Partial Lyapunov Strictification: Dual Quaternion based Observer for 6-DOF Tracking Control

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***Abstract*—Based on the dual-quaternion description, a smooth six-degree-of-freedom observer is proposed to estimate the incorporating linear and angular velocity, called the dual angular velocity, for a rigid body. To establish the observer, some important properties of dual vectors and dual quaternions are established, additionally, the kinematics of dual transformation matrices is deduced, and the transition relationship between dual quaternions and dual transformation matrices is subsequently analyzed. An important feature of the observer is that all estimated states are ensured to be $\mathcal{C}^{\infty}$ continuous, and estimation errors are shown to exhibit asymptotic convergence. Furthermore, to achieve tracking control objectives, the proposed observer is combined with an independently designed proportional-derivative-like feedback control law (using full-state feedback), and a special Lyapunov "strictification" process is employed to ensure a separation property between the observer and the controller, which further guarantees almost global asymptotic stability of the closed-loop dynamics. Numerical simulation results for a prototypical spacecraft pose tracking mission application are presented to illustrate the effectiveness and robustness of the proposed method.**

***Index Terms*—Dual quaternion, Observer, Lyapunov strictification, 6-DOF control, Tracking control.**

I. Introduction

FOR various motion control problems of practical importance, such as on-orbit missions of spacecrafts (including monitoring, surveillance, refueling, on-orbit assembly), a follower spacecraft is often required to track both the time-varying relative positions and the reference attitude trajectories accurately and synchronously with respect to a leader spacecraft (i.e., six-degree-of-freedom (6-DOF) tracking or pose tracking). Beard et al. [1] proposed a coordination architecture for spacecraft formation tracking control problems, in which the leader-following strategy and the virtual-structure approach are introduced. A 6-DOF synchronization scheme of spacecraft formations for deep-space mission applications was presented in Ref. [2]. Kristiansen et al. [3] introduced backstepping and passivity control methods for 6-DOF tracking problems, and Lv et al. [4] addressed the input constraint problem under the similar background.

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Six-degree-of freedom spacecraft controllers based on the dual-number/dual-quaternion description have recently drawn significant attention in the literature. Qualitatively speaking, dual quaternions are the extension of the traditional Euler quaternions, but different in the sense that they can be used to describe not only the rotational motion of a rigid body, but also the translational motion synchronously. Compared to other 6-DOF description methods, dual quaternions have clear physical meanings and compact forms, and the intrinsic couplings of the relative translational motion with the rotational one are taken into account automatically in the dual-quaternion-based model. Another appealing property of dual quaternions is that the algebraic similarity between dual quaternions and quaternions could help designers a lot in the design process of control methods. Wang et al. [5], [6] provided detailed discussions of the dual-quaternion-based modeling process for the integrated 6-DOF kinematics and dynamics of rigid bodies, and they also proposed several finite-time control laws by using sliding mode methods. Based on the dual quaternion formulation, Filipe and Tsiotras [7] presented a robust control method with additional mass and inertia identification mechanisms. Refs. [8] and [9] further introduced several fault-tolerant control schemes for 6-DOF tracking operations of spacecrafts in the presence of actuator faults.

From a practical standpoint, due to the slow sampling rates and inherent noise characteristics of sensors, and also possible strict constraints on the cost and space (volume) available for sensors installation in case of cube/nano-satellites, reliable linear and angular velocity measurements may not always be feasible/available. For attitude tracking control problems, when angular velocities are unavailable, the passivity property of the rigid-body attitude dynamics allows control objectives to still be accomplished using only output (dynamic) feedback [10], [11], [12]. Another widely-studied method to solve this problem is designing observers to estimate angular velocities and establishing proper observer-controller architectures to achieve tracking control goals [13], [14], [15]. To our best knowledge, the first nonlinear angular velocity observer for rigid bodies under the quaternion kinematics description was proposed by Salcudean [16]. Zou derived distributed [17] and finite-time stable [18] observers for attitude tracking control problems. But because the fact that separation properties are usually difficult to be established for nonlinear systems, stability analyses of closed-loop systems with observer-controller architectures always involved in nontrivial theoretical complexities. Nicosia et al. [19] presented a nonlinear observer and guaranteed asymptotically bounded stability for the closed-loop system.

Recently, based on a switching logic, the separation property of the attitude tracking system with a nonlinear observer and an independently designed proportional-derivative-like (PD-like) controller was synthesized in Ref. [20], and the closed-loop system was rendered to be almost global stable. Ref. [21] further extended this result and guaranteed $\mathcal{C}^{\infty}$ continuity for all estimated states by the way of circumventing the need for switching within the angular velocity observer structure. However, when concerning the controller development and observer design for 6-DOF tracking control problems without both linear and angular velocities measurements, relevant studies are relatively limited. Ref. [22] extended the attitude-only result given in [12], and proposed a velocity-free output-feedback controller under the dual quaternion formulation. Based on the vectrix formalism, a 6-DOF output feedback control law was introduced in Ref. [23], and the closed-loop dynamics was guaranteed to be locally asymptotically stable. But these results are all given upon the passivity control strategy, so can't provide real-time linear/angular velocities estimations.

In this paper, significantly building upon former results given in Refs. [20] and [21], a novel 6-DOF observer under the dual-quaternion-based description is proposed to estimate the incorporating linear and angular velocity (i.e., the dual angular velocity) of a rigid body. The special structure of the observer ensures $\mathcal{C}^{\infty}$ continuity of all estimated states, and guarantees the global asymptotic convergence of estimation errors irrespective of prescribed control inputs. To establish the observer, some important mathematical properties of dual vectors, dual quaternions and dual transformation matrices are presented and proved, the kinematics of dual transformation matrices and the transition relationship between dual quaternions and dual transformation matrices are subsequently deduced. These results provide important building blocks for the establishment of all the main results in the paper.

Furthermore, the proposed observer is shown to satisfy a "separation" property when combined with an independently designed PD-like controller. This is achieved by utilizing a partial Lyapunov "strictification" strategy [24], [25], [26], in which a nonstrict Lyapunov function could be transformed into a strict one whose derivative contains additional non-positive terms of system states. Specifically, a strictification-like analysis is carried out for the controller part Lyapunov-like function to guarantee the boundedness of all tracking state errors. Subsequently, this result is employed to conduct a further strictification process to obtain a partially strict Lyapunov-like function for observer. The word "partially" is used to emphasize that, for a dual quaternion, only the vector component of it is contained in the "strictified" Lyapunov-like function's time derivative. Finally, by employing a composite function, consisting of both the new partially strict observer Lyapunov-like function and the controller one, the combined observer-controller scheme is proved to render almost global[1] asymptotic convergence of all tracking errors, and the separation property is guaranteed accordingly.

[1]Due to topological obstructions (noncontractible) of the configuration space of attitude motion $SO(3)$, it is impossible for any continuous state-feedback controller to render global asymptotic stability [27], [28], so the notion "almost global" is adopted here to imply the stability of attitude motion over an open and dense set in $SO(3)$.

It is noteworthy that, Refs. [20] and [21] (in which only 3-DOF attitude tracking problems are considered), especially the strictification process in Ref. [21], highly rely on the boundedness property of traditional quaternions ($\|\boldsymbol{q}\| = 1$), but dual quaternions don't inherit this property due to containing position information. This fact and also the intrinsic couplings between the orientational motion and the translational motion increase the difficulties and complexities for both the observer design and the subsequent controller development. From these points of view, the strictification process and the observer-controller construction presented in this paper are more general. To the best knowledge of the authors, this is the first time a dual angular velocity observer is designed, and also the first time a separation property is established for the rigid-body dynamics under the dual-quaternion formulation.

