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# Schur Lemma and Uniform Convergence of Series through Convergence Methods

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**Abstract:** In this note, we prove a Schur-type lemma for bounded multiplier series. This result allows us to obtain a unified vision of several previous results, focusing on the underlying structure and the properties that a summability method must satisfy in order to establish a result of Schur's lemma type.

**Keywords:** Schur lemma; unconditionally Cauchy series; completeness; summability methods

**MSC:** 40H05; 40A35

## 1. Introduction

Throughout this paper,  $\mathbb{N}$  will denote the set of natural numbers. If  $X$  is a normed space and  $\mathcal{R} : D_{\mathcal{R}} \subset X^{\mathbb{N}} \rightarrow X$  is a linear map which assigns limits to a sequence, we will say that  $\mathcal{R}$  is a convergence method (or summability method) and  $D_{\mathcal{R}}$  is the convergence domain of  $\mathcal{R}$ .

With the development of Fourier theory, other convergence methods of the series were studied which are interesting in their own right. Convergence methods have generated so much interest in Approximation Theory and Applied Mathematics that different monographs have appeared in the literature [1–4]; moreover, this is a very active field of research with many contributors.

A good source of problems consists in considering a result on convergence of series which is true for the usual convergence and, to try to prove it, replacing the usual convergence by other convergence methods [5–9]. In this way, it is possible to see a classical result from a new point of view. Sometimes [10,11], it is possible to characterize those summability methods for which these classical results hold. For instance, in [10], the summability methods for which the classical Orlicz–Pettis's result is true are characterized, namely, it is possible to obtain a version of the Orlicz–Pettis's theorem for any regular convergence method.

Schur lemma is one of the best known and most useful results in Functional Analysis, so that it has attracted the interest of many people. One of the classical versions [12] states that a sequence in  $\ell_1$  is weakly convergent if and only if it is norm convergent. This result was sharpened by Antosik and Swartz using the Basic Matrix Theorem (see [13]); moreover, Swartz [4,14] obtained a version of the Schur lemma for bounded multiplier convergent series. In this note, we aim to unify different versions of Swartz's result that incorporated summability methods. Of course, Swartz's result is not true for a general summability method; we analyze those summability methods for which Swartz's

result continues being true. In the way, we show up some properties of summability methods that have been not treated and that deserves subsequent studies.

We continued the research line started in [10,11], and we aim to unify different versions of Swartz’s result [5,7,15]. For instance, Schur type results were obtained for any regular matrix summability method [15] and for the Banach–Lorentz convergence [5].

This paper is structured as follows. In Section 2, we are going to point out four properties for a general summability method that will be hypotheses in our results and we will study their basic properties. In Section 3, we will put into practice what we learned on general summability in Section 3. We will see a result on unconditionally convergent series in the context of any general summability method. This result is independent of the main result in Section 4.

In Section 4, we will obtain a general Schur-type lemma for general summability methods; thus, we unify results appeared in [5,15], and finally we close the paper with a brief section with concluding remarks and open questions.

## 2. Some Preliminary Results

For simplicity, we will suppose throughout the paper that  $(X, \| \cdot \|)$  is a real Banach space. Let us denote by  $\ell_\infty(X)$  the space of all bounded sequences in  $X$  provided with the supremum norm (which we will denote sometimes abusively by  $\| \cdot \|$ ):

$$\|(x_k)\|_{\ell_\infty(X)} = \sup\{\|x_n\| : n \in \mathbb{N}\}.$$

A linear summability method in  $X$  will be denoted by  $\mathcal{R}$ ; that is,  $\mathcal{R}$  will be a linear map  $\mathcal{R} : D_{\mathcal{R}} \subset X^{\mathbb{N}} \rightarrow X$  (here  $D_{\mathcal{R}}$  denotes the domain of  $\mathcal{R}$ ). Thus, a sequence  $(x_n) \in X^{\mathbb{N}}$  is  $\mathcal{R}$ -convergent to  $L$  (and it will be denoted by  $x_n \xrightarrow{\mathcal{R}} L$ ) provided  $\mathcal{R}((x_n)) = L$ . We will require on  $\mathcal{R}$  that the limit assignment does not depend on the first terms, that is, for any  $(x_n) \in D_{\mathcal{R}}$  such that  $\mathcal{R}((x_n)_{n \geq 1}) = L$  and, for any  $n_0 \in \mathbb{N}$ , we also require that  $(x_n)_{n \geq n_0} \in D_{\mathcal{R}}$  and  $\mathcal{R}((x_n)_{n \geq n_0}) = L$ .

A sequence  $(x_n) \in \ell_\infty(X)$  is said to be  $\mathcal{R}$ -Cauchy if for any  $\varepsilon > 0$  there exists  $n_0 \in \mathbb{N}$ , such that,  $(x_n - x_{n_0})_{n \geq n_0} \in D_{\mathcal{R}} \cap \ell_\infty(X)$  and  $\|\mathcal{R}((x_n - x_{n_0})_{n \geq n_0})\| < \varepsilon$ .

