}{dx}, \quad y'' \equiv \frac{d^2y}{dx^2}, \cdots$$

etc, we need to know how derivatives transform under the infinitesimal transformation

$$\bar{x} = x + \varepsilon\xi(x, y)$$

$$\bar{y} = y + \varepsilon\eta(x, y) \tag{2.3}$$

which has a generator given by

$$G = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \tag{2.4}$$

In terms of the quantities $\bar{x}$ and $\bar{y}$ we have, for the first derivative,

$$\begin{aligned}
\frac{d\bar{y}}{d\bar{x}} &= \frac{d(y + \varepsilon\eta)}{d(x + \varepsilon\xi)} \\
&= \frac{\dfrac{dy}{dx} + \varepsilon\dfrac{d\eta}{dx}}{1 + \varepsilon\dfrac{d\xi}{dx}} \\
&= \left(y' + \varepsilon\eta'\right)\left(1 - \varepsilon\xi' + \varepsilon^2\xi'^2 - \cdots\right) \\
&= y' + \varepsilon\left(\eta' - y'\xi'\right)
\end{aligned}$$

which we have terminated at $O(\varepsilon^2)$. Note that primes refer to total differentiation with respect to x. For the second derivative we have

$$\begin{aligned}
\frac{d^2\bar{y}}{d\bar{x}^2} &= \frac{d}{d\bar{x}}\left(\frac{d\bar{y}}{d\bar{x}}\right) \\
&= \frac{d[y' + \varepsilon\left(\eta' - y'\xi'\right)]}{d\left(x + \varepsilon\xi\right)}
\end{aligned}$$

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$$= \frac{\frac{dy'}{dx} + \varepsilon \frac{d}{dx} \left(\eta' - y'\xi' \right)}{1 + \varepsilon \xi'}$$

$$= y'' + \varepsilon \left(\eta'' - 2y''\xi' - y'\xi'' \right).$$

Similarly, we obtain

$$\frac{d^3 \bar{y}}{d\bar{x}^3} = y''' + \varepsilon \left(\eta''' - 3y'''\xi' - 3y''\xi'' - y'\xi''' \right)$$

$$\frac{d^4 \bar{y}}{d\bar{x}^4} = y^{iv} + \varepsilon \left(\eta^{iv} - 4y^{iv}\xi' - 6y'''\xi'' - 4y''\xi''' - y'\xi^{iv} \right)$$

and so on. In general we generate the formula (Leach 1995)

$$\frac{d^n \bar{y}}{d\bar{x}^n} = y^{(n)} + \varepsilon \left(\eta^{(n)} - \sum_{i=1}^{n} C_i^n y^{(i+1)} \xi^{(n-i)} \right) \tag{2.5}$$

where the superscript $^{(i)}$ denotes $\dfrac{d^i}{dx^i}$ and C_i^n is the number of combinations of n objects taken i at a time.

To deal with the infinitesimal transformations of equations and functions involving derivatives, we need the extensions of the generator G. We indicate that a generator G has been extended by writing

$$G^{[1]} = G + \left(\eta' - y'\xi' \right) \frac{\partial}{\partial y'} \tag{2.6}$$

$$G^{[2]} = G^{[1]} + \left(\eta'' - 2y''\xi' - y'\xi'' \right) \frac{\partial}{\partial y''} \tag{2.7}$$

for the first and the second extensions respectively. When generating an extension of G we have to extend G such that all the derivatives appearing in the equation or function are included in the extension. For an nth order differential equation, the nth extension is of the form (Mahomed and Leach 1990)

$$G^{[n]} = G + \sum_{i=1}^{n} \left\{ \eta^{(i)} - \sum_{j=1}^{i} \binom{i}{j} y^{(i+1-j)} \xi^{(j)} \right\} \frac{\partial}{\partial y^{(i)}}.$$

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The generator

$$G = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \qquad (2.8)$$

is a symmetry of the differential equation

$$E\left(x, y, y', y'', ..., y^{(n)}\right) = 0 \qquad (2.9)$$

if and only if

$$G^{[n]} E|_{E=0} = 0 \qquad (2.10)$$

which means that the action of the nth extension of G on E is zero when the original equation is satisfied.

We next illustrate the method of finding the symmetries of some ordinary differential equations.

2.2   Examples

As a first example, we determine the symmetries of the following second order ordinary differential equation

$$y'' + 3yy' + y^3 = 0. \qquad (2.11)$$

which is a special case of the modified Painlevé-Ince equation given by (Abraham-Shrauner 1993)

$$y'' + \sigma yy' + \beta y^3 = 0.$$

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Equation (2.11) also arises in the study of the modified Emden equation (Abraham-Shrauner 1993 and Leach et al. 1988). We use the second extension

$$G^{[2]} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + (\eta' - y'\xi')\frac{\partial}{\partial y'} + (\eta'' - 2y''\xi' - y'\xi'')\frac{\partial}{\partial y''} \qquad (2.12)$$

of G since the equation (2.11) is a second order differential equation. When $G^{[2]}$ acts on the differential equation (2.11) we obtain

$$3y'\eta + 3y^2\eta + 3y\eta' - 3yy'\xi' + \eta'' - 2y''\xi' - y'\xi'' = 0 \qquad (2.13)$$

which is subject to

$$y'' = -3yy' - y^3. \qquad (2.14)$$

We recall that primes in (2.13) refer to total derivatives and so the first and the second total derivatives of ξ and η can be expressed in terms of partial derivatives as follows

$$\xi' = \frac{\partial\xi}{\partial x} + y'\frac{\partial\xi}{\partial y} \qquad (2.15)$$

$$\xi'' = \frac{\partial^2\xi}{\partial x^2} + 2y'\frac{\partial^2\xi}{\partial x\partial y} + y'^2\frac{\partial^2\xi}{\partial y^2} + y''\frac{\partial\xi}{\partial y} \qquad (2.16)$$

$$\eta' = \frac{\partial\eta}{\partial x} + y'\frac{\partial\eta}{\partial y} \qquad (2.17)$$

$$\eta'' = \frac{\partial^2\eta}{\partial x^2} + 2y'\frac{\partial^2\eta}{\partial x\partial y} + y'^2\frac{\partial^2\eta}{\partial y^2} + y''\frac{\partial\eta}{\partial y}. \qquad (2.18)$$

Substituting (2.14) − (2.18) in (2.13) and simplifying gives

$$3y'\eta + 3y^2\eta + 3y\frac{\partial\eta}{\partial x} - 3yy'^2\frac{\partial\xi}{\partial y} + \frac{\partial^2\eta}{\partial x^2} + 2y'\frac{\partial^2\eta}{\partial x\partial y} + y'^2\frac{\partial^2\eta}{\partial y^2}$$
$$-y^3\frac{\partial\eta}{\partial y} + 6yy'\frac{\partial\xi}{\partial x} + 2y^3\frac{\partial\xi}{\partial x} + 6yy'^2\frac{\partial\xi}{\partial y} + 2y^3y'\frac{\partial\xi}{\partial y} - y'\frac{\partial^2\xi}{\partial x^2}$$
$$-2y'^2\frac{\partial^2\xi}{\partial x\partial y} - y'^3\frac{\partial^2\xi}{\partial y^2} + y^3y'\frac{\partial\xi}{\partial y} = 0. \qquad (2.19)$$

Equation (2.19) is an identity in x, y and y', i.e. it holds for any arbitrary choice of x, y and y' (Dresner 1999). Since ξ and η are functions of x and y only, we must equate

the coefficients of the powers of y' to zero. We obtain the following system of partial differential equations known as the determining equations (Dresner 1999, Hydon 2000):

$$y'^3 \quad : \quad \frac{\partial^2 \xi}{\partial y^2} = 0 \tag{2.20}$$

$$y'^2 \quad : \quad \frac{\partial^2 \eta}{\partial y^2} + 6y\frac{\partial \xi}{\partial y} - 2\frac{\partial^2 \xi}{\partial x \partial y} = 0 \tag{2.21}$$

$$y'^1 \quad : \quad 3\eta + 3y\frac{\partial \xi}{\partial x} + 2\frac{\partial^2 \eta}{\partial x \partial y} + 3y^3\frac{\partial \xi}{\partial y} - \frac{\partial^2 \xi}{\partial x^2} = 0 \tag{2.22}$$

$$y'^0 \quad : \quad 3y^2\eta + 3y\frac{\partial \eta}{\partial x} + \frac{\partial^2 \eta}{\partial x^2} - y^3\frac{\partial \eta}{\partial y} + 2y^3\frac{\partial \xi}{\partial x} = 0. \tag{2.23}$$

Integrating equation (2.20), we find that

$$\xi = ay + b \tag{2.24}$$

where a and b are arbitrary functions of x. We substitute (2.24) in (2.21) and solve to obtain

$$\eta = a'y^2 - ay^3 + cy + d \tag{2.25}$$

where c and d are also arbitrary functions of x.

Substituting (2.24) and (2.25) in (2.22) we have

$$3cy + 3d + 3b'y + 2c' + 3a''y - b'' = 0. \tag{2.26}$$

Since a, b, c and d depend on x only, we can now equate the coefficients of powers of y to zero. This yields

$$y^1 \quad : \quad a'' + b' + c = 0 \tag{2.27}$$

$$y^0 \quad : \quad b'' - 2c' - 3d = 0. \tag{2.28}$$

Now we substitute (2.24) and (2.25) in (2.23) to obtain

$$3cy^3 + 3dy^2 + 3a''y^3 + 3c'y^2 + 3d'y + y^2a''' - a''y^3 + c''y + d'' - cy^3 + 2b'y^3 = 0. \tag{2.29}$$