The remainder of the paper is organized as follows. In Section II, commonly-used definitions and operations of dual numbers and dual quaternions are reviewed, and some important properties of them are introduced and proved, the kinematics of dual transformation matrices and the transit relationship between dual quaternions and dual transformation matrices are also derived, and then the dual-quaternion-based relative kinematics and dynamics of rigid bodies are introduced. Based on these mathematical preliminaries, a novel dual angular velocity observer is proposed in Section III, and the convergence of estimation errors is analyzed. In Section IV, the observer is combined with a PD-like controller, by a special strictification process, the proof of the separation property and the stability analysis for the closed-loop system are presented. Then, numerical simulations for a prototypical pose tracking task of spacecrafts are presented in Section V to illustrate the effectiveness of the proposed method. Finally the paper is completed with some conclusions in Section VI.

Throughout the paper, $\mathbb{R}$ and $\hat{\mathbb{R}}$ are employed to denote the sets of real numbers and dual numbers, respectively. Superscript $\hat{\cdot}$ implies the corresponding quantity belongs to the dual number set. Right subscripts $\cdot_r$ and $\cdot_d$ denote the real part and the dual part of a dual number, respectively. Right superscript $\cdot^x$ means the corresponding vector is expressed in a frame $\mathcal{X}$. $\mathbb{H}$ and $\hat{\mathbb{H}}$ are the sets of quaternions and dual quaternions, respectively. $\hat{\mathbb{H}}_s$ denotes the scalar-part set of dual quaternions, while $\hat{\mathbb{H}}_v$ is the vector-part set. Note that $\hat{\mathbb{H}}_s \subset \hat{\mathbb{R}}$ and $\hat{\mathbb{H}}_v \subset \hat{\mathbb{R}}^3$. The notation $\|\cdot\|$ refers to the Euclidean norm, and $\cdot^{\mathrm{T}}$ refers to the transpose of vectors and matrices.

II. Mathematical Preliminaries

A. Dual Numbers and Dual Vectors

The concept of dual numbers was first introduced by Clifford [29], then named and perfected by Study [30]. The definition of a dual number is given as follows.

$$\hat{a} = a_r + \varepsilon a_d \tag{1}$$

where $\hat{a} \in \hat{\mathbb{R}}$ is a dual number, $a_r,\ a_d \in \mathbb{R}$ are called the real part and the dual part of $\hat{a}$, respectively. ε is referred to a special dual unit with rules:

$$\varepsilon^2 = 0 \text{ but } \varepsilon \neq 0 \tag{2}$$

Throughout the paper, $\hat{a}_1 \geq \hat{a}_2$ implies both the real part and the dual part of $\hat{a}_1$ are larger or equal to $\hat{a}_2$, where $\hat{a}_1, \hat{a}_2 \in \hat{\mathbb{R}}$.

Dual vectors are a class of dual numbers whose real and dual parts are vectors:

$$\hat{\boldsymbol{a}} = \boldsymbol{a}_r + \varepsilon \boldsymbol{a}_d \tag{3}$$

where $\boldsymbol{a}_r,\ \boldsymbol{a}_d \in \mathbb{R}^n$ are the real part and dual part of $\hat{\boldsymbol{a}}$, respectively. The zero dual vector $\hat{\mathbf{0}}_n \in \hat{\mathbb{R}}^n$ is defined as $\hat{\mathbf{0}}_n = \mathbf{0}_n + \varepsilon \mathbf{0}_n$, where $\mathbf{0}_n$ denotes the n-dimensional zero vector.

Some basic operations and properties of dual numbers and dual vectors are introduced as follows.

$$\lambda \hat{\boldsymbol{a}} = \lambda \boldsymbol{a}_r + \varepsilon \lambda \boldsymbol{a}_d \tag{4}$$

$$\hat{\boldsymbol{a}}^{\mathrm{T}} = \boldsymbol{a}_r^{\mathrm{T}} + \varepsilon \boldsymbol{a}_d^{\mathrm{T}} \tag{5}$$

$$\hat{a}\hat{\boldsymbol{a}} = a_r \boldsymbol{a}_r + \varepsilon(a_r \boldsymbol{a}_d + a_d \boldsymbol{a}_r) \tag{6}$$

$$\hat{\boldsymbol{a}}^s = \boldsymbol{a}_d + \varepsilon \boldsymbol{a}_r,\ (\hat{\boldsymbol{a}}^s)^s = \hat{\boldsymbol{a}} \tag{7}$$

$$\|\hat{\boldsymbol{a}}\| = \|\boldsymbol{a}_r\| + \varepsilon \|\boldsymbol{a}_d\|, \tag{8}$$

$$\hat{\boldsymbol{a}}_1 \pm \hat{\boldsymbol{a}}_2 = \boldsymbol{a}_{r1} \pm \boldsymbol{a}_{r2} + \varepsilon(\boldsymbol{a}_{d1} \pm \boldsymbol{a}_{d2}) \tag{9}$$

$$\hat{\boldsymbol{a}}_1 \cdot \hat{\boldsymbol{a}}_2 = \hat{\boldsymbol{a}}_1^{\mathrm{T}} \hat{\boldsymbol{a}}_2 = \boldsymbol{a}_{1r} \cdot \boldsymbol{a}_{2r} + \varepsilon(\boldsymbol{a}_{1r} \cdot \boldsymbol{a}_{2d} + \boldsymbol{a}_{1d} \cdot \boldsymbol{a}_{2r}) \tag{10}$$

$$\hat{\boldsymbol{A}}\hat{\boldsymbol{a}} = \boldsymbol{A}_r \boldsymbol{a}_r + \varepsilon(\boldsymbol{A}_r \boldsymbol{a}_d + \boldsymbol{A}_d \boldsymbol{a}_r) \tag{11}$$

$$\hat{\boldsymbol{a}}_1 \circ \hat{\boldsymbol{a}}_2 = \boldsymbol{a}_{1r}^{\mathrm{T}} \boldsymbol{a}_{2r} + \boldsymbol{a}_{1d}^{\mathrm{T}} \boldsymbol{a}_{2d},\ \hat{\boldsymbol{a}}_1^s \circ \hat{\boldsymbol{a}}_2^s = \hat{\boldsymbol{a}}_1 \circ \hat{\boldsymbol{a}}_2 \tag{12}$$

where $\lambda \in \mathbb{R}$, $\hat{\boldsymbol{a}}_1, \hat{\boldsymbol{a}}_2 \in \hat{\mathbb{R}}^n$, and $\hat{\boldsymbol{A}} = \boldsymbol{A}_r + \varepsilon \boldsymbol{A}_d \in \hat{\mathbb{R}}^{m \times n}$ is called a dual matrix. When $\hat{\boldsymbol{a}}_1,\ \hat{\boldsymbol{a}}_2 \in \hat{\mathbb{R}}^3$, one further has

$$\begin{aligned} \hat{\boldsymbol{a}}_1 \times \hat{\boldsymbol{a}}_2 &= -\hat{\boldsymbol{a}}_2 \times \hat{\boldsymbol{a}}_1 \\ &= \boldsymbol{a}_{1r} \times \boldsymbol{a}_{2r} + \varepsilon(\boldsymbol{a}_{1r} \times \boldsymbol{a}_{2d} + \boldsymbol{a}_{1d} \times \boldsymbol{a}_{2r}) \end{aligned} \tag{13}$$

