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Union of Sets of Lengths of Numerical Semigroups

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Abstract: Let $S = \langle a_1, \ldots, a_p \rangle$ be a numerical semigroup, let $s \in S$ and let $\mathsf{Z}(s)$ be its set of factorizations. The set of lengths is denoted by $\mathcal{L}(s) = \{\mathsf{L}(x_1, \ldots, x_p) \mid (x_1, \ldots, x_p) \in \mathsf{Z}(s)\}$, where $\mathsf{L}(x_1, \ldots, x_p) = x_1 + \cdots + x_p$. The following sets can then be defined: $\mathsf{W}(n) = \{s \in S \mid \exists x \in \mathsf{Z}(s) \text{ such that } \mathsf{L}(x) = n\}$, $\nu(n) = \bigcup_{s \in \mathsf{W}(n)} \mathcal{L}(s) = \{l_1 < l_2 < \cdots < l_r\}$ and $\Delta\nu(n) = \{l_2 - l_1, \ldots, l_r - l_{r-1}\}$. In this paper, we prove that the function $\Delta\nu : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ is almost periodic with period $\operatorname{lcm}(a_1, a_p)$.

Keywords: delta-set; non-unique factorization; numerical monoid; numerical semigroup

MSC: 11A51; 20M14; 11B75

1. Introduction

A numerical semigroup (or numerical monoid) is a finitely generated subsemigroup of the set of nonnegative integers $\mathbb{N}$, such that the group generated by it is the set of all integers $\mathbb{Z}$. Every numerical semigroup is finitely generated and their elements might be expressed in different ways as a linear combination with non-negative integer coefficients of its generators. Each such expression is usually known as a factorization of the element.

For many rings and semigroups, their elements can be written as finite products (or sums) of other elements, but in general such factorizations are not unique, which is not the case for the ring of integer numbers. Non-unique factorization theory describes and classifies these properties using invariants of the algebraic structure in question (see [1] for further background). From among the relevant parameters, we can highlight the ω-primality, the tame degree, the Δ-set and the elasticity. What these try to measure, in one way or another, is how far a semigroup or a ring is from having unique factorization, and if factorization is not unique, they explain its behaviour. For example, if the Δ-set of an element is the empty set, this means that all its factorizations have the same length. Computation of these parameters is not trivial, however, because, in general, although their definitions might not be complicated, to establish appropriate and effective algorithms and relevant examples, it is necessary to have knowledge of a variety of properties (bounds, periodicity, etc.).

In recent years, two structures for which these parameters have been well studied are numerical and affine semigroups. We highlight, for example, the library "NumericalSgps" made in GAP [2], where functions are implemented to compute some of these parameters. Along the same line, we mention the work in [3–5] and many of the references cited therein.

In this paper, we start from the definition of the Δ-set of the elements of a numerical semigroup and we define Δ of the union of sets of elements. This parameter has been discussed widely in the literature. Generalized sets of lengths were studied in Dedekind domains by Chapman and Smith [6], who had earlier determined their asymptotic behaviour [7]. Amos et al. [8] obtained some properties of the set ν_n for numerical semigroups generated by an arithmetic progression. Baginski et al. [9] computed the set $\Delta\nu(M)$ for several monoids and also studied the asymptotic behaviour of $\Delta\nu_n$. This invariant was also analysed by Chapman et al. [10]. More recently, Geroldinger [11] surveyed some parameters and proved some results on the structure of ν_n, using the fact that $d = \min(\Delta(S)) = \gcd\{a_{i+1} - a_i \mid i = 1, \ldots, p-1\}$. These sets are almost arithmetic progressions and therefore $\Delta\nu(S) \subset \{d, 2d, 3d, \ldots\}$.

The main goal of this work is to give properties of the set of lengths of a numerical semigroup and to obtain algorithms that allow computation of the function $\Delta\nu$. We prove that for its computation, we do not need to calculate the Δ-set of all the elements involved and thus we improve its computation in a remarkable way. We also show that this function is almost periodic and we use this period and its bound for obtaining the function $\Delta\nu$ for any numerical semigroup. We provide some examples that illustrate these algorithms. The software developed and all the associated examples can be downloaded from [12].