Again we equate the coefficients of powers of y to zero and obtain

$$y^3 \quad : \quad a'' + b' + c = 0 \tag{2.30}$$

$$y^2 \quad : \quad a''' + 3c' + 3d = 0 \tag{2.31}$$

$$y^1 \quad : \quad c'' + 3d' = 0 \tag{2.32}$$

$$y^0 \quad : \quad d'' = 0. \tag{2.33}$$

It is now possible to solve the differential equations (2.33), (2.32), (2.28) and (2.27), in that order. For simplicity we use (2.27) and (2.28) to find a and b instead of (2.30) and (2.31). From (2.33) we have

$$d = A_1 x + A_2. \tag{2.34}$$

We substitute (2.34) in (2.32) and solve to obtain

$$c = -\frac{3A_1}{2}x^2 + A_3 x + A_4. \tag{2.35}$$

Substituting (2.34) and (2.35) in (2.28) and solving yields

$$b = \frac{A_1}{2}x^3 + \frac{3A_2}{2}x^2 + A_3 x^2 + A_5 x + A_6. \tag{2.36}$$

Finally we substitute (2.35) and (2.36) in (2.27) and solve to obtain

$$a = \frac{A_1}{4}x^4 - \frac{A_2}{2}x^3 - \frac{A_3}{2}x^3 - \frac{A_4}{2}x^2 - \frac{A_5}{2}x^2 + A_7 x + A_8, \tag{2.37}$$

where A_1, A_2, A_3, A_4, A_5, A_6, A_7 and A_8 are arbitrary constants of integration. We then substitute (2.36) and (2.37) in (2.24) to find

$$
\begin{aligned}
\xi(x,y) \quad = \quad & \frac{A_1}{4}x^4 y - \frac{A_2}{2}x^3 y - \frac{A_3}{2}x^3 y - \frac{A_4}{2}x^2 y - \frac{A_5}{2}x^2 y + A_7 xy \\
& + A_8 y - \frac{A_1}{2}x^3 + \frac{3A_2}{2}x^2 + A_3 x^2 + A_5 x + A_6
\end{aligned} \tag{2.38}
$$

and we substitute (2.34), (2.35) and (2.37) in (2.25) to produce

$$
\begin{aligned}
\eta(x,y) \;=\;& A_1 x^3 y^2 - \frac{3A_2}{2}x^2 y^2 - \frac{3A_3}{2}x^2 y^2 - A_4 xy^2 - A_5 xy^2 + A_7 y^2 \\
& - \frac{A_1}{4}x^4 y^3 + \frac{A_2}{2}x^3 y^3 + \frac{A_3}{2}x^3 y^3 + \frac{A_4}{2}x^2 y^3 + \frac{A_5}{2}x^2 y^3 - A_7 xy^3 \\
& - A_8 y^3 - \frac{3A_1}{2}x^2 y + A_3 xy + A_4 y + A_1 x + A_2 .
\end{aligned}
\tag{2.39}
$$

As a result the generator, G, of the infinitesimal transformation is

$$
\begin{aligned}
G \;=\;& A_1 \left(\frac{1}{4}x^4 y \frac{\partial}{\partial x} - \frac{1}{2}x^3 \frac{\partial}{\partial x} + x^3 y^2 \frac{\partial}{\partial y} - \frac{1}{4}x^4 y^3 \frac{\partial}{\partial y} - \frac{3}{2}x^2 y \frac{\partial}{\partial y} + x \frac{\partial}{\partial y} \right) \\
& + A_2 \left(-\frac{1}{2}x^3 y \frac{\partial}{\partial x} + \frac{3}{2}x^2 \frac{\partial}{\partial x} - \frac{3}{2}x^2 y^2 \frac{\partial}{\partial y} + \frac{1}{2}x^3 y^3 \frac{\partial}{\partial y} + \frac{\partial}{\partial y} \right) \\
& + A_3 \left(-\frac{1}{2}x^3 y \frac{\partial}{\partial x} + x^2 \frac{\partial}{\partial x} - \frac{3}{2}x^2 y^2 \frac{\partial}{\partial y} + \frac{1}{2}x^3 y^3 \frac{\partial}{\partial y} + xy \frac{\partial}{\partial y} \right) \\
& + A_4 \left(-\frac{1}{2}x^2 y \frac{\partial}{\partial x} - xy^2 \frac{\partial}{\partial y} + \frac{1}{2}x^2 y^3 \frac{\partial}{\partial y} + y \frac{\partial}{\partial y} \right) \\
& + A_5 \left(-\frac{1}{2}x^2 y \frac{\partial}{\partial x} + x \frac{\partial}{\partial x} - xy^2 \frac{\partial}{\partial y} + \frac{1}{2}x^2 y^3 \frac{\partial}{\partial y} \right) + A_6 \left(\frac{\partial}{\partial x} \right) \\
& + A_7 \left(xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} - xy^3 \frac{\partial}{\partial y} \right) + A_8 \left(y \frac{\partial}{\partial x} - y^3 \frac{\partial}{\partial y} \right)
\end{aligned}
\tag{2.40}
$$

which is an eight-parameter symmetry.

Any n-parameter symmetry may be separated into n one-parameter symmetries by letting particular parameters take on specific values. Usually we set one parameter equal to one and the rest equal to zero in turn. If we do this in (2.40) we generate the following eight one-parameter symmetries (Mahomed and Leach 1985)

$$
\begin{aligned}
G_1 \;=\;& \frac{\partial}{\partial x} \\
G_2 \;=\;& y \frac{\partial}{\partial x} - y^3 \frac{\partial}{\partial y} \\
G_3 \;=\;& xy \frac{\partial}{\partial x} + \left(y^2 - xy^3 \right) \frac{\partial}{\partial y}
\end{aligned}
$$

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$$G_4 = \left(x - \frac{1}{2}x^2y\right)\frac{\partial}{\partial x} + \left(\frac{1}{2}x^2y^3 - xy^2\right)\frac{\partial}{\partial y}$$

$$G_5 = -\frac{1}{2}x^2y\frac{\partial}{\partial x} + \left(\frac{1}{2}x^2y^3 - xy^2 + y\right)\frac{\partial}{\partial y}$$

$$G_6 = \left(\frac{3}{2}x^2 - \frac{1}{2}x^3y\right)\frac{\partial}{\partial x} + \left(\frac{1}{2}x^3y^3 - \frac{3}{2}x^2y^2 + 1\right)\frac{\partial}{\partial y}$$

$$G_7 = \left(x^2 - \frac{1}{2}x^3y\right)\frac{\partial}{\partial x} + \left(\frac{1}{2}x^3y^3 - \frac{3}{2}x^2y^2 + xy\right)\frac{\partial}{\partial y}$$

$$G_8 = \left(\frac{1}{4}x^4y - \frac{1}{2}x^3\right)\frac{\partial}{\partial x} + \left(x^3y^2 - \frac{1}{4}x^4y^3 - \frac{3}{2}x^3y + x\right)\frac{\partial}{\partial y}. \tag{2.41}$$

From (2.41) we can generate, after some tedious calculations, the following nonzero Lie brackets

$$[G_1, G_2] = G_2$$

$$[G_1, G_4] = G_1 - G_3$$

$$[G_1, G_5] = -G_3$$

$$[G_1, G_6] = 3G_4$$

$$[G_1, G_7] = 2G_4 + G_5$$

$$[G_1, G_8] = G_6 - 3G_7$$

$$[G_2, G_4] = G_2$$

$$[G_2, G_5] = -G_2$$

$$[G_2, G_6] = 3G_3 - G_1$$

$$[G_2, G_7] = G_3$$

$$[G_2, G_8] = G_5 - G_4$$

$$[G_3, G_5] = -G_3$$

$$[G_3, G_6] = -G_4 - 2G_5$$

$$[G_3, G_8] = -G_7$$

$$[G_4, G_6] = 2G_7$$

$$[G_4, G_7] = G_7$$

$$[G_4, G_8] = G_8$$

$$[G_5, G_6] = 2G_7 - G_6$$

$$[G_5, G_8] = -G_8$$

$$[G_6, G_7] = G_8$$

for the corresponding Lie algebra. We deduce that the Lie algebra is $sl(3, R)$ and that (2.11) is linearisable to (Mahomed and Leach 1985)

$$y'' = 0. \tag{2.42}$$

(It is well-known that all second order equations admitting an eight-dimensional Lie algebra are linearisable).

As a second example we determine the symmetries of the following differential equation (Spiegel 1958)

$$y'' + 2yy'^3 = 0. \tag{2.43}$$

The action of $G^{[2]}$ on this equation results in

$$2y'^3\eta + 6yy'^2\eta' - 6yy'^3\xi' + \eta'' - 2y'' - y'\xi'' = 0 \tag{2.44}$$

and we must take (2.43) into account. This leads to

$$2y'^3\eta + 6yy'^2\frac{\partial\eta}{\partial x} + 4yy'^3\frac{\partial\eta}{\partial y} - 2yy'^3\frac{\partial\xi}{\partial x} + \frac{\partial^2\eta}{\partial x^2} + 2y'\frac{\partial^2\eta}{\partial x\partial y}$$

$$+y'^2\frac{\partial^2\eta}{\partial y^2} - y'\frac{\partial^2\xi}{\partial x^2} - 2y'^2\frac{\partial\xi}{\partial x\partial y} - y'^3\frac{\partial^2\xi}{\partial y^2} = 0. \tag{2.45}$$

Equating the coefficients of powers of y' to zero leads to the following set of partial differential equations:

$$y'^3 \quad : \quad 2\eta + 4y\frac{\partial \eta}{\partial y} - 2y\frac{\partial \xi}{\partial x} - \frac{\partial^2 \xi}{\partial y^2} = 0 \tag{2.46}$$

$$y'^2 \quad : \quad 6y\frac{\partial \eta}{\partial x} + \frac{\partial^2 \eta}{\partial y^2} - 2\frac{\partial^2 \xi}{\partial x \partial y} = 0 \tag{2.47}$$

$$y'^1 \quad : \quad 2\frac{\partial^2 \eta}{\partial x \partial y} - \frac{\partial^2 \xi}{\partial x^2} = 0 \tag{2.48}$$

$$y'^0 \quad : \quad \frac{\partial^2 \eta}{\partial x^2} = 0. \tag{2.49}$$

Integrating equations (2.49) and (2.48) gives

$$\eta \;=\; ax + b \tag{2.50}$$

$$\xi \;=\; a'x^2 + cx + d \tag{2.51}$$

where, in this case, a, b, c and d are arbitrary functions of y. Substituting (2.50) and (2.51) into (2.46) and (2.47) yields (after equating the coefficients of powers of x to zero) the following set of equations:

$$a'' \;=\; 0 \tag{2.52}$$

$$6ay + b'' - 2c \;=\; 0 \tag{2.53}$$

$$2a - c'' \;=\; 0 \tag{2.54}$$

$$2b + 4by - 2cy - d'' \;=\; 0. \tag{2.55}$$

The solution of the system (2.52) − (2.55) is

$$a \;=\; A_1 y + A_2 \tag{2.56}$$

$$c \;=\; \frac{A_1}{3}y^3 + A_2 y^2 + A_3 y + A_4 \tag{2.57}$$

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$$b = -\frac{A_1}{3}y^4 - \frac{A_2}{3}y^3 + A_3y^2 + A_5y + A_6 \tag{2.58}$$

$$d = -\frac{2A_1}{9}y^6 - \frac{A_2}{3}y^5 + \frac{2A_3}{3}y^4 + A_5y^3 - \frac{A_4}{3}y^3 + A_6y^2 + A_7y + A_8. \tag{2.59}$$