Another important property of dual vectors is that the dual cross product can also be written as the multiplication of a skew-symmetric matrix and a vector: $\hat{\boldsymbol{a}} \times \hat{\boldsymbol{b}} = \hat{\boldsymbol{S}}(\hat{\boldsymbol{a}})\hat{\boldsymbol{b}}$, where $\hat{\boldsymbol{S}}(\hat{\boldsymbol{a}}) = -\hat{\boldsymbol{S}}^{\mathrm{T}}(\hat{\boldsymbol{a}}) = \boldsymbol{S}(\boldsymbol{a}_r) + \varepsilon \boldsymbol{S}(\boldsymbol{a}_d)$ is called the dual skew-symmetric matrix of the dual vector $\hat{\boldsymbol{a}}$, and satisfies

$$\begin{aligned} \hat{\boldsymbol{S}}(\hat{\boldsymbol{a}}) &= \begin{bmatrix} 0 & -\hat{a}_3 & \hat{a}_2 \\ \hat{a}_3 & 0 & -\hat{a}_1 \\ -\hat{a}_2 & \hat{a}_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a_{3r} & a_{2r} \\ a_{3r} & 0 & -a_{1r} \\ -a_{2r} & a_{1r} & 0 \end{bmatrix} \\ &\quad + \varepsilon \begin{bmatrix} 0 & -a_{3d} & a_{2d} \\ a_{3d} & 0 & -a_{1d} \\ -a_{2d} & a_{1d} & 0 \end{bmatrix} \end{aligned} \tag{14}$$

and here $\hat{a}_i = a_{ir} + \varepsilon a_{id}, i = 1, 2, 3$ are entries of $\hat{\boldsymbol{a}}$, with $\hat{\boldsymbol{a}} = [\hat{a}_1, \hat{a}_2, \hat{a}_3]^{\mathrm{T}}$.

Some other important properties of dual vectors are presented as follows.

Lemma 1: $\hat{\boldsymbol{a}} \times (\hat{\boldsymbol{b}} \times \hat{\boldsymbol{c}}) + \hat{\boldsymbol{b}} \times (\hat{\boldsymbol{c}} \times \hat{\boldsymbol{a}}) + \hat{\boldsymbol{c}} \times (\hat{\boldsymbol{a}} \times \hat{\boldsymbol{b}}) = \hat{\mathbf{0}}_3$.

Lemma 2: $\hat{\boldsymbol{a}} \times (\hat{\boldsymbol{b}} \times \hat{\boldsymbol{c}}) = \hat{\boldsymbol{b}}(\hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{c}}) - \hat{\boldsymbol{c}}(\hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}})$.

Lemma 3: $(\hat{\boldsymbol{a}} \times \hat{\boldsymbol{b}}) \cdot \hat{\boldsymbol{a}} = \hat{\mathbf{0}}_3$, $\hat{\boldsymbol{a}} \cdot (\hat{\boldsymbol{a}} \times \hat{\boldsymbol{b}}) = \hat{\mathbf{0}}_3$.

Lemma 4: $\hat{\boldsymbol{a}}^s \circ (\hat{\boldsymbol{a}} \times \hat{\boldsymbol{b}}) = \hat{\mathbf{0}}_3$.

Proof of Lemmas 1-4: See Appendix A.

Lemma 5 [22]: $\hat{\boldsymbol{a}} \circ (\hat{\boldsymbol{b}} \times \hat{\boldsymbol{c}}) = \hat{\boldsymbol{b}}^s \circ (\hat{\boldsymbol{c}} \times \hat{\boldsymbol{a}}^s) = \hat{\boldsymbol{c}}^s \circ (\hat{\boldsymbol{a}}^s \times \hat{\boldsymbol{b}})$.

B. Quaternion and Dual Quaternion

The unit quaternion is the most commonly-used method to describe the relative attitude motion between two reference frames. The definition of it is $\boldsymbol{q} = [\eta, \boldsymbol{\xi}^{\mathrm{T}}]^{\mathrm{T}}$, where $\eta \in \mathbb{R}$ and $\boldsymbol{\xi} = [\xi_1, \xi_2, \xi_3]^{\mathrm{T}} \in \mathbb{R}^3$ are called the scalar part and the vector part of $\boldsymbol{q}$, respectively, and satisfies $\eta^2 + \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi} = 1$. The multiplication of two quaternions $\boldsymbol{q}_1 = [\eta_1, \boldsymbol{\xi}_1^{\mathrm{T}}]^{\mathrm{T}}$ and $\boldsymbol{q}_2 = [\eta_2, \boldsymbol{\xi}_2^{\mathrm{T}}]^{\mathrm{T}}$ is defined as:

$$\boldsymbol{q}_1 \otimes \boldsymbol{q}_2 = [\eta_1\eta_2 - \boldsymbol{\xi}_1^{\mathrm{T}}\boldsymbol{\xi}_2, (\eta_1\boldsymbol{\xi}_2 + \eta_2\boldsymbol{\xi}_1 + \boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2)^{\mathrm{T}}]^{\mathrm{T}} \tag{15}$$

Based on the definition of quaternions, the coordinates of any real vector $\boldsymbol{a} \in \mathbb{R}^3$ in a frame $\mathcal{Y}$ can be calculated from the coordinates of another frame $\mathcal{X}$ of that same vector [31]

$$[0, (\boldsymbol{a}^y)^{\mathrm{T}}]^{\mathrm{T}} = \boldsymbol{q}_{yx}^* \otimes [0, (\boldsymbol{a}^x)^{\mathrm{T}}]^{\mathrm{T}} \otimes \boldsymbol{q}_{yx} \tag{16}$$

In which $\boldsymbol{q}_{yx}$ is the quaternion of the frame $\mathcal{Y}$ with respect to the frame $\mathcal{X}$, $\boldsymbol{q}_{yx}^* = [\eta_{yx}, -\boldsymbol{\xi}_{yx}^{\mathrm{T}}]^{\mathrm{T}}$ is the conjugate quaternion of $\boldsymbol{q}_{yx}$. Eq. (16) can also be represented by

$$\boldsymbol{a}^y = \boldsymbol{C}(\boldsymbol{q}_{yx})\boldsymbol{a}^x \tag{17}$$

and here

$$\boldsymbol{C}(\boldsymbol{q}_{yx}) = \mathbf{I}_{3\times3} - 2\eta_{yx}\boldsymbol{S}(\boldsymbol{\xi}_{yx}) + 2\boldsymbol{S}(\boldsymbol{\xi}_{yx})\boldsymbol{S}(\boldsymbol{\xi}_{yx}) \tag{18}$$

is called the transformation matrix from $\mathcal{X}$ to $\mathcal{Y}$.

The definition of dual quaternions are based on both quaternions and dual vectors, as a generic example, the dual quaternion of $\mathcal{Y}$ with respect to $\mathcal{X}$ is [6], [32]

$$\hat{\boldsymbol{q}}_{yx} = \boldsymbol{q}_{yx} + \varepsilon \frac{1}{2} \boldsymbol{q}_{yx} \otimes [0, (\boldsymbol{r}_{yx}^y)^{\mathrm{T}}]^{\mathrm{T}} \tag{19}$$

where $\boldsymbol{r}_{yx}^y$ is the position vector from the origin of frame $\mathcal{X}$ to the origin of frame $\mathcal{Y}$ (and expressed in frame $\mathcal{Y}$). $\hat{\boldsymbol{q}}_{yx}$ can also be written as the combination of a dual scalar and a dual vector: $\hat{\boldsymbol{q}}_{yx} = [\hat{\eta}_{yx}, \hat{\boldsymbol{\xi}}_{yx}^{\mathrm{T}}]^{\mathrm{T}}$, where $\hat{\eta}_{yx} \in \hat{\mathbb{H}}_s$ and $\hat{\boldsymbol{\xi}}_{yx} \in \hat{\mathbb{H}}_v$ are called the scalar part and the vector part of $\hat{\boldsymbol{q}}_{yx}$, respectively.