In Section 2, we give some basic definitions and introduce the notation that we use through this paper. Section 3 is devoted to explaining the behaviour of the function $\Delta\nu$, and an improved algorithm for computing it is also given there. Finally, in Section 4, we study the periodicity of $\Delta\nu$ and provide some examples.

2. Definitions and Notation

Denote by $\mathbb{N}$ the set of non-negative integers. In this work, S denotes a primitive numerical monoid (or numerical semigroup). Since every numerical monoid is finitely generated, there exist $a_1, \ldots, a_p \in \mathbb{N}$ such that $S = \langle a_1 < \cdots < a_p \rangle = \{\sum_{i=1}^{p} \lambda_i a_i \mid \lambda_1, \ldots, \lambda_p \in \mathbb{N}\}$. If M is the subgroup of $\mathbb{Z}^p$ defined by the equation $a_1 x_1 + \cdots + a_p x_p = 0$ and $\sim_M$ is the equivalence relation on $\mathbb{N}^p$ defined by $z \sim_M z'$ if $z - z' \in M$, then the semigroup S is isomorphic to the quotient $\mathbb{N}^p / \sim_M$.

Let s be an element of S. If $(x_1, \ldots, x_p) \in \mathbb{N}^p$ satisfies $\sum_{i=1}^{p} x_i a_i = s$, then we say that $(x_1, \ldots, x_p)$ is a factorization of s. We denote by $\mathtt{Z}(s)$ the set $\{(x_1, \ldots, x_p) \in \mathbb{N}^p \mid \sum_{i=1}^{p} x_i a_i = s\}$ and we call it the set of factorizations of s.

Define the linear function $\mathtt{L} : \mathbb{Q}^p \to \mathbb{Q}$ as $\mathtt{L}(x_1, \ldots, x_p) = x_1 + \cdots + x_p$. The length of a factorization x of $s \in S$ is the number $\mathtt{L}(x)$.

The following definition is found in [4,13].

Definition 1. *Given $s \in S$ and $S = \langle a_1, \ldots, a_p \rangle$, the set $\mathcal{L}(s) = \{\mathtt{L}(x_1, \ldots, x_p) \mid (x_1, \ldots, x_p) \in \mathtt{Z}(s)\}$ is called the set of lengths of s in S. Since S is a numerical monoid, it is not hard to prove that this set of lengths is bounded, and so there exist some positive integers $l_1 < \cdots < l_k$ such that $\mathcal{L}(s) = \{l_1, \ldots, l_k\}$. The set*

$$\Delta(s) = \{l_i - l_{i-1} : 2 \leq i \leq k\}$$

is called the Δ-set of s.

The set

$$\Delta(S) = \bigcup_{s \in S} \Delta(s)$$

is called the Δ-set of S.

In [4], it was proved that for every numerical semigroup S, the function $\Delta : S \to \mathcal{P}(\mathbb{N})$ is almost periodic. The following definition is found in [8,9,11,14].

Definition 2. *Let* $S = \langle a_1, \ldots, a_p \rangle$ *and* $n \in \mathbb{N}$.

- *Define* $\mathsf{W}(n) = \{s \in S \mid \exists x \in \mathsf{Z}(s) \text{ such that } \mathsf{L}(x) = n\}$.
- *Define* $\nu(n) = \bigcup_{s \in \mathsf{W}(n)} \mathcal{L}(s)$.

If $\nu(n) = \{l_1 < l_2 < l_3 < \cdots < l_r\}$, *then*

$$\Delta\nu(n) = \{l_2 - l_1, l_3 - l_2, \ldots, l_r - l_{r-1}\}$$

and

$$\Delta\nu(S) = \bigcup_{n \in \mathbb{N}} \Delta\nu(n).$$

Clearly, for every $n \in \mathbb{N}$, the set $\Delta\nu(n)$ is a subset of $\mathbb{N}$. Thus, for a S numerical semigroup, we define $\Delta\nu$ as follows:

$$\begin{aligned} \Delta\nu : \mathbb{N} &\to \mathcal{P}(\mathbb{N}), \\ n &\to \Delta\nu(n). \end{aligned}$$

The main aim of this work is to prove that this function is almost periodic and that its period is a divisor of $\mathrm{lcm}(a_1, a_p)$.

An unrefined method for computing $\Delta\nu(n)$ is presented in Algorithm 1.

