Stellar structure and Accretion in gravitating systems

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Publication 1

An exact isotropic solution, A J John and S D Maharaj *Il Nuovo Cimento B* 27-33 **121** (2006) (There were regular meetings between my supervisor and I to discuss research material for publication. The outline of the research paper and discussion of the significance of the results were jointly done. The paper was mainly written by me with some input from my supervisor.)

Publication 2

Relativistic stellar models, A J John and S D Maharaj *Pramana - J. Phys.* 461-468 **77** (2011)

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Publication 3

Accretion onto a higher dimensional black hole, *Astrophys. Space Sci.* in preparation (2012)

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Publication 4

Chaplygin gas accretion, *Phys. Rev. D.* in preparation (2012)

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Abstract

In this thesis we study classes of static spherically symmetric solutions to the Einstein and Einstein–Maxwell equations that may be used to model the interior of compact stars. We also study the spherical accretion of fluids on to bodies in both general relativity and the Newtonian theory of gravity. The condition for pressure isotropy is obtained upon specifying one of the gravitational potentials and the electric field intensity. A series solution was found after specifying a cubic form for the potential. The pressure and energy density appear to be non–singular and continuous inside the star. This solution admits an explicit equation of state that, in regions close to the stellar centre, may be approximated by a polytrope. Another class of exact solutions to the Einstein–Maxwell solutions was found with charge. These solutions are in the form of hypergeometric functions with two free parameters. For particular parameter values we recovered two previously known exact solutions that are reasonable models for the interior of compact stars. We demonstrated two new solutions for other choices of the parameters. One of these has well behaved pressure, energy density and electric field intensity variables within the star. The other was rejected as unphysical on the grounds that it has a negative energy density. This violates the energy conditions. We obtained the mass accretion rate and critical radius of a polytrope accreting onto a D–dimensional Schwarzschild black hole. The accretion rate, $\dot{M}$, is an explicit function of the black hole mass, M, as well as the gas boundary conditions and the dimensionality, D, of the spacetime. We also found the asymptotic compression ratios and temperature profiles below the accretion radius and at the event horizon. This generalises the Newtonian expressions of Giddings and Mangano (2008) which examined the accretion of TeV black holes. We obtained the critical radius and accretion rates of a generalised Chaplygin gas accreting on to body under a Newtonian potential. The accretion rate is about 2 - 4 times greater than that for neutral hydrogen. The Rankine–Hugoniot relations for shocked GCG flow were also found. We found general expressions for the pressure and density compression ratios. Some post shock states imply negative volumes. We suspect that these may be thermodynamically forbidden.

Dedicated to Aaron Swartz (1986 – 2013), pioneer and champion of Open Access

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A number of events and individuals proved to be influential in my worldline. The works of Carl Sagan, Arthur C. Clarke and David Goodstein ignited my early interest in science. Stephen Hawking, Richard Feynman and Marcus Chown wrote exceptional popular science books that emboldened me to pursue a career in physics. I strongly encourage every graduate student to read PhD Comics and xkcd penned by the insightful Jorge Cham and Randall Munroe respectively. These outstanding comics succeed in capturing the humanity of scientists. They will keep you company during the long, lonely months of research and remain an inspiration.

Finally I would like to thank Monique James for her help, support and friendship throughout an often challenging, penniless period. You are truly a being of singular beauty. Thank you for the magic.

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Chapter 1

Introduction

"There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced with something even more bizarre and inexplicable. There is another theory which states that this has already happened." – Douglas Adams, The Restaurant at the End of the Universe.

The discovery of dark energy, an unknown mechanism responsible for the late time acceleration of the universe poses serious problems for the standard model of cosmology. It appears that a successful explanation of this phenomenon requires one to either introduce an exotic component into the energy budget of the universe, or dethrone orthodox general relativity as the reigning theory of gravity.

In this thesis we explore two main themes. Firstly we looked at the existence and physical reasonableness of exact solutions to the Einstein equations of general relativity applicable to stellar interiors. Here we operated firmly in the arena of classical general relativity. The second theme sees us explore extensions to this paradigm by considering changes to the geometric sector of the theory as well as augmentations of the matter sector. In this part we look at various modified scenarios where matter accretes onto a massive object. These modifications were of two types. First we examine accretion in the context of D–dimensional general relativity. Next we look at the accretion of a generalised Chaplygin gas (GCG). The GCG is a promising candidate for dark matter and/or dark energy. Since the dark sector's existence was necessitated by astronomical observations we model the accreting process with a Newtonian field. This facilitates comparison of data from accreting stars that do not generate extreme spacetime curvature.

We briefly review the work conducted here. Chapters 2 and 3 form our first theme and are concerned with finding and analysing exact solutions of the Einstein and Einstein–Maxwell equations. These solutions are used to model the interior of static, spherically symmetric stars. These solutions can be matched at the stellar boundary to the Schwarzschild or the Reissner–Nordström solution. The matter content is taken to a perfect fluid with isotropic pressure. This fluid may or may not be charged. These systems are, in general, underdetermined and we specify a form for one or more of the variables in order to obtain an exact solution.

In Chapter 2 we specified a cubic form for one of the gravitational potentials viz. Z, and attempted to find new solutions. Using the method of Frobenius we obtained a series solution to the pressure isotropy equation. The metric potentials, pressure and energy density are non–singular and continuous. The behaviour of the pressure and energy density is depicted graphically in Figs. 2.1 and 2.2. This solution has the interesting feature of admitting an explicit equation of state. Close to the stellar core this equation resembles a polytrope.

In Chapter 3 we considered charged, static stars and attempt to solve the Einstein-Maxwell system by prescribing the potential Z and the electric field intensity E. We obtained a class of charged solutions expressed in terms of hypergeometric functions with two free parameters, viz. α and K. When α vanishes our solution describes uncharged fluids. For certain parameter values the hypergeometric functions reduce to algebraic expressions. For particular choices of K and α we regained some known stellar models. We then obtained two new charged solutions. These were expressed in terms of elementary functions. The first of these appears to be physically reasonable and we illustrate the behaviour of the pressure, energy density and electric field in Figs. 3.1, 3.2 and 3.3. The second new solution admits negative energy density and must therefore be rejected as unphysical as it violates the strong and weak energy conditions.

Chapters 4 and 5 comprise the second major theme of this thesis. They are devoted to extending studies of steady, spherically symmetric accretion by considering an alternative theory of gravity as well as an exotic form of matter. In chapter 4 we formulated the problem of a polytropic gas accreting onto a Schwarzschild black hole in arbitrary dimensions. The gravitational model used is D–dimensional general relativity. We determine analytically the critical radius, fluid velocity and sound speed and subsequently the mass accretion rate. We then obtained expressions for the asymptopic behaviour of the fluid density and temperature near the event horizon.

In chapter 5 we examined a generalised Chaplygin gas (GCG) accreting onto a body in the Newtonian theory of gravity. The GCG is a dark matter and dark energy candidate. We find analytical expressions for the critical radius and mass accretion rate, $\dot{M}$. These are compared to the rates for hydrogen accretion on to a star. We also consider the possibility of shock fronts arising in the flow. The Rankine–Hugoniot conditions for a GCG are obtained and analysed to express the post–shock behaviour of the gas in terms of its pre–shock values.

The results obtained in this thesis, are summarised in the concluding chapter, wherein suggestions for future work are outlined.

Chapter 2

An exact isotropic solution

2.1 Introduction

Static solutions of the Einstein field equations for spherically symmetric manifolds are important in the description of relativistic spheres in astrophysics. The models generated may be used to describe highly compact objects where the gravitational field is strong as in neutron stars. It is for this reason that many investigators use a variety of mathematical techniques to attain exact solutions. One of the first models, satisfying all the physical requirements for a neutron star, was found by Durgapal and Bannerji (1983). Now there exist a number of comprehensive collections eg. Stephani et al. (2003), Skea (1996), Delgaty and Lake (1998) of static, spherically symmetric solutions which provide a useful guide to the literature. It is important to note that only a few of these solutions correspond to nonsingular metric functions with a physically acceptable energy momentum tensor.

In this chapter we seek a new exact solution to the field equations which can be used to describe the interior of a relativistic sphere. We rewrite the Einstein equations as a new set of differential equations which facilitates the integration process in §2.2. We choose a cubic form for one of the gravitational potentials, which we believe has not been studied before, which enables us to simplify the condition of pressure isotropy in §2.3. This yields a third order recurrence relation, which we manage to solve from first principles. It is then possible to exhibit a new exact solution to the Einstein field equations. We then show in §2.4 that the curvature and thermodynamical variables appear to be well-behaved. We also demonstrate the existence of an explicit barotropic equation of state. For small values of the radial coordinate close to the stellar core the equation of state approximates a polytrope. We believe that a detailed physical analysis of our solution is likely to lead to a realistic model for compact objects. Some general comments relating to this exact solution are made in §2.5.

2.2 Static spacetimes

Since our intention is to study relativistic stellar objects it seems reasonable, on physical grounds, to assume that spacetime is static and spherically symmetric. This is clearly consistent with models utilised to study physical processes in compact objects. The generic line element for static, spherically symmetric spacetimes is given by

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{2.1}$$

in Schwarzschild coordinates.

For neutral perfect fluids the Einstein field equations can be written in the form

$$\frac{1}{r^2}[r(1-e^{-2\lambda})]' = \rho \tag{2.2a}$$

$$-\frac{1}{r^2}\,(1-e^{-2\lambda}) + \frac{2\nu'}{r}\,e^{-2\lambda} = p \tag{2.2b}$$

$$e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) = p \tag{2.2c}$$

for the line element (2.1) where the energy density ρ and the pressure p are measured relative to the comoving fluid 4–velocity $u^a = e^{-\nu}\delta^a_0$ and primes denote differentiation with respect to the radial coordinate r. In the field equations (2.2) we are using units where the coupling constant $\frac{8\pi G}{c^4} = 1$ and the speed of light is $c = 1$. An equivalent form of the field equations is obtained if we use the transformation

$$A^2y^2(x) = e^{2\nu(r)}, Z(x) = e^{-2\lambda(r)}, x = Cr^2 \tag{2.3}$$

due to Durgapal and Bannerji (1983), where A and C are arbitrary constants. Under the transformation (2.3), the system (2.2) becomes

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} \tag{2.4a}$$

$$4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} = \frac{p}{C} \tag{2.4b}$$

$$4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + (\dot{Z}x - Z + 1)y = 0 \tag{2.4c}$$

where the overdot denotes differentiation with respect to the variable x. Note that (2.4) is a system of three equations in the four unknowns ρ, p, y and Z. The advantage of this system lies in the fact that a solution can, upon a suitable specification of $Z(x)$, be readily obtained by integrating (2.4c) which is second order and linear in y.

2.3 A new series solution

A large number of exact solutions are known for the system of equations (2.4) that model a relativistic star with no charge. Many of these are listed by Stephani et al. (2003) and Skea (1996). A comprehensive list of static solutions, that satisfy stringent conditions for spherically symmetric perfect fluids, was compiled by Delgaty and Lake (1998). The Einstein field equations in the form (2.4) are under-determined. From inspection it is clear that the simplest solutions to the system (2.4) correspond to polynomials forms for $Z(x)$. As far as we are aware all exact solutions found previously correspond to forms of the gravitational potential $Z(x)$ which are linear or quadratic in the independent variable x. Our approach here is to specify the gravitational potential $Z(x)$ and attempt to solve (2.4c) for the potential y. In an attempt to obtain a new solution to the system (2.4) we make the choice

$$Z = ax^3 + 1 \tag{2.5}$$

where a is a constant. We suspect that the cubic form (2.5) has not been considered before because the resulting differential equation in the dependent variable y is difficult to solve; quadratic forms for Z are listed by Delgaty and Lake (1998). The quadratic form for the potential Z is simpler to handle and contains the familiar Tolman models. With the specified function Z, the condition of pressure isotropy (2.4c) becomes

$$2(ax^3 + 1)\ddot{y} + 3ax^2\dot{y} + axy = 0. \tag{2.6}$$

The linear second order differential equation (2.6) is difficult to solve when $a \neq 0$. We have not found a solution for $a \neq 0$ in standard handbooks of differential equations. Software packages such as Mathematica have also not been helpful as they generate a solution in terms of hypergeometric functions with complex arguments.

We attempt to find a series solution to (2.6) using the method of Frobenius. As the point $x = 0$ is a regular point of (2.6), there exist two linearly independent solutions of the form of a power series with centre $x = 0$. We therefore can write

$$y(x) = \sum_{n=0}^{\infty} c_n x^n \tag{2.7}$$

where the c_n are the coefficients of the series. For a legitimate solution we need to determine the coefficients c_n explicitly.