Dual quaternions follow the operations of dual vectors, and some other special operations and properties used in this paper are given as follows.

$$\mathrm{vec}(\hat{\boldsymbol{q}}) = \hat{\boldsymbol{\xi}} \tag{20}$$

$$[\hat{0}, \mathrm{vec}(\hat{\boldsymbol{q}})^{\mathrm{T}}]^{\mathrm{T}} \circ \hat{\boldsymbol{q}} = \mathrm{vec}(\hat{\boldsymbol{q}}) \circ \mathrm{vec}(\hat{\boldsymbol{q}}) \tag{21}$$

$$\hat{\boldsymbol{q}}_1 \otimes \hat{\boldsymbol{q}}_2 = [\hat{\eta}_1\hat{\eta}_2 - \hat{\boldsymbol{\xi}}_1^{\mathrm{T}}\hat{\boldsymbol{\xi}}_2, (\hat{\eta}_1\hat{\boldsymbol{\xi}}_2 + \hat{\eta}_2\hat{\boldsymbol{\xi}}_1 + \hat{\boldsymbol{\xi}}_1 \times \hat{\boldsymbol{\xi}}_2)^{\mathrm{T}}]^{\mathrm{T}} \tag{22}$$

$$\hat{\boldsymbol{q}}^* = [\hat{\eta}, -\hat{\boldsymbol{\xi}}^{\mathrm{T}}]^{\mathrm{T}}, \hat{\boldsymbol{q}}^* \otimes \hat{\boldsymbol{q}} = \hat{\boldsymbol{q}}_I \tag{23}$$

$$\hat{\boldsymbol{q}}_1 \circ (\hat{\boldsymbol{q}}_2 \otimes \hat{\boldsymbol{q}}_3) = \hat{\boldsymbol{q}}_3^s \circ [\hat{\boldsymbol{q}}_2^* \otimes (\hat{\boldsymbol{q}}_1^s)] \tag{24}$$

In which $\hat{\boldsymbol{q}}_1, \hat{\boldsymbol{q}}_2, \hat{\boldsymbol{q}}_3 \in \hat{\mathbb{H}}$, $\hat{\boldsymbol{q}}_I = [1, 0, 0, 0]^{\mathrm{T}} + \varepsilon[0, 0, 0, 0]^{\mathrm{T}}$.

Furthermore, the kinematics of the dual quaternion $\hat{\boldsymbol{q}}_{yx}$ satisfies

$$\dot{\hat{\boldsymbol{q}}}_{yx} = \frac{1}{2}\hat{\boldsymbol{q}}_{yx} \otimes [\hat{0}, (\hat{\boldsymbol{\omega}}_{yx}^y)^{\mathrm{T}}]^{\mathrm{T}} = \frac{1}{2}\hat{\boldsymbol{E}}(\hat{\boldsymbol{q}}_{yx})\hat{\boldsymbol{\omega}}_{yx}^y \tag{25}$$

where function

$$\hat{\boldsymbol{E}}(\hat{\boldsymbol{q}}_{yx}) = \begin{bmatrix} -\hat{\boldsymbol{\xi}}_{yx}^{\mathrm{T}} \\ \hat{\eta}_{yx}\mathbf{I}_{3\times3} + \hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_{yx}) \end{bmatrix} \tag{26}$$

and $\hat{\boldsymbol{\omega}}_{yx}^y = \boldsymbol{\omega}_{yx}^y + \varepsilon(\boldsymbol{v}_{yx}^y + \boldsymbol{\omega}_{yx}^y \times \boldsymbol{r}_{yx}^y)$ is called the dual angular velocity, and here $\boldsymbol{\omega}_{yx}^y$ and $\boldsymbol{v}_{yx}^y$ are the angular velocity and linear velocity of $\mathcal{Y}$ with respect to $\mathcal{X}$, respectively.

Remark 1: It is noteworthy that $\boldsymbol{v}^y_{yx} = \dot{\boldsymbol{r}}^y_{yx}$, which is the time derivative of $\boldsymbol{r}_{yx}$ with respect to the frame $\mathcal{Y}$ and then expressed in the frame $\mathcal{Y}$. As a counterpart, $\boldsymbol{v}^x_{yx} = \dot{\boldsymbol{r}}^x_{yx}$ is the time derivative of $\boldsymbol{r}_{yx}$ with respect to the frame $\mathcal{X}$ and then expressed in the frame $\mathcal{X}$. So actually the rigorous expressions of them should be ${}^{(y)}\boldsymbol{v}^y_{yx}$ and ${}^{(x)}\boldsymbol{v}^x_{yx}$, respectively, where the left-superscripts denote in which frames these derivatives are obtained. But because throughout the paper, we never define a linear velocity vector in the form ${}^{(x)}\boldsymbol{v}^y$ (which implies this derivative is got with respect to a frame $\mathcal{X}$ but then expressed in a frame $\mathcal{Y}$), so $\boldsymbol{v}^y \doteq {}^{(y)}\boldsymbol{v}^y$ and $\boldsymbol{v}^x \doteq {}^{(x)}\boldsymbol{v}^x$ are employed for ease of notation, where $\boldsymbol{v}$ could be any linear velocity vector. Furthermore, for this special notation, the relationship between $\boldsymbol{v}^y$ and $\boldsymbol{v}^x$ doesn't follow the one given in (17), instead, it should be deduced from the original relationship $\boldsymbol{r}^y = C(\boldsymbol{q}_{yx})\boldsymbol{r}^x$, by taking time derivative for both sides, one can get $\boldsymbol{v}^y = -\boldsymbol{\omega}^y_{yx} \times \boldsymbol{r}^y + C(\boldsymbol{q}_{yx})\boldsymbol{v}^x$.

C. Dual-Quaternion-based Transformation

Dual quaternions can be used to describe 6-DOF transformation. For any 3-dimensional dual vector $\hat{\boldsymbol{a}}^x = \boldsymbol{a}^x_r + \varepsilon \boldsymbol{a}^x_d \in \hat{\mathbb{R}}^3$ (which is initially expressed in the frame $\mathcal{X}$), the following equation holds [33],

$$\mathrm{vec}(\hat{\boldsymbol{q}}^*_{yx} \otimes [\hat{0}, (\hat{\boldsymbol{a}}^x)^{\mathrm{T}}]^{\mathrm{T}} \otimes \hat{\boldsymbol{q}}_{yx}) = \boldsymbol{a}^y_r + \varepsilon \boldsymbol{a}^y_d + \varepsilon(\boldsymbol{a}^y_r \times \boldsymbol{r}^y_{yx}) \quad (27)$$

where $\boldsymbol{a}^y_r = \boldsymbol{C}(\boldsymbol{q}_{yx})\boldsymbol{a}^x_r$, $\boldsymbol{a}^y_d = \boldsymbol{C}(\boldsymbol{q}_{yx})\boldsymbol{a}^x_d$. Eq. (27) indicates that the transformation rule upon dual quaternions is a little different with the property given in (16). Specifically, an additional cross term shows up, which stems from the unique structure of dual quaternions.

Based on (27), the definition and properties of dual transformation matrices are given in the following proposition.

Proposition 1: The dual quaternion transformation rule given in (27) can be expressed by

$$\mathrm{vec}(\hat{\boldsymbol{q}}^*_{yx} \otimes [\hat{0}, (\hat{\boldsymbol{a}}^x)^{\mathrm{T}}]^{\mathrm{T}} \otimes \hat{\boldsymbol{q}}_{yx}) = \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx})\hat{\boldsymbol{a}}^x \quad (28)$$

and here

$$\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx}) = \mathbf{I}_{3\times 3} - 2\hat{\eta}_{yx}\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_{yx}) + 2\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_{yx})\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_{yx}) \quad (29)$$

is called the dual transformation matrix of the frame $\mathcal{Y}$ with respect to the frame $\mathcal{X}$, which satisfies the following properties:

1) $\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}^*_{yx}) = \hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_{yx})$;
2) $\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_{yx})\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx}) = \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx})\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_{yx}) = \mathbf{I}_{3\times 3} + \varepsilon \mathbf{0}_{3\times 3}$;
3) For any dual vectors $\hat{\boldsymbol{a}}_1, \hat{\boldsymbol{a}}_2 \in \hat{\mathbb{R}}^3$,

$$\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx})(\hat{\boldsymbol{a}}^x_1 \times \hat{\boldsymbol{a}}^x_2) = [\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx})\hat{\boldsymbol{a}}^x_1] \times [\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx})\hat{\boldsymbol{a}}^x_2]$$

4) $\mathrm{d}[\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx})]/\mathrm{d}t = -\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}^y_{yx})\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx})$.