Algorithm 1 Sketch of the algorithm to compute $\Delta\nu(n)$.

INPUT: $S = \langle a_1, \ldots, a_p \rangle$ a numerical semigroup and $n \in \mathbb{N}$.

OUTPUT: $\Delta\nu(n)$.

1: $A := \left\{(x_1, \ldots, x_p) \mid \sum_{i=1}^{p} x_i a_i = n\right\}$.
2: $\mathsf{W}(n) := \left\{\sum_{i=1}^{p} x_i a_i \mid (x_1, \ldots, x_p) \in A\right\}$.
3: $\mathfrak{L} = \bigcup_{s \in \mathsf{W}(n)} \mathcal{L}(s)$.
4: **return** $\Delta\mathfrak{L}$.

The tuples $(n, 0, 0, \ldots, 0)$, $(n-1, 1, 0, \ldots, 0)$, $\ldots$, $(0, n, 0, \ldots, 0)$ are factorizations of different elements. So, $\lim_{n \to +\infty} \#\mathsf{W}(n) = \infty$.

Example 1. *Let* $S = \langle 5, 9, 11 \rangle$ *and* $n = 100$. *The cardinality of* $\mathsf{W}(100)$ *is* 300 *and for the computation of* $\Delta\nu(100)$ *using Algorithm 1, it is necessary to know the factorizations of all of the elements of* $\mathsf{W}(100)$. *In the following section, we prove that for any* $n \in \mathbb{N}$, *it is only necessary to calculate the factorizations of* 220 *elements for computing* $\Delta\nu(n)$.

This number increases with n. For instance, if $n = 200$, *the cardinality of* $\mathsf{W}(200)$ *is* 600, *but with Algorithm 2 it is again only necessary to compute the factorizations of* 220 *of the elements of* $\mathsf{W}(200)$.

Algorithm 2 Sketch of the algorithm to compute $\Delta\nu(n)$.

INPUT: $S = \langle a_1, \ldots, a_p \rangle$ a numerical semigroup and $n \in \mathbb{N}$.

OUTPUT: $\Delta\nu(n)$.

$d := \gcd(a_2 - a_1, \ldots, a_p - a_{p-1})$.
Compute N_S as in §3 [4].
$C_1 := (a_p - a_{p-1})N_S a_{p-1}$, $C_2 := (a_1 - a_2)N_S a_2$,
$C_3 := \left(-\frac{a_p}{a_1} + \frac{a_p}{a_2} - \frac{a_p}{a_{p-1}} + 1\right)N_S$, $C_4 := \left(\frac{a_1}{a_{p-1}} - \frac{a_1}{a_p} - \frac{a_1}{a_2} + 1\right)N_S$.
$\lambda_1 := \max(C_1, C_4)$, $\lambda_2 := -\min(C_2, C_3)$.
Compute N_0 as in Proposition 1.
if $n \leq N_0$ **then**
    Compute $\Delta\nu(n)$ using Algorithm 1.
    return $\Delta\nu(n)$.
$x_1 := na_1 + \lceil \lambda_1 \rceil$.
$x_2 := na_p - \lfloor \lambda_2 \rfloor$.
$W_3 := \mathsf{W}(n) \cap [na_1, x_1]$.
$B_3(n) := \left\{x \in \bigcup_{s \in W_3} \mathcal{L}(s) \mid x \leq \frac{x_1}{a_p}\right\}$.
$W_1 := \mathsf{W}(n) \cap [x_2, na_p]$.
$B_1(n) := \left\{x \in \bigcup_{s \in W_1} \mathcal{L}(s) \mid x \geq \frac{x_2}{a_1}\right\}$.
Compute $\Delta B_3(n)$.
Compute $\Delta B_1(n)$.
return $\Delta B_3(n) \cup \{d\} \cup \Delta B_1(n)$.

3. Computation of $\Delta\nu(n)$

In [4], it is proved that there exist $\delta \in \mathbb{N}$ and a bound $N_S \in \mathbb{N}$ such that $\delta | \operatorname{lcm}(a_1, a_p)$ and, for every $s \in S$ with $s \geq N_S$, we have $\Delta(s + \delta) = \Delta(s)$.