A series  $\sum_i x_i$  in a real Banach space  $X$  is called weakly unconditionally Cauchy (wuc) if  $\sum_i |f(x_i)| < \infty$  for every  $f \in X^*$ , and  $\sum_i x_i$  is called unconditionally convergent (uc) if  $\sum_i x_{\pi(i)}$  is convergent for every permutation  $\pi$  of  $\mathbb{N}$ .

Let us denote by  $c_0$  the Banach space of all sequences  $(a_n) \in \mathbb{C}^{\mathbb{N}}$  such that  $\lim_n a_n = 0$  endowed with the canonical norm and by  $B_{\ell_\infty}$  the unit ball of the space  $\ell_\infty$  of bounded sequences of complex numbers. It is well known that a series  $\sum_i x_i$  is (wuc) if and only if  $\sum_i a_i x_i$  is convergent for every sequence  $(a_i) \in c_0$ , or equivalently  $\{\sum_{i=1}^n a_i x_i : (a_i) \in B_{\ell_\infty}, n \in \mathbb{N}\}$  is bounded in the normed space  $X$ . It is also known that a series  $\sum_i x_i$  is (uc) if and only if  $\sum_i a_i x_i$  is convergent for every  $(a_i) \in \ell_\infty$ .

Let us denote by  $X(c_0)$  the (wuc) series and  $X(\ell_\infty)$  will denote the space of all (uc) series. Both spaces are real Banach spaces, endowed with the norm:

$$\|(x_k)_{k \in \mathbb{N}}\|_s = \sup \left\{ \left\| \sum_{i=1}^n a_i x_i \right\| : |a_i| \leq 1, i \in \{1, \dots, n\}, n \in \mathbb{N} \right\}. \tag{1}$$

We aim to extend the following striking result by Swartz [14] which is a version of Schur lemma for bounded multiplier convergent series:

**Theorem 1** (Swartz-1983). *Let  $(x_n)_{n \in \mathbb{N}} = (x_n(k))$  be a sequence in  $X(\ell_\infty)$  such that, for every  $(a(k)) \in \ell_\infty$ ,  $\lim_{n \rightarrow \infty} \sum_{k=1}^\infty a(k)x_n(k)$  exists. Then, there exists  $x_0 \in X(\ell_\infty)$  such that  $\lim_{n \rightarrow \infty} \|x_n - x_0\|_s = 0$ .*

Let  $S$  be a closed subspace of  $\ell_\infty$  containing  $c_0$ . Let us consider  $\mathcal{R} : D_{\mathcal{R}} \subset X^{\mathbb{N}} \rightarrow X$  a summability method defined on a real Banach space  $X$ , and let us consider the following vector spaces:

$$X(S, \mathcal{R}) = \left\{ (x_k)_{k \in \mathbb{N}} : \mathcal{R} \left( \left( \sum_{k=1}^n a_k x_k \right) \right) \text{ exists for every } (a_k)_{k \in \mathbb{N}} \in S \right\}.$$

Of course, we will need to place some limits on the summability methods  $\mathcal{R}$  because Theorem 1 is not true for every summability method  $\mathcal{R}$ . We will consider the following properties:

- (h1) Regularity.** That is, for any sequence  $(x_n)$  convergent in  $X$  that is,  $\lim_{n \rightarrow \infty} x_n = L \in X$ , it is satisfied that  $(x_n) \in D_{\mathcal{R}}$  and  $\mathcal{R}((x_n)) = L$ .
- (h2)  $\mathcal{R}$ -weak convergence.** For every  $(x_n) \in D_{\mathcal{R}} \cap \ell_\infty(X)$  such that  $\mathcal{R}((x_n)) = L$ , it is satisfied that  $\sup_{f \in B_{X^*}} |f(x_n) - f(L)| = 0$ .
- (h3) Boundedness.** In the following sense, there exists  $M > 0$  such that  $\|\mathcal{R}((x_n))\| \leq M \|(x_n)\|_{\ell_\infty(X)}$  for all  $(x_n) \in D_{\mathcal{R}} \cap \ell_\infty(X)$ .
- (h4)  $\mathcal{R}$ -completeness.** That is, a sequence  $(x_n) \in D_{\mathcal{R}} \cap \ell_\infty(X)$  if and only if  $(x_n)$  is  $\mathcal{R}$ -Cauchy.

Another property of a summability method  $\mathcal{R}$  that plays an important role, and which is weaker than property **(h1)**, is to be *regular on constant*, that is,  $D_{\mathcal{R}}$  contains the constant sequences.

Property **(h2)** will be used only on Theorem 2. It requires that the sequences in  $D_{\mathcal{R}} \cap \ell_\infty(X)$  must be weakly convergent. In fact, as we will see in the proof of Theorem 2 (see Remark 1), we need a weak version of **(h2)**. Specifically:

- (h2')** For any sequence  $(x_n) \in D_{\mathcal{R}} \cap \ell_\infty(X)$  and  $f \in X^*$ , the sequence  $f(x_n)$  converges to some  $L_f \in \mathbb{R}$ .