Substituting (2.7) into (2.6) yields

$$\begin{aligned} 4c_2 + (12c_3 + ac_0)x + 4(6c_4 + ac_1)x^2 + \\ \sum_{n=2}^{\infty}\{a[2n^2 + n + 1]c_n + 2(n+3)(n+2)c_{n+3}\}x^{n+1} &= 0. \end{aligned}$$

For this equation to hold true for all x we require

$$
\begin{aligned}
4c_2 &= 0 & \text{(2.8a)}\\
12c_3 + ac_0 &= 0 & \text{(2.8b)}\\
6c_4 + ac_1 &= 0 & \text{(2.8c)}\\
a[2n^2+n+1]c_n + 2(n+3)(n+2)c_{n+3} &= 0,\ n \geq 2. & \text{(2.8d)}
\end{aligned}
$$

Equation (2.8d) is a linear recurrence relation with variable, rational coefficients of order three. General techniques of solution for difference equations are limited to the simplest cases and (2.8d) does not fall into the known classes. However it is possible to solve (2.8d) from first principles. Equations (2.8a) and (2.8d) imply

$$c_2 = c_5 = c_8 = \cdots = 0. \tag{2.9}$$

From (2.8b) and (2.8d) we generate the expressions

$$
\begin{aligned}
c_3 &= -\frac{a}{2}\frac{1}{3.2}c_0\\
c_6 &= \frac{a^2}{2^2}\frac{2.3^2+3+1}{6.3}\frac{1}{5.2}c_0\\
c_9 &= -\frac{a^3}{2^3}\frac{2.6^2+6+1}{9.6.3}\frac{2.3^2+3+1}{8.5.2}c_0.
\end{aligned}
$$

It is clear that the coefficients $c_3, c_6, c_9, \ldots$ can all be written in terms of the coefficient c_0. These coefficients generate a pattern and we can write

$$
\begin{aligned}
c_{3n+3} = {} & (-1)^{n+1}\left(\frac{a}{2}\right)^{n+1} \times \\
& \frac{[2(3n)^2+3n+1]\cdots[2(3.1)^2+3.1+1][2(3.0)^2+3.0+1]}{\{(3n+3)\cdots(3.1+3)(3.0+1)\}\{(3n+2)\cdots(3.1+2)(3.0+2)\}}c_0.
\end{aligned} \tag{2.10}
$$

We can rewrite this in the form

$$c_{3n+3} = (-1)^{n+1}\left(\frac{a}{2}\right)^{n+1}\prod_{k=0}^{n}\frac{2(3k)^2+3k+1}{(3k+3)(3k+2)}c_0 \tag{2.11}$$

where we have utilised the conventional symbol $\prod$ to denote multiplication. We can obtain a similar formula for the coefficients $c_4, c_7, c_{10}, \ldots$ From (2.8c) and (2.8d) we have

$$
\begin{aligned}
c_4 &= -\frac{a}{2}\frac{2.1^2+1+1}{4.3}c_1 \\
c_7 &= \frac{a^2}{2^2}\frac{2.4^2+4+1}{7.4}\frac{2.1^2+1+1}{6.3}c_1 \\
c_{10} &= -\frac{a^3}{2^3}\frac{2.7^2+7+1}{10.7.4}\frac{2.4^2+4+1}{9.6.3}\frac{2.1^2+1+1}{1}c_1.
\end{aligned}
$$

The coefficients $c_4, c_7, c_{10}, \ldots$ can all be written in terms of the coefficient c_1. These coefficients generate a pattern which is clearly of the form

$$
\begin{aligned}
c_{3n+4} &= (-1)^{n+1}\left(\frac{a}{2}\right)^{n+1}\frac{[2(3.0+1)^2+(3.0+1)+1][2(3.1+1)^2+(3.1+1)+1]}{\{(3n+4)\cdots(3.1+4)(3.0+4)\}} \times \\
&\quad \frac{[2(3.2)^2+(3.2+2)+1]\cdots[2(3n+1)^2+(3n+1)+1]}{\{(3n+3)\cdots(3.1+3)(3.0+3)\}}c_1. \qquad (2.12)
\end{aligned}
$$

This may be expressed as

$$
c_{3n+4} = (-1)^{n+1}\left(\frac{a}{2}\right)^{n+1}\prod_{k=0}^{n}\frac{2(3k+1)^2+(3k+1)+1}{(3k+4)(3k+3)}c_1 \qquad (2.13)
$$

where $\prod$ denotes multiplication.

From (2.9) we observe that the coefficients $c_2, c_5, c_8, \ldots$ all vanish. The coefficients $c_3, c_6, c_9, \ldots$ are generated from (2.11). The coefficients $c_4, c_7, c_{10}, \ldots$ are generated from (2.13). Hence the difference equation (2.8d) has been solved and all non-zero coefficients are expressible in terms of the leading coefficients c_0 and c_1. We can write the series (2.7) as

$$\begin{aligned}
y(x) &= c_0 + c_1x^1 + c_3x^3 + c_4x^4 + c_6x^6 + c_7x^7 + c_9x^9 + c_{10}x^{10} + \cdots \\
&= c_0\left(1 + \sum_{n=0}^{\infty} c_{3n+3}x^{3n+3}\right) + c_1\left(x + \sum_{n=0}^{\infty} c_{3n+4}x^{3n+4}\right) \\
&= c_0\left(1 + \sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{a}{2}\right)^{n+1}\prod_{k=0}^{n}\frac{2(3k)^2+3k+1}{(3k+3)(3k+2)}x^{3n+3}\right) + \\
& \quad c_1\left(x + \sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{a}{2}\right)^{n+1}\prod_{k=0}^{n}\frac{2(3k+1)^2+(3k+1)+1}{(3k+4)(3k+3)}x^{3n+4}\right) \qquad (2.14)
\end{aligned}$$

where c_0 and c_1 are arbitrary constants. Clearly (2.14) is of the form

$$y(x) = c_0y_1(x) + c_1y_2(x) \qquad (2.15)$$

where

$$\begin{aligned}
y_1(x) &= \left(1 + \sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{a}{2}\right)^{n+1}\prod_{k=0}^{n}\frac{2(3k)^2+3k+1}{(3k+3)(3k+2)}x^{3n+3}\right) \\
y_2(x) &= \left(x + \sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{a}{2}\right)^{n+1}\prod_{k=0}^{n}\frac{2(3k+1)^2+(3k+1)+1}{(3k+4)(3k+3)}x^{3n+4}\right)
\end{aligned}$$

are linearly independent solutions of (2.6). Therefore we have found the general solution to the differential equation (2.6) for the particular gravitational potential Z given in (2.5). The advantage of the solutions in (2.15) is that they are expressed in terms of a series with real arguments unlike the complex arguments given by software packages.

2.4 Physical models

From (2.15) and the Einstein field equations (2.4) we generate the exact solution

$$e^{2\lambda} = \frac{1}{ax^3+1} \tag{2.16a}$$

$$e^{2\nu} = A^2y^2 \tag{2.16b}$$

$$\frac{\rho}{C} = -7ax^2 \tag{2.16c}$$

$$\frac{p}{C} = 4(ax^3+1)\frac{\dot{y}}{y} + ax^2 \tag{2.16d}$$

In the above the quantity y is given by (2.15), $x = Cr^2$ and a is a constant. This solution has a simple form and is expressed completely in terms of elementary functions. The expressions given above have the advantage of simplifying the analysis of the physical features of the solution, and will assist in the description of relativistic compact bodies such as neutron stars.

Consider a relativistic sphere where $0 \leq x \leq R$. We note that the functions ν and λ have constant values at the centre $x = 0$, as do the functions ρ and p. Hence the gravitational potentials and the matter variables are finite at the centre. Since $y(x) = c_0y_1(x) + c_1y_2(x)$ is a well defined series on the interval $[0, R]$ the quantities ν, λ, ρ and p are nonsingular and continuous. If $a < 0$ then the energy density $\rho > 0$. The constants c_0 and c_1 can be chosen such that the pressure $p > 0$. The physical reasonableness of ρ and p is demonstrated in Figs. 2.1 and 2.2.

Consequently the energy density and the pressure are positive on the interval $[0, R]$. At the boundary $x = CR^2$ we must have

$$e^{-2\lambda(R)} = aC^3R^6 + 1 = 1 - \frac{2M}{R}$$

for a sphere of mass M; this ensures that the interior spacetime matches smoothly to the Schwarzschild exterior. For the speed of sound to be less than the speed of light we require that

$$0 \leq \frac{dp}{d\rho} \leq 1$$

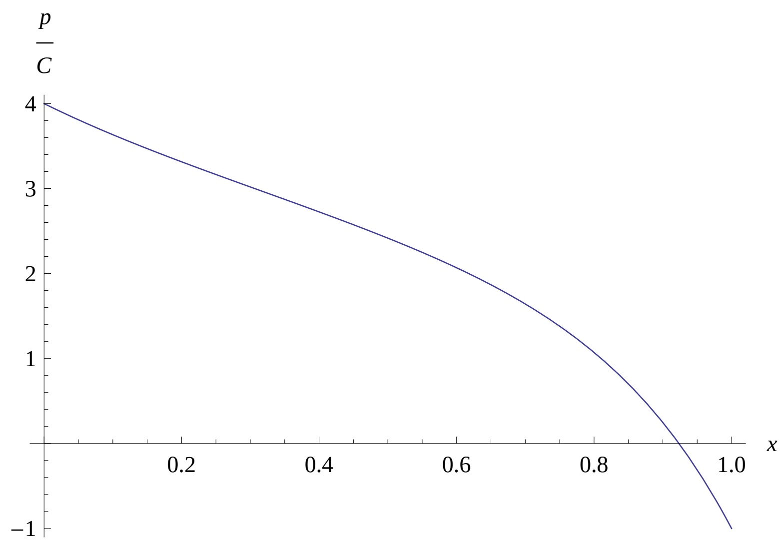

Figure 2.1: Pressure, p, vs radial coordinate, x (for $c_1 = c_0 = -a = 1$).

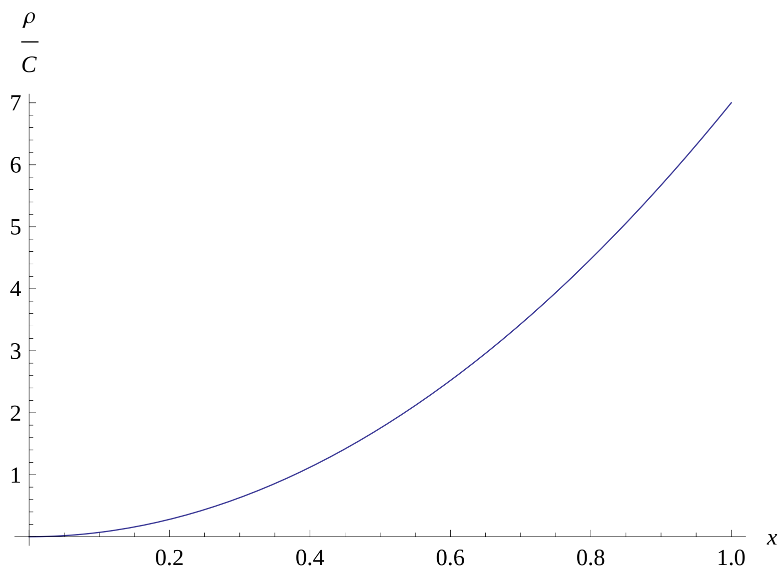

Figure 2.2: Density, ρ, vs radial coordinate, x (for $c_1 = c_0 = -a = 1$).

in our units. This inequality will constrain the values of the constants a, c_0, c_1, A and C. From this qualitative analysis we believe that the solution found can be used as a basis to describe realistic relativistic stars. We believe that a detailed physical analysis is likely to lead to realistic models for compact objects.

Our solution has the interesting feature of admitting an explicit barotropic equation of state. We observe from (2.16c) that

$$x = \sqrt{\frac{\rho}{-7aC}}, \quad a < 0$$

and the variable x can be written in terms of ρ only. The function y in (2.14) can be expressed in terms of ρ and the variable x is eliminated. Consequently the pressure p in (2.16d) is expressible in terms of ρ only, and we can write

$$p = p(\rho).$$

Thus the solution in (2.16) obeys a barotropic equation of state. This highly desirable feature is unusual for most exact solutions as pointed out in Stephani et al. (2003). For small values of x close to the stellar centre we have $y \approx c_0 + c_1 x$. Then from (2.16d) we have the approximation

$$\frac{p}{C} \approx \frac{4c_1}{c_0 + c_1\sqrt{\frac{\rho}{-7aC}}} \tag{2.17}$$

Therefore for small values of x close to stellar centre (2.17) implies that we have the approximate equation of state

$$p \propto \rho^{-1/2}$$

which is of the form of a polytrope.

We point out that the solutions presented in this paper may be extended to anisotropic matter. In recent years a number of researchers have proposed models corresponding to anisotropic matter where the radial component of the pressure differs from the angular component. The physical motivation for the analysis of anisotropic matter is that anisotropy affects the critical mass, critical surface redshift and stability of highly compact bodies. These investigations are contained in the papers of Chaisi and Maharaj (2005), Dev and Gleiser (2002, 2003), Herrera et al. (2002, 2004), Ivanov (2002), Mak and Harko (2002, 2003), among others. It appears that anisotropy may be important in fully understanding the gravitational behaviour of boson stars and the role of strange matter with densities higher than neutron stars. Mak and Harko (2002) and Sharma and Mukherjee (2002) have observed that anisotropy is a crucial ingredient in the analysis of dense stars with strange matter. The simple form of our solutions allows for extension to study such matter by adapting the energy momentum tensor to include both radial and tangential pressures.

2.5 Discussion

The condition for pressure isotropy is reduced to a recurrence equation with variable, rational coefficients of order three. We prove that this difference equation can be solved in general. Consequently we can find an exact solution to the field equations corresponding to a static spherically symmetric gravitational potential in terms of elementary functions. The metric functions, the energy density and the pressure are continuous and well behaved which implies that this solution could be used to model the interior of a relativistic sphere. The model satisfies a barotropic equation of state in general which approximates a polytrope close to the stellar centre. The approach used in this chapter has proved to be useful in other gravitational studies of compact relativistic objects. Maharaj and Thirukkanesh (2006) considered isotropic matter configurations with power law functions. Thirukkanesh and Maharaj (2006) analysed charged isotropic compact bodies. Maharaj and Komathiraj (2007) found new Einstein–Maxwell models consistent with neutron star densities. Thirukkanesh and Maharaj (2009) specified one of the gravitational potentials and the electric field to generate physically reasonable charged spheres. Clearly these, and other, investigations depend crucially on the solution of the relevant recurrence relation resulting from the method of Frobenius.