Proof: See Appendix B.

Remark 2: In Refs. [34] and [35], Condurache and Burlacu introduced a special dual orthogonal tensor-based construction and showed the similar results as given in Proposition 1. By comparison, in this paper, these important facts are proved from a different but straightforward way, in which the relationship between dual quaternions and dual transformation matrices is emphasized, and then it is utilized to deduct the properties of dual transformation matrices. Furthermore, one can also use the kinematics of dual transformation matrices given in Proposition 1 to deduce the kinematics of dual quaternions shown in (25), the fundamental reason is that dual quaternions and dual transformation matrices are two equivalent methods to describe the 6-DOF motion between two arbitrary frames, so they can be converted to each other.

D. Dual-Quaternion based Relative Kinematics and Dynamics

In this paper, the 6-DOF relative pose tracking control problem of a leader-follower spacecraft formation is considered. As shown in Fig. 1, three reference frames are employed: the inertial frame, the body-fixed frame of the leader and the body-fixed frame of the follower, denoted by $\mathcal{I} = \{X_I, Y_I, Z_I\}$, $\mathcal{L} = \{X_L, Y_L, Z_L\}$ and $\mathcal{B} = \{X_B, Y_B, Z_B\}$, respectively. Within this context, compared with the "absolute" positions and velocities with respect to the inertia frame $\mathcal{I}$, the relative positions and velocities between the leader and the follower are much smaller, and to implement precisely 6-DOF tracking control, the follower should have the ability to measure sufficient relative motion information with respect to the leader. Furthermore, instead of directly tracking the body-fixed frame of the leader (which is actually impractical, because superimposing the frame $\mathcal{B}$ onto the frame $\mathcal{L}$ will lead to collision), to extend the design flexibility of control objectives, a frame $\mathcal{T}$ (named as the target frame) is defined to describe the desired relative pose of the follower with respect to the leader (in Fig. 1, the frame $\mathcal{T}$ is blurred to indicate it is a virtual and user-designed frame).

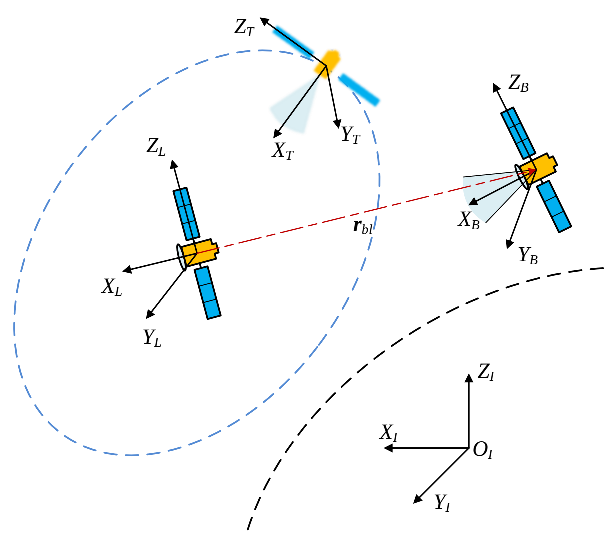

Fig. 1: Reference frames

By (19) and (25), the dual quaternion of the frame $\mathcal{B}$ with respect to the frame $\mathcal{L}$ is

$$\hat{\boldsymbol{q}}_{bl} \doteq [\hat{\eta}_{bl}, \hat{\boldsymbol{\xi}}^{\mathrm{T}}_{bl}]^{\mathrm{T}} = \boldsymbol{q}_{bl} + \varepsilon \frac{1}{2}\boldsymbol{q}_{bl} \otimes \boldsymbol{r}^b_{bl} \quad (30)$$

where $\boldsymbol{q}_{bl}$ is the quaternion of $\mathcal{B}$ with respect to $\mathcal{L}$, and $\boldsymbol{r}^b_{bl} \in \mathbb{R}^3$ is the relative position vector from the origin of $\mathcal{L}$ to the origin of $\mathcal{B}$. Then the 6-DOF relative kinematics and dynamics can be presented as [5], [6], [7],

$$\dot{\hat{\boldsymbol{q}}}_{bl} = \frac{1}{2}\hat{\boldsymbol{E}}(\hat{\boldsymbol{q}}_{bl})\hat{\boldsymbol{\omega}}_{bl}^{b} \tag{31}$$

$$\begin{aligned}\hat{\boldsymbol{M}}_b\dot{\hat{\boldsymbol{\omega}}}_{bl}^{b} =& -\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bi}^{b})(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_{bi}^{b}) \\ &+ \hat{\boldsymbol{M}}_b[\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bl}^{b})\hat{\boldsymbol{\omega}}_{li}^{b} - \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{bl})\dot{\hat{\boldsymbol{\omega}}}_{li}^{l}] + \hat{\boldsymbol{u}}^b\end{aligned} \tag{32}$$

where $\hat{\boldsymbol{\omega}}_{bi}^{b}$ and $\hat{\boldsymbol{\omega}}_{li}^{l}$ are the dual angular velocities of $\mathcal{B}$ and $\mathcal{L}$ with respect to $\mathcal{I}$, respectively. A reasonable assumption is that both $\hat{\boldsymbol{\omega}}_{li}^{l}$ and $\dot{\hat{\boldsymbol{\omega}}}_{li}^{l}$ are bounded and available. $\hat{\boldsymbol{\omega}}_{bl}^{b} = \hat{\boldsymbol{\omega}}_{bi}^{b} - \hat{\boldsymbol{\omega}}_{li}^{b}$, and here $\hat{\boldsymbol{\omega}}_{li}^{b} = \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{bl})\hat{\boldsymbol{\omega}}_{li}^{l}$. Notice by (27) and remark 1, because additional cross terms show up, $\hat{\boldsymbol{\omega}}_{li}^{b}$ doesn't transform every component of $\hat{\boldsymbol{\omega}}_{li}^{l}$ onto the frame $\mathcal{B}$. But these cross terms actually play an important role to establish $\hat{\boldsymbol{\omega}}_{bl}^{b}$, and it can be proved that $\hat{\boldsymbol{\omega}}_{bl}^{b}$ fully obey the definition of general dual angular velocities:

$$\hat{\boldsymbol{\omega}}_{bl}^{b} = \boldsymbol{\omega}_{bl}^{b} + \varepsilon(\boldsymbol{v}_{bl}^{b} + \boldsymbol{\omega}_{bl}^{b} \times \boldsymbol{r}_{bl}^{b}) \tag{33}$$

where $\boldsymbol{\omega}_{bl}^{b}$ and $\boldsymbol{v}_{bl}^{b}$ are the relative angular velocity and the relative linear velocity of $\mathcal{B}$ with respect to $\mathcal{L}$, respectively. Furthermore, $\hat{\boldsymbol{u}}^b = \boldsymbol{f}^b + \varepsilon\boldsymbol{\tau}^b$ is the total dual input applied to the rigid body, and here $\boldsymbol{f}^b \in \mathbb{R}^3$ and $\boldsymbol{\tau}^b \in \mathbb{R}^3$ are the total force and the total torque applied to the rigid body, respectively. $\hat{\boldsymbol{M}}_b = m_b\mathbf{I}_{3\times3}\frac{\mathrm{d}}{\mathrm{d}\varepsilon} + \varepsilon\boldsymbol{J}_b$ is called the dual mass matrix, with $m_b \in \mathbb{R}$ and $\boldsymbol{J}_b \in \mathbb{R}^{3\times3}$ are the mass and the inertia (expressed in the frame $\mathcal{B}$) of the rigid body, respectively.