Let us observe that, if  $\mathcal{R}$  is regular on constant sequences, then  $D_{\mathcal{R}}$  is invariant by translations; therefore, condition  $(x_n - x_{n_0})_{n \geq n_0} \in D_{\mathcal{R}} \cap \ell_\infty(X)$  is automatically satisfied in **(h4)**.

**Proposition 1.** *Let  $\mathcal{R}$  be a linear summability method such that  $\mathcal{R}$  is regular on the constant and  $\mathcal{R}$  satisfies **(h3)**. If  $(x_n)$  is a Cauchy sequence, then  $(x_n)$  is  $\mathcal{R}$ -Cauchy.*

**Proof.** Indeed, if  $(x_n)$  is Cauchy, then  $(x_n)$  is bounded and, for any  $\varepsilon > 0$ , there exists  $n_0$  such that, if  $n, m \geq n_0$ , then  $\|x_n - x_m\| < \varepsilon / M$  ( $M$  is the constant guaranteed by **(h3)**). Hence,

$$\|\mathcal{R}((x_n - x_{n_0})_{n \geq n_0})\| \leq M \|x_n - x_{n_0}\|_{\ell_\infty(X)} \leq \varepsilon,$$

which gives the desired result.  $\square$

As a consequence,

**Corollary 1.** *let  $\mathcal{R}$  be a linear summability method satisfying **(h3)** and **(h4)**. The convergence method  $\mathcal{R}$  is regular on the constant if and only if  $\mathcal{R}$  is regular.*

**Example 1.** *Let us observe that hypothesis **(h3)** does not imply regularity. Indeed, we will say that a sequence  $(x_n) \in \mathbb{R}^{\mathbb{N}}$  is  $\rho$ -convergent to  $x_0$  if  $\lim_{n \rightarrow \infty} \frac{x_n}{n^2} = x_0$ . Then, clearly  $\rho|_{\ell_\infty} = 0$ . Hence,  $\rho$  is not regular. However,  $\rho$  satisfies trivially **(h3)**.*

The notion of *induced summability* was introduced in [10] and it allows us to unify several results that incorporate different types of weak convergence. Let  $\rho : D_\rho \subset \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}$  be a summability method on  $\mathbb{R}$ . The summability method  $\rho$  could induce a summability method  $\mathcal{R}$  on every normed space  $X$  as follows. A sequence  $(x_n)_n \in X^{\mathbb{N}}$  is said to be  $\mathcal{R}$  convergent to  $L$  if for any  $f \in X^*$  the sequence

$f(x_n) \xrightarrow{\rho} f(L)$ . The summability method  $\mathcal{R}$  is called a *summability method induced by  $\rho$*  on the space  $X$ . Let us define the following space:

$$X_{\omega}(S, \rho) = \left\{ (x_k) \in X^{\mathbb{N}} : \sum_{i=1}^n a_i f(x_i) \text{ } \rho\text{-converges } \forall (a_i) \in S, \forall f \in X^* \right\}.$$

In particular, when the summability method is induced by the usual convergence on  $\mathbb{R}$ , then we denote:  $X_{\omega}(S) = X_{\omega}(S, |\cdot|)$ . Let us show some basic properties of these spaces.

**Proposition 2.** *If  $\mathcal{R}$  is a summability method induced by  $\rho$ , then  $X(S, \mathcal{R}) \subset X_{\omega}(S, \rho)$ .*

**Proof.** Indeed, if  $(x_k) \in X(S, \mathcal{R})$ , then, for all  $a = (a_j) \in S$ , there exists  $L_a \in X$  such that  $\sum_{i=1}^n a_i x_i \xrightarrow{\mathcal{R}} L_a$ ; therefore, for all  $f \in X^*$ , we have that  $\sum_{i=1}^n a_i f(x_i) \xrightarrow{\rho} f(L_a)$ , that is,  $(x_k) \in X_{\omega}(S, \rho)$ .  $\square$

**Proposition 3.** *Let  $\rho_1$  and  $\rho_2$  be two summability methods on  $\mathbb{R}$ . If  $D_{\rho_1} \subset D_{\rho_2}$  then  $X_{\omega}(S, \rho_1) \subset X_{\omega}(S, \rho_2)$ . In particular, if  $\rho$  is regular, then  $X_{\omega}(S, \rho) \subset X_{\omega}(S)$ .*

**Proposition 4.** *If  $\rho$  is a regular summability method in  $\mathbb{R}$  and  $\mathcal{R}$  is an induced summability method in a normed space  $X$ , then  $\mathcal{R}$  is non-trivial, and  $\mathcal{R}$  is also regular.*

**Proof.** Indeed, if  $x_k \xrightarrow{\|\cdot\|} L$ , then, for every  $f \in X^*$ , we have  $f(x_n) \xrightarrow{|\cdot|} f(L)$ . Since  $\rho$  is regular, we obtain that  $f(x_n) \xrightarrow{\rho} f(L)$ , for any  $f \in X^*$ . Hence,  $x_n \xrightarrow{\mathcal{R}} L$ , which yields the desired result.  $\square$