Chapter 3

Relativistic stellar models

3.1 Introduction

We obtain a class of solutions to the Einstein–Maxwell equations describing charged static spheres. Upon specifying particular forms for one of the gravitational potentials and the electric field intensity the condition for pressure isotropy is transformed into a hypergeometric equation with two free parameters. For particular parameter values we recovered uncharged solutions corresponding to specific neutron star models. We obtained two new charged solutions in terms of elementary functions for other parameter choices. The first of these solutions is physically reasonable as the metric and thermodynamic variables remain nonsingular, finite and continuous for a wide range of values of the radial coordinate. The second new solution admits a negative energy density and violates the weak and strong energy conditions.

A variety of static solutions to the Einstein–Maxwell system of field equations has been found with isotropic matter distributions with spherical symmetry. These exact solutions need to be matched at the boundary of the relativistic star to the Reissner–Nordström exterior spacetime. The matching of nonstatic charged perfect fluid spheres to the Reissner–Nordström exterior is restricted by the Bianchi identities as shown by Mahomed et al. (2003). Static exact solutions may be used to model the interior of neutron stars as indicated in the treatments of Tikekar (1990), Maharaj and Leach

(1996) and Komathiraj and Maharaj (2007). Charged stars with a spheroidal spatial geometry have been analysed by Sharma et al. (2001), Gupta and Kumar (2005) and Karmakar et al. (2007). Static charged solutions of the Einstein–Maxwell system may be used to model cold compact objects [Sharma et al. (2006)], strange matter and binary pulsars [Sharma and Mukherjee (2002)], and quark–diquark mixtures in equilibrium [Sharma and Mukherjee (2001)]. Charged relativistic matter has been shown to be consistent with the modelling of core–envelope stellar systems by Thomas et al. (2005), Tikekar and Thomas (1998) and Paul and Tikekar (2005). The recent treatments of Varela et al. (2010) and Thirukkanesh and Maharaj (2008) show that a barotropic equation of state is consistent with dark energy stars and charged quark matter. Some exact models with nonlinear quadratic equations of state have been found by Maharaj and Mafa Takisa (2012), Mafa Takisa and Maharaj (2013) and Feroze and Siddiqui (2011).

General methods have been proposed to solve the Einstein–Maxwell system for static gravitational fields. Thirukkanesh and Maharaj (2009) generated a new class of solutions in closed form by using a systematic series analysis. This approach generates a number of difference equations which must be solved explicitly from first principles. Solutions in terms of elementary functions are regainable by placing restrictions on the parameters. In this work we reduce the solutions of the field equations to a hypergeometric equation. We demonstrate that particular solutions, both with charged and uncharged matter, may be extracted from the hypergeometric equation. Some advantages of our approach compared to the series method are highlighted and we illustrate the restrictions on solutions allowed on physical grounds.

In §3.2 we list the Einstein–Maxwell equations for a charged static fluid in a spherically symmetric spacetime. This nonlinear system is transformed into a more tractable set of equations. A particular form is chosen for one of the metric potentials as well as the electric field intensity. The system can be integrated easily and the equation governing pressure isotropy is reduced to a hypergeometric differential equation with two free parameters. In §3.3 we recover two known uncharged solutions that correspond to the neutron star models of Durgapal and Bannerji (1983) and Maharaj and Mkhwanazi (1996). We then obtain two new exact solutions that describe charged static spheres. Whilst these solutions are, in principle, incorporated in the class generated by Thirukkanesh and Maharaj (2008) establishing this correspondence is non–trivial due to the series form of their solutions. Moreover it is not obvious whether an infinite series will be nonsingular, bounded and continuous. In order for a charged stellar model to be physically viable the gravitational functions must also match the Reissner–Nordström metric at the star's boundary and the sound speed must be subluminal. The solutions presented here are special cases of the hypergeometric function that can be represented by elementary functions. The simple form of our solutions greatly facilitates a detailed analysis of their physical properties. Our first new solution appears to be physically well behaved and the pressure, energy density and electric field intensity are plotted. The second solution violates the strong and weak energy conditions as the energy density remains negative throughout its domain. Our findings are summarised in §3.4.

3.2 Basic equations

The interior of a dense star is described by the line element

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{3.1}$$

in comoving coordinates $(x^a) = (t, r, \theta, \phi)$. It is convenient to introduce coordinates

$$x = Cr^2, Z(x) = e^{-2\lambda(r)}, A^2y^2(x) = e^{2\nu(r)}$$

suggested by Durgapal and Bannerji (1983). Then (3.1) can be written as

$$ds^2 = -A^2y^2dt^2 + \frac{1}{4CxZ}dx^2 + \frac{x}{C}\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{3.2}$$

where A and C are constants. Then the Einstein–Maxwell system of field equations becomes

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} + \frac{E^2}{2C} \tag{3.3a}$$

$$4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} = \frac{p}{C} - \frac{E^2}{2C} \tag{3.3b}$$

$$4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + \left(\dot{Z}x - Z + 1 - \frac{E^2x}{C}\right)y = 0 \tag{3.3c}$$

$$\frac{\sigma^2}{C} = \frac{4Z}{x}\left(x\dot{E} + E\right)^2. \tag{3.3d}$$

In the above ρ is the energy density, p is the isotropic pressure, E is the electric field intensity and σ is the charge density. Overdots indicate differentiation with respect to the variable x.

We make the particular choice

$$Z(x) = \frac{1+kx}{1+x} \tag{3.4}$$

where k is a real constant. In (3.4) we take $k \neq 1$ to avoid negative energy densities which are unphysical for barotropic stars. The choice (3.4) was also made by Maharaj and Mkhwanazi (1996) in their analysis of uncharged relativistic stars. Upon substituting (3.4) into (3.3c) we obtain

$$4(1+kx)(1+x)\ddot{y} + 2(k-1)\dot{y} + \left(1 - k - \frac{E^2(1+x)^2}{Cx}\right)y = 0. \tag{3.5}$$

When $E = 0$, equation (3.5) is valid for uncharged stars. It is now convernient to introduce a new independent variable X which helps to simplify the second order equation (3.5). The relevant transformation is given by

$$1 + x = KX, K = \frac{k-1}{k}, Y(X) = y(x).$$

Then equation (3.5) can be written as

$$X(1-X)\frac{d^2Y}{dX^2}-\frac{1}{2}\frac{dY}{dX}+\left(\frac{K}{4}+\frac{K^2(1-K)E^2X^2}{C(KX-1)}\right)Y=0. \tag{3.6}$$

We observe that (3.6) is simplified if we make the choice

$$E^2=\frac{\alpha}{4}\frac{C}{K^2(1-K)}\frac{KX-1}{X^2} \tag{3.7}$$

where α is a constant. The electric field intensity E in (3.7) vanishes at the centre of the star, and remains continuous and bounded in the interior of the star for a wide range of values of the parameter K. Thus this choice for E is physically reasonable and is a useful form to study the gravitational behaviour of charged stars. Equation (3.6) now assumes the simpler form

$$X(1-X)\frac{d^2Y}{dX^2}-\frac{1}{2}\frac{dY}{dX}+\left(\frac{K}{4}+\frac{\alpha}{4}\right)Y=0. \tag{3.8}$$

for the choice (3.7). Note that (3.8) is a special case of the hypergeometric differential equation.

3.3 Particular models

A variety of new solutions, in terms of elementary and special functions, are obtainable from (3.8) for particular values of α and K. Some values may reduce (3.8) to solutions that have already been documented. Here we regain neutral stars with no electromagnetic field.

As a first example we take $\alpha=0$ and $K=-1(\Leftrightarrow k=\frac{1}{2})$. Then (3.8) becomes

$$X(1-X)\frac{d^2Y}{dX^2}-\frac{1}{2}\frac{dY}{dX}-\frac{1}{4}Y=0. \tag{3.9}$$

This equation admits the two linearly independent hypergeometric functions

$$\begin{aligned} Y_1 &= F\left(-\frac{1}{2} - \frac{1}{2}; -\frac{1}{2}; X\right), \\ Y_2 &= X^{3/2} F\left(1, 1; \frac{5}{2}; X\right). \end{aligned}$$

It is possible to express these solutions in terms of elementary functions and we get

$$y_1(x) = (2+x)^{1/2} \tag{3.10a}$$

$$y_2(x) = (2+x)^{1/2} \ln\left[(1+x)^{1/2} + (2+x)^{1/2}\right]^2 - 2(1+x)^{1/2} \tag{3.10b}$$

in terms of the variables x and y used earlier. This solution was found previously by Maharaj and Mkhwanazi (1996). As a second example we take $\alpha = 0$ and $K = 3(\Leftrightarrow k = -\frac{1}{2})$. Then (3.8) becomes

$$X(1-X)\frac{d^2Y}{dX^2} - \frac{1}{2}\frac{dY}{dX} + \frac{3}{4}Y = 0. \tag{3.11}$$

In this case the two linearly independent hypergeometric functions are

$$\begin{aligned} Y_1 &= F\left(\frac{1}{2}, -\frac{3}{2}; -\frac{1}{2}; X\right), \\ Y_2 &= X^{3/2} F\left(2, 0; \frac{5}{2}; X\right). \end{aligned}$$

These quantities are equivalent to the elementary functions

$$y_1(x) = (2-x)^{1/2}(2x+5) \tag{3.12a}$$

$$y_2(x) = (1+x)^{3/2} \tag{3.12b}$$

which correspond to the neutron star model of Durgapal and Bannerji (1983). We have regained two exact solutions studied previously for particularly choices of the parameter K. Other values of K will correspond to new solutions of the Einstein field equations for uncharged matter.

It is possible, upon integrating the hypergeometric equation (3.8), to generate new solutions to the system (3.3) corresponding to a charged star where the hypergeometric functions can be written in terms of elementary functions. We choose the parameter values $K = -2 (\Leftrightarrow k = \frac{1}{3})$ and $\alpha = 1$. The general solution to equation (3.8) is given by

$$Y = c_1 F\left(-\frac{1}{2}, -\frac{1}{2}; -\frac{1}{2}; X\right) + c_2 X^{-3/2} F\left(1, 1; \frac{5}{2}; X\right)$$

where $F\left(-\frac{1}{2}, -\frac{1}{2}; -\frac{1}{2}; X\right)$ and $X^{-3/2} F\left(1, 1; \frac{5}{2}; X\right)$ are linearly independent hypergeometric functions and c_1 and c_2 are constants. In terms of the variables x and y we can rewrite the solution as

$$y(x) = (2+x)^{1/2}\left(c_1 + 2c_2 \ln\left[(1+x)^{1/2} + (2+x)^{1/2}\right]\right) - 2c_2(1+x)^{1/2}. \tag{3.13}$$

Then the exact solution to the Einstein–Maxwell system (3.3) is given by

$$e^{2\lambda} = \frac{1+x}{1+\frac{1}{3}x} \tag{3.14a}$$

$$e^{2\nu} = A^2\left((2+x)^{1/2}\left(c_1 + c_2 \ln\left[(1+x)^{1/2} + (2+x)^{1/2}\right]\right) - 2c_2(1+x)^{1/2}\right)^2 \tag{3.14b}$$

$$\frac{\rho}{C} = \frac{28+15x}{24(1+x)^2} \tag{3.14c}$$

$$\frac{p}{C} = \frac{-(15x+16)}{24(1+x)^2} + \frac{1}{2(2+x)^{1/2}} \times \frac{c_1 + 2c_2 \ln\left[(1+x)^{1/2} + (2+x)^{1/2}\right]}{c_1(2+x)^{1/2} - 2c_2(1+x)^{1/2} + 2c_2(2+x)^{1/2} \ln\left[(1+x)^{1/2} + (2+x)^{1/2}\right]} \tag{3.14d}$$

$$\frac{E^2}{C} = \frac{1}{12}\frac{x}{(1+x)^2} \tag{3.14e}$$

$$\frac{\sigma^2}{C} = \frac{(3+x)^3}{36(1+x)^5}. \tag{3.14f}$$

The gravitational potentials ν and λ are well behaved and continuous in the interior of the star. This is also true for the energy density ρ, the pressure, p, the electric field intensity E and the charge density σ. These quantities remain finite and nonsingular. This behaviour is depicted in Figs. 3.1, 3.2 and 3.3. In Fig. 3.3 we have plotted

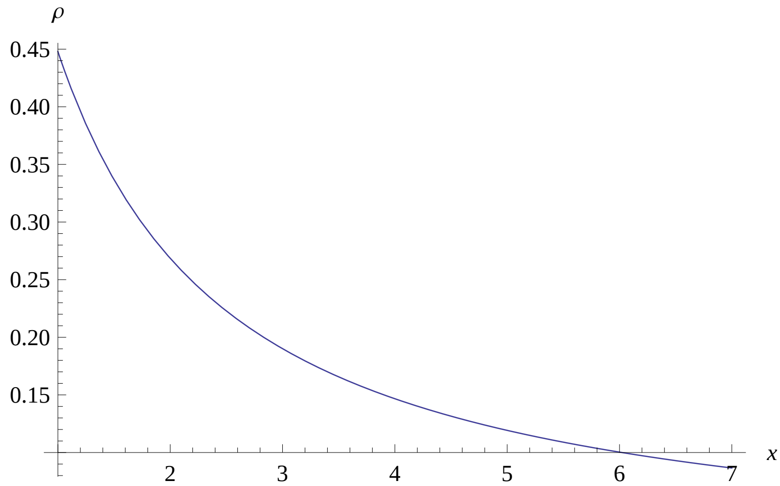

Figure 3.1: Energy density, ρ, vs radial coordinate, x (for $c_1 = c_2 = A = C = 1$)