We are now in a position to find the coefficient functions

$$\begin{aligned}\xi(x,y) &= A_1x^2 + \frac{A_1}{3}xy^3 + A_2xy^2 + A_3xy + A_4x - \frac{2A_1}{9}y^6 - \frac{A_2}{3}y^5 \\ &\quad + \frac{2A_3}{3}y^4 + A_5y^3 - \frac{A_4}{3}y^3 + A_6y^2 + A_7y + A_8\end{aligned} \tag{2.60}$$

and

$$\eta(x,y) = A_1xy + A_2x - \frac{A_1}{3}y^4 - \frac{A_2}{3}y^3 + A_3y^2 + A_5y + A_6. \tag{2.61}$$

The generator of the infinitesimal transformation is

$$\begin{aligned}G &= A_1\left(x^2\frac{\partial}{\partial x} + \frac{1}{3}xy^3\frac{\partial}{\partial x} - \frac{2}{9}y^6\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} - \frac{1}{3}y^4\frac{\partial}{\partial y}\right) \\ &\quad + A_2\left(xy^2\frac{\partial}{\partial x} - \frac{1}{3}y^5\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - \frac{1}{3}y^3\frac{\partial}{\partial y}\right) \\ &\quad + A_3\left(+xy\frac{\partial}{\partial x} + \frac{2}{3}y^4\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right) + A_4\left(x\frac{\partial}{\partial x} - \frac{1}{3}y^3\frac{\partial}{\partial x}\right) \\ &\quad + A_5\left(y^3\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + A_6\left(y^2\frac{\partial}{\partial y} + \frac{\partial}{\partial y}\right) \\ &\quad + A_7y\frac{\partial}{\partial x} + A_8\frac{\partial}{\partial x}\end{aligned} \tag{2.62}$$

which is an eight-parameter symmetry. Consequently the eight one-parameter symmetries are

$$\begin{aligned}G_1 &= \frac{\partial}{\partial x} \\ G_2 &= y\frac{\partial}{\partial x} \\ G_3 &= y^2\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \\ G_4 &= y^3\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\end{aligned}$$

$$G_5 = \left(x - \frac{1}{3}y^3\right)\frac{\partial}{\partial x}$$

$$G_6 = \left(xy + \frac{2}{3}y^4\right)\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$$

$$G_7 = \left(xy^2 - \frac{1}{3}\right)\frac{\partial}{\partial x} + \left(x - \frac{1}{3}y^3\right)\frac{\partial}{\partial y}$$

$$G_8 = \left(x^2 + \frac{1}{3}xy^3 - \frac{2}{9}y^6\right)\frac{\partial}{\partial x} + \left(xy - \frac{1}{3}y^4\right)\frac{\partial}{\partial y} \tag{2.63}$$

with nonzero Lie brackets given by

$$[G_1, G_5] = G_1$$

$$[G_1, G_6] = G_2$$

$$[G_1, G_7] = G_3$$

$$[G_1, G_8] = G_4 + 2G_5$$

$$[G_2, G_5] = G_2$$

$$[G_2, G_7] = G_4 - G_5$$

$$[G_2, G_8] = G_6$$

$$[G_3, G_4] = G_3$$

$$[G_3, G_6] = 2G_4 + G_5$$

$$[G_3, G_8] = G_7$$

$$[G_4, G_6] = G_6$$

$$[G_4, G_7] = -G_7$$

$$[G_5, G_7] = G_7$$

$$[G_5, G_8] = G_8$$

$$[G_6, G_7] = -G_8 \tag{2.64}$$

for the corresponding Lie algebra which is, again, $sl(3, R)$. Thus the equation (2.43) can be linearised.

We notice that the procedure of finding symmetries of ordinary differential equations is highly systematic and thus amenable to programming with symbol manipulation software. A variety of software tools are available for analysing symmetries of differential equations (Hereman 1994). In this thesis we use the package Program LIE (Head 1993), hereinafter referred to as LIE.

2.3 Hidden symmetries

Few equations admit the required number of point symmetries to enable reduction to quadratures. In an attempt to overcome this limitation, various extensions of the classical Lie approach have been devised. One such extension is due to the observance of the so-called hidden symmetries - point symmetries that arise unexpectedly due to decreasing and/or increasing the order of a differential equation (Edelstein et al. 2001).

Hidden symmetries have been shown to lead to the solutions of a number of equations that do not possess sufficient Lie point symmetries with the appropriate Lie algebras. Increasing the order of an equation can give rise to Type I hidden symmetries and the reduction of order can give rise to Type II hidden symmetries (Abraham-Shrauner 1993). We illustrate the techniques of the reduction and the increase of order by means of examples.

2.3.1 Reduction of order

If a differential equation

$$E\left(x, y, \dots\dots, y^{(n)}\right) = 0 \tag{2.65}$$

has a symmetry

$$G = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{2.66}$$

we can obtain an equation of order $(n-1)$ in a systematic manner. This is achieved by using the zeroth order and first order differential invariants which are two characteristics associated with $G^{[1]}$. The characteristics are obtained by solving the following system of ordinary differential equations

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy'}{\eta' - y'\xi'}. \tag{2.67}$$

If we integrate the equation involving the first two terms we obtain the characteristic $u = f(x, y)$ and the equation involving the first and the third (equally the second and the third) terms gives the characteristic $v = g(x, y, y')$. Since $Gu = 0$ we call u the zeroth order invariant. Similarly v is called the first order differential invariant since $G^{[1]}v = 0$. A key feature of the Lie method is that all higher derivatives can be expressed in terms of u, v and the derivatives of v with respect to u. As a result equation (2.65) reduces to

$$F\left(u, v, \dots\dots, v^{(n-1)}\right) = 0, \tag{2.68}$$

i.e. it reduces to an equation of order one less than the original. If the reduced equation has a symmetry, the order of the equation can be reduced again. The process can be repeated until the original differential equation is reduced to an algebraic equation. This

reduction of order reduces an nth order equation to a set of n first order equations *provided* there is a sufficient number of symmetries with the appropriate Lie algebra.

As an example we reduce the order of

$$y^3 y'' - 1 = 0, \qquad (2.69)$$

a simplified Ermakov-Pinney equation (Ermakov 1880 and Pinney 1950). The equation (2.69) has the following three symmetries

$$G_1 = \frac{\partial}{\partial x} \qquad (2.70)$$

$$G_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \qquad (2.71)$$

$$G_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \qquad (2.72)$$

with nonzero commutation relations

$$[G_1, G_2] = 2G_1$$

$$[G_1, G_3] = G_2$$

$$[G_2, G_3] = 2G_3 \qquad (2.73)$$

and hence form the Lie algebra $sl(2, R)$. If we use G_1 to reduce the order we obtain the following system

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dy'}{0}. \qquad (2.74)$$

In this case it is trivial to identify

$$u = y$$

$$v = y' \qquad (2.75)$$

Now we can write

$$
\begin{aligned}
\frac{dv}{du} &= \frac{dv}{dx} \bigg/ \frac{du}{dx} \\
&= \frac{y''}{y'} \\
&= \frac{y''}{v}
\end{aligned}
\tag{2.76}
$$

and so we have, for the second derivative,

$$
y'' = v\frac{dv}{du}.
\tag{2.77}
$$

Thus if we adopt the zeroth order invariant u and the first differential invariant v as new variables (Dresner 1999), equation (2.69) is reduced to

$$
\frac{dv}{du} = \frac{u^{-3}}{v}
\tag{2.78}
$$

which is a first order ordinary differential equation and is easily solved by separating the variables.

Similarly, if we reduce the order of (2.69) using G_3 we obtain the invariants

$$
u = \frac{y}{x}
\tag{2.79}
$$

$$
v = xy' - y.
\tag{2.80}
$$

The reduced equation is then given by

$$
\frac{dv}{du} = \frac{u^{-3}}{v}
\tag{2.81}
$$

which is (2.78) again.

If we use G_2 to reduce the order of (2.69), the invariants are

$$u = \frac{y}{x^{1/2}} \qquad (2.82)$$

$$v = y'x^{1/2}. \qquad (2.83)$$

From the invariants (2.82) and (2.83), the reduced equation is

$$\frac{dv}{du} = \frac{u^{-3} + \dfrac{1}{2v}}{v - \dfrac{1}{2u}}. \qquad (2.84)$$

which cannot be easily solved.

This example illustrates the importance of choosing the correct symmetry to reduce the order of an equation. If an equation admits two symmetries with the Lie algebra

$$[G_i, G_j] = \lambda G_i, \qquad (2.85)$$

where λ is a nonzero constant usually scaled to one, then reduction of order via G_j will usually result in the reduced equation being difficult to solve (this is due to the fact that the transformed form of G_i is not a symmetry of the reduced equation). However, reduction via G_i will usually result in an equation that is easy to solve as the transformed form of G_j is a symmetry of the reduced equation. This serves to make an important point - it is not just the number of symmetries that determines whether an equation can be reduced to quadratures but rather the Lie algebra of the symmetries.

We also note that although the existence of a symmetry enables the order of an equation to be reduced, this does not necessarily mean that we shall be able to find an expression for the solution of the reduced equation. However, in the case of a first order differential equation, in theory the solution exists.

2.3.2 Increasing the order

Unlike in the reduction of order, for the increase of order we are not governed by the symmetries of the equation under analysis. In fact, the approach is to increase the order in a manner that imposes a particular symmetry on the new higher order equation. Thus we are guaranteed that the new equation will have at least one symmetry.

As a first example we revisit equation (2.43)

$$y'' + 2yy'^3 = 0. \tag{2.86}$$

We use the standard transformation

$$
\begin{aligned}
x &= p \\
y &= \frac{q'}{q}.
\end{aligned}
\tag{2.87}
$$

As a result we expect the new equation to admit

$$G_1 = q\frac{\partial}{\partial q} \tag{2.88}$$

as a symmetry (Abraham-Shrauner et al. 1995). From (2.87) we find that

$$y' = \frac{q''}{q} - \frac{q'^2}{q^2} \tag{2.89}$$

and

$$y'' = \frac{q'''}{q} - 3\frac{q''q'}{q^2} + 2\frac{q'^3}{q^3}. \tag{2.90}$$

We substitute (2.89) and (2.90) in (2.86) to obtain

$$\frac{q'''}{q} - 3\frac{q''q'}{q^2} + 2\frac{q'^3}{q^3} + 2\frac{q'}{q}\left(\frac{q''}{q} - \frac{q'^2}{q^2}\right)^3 = 0 \tag{2.91}$$

which "simplifies" to the third order nonlinear ordinary differential equation

$$q^6 q''' - 3q^5 q'' q' + 2q^4 q'^3 + 2q^3 q''^3 q' - 6q^2 q''^2 q'^3 + 6qq'' q'^5 - 2q'^7 = 0. \qquad (2.92)$$

The form (2.92) is more complicated than the original differential equation (2.86). This indicates that raising the order does not necessarily lead to simplification.