**Proposition 5.** *If  $\mathcal{R}$  is a linear summability method satisfying (h2), then  $X(S, \mathcal{R}) \subset X_{\omega}(S)$ .*

**Proof.** Indeed, if  $(x_k) \in X(S, \mathcal{R})$ , then

$$\mathcal{R} \left( \left( \sum_{i=1}^n a_i x_i \right) \right) = L_a$$

for each  $a \in S$ . Since  $\mathcal{R}$  satisfies (h2), for any  $f \in X^*$ , we have that  $\sum_{i=1}^n a_i f(x_i) \xrightarrow{|\cdot|} f(L_a)$ ; hence,  $(x_k) \in X_{\omega}(S)$  as we desired.  $\square$

### 3. Completeness of a Normed Space through Summability Methods

We define in an abstract way the  $\mathcal{R}$ -sequence space associated with a (wuc)-series  $\sum_i x_i$  as follows:

$$S_{\mathcal{R}} \left( \sum_i x_i \right) = \left\{ (a_i) \in \ell_{\infty} : \sum_{i=1}^n a_i x_i \text{ is } \mathcal{R} \text{ convergent in } X \right\}.$$

Given a summability method  $\mathcal{R}$  and a series  $\sum_i x_i$ , under certain conditions, it is possible to obtain when  $\sum_i x_i$  is (wuc) in terms of the completeness of the space  $S_{\mathcal{R}}(\sum_i x_i)$ . Moreover, when  $S_{\mathcal{R}}(\sum_i x_i)$  is closed in  $\ell_{\infty}$  for each (wuc) series, it is possible to characterize the completeness of  $X$ .

This kind of result has been obtained for many summability methods. The results in [11] try to unify all known results. In fact, the results in [11] are true for any summability method defined by a non-trivial regular ideal. However, not all summability methods can be defined by means of an *ideal convergence*; for instance, this result was obtained in [16] in terms of the lacunary statistical convergence. In the following statement, we are going to put into practice what we learned from general summability methods in Section 3, and we are going to obtain the following general result.

**Theorem 2.** Let  $\mathcal{R}$  be a convergence method on a Banach space  $X$  satisfying **(h1)**-**(h4)**. The following conditions are equivalent:

1. The series  $\sum_i x_i$  is a weakly unconditionally Cauchy (wuc).
2. The subspace  $S_{\mathcal{R}}(\sum_i x_i)$  is closed in  $\ell_\infty$ .
3.  $c_0 \subset S_{\mathcal{R}}(\sum_i x_i)$ .

**Proof.** To prove (1) $\Rightarrow$ (2), let us consider the supremum

$$H = \sup \left\{ \left\| \sum_{i=1}^n a_i x_i \right\| : |a_i| \leq 1, 1 \leq i \leq n, n \in \mathbb{N} \right\} < \infty,$$

which is finite because the series  $\sum_i x_i$  is (wuc). Moreover, let us consider  $(a^m)_m \in S_{\mathcal{R}}^{\mathbb{N}}(\sum_i x_i)$  satisfying  $\lim_{m \rightarrow \infty} \|a^m - a^0\|_\infty = 0$  for some  $a^0 \in \ell_\infty$ . We will show that  $a^0 \in S_{\mathcal{R}}(\sum_i x_i)$ ; that is, the sequence  $S_n^0 = \sum_{i=1}^n a_i^0 x_i$  is  $\mathcal{R}$ -convergent.

Indeed, let  $m_0 \in \mathbb{N}$  be large enough such that  $\|a^m - a^0\|_\infty \leq \frac{\varepsilon}{3H}$  for all  $m \geq m_0$ . In particular, for  $m \geq m_0$

$$\left\| \sum_{i=1}^n \frac{3H}{\varepsilon} (a_i^m - a_i^0) x_i \right\| \leq H,$$

that is,

$$\|S_n^m - S_n^0\| \leq \frac{\varepsilon}{3M}, \tag{2}$$

for any  $m \geq m_0$  and  $n \in \mathbb{N}$ , where  $M$  denotes the constant guaranteed by hypothesis **(h3)**. On the other hand, since  $a^m \in S_{\mathcal{R}}(\sum_i x_i)$ , then  $\mathcal{R}(\sum_{i=1}^n a_i^m x_i) = L_m \in X$ , for some  $L_m \in X$ . In addition, since  $\sum_i x_i$  is (wuc), we have that  $(S_n^m)_n \in \ell_\infty(X)$ . In particular, since  $\mathcal{R}$  satisfies hypothesis **(h4)**, the sequences  $(S_n^m)_n$  is  $\mathcal{R}$ -Cauchy. Thus, for any  $\varepsilon > 0$ , there exists  $n_0$  such that, if  $n \geq n_0$ , then

$$\|\mathcal{R}((S_n^m - S_{n_0}^m)_{n \geq n_0})\| \leq \frac{\varepsilon}{3}. \tag{3}$$