It is straightforward to prove that $\min \mathsf{W}(n) = na_1$ and $\max \mathsf{W}(n) = na_p$. We use the notation of [4], and the definitions of the elements N_S, $\vec{w}$ and $\vec{w}'$ can also be found there. We recall that, explicitly,

$$
\begin{aligned}
d &= \gcd\{a_{i+1} - a_i \mid i = 1, \ldots, p-1\}, \\
S_i &= -\frac{a_2\left(a_1 d \gcd\left(a_i - a_1, a_1 - a_p, a_p - a_i\right) + (p-2)\left(a_1 - a_i\right)\left(a_1 - a_p\right)\right)}{\left(a_1 - a_2\right)\gcd\left(a_i - a_1, a_1 - a_p, a_p - a_i\right)}, \\
S_i' &= \frac{a_{p-1}\left((p-2)\left(a_1 - a_p\right)\left(a_p - a_i\right) - d a_p \gcd\left(a_i - a_1, a_1 - a_p, a_p - a_i\right)\right)}{\left(a_{p-1} - a_p\right)\gcd\left(a_i - a_1, a_1 - a_p, a_p - a_i\right)}, \\
N_S &= \lceil \max(\{S_i \mid i = 2, \ldots, p-1\} \cup \{S_i' \mid i = 2, \ldots, p-1\}) \rceil, \\
\vec{w} &= \frac{N_S(a_2 - a_p)}{a_2(a_1 - a_p)} e_1 + \frac{N_S(a_1 - a_2)}{a_2(a_1 - a_p)} e_p - \frac{N_S}{a_1} e_1, \\
\vec{w}' &= \frac{N_S(a_{p-1} - a_p)}{a_{p-1}(a_1 - a_p)} e_1 + \frac{N_S(a_1 - a_{p-1})}{a_{p-1}(a_1 - a_p)} e_p - \frac{N_S}{a_p} e_p.
\end{aligned}
$$

Lemma 1. *Let S be a numerical semigroup and let N_S be the bound of [4]. Then, there exists $N_S' \in \mathbb{N}$ such that for every $n \geq N_S'$, we have $\min \mathsf{W}(n) \geq N_S$.*

Proof. The minimum of $\mathsf{W}(n)$ is equal to na_1. It is enough to take $N_S' \geq N_S / a_1$. □

Definition 3 (Definition 15 [4]). *Let $S = \langle a_1, \ldots, a_p \rangle$ be a numerical monoid. For every $s \in \mathbb{N}$ such that $s \geq N_S$, define*

- $\mathsf{Z}_1(s)$ *the set of elements $x = (x_1, \ldots, x_p) \in \mathsf{Z}(s)$ satisfying $s/a_1 + \mathsf{L}(\overrightarrow{w}) < \mathsf{L}(x) \leq s/a_1$;*
- $\mathsf{Z}_2(s)$ *the set of elements $x = (x_1, \ldots, x_p) \in \mathsf{Z}(s)$ satisfying $s/a_p + \mathsf{L}(\overrightarrow{w}') - d \leq \mathsf{L}(x) \leq s/a_1 + \mathsf{L}(\overrightarrow{w}) + d$;*
- $\mathsf{Z}_3(s)$ *the set of elements $x = (x_1, \ldots, x_p) \in \mathsf{Z}(s)$ satisfying $s/a_p \leq \mathsf{L}(x) < s/a_p + \mathsf{L}(\overrightarrow{w}')$.*

Note that

$$\mathsf{L}(\vec{w}) = \frac{(a_1 - a_2)N_S}{a_1 a_2}, \qquad \mathsf{L}(\vec{w}') = \frac{(a_p - a_{p-1})N_S}{a_p a_{p-1}}.$$

Let C_i be the following values:

$$C_1 = \frac{(a_p - a_{p-1})N_S}{a_{p-1}}, \qquad C_2 = \frac{(a_1 - a_2)N_S}{a_2},$$

$$C_3 = \left(-\frac{a_p}{a_1} + \frac{a_p}{a_2} - \frac{a_p}{a_{p-1}} + 1\right)N_S, \qquad C_4 = \left(\frac{a_1}{a_{p-1}} - \frac{a_1}{a_p} - \frac{a_1}{a_2} + 1\right)N_S.$$

Define $\lambda_1 = \max(C_1, C_4)$ and $\lambda_2 = -\min(C_2, C_3)$.