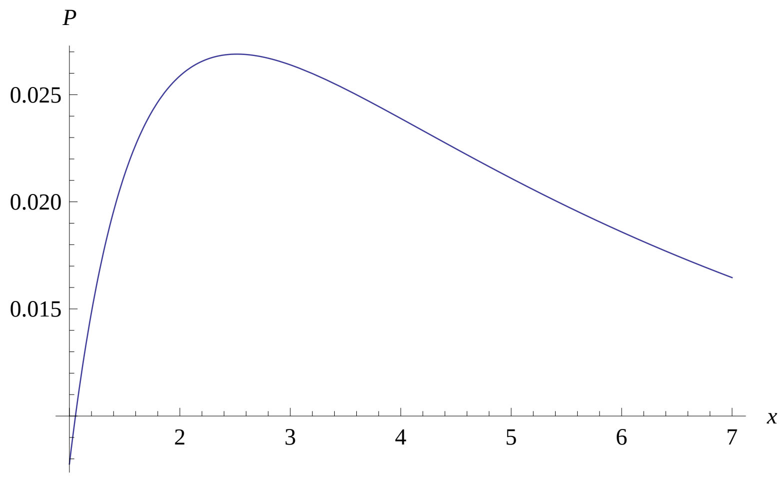

Figure 3.2: Pressure, p, vs radial coordinate, x (for $c_1 = c_2 = A = C = 1$)

the electric field intensity E^2 close to the centre and note that after reaching a local maximum the function decreases as the boundary is approached. The simple form of this solution makes a detailed analysis of the physical features of the model feasible.

It is important to observe that not all exact solutions derivable from (3.8) will be physically reasonable. For example if we take $K = \frac{1}{2}(\Leftrightarrow 2), \alpha = \frac{5}{2}$ then we find

$$Y = c_1 F\left(\frac{1}{2}, -\frac{3}{2}; -\frac{1}{2}; X\right) + c_2 X^{3/2} F\left(2, 0; \frac{5}{2}; X\right)$$

or in terms of the variables x and y:

$$y(x) = c_1(2-x)^{1/2}(2x+5) + c_2(1+x)^{3/2}. \tag{3.15}$$

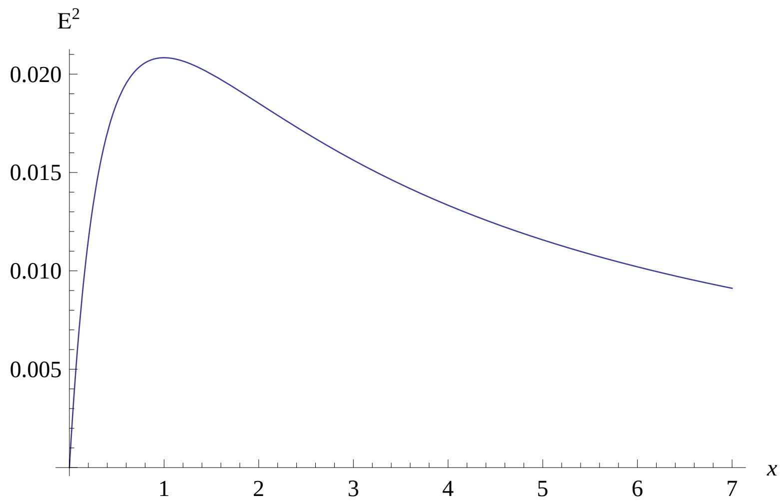

Figure 3.3: Square of electric field intensity, E^2, vs radial coordinate, x (for $c_1 = c_2 = A = C = 1$)

The energy density in this case is given by

$$\frac{\rho}{C} = -\frac{3 + \frac{13}{8}x}{(1+x)^2}.$$

Therefore the energy density ρ is negative. Consequently this solution is not very useful for matter that has to satisfy the weak and strong energy conditions. This example demonstrates the difficulty of finding Einstein–Maxwell solutions that satisfy all the conditions for physical acceptability for a dense relativistic star.

3.4 Discussion

The solution of the Einstein–Maxwell system of field equations describing charged static spheres was reduced to solving the equation governing pressure isotropy. Upon specifying one of the gravitational potentials, $Z(x)$, and the electric field intensity, $E(x)$, this equation can be solved in terms of hypergeometric functions with two free parameters, K and α. For specific parameter values the hypergeometric functions reduce to elementary functions. Charged and uncharged solutions were extracted. In particular, the uncharged solutions of Maharaj and Mkhwanazi (1996) and Durgapal and Bannerji (1983) were recovered by setting $\alpha = 0, K = -1$ and $\alpha = 0, K = 3$

respectively.

Two new charged solutions were obtained. For $\alpha = 1, K = -2$ we obtained an exact solution in terms of algebraic functions. The gravitational potentials, pressure, energy density, electric field intensity and charge density are non–singular, finite and continuous over a wide range of values of x, the transformed radial coordinate. This behaviour is depicted in Figs 3.1 and 3.2. The behaviour of the electric field intensity is physically acceptable as illustrated in Fig. 3.3.

The other new charged solution was found by setting $\alpha = \frac{5}{2}, K = \frac{1}{2}$. The corresponding energy density is negative and this model is unphysical as it violates the weak and strong energy conditions. This caveat serves as a reminder to practitioners that not all exact solutions of the Einstein–Maxwell system are physically reasonable despite the under–determined character of this problem.

Our technique employed in this chapter can be regarded as complementary to the series solution method used by Thirukkanesh and Maharaj (2009). The simple, algebraic form of the exact solutions admitted by this treatment simplifies the task of determining the physical viability of these stellar models. We believe that this method can be generalised for other specified forms of the gravitational potential.

Chapter 4

Accretion onto a higher–dimensional black hole

4.1 Introduction

The problem of spherical accretion [Frank et al. (2002), Shu (1992)] of a perfect fluid onto a Schwarzschild black hole [Shapiro and Teukolsky (1983), Michel (1972)] is generalised to D–dimensions. In a seminal paper Bondi (1952) solved the problem of a polytropic gas accreting onto a central object under the influence of gravity. This work generalises the earlier results of Bondi and Hoyle (1944) and Hoyle and Lyttleton (1939) which investigated pressure–free gas being dragged onto a massive central object. There has been some confusion in distinguishing these cases in the literature. The latter case is usually referred to as Lyttleton–Hoyle accretion whilst the former is termed Bondi accretion. The key distinction between the two cases is that the gas and the accretor are in the same inertial rest–frame in Bondi accretion whilst in Lyttleton–Hoyle accretion the gas has a finite velocity at infinity (see Edgar (2004)). Both studies are performed in the regime of Newtonian gravity. Detailed treatments of the accretion problem can be found in any of the standard texts by Shapiro and Teukolsky (1983), Shu (1992) or Frank et al. (2002) The first study of accretion in a general relativistic context was undertaken by Michel (1972). This was followed by more comprehensive treatments determining the luminosity and frequency spectrum [Shapiro (1973a)] the influence of an interstellar magnetic field on the accretion of ionized gas [Shapiro (1973b)] and accretion onto a rotating black hole [Shapiro (1974)]. Accretion onto a charged black hole was considered in Michel (1972) and more fully investigated by de Freitas Pacheco (2011). Spherical winds and shock transitions were studied in Blumenthal and Mathews (1976).

Higher–dimensional accretion onto TeV black holes was studied in Giddings and Mangano (2008). Their treatment, however, was restricted to D–dimensional Newtonian gravity. Higher dimensional theories arise from extensions to the standard model of particle physics that are believed to lead to the unification of all four fundamental forces.

Since general relativity, and indeed Newtonian gravity, is a classical, low energy theory it is unclear whether it will still provide an accurate description of gravitational interactions at extremely high energies. Semi–classical theories simply assume the validity of some gravitational theory at high energies. This is the rationale behind exploring general relativity in higher dimensional spacetimes. Accordingly we extend the analysis of Giddings and Mangano (2008) to D–dimensional general relativity which may be a more appropriate gravity model for TeV black holes. Other gravitational theories have been postulated, and Lovelock and Gauss–Bonnet gravity in particular have been demonstrated to be low energy limits of various string theories. A future extension of this work will be to examine accretion onto higher dimension black holes in these particular theories.

The accretion of phantom matter in 5–dimensions was studied by Sharif and Abbas (2011). In this chapter we restrict our attention to steady, spherical accretion of the more conventional polytropic gas onto a point mass in D–dimensional general relativity.

4.1.1 Units and conventions

We use the following values for physical constants and the accreting system's parameters:

$$\begin{aligned}
c &= 3.00 \times 10^{10}\mathsf{cm.s}^{-1} \\
G &= 6.674 \times 10^{-8}\mathsf{cm}^3.\mathsf{g.}^{-1}\mathsf{s}^{-2} \\
k_B &= 1.380 \times 10^{-16}\mathsf{erg.K}^{-1} \\
M = M_\odot &= 1.989 \times 10^{33}\mathsf{g} \\
m = m_p &= 1.67 \times 10^{-28}\mathsf{g} \\
a_\infty &= 10^{6}\mathsf{cm.s}^{-1} \\
n_\infty &= 1\mathsf{cm}^{-3} \\
T_\infty &= 10^{2}\mathsf{K}
\end{aligned} \tag{4.1}$$

4.2 Basic equations

The spacetime exterior to a Schwarzschild black hole in D–dimensions is described by the line element

$$ds^2 = -A(r)dt^2 + A^{-1}(r)dr^2 + r^2 d\Omega^2_{(D-2)} \tag{4.2}$$

where

$$A(r) := \left(1 - \frac{2M}{(D-3)r^{D-3}}\right) \tag{4.3}$$

and $d\Omega^2_{(D-2)}$ is the line element on a unit $D-2$ sphere, viz.

$$d\Omega^2_{(D-2)} = d\theta^2_{(1)} + \sum_{n=2}^{D-2} d\theta^2_{(n)} \left(\prod_{m=2}^{n} \sin^2\theta_{(m-1)}\right) \tag{4.4}$$

We use comoving coordinates $(x^a) = (t, r, \theta_1, \theta_2, \cdots, \theta_{D-2})$ and units where $c = 1$ and $(D-3)G_D = 1$. The D–dimensional Newton's constant is defined as

$$G_D = V_{D-4}G = \frac{2\pi^{(D-4)/2}}{\Gamma((D-4)/2)} \frac{r^{D-4}}{D-4} G$$

where Γ is the gamma function. Equation (3.1) is adapted from an extension of the Schwarzschild black hole into D–dimensions [Tangherlini (1963)].

We consider the steady–state radial inflow of gas onto the central mass M. The gas is approximated as a perfect fluid described by the energy–momentum tensor

$$T^{ab} = (\rho + p)\, u^a u^b + p g^{ab} \tag{4.5}$$

where the fluid D–velocity is

$$u^a = dx^a/ds \tag{4.6}$$

and ρ and p are the fluid proper energy density and pressure respectively. We also define the proper density of rest mass n and the flux of rest mass $J^a = nu^a$. All these quantities are evaluated in the local inertial rest frame of the fluid. The spacetime curvature is dominated by the compact object and we ignore the self–gravity of the fluid. If no particles are created or destroyed then particle number is conserved and

$$\nabla_a J^a = \nabla_a (nu^a) = 0 \tag{4.7}$$

where ∇_a denotes the covariant derivative with respect to the coordinate x^a. Conservation of energy and momentum is governed by

$$\nabla_a T^a_b = 0. \tag{4.8}$$

We define the radial component of the D–velocity, $v(r) := u^1 = dr/ds$. Since $u_a u^a = -1$ and the velocity components vanish for $a > 1$ we have

$$(u^0)^2 = \frac{v^2 + A}{A^2}. \tag{4.9}$$

Equation (4.7) for our problem is

$$\frac{1}{r^{D-2}} \frac{d}{dr} \left(r^{D-2} n v \right) = 0. \tag{4.10}$$

The $a = 0$ component of (4.8) is

$$\frac{1}{r^{D-2}}\frac{d}{dr}\left(r^{D-2}(\rho+p)v\left(A+v^2\right)^{1/2}\right) = 0. \tag{4.11}$$

The $a = 1$ component can be simplified to

$$v\frac{dv}{dr} = -\frac{dp}{dr}\frac{A+v^2}{\rho+p} - \frac{M}{r^{D-2}}. \tag{4.12}$$