Since  $\sum_i x_i$  is (wuc), we also have that  $S_n^0 = \sum_{i=1}^n a_i^0 x_i \in \ell_\infty(X)$ . Therefore, in order to show that  $(S_n^0)$  is  $\mathcal{R}$ -convergent, we will show that  $(S_n^0)$  is  $\mathcal{R}$ -Cauchy. Now, let us suppose that  $m \geq m_0$ , using the linearity of  $\mathcal{R}$  and the triangular inequality, we get:

$$\begin{aligned} \left\| \mathcal{R} \left( (S_n^0 - S_{n_0}^0)_{n \geq n_0} \right) \right\| &= \left\| \mathcal{R} \left( (S_n^0 - S_{n_0}^0 - S_n^m + S_n^0 + S_{n_0}^m - S_{n_0}^0)_{n \geq n_0} \right) \right\| \\ &\leq \left\| \mathcal{R} \left( (S_n^0 - S_n^m)_{n \geq n_0} \right) \right\| + \left\| \mathcal{R} \left( (S_{n_0}^m - S_{n_0}^0) \right) \right\| + \left\| \mathcal{R} \left( (S_n^m - S_{n_0}^m)_{n \geq n_0} \right) \right\| \\ &\leq M \sup_{n \geq n_0} \|S_n^0 - S_n^m\| + M \|S_{n_0}^m - S_{n_0}^0\| + \frac{\varepsilon}{3} \\ &\leq \frac{\varepsilon}{3} + M \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

We used that the constant sequences  $x_n = L$  are  $\mathcal{R}$  convergent to  $L$  for all  $L \in X$ , which is guaranteed by **(h1)**. In the second inequality, we used that  $\mathcal{R}$  satisfies hypothesis **(h3)**, and we also used Equations (2) and (3). Hence, we have shown that  $(S_n^0)$  is a  $\mathcal{R}$ -Cauchy sequence; therefore, it is  $\mathcal{R}$ -convergent as we desired.

To establish (2) $\Rightarrow$ (3), it is sufficient to observe that, since  $\mathcal{R}$  is regular (hypothesis **(h1)**), the space  $c_{00}$  of eventually zero sequences is contained in  $S_{\mathcal{R}}(\sum_i x_i)$ . Since  $S_{\mathcal{R}}(\sum_i x_i)$  is a closed subspace of  $\ell_\infty$ , we get that  $c_0 \subset S_{\mathcal{R}}(\sum_i x_i)$ .

Finally, to prove (3) $\Rightarrow$  (1), if  $(x_n)$  is not a (wuc) series, then there exists  $f \in X^*$  such that  $\sum_{i=1}^\infty |f(x_i)| = \infty$ . In such a case, we will show that it is possible to find a sequence  $(a_n) \in c_0$  such that  $\sum_{i=1}^\infty a_i f(x_i) = \infty$ . Indeed, we can select a sequence  $(b_n)$  of positive terms converging to 0

slowly enough such that  $\sum_{i=1}^{\infty} b_i |f(x_i)| = \infty$ . Then, the sequence  $a_n = b_n \text{sign}(f(x_n))$  and satisfies that  $\sum_{i=1}^{\infty} a_i f(x_i) = \infty$ . Let us observe that the sequence  $(a_n) \notin S_{\mathcal{R}}(\sum_i x_i)$ . Indeed, if  $(a_n) \in S_{\mathcal{R}}(\sum_i x_i)$ , then  $S_n = \sum_{i=1}^n a_i x_i$  should be  $\mathcal{R}$ -convergent to some  $L \in X$ . Hence, since  $\mathcal{R}$  satisfies condition **(h2)**, we get that  $f(S_n) = \sum_{i=1}^n a_i f(x_i)$  converges to some  $L_f \in \mathbb{R}$  a contradiction. Thus, we have shown that there exists  $a = (a_n) \in c_0$  such that  $a \notin S_{\mathcal{R}}(\sum_i x_i)$ , which contradicts our hypothesis (3), and it yields the desired result.  $\square$

**Remark 1.** As we mentioned before, to prove (3) $\Rightarrow$ (1) above, we need only the hypothesis **(h2')**. To prove (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3), we do not need completeness on  $X$ .

**Remark 2.** Let us see that Theorem 2 can be used to characterize the completeness of  $X$  through the completeness of the sequences spaces  $S_{\mathcal{R}}(\sum_i x_i)$ . Indeed, following the ideas of Theorem 3 in [11], we can show the following result. Let  $\mathcal{R}$ -be a summability method satisfying **(h1)**–**(h4)** then  $X$  is complete if and only if  $S_{\mathcal{R}}(\sum_i x_i)$  is closed in  $\ell_{\infty}$  for each (wuc) series  $\sum_i x_i$ .