Proposition 1. *For every $n \geq N_0 = \max\left(\frac{N_S}{a_1}, \frac{a_p - a_1 + \lambda_1 + \lambda_2}{a_p - a_1}\right)$, we have*

$$\Delta\nu(n) = \Delta\left(\bigcup_{x \in [na_1, na_1 + \lambda_1] \cup [na_p - \lambda_2, na_p]} \mathsf{Z}(x)\right).$$

Proof. Let $n \geq N_0$. Then, by Lemma 1, we obtain that $x \geq N_S$ for all $x \in \mathsf{W}(n)$.

Using the properties of the sets Z_i (Definition 3), for every $x \in \mathsf{W}(n)$ with $x \geq N_0$, there exists $c_1 \in \mathsf{Z}_1(x)$ such that $\mathsf{L}(c_1) = \min\{\mathsf{L}(x) \mid x \in \mathsf{Z}_1(x)\}$ and $b_1 \in \mathsf{Z}_1(x)$ such that $\mathsf{L}(b_1) = \max\{\mathsf{L}(x) \mid x \in \mathsf{Z}_1(x)\}$. We have that $\mathsf{L}(b_1) \leq x/a_1$ and that $x/a_1 + \mathsf{L}(\vec{w}) \leq \mathsf{L}(c_1)$. Analogously, there exists $c_2 \in \mathsf{Z}_3(x)$ such that $\mathsf{L}(c_2) = \min\{\mathsf{L}(x) \mid x \in \mathsf{Z}_3(x)\}$ and $b_2 \in \mathsf{Z}_3(x)$ such that $\mathsf{L}(b_2) = \max\{\mathsf{L}(x) \mid x \in Z_3(x)\}$. Thus, $x/a_p \leq \mathsf{L}(c_2)$ and $\mathsf{L}(b_2) \leq x/a_p + \mathsf{L}(\vec{w}')$.

The following system of inequalities is obtained:

$$\frac{x}{a_p} > \frac{na_1}{a_p} + \mathsf{L}(\vec{w}'), \tag{1}$$

$$\frac{x}{a_1} < \frac{na_p}{a_1} + \mathsf{L}(\vec{w}), \tag{2}$$

$$\frac{x}{a_p} + \mathsf{L}(\vec{w}') < n + \mathsf{L}(\vec{w}), \tag{3}$$

$$\frac{x}{a_1} + \mathsf{L}(\vec{w}) > n + \mathsf{L}(\vec{w}'). \tag{4}$$

These inequalities can be summarized as follows:

$$na_1 + \lambda_1 < x < na_p - \lambda_2. \tag{5}$$

If (1) and (3) are satisfied, then we get $\mathtt{L}(\mathsf{Z}_1(x)) \subset \mathtt{L}(\mathsf{Z}_2(na_1))$. With (2) and (4), we obtain $\mathtt{L}(\mathsf{Z}_3(x)) \subset \mathtt{L}(\mathsf{Z}_2(na_p))$. From (3) and (4), we get $\mathtt{L}(\mathsf{Z}_1(na_1)) \subset \mathtt{L}(\mathsf{Z}_2(x))$ and $\mathtt{L}(\mathsf{Z}_3(na_p)) \subset \mathtt{L}(\mathsf{Z}_2(x))$. Finally, $\mathtt{L}(\mathsf{Z}_1(x) \cup \mathsf{Z}_3(x)) \subset \mathtt{L}(\mathsf{Z}_2(na_1) \cup \mathsf{Z}_2(na_p))$ and $\mathtt{L}(\mathsf{Z}_1(na_1) \cup \mathsf{Z}_3(na_p) \subset \mathtt{L}(\mathsf{Z}_2(x))$. Therefore, if there exists a solution of (5), we obtain that $\Delta(\cup\{\mathsf{Z}(x) | x \in (na_1 + \lambda_1, na_p - \lambda_2)\}) = \{d\}$.