These expressions generalise those obtained by Michel (1972) for spherical accretion onto a Schwarzschild black hole. Those equations are naturally recovered when $D = 4$:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 nv\right) = 0 \tag{4.13a}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2(\rho+p)v\left(\tilde{A}+v^2\right)^{1/2}\right) = 0 \tag{4.13b}$$

$$v\frac{dv}{dr} = -\frac{dp}{dr}\frac{\tilde{A}+v^2}{\rho+p} - \frac{M}{r^2} \tag{4.13c}$$

where

$$\tilde{A} := 1 - \frac{2M}{r^2}.$$

4.3 Analysis

In the spirit of the original calculation [Bondi (1952)] we obtain the mass accretion rate from a qualitative analysis of (4.10) and (4.12). For an adiabatic fluid there is no entropy production and the conservation of energy is governed by

$$Tds = 0 = d\left(\frac{\rho}{n}\right) + pd\left(\frac{1}{n}\right) \tag{4.14}$$

which implies the relation

$$\frac{d\rho}{dn} = \frac{\rho+p}{n}. \tag{4.15}$$

The adiabatic sound speed, a, is

$$\begin{aligned} a^2 &:= \frac{dp}{d\rho} \\ &= \frac{dp}{dn}\frac{n}{\rho+p}. \end{aligned} \tag{4.16}$$

We rewrite the continuity and momentum equations as

$$\frac{1}{v}v' + \frac{1}{n}n' = -\frac{(D-2)}{r} \tag{4.17}$$

$$vv' + (A+v^2)\frac{a^2}{n}n' = -\frac{M}{r^{D-2}} \tag{4.18}$$

This implies

$$\begin{aligned} v' &= \frac{N_1}{N} \\ n' &= -\frac{N_2}{N} \end{aligned} \tag{4.19}$$

where

$$N_1 = \frac{1}{n}\left((A+v^2)\frac{(D-2)a^2}{r} - \frac{M}{r^{D-2}}\right), \tag{4.20a}$$

$$N_2 = \frac{1}{v}\left((D-2)\frac{v^2}{r} - \frac{M}{r^{D-2}}\right) \tag{4.20b}$$

$$N = \frac{v^2-(A+v^2)a^2}{vn}. \tag{4.20c}$$

At large r we demand the flow be subsonic i.e. $v < a$ and since the sound speed is always subluminal i.e. $a < 1$ and we have $v^2 \ll 1$. The denominator is thus

$$N \approx \frac{v^2-a^2}{vn} \tag{4.21}$$

and so $N > 0$ as $r \to \infty$. At the event horizon $r_H = (\frac{2M}{D-3})^{1/D-3}$ and

$$N = \frac{v^2(1-a^2)}{vn}. \tag{4.22}$$

Since $a < 1$ we have $N < 0$, and we must have $N = 0$ for some r_s where $r_H < r_s < \infty$. In order to avoid discontinuities in the flow we must have $N = N_1 = N_2 = 0$ at $r = r_s$ i.e.

$$N_1 = \frac{1}{n_s}\left((A_s + v_s^2)\frac{(D-2)a_s^2}{r_s} - \frac{M}{r_s^{D-2}}\right) = 0 \tag{4.23a}$$

$$N_2 = \frac{1}{v_s}\left((D-2)\frac{v_s^2}{r_s} - \frac{M}{r_s^{D-2}}\right) = 0 \tag{4.23b}$$

$$N = \frac{v_s^2 - (A_s + v_s^2)a_s^2}{v_s n_s} = 0 \tag{4.23c}$$

where $v_s := v(r_s), a_s := a(r_s)$ etc. At the critical point that satisfies equations (4.23) we have

$$v_s^2 = \frac{a_s^2}{1 + (D-1)/(D-3)a_s^2} \tag{4.24}$$

$$= \frac{M}{D-2}\frac{1}{r_s^{D-3}}. \tag{4.25}$$

To obtain the mass accretion rate we write the continuity equation explicitly in the form of a conservation equation. Integrating equation (4.10) over a $(D-1)$–dimensional volume we obtain

$$\frac{2\pi^{(D-1)/2}}{\Gamma\left((D-1)/2\right)} r^{D-2} mnv = \dot{M} \tag{4.26}$$

where $\dot{M}$ is an integration constant, independent of r, having dimensions of mass per unit time. $\dot{M}$ is the higher dimensional generalization of the Bondi accretion rate. Equations (4.10) and (4.11) can be combined to yield

$$\left(\frac{\rho + p}{n}\right)^2 \left(1 - \frac{2M}{(D-3)r^{D-3}} + v^2\right) = \left(\frac{\rho_\infty + p_\infty}{n_\infty}\right)^2, \tag{4.27}$$

which is the D–dimensional generalization of the relativistic Bernoulli equation. We now introduce the polytrope equation of state

$$p = Kn^\gamma \tag{4.28}$$

where the adiabatic index γ satisfies $1 < \gamma < \frac{5}{3}$. The energy equation (4.14) can be integrated to obtain

$$\rho = \frac{K}{\gamma - 1} n^\gamma + mn \tag{4.29}$$

where mn is the rest–energy density. Using (4.16) we re–write the Bernoulli equation (4.27) as

$$\left(1 + \frac{a^2}{\gamma - 1 - a^2}\right)^2 \left(1 - \frac{2M}{(D-3)r^{D-3}} + v^2\right) = \left(1 + \frac{a_\infty^2}{\gamma - 1 - a_\infty^2}\right)^2. \tag{4.30}$$

At the critical point r_s this must satisfy

$$\left[\frac{(D-3) + (D-1)a_s^2}{(D-3)}\right] \left(1 - \frac{a_s^2}{\gamma - 1}\right)^2 = \left(1 - \frac{a_\infty^2}{\gamma - 1}\right)^2 \tag{4.31}$$

where we have used the critical velocity and sound speed viz. equations (4.24) and (4.25). For large, but finite r i.e. $r \geq r_s$ the baryons should still be non–relativistic i.e. $T \ll mc^2/k = 10^{13} K$ for neutral hydrogen. In this regime we expect $a \leq a_s \ll 1$. Expanding (4.31) to leading order in a_s and a_∞ we obtain

$$a_s^2 \approx \frac{2(D-3)}{(3D-7) - \gamma(D-1)} a_\infty^2. \tag{4.32}$$

We thus obtain the critical radius r_s in terms of the black hole mass M and the boundary condition a_∞:

$$\begin{aligned} r_s^{D-3} &= \frac{M}{D-2} \frac{1 + (D-1)/(D-3)a_s^2}{a_s^2} \\ &\approx \left[\frac{(3D-7) - (D-1)\gamma}{2(D-2)(D-3)}\right] \frac{M}{a_\infty^2} \end{aligned} \tag{4.33}$$

Reintroducing the normalised constants this reads,

$$r_s^{D-3} \approx \left[\frac{(3D-7) - (D-1)\gamma}{2(D-2)(D-3)}\right] \frac{(D-3)G_D M}{a_\infty^2} \tag{4.34}$$

where $G_D = V_{D-4}G = \frac{2\pi^{(D-4)/2}}{\Gamma((D-4)/2)}\frac{r^{D-4}}{D-4}G$ is the D–dimensional Newton's constant. From (4.16), (4.28) and (4.29) we have

$$\gamma K n^{\gamma-1} = \frac{ma^2}{1 - a^2/(\gamma-1)}. \tag{4.35}$$

For $a^2/(\gamma-1) \ll 1$ we have $n \sim a^{2/(\gamma-1)}$ and

$$\frac{n_s}{n_\infty} \approx \left(\frac{a_s}{a_\infty}\right)^{2/(\gamma-1)}. \tag{4.36}$$

We are now in a position to evaluation the accretion rate, $\dot{M}$. Since $\dot{M}$ is independent of r, equation (4.26) must also hold for $r = r_s$. We use the critical point to determine the D–dimensional Bondi accretion rate,

$$\begin{aligned}\dot{M} &= \frac{2\pi^{(D-1)/2}}{\Gamma\left[(D-1)/2\right]} r^{D-2} mnv \\ &= \frac{2\pi^{(D-1)/2}}{\Gamma\left[(D-1)/2\right]} r_s^{D-2} mn_s v_s \\ &= \frac{2\pi^{(D-1)/2}}{\Gamma\left[(D-1)/2\right]} \lambda m n_\infty M^{(D-2)/(D-3)} a_\infty^{(1-D)/(D-3)}\end{aligned} \tag{4.37}$$

where we have defined the dimensionless accretion eigenvalue

$$\lambda := \left(\frac{1}{D-2}\right)^{(D-2)/(D-3)} \left[\frac{(3D-7)-(D-1)\gamma}{2(D-3)}\right]^{-[(3D-7)-(D-1)\gamma]/[2(D-3)(\gamma-1)]}. \tag{4.38}$$

We rewrite the D–dimensional accretion rate explicitly in terms of G_D, the gravitational constant,

$$\dot{M} = \frac{2\pi^{(D-1)/2}}{\Gamma\left[(D-1)/2\right]} \lambda m n_\infty \left[(D-3)G_D M\right]^{(D-2)/(D-3)} a_\infty^{(1-D)/(D-3)} \tag{4.39}$$

Note that the accretion rate scales as $\dot{M} \sim M^{(D-2)/(D-3)}$. This extends the familiar result of Bondi (1952) where $\dot{M} \sim M^2$ and suggests potentially observable hints of the presence of higher dimesions.

4.4 Asymptotic behaviour

We obtain the flow characteristics for $r_h < r \ll r_s$ and at the event horizon $r = r_h$.

4.4.1 Sub–Bondi radius

The gas is supersonic at distances below the Bondi radius so $v > a$ when $r_h < r \ll r_s$. From (4.30) we obtain an upper bound on the radial dependence of the gas velocity viz.

$$v^2 \approx \frac{2M}{(D-3)r^{D-3}}. \tag{4.40}$$

We now estimate the gas compression on these scales using (4.26), (4.39) and (4.40):

$$\frac{n(r)}{n_\infty} = \lambda \left(\frac{D-3}{2}\right)^{1/2} \left[\frac{(D-3)G_D M}{a_\infty^2 r^{D-3}}\right]^{(D-1)/2(D-3)}. \tag{4.41}$$

Assuming a Maxwell–Boltzmann gas, $p = nk_BT$, we find the adiabatic temperature profile using (4.28) and (4.41):

$$\frac{T(r)}{T_\infty} = \lambda^{\gamma-1} \left(\frac{D-3}{2}\right)^{\gamma-1/2} \left[\frac{(D-3)G_D M}{a_\infty^2 r^{D-3}}\right]^{(D-1)(\gamma-1)/2(D-3)}. \tag{4.42}$$

4.4.2 Event horizon

At the event horizon $r = r_H = \left(\frac{2M}{D-3}\right)^{1/(D-3)}$. Since the flow is supersonic as we are well below the Bondi radius, the fluid velocity is still described by $v^2 \approx \frac{2M}{(D-3)r^{D-3}}$. At r_H, $v_H^2 := v^2(r_H) \approx 1$, i.e. the flow speed at the horizon equals the speed of light. Using (4.26), (4.39) and (4.40) we obtain the gas compression at the event horizon:

$$\frac{n_H}{n_\infty} = \lambda \left(\frac{D-3}{2}\right)^{(D-2)/(D-3)} \left(\frac{c}{a_\infty}\right)^{(D-1)/(D-3)}. \tag{4.43}$$

Again assuming a Maxwell–Boltzmann gas, $p = nk_B T$, we find the adiabatic temperature profile at the event horizon using (4.28) and (4.43):

$$\frac{T_H}{T_\infty} = \left[\lambda \left(\frac{D-3}{2}\right)^{(D-2)/(D-3)} \left(\frac{c}{a_\infty}\right)^{(D-1)/(D-3)}\right]^{\gamma-1} \tag{4.44}$$

4.5 Discussion

We have determined the Bondi radius and accretion rate for a polytropic gas accreting onto a D–dimensional Schwarzschild black hole. Our expressions are fully general–relativistic and can be compared to the higher dimensional Newtonian terms obtained by Giddings and Mangano (2008). We have not considered compactification of higher dimensions and leave this as a future project. Upper bounds for higher dimensions have been established in the literature and their effects on black hole accretion, as well as other physical processes, will be restricted to the compactification scale. Beyond this length scale we expect conventional 4–dimensional physics to dominate. The luminosity, frequency spectrum and energy conversion efficiency of D–dimensional accretion should also be determined. The effects of black hole rotation and the presence of magnetic fields can also be included. At the energy levels relevant to higher dimensional black holes, the environment may consist of more exotic matter than polytropic gases. Scalar field accretion, for example, could be investigated. In addition it is unclear whether Einstein gravity is the appropriate low energy limit of higher–dimensional theories. In this light an investigation of accretion in Einstein–Gauss–Bonnet and Lovelock gravity may be instructive.

Chapter 5

Chaplygin gas accretion

5.1 Introduction

In this chapter we consider the accretion of a generalised Chaplygin gas (GCG) onto a star. We determine the critical point of the gas flow and obtain the mass accretion rate, $\dot{M}$. We then examine the behaviour of the gas when it experiences a shock. After obtaining the Rankine–Hugoniot conditions we determine the post–shock behaviour of the gas in terms of its pre–shock properties. The flow properties during a shock are then obtained.

We treat the gas as a fluid and the terms will be used interchangeably. It should be evident that we are not attempting a kinetic theory formulation of Chaplygin gas dynamics but are content with regarding the gas as a fluid.

5.1.1 Chaplygin gases and their generalisations

The discovery of the late–time acceleration of the universe led to the coincidence problem [Zlatev et al. (1999)] viz. why does dark energy start to dominate the energy budget of the universe at low redshifts? Invoking the cosmological constant as a candidate is problematic due to its tenuous interpretation as the vacuum energy density, a quantity predicted to be 120 orders of magnitude larger than the observed dark energy density!

Quintessence models prescribe a class of dynamically evolving scalar fields to account for dark energy. These models are, however, generically plagued by the need to fine tune the scalar field potentials at the time of matter–radiation equality.