**Remark 3.** We tried to give an overview of all methods of summability for which it is possible to establish Theorem 2. Of course, there exist summability methods that satisfy the properties **(h1)**–**(h4)**. For instance, the results in [11] establish Theorem 2 when the summability method  $\mathcal{R}$  is induced by a non-trivial ideal  $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ , that is, the  $\mathcal{I}$ -convergence provided  $\mathcal{I}$  is regular, that is,  $\mathcal{I}$  contains the finite subsets. However, not every summability method is induced by an ideal, for instance, the lacunary statistical convergence. Theorem 2 was established for the lacunary statistical convergence in [16]. For the lacunary statistical convergence, the hypothesis **(h1)** and **(h4)** were established in [16] Theorem 1 and Theorem 3, and the hypothesis **(h2')** and **(h3)** can be established also easily.

#### 4. Schur Lemma through Summability Methods

Hypothesis **(h3)** will guarantee that  $X(S, \mathcal{R})$  is a closed subspace of  $X(c_0)$  endowed with the norm  $\|\cdot\|_s$ ; this is our first result in this section.

**Theorem 3.** Let  $\mathcal{R}$  be a convergence method on a Banach space  $X$  satisfying **(h3)**. Then,  $X(S, \mathcal{R})$  is a closed subspace of  $X(c_0)$  endowed with the norm  $\|\cdot\|_s$ .

**Proof.** Let  $(x^n) \in X^{\mathbb{N}}(S, \mathcal{R})$  satisfying  $\lim_{n \rightarrow \infty} \|x^n - x^0\|_s = 0$  for some  $x^0 = (x_i^0) \in X(c_0)$  and let us show that  $x^0 \in X(S, \mathcal{R})$ ; that is, for all  $(a_k) \in S$ , we have that  $\sum_{k=1}^n a_k x_k^0$  is  $\mathcal{R}$ -convergent.

By hypothesis,  $\mathcal{R}$  satisfies **(h3)**; therefore, there exists  $M > 0$  such that

$$\|\mathcal{R}((x_k)) - \mathcal{R}((y_k))\| \leq M \|(x_k - y_k)\|_{\ell_{\infty}(X)}$$

for all  $(x_n), (y_m) \in \ell_{\infty}(X)$ .

Since  $(x^n)$  is a Cauchy sequence, for each  $\varepsilon > 0$ , there exists  $k_0$ , such that, for all  $p, q \geq k_0$ ,  $\|x^p - x^q\|_s < \varepsilon/M$ .

Let us fix  $(a_k)$  in the unit ball of  $S$ . Since  $x^m \in X(S, \mathcal{R})$ , we obtain that the partial sums  $\sum_{k=1}^n a_k x_k^m$  are  $\mathcal{R}$ -convergent to some  $y_m \in X$ . Then, for  $p, q \geq k_0$ ,

$$\begin{aligned} \|y_p - y_q\| &= \left\| \mathcal{R} \left( \left( \sum_{k=1}^n a_k x_k^q \right) \right) - \mathcal{R} \left( \left( \sum_{k=1}^n a_k x_k^p \right) \right) \right\| \\ &\leq M \sup_n \left\| \sum_{k=1}^n (a_k^p - a_k^q) x_k \right\| \\ &\leq M \|x^p - x^q\|_s \leq \varepsilon. \end{aligned}$$

Thus,  $(y_m)$  is a Cauchy sequence. Since  $X$  is complete, let  $y_0$  be its limit. We claim that  $\mathcal{R} \left( \sum_{k=1}^n a_k x_k^0 \right) = y_0$ . Indeed, for any  $\varepsilon > 0$ , there exists  $p$  such that  $\|y_p - y_0\| \leq \frac{\varepsilon}{2}$  and  $\|x^p - x^0\|_s \leq \frac{\varepsilon}{2M}$ . Since  $\mathcal{R}$  satisfies **(h3)**:

$$\left\| \mathcal{R} \left( \left( \sum_{k=1}^n a_k x_k^p \right) \right) - \mathcal{R} \left( \left( \sum_{k=1}^n a_k x_k^0 \right) \right) \right\| \leq M \|x^p - x^0\|_s \leq \frac{\varepsilon}{2}.$$

Hence,

$$\begin{aligned} \left\| \mathcal{R} \left( \left( \sum_{k=1}^n a_k x_k^0 \right) \right) - y_0 \right\| &= \left\| \mathcal{R} \left( \left( \sum_{k=1}^n a_k x_k^0 \right) \right) - y_p + y_p - y_0 \right\| \\ &\leq \left\| \mathcal{R} \left( \left( \sum_{k=1}^n a_k x_k^p \right) \right) - \mathcal{R} \left( \left( \sum_{k=1}^n a_k x_k^0 \right) \right) \right\| + \|y_p - y_0\| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since  $\varepsilon$  was arbitrary, we obtain that  $\mathcal{R} \left( \sum_{k=1}^n a_k x_k^0 \right) = y_0$  as we desired.  $\square$

**Remark 4.** Thus, using Proposition 5, if  $\mathcal{R}$  is a linear convergence method satisfying **(h2)**, then the following chain of inclusions are true:  $X(\ell_\infty) \subset X(S, \mathcal{R}) \subset X_\omega(S, \rho) \subset X(c_0)$ .