However, there are situations where the higher order differential equation is indeed in a simpler form. To illustrate this we consider the nonlinear second order differential equation

$$y'' + 3yy' + y^3 = 0 \qquad (2.93)$$

again. Using (2.87), (2.89) and (2.90), (2.93) becomes

$$\frac{q'''}{q} - 3\frac{q'' q'}{q^2} + 2\frac{q'^3}{q^3} + 3\frac{q'}{q}\left(\frac{q''}{q} - \frac{q'^2}{q^2}\right) + \frac{q'^3}{q^3} = 0 \qquad (2.94)$$

which simplifies to

$$q''' = 0, \qquad (2.95)$$

a third order linear differential equation.

2.3.3 Nonlocal symmetries

Hidden symmetries manifest themselves as nonlocal symmetries of the original equation. Nonlocal symmetries are those in which the infinitesimals depend on integrals containing the dependent (and the independent) variables. These symmetries are important as they have been linked with with integrable models (Olver 1993).

2.4 Transformation of symmetries

Besides the number of symmetries of a differential equation, we also need to identify the Lie algebra of the symmetries. After finding the symmetries we work out the Lie brackets so as to identify the Lie algebra. Sometimes the Lie algebra obtained from the symmetries of a particular differential equation cannot be easily identified as belonging to a particular Lie group. To classify the Lie algebra we often need to make a change of basis (Govinder 1993). Lie algebras have standard representations that have already been worked out (Edelstein et al 2001). For example, in the case of $sl(2, R)$ the standard representation is

$$
\begin{aligned}
G_1 &= \frac{\partial}{\partial x} \\
G_2 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\
G_3 &= x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}.
\end{aligned}
\tag{2.96}
$$

Let us assume that the Lie algebra has been identified. We want to transform the symmetries revealed by a particular equation into the standard representation. This transformation is then used to transform the equation into the standard representative equation for that Lie algebra, a form in which we hope that the solution is more evident. We illustrate the symmetry transformation procedure with a simple example.

We want to transform the two equivalent representations of $sl(2, R)$ into each other, i.e.

$$
\begin{aligned}
R_1 &= \frac{\partial}{\partial x} \\
R_2 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}
\end{aligned}
$$

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$$R_3 = x^2 \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y} \tag{2.97}$$

into

$$T_1 = \frac{\partial}{\partial X}$$

$$T_2 = 2X \frac{\partial}{\partial X} + Y \frac{\partial}{\partial Y}$$

$$T_3 = X^2 \frac{\partial}{\partial X} + XY \frac{\partial}{\partial Y}. \tag{2.98}$$

Since we are looking for a point transformation we set

$$X = F(x, y)$$

$$Y = G(x, y). \tag{2.99}$$

We operate on (2.99) with R_1 to obtain

$$\frac{\partial F}{\partial x} \frac{\partial}{\partial x} + \frac{\partial G}{\partial x} \frac{\partial}{\partial y} \tag{2.100}$$

To obtain T_1 we must set

$$\frac{\partial F}{\partial x} = 1 \tag{2.101}$$

$$\frac{\partial G}{\partial x} = 0. \tag{2.102}$$

From (2.101) we have

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dF}{1} \tag{2.103}$$

from which

$$F = f(y) + x. \tag{2.104}$$

From (2.102) we have

$$G = g(y). \tag{2.105}$$

We now use these forms of F and G in our subsequent calculations to transform R_2 into T_2. The relevant equations are

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = F \tag{2.106}$$

$$x\frac{\partial G}{\partial x} + y\frac{\partial G}{\partial y} = \frac{1}{2}G. \tag{2.107}$$

Substituting (2.104) in (2.106) we obtain

$$x + yf'(y) = f(y) + x$$

which gives

$$f(y) = ky$$

where k is an arbitrary constant of integration. Similarly, we find that

$$g(y) = my^{\frac{1}{2}}$$

with m being a constant. Therefore (2.104) and (2.105) become

$$F = ky + x \tag{2.108}$$

$$G = my^{\frac{1}{2}}. \tag{2.109}$$

We now transform R_3 into T_3. The relevant equations are

$$x^2\frac{\partial F}{\partial x} + 2xy\frac{\partial F}{\partial y} = F^2 \tag{2.110}$$

$$x^2\frac{\partial G}{\partial x} + 2xy\frac{\partial G}{\partial y} = FG. \tag{2.111}$$

Substituting (2.108) in (2.110) we obtain

$$x^2 + 2kxy = (ky + x)^2$$

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which is equivalent to

$$k^2 y^2 = 0$$

which implies

$$k = 0.$$

Equation (2.111) is identically satisfied. Therefore we have

$$F(x, y) \;\; = \;\; x \tag{2.112}$$

$$G(x, y) \;\; = \;\; y^{\frac{1}{2}} \tag{2.113}$$

if we let $m = 1$. This implies

$$X \;\; = \;\; x \tag{2.114}$$

$$Y \;\; = \;\; y^{\frac{1}{2}} \tag{2.115}$$

can be used to transform (2.97) into (2.98).

We can use the above approach (although it is far more tedious) to linearise (2.11) and (2.43) to $y'' = 0$. Having established the necessary tools, we are now in a position to analyse some physically interesting differential equations.

Chapter 3

Lie analysis of an equation arising in astrophysics

3.1 Stellar model

Systems of nonlinear differential equations appear in modelling physical phenomena arising in relativistic astrophysics. Exact solutions to these equations are essential to describe the gravitational interactions, and to study the physical features of the model. For the relevant background to the Einstein field equations and relativistic astrophysics, we refer the reader to the standard references in Misner et al. (1973), Schutz (1980) and Shapiro and Teukolsky (1983). Particular exact solutions applicable to problems arising in astrophysics are listed by Delgaty and Lake (1998), Krasinski (1997) and Stephani et al. (2003). A particular application of physical significance is the description of a dense

star where general relativistic effects cannot be ignored; further details can be found in Maharaj and Leach (1996), Mukherjee et al. (1997) and Sharma et al. (2001) amongst others.

The behaviour of a relativistic star is described by the following system of equations

$$\frac{1}{r^2}\left[r\left(1-e^{-2\lambda}\right)\right]' = \rho \tag{3.1}$$

$$-\frac{1}{r^2}\left(1-e^{-2\lambda}\right) + \frac{2\nu'}{r}e^{-2\lambda} = p \tag{3.2}$$

$$e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) = p \tag{3.3}$$

in a spherically symmetric space-time which is static. Primes denote differentiation with respect to the radial coordinate r. The functions $\nu = \nu(r)$ and $\lambda = \lambda(r)$ represent the gravitational potentials; $\rho = \rho(r)$ and $p = p(r)$ are the energy density and pressure respectively. The equations $(3.1) - (3.3)$ may be represented in a number of equivalent forms to make the integration easier. It is convenient to use the transformation

$$x = Cr^2 \tag{3.4}$$

$$Z(x) = e^{-2\lambda(r)} \tag{3.5}$$

$$A^2 y^2(x) = e^{2\nu(r)} \tag{3.6}$$

due to Durgapal and Bannerji (1983) where A and C are constants. Under the transformation $(3.4) - (3.6)$, the Einstein field equations $(3.1) - (3.3)$ take the form

$$\frac{1-Z}{x} - 2\frac{dZ}{dx} = \frac{\rho}{C} \tag{3.7}$$

$$4Z\frac{1}{y}\frac{dy}{dx} + \frac{Z-1}{x} = \frac{p}{C} \tag{3.8}$$

$$4Zx^2\frac{d^2y}{dx^2} + 2x^2\frac{dZ}{dx}\frac{dy}{dx} + \left(x\frac{dZ}{dx} - Z + 1\right)y = 0 \tag{3.9}$$

We utilise the form $(3.7) - (3.9)$ in our analysis of a dense star.

For a realistic stellar model it is necessary to impose conditions on the matter distribution. On observational grounds we often impose the barotropic equation of state

$$p = p(\rho) \tag{3.10}$$

relating the energy density ρ to the pressure p. A special case of (3.10) is

$$p = \rho \tag{3.11}$$

which is called the stiff equation of state. If (3.11) holds then the speed of sound is equal to the speed of light, and this equation of state is physically relevant (Stephani et al. 2003). We observe that apart from the case $\rho \propto r^{-2}$ with a linear equation of state in (3.10), closed form solutions to the field equations are not known. It is for this reason that we perform a Lie analysis in this model in an attempt to obtain a deeper understanding of the underlying differential equations.

3.2 Standard representation for a third order equation

It is possible to eliminate Z from the system of equations $(3.7) - (3.9)$, when (3.11) is true, and we generate an ordinary differential equation in y only.

With the assistance of (3.11) in $(3.7) - (3.9)$ we can find the following expressions for Z

and $\dfrac{dZ}{dx}$ in terms of y and its derivatives:

$$Z = \frac{xyy' + y^2}{y^2 + 2xyy' + 2x^2y'^2 - 2x^2yy''} \tag{3.12}$$

$$\frac{dZ}{dx} = \frac{2xyy'' + yy'}{2x^2yy'' - 2x^2y'^2 - 2xyy' - y^2}. \tag{3.13}$$

where primes denote differentiation with respect to x. Differentiating (3.12) and equating to (3.13) generates the following third order nonlinear differential equation

$$2x^2y^3y''' + 2x^3y^2y'y''' - xy^2y'^2 + 4x^2yy'^3 + 2x^3yy'^4$$

$$+5xy^3y'' - 2x^2y^2y'y'' + 2x^3yy'^2y'' - 6x^3y^2y''^2 = 0. \tag{3.14}$$

Equation (3.14) is the master equation governing the evolution of the model.

We analyse (3.14) for symmetries using LIE and we find that the symmetries are

$$G_1 = x\frac{\partial}{\partial x} \tag{3.15}$$

$$G_2 = y\frac{\partial}{\partial y}. \tag{3.16}$$

While these symmetries do commute, since the master equation is third order, the number of symmetries is insufficient. We therefore transform equation (3.14) to the standard representative equation in which form we hope that the solution will be possible to find.

The Lie bracket of the symmetries (3.15) and (3.16) is

$$[G_1, G_2] = 0 \tag{3.17}$$

so that the Lie algebra is $2A_1$. There are two standard representations (Edelstein et al. 2001), viz.

$$U_1 = \frac{\partial}{\partial Y} \tag{3.18}$$

$$U_2 = X\frac{\partial}{\partial Y} \tag{3.19}$$

and

$$U_3 = \frac{\partial}{\partial X} \tag{3.20}$$

$$U_4 = \frac{\partial}{\partial Y}. \tag{3.21}$$

We will have to consider each representation in turn to determine the most appropriate one.