The coincidence problem may be resolved by a change in the dark energy equation of state. This has the advantage of obviating the need to fine tune model parameters. The generalised Chaplygin gas (GCG), proposed by Bento et al. (2002), is a fluid with a dynamical equation of state given by

$$p = -\frac{A}{\rho^{\alpha}} \tag{5.1}$$

where A is a positive constant and α is a dimensionless parameter confined to the range $0 \leq \alpha \leq 1$. The pressure, p, and the energy density, ρ, are functions of the scale factor, $a(t)$. The GCG has a number of highly attractive features. It smoothly interpolates between an early, dust–dominated phase, where $\rho \sim a^{-3}$, and a later de Sitter phase where, $p \sim \rho$. In the intervening period the universe experiences a stiff phase, where $p = \rho$. The case $\alpha = 1$ corresponds to the original Chaplygin gas. Bilić et al. (2002) developed an inhomogenous generalisation of the Chaplygin gas which may provide a unifying account of dark matter and dark energy that remains consistent with structure formation scenarios. The equation of state arising from the action of a Nambu–Goto d–brane in a $(d+1,1)$–dimensional spacetime has the same form as that of a GCG [Bento et al. (2002)]. The Chaplygin gas is the only gas known to admit a supersymmetric generalisation [Jackiw (2000)]. The GCG thus provides a promising avenue to investigate the phenomenology of string and brane–inspired cosmologies.

Bhattacharya and Debnath (2012) studied the thermodynamic behaviour of a generalisation of the GCG called the modified Chaplygin gas (MCG). This has the equation of state

$$p = -\frac{A}{\rho^{\alpha}} + B\rho \tag{5.2}$$

where A and B are positive constants. A number of authors have studied the accretion of Chaplygin gases. Bhadra and Debnath (2012) investigated Schwarzschild and Kerr–

Newman black hole accretion of two extended classes, viz. the new variable modified Chaplygin gas (NVMCG) and the generalized cosmic Chaplygin gas (GCCG). The NVMCG has the equation of state

$$p = A(a) - \frac{B(a)}{\rho} \tag{5.3}$$

where the coefficients A and B are functions of the scale factor $a(t)$. The GCGG obeys the equation of state

$$p = -\frac{1}{\rho^{\alpha}} \left[C + \left(\rho^{1+\alpha} - C \right)^{-w} \right] \tag{5.4}$$

where the constant C is defined as

$$C = \frac{A}{1+w} - 1. \tag{5.5}$$

Babichev et al. (2011) studied the accretion of a Chaplygin gas onto a Reissner–Nordström black hole. If a naked singularity is present here then no steady accretion is possible and the gas forms a static fluid atmosphere.

Jamil (2009) looked at a GCG with a bulk viscous pressure contribution, i.e. $p = \frac{A}{\rho^{\alpha}} + \Pi$, where Π is the bulk viscous term. This gas exhibits phantom–like behaviour and will lead to a decrease in the mass of the accreting black hole provided the generalised second law of thermodynamics (GSL) is broken. If the GSL remains true then the black hole mass will increase.

Kremer (2003) investigated the dynamics of a Chaplygin gas model of dark energy while Zhai et al. (2006) considered a GCG with bulk viscosity as a dark energy candidate.

5.1.2 Constraints on Chaplygin gas models

Bedran et al. (2008) investigated the temperature evolution of a universe sourced by a MCG. The current temperature of the microwave background, T_0, and its temperature at decoupling, T_{dec}, imply that the Chaplygin parameter should satisfy $\alpha = \frac{1}{4}$.

Bento et al. (2003) obtained constraints on the GCG via the locations of the peaks and troughs of the CMB power spectrum from WMAP and BOOMERanG data. In order for the sound speed to remain bounded by the speed of light we require $0 < \alpha \leq 1$. Sandvik et al. (2004) point out that a GCG cannot, by itself, account for the observed matter power spectrum. Beca et al. (2003) however included an additional baryonic matter component and succeeded in reproducing the 2dF large scale structure data. Silva and Bertolami (2008) determined the expected constraints on the GCG from future Type 1a supernovae and gravitational lensing data. Bento et al. (2003) show that the existing data from the CMB is sensitive to the values of the Hubble parameter, h, and spectral index, n_s. The derived constraint $\alpha \leq 0.6$ rules out a pure Chaplygin gas model which is specified by $\alpha = 1$. Similar conclusions were reached by Amendola et al. (2003) using a more comprehensive likelihood analysis of WMAP data.

The adiabatic accretion of dark matter on black hole seeds in galaxy haloes was examined by Peirani and de Freitas Pacheco (2008). Under typical dark halo conditions the critical radius, r_s, is about $30 - 150$ times larger than the event horizon radius, r_h, and the accretion rate is about 5 times larger than the Bondi accretion rate for non–relativistic and non–interacting particles. Cold dark matter thus comprises at most 10% of the mass accreted on to a halo black hole and the bolometric quasar luminosity function is largely determined by the baryonic accretion history.

The accretion of dark matter onto intermediate–mass black holes (IMBHs) was investigated by Pepe et al. (2012). IMBHs are hypothetical objects with masses of the order $(10^2 - 10^4)\, M_\odot$. These objects were proposed as means of accounting for the super–Eddington luminosities of ultraluminous X–ray sources (ULXs) and are believed to exist in the centre of globular clusters. Pepe et al. (2012) showed that if dark matter is collisionless or has a very low sound speed the accretion rate no longer scales like $\dot{M} \sim M^2$ as in the original case [Bondi (1952)] but rather as the square of the mass enclosed by the critical radius. Since this radius is often well outside the cluster core the mass can be much greater than the IMBH mass and the accretion rate is, accordingly, enhanced by factors of $10^4 - 10^6$. This larger accretion rate will lead to IMBHs with masses much greater than $10^4 M_\odot$. These masses are well beyond the upper limits determined from observations. Pepe et al. (2012) conclude that either IMBHs do not exist, or dark matter must have a sound speed of at least the order $a \sim 10 km.s^{-1}$. Guzmán and Lora-Clavijo (2011) reach the same conclusion using numerical simulations of time–dependent accretion on to supermassive black holes. Conclusive evidence of the existence of IMBHs has implications for the nature of dark matter. In particular the quoted lower bound of the sound speed must be obeyed by the collisional dark matter candidates like the GCG.

The possibility of collisional dark matter should be thoroughly investigated and in this regard we further consider the formation of shocks as a GCG is accreted. We use the Newtonian theory of gravity as it has a wide range of applicability to astrophysical objects and permits shocked flows when the supersonic gas impacts on the stellar surface.

5.2 The generalised Chaplygin gas

Consider the barotropic equation of state

$$p = -\frac{A}{\rho^\alpha} \tag{5.6}$$

where $A > 0$ and α is a dimensionless parameter confined to the range $0 < \alpha \leq 1$. Here p is the pressure and we take ρ to be the mass density. These are both functions of position, r, and (5.6) is an inhomogeneous fluid. The equation of state (5.6) behaves in a similar manner to the GCG, (5.1), defined by Bento et al. (2002). This motivates our use of the name GCG for the fluid defined above. When $\alpha = 1$ this describes the original Chaplygin gas. The parameter α allows one to interpolate between cold dark matter and dark energy. When $\alpha = 0$ the pressure is a negative constant and the Chaplygin gas mimics the behaviour of dark energy and may be used to describe the late time acceleration of the universe on very large scales. When $\alpha = 1$ the pressure tends to vanish at very high densities. This is characteristic of cold dark matter, a pressure–free fluid which clumps and creates steep gravitational potentials resulting in the formation of large scale structure, clusters and galaxies. The GCG provides a relatively simple means to unify the dark sector of the universe.

The sound speed, a, is defined by

$$
\begin{aligned}
a^2 &:= \frac{dp}{d\rho} &\text{(5.7a)}\\
&= \alpha\frac{A}{\rho^{\alpha+1}} &\text{(5.7b)}\\
&= -\alpha\frac{p}{\rho} &\text{(5.7c)}
\end{aligned}
$$

and is always positive. The specific internal energy, ε, can be found from the first law of thermodynamics, viz.

$$d\varepsilon + pdV = TdS \tag{5.8}$$

Here V is the specific volume, T is the temperature, S is the specific entropy and ε has dimensions of $\mathsf{energy.mass}^{-1}$. If the gas flow is adiabatic there is no entropy generation and $dS = 0$. After rewriting the specific volume, V, in terms of the density, $\rho = V^{-1}$, equation (5.8) reads

$$d\varepsilon - \frac{p}{\rho^2}d\rho = 0. \tag{5.9}$$

We introduce the GCG equation of state and integrate (5.9) to find the specific internal energy

$$
\begin{aligned}
\varepsilon &= \frac{1}{\alpha+1}\frac{A}{\rho^{\alpha+1}} &\text{(5.10a)}\\
&= -\frac{1}{\alpha+1}\frac{p}{\rho}. &\text{(5.10b)}
\end{aligned}
$$

The specific enthalpy, w, follows from the thermodynamic relation

$$dw = TdS + Vdp \tag{5.11}$$

which, for a GCG, is given by

$$
\begin{aligned}
w &= -\frac{\alpha}{\alpha+1}\frac{A}{\rho^{\alpha+1}} && (5.12a)\\
&= \frac{\alpha}{\alpha+1}\frac{p}{\rho}. && (5.12b)
\end{aligned}
$$

5.3 Accretion

Consider the steady, spherically symmetric accretion of the GCG onto a body of mass, M. The gas velocity u, pressure p and mass density ρ are functions of distance, r, from the central mass. The continuity and Euler equations for the flow are

$$
\begin{aligned}
\frac{1}{r^2}\frac{d}{dr}\left(\rho u r^2\right) &= 0 && (5.13)\\
\rho u \frac{du}{dr} &= -\frac{dp}{dr} - \rho\frac{GM}{r^2}. && (5.14)
\end{aligned}
$$

These encompass conservation of mass and momentum in a Newtonian accreting system. If the gas is at rest far away from the body we have the boundary condition

$$u_\infty = 0. \tag{5.15}$$

The gravitational field of the mass, M, accelerates the gas from zero initial velocity. The self–gravity of the gas is negligible and the mass gain of the central object is insignificant. By invoking the identity

$$
\begin{aligned}
\frac{dp}{dr} &= \frac{dp}{d\rho}\frac{d\rho}{dr}\\
&= a^2\frac{d\rho}{dr}
\end{aligned}
$$

we can re–write (5.13) and (5.14) as

$$
\begin{aligned}
\frac{u'}{u} + \frac{\rho'}{\rho} &= -\frac{2}{r} && (5.16)\\
uu' + a^2\frac{\rho'}{\rho} &= -\frac{GM}{r^2} && (5.17)
\end{aligned}
$$

where primes denotes derivatives with respect to r. We obtain a formal solution of this system, viz.

$$u' = \frac{u}{u^2 - a^2}\left(\frac{2a^2}{r} - \frac{GM}{r^2}\right) \tag{5.18}$$

$$\rho' = \frac{\rho}{u^2 - a^2}\left(\frac{-2u^2}{r} + \frac{GM}{r^2}\right). \tag{5.19}$$

We seek the solution where the flow velocity, $u(r)$, reaches its local sound speed, $a(r)$, at some critical point $r = r_s$. This critical point is hence known as the sonic point of the flow and is defined by the condition

$$\left(u^2 - a^2\right)|_{r_s} = 0 \tag{5.20}$$

at $r = r_s$. In order to avoid singularities in the flow we impose the following conditions

$$\frac{2a^2}{r} - \frac{GM}{r^2} = 0 \tag{5.21}$$

$$-\frac{2u^2}{r} + \frac{GM}{r^2} = 0 \tag{5.22}$$

at the sonic point $r = r_s$. These sonic point conditions imply that

$$u_s^2 = a_s^2 = \frac{GM}{2r_s} \tag{5.23}$$

where $u_s := u(r_s)$ and $a_s := a(r_s)$.