To finish the proof, we now prove the existence of solutions of (5). Note that there exists n such that $na_p - \lambda_2 > na_1 + \lambda_1$ and $(na_p - \lambda_2) - (na_1 + \lambda_1) > a_p - a_1$. Thus, there exists $k \in \mathbb{N}$ with $k \leq n$ such that $na_1 + \lambda_1 < na_1 + k(a_p - a_1) < na_p - \lambda_2$ and the element $na_1 + k(a_p - a_1)$ belongs to $\mathsf{W}(n)$. This is fulfilled if $(na_p - \lambda_2) - (na_1 + \lambda_1) > a_p - a_1$, which is satisfied if and only if

$$n > \frac{a_p - a_1 + \lambda_1 + \lambda_2}{a_p - a_1}.$$

Thus, we assert that there exists $x \in \mathsf{W}(n)$ satisfying (5). □

With the notation of Proposition 1, we give the following definitions.

Definition 4. *Let $n \geq N_0$. Consider three zones in $\nu(n)$: $B_3(n)$, $B_2(n)$ and $B_1(n)$, given by*

$$B_3(n) = \left\{ x \in \nu(n) \;\middle|\; x < \frac{na_1 + \lambda_1}{a_p} \right\},$$

$$B_1(n) = \left\{ x \in \nu(n) \;\middle|\; x > \frac{na_p - \lambda_2}{a_1} \right\},$$

$$B_2(n) = \nu(n) \setminus (B_1 \cup B_3).$$

Remark 1. *From the construction given in Proposition 1, we have that $\Delta\nu(n) = \Delta B_1(n) \cup \Delta B_2(n) \cup \Delta B_3(n)$ and $\Delta B_2(n) = \{d\}$.*

Example 2. *Let S be the numerical semigroup generated by $\langle 4, 9, 10, 15\rangle$. In this case, $N_0 = 73$, which means that if we compute $\Delta\nu(n)$ with n greater than 73, for example $n = 130$, we can save a lot of computation. In this case, $W(130) \subset [520, 1950]$, $\lambda_1 = 203$, $\lambda_2 = 759$, $x_1 = 723$ and $x_2 = 1191$. Therefore, by using Algorithm 2, we have 468 values that we can skip.*

The attractive aspect of this algorithm is that even if we increase the value of n, we only have to compute the same number of elements. For instance for $n = 150$, $W(150) \subset [600, 2250]$, but since λ_1 and λ_2 do not depend on n, we save 688 evaluations.

4. Periodicity of $\Delta\nu : \mathbb{N} \to \mathcal{P}(\mathbb{N})$

The main result of this work is presented in this section. This result allows us to give some examples where we compute the function $\Delta\nu$ for some numerical semigroups.

Proposition 2. *Let $n \geq N_0$. Then, $\Delta B_1(n) = \Delta B_1(n + \mu a_1)$, $\Delta B_3(n) = \Delta B_3(n + \mu a_p)$ and $\Delta B_2(n) = \Delta B_2(n + \mu a_i)$ for all $i \in \{1, \ldots, p\}$ and for all $\mu \in \mathbb{N}$.*

Proof. Trivially, $\Delta B_2(n) = \Delta B_2(n + \mu a_i) = \{d\}$ for every $n \geq N_0$.

Let $x \in \Delta B_3(n)$. Then, there exist $s_1, s_2 \in [na_1, na_1 + \lambda_1] \cap \mathsf{W}(n)$, $z_1 \in \mathsf{Z}(s_1)$ and $z_2 \in \mathsf{Z}(s_2)$ with $\mathtt{L}(z_1), \mathtt{L}(z_2) < (na_1 + \lambda_1)/a_p$ satisfying $\mathtt{L}(z_1) - \mathtt{L}(z_2) = x$, and there is no $z \in \nu(n)$ such that $\mathtt{L}(z_2) < \mathtt{L}(z) < \mathtt{L}(z_1)$. Let $\tilde{s}_1 = s_1 + \mu a_p$ and $\tilde{s}_2 = s_2 + \mu a_p$. We have that $z_1 + \mu e_p \in \mathsf{Z}(\tilde{s}_1)$ and $z_2 + \mu e_p \in \mathsf{Z}(\tilde{s}_2)$ satisfying $\mathtt{L}(\tilde{z}_1) - \mathtt{L}(\tilde{z}_2) = x$. Furthermore, $\tilde{s}_1, \tilde{s}_2$ belong to $[na_1 + \mu a_p, na_1 + \lambda_1 + \mu a_p] \cap \mathsf{W}(n + \mu a_p)$.