Using the point transformation

$$X = F(x, y) \tag{3.22}$$

$$Y = G(x, y) \tag{3.23}$$

we want to transform (3.15) and (3.16) into (3.18) and (3.19), i.e. $G_1 \to U_1$ and $G_2 \to U_2$.

Proceeding as in the previous chapter we begin with U_1 and G_1. The relevant equations to solve are

$$x\frac{\partial F}{\partial x} + 0\frac{\partial F}{\partial y} = 0 \tag{3.24}$$

$$x\frac{\partial G}{\partial x} + 0\frac{\partial G}{\partial y} = 1. \tag{3.25}$$

From (3.24) we have the following characteristic equations

$$\frac{dx}{x} = \frac{dy}{0} = \frac{dF}{0} \tag{3.26}$$

and the characteristics are $y = c_1$ and $F = c_2$. Hence we can write

$$F = f(y). \tag{3.27}$$

From (3.25) we have the following characteristic equations

$$\frac{dx}{x} = \frac{dy}{0} = \frac{dG}{1} \tag{3.28}$$

with the characteristics $y = c_3$ and $G - \log x = c_4$. This implies the relationship

$$G = g(y) + \log x. \tag{3.29}$$

Using U_2 and G_2 we have:

$$0\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = 0 \tag{3.30}$$

$$0\frac{\partial G}{\partial x} + y\frac{\partial G}{\partial y} = F \tag{3.31}$$

which gives

$$F = \text{constant}$$

and therefore

$$X = \text{constant}.$$

This implies that in this case we cannot transform (3.15) and (3.16) into (3.18) and (3.19).

Now we want to transform (3.15) and (3.16) into (3.20) and (3.21), i.e. $G_1 \to U_3$ and $G_2 \to U_4$. Using G_1 and U_3 we obtain

$$x\frac{\partial F}{\partial x} + 0\frac{\partial F}{\partial y} = 1 \tag{3.32}$$

$$x\frac{\partial G}{\partial x} + 0\frac{\partial G}{\partial y} = 0. \tag{3.33}$$

From (3.32) we get the following characteristic equations

$$\frac{dx}{x} = \frac{dy}{0} = \frac{dF}{1} \tag{3.34}$$

which generate the characteristics $y = c_1$ and $F - \log x = c_2$. Therefore we can write

$$F = f(y) + \log x. \tag{3.35}$$

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To determine G we use (3.33) and we get the following characteristic equations

$$\frac{dx}{x} = \frac{dy}{0} = \frac{dG}{0} \tag{3.36}$$

which give $y = c_3$ and $G = c_4$. Consequently we can write

$$G = g(y). \tag{3.37}$$

Using G_2 and U_4 we have

$$0\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = 0 \tag{3.38}$$

$$0\frac{\partial G}{\partial x} + y\frac{\partial G}{\partial y} = 1. \tag{3.39}$$

Equations (3.35) and (3.38) give

$$f'(y) = 0$$

and therefore

$$f(y) = \text{constant.} \tag{3.40}$$

On setting this integration constant equal to zero we get, from (3.35),

$$F = \log x. \tag{3.41}$$

Equations (3.37) and (3.39) give

$$g(y) = \log y + \text{constant.} \tag{3.42}$$

On setting this constant to vanish we get

$$G = \log y. \tag{3.43}$$

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From (3.41) and (3.43) we conclude that

$$X = \log x \tag{3.44}$$

$$Y = \log y \tag{3.45}$$

is the relevant transformation for the symmetries. We now wish to transform the equation.

From the transformation $(3.44) - (3.45)$ we obtain the expressions

$$Y' = \frac{xy'}{y} \tag{3.46}$$

$$Y'' = Y' - y'^2 + \frac{x^2 y''}{y}. \tag{3.47}$$

These give the forms

$$y' = \frac{y}{x} Y' \tag{3.48}$$

$$y'' = \frac{y}{x^2} \left(Y'' + y'^2 - Y' \right). \tag{3.49}$$

From (3.47), and using (3.48) and (3.49) we find

$$Y''' = 3Y'' - 3Y'Y'' - Y'^3 + 3Y'^2 - 2Y' + \frac{x^3}{y} y'''. \tag{3.50}$$

From (3.14), and using (3.48) and (3.49), we get

$$y''' = \frac{y}{2x^3 (1 + Y')} \left(6Y''^2 + 10Y'^2 Y'' - 10Y'Y'' + 2Y'^4 - 12Y'^3 - 5Y'' + 5Y' \right). \tag{3.51}$$

Eliminating y''' in (3.50) and (3.51) finally gives

$$2Y'Y''' + 2Y''' - 6Y''^2 - 4Y'^2 Y'' + 10Y'Y'' - Y'' + 8Y'^3 - 2Y'^2 - Y' = 0 \tag{3.52}$$

which is a third order nonlinear equation in Y. We note here that (3.52) is simpler than (3.14).

When (3.52) is analysed for symmetries, LIE provides the following symmetries

$$G_1 = \frac{\partial}{\partial X} \tag{3.53}$$

$$G_2 = \frac{\partial}{\partial Y}, \tag{3.54}$$

i.e. (3.20) and (3.21). This is a good confirmation that our transformation is correct. Unfortunately the new form of the equation is not any easier to analyse.

We note here that for both the master equation (3.14) and the transformed equation (3.52) the required number of symmetries to enable reduction to quadratures is not attained since an nth order equation must possess at least n symmetries for this reduction. To overcome this limitation we reduce and/or increase the order of the transformed equation using its symmetries in an attempt to reveal any hidden symmetries.

3.3   Reduction of order of a third order equation

We now consider the transformed third order equation

$$2Y'Y''' + 2Y''' - 6Y''^2 - 4Y'^2Y'' + 10Y'Y'' - Y'' + 8Y'^3 - 2Y'^2 - Y' = 0 \tag{3.55}$$

rather than the original equation (3.14). We apply reduction of order to generate a second order equation which, hopefully, will reveal a sufficient number of symmetries, in which case this equation may be integrable (see section 2.3.1).

Firstly we use the symmetry

$$G_1 = \frac{\partial}{\partial X}. \tag{3.56}$$

The reduction variables obtained via G_1 are

$$u = Y$$

$$v = Y'. \qquad (3.57)$$

In terms of the new variables u and v we obtain

$$2v^2v'' + 2vv'' - 4vv'^2 + 2v'^2 - 4v^2v' + 10vv' - v' + 8v^2 - 2v - 1 = 0 \qquad (3.58)$$

which is a second order nonlinear equation. When we analyse (3.58) for symmetries, LIE only gives the symmetry

$$X_1 = \frac{\partial}{\partial u}, \qquad (3.59)$$

i.e. the transformed form of G_2

We now reduce the order of (3.55) using the symmetry

$$G_2 = \frac{\partial}{\partial Y}. \qquad (3.60)$$

This time the reduction variables are

$$u = X$$

$$v = Y' \qquad (3.61)$$

The reduced equation, in terms of the variables u and v, is

$$2vv'' + 2v'' - 6v'^2 - 4v^2v' + 10vv' - v' + 8v^3 - v = 0. \qquad (3.62)$$

We analyse (3.62) for symmetries and LIE gives only one symmetry

$$X_1 = \frac{\partial}{\partial u}, \qquad (3.63)$$

i.e. the transformed form of G_1.

We notice here that both reduced second order equations (3.58) and (3.62) do not reveal a sufficient number of symmetries and therefore they cannot be reduced to quadratures using Lie point symmetries.

3.4 Increase of order

We now attempt to increase the order of the reduced second order equations (3.58) and (3.62). This approach is influenced by the results of Edelstein et al. (2001).

We firstly consider equation (3.58):

$$2v^2v'' + 2vv'' - 4vv'^2 + 2v'^2 - 4v^2v' + 10vv' - v' + 8v^2 - 2v - 1 = 0. \tag{3.64}$$

As before we let

$$\begin{aligned} u &= p \\ v &= \frac{q'}{q}, \end{aligned} \tag{3.65}$$

to ensure that the new equation has

$$q\frac{\partial}{\partial q}$$

as a symmetry. In terms of the new dependent variable q, (3.64) becomes

$$2qq'^2q''' + 2q^2q'q''' - 4qq'q''^2 + 2q^2q''^2 + 2q'^3q'' - 14qq'^2q'' + 10q^2q'q'' - q^3q''$$

$$+10q'^4 - 10qq'^3 + 9q^2q'^2 - 2q^3q' - q^4 = 0 \tag{3.66}$$

which is a third order nonlinear equation. When analysing (3.66) for symmetries LIE gives the following two symmetries

$$X_1 = \frac{\partial}{\partial p} \tag{3.67}$$

$$X_2 = q\frac{\partial}{\partial q}. \tag{3.68}$$

Next we consider the equation (3.62):

$$2vv'' + 2v'' - 6v'^2 - 4v^2v' + 10vv' - v' + 8v^3 - v = 0. \tag{3.69}$$

Again we utilise the transformation (3.65) and generate the following third order non-linear equation

$$2q^2q'q''' + 2q^3q''' - 6q^2q''^2 + 2qq'^2q'' + 4q^2q'q'' - q^3q'' + 2q'^4 + 2qq'^3 - q^2q'^2 - q^3q' = 0. \tag{3.70}$$

When analysing (3.70) for symmetries LIE gives the following two symmetries

$$X_1 = \frac{\partial}{\partial p} \tag{3.71}$$

$$X_2 = q\frac{\partial}{\partial q}. \tag{3.72}$$

The number of symmetries for both third order differential equations is insufficient.