Similarly, for the frame $\mathcal{B}$ and the target frame $\mathcal{T}$,

$$\dot{\hat{\boldsymbol{q}}}_{bt} = \frac{1}{2}\hat{\boldsymbol{E}}(\hat{\boldsymbol{q}}_{bt})\hat{\boldsymbol{\omega}}_{bt}^{b} \tag{34}$$

$$\begin{aligned}\hat{\boldsymbol{M}}_b\dot{\hat{\boldsymbol{\omega}}}_{bt}^{b} =& -\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bi}^{b})(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_{bi}^{b}) \\ &+ \hat{\boldsymbol{M}}_b[\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bt}^{b})\hat{\boldsymbol{\omega}}_{ti}^{b} - \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{bt})\dot{\hat{\boldsymbol{\omega}}}_{ti}^{t}] + \hat{\boldsymbol{u}}^b\end{aligned} \tag{35}$$

Notice that since frame $\mathcal{T}$ is designed based upon $\mathcal{L}$, $\hat{\boldsymbol{\omega}}_{tl}$ is available. And in (35), $\hat{\boldsymbol{\omega}}_{ti}^{b} = \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{bt})\hat{\boldsymbol{\omega}}_{ti}^{t}$, $\hat{\boldsymbol{\omega}}_{ti}^{t} = \hat{\boldsymbol{\omega}}_{tl}^{t} + \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{tl})\hat{\boldsymbol{\omega}}_{li}^{l}$, $\hat{\boldsymbol{\omega}}_{bt}^{b} = \hat{\boldsymbol{\omega}}_{bi}^{b} - \hat{\boldsymbol{\omega}}_{ti}^{b} = \hat{\boldsymbol{\omega}}_{bl}^{b} - \hat{\boldsymbol{\omega}}_{tl}^{b}$, and here $\hat{\boldsymbol{\omega}}_{tl}^{b} = \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{bt})\hat{\boldsymbol{\omega}}_{tl}^{t}$. The control objective is to superimpose the frame $\mathcal{B}$ onto the frame $\mathcal{T}$.

E. Proportional-Derivative-Like Controller

Similar with the traditional attitude tracking control problem, Ref. [22] shows that one can also use a PD-like controller to achieve the 6-DOF tracking goal for the system in (34-35), as summarized in Proposition 2.

Proposition 2 [22]: Consider the system given in (34-35), design the total dual input applied to the rigid body as

$$\begin{aligned}\hat{\boldsymbol{u}}^b =& -k_p\mathrm{vec}(\hat{\boldsymbol{q}}_{bt}^{*} \otimes (\hat{\boldsymbol{q}}_{bt} - \hat{\boldsymbol{q}}_I)^s) - k_d(\hat{\boldsymbol{\omega}}_{bt}^{b})^s \\ &+ \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{bt})\dot{\hat{\boldsymbol{\omega}}}_{ti}^{t}) + \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{ti}^{b})(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_{ti}^{b})\end{aligned} \tag{36}$$

with $k_p, k_d > 0$. It can be guaranteed for any $\hat{\boldsymbol{q}}_{bt}(0) \in \hat{\mathbb{H}}$, when $t \to \infty$, $\hat{\boldsymbol{\xi}}_{bt}(t) \to \hat{\mathbf{0}}_3$, and $\hat{\boldsymbol{\omega}}_{bt}^{b}(t) \to \hat{\mathbf{0}}_3$.

Remark 3: As discussed in Sec. II.D, $\hat{\boldsymbol{\omega}}_{bt}^{b}$ can be obtained by either an "absolute" way $\hat{\boldsymbol{\omega}}_{bi}^{b} - \hat{\boldsymbol{\omega}}_{ti}^{b}$ or a "relative" way $\hat{\boldsymbol{\omega}}_{bl}^{b} - \hat{\boldsymbol{\omega}}_{tl}^{b}$. As aforementioned, for the pose tracking control problem of spacecrafts, the relative velocities between spacecrafts has a much smaller order of magnitude than their absolute velocities with respect to the inertia frame. So it is not reasonable to use the "absolute" way to calculate $\hat{\boldsymbol{\omega}}_{bt}^{b}$, because it will introduce big measurement errors and lead to poor control precisions. Instead, the follower should directly measure the relative pose and linear/angular velocities with respect to the leader. And when sensors for relative linear/angular velocity measurements are unavailable, observers could be established to estimate $\hat{\boldsymbol{\omega}}_{bl}^{b}$ instead of $\hat{\boldsymbol{\omega}}_{bi}^{b}$. To this end, a smooth observer is designed in the following section.

III. Dual Angular Velocity Observer Development

Consider an observer-related reference frame $\mathcal{O}$, in which the estimates for $\hat{\boldsymbol{\omega}}_{bl}^{b}$ are generated. Let $\hat{\boldsymbol{q}}_{ol} = [\hat{\eta}_{ol}, \hat{\boldsymbol{\xi}}_{ol}^{\mathrm{T}}]^{\mathrm{T}} \in \hat{\mathbb{H}}$ and $\hat{\boldsymbol{\omega}}_{ol}^{o} \in \hat{\mathbb{R}}^3$ denote the estimated values of $\hat{\boldsymbol{q}}_{bl}$ and $\hat{\boldsymbol{\omega}}_{bl}^{b}$, respectively. Furthermore, define $\hat{\boldsymbol{q}}_e = [\hat{\eta}_e, \hat{\boldsymbol{\xi}}_e^{\mathrm{T}}]^{\mathrm{T}}$ and $\hat{\boldsymbol{\omega}}_e^{b}$ as the estimation error states:

$$\hat{\boldsymbol{q}}_e = \hat{\boldsymbol{q}}_{ol}^{*} \otimes \hat{\boldsymbol{q}}_{bl} = \boldsymbol{q}_e + \varepsilon\frac{1}{2}\boldsymbol{q}_e \otimes \boldsymbol{r}_e^{b} \tag{37}$$

$$\hat{\boldsymbol{\omega}}_e^{b} = \hat{\boldsymbol{\omega}}_{bl}^{b} - \hat{\boldsymbol{\omega}}_{ol}^{b} = \hat{\boldsymbol{\omega}}_{bi}^{b} - \hat{\boldsymbol{\omega}}_{oi}^{b} \tag{38}$$

where $\boldsymbol{q}_e = \boldsymbol{q}_{ol}^{*} \otimes \boldsymbol{q}_{bl}$ and $\boldsymbol{r}_e^{b} = \boldsymbol{r}_{bl}^{b} - \boldsymbol{r}_{ol}^{b}$ are the estimation errors of the relative quaternion and position, respectively, $\hat{\boldsymbol{\omega}}_{ol}^{b} = \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)\hat{\boldsymbol{\omega}}_{ol}^{o}$ is defined for ease of notation, and $\hat{\boldsymbol{\omega}}_{oi}^{b} = \hat{\boldsymbol{\omega}}_{ol}^{b} + \hat{\boldsymbol{\omega}}_{li}^{b}$.

Then, one of the main results of this paper, a smooth observer which guarantees the asymptotic convergence of estimation errors, is summarized in the following theorem.