If there is an element $\tilde{s} \in \mathtt{W}(n + \mu a_p)$ with $\tilde{z} \in \mathtt{Z}(\tilde{s})$ such that $\mathtt{L}(\tilde{z}_2) < \mathtt{L}(\tilde{z}) < \mathtt{L}(\tilde{z}_1)$, then, when we consider the element $\tilde{s} - \mu a_p$, we obtain that this element has a factorization z that satisfies $\mathtt{L}(z_2) < \mathtt{L}(z) < \mathtt{L}(z_1)$, which is a contradiction. Thus, we have proved that $\Delta B_3(n) \subset \Delta B_3(n + \mu a_p)$. In the same way, the other inclusion can be proved, and so $\Delta B_3(n) = \Delta B_3(n + \mu a_p)$.

The proof that $\Delta B_1(n) = \Delta B_1(n + \mu a_1)$ is analogous. □

Theorem 1. *Let S be a numerical semigroup. The function $\Delta\nu : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ is almost periodic with period $\delta = \mathrm{lcm}(a_1, a_p)$. A bound from which this function is periodic is N_0.*

Proof. From Proposition 2, $\Delta B_2(n) = \{d\}$. On the other hand, B_1 and B_3 are periodic with periods a_p and a_1, respectively, so ΔB_1 and ΔB_3 have the same period. We now use the fact that $\Delta\nu(n) = \Delta B_1(n) \cup \Delta B_2(n) \cup \Delta B_3(n)$ to obtain that $\Delta\nu$ has period $\mathrm{lcm}(a_1, a_p)$. □

Finally, we illustrate the results of this work with some examples. In these examples, we show how we can compute $\Delta\nu(n)$ for several semigroups for all values of n. To do this, we use a supercomputer [15] to check the tree of numerical semigroups, in a parallel way, ordering these semigroups by their genus and examining them. We discard those semigroups of the form $\langle m, m+k, \ldots, m+qk\rangle$ with $k, q \in \mathbb{N}$, since they have already been studied in [8].

Example 3. *Here we have a collection of numerical semigroups with non-constant $\Delta\nu$.*

- *It is quite easy to find semigroups whose $\Delta\nu$ have constant periodic parts. For example, if S is the semigroup $\langle 3, 10, 11\rangle$, we have that $N_0 = 82$ and $\delta = 33$. Therefore, we only have to compute the first 115 values of $\Delta\nu$ to find all its values. After performing these computations, we have the following results: $\Delta\nu(1) = \emptyset$, $\Delta\nu(2) = \Delta\nu(3) = \Delta\nu(4) = \Delta\nu(7) = \{1, 2\}$ and $\Delta\nu(n) = \{1\}$ for $n \in \{5, 6\} \cup [8, 33]$. So, the real periodicity of this function is 1, and because of this, if $n \geq 34$, then $\Delta\nu(n) = \{1\}$. Further semigroups having $\Delta\nu$ with this behaviour are $\langle 10, 13, 15\rangle$, $\langle 4, 7, 9\rangle$ and $\langle 6, 8, 9, 11\rangle$.*
- *A more interesting semigroup is the following one. If $S = \langle 3, 10, 14\rangle$, then we only need to compute 102 values of $\Delta\nu$, since $N_0 = 60$ and $\delta = 42$. The results are*

$$\emptyset, \{1,4\}, \{1,3,4\}, \{1,3\}, \{1,3\}, \{1,4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2\}, \ldots .$$

  If $n \in [5, 59]$, then we have $\Delta\nu(n) = \{1, 4\}$ if $n \equiv 0 \bmod 3$, $\Delta\nu(n) = \{1, 2\}$ if $n \equiv 1 \bmod 3$ and $\Delta\nu(n) = \{1, 3\}$ if $n \equiv 2 \bmod 3$. If $n \geq 60$, then $\Delta\nu(n) = \{1, 2\}$ if $n \equiv 0 \bmod 3$, $\Delta\nu(n) = \{1, 3\}$ if $n \equiv 1 \bmod 3$ and $\Delta\nu(n) = \{1, 4\}$ if $n \equiv 2 \bmod 3$. The other values are $\Delta\nu(1) = \emptyset$, $\Delta\nu(2) = \{1, 4\}$, $\Delta\nu(3) = \{1, 3, 4\}$ and $\Delta\nu(4) = \{1, 3\}$. Hence, the real period is just 3. Other examples with non-constant periodic part are $\langle 5, 12, 16\rangle$, $\langle 6, 13, 17\rangle$, $\langle 10, 17, 21\rangle$, $\langle 17, 24, 28\rangle$ and $\langle 4, 9, 10, 15\rangle$.