As this approach has failed we now again consider the transformed third order equation (3.52):

$$2Y'Y''' + 2Y''' - 6Y''^2 - 4Y'^2Y'' + 10Y'Y'' - Y'' + 8Y'^3 - 2Y'^2 - Y' = 0 \tag{3.73}$$

and apply an increase of order to generate a fourth order equation using the transformation

$$X = p$$

$$Y = \frac{q'}{q}. \qquad (3.74)$$

The result is the following fourth order nonlinear equation in terms of q:

$$2q^5 q'' q'''' - 2q^4 q'^2 q'''' + 2q^6 q'''' - 6q^5 q'''^2 - 4q^4 q''^2 q''' + 8q^3 q'^2 q'' q'''$$

$$+28q^4 q' q'' q''' + 10q^5 q'' q''' - 4q^2 q'^4 q''' - 16q^3 q'^3 q''' - 10q^4 q'^2 q''' - 8q^5 q' q'''$$

$$-q^6 q''' + 12q^3 q' q''^3 + 2q^4 q''^3 - 32q^2 q'^3 q''^2 - 48q^3 q'^2 q''^2 - 30q^4 q' q''^2$$

$$-8q^5 q''^2 + 28qq'^5 q'' + 60q^2 q'^4 q'' + 50q^3 q'^3 q'' + 28q^4 q'^2 q'' + 3q^5 q' q''$$

$$-q^6 q'' - 8q'^7 - 20qq'^6 - 20q^2 q'^5 - 14q^3 q'^4 - 2q^4 q'^3 + q^5 q'^2 = 0. \qquad (3.75)$$

When we analyse (3.75) for symmetries, LIE gives the following three symmetries

$$X_1 = \frac{\partial}{\partial p} \qquad (3.76)$$

$$X_2 = q\frac{\partial}{\partial q} \qquad (3.77)$$

$$X_3 = pq\frac{\partial}{\partial q}. \qquad (3.78)$$

Not only is the number of symmetries too few but from the nonzero commutation relations of the symmetries (3.76) − (3.78), given by

$$[X_1, X_2] = 0$$

$$[X_1, X_3] = X_2$$

$$[X_2, X_3] = 0, \qquad (3.79)$$

we see that the algebra is not solvable.

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3.5 Reduction of order of a fourth order equation

While the number of symmetries is still too few, we embark on reduction of order with the hope of finding type II hidden symmetries. We consider the fourth order equation (3.75):

$$2q^5q''q'''' - 2q^4q'^2q'''' + 2q^6q'''' - 6q^5q'''^2 - 4q^4q''^2q''' + 8q^3q'^2q''q'''$$

$$+28q^4q'q''q''' + 10q^5q''q''' - 4q^2q'^4q''' - 16q^3q'^3q''' - 10q^4q'^2q''' - 8q^5q'q'''$$

$$-q^6q''' + 12q^3q'q''^3 + 2q^4q''^3 - 32q^2q'^3q''^2 - 48q^3q'^2q''^2 - 30q^4q'q''^2$$

$$-8q^5q''^2 + 28qq'^5q'' + 60q^2q'^4q'' + 50q^3q'^3q'' + 28q^4q'^2q'' + 3q^5q'q''$$

$$-q^6q'' - 8q'^7 - 20qq'^6 - 20q^2q'^5 - 14q^3q'^4 - 2q^4q'^3 + q^5q'^2 = 0 \qquad (3.80)$$

and apply the reduction of order using the symmetry

$$X_1 = \frac{\partial}{\partial p} \qquad (3.81)$$

given by (3.76). This suggests that we introduce new variables u and v such that

$$q = u$$

$$q' = v. \qquad (3.82)$$

Consequently we obtain a third order nonlinear differential equation

$$2u^5v^3v'v''' - 2u^4v^4v''' + 2u^6v^2v''' - 6u^5v^3v''^2 - 4u^4v^3v'^2v'' - 4u^5v^2v'^2v''$$

$$+8u^3v^4v'v'' + 20u^4v^3v'v'' + 10u^5v^2v'v'' + 8u^6vv'v'' - 4u^2v^5v'' - 16u^3v^4v''$$

$$-10u^4v^3v'' - 8u^5v^2v'' - u^6vv'' - 4u^4v^2v'^4 - 4u^5vv'^4 + 20u^3v^3v'^3 + 28u^4v^2v'^3$$

$$+10u^5vv'^3 + 2u^6v'^3 - 36u^2v^4v'^2 - 64u^3v^3v'^2 - 40u^4v^2v'^2 - 16 - 16u^5vv'^2$$

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$$-u^6v'^2 + 28uv^5v' + 60u^2v^4v' + 50u^3v^3v' + 28u^4v^2v' + 3u^5vv' - u^6v' - 8v^6$$

$$-20uv^5 - 20u^2v^4 - 14u^3v^3 - 2u^4v^2 + u^5v = 0. \tag{3.83}$$

When we analyse (3.83) for symmetries, LIE gives only one symmetry

$$V_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}. \tag{3.84}$$

We now reduce the order of (3.83) using the symmetry (3.84). Here we use the transformation

$$
\begin{aligned}
v &= us \\
v' &= t. \tag{3.85}
\end{aligned}
$$

We obtain the following second order nonlinear differential equation

$$2s^3t^3t'' - 6s^4t^2t'' + 2s^2t^2t'' + 6s^5tt'' - 4s^3tt'' - 2s^6t'' + 2s^4t'' - 4s^3t^2t'^2$$

$$+8s^4tt'^2 + 2s^2tt'^2 - 4s^5t'^2 - 2s^3t'^2 - 4s^3t^3t' - 4s^2t^3t' + 12s^4t^2t' + 20s^3t^2t'$$

$$+10s^2t^2t' + 8st^2t' - 12s^5tt' - 28s^4tt' - 20s^3tt' - 20s^2tt' - stt' + 4s^6t'$$

$$+12s^5t' + 10s^4t' + 12s^3t' + s^2t' - 4s^2t^4 - 4st^4 + 20s^3t^3 + 28s^2t^3 + 10st^3$$

$$+2t^3 - 36s^4t^2 - 64s^3t^2 - 40s^2t^2 - 16st^2 - t^2 + 28s^5t + 60s^4t + 50s^3t$$

$$+28s^2t + 3st - t - 8s^6 - 20s^5 - 20s^4 - 14s^3 - 2s^2 + s = 0. \tag{3.86}$$

When analysing (3.86) for symmetries, LIE gives no symmetry.

We again consider the fourth order equation (3.75)

$$2q^5q''q'''' - 2q^4q'^2q'''' + 2q^6q'''' - 6q^5q'''^2 - 4q^4q''^2q''' + 8q^3q'^2q''q'''$$

$$+28q^4q'q''q''' + 10q^5q''q''' - 4q^2q'^4q''' - 16q^3q'^3q''' - 10q^4q'^2q''' - 8q^5q'q'''$$

$$-q^6q''' + 12q^3q'q''^3 + 2q^4q''^3 - 32q^2q'^3q''^2 - 48q^3q'^2q''^2 - 30q^4q'q''^2$$

$$-8q^5q''^2 + 28qq'^5q'' + 60q^2q'^4q'' + 50q^3q'^3q'' + 28q^4q'^2q'' + 3q^5q'q''$$

$$-q^6q'' - 8q'^7 - 20qq'^6 - 20q^2q'^5 - 14q^3q'^4 - 2q^4q'^3 + q^5q'^2 = 0 \qquad (3.87)$$

and reduce the order using the symmetry

$$X_3 = pq\frac{\partial}{\partial q} \qquad (3.88)$$

given by (3.78). As this is not as simple as before, we provide a few details. The first extension of X_3 is

$$X_3^{[1]} = pq\frac{\partial}{\partial q} + (q + pq')\frac{\partial}{\partial q'}. \qquad (3.89)$$

The characteristic equations are

$$\frac{dp}{0} = \frac{dq}{pq} = \frac{dq'}{q + pq'}. \qquad (3.90)$$

The first term provides the zeroth order invariant

$$u = p. \qquad (3.91)$$

Using the theory of quasi-linear differential equations (Raisinghania and Aggarwal 1981, Zachmanoglou and Thoe 1986) we combine q' times the second term and $-q$ times the third term to obtain

$$\frac{q'dq - qdq'}{pqq' - q^2 - pqq'} = \frac{q'dq - qdq'}{-q^2}$$

$$= d\left(\frac{q'}{q}\right). \qquad (3.92)$$

Now, this and the second term in (3.90) give

$$\frac{1}{p}\cdot\frac{dq}{q} = d\left(\frac{q'}{q}\right). \qquad (3.93)$$

From (3.93) we get the following first differential invariant

$$v = \frac{q'}{q} - \frac{1}{p} \log q. \tag{3.94}$$

Using (3.94) and (3.91) we obtain

$$
\begin{aligned}
v' &= \frac{q''}{q} - \frac{q'^2}{q^2} + \frac{1}{p^2} \log q - \frac{q'}{pq} \\
v'' &= \frac{q'''}{q} - 3\frac{q''q'}{q^2} - \frac{q''}{pq} + \frac{q'^2}{pq^2} + 2\frac{q'}{p^2q} - \frac{2}{p^3} \log q + 2\frac{q'^3}{q^3} \\
v''' &= \frac{q''''}{q} - 4\frac{q'''q'}{q^2} - \frac{q'''}{pq} - 3\frac{q''^2}{q^2} + 12\frac{q''q'^2}{q^3} + 3\frac{q''}{p^2q} + 3\frac{q''q'}{pq^2} \\
&\quad -6\frac{q'^4}{q^4} - 2\frac{q'^3}{pq^3} - 3\frac{q'^2}{p^2q^2} - 6\frac{q'}{p^3q} + 6\frac{1}{p^4} \log q. \tag{3.95}
\end{aligned}
$$

From (3.94) and (3.95) we obtain

$$q' = qv + \frac{q}{p} \log q \tag{3.96}$$

$$q'' = qv' + \frac{q'^2}{q} + \frac{q'}{p} - \frac{q}{p^2} \log q \tag{3.97}$$

$$q''' = qv'' + 3\frac{q''q'}{q} + \frac{q''}{p} - \frac{q'^2}{pq} - 2\frac{q'}{p^2} - 2\frac{q'^3}{q^2} + 2\frac{q}{p^3} \log q \tag{3.98}$$

$$
\begin{aligned}
q'''' &= qv''' + 4\frac{q'''q'}{q} + \frac{q'''}{p} + 3\frac{q''^2}{q} - 12\frac{q''q'^2}{q^2} - 3\frac{q''}{p^2} - 3\frac{q''q'}{pq} \\
&\quad +2\frac{q'^3}{pq^2} + 3\frac{q'^2}{p^2q} + 6\frac{q'}{p^3} + 6\frac{q'^4}{q^3} - 6\frac{q}{p^4} \log q. \tag{3.99}
\end{aligned}
$$

Using (3.96) − (3.99) in (3.87) we obtain the following third order nonlinear differential equation

$$2u^4 v'v''' + 2u^3 vv''' + 2u^4 v''' - 6u^4 v''^2 - 4u^4 v'^2 v'' - 8u^3 vv'v''$$

$$+10u^4 v'v'' - 10u^3 v'v'' - 4u^3 v^2 v'' + 10u^3 vv'' + 14u^2 vv'' - u^4 v'' + 2u^3 v''$$

$$+8u^4 v'^3 - 4u^3 v'^3 + 24u^3 vv'^2 - 4u^2 vv'^2 - 2u^4 v'^2 + 10u^3 v'^2 - 10u^2 v'^2$$

$$+24u^2 v^2 v' + 4uv^2 v' - 4u^3 vv' + 12uvv' - u^4 v' - u^3 v' - 4u^2 v' + 8uv^3$$

$$+4v^3 - 2u^2 v^2 - 10uv^2 - 2v^2 - u^3 v + u^2 v + 4uv = 0. \tag{3.100}$$

We analyse (3.100) for symmetries and LIE gives only one symmetry

$$V_1 = \frac{1}{u} \frac{\partial}{\partial v}.$$
(3.101)

We now reduce the order of (3.100) using the symmetry (3.101). The zeroth order invariant is

$$s = u$$
(3.102)

and the first differential invariant is

$$t = uv' + v.$$
(3.103)

In terms of the new variables s and t we obtain the following second order nonlinear equation

$$2s^2 tt'' + 2s^3 t'' - 6s^2 t'^2 - 4st^2 t' + 10s^2 tt' + 8stt' - s^3 t' - 4s^2 t'$$

$$+8st^3 + 4t^3 - 2s^2 t^2 - 10st^2 - 2t^2 - s^3 t + s^2 t + 4st = 0.$$
(3.104)

We analyse (3.104) for symmetries and LIE gives no symmetry.