Theorem 1: Consider the following smooth observer construction,

$$\dot{\hat{\boldsymbol{q}}}_{ol} = \frac{1}{2}\hat{\boldsymbol{E}}(\hat{\boldsymbol{q}}_{ol})[\hat{\boldsymbol{\omega}}_{ol}^{o} + \lambda\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_e)\mathrm{vec}(\hat{\boldsymbol{q}}_e^{*} \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s)^s] \tag{39}$$

$$\begin{aligned}\dot{\hat{\boldsymbol{\omega}}}_{ol}^{o} =& \hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_e)\hat{\boldsymbol{M}}_b^{-1}\cdot \\ &[\gamma\mathrm{vec}(\hat{\boldsymbol{q}}_e^{*} \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s) - \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{oi}^{b})(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_{oi}^{b}) - ... \\ &\lambda\hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\mathrm{vec}(\hat{\boldsymbol{q}}_e^{*} \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s)^s)\hat{\boldsymbol{\omega}}_{ol}^{b}) + ... \\ &\hat{\boldsymbol{M}}_b(\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{bl})\dot{\hat{\boldsymbol{\omega}}}_{li}^{l}) + \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{ol}^{b})\hat{\boldsymbol{\omega}}_{li}^{l}) + \hat{\boldsymbol{u}}^b]\end{aligned} \tag{40}$$

where $\lambda, \gamma > 0$ and $\hat{\boldsymbol{M}}_b^{-1} = \boldsymbol{J}_b^{-1}\frac{\mathrm{d}}{\mathrm{d}\varepsilon} + \frac{1}{m_b}\mathbf{I}_{3\times3}\varepsilon$ is the inverse of $\hat{\boldsymbol{M}}_b$. Then for any $\hat{\boldsymbol{q}}_{ol}(0) \in \hat{\mathbb{H}}$, and $\hat{\boldsymbol{\omega}}_{ol}^{o}(0) \in \hat{\mathbb{R}}^3$, it can be guaranteed $\hat{\boldsymbol{\xi}}_e(t) \to \hat{\mathbf{0}}_3$ and $\hat{\boldsymbol{\omega}}_e^{b}(t) \to \hat{\mathbf{0}}_3$, when $t \to \infty$.

Proof: By the definition of $\hat{\boldsymbol{q}}_e$, one has,

$$\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e) = \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{bl})\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_{ol}) \tag{41}$$

Substituting (39) into the time derivative of (41) yields,

$$\begin{aligned}\frac{\mathrm{d}(\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e))}{\mathrm{d}t} =& -\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bl}^{b})\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e) \\ &+ \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)\hat{\boldsymbol{S}}[\hat{\boldsymbol{\omega}}_{ol}^{o} + \lambda\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_e)\mathrm{vec}(\hat{\boldsymbol{q}}_e^{*} \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s)^s]\end{aligned} \tag{42}$$

Recall the Property 3 given in Proposition 1, for any dual vector $\hat{\boldsymbol{a}} \in \hat{\mathbb{R}}_3$, one has

$$\begin{aligned}&\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)\hat{\boldsymbol{S}}[\hat{\boldsymbol{\omega}}_{ol}^{o} + \lambda\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_e)\mathrm{vec}(\hat{\boldsymbol{q}}_e^{*} \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s)^s]\hat{\boldsymbol{a}}^o \\ &= \hat{\boldsymbol{S}}[\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)(\hat{\boldsymbol{\omega}}_{ol}^{o} + \lambda\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_e)\mathrm{vec}(\hat{\boldsymbol{q}}_e^{*} \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s)^s)] \\ &\quad \cdot \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)\hat{\boldsymbol{a}}^o\end{aligned} \tag{43}$$

Due to the arbitrariness of $\hat{\boldsymbol{a}}^o$, one can further get,

$$\begin{aligned}&\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)\hat{\boldsymbol{S}}[\hat{\boldsymbol{\omega}}_{ol}^o+\lambda\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_e)\mathrm{vec}(\hat{\boldsymbol{q}}_e^*\otimes(\hat{\boldsymbol{q}}_e-\hat{\boldsymbol{q}}_I)^s)^s]\\&=\hat{\boldsymbol{S}}[\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)(\hat{\boldsymbol{\omega}}_{ol}^o+\lambda\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_e)\mathrm{vec}(\hat{\boldsymbol{q}}_e^*\otimes(\hat{\boldsymbol{q}}_e-\hat{\boldsymbol{q}}_I)^s)^s)]\\&\quad\cdot\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)\end{aligned}\tag{44}$$

Substitute (44) into (42) and consider (38), one has:

$$\frac{\mathrm{d}(\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e))}{\mathrm{d}t}=-\hat{\boldsymbol{S}}[\hat{\boldsymbol{\omega}}_e^b-\lambda\mathrm{vec}(\hat{\boldsymbol{q}}_e^*\otimes(\hat{\boldsymbol{q}}_e-\hat{\boldsymbol{q}}_I)^s)^s]\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_e)\tag{45}$$

As discussed in Remark 2, (45) can be converted to a equivalent dual-quaternion kinematics:

$$\begin{aligned}\dot{\hat{\boldsymbol{q}}}_e&=\frac{1}{2}\hat{\boldsymbol{E}}(\hat{\boldsymbol{q}}_e)[\hat{\boldsymbol{\omega}}_e^b-\lambda\mathrm{vec}(\hat{\boldsymbol{q}}_e^*\otimes(\hat{\boldsymbol{q}}_e-\hat{\boldsymbol{q}}_I)^s)^s]\\&=\frac{1}{2}\hat{\boldsymbol{q}}_e\otimes[\hat{\boldsymbol{\omega}}_e^b-\lambda\mathrm{vec}(\hat{\boldsymbol{q}}_e^*\otimes(\hat{\boldsymbol{q}}_e-\hat{\boldsymbol{q}}_I)^s)^s]\end{aligned}\tag{46}$$

Based on these analyses, consider the following Lyapunov-like function candidate:

$$V_o=\gamma(\hat{\boldsymbol{q}}_e-\hat{\boldsymbol{q}}_I)\circ(\hat{\boldsymbol{q}}_e-\hat{\boldsymbol{q}}_I)+\frac{1}{2}(\hat{\boldsymbol{\omega}}_e^b)^s\circ(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_e^b)\tag{47}$$

By the definition of "$\circ$" product, it is easy to show $V_o(\hat{\boldsymbol{q}}_e=\hat{\boldsymbol{q}}_I,\hat{\boldsymbol{\omega}}_e^b=\hat{\boldsymbol{0}}_3)=0$, and $V_o>0$ for any $\hat{\boldsymbol{q}}_e\neq\hat{\boldsymbol{q}}_I$ and $\hat{\boldsymbol{\omega}}_e^b\neq\hat{\boldsymbol{0}}_3$, so V_o is a valid Lyapunov-like candidate. The time differential of V_o is,

$$\dot{V}_o=2\gamma(\hat{\boldsymbol{q}}_e-\hat{\boldsymbol{q}}_I)\circ\dot{\hat{\boldsymbol{q}}}_e+(\hat{\boldsymbol{\omega}}_e^b)^s\circ(\hat{\boldsymbol{M}}_b\dot{\hat{\boldsymbol{\omega}}}_e^b)\tag{48}$$

By (32), (38), and (40), we have

$$\begin{aligned}\hat{\boldsymbol{M}}_b\dot{\hat{\boldsymbol{\omega}}}_e^b=&-\gamma\mathrm{vec}(\hat{\boldsymbol{q}}_e^*\otimes(\hat{\boldsymbol{q}}_e-\hat{\boldsymbol{q}}_I)^s)\\&-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bi}^b)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_{bi}^b)+\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{oi}^b)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_{oi}^b)\\&+\hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^b)\hat{\boldsymbol{\omega}}_{li}^b)+\hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^b)\hat{\boldsymbol{\omega}}_{ol}^b)\\=&-\gamma\mathrm{vec}(\hat{\boldsymbol{q}}_e^*\otimes(\hat{\boldsymbol{q}}_e-\hat{\boldsymbol{q}}_I)^s)-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^b)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_{bi}^b)\\&-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{oi}^b)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_e^b)+\hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^b)\hat{\boldsymbol{\omega}}_{oi}^b)\end{aligned}\tag{49}$$