Thanks to our software (available in [12]), it is not difficult to obtain semigroups with non-constant $\Delta\nu$ and even with non-constant periodic part. This software has been developed in C++ to achieve the maximum speed. However, we have provided a user-friendly interface for Python3 and IPython3 [16] notebooks using SWIG [17]. Therefore, the user can load our library in a Jupyter notebook and use its Python functions, which actually call our pre-compiled functions in C++, thereby mixing the efficiency of C++ with the user-friendliness of Python.

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References

1. Chapman, S.T.; Fontana, M.; Geroldinger, A.; Olberding, B. (Eds.) *Multiplicative Ideal Theory and Factorization Theory: Commutative and Non-Commutative Perspectives*; Springer: Berlin/Heidelberg, Germany, 2016.
2. Delgado, M.; García-Sánchez, P.A. Numericalsgps—A GAP Package. Version 1.1.10. Available online: https://www.gap-system.org/Packages/numericalsgps.html (accessed on 1 September 2020).
3. Chapman, S.T.; García-Sánchez, P.A.; Llena, D.; Ponomarenko, V.; Rosales, J.C. The catenary and tame degree in finitely generated commutative cancellative monoids. *Manuscripta Math.* **2006**, *120*, 253–264. [CrossRef]
4. García-García, J.I.; Moreno-Frías, M.A.; Vigneron-Tenorio, A. Computation of delta sets of numerical monoids. *Monatshefte FÜr Math.* **2015**, *178*, 457–472. [CrossRef]
5. García-García, J.I.; Moreno-Frías, M.A.; Vigneron-Tenorio, A. Computation of the ω-primality and asymptotic ω-primality with applications to numerical semigroups. *Isr. J. Math.* **2015**, *206*, 395–411. [CrossRef]
6. Chapman, S.T.; Smith, W.W. Generalized sets of lengths. *J. Algebra* **1998**, *200*, 449–471. [CrossRef]
7. Chapman, S.T.; Smith, W.W. On the lengths of factorizations of elements in an algebraic number ring. *J. Number Theory* **1993**, *43*, 24–30. [CrossRef]
8. Amos, J.; Chapman, S.T.; Hine, N.; Paixão, J. Sets of lengths do not characterize numerical monoids. *Integers* **2007**, *7*, A50.
9. Baginski, P.; Chapman, S.T.; Hine, N.; Paixão, J. On the asymptotic behavior of unions of sets of lengths in atomic monoids. *Involve* **2008**, *1*, 101–110. [CrossRef]
10. Chapman, S.T.; Freeze, M.; Smith, W.W. On generalized lengths of factorizations in Dedekind and Krull domains. In *Non-Noetherian Commutative Ring Theory*; Springer: Berlin/Heidelberg, Germany, 2000; pp. 117–137.
11. Geroldinger, A. Sets of lengths. *Am. Math. Mon.* **2016**, *123*, 960. [CrossRef]
12. Marín-Aragón, D.; García-García, I. D-Marina/Commutativemonoids: Draft of the Project. 2019. Available online: https://zenodo.org/record/3237400 (accessed on 1 September 2020).
13. Chapman, S.T.; Daigle, J.; Hoyer, R.; Kaplan, N. Delta Sets of Numerical Monoids Using Nonminimal Sets of Generators. *Commun. Algebra* **2010**, *38*, 2622–2634. [CrossRef]
14. Freeze, M.; Geroldinger, A. Unions of sets of lengths. *Funct. Approx. Comment. Math.* **2008**, *39*, 149–162. [CrossRef]
15. UCA Supercomputer Service. Available online: http://supercomputacion.uca.es/ (accessed on 1 September 2020).
16. Python Software Foundation. Python Language Reference, Version 3.6. Available online: http://www.python.org/ (accessed on 1 September 2020).
17. SWIG. Simplified Wrapper and Interface Generator. Available online: http://www.swig.org (accessed on 1 September 2020).