Note that we do not reduce the order of the fourth order equation (3.75) via (3.77) as this will result in the original third order equation (3.52).

3.6   Discussion

It is remarkable that, while the Lie theory of symmetry analysis is (regarded as) the most important and widely applicable method for finding solutions to nonlinear problems

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(Cantwell 2002), it could not in this case be used to solve the equations considered in this chapter. Also through the reduction and the increase of order, no hidden symmetries could be revealed. Govinder (1993) does highlight the fact that it is not always possible to find the analytical solutions to equations. Equation (3.14)

$$2x^2y^3y''' + 2x^3y^2y'y''' - xy^2y'^2 + 4x^2yy'^3 + 2x^3yy'^4$$

$$+5xy^3y'' - 2x^2y^2y'y'' + 2x^3yy'^2y'' - 6x^3y^2y''^2 = 0 \qquad (3.105)$$

and hence the system of Einstein field equations

$$\frac{1-Z}{x} - 2\frac{dZ}{dx} = \frac{\rho}{C} \qquad (3.106)$$

$$4Z\frac{1}{y}\frac{dy}{dx} + \frac{Z-1}{x} = \frac{p}{C} \qquad (3.107)$$

$$4Zx^2\frac{d^2y}{dx^2} + 2x^2\frac{dZ}{dx}\frac{dy}{dx} + \left(x\frac{dZ}{dx} - Z + 1\right)y = 0 \qquad (3.108)$$

are therefore some examples of ordinary differential equations for which the analytical solutions are not easy to find in a systematic manner. This may explain why so few exact solutions exist in the literature.

Chapter 4

Lie analysis of an equation arising in cosmology

4.1 Early universe models

Nonlinear ordinary differential equations often arise in modelling the early universe in cosmology. It is necessary to obtain exact solutions to such equations to describe the behaviour of gravity and to study the temporal evolution of the cosmological model. A special application of physical interest is the profile of the scale factor in the very early universe when the gravitational field is strong. This occurs in the creation of an open inflationary universe (Dadhich and Kembhavi 2000). For relevant background to cosmology and the early universe the reader is referred to Bailin and Love (1999), Narlikar (2002), Peacock et al. (1990) and Straumann (2004).

In the early universe it is necessary to incorporate a scalar field into a generalised action to fully describe rapid inflationary growth. To investigate gravity with a scalar field we take the generalised action to be

$$I = \int f(R)\sqrt{g}d^4x$$

where $f(R)$ is a function of the scalar curvature R (Dadhich and Kembhavi 2000). In the search for solutions to the field equations researchers often choose

$$f(R) = R + \alpha R^2 - 2\lambda$$

where α couples gravity to the scalar field, and the parameter λ is the gravitational constant. This assumption leads to a higher derivative theory which describes the creation of an open inflationary model (Coleman and de Luccia 1980, Hartle and Hawking 1983). Of physical interest here is the behaviour of the scale factor in this theory of gravity coupled to a scalar field.

A typical ordinary differential equation arising from the field equations in this scenario in early universe cosmological models is of the form

$$2HH'' + 6H^2H' - H'^2 + aH^2 = b \tag{4.1}$$

where $H = H(t)$ is related to the scale factor and the parameters a and b are constants.

4.2 Lie analysis

We now perform a Lie analysis on the nonlinear ordinary differential equation (4.1)

$$2HH'' + 6H^2H' - H'^2 + aH^2 = b \tag{4.2}$$

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in an attempt to obtain an analytic form for $H(t)$. When we analyse (4.2) for symmetries, LIE gives only one symmetry

$$G_1 = \frac{\partial}{\partial t}. \tag{4.3}$$

Due to the lack of symmetries we search for hidden symmetries.

4.2.1 Increase of order

We consider the second order equation (4.2)

$$2HH'' + 6H^2H' - H'^2 + aH^2 = b. \tag{4.4}$$

We apply the increase of order to generate a third order equation which may hopefully reveal a sufficient number of symmetries. We use the transformation

$$
\begin{aligned}
t &= S \\
H &= \frac{R'}{R}.
\end{aligned} \tag{4.5}
$$

We obtain the following third order nonlinear equation in terms of the variable R:

$$2R^2R'R''' - R^2R''^2 + 2RR'^2R'' - 3R'^4 + aR^2R'^2 - bR^4 = 0. \tag{4.6}$$

When we analyse (4.6) for symmetries, LIE gives the following two symmetries

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial S} \tag{4.7} \\
X_2 &= R\frac{\partial}{\partial R} \tag{4.8}
\end{aligned}
$$

which is an insufficient number of symmetries.

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4.2.2 Reduction of order

We now consider the third order equation (4.6)

$$2R^2R'R''' - R^2R''^2 + 2RR'^2R'' - 3R'^4 + aR^2R'^2 - bR^4 = 0. \tag{4.9}$$

We apply the reduction of order to generate a second order equation using the symmetry

$$X_1 = \frac{\partial}{\partial S} \tag{4.10}$$

given by (4.7). It is convenient to introduce new variables u and v defined by

$$u = R \tag{4.11}$$

$$v = R'. \tag{4.12}$$

The resulting equation in terms of the variables u and v is

$$2u^2v^2v'' + u^2v^2v'^2 + 2uv^3v' - 3v^4 + au^2v^2 - bu^4 = 0. \tag{4.13}$$

When we analyse this equation via LIE we obtain only one symmetry

$$X_1 = S\frac{\partial}{\partial S} + R\frac{\partial}{\partial R}. \tag{4.14}$$

Again in this case the number of symmetries is insufficient. As we have pointed out before, we do not use (4.8) to reduce the order of (4.6) since this will result in the original second order equation (4.1).

We note that the Lie approach has not yielded any informative results. We therefore undertake another test for integrability.

4.3 The Painlevé integrability test

Firstly we investigate rescaling to reduce the number of parameters to a minimum in

$$2HH'' + 6H^2H' - H'^2 + aH^2 = b. \tag{4.15}$$

We let $H \to \alpha y$ and $t \to \beta x$. Then after division by $\dfrac{\alpha^2}{\beta^2}$, (4.15) becomes

$$2yy'' - y'^2 + 6\alpha\beta y^2 y' + \alpha\beta^2 y^2 = \frac{b\beta^2}{\alpha^2}. \tag{4.16}$$

If we set $\beta^2 = \dfrac{1}{a}$, $\alpha^2 = \dfrac{b}{a}$ and $\gamma = 6\alpha\beta$, (4.16) may be written in the form

$$2yy'' - y'^2 + \gamma y^2 y' + y^2 - 1 = 0 \tag{4.17}$$

which contains only one parameter γ.

As (4.17) is an autonomous nonlinear ordinary differential equation, we subject it to the Painlevé test. We implement the Painlevé test in terms of the standard ARS algorithm as summarised in standard sources such as Ablowitz et al. (1978, 1980a, 1980b). We recall that the Painlevé test consists of three parts. The first is the determination of the leading order behaviour of the dependent variable (we speak in terms of a single ordinary differential equation as is the case for (4.17)). This establishes those terms of the equation which contribute to the leading order behaviour. In the case that the leading order behaviour reveals a movable singularity, be it polelike or branchlike, an independent arbitrary constant is introduced. For an nth order equation the general solution requires additional $n - 1$ constants of integration. On the assumption of a series expansion about the singularity in terms of a power series of either integral or rational powers the additional arbitrary constants initially appear in linear combination

with the leading order term already determined. This fortunate coincidence simplifies the calculation of the possible powers at which these $n-1$ arbitrary constants intrude. Provided the resonances, as these powers are termed, are integral or rational, the next stage of the test begins.

A series commencing at the leading order power and expanding in powers commensurate with the resonances as well as the leading order term is substituted into the full equation to ensure that there is consistency. The analysis for the resonances introduces arbitrary coefficients in an expansion encompassing the leading order terms only. The complete test requires that this arbitrariness persists when all of the terms in the equation are considered.

If the equation satisfies the requirements of the test, there is sufficient cause to believe that it is integrable in terms of analytic functions (algebraic functions in the case that the leading order power or resonances are rational). The reader is referred to the extensive analysis required to establish integrability provided in Ince (1956).

To determine the leading order behaviour we make the substitution

$$y = \alpha\chi^{p-1}$$

$$\chi = x - x_0 \tag{4.18}$$

where x_0 is the location of the putative movable singularity, into (4.17). We require that to the leading order

$$p(p-2)\alpha^2\chi^{2p-2} + \gamma p\alpha^3\chi^{3p-1} + \alpha^2\chi^{2p} - 1 = 0 \tag{4.19}$$

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and it is evident that just the first and the second terms are involved. We have

$$p = -1 \tag{4.20}$$

$$\alpha = \frac{3}{\gamma}. \tag{4.21}$$

To determine the resonance we substitute

$$y = \alpha\chi^{-1} + \mu\chi^{r-1} \tag{4.22}$$

into the dominant terms. For μ to be arbitrary we require that

$$2r^2 - r - 3 = 0 \qquad \Leftrightarrow \qquad r = -1, \frac{3}{2}. \tag{4.23}$$

The resonance $r = -1$ is a characteristic of the analysis. The second resonance indicates

that a series expansion in half-integral powers of χ is required, i.e. we look to the

possession by (4.17) of the weak Painlevé property.