As a Corollary of Theorem 3, we get:

**Theorem 4.** Let  $\rho$  be a convergence method on  $\mathbb{R}$  and  $\mathcal{R}$  its induced convergence method in  $X$ . If  $\mathcal{R}$  satisfies hypothesis **(h3)**, then  $X_\omega(S, \rho) \subset X(c_0)$  is closed.

The key to the proof of Theorem 5 is to ensure that  $R$  induces a bounded linear operator; we can guarantee this condition thanks to the hypothesis **(h3)**.

**Lemma 1.** Let  $X$  be a Banach space and let  $\mathcal{R}$  be a convergence method satisfying **(h3)**. For each closed subspace  $S, c_0 \subset S \subset \ell_\infty$  and  $x = (x_k) \in X(S, \mathcal{R})$ , the linear operator  $\Sigma_x : S \rightarrow X$ , defined by

$$\Sigma_x((a_k)_k) = \mathcal{R} \left( \left( \sum_{k=1}^n a_k x_k \right) \right),$$

is bounded.

**Proof.** Indeed, if  $(a_k) \in S$ , then  $\mathcal{R}(\sum_k a_k x_k)$  exists, therefore the mapping  $\Sigma_x$  is well defined. Since  $(x_k) \in X(\ell_\infty)$ , let us denote by  $M = \|x\|_s$ . For every  $(a_k) \in S \subset \ell_\infty$ , we obtain:

$$\begin{aligned} \Sigma_x((a_k)_k) &= \mathcal{R} \left( \left( \sum_{k=1}^n a_k x_k \right) \right) \\ &= \mathcal{R} \left( \left( \| (a_k) \|_\infty \sum_{k=1}^n \frac{1}{\| (a_k) \|_\infty} a_k x_k \right) \right) \\ &= \| (a_k) \|_\infty \mathcal{R} \left( \left( \sum_{k=1}^n \frac{1}{\| (a_k) \|_\infty} a_k x_k \right) \right), \end{aligned} \tag{4}$$

in the last equality, we have used that  $\mathcal{R}$  is linear. On the other hand, since  $\mathcal{R}$  satisfies **(h3)**, there exists  $M > 0$  such that

$$\left\| \mathcal{R} \left( \left( \sum_{k=1}^n k \frac{1}{\|(a_k)\|_\infty} a_k x_k \right) \right) \right\| \leq M \sup_n \left\{ \left\| \sum_{k=1}^n \frac{1}{\|(a_k)\|_\infty} a_k x_k \right\| \right\} \leq M \|x\|_s.$$

Therefore, using the last inequality in Equation (4), we get:

$$\|\Sigma_x((a_k)_k)\| \leq M \|x\|_s \|(a_k)\|_\infty;$$

that is, the linear operator  $\Sigma_x$  is bounded as we desired.  $\square$

A vector subspace  $M$  of the dual  $X^{**}$  of a real Banach space  $X$  is called a  $M$ -Grothendieck space if every sequence in  $X^*$  which is  $\sigma(X^*, X)$  convergent is also  $\sigma(X^*, M)$  convergent. In particular,  $X$  is said to be Grothendieck if it is  $X^{**}$ -Grothendieck, that is, every weakly- $\star$  convergent sequence in the dual space  $X^*$  converges with respect to the weak topology of  $X^*$ .

There are many summability methods for which satisfy **(h3)** in Theorem 5. For instance, of course the usual convergence, the statistical convergence, lacunary statistical convergence, the uniform almost convergence, any regular bounded matrix summability method, etc.

Next, we will prove the main result that, in a way, mimics some ideas that appear in [5,15]. It is surprising how this result unifies all known results and it applies to most summability methods.

**Theorem 5.** *Let  $X$  be a real Banach space, and let  $\mathcal{R}$  be a summability method satisfying **(h2)**, **(h3)**. Let  $(x^n)$  be a sequence in  $X(c_0)$ . Let  $S$  be a closed subspace of  $\ell_\infty$  containing  $c_0$  and assume that  $S$  is  $\ell_\infty$ -Grothendieck. If, for each  $(a_k) \in S$  the sequence  $y_n = \sum_{k=1}^\infty a_k x_k^n$   $\mathcal{R}$ -converges, then  $(x^n)$  converges in  $X(c_0)$ .*

**Proof.** Suppose the result is false. Then, there exist  $\delta > 0$  and a subsequence  $\{n_m\}$  such that  $\|x^{n_m} - x^{n_{m+1}}\|_s > \delta$  for all  $m \in \mathbb{N}$ . For each  $k \in \mathbb{N}$ , let us denote  $z^m = x^{n_m} - x^{n_{m+1}}$ , that is,  $z^m = (z_i^m)_i$  and  $z_i^m = x_i^{n_m} - x_i^{n_{m+1}}$ , for every  $i \in \mathbb{N}$ .