5.3.1 Sonic point evaluation via the Bernoulli equation

We obtain explicit expressions for the sonic point, r_s, and critical velocity, $u_s(= a_s)$, by integrating the Euler equation (5.14):

$$\begin{aligned} u\frac{du}{dr} + \frac{1}{\rho}\frac{dp}{dr} + \frac{GM}{r^2} &= 0 \\ u\frac{du}{dr} + \frac{a^2}{\rho}\frac{d\rho}{dr} + \frac{GM}{r^2} &= 0 \\ \frac{1}{2}u^2 + \int \frac{a^2}{\rho}d\rho - \frac{GM}{r} &= E \end{aligned} \tag{5.24}$$

The integration constant E is independent of r and equation (5.24) is in essence the Bernoulli equation of fluid dynamics. At this stage we have not used any properties peculiar to the GCG, and (5.24) is quite general. The nature of the gas being accreted will enter the problem via the sound speed. For a GCG we have

$$\begin{aligned} \int \frac{a^2}{\rho}d\rho &= -\frac{\alpha}{\alpha+1}\frac{A}{\rho^{\alpha+1}} \\ &= \frac{-1}{\alpha+1}a^2. \end{aligned} \tag{5.25}$$

The Bernoulli number, a constant of the motion, is thus

$$E = \frac{1}{2}u^2 - \frac{1}{\alpha+1}a^2 - \frac{GM}{r}. \tag{5.26}$$

At large distances from the central mass the gas is at rest, $u_\infty = 0$ and the gravitational force is negligible. The Bernoulli number is hence

$$\begin{aligned} E &= \frac{1}{2}u_\infty^2 - \frac{1}{\alpha+1}a_\infty^2 - \frac{GM}{r_\infty} \\ &= -\frac{1}{\alpha+1}a_\infty^2 \end{aligned} \tag{5.27}$$

We thus obtain the Bernoulli equation for GCG accretion

$$\frac{1}{2}u^2 - \frac{1}{\alpha+1}a^2 - \frac{GM}{r} = -\frac{1}{\alpha+1}a_\infty^2, \tag{5.28}$$

which holds true for all values of r and is an expression of energy conservation. We obtain the critical velocity, $u_s(= a_s)$, by evaluating (5.28) at the sonic point, r_s:

$$\frac{1}{2}u_s^2 - \frac{1}{\alpha+1}a_s^2 - \frac{GM}{r_s} = -\frac{1}{\alpha+1}a_\infty^2, \tag{5.29}$$

After substituting (5.23) into (5.29) we find the critical velocity,

$$u_s^2 = a_s^2 = \frac{2}{3\alpha+5}a_\infty^2. \tag{5.30}$$

After substituting (5.30) into (5.23) we obtain the sonic point,

$$r_s = \frac{3\alpha+5}{4}\frac{GM}{a_\infty^2}. \tag{5.31}$$

Observe that the sonic point and critical velocity are determined solely by the mass of the accretor, the gas sound speed – or, equivalently, its density – and the GCG parameter, α. The theoretical range of α constrains the critical radius for GCG accretion to lie between

$$\frac{5}{4}\frac{GM}{a_\infty^2} \leq r_s \leq 2\frac{GM}{a_\infty^2}. \tag{5.32}$$

The critical radius for polytrope accretion onto a mass is given by Bondi (1952), and Shapiro and Teukolsky (1983). Since the polytrope index satisfies $1 \leq \gamma \leq 5/3$ this critical radius falls within the range

$$0 \leq r_s \leq \frac{1}{2}\frac{GM}{a_\infty^2}. \tag{5.33}$$

The lower limit, where the sonic point shrinks to zero, represents the accretion of monatomic hydrogen($\gamma = \frac{5}{3}$), which always occurs at subsonic velocities. This is quantitatively distinct from the accretion of a GCG which always possesses a finite critical radius and, hence, a transonic solution.

5.3.2 The accretion rate, $\dot{M}$

Integrating (5.13) over a volume in spherical polar coordinates yields the expression

$$4\pi r^2 \rho u = \dot{M} \tag{5.34}$$

where $\dot{M}$ is an integration constant with dimensions of $\mathsf{mass.time}^{-1}$. This is the accretion rate of the system and is independent of r. The use of the symbol $\dot{M}$ is purely historical and represents the flux of mass flowing through a volume element. It is not to be confused with the rate of change of mass of the accretor, $\frac{dM}{dt}$, which we have assumed to be negligible from the outset. The accretion rate, $\dot{M}$, is equivalent to a quantity arising in fluid dynamics known as the discharge, [Landau and Lifshitz (1987)].

We determine the accretion rate by evaluating (5.34) at the sonic point, r_s,

$$\dot{M} = 4\pi r_s^2 u_s \rho_s. \tag{5.35}$$

From the definition of the GCG sound speed, (5.7), we have

$$\begin{aligned} \rho &\propto a^{-2/(\alpha+1)} \\ \frac{\rho}{\rho_\infty} &= \left(\frac{a}{a_\infty}\right)^{-2/(\alpha+1)}. \end{aligned} \tag{5.36}$$

Substituting (5.30), (5.31) and (5.36) into (5.35) we obtain

$$\dot{M} = 4\pi \lambda_s \rho_\infty a_\infty \left(\frac{GM}{a_\infty^2}\right)^2 \tag{5.37}$$

where we define the dimensionless accretion eigenvalue,

$$\lambda_s = \left(\frac{1}{2}\right)^{(\alpha-1)/2(\alpha+1)} \left(\frac{3\alpha+5}{4}\right)^{(3\alpha+5)/2(\alpha+1)}. \tag{5.38}$$

Equations (5.37) and (5.38) quantify the accretion rate of a GCG onto a central mass under the influence of Newtonian gravity. It extends the mass accretion formula of Bondi to Chaplygin gases and may be used to model the growth of astrophysical objects in non–baryonic environments. Our result may be used to constrain the Chaplygin parameter α and, in principle, falsify a class of unified dark sector candidates. The accretion eigenvalue for GCG lies in the range

$$2.47 \leq \lambda_{s,GCG} \leq 4. \tag{5.39}$$

The corresponding limits for the accretion eigenvalue of a polytrope (see Bondi (1952), Shapiro and Teukolsky (1983)) are

$$0.25 \leq \lambda_{s,Bondi} \leq 1.12. \tag{5.40}$$

We now compare the accretion rates for GCG and polytropic gases accreting on to a star of mass M. In both cases the accretion rate, $\dot{M}$ scales as

$$\dot{M} \sim \rho a^{-3}(GM)^2 \tag{5.41}$$

For a polytrope this is

$$\dot{M} \sim \rho^{(5-3\gamma)/2}(GM)^2 \tag{5.42}$$

and for hydrogen ($\gamma = \frac{5}{3}$) we have

$$\dot{M} \sim (GM)^2. \tag{5.43}$$

Equation (5.41) for a GCG is

$$\dot{M} \sim \rho^{(3\alpha+5)/2}(GM)^2 \tag{5.44}$$

Ignoring coefficients of order unity we find

$$\dot{M}_{\alpha=0} \sim \rho^{5/2}(GM)^2 \tag{5.45a}$$

$$\dot{M}_{\alpha=1} \sim \rho^{4}(GM)^2 \tag{5.45b}$$

The accretion rate of a GCG on to a star of mass M is approximately a factor of $\rho_\infty^{5/2} - \rho_\infty^4$ times greater than the rate for hydrogen.

5.4 Shock waves

The transonic solution outlined above describes a GCG accelerating from rest towards a star under the influence of Newtonian gravity. After passing through the critical point the gas flow becomes supersonic. Upon impacting on the surface of the star the gas will experience a rapid deceleration and will flow subsonically. This deceleration from supersonic to subsonic speeds is a definitive characteristic of shocked flows. An accreting gas in a Newtonian potential will necessarily experience a shock. This behaviour is in contrast to accretion under general relativity where the gas is expected to pass through the event horizon at supersonic speeds. Observable shock waves may only arise if there is further structure between the event horizon and the critical point. The presence of an accretion disc is one such scenario where shocks are expected.

5.4.1 The Rankine–Hugoniot conditions

An elegant account of the the physics of shock waves can be found in Landau and Lifshitz (1987). Our analysis largely follows this treatment. An authoritative study of shock wave phenomena can be found in Zel'dovich and Raizer (2002).

Shock fronts typically occur in regions significantly smaller than the characteristic length scales of a system. As a supersonic fluid gets compressed its density profile steepens. Eventually the density length scale reaches the order of the mean free path of the fluid and the steepening cannot continue. At this point the bulk viscosity of the fluid becomes significant and irreversible, dissipative processes begin. Ordered kinetic energy is converted into chaotic thermal motion and the fluid velocity rapidly becomes subsonic, resulting in the formation of a shock front [Frank et al. (2002), Shu (1992)]. This front is usually approximated as a discontinuity in the fluid flow. The dynamics of a shock front are determined, in the most general case, by the Navier–Stokes system of equations. The inviscid form of this system is known as the Euler system of equations. These laws can be expressed as total differentials depicting the conservation of mass flux, momentum flux and energy flux. One then integrates these expressions over an infinitesimal distance to relate the pre–shock and post–shock gas properties. The integrated equations are known as the Rankine–Hugoniot equations. They have more general validity than the differential Euler equations which describe the fluid flow locally. The presence of a shock results in the increase of entropy and viscosity even if the gas was initially adiabatic and inviscid. The post–shock flow characteristics arise in order to balance the energy budget due to the generation of entropy and viscosity. Eventually the gas will relax to an adiabatic, inviscid flow.

Since we are approximating the shock front as a sharp discontinuity we cannot describe the details of the dissipitative processes that occur (fluid variables experience infinite gradients here). It is sufficient here to describe the jumps in pressure, density and velocity. A more detailed formulation of the shock front problem must include a precise description of the entropy and viscosity generating mechanisms and treat the shock front as a layer with finite thickness. In this case the Euler equations do not hold locally at all points of the flow. It is for this reason that the Rankine–Hugoniot equations, being integrals, are more general descriptions of shocked gases.

The Euler equations for a fluid accreting onto a central mass, M, are

$$\frac{1}{r^2}\frac{d}{dr}\left(\rho u r^2\right) = 0 \tag{5.46a}$$

$$\rho u\frac{du}{dr} = -\frac{dp}{dr} - \rho\tfrac{GM}{r^2} \tag{5.46b}$$

$$\frac{1}{r^2}\frac{d}{dr}\left[\rho u r^2\left(\varepsilon + \frac{1}{2}u^2\right) + p u r^2\right] = -\tfrac{GM}{r^2}\rho u \tag{5.46c}$$

where the symbols have their usual meaning. After substituting (5.46a) into (5.46c) we obtain

$$\rho u \frac{d}{dr}\left(\varepsilon + \frac{1}{2}u^2 + \frac{p}{\rho} - \frac{GM}{r}\right) = 0 \tag{5.47}$$

From (5.10b) the specific internal energy, ε, of a GCG can be expressed as a function of its pressure, p, and density, ρ, viz.

$$\varepsilon = -\frac{1}{\alpha+1}\frac{p}{\rho}.$$

Thus (5.47) can be written as the total differential,

$$\frac{d}{dr}\left(\frac{\alpha}{\alpha+1}\frac{p}{\rho} + \frac{1}{2}u^2 - \frac{GM}{r}\right) = 0 \tag{5.48}$$

or

$$\frac{d}{dr}\left(w + \frac{1}{2}u^2 - \frac{GM}{r}\right) = 0 \tag{5.49}$$

where we have introduced the specific enthalpy, w, using (5.10b). The energy equation (5.49) can also be written in so–called thermodynamic form

$$-\frac{1}{\alpha+1}\frac{1}{\rho}\left(\frac{dp}{dr} + \alpha\frac{p}{\rho}\frac{d\rho}{dr}\right) = 0 \tag{5.50}$$

or

$$-\frac{1}{\alpha+1}\frac{1}{\rho}\left(\frac{dp}{dr} - a^2\frac{d\rho}{dr}\right) = 0 \tag{5.51}$$

which emphasises the role of the GCG sound speed, a, defined by (5.7c).

Given that we are interested in the behaviour in a small neighbourhood about the infinitesimally thin shock front we can neglect the spherical geometry on this scale and treat the gas flow as one–dimensional. Introducing x, the distance coordinate along the direction of the gas flow, which we define to be orthogonal to the shock front, equations (5.46a), (5.46b) and (5.49) become

$$\frac{d}{dx}(\rho u) = 0 \tag{5.52a}$$

$$\frac{d}{dx}\left(\rho u^2 + p - \frac{GM}{x}\right) = 0 \tag{5.52b}$$

$$\frac{d}{dx}\left(w + \frac{1}{2}u^2 - \frac{GM}{x}\right) = 0 \tag{5.52c}$$

respectively. The shock front is defined to lie at the origin of the x–axis. We can integrate a variable $\frac{dQ}{dx}$ over a small distance centred on $x = 0$ and consider the limit as the length shrinks to zero i.e.

$$\begin{aligned} [Q]_1^2 &:= Q_2 - Q_1 \\ &= \lim_{\epsilon \to 0} \int_{0-\epsilon}^{0+\epsilon} \frac{dQ}{dx} dx. \end{aligned} \tag{5.53}$$

If Q remains continuous at $x = 0$, i.e. at the shock front, then $[Q]_1^2 = 0$. A shock will occur when Q experiences a discontinuous change from a pre–shock value, Q_1, to its post–shock value, Q_2. We integrate (5.52) accordingly to obtain the Rankine–Hugoniot equations,

$$[\rho u]_1^2 = 0 \tag{5.54a}$$

$$\left[p + \rho u^2\right]_1^2 = 0 \tag{5.54b}$$

$$\left[w + \frac{1}{2}u^2\right]_1^2 = 0. \tag{5.54c}$$

Note that the gravitational field does not explicitly enter into (5.54) as it is an external force imposed on the gas and will remain continuous throughout the shock. The gravitational field does, however, play a crucial in determining the dynamics of the gas flow. (Note that the self–gravity of the gas is a priori insignificant compared to the gravitational field of the accreting mass.) Expanding (5.54) we find

$$
\begin{aligned}
\rho_1 u_1 &= \rho_2 u_2 && (5.55a)\\
p_1 + \rho_1 u_1^2 &= p_2 + \rho_2 u_2^2 && (5.55b)\\
w_1 + \frac{1}{2}u_1^2 &= w_2 + \frac{1}{2}u_2^2 && (5.55c)
\end{aligned}
$$

which expresses the fact that mass flux, momentum flux and energy flux are conserved in shocks. The Rankine–Hugoniot relations (5.55) are valid for general fluid types.