A critical feature of the Painlevé test is the establishment that the non-dominant terms

do not disrupt the consistency of the preceding analysis. In the case of (4.17) this happens

not to be a problem since the non-dominant terms enter at χ^{-2} and χ^0 and the single

arbitrary constant required in the expansion enters at $\chi^{-5/2}$. If we make the substitution

$$y = \sum_{i=0}^{\infty} a_i \chi^{i-1} \tag{4.24}$$

into (4.17), we find that

$$a_0 = \frac{3}{\gamma}, \quad a_1 = a_2 = 0, \quad a_3 \text{ is arbitrary}, \quad a_4 = \frac{1}{3\gamma}, \text{etc.} \tag{4.25}$$

The expansion indicated by the dominant terms is consistent with the full equation. We

therefore conjecture that the equation(4.17) is integrable in terms of algebraic functions.

This success suggests that the symmetry approach needs to be revisited.

4.4 Reducing the order using nonlocal symmetries

We have conjectured that equation (4.17) is integrable, so we now use nonlocal symmetries to reduce it to quadratures. Govinder and Leach (1995) give an algorithm that is used to determine nonlocal symmetries.

The determination of integrability in practice is facilitated by the Lie analysis for the existence of Lie symmetries. We examine (4.17)

$$2yy'' - y'^2 + \gamma y^2 y' + y^2 - 1 = 0 \qquad (4.26)$$

for Lie point symmetries using LIE and the following single symmetry obtained is

$$V_1 = \frac{\partial}{\partial x} \qquad (4.27)$$

i.e. the transformed form of (4.3). The symmetry (4.27) is sufficient only to reduce (4.26) to the first order ordinary differential equation

$$2uvv' - v^2 + \gamma u^2 v + u^2 - 1 = 0, \qquad (4.28)$$

after using the reduction variables

$$
\begin{aligned}
u &= y \\
v &= y'
\end{aligned} \qquad (4.29)
$$

obtained from V_1. Equation (4.28) is not obviously reducible further.

In the absence of a second Lie point symmetry we examine (4.17) for the existence of nonlocal symmetries. Since (4.17) is autonomous, we assume the specific structure

$$\Gamma_2 = \xi \frac{\partial}{\partial x} \qquad (4.30)$$

in which the functional dependence of ξ is not restricted (Govinder and Leach 1995).

The condition that Γ_2 be a symmetry of (4.17) is given by

$$\Gamma_2^{[2]}(\text{LHS of (4.17)})|_{(4.17)=0} = 0 \tag{4.31}$$

and generates the equation

$$4yy''\xi' + 2yy'\xi'' - 2y'^2\xi' + \gamma y^2 y'\xi' = 0 \tag{4.32}$$

which is a linear second order ordinary differential equation in ξ. We have

$$\frac{\xi''}{\xi'} = -2\frac{y''}{y'} + \frac{y'}{y} - \frac{\gamma}{2}y \tag{4.33}$$

which can be integrated to yield

$$\xi = A_0 + A_1 \int \frac{y}{y'^2}\exp\left[-\frac{1}{2}\gamma \int y dx\right] dx, \tag{4.34}$$

where A_0 and A_1 are constants of integration. Thus in addition to the Lie point symmetry (4.27), we have the nonlocal symmetry

$$\Gamma_2 = \left(\int \frac{y}{y'^2}\exp\left[-\frac{1}{2}\gamma \int y dx\right] dx\right) \frac{\partial}{\partial x}. \tag{4.35}$$

In the reduction of order of (4.17) using (4.27), (4.35) becomes the exponential nonlocal symmetry

$$\Lambda_2 = \frac{u}{v}\exp\left[-\frac{1}{2}\gamma \int \frac{u}{v}du\right] \frac{\partial}{\partial v}, \tag{4.36}$$

in which the nonlocal part of (4.35) is expressed in terms of the invariants of V_1 as

$$\int y dx = \int y\frac{dx}{dy}dy = \int \frac{u}{v}du. \tag{4.37}$$

The exponential nonlocal symmetry (4.36) may be used for a further reduction of order of (4.28). The associated Lagrange's system for the invariants of (4.36) is

$$\frac{du}{0} = \frac{dv}{\left(\dfrac{u}{v}\right)} = \frac{dv'}{\dfrac{1}{v} - \dfrac{uv'}{v^2} - \dfrac{1}{2}\gamma\dfrac{u}{v}} \tag{4.38}$$

and the invariants are

$$
\begin{aligned}
p &= u \\
q &= vv' - \frac{v^2}{2u} + \frac{1}{2}\gamma uv.
\end{aligned} \tag{4.39}
$$

The algebraic equation resulting from the second reduction of order of (4.28) using (4.39) is simply

$$p^2 + 2pq - 1 = 0, \tag{4.40}$$

i.e. the original second order ordinary differential equation

$$2HH'' + 6H^2H' - H'^2 + aH^2 = b \tag{4.41}$$

is reducible to an algebraic equation. However, reversing the various transformations to obtain a solution in terms of the original variables H and t is not a trivial exercise. This is expected as the equation passes the weak Painlevé Test and not the full version.

Chapter 5

Conclusion

The main focus of this dissertation was to systematically explore the analytical solutions of some physically relevant differential equations arising in Mathematical Physics. We undertook a Lie point symmetry analysis of the system of field equations

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} \tag{5.1}$$

$$4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} = \frac{p}{C} \tag{5.2}$$

$$4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + \left(\dot{Z}x - Z + 1\right)y = 0 \tag{5.3}$$

arising in astrophysics and the second order ordinary differential equation

$$2H\ddot{H} + 6H^2\dot{H} - \dot{H}^2 + aH^2 = b \tag{5.4}$$

which arises in cosmology.

In Chapter 1 we summarised the life story of Sophus Lie (according to Cantwell 2002 and Yaglom 1988) as one of the pioneers of the approach adopted in this dissertation.

Definitions of some concepts were also given with the aim of clarifying the approach.

In Chapter 2 we outlined some basic ideas behind the Lie symmetry analysis. The approach adopted in this dissertation makes extensive use of Lie point symmetries. We therefore began by illustrating the method of finding this type of symmetries. To be able to reduce an nth order ordinary differential equation to quadratures using Lie point symmetries the equation must have at least n point symmetries. Due to a limitation that some equations have an insufficient number of point symmetries, some extensions of the classical Lie point symmetry analysis that attempt to force equations to reveal the so-called hidden symmetries have been devised, viz. the reduction of order and the increase of the order of a differential equation. Some simple examples were therefore used to illustrate the reduction and the increase of orders of equations.

The beauty of the method of the reduction of order is that if an ordinary differential equation admits an m-parameter Lie group of transformations whose Lie algebra is solvable, then its order can be reduced by m. In the various increases of order we used the transformation

$$
\begin{aligned}
u &= p \\
v &= \frac{q'}{q}.
\end{aligned}
\tag{5.5}
$$

This transformation guaranteed that the resulting higher order equation in p and q (and derivatives of q with respect to p) would admit the Lie point symmetry

$$
G_1 = q\frac{\partial}{\partial q}.
\tag{5.6}
$$

It is valid to ponder the importance of requiring the presence of (5.6) as opposed to any

other symmetry.

After all, since we are dealing with just a single symmetry, we could require any form. The reason for insisting on (5.6) is due to its occurrence as the only symmetry for nth order linear equations which does not require a solution of the equation. All other symmetries of a linear equation rely on the solution of the equation. Thus if we require a symmetry of the form

$$G_2 = \frac{\partial}{\partial p}$$

(i.e. the resulting equation is required to be autonomous) we could be restricting the possible outcomes to linear equations which have a constant as a particular solution. Clearly this is far too restrictive. (See Abraham-Shrauner et al (1995) for an illustrative example).

In Chapter 3 we considered the system of equations $(5.1) - (5.3)$. We eliminated Z and this resulted in the new master equation in y only governing the evolution of the stellar model

$$2x^2y^3y''' + 2x^3y^2y'y''' - xy^2y'^2 + 4x^2yy'^3 + 2x^3yy'^4$$

$$+5xy^3y'' - 2x^2y^2y'y'' + 2x^3yy'^2y'' - 6x^3y^2y''^2 = 0. \tag{5.7}$$

Due to the lack of a sufficient number of symmetries we then transformed (5.7) into a standard representative equation of the Lie algebra $2A_1$ and obtained

$$2Y'Y''' + 2Y''' - 6Y''^2 - 4Y'^2Y'' + 10Y'Y'' - Y'' + 8Y'^3 - 2Y'^2 - Y' = 0. \tag{5.8}$$

We then considered the transformed equation (5.8) for the rest of the symmetry analysis

undertaken. We reduced and increased the order of this equation but none of the resulting equations had a sufficient number of point symmetries.

We remarked that finding the analytical solutions of the new master equation (5.7) and therefore of the system of Einstein field equations $(5.1) - (5.3)$ in a systematic manner is quite an involved exercise and concluded that the existence of so few exact solutions to this system of equations in the literature may therefore be attributed to this fact. The future challenge is to devise and / or employ other approaches in an attempt to overcome the limitations of the Lie point symmetry approach.

In Chapter 4 we considered a second order nonlinear ordinary differential equation, viz.

$$2H\ddot{H} + 6H^2\dot{H} - \dot{H}^2 + aH^2 = b. \tag{5.9}$$

When we performed a Lie symmetry analysis of this equation, LIE only provided one symmetry, viz.

$$G_1 = \frac{\partial}{\partial t}. \tag{5.10}$$

We then searched for any hidden symmetries by increasing the order of (5.9) and thereafter decreasing the order of the resulting third order equation. In both cases the number of point symmetries obtained was insufficient. Due to the failure of the Lie approach, we transformed equation (5.9) into a one parameter differential equation

$$2yy'' - y'^2 + \gamma y^2 y' + y^2 - 1 = 0. \tag{5.11}$$

For this equation LIE provided one point symmetry, viz.

$$V_1 = \frac{\partial}{\partial x} \tag{5.12}$$

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which is a transformed form of G_1. We undertook the Painlevé integrability test on the second order equation (5.11). The equation passed the weak version of the test and we therefore conjectured that the equation was integrable in terms of algebraic functions.

Integrability suggested the use of nonlocal symmetries since this type of symmetries has been linked with integrable models. The local symmetry obtained was

$$\Gamma_2 = \left(\int \frac{y}{y'^2} \exp \left[\frac{1}{2} \gamma \int y dx \right] dx \right) \frac{\partial}{\partial x} \tag{5.13}$$

and this was a second symmetry of the second order equation (5.11).

Using the two symmetries (5.12) and (5.13) we could reduce the equation

$$2yy'' - y'^2 + \gamma y^2 y' + y^2 - 1 = 0 \tag{5.14}$$

and therefore the original equation

$$2H\ddot{H} + 6H^2\dot{H} - \dot{H}^2 + aH^2 = b$$

into an algebraic equation

$$p^2 + 2pq - 1 = 0. \tag{5.15}$$

From (5.15) it was obvious that generating a solution in terms of the original variables H and t was not a simple exercise.

Chapter 6

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