Since

$$y_{n_m} - y_{n_{m+1}} = \sum_{j=1}^\infty a_j (x_j^{n_m} - x_j^{n_{m+1}}) = \sum_{j=1}^\infty a_j z_j^m$$

and  $\mathcal{R}(y_n) = L_a$ , for each  $a = (a_j) \in S$ . Using the linearity of  $\mathcal{R}$ , we get:

$$0 = \mathcal{R}((y_{n_m} - y_{n_{m+1}})) = \mathcal{R} \left( \left( \sum_{j=1}^\infty a_j z_j^m \right) \right). \tag{5}$$

On the other hand, since

$$0 < \delta < \|z^m\|_s = \sup_{f \in B_{X^*}} \sum_{j=1}^\infty |f(z_j^m)|,$$

we obtain that, for any  $m$ , there exists  $f_m \in B_{X^*}$  such that

$$\sum_{i=1}^\infty |f_m(z_i^m)| > \delta. \tag{6}$$

Let us consider the family of linear operators  $\Sigma_{z^m} : S \rightarrow X$ , defined by

$$\Sigma_{z^m}(a_i) = \mathcal{R} \left( \left( \sum_{i=1}^n a_i z_i^m \right) \right).$$



Since the operators  $\Sigma_{z^m}$  are bounded  $f_m \circ \Sigma_{z^m} \in S^*$ . Moreover, using Equation (5) and (h2), for each  $(a_j) \in S$  and for any  $f \in B_{X^*}$ , we get

$$\lim_n |f \circ \Sigma_{z^m}((a_i)_i)| = \lim_n \left| f \circ \mathcal{R} \left( \left( \sum_{j=1}^n a_j z_j^m \right) \right) \right| = \left| \sum_{j=1}^\infty a_j f(z_j^m) \right|.$$

Thus, for each  $(a_j) \in S$ ,

$$\begin{aligned} \lim_m |f_m \circ \Sigma_{z^m}(a_j)| &= \lim_m \left| f_m \circ \mathcal{R} \left( \left( \sum_{j=1}^n a_j z_j^m \right) \right) \right| \\ &= \lim_m \left| \lim_n \sum_{j=1}^n a_j f_m(z_j^m) \right| \\ &= \lim_m \left| \sum_{j=1}^\infty a_j f_m(z_j^m) \right| \\ &= \lim_m \left| f_m \circ \mathcal{R} \left( \left( \sum_{j=1}^\infty a_j z_j^m \right) \right) \right| = 0 \end{aligned}$$

Hence,  $f_m \circ \Sigma_{z^m}((a_i)_i)$  is a weakly star convergent sequence in  $S^*$ , which converges to 0. Since  $S$  is  $\ell_\infty$ -Grothendieck, we obtain that for any  $h = (a_j) \in \ell_\infty$ :

$$\lim_{m \rightarrow \infty} h(f_m \circ \Sigma_{z^m}) = \lim_{m \rightarrow \infty} \sum_{j=1}^\infty a_j f_m(z_j^m) = 0.$$

That is, the sequence  $(f_m(z_i^m))_i$  is a null weakly convergent sequence in  $\ell_1$ ; therefore, the sequence  $(f_m(z_i^m))_i$  is norm convergent to 0 in  $\ell_1$ . This contradicts (6), and we obtain the desired result.  $\square$

**Remark 5.** Let  $S$  be a subspace of  $\ell_\infty$  containing  $c_0$ . We consider the inclusion map  $\iota : c_0 \rightarrow S$  and the corresponding bidual map which is an isometry from  $c_0^{**} = \ell_\infty$  into  $S^{**}$ . As a consequence, it is intriguing to characterize the subspaces  $S$  which are  $\ell_\infty$ -Grothendieck. Theorem 5 is true for  $S = \ell_\infty$ , but it also continues true for every subspace  $S \subset \ell_\infty$  which are  $\ell_\infty$ -Grothendieck.

There are non-trivial subspaces of  $\ell_\infty$ , which are  $\ell_\infty$ -Grothendieck. As it was noted in [5] Remark 4.1, Haydon constructed, using transfinite induction, a Boolean Algebra  $\mathcal{F}$  containing the sets  $\{\{i\} : i \in \mathbb{N}\}$  whose corresponding space  $C(\mathcal{F})$  can be seen as a proper subspace of  $\ell_\infty$ , contains  $c_0$  and is also Grothendieck. We refer to the interested reader to a forthcoming paper ([17]) where we analyze the property  $\ell_\infty$ -Grothendieck and we obtain natural examples of such subspaces of  $\ell_\infty$ .

### 5. Conclusions

In this section, we are going to discuss a little bit about summability methods in general. Summability methods are a great tool that provides many applications in Applied Mathematics. In fixed point theory for non-expansive mappings, there are, for some classes of non-expansive mappings, iterative methods that converge to some fixed point [18,19]. A connection of these results with different convergence methods will be interesting.

There is a strong connection between summability methods and approximation theory—for instance, a connection with Korovkin-type approximation theorems [20,21]. It would be very interesting to describe those properties that should exhibit a general a summability method in order to obtain Korovkin-type approximation theorems.

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