5.4.2 Chaplygin gas shocks

We introduce the specific volume, $V := \rho^{-1}$, and define the conserved mass flux

$$J := \rho_1 u_1 = \rho_2 u_2. \tag{5.56}$$

From (5.56) we can express the pre and post–shock velocities in terms of J and V i.e.

$$
\begin{aligned}
u_1 &= JV_1 && (5.57a)\\
u_2 &= JV_2 && (5.57b)
\end{aligned}
$$

From (5.55b) we have

$$p_1 + \rho_1 u_1^2 = p_1 + J^2 V_1. \tag{5.58}$$

Similarly

$$p_2 + \rho_2 u_2^2 = p_2 + J^2 V_2. \tag{5.59}$$

Combining (5.55b), (5.58) and (5.59) we find

$$J^2 = \frac{p_2 - p_1}{V_1 - V_2} \tag{5.60}$$

which relates the pre and post–shock pressures and volumes via the conserved mass flux, J. Equation (5.55c) can now be rewritten as

$$\begin{aligned} w_1 - w_2 + \frac{1}{2}\left(u_1^2 - u_2^2\right) &= 0 \\ w_1 - w_2 + \frac{1}{2}J^2\left(V_1^2 - V_2^2\right) &= 0 \\ w_1 - w_2 + \frac{1}{2}\left(\frac{p_2 - p_1}{V_1 - V_2}\right)\left(V_1^2 - V_2^2\right) &= 0 \\ w_1 - w_2 + \frac{1}{2}\left(p_2 - p_1\right)\left(V_1 + V_2\right) &= 0. \end{aligned} \tag{5.61}$$

This is known as the shock or Hugoniot relation. From the definition

$$w = \varepsilon + pV \tag{5.62}$$

we can find an alternative formulation of (5.61) in terms of the energy difference, viz.

$$\varepsilon_1 - \varepsilon_2 + \frac{1}{2}\left(p_1 + p_2\right)\left(V_1 - V_2\right) = 0. \tag{5.63}$$

Recall that the specific enthalpy of a GCG is given by (5.12b) which we now rewrite as

$$w = \frac{\alpha}{\alpha + 1}pV. \tag{5.64}$$

Substituting (5.64) into (5.61) and re–arranging terms we obtain

$$\frac{V_2}{V_1} = \frac{(\alpha - 1)p_1 + (\alpha + 1)p_2}{(\alpha + 1)p_1 + (\alpha - 1)p_2} \tag{5.65}$$

which is the Hugoniot relation for a GCG. Observe that (5.65) depicts the trajectory of a shocked system from an initial point (p_1, V_1).

From (5.60) and (5.65) we obtain the mass flux density, J, viz.

$$J^2 = -\frac{1}{2V_1}\left[(\alpha + 1)p_1 + (\alpha - 1)p_2\right] \tag{5.66}$$

which we use to find the propagation velocities of the shock, relative to the gas.

Upstream of the shock wave we have

$$\begin{aligned}
u_1^2 &= J^2 V_1^2 \\
&= -\frac{1}{2}\left(p_1 V_1\right)\left[(\alpha+1)+(\alpha-1)\frac{p_2}{p_1}\right] \\
&= -\frac{1}{2}\left(\frac{p_1}{\rho_1}\right)\left[(\alpha+1)+(\alpha-1)\frac{p_2}{p_1}\right] \\
&= \frac{1}{2}\left(\frac{a_1^2}{\alpha}\right)\left[(\alpha+1)+(\alpha-1)\frac{p_2}{p_1}\right] \qquad (5.67)
\end{aligned}$$

where we have used the pre–shock sound speed (5.7c) viz.

$$c_1^2 = -\alpha\frac{p_1}{\rho_1}.$$

We similarly obtain the post–shock velocity,

$$\begin{aligned}
u_2^2 &= J^2 V_2^2 \\
&= -\frac{1}{2}\left(p_2 V_2\right)\left[(\alpha+1)+(\alpha-1)\frac{p_1}{p_2}\right] \\
&= -\frac{1}{2}\left(\frac{p_2}{\rho_2}\right)\left[(\alpha+1)+(\alpha-1)\frac{p_1}{p_2}\right] \\
&= \frac{1}{2}\left(\frac{a_2^2}{\alpha}\right)\left[(\alpha+1)+(\alpha-1)\frac{p_1}{p_2}\right]. \qquad (5.68)
\end{aligned}$$

The ratios of post to pre–shock densities and pressures can be expressed succinctly in terms of the pre–shock Mach number,

$$\mathcal{M}_1 := \frac{u_1}{a_1}. \qquad (5.69)$$

Substituting (5.69) into (5.67) yields

$$\frac{p_2}{p_1} = \frac{2\alpha}{\alpha-1}\mathcal{M}_1^2 - \frac{\alpha+1}{\alpha-1}. \qquad (5.70)$$

Substituting this result into the Hugoniot relation (5.65) yields the density ratio,

$$\begin{aligned}\frac{\rho_2}{\rho_1} &= \frac{V_1}{V_2} \\ &= \frac{(\alpha+1)p_1 + (\alpha-1)p_2}{(\alpha-1)p_1 + (\alpha+1)p_2} \\ &= \frac{(\alpha+1) + (\alpha-1)p_2/p_1}{(\alpha-1) + (\alpha+1)p_2/p_1} \\ &= \frac{(\alpha-1)\mathcal{M}_1^2}{(\alpha+1)\mathcal{M}_1^2 - 2}. \end{aligned} \tag{5.71}$$

From (5.68) and (5.70) we can express the post–shock Mach number $\mathcal{M}_2 := \frac{u_2}{a_2}$ in terms of $\mathcal{M}_1$,

$$\mathcal{M}_2^2 = \frac{(\alpha+1)\mathcal{M}_1^2 - 2}{2\alpha\mathcal{M}_1^2 - (\alpha+1)}. \tag{5.72}$$

Upon inspection of (5.71) it is clear that negative specific volumes are admitted in the limit of very strong shocks i.e. $\frac{\rho_2}{\rho_1} \to \frac{\alpha-1}{\alpha+1}$ as $\mathcal{M}_1^2 \to \infty$. It is unclear how to interpret this statement. Whilst it is mathematically precise its physical content may be questioned. As we are considering a fluid with negative pressure some caution with the analysis is advised. Gao and Law (2012) studied a broad class of phenomena described by Rankine–Hugoniot relations for relativistic combustion waves. They obtained systems with negative pressure and imaginary volumes downstream. A more rigorous analysis of the formation of weak shocks determined that these states were indeed unphysical as they violated the second law of thermodynamics. Another state was found to be consistent with the the entropy increase law if the gas developed a rarefaction shock. Shock waves are usually compressive and the presence of a rarefaction shock required an unusual feature like a negative pressure. We suspect that a detailed analysis of the thermodynamic properties of GCG shocks is necessary in order to assess the range of validity of our results. This is beyond the scope of this thesis and will be explored in future work.

5.5 Discussion

In this chapter we examined the spherical accretion of a GCG onto an astrophysical body under the regime of Newtonian gravity. The GCG was treated as a unified model for dark matter and dark energy. We obtained analytical expressions for the critical radius, mass accretion rate and eigenvalues. These were compared with the corresponding values for the accretion of a polytropic gas. A GCG accretes onto a star at a rate approximately 2 - 4 times greater than neutral hydrogen.

We then considered the possibility of shocks in GCG accretion. The Rankine–Hugoniot relations for this problem were derived. The Hugoniot relation describing the relationship between pre and post–shock pressures and specific volumes were deduced. The pressure and density compression ratios for GCG shocks were derived. These results require careful interpretation as some of these states may only arise from thermodynamically forbidden processes. We leave a detailed study of the entropy increase of weakly shocked GCGs as a task for future investigation. If the speed of sound becomes sufficiently high the GCG may start to mimic the behaviour of relativistic gases like those analysed by Gao and Law (2012). We suspect that this investigation will constrain the permissable values of the GCG parameters and provide hints as to the viability of Chaplygin gas models as dark sector candidates. A number of further avenues of investigation may be explored. If a stellar object has non–zero angular momentum then accreting matter will, in general, form an accretion disk. The properties of rotating accretion systems are distinct from those of the simpler, spherical case. This problem lies beyond the scope of this thesis and we leave it as a project for future work. In our formulation of the shock wave problem we have assumed that the GCG is accreted at relatively low velocities. If the sound speed became sufficiently high our model would break down. We would then have to utilise the special relativistic form of the Rankine–Hugoniot equations, outlined in Taub (1948) and Thorne (1973). Our task here was to investigate the properties of shocked GCG flow and a detailed analysis of relativistic accretion would form a natural extension of this work.

Chapter 6

Conclusion

One of our aims was to find and analyse new exact solutions of the Einstein and Einstein–Maxwell equations. These solutions may be used to model the interior of compact stars. The condition for pressure isotropy was central to our investigations. As our systems were under–determined we specified forms for one of the gravitational potentials, Z and, where applicable, the electric field intensity, E. We utilised two, complementary techniques to solve the pressure isotropy condition, viz. the method of Frobenius, and transformation to a hypergeometric function.

We assumed the spacetime was static and spherically symmetric. We also explored the possibility of unifying seemingly disparate exact solutions as special cases of more general classes of solutions. We generated new solutions and demonstrated their physical reasonableness by plotting their behaviour. We believe that these solutions are new and may provide realistic models for dense, static stars.

Our second major aim was the exploration of accretion in cases with either exotic matter or exotic gravity. For the first case we considered a polytropic gas accreting onto a D–dimensional Schwarzschild black hole. In the second case we looked at a generalised Chaplygin gas (GCG) accreting onto a body with a gravitational field described by Newtonian theory. The GCG was used as a proxy for dark matter and dark energy as it has been shown to demonstrated promise in accounting for these phenomena. We further explored the possibility of shock waves in the GCG flow. In both studies we obtained analytical expressions for the gas accretion rates and critical radii.

In particular we point out our specific results:

- We specified a cubic form for one of the gravitational potentials. We solved the differential equation governing pressure isotropy by assuming the existence of a power series solution. The pressure isotropy condition was thus reduced to a third order difference equation. The pressure and energy density appear to be continuous and non–singular over a large range of radial distances. This behaviour was depicted in Figs. 2.1 and 2.2 for fiducial values of the solution's parameters. Our model satisfies a barotropic equation of state which can be approximated as a polytrope for radial distances close to the stellar centre. The method of Frobenius was shown to be a powerful technique for extracting exact solutions of the Einstein field equations. We believe that our solution is original.

- We prescribed the metric potential Z as well as the electric field E. This transformed the condition for pressure isotropy into a hypergeometric differential equation. The general solution to this equation is written in terms of hypergeometric functions. These particular special functions possess two free parameters, K and α, corresponding to the metric potential, Z, and electric field, E, respectively. For particular choices of these parameters the hypergeometric functions are reduced to algebraic functions. We recovered two known solutions, viz. Durgapal and Bannerji (1983) and Maharaj and Mkhwanazi (1996). We also obtained two new exact solutions expressed in terms of algebraic functions. The first of these, parametrised by $K = -2$ and $\alpha = 1$ appears to be well behaved. The pressure, energy density and electric field were plotted in Figs. 3.1, 3.2 and 3.3. We believe this solution is original and may serve as a model for a charged, compact star. The second new solution, described by $K = \frac{1}{2}$ and $\frac{5}{2}$, is unphysical. The energy density is negative and this violates the strong and weak energy conditions. The simple form of the algebraic functions greatly facilitated the physical analysis of these new solutions. These results also demonstrate that, despite the freedom of these under–determined systems, unphysical solutions may often occur.

- We determined the critical radius and mass accretion rate for a polytrope accreting onto a D–dimensional Schwarzschild black hole. We also found explict expressions for the gas compression and temperature profile both below the critical radius and at the event horizon. The accretion rate $\dot{M}$ is clearly dependent on the mass and dimensionality of the black hole. This is to be contrasted with the result of Bondi (1952) which showed that $\dot{M} \sim M^2$. Our result also generalises the study of Giddings and Mangano (2008) which obtained the mass–dependent accretion rate of matter accreting via the Newtonian gravity potential of a D–dimensional TeV black hole.

- We obtained analytical expressions for the critical velocity, radius and mass accretion rate of a GCG under the influence of a Newtonian potential. By comparison with values typical for neutral hydrogen we showed that a GCG will accrete approximately 2 - 4 times faster onto a star. We derived the Rankine–Hugoniot conditions relating GCG parameters before and after a shock wave. The Hugoniot relation describing the relationship between pre and post shock pressure and specific volume was determined. We obtained the pressure and density compression ratios for GCG shocks in general and examined the case of strongly shocked flows. We suspect some of these states may be energetically forbidden as they predict negative volumes.

There is great scope for extending our study of exact solutions modelling compact stars. One can specify other forms for the gravitational potentials to generate series solutions of the pressure isotropy condition. These series solutions may then be subjected to stringent physical analysis. The stability of these solutions needs to be determined. A growing body of work in this direction is already underway eg. Thirukkanesh and Maharaj (2006), Maharaj and Thirukkanesh (2006), Maharaj and Komathiraj (2007).

A number of extensions to our study of higher dimensional accretion are possible. One can attempt to work out the effect of extra dimensions on the luminosity, frequency spectrum and energy conversion efficiency of the the accretion flow. More exotic matter, like a scalar field, cound be investigated. Unlike general relativity, Lovelock gravity and its special case, Einstein–Gauss–Bonnet gravity, have been demonstrated to be low energy limits of particular string theories. It may be feasible to study the effects of accretion on to higher dimensional black holes described by those gravity theories.

If a star is rotating then matter falling in its potential will form an accretion disk. It should be instructive to extend our study of spherical GCG accretion to systems with non–zero angular momentum. We have assumed non–relativistic energies for the GCG. If the sound speed becomes significantly large then our description of shock waves becomes invalid. One can expand our study of GCG shocks by determing the appropriate special relativistic Rankine–Hugoniot conditions.

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