Considering Lemma 4 and also the property given in (24), then substituting (49) into (48) yields,

$$\begin{aligned}\dot{V}_o=&-\gamma\lambda\mathrm{vec}(\hat{\boldsymbol{q}}_e^*\otimes(\hat{\boldsymbol{q}}_e-\hat{\boldsymbol{q}}_I)^s)\circ\mathrm{vec}(\hat{\boldsymbol{q}}_e^*\otimes(\hat{\boldsymbol{q}}_e-\hat{\boldsymbol{q}}_I)^s)\\&+(\hat{\boldsymbol{\omega}}_e^b)^s\circ[-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{oi}^b)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_e^b)+\hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^b)\hat{\boldsymbol{\omega}}_{oi}^b)]\end{aligned}\tag{50}$$

Furthermore,

$$\begin{aligned}&-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{oi}^b)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_e^b)+\hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^b)\hat{\boldsymbol{\omega}}_{oi}^b)\\&=-m_b\boldsymbol{\omega}_{oir}^b\times\boldsymbol{\omega}_{ed}^b-\varepsilon[m_b\boldsymbol{\omega}_{oid}^b\times\boldsymbol{\omega}_{ed}^b+\boldsymbol{\omega}_{oir}^b\times(\boldsymbol{J}_b\boldsymbol{\omega}_{er}^b)]\\&\quad+\varepsilon\boldsymbol{J}_b(\boldsymbol{\omega}_{er}^b\times\boldsymbol{\omega}_{oir}^b)+m_b(\boldsymbol{\omega}_{er}^b\times\boldsymbol{\omega}_{oid}^b+\boldsymbol{\omega}_{ed}^b\times\boldsymbol{\omega}_{oir}^b)\end{aligned}\tag{51}$$

where $\boldsymbol{\omega}_{oir}^b$ and $\boldsymbol{\omega}_{oid}^b$ are the real part and dual part of $\hat{\boldsymbol{\omega}}_{oi}^b$, respectively, while $\boldsymbol{\omega}_{er}^b$ and $\boldsymbol{\omega}_{ed}^b$ are the real part and dual part of $\hat{\boldsymbol{\omega}}_e^b$, respectively. Then

$$\begin{aligned}&(\hat{\boldsymbol{\omega}}_e^b)^s\circ[-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{oi}^b)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_e^b)+\hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^b)\hat{\boldsymbol{\omega}}_{oi}^b)]\\&=m_b\boldsymbol{\omega}_{ed}^b\cdot(\boldsymbol{\omega}_{er}^b\times\boldsymbol{\omega}_{oid}^b)-m_b\boldsymbol{\omega}_{er}^b\cdot(\boldsymbol{\omega}_{oid}^b\times\boldsymbol{\omega}_{ed}^b)\\&\quad-\boldsymbol{\omega}_{er}^b\cdot(\boldsymbol{\omega}_{oir}^b\times(\boldsymbol{J}_b\boldsymbol{\omega}_{er}^b))+\boldsymbol{\omega}_{er}^b\cdot(\boldsymbol{J}_b(\boldsymbol{\omega}_{er}^b\times\boldsymbol{\omega}_{oir}^b))\\&=-\boldsymbol{\omega}_{er}^b\cdot(\boldsymbol{S}(\boldsymbol{\omega}_{oir}^b)\boldsymbol{J}_b\boldsymbol{\omega}_{er}^b)-\boldsymbol{\omega}_{er}^b\cdot(\boldsymbol{J}_b\boldsymbol{S}(\boldsymbol{\omega}_{oir}^b)\boldsymbol{\omega}_{er}^b)\end{aligned}\tag{52}$$

Notice $\boldsymbol{S}(\boldsymbol{\omega}_{oir}^b)\boldsymbol{J}_b+\boldsymbol{J}_b\boldsymbol{S}(\boldsymbol{\omega}_{oir}^b)$ is a skew-symmetric matrix, so that:

$$\begin{aligned}&-\boldsymbol{\omega}_{er}^b\cdot(\boldsymbol{S}(\boldsymbol{\omega}_{oir}^b)\boldsymbol{J}_b\boldsymbol{\omega}_{er}^b)-\boldsymbol{\omega}_{er}^b\cdot(\boldsymbol{J}_b\boldsymbol{S}(\boldsymbol{\omega}_{oir}^b)\boldsymbol{\omega}_{er}^b)=\\&-\boldsymbol{\omega}_{er}^b\cdot[(\boldsymbol{S}(\boldsymbol{\omega}_{oir}^b)\boldsymbol{J}_b+\boldsymbol{J}_b\boldsymbol{S}(\boldsymbol{\omega}_{oir}^b))\boldsymbol{\omega}_{er}^b]=0\end{aligned}\tag{53}$$

Furthermore,

$$\begin{aligned}\dot{V}_o=&-\gamma\lambda\mathrm{vec}(\hat{\boldsymbol{q}}_e^*\otimes(\hat{\boldsymbol{q}}_e-\hat{\boldsymbol{q}}_I)^s)\circ\mathrm{vec}(\hat{\boldsymbol{q}}_e^*\otimes(\hat{\boldsymbol{q}}_e-\hat{\boldsymbol{q}}_I)^s)\\=&-\gamma\lambda(\frac{1}{2}\boldsymbol{r}_e^b+\varepsilon\boldsymbol{\xi}_e)\circ(\frac{1}{2}\boldsymbol{r}_e^b+\varepsilon\boldsymbol{\xi}_e)\\=&-\gamma\lambda(\frac{1}{4}\|\boldsymbol{r}_e^b\|^2+\|\boldsymbol{\xi}_e\|^2)\end{aligned}\tag{54}$$

Because $V_o(t)\geq 0$ and $\dot{V}_o(t)\leq 0$, so $V_o\in\mathcal{L}_\infty$, by the definition of V_o, we have $\hat{\boldsymbol{q}}_e,\hat{\boldsymbol{\omega}}_e^b\in\mathcal{L}_\infty$, which implies $\boldsymbol{\xi}_e,\boldsymbol{r}_e^b,\boldsymbol{v}_e^b,\boldsymbol{\omega}_e^b\in\mathcal{L}_\infty$. Eq. (54) also shows that $\int_0^\infty\dot{V}_o(\sigma)\mathrm{d}\sigma$ exists, and $\boldsymbol{\xi}_e,\boldsymbol{r}_e^b\in\mathcal{L}_2$. Furthermore, by (46), one has $\dot{\boldsymbol{\xi}}_e,\dot{\boldsymbol{r}}_e^b\in\mathcal{L}_\infty$. Actually to the corollary of Barbalat's lemma, one can guarantee $\lim_{t\to\infty}\boldsymbol{\xi}_e(t)=\boldsymbol{0}_3$ and $\lim_{t\to\infty}\boldsymbol{r}_e^b(t)=\boldsymbol{0}_3$, and accordingly, $\lim_{t\to\infty}\hat{\boldsymbol{\xi}}_e(t)=\boldsymbol{0}_3$.

Recall (46), the vector part of $\dot{\hat{\boldsymbol{q}}}_e$ can be written as:

$$\dot{\hat{\boldsymbol{\xi}}}_e=-\frac{\lambda}{2}\hat{\boldsymbol{\xi}}_e+\hat{\boldsymbol{\delta}}\tag{55}$$

where

$$\begin{aligned}\hat{\boldsymbol{\delta}}=&\frac{\lambda}{2}\hat{\boldsymbol{\xi}}_e-\frac{\lambda}{2}(\hat{\eta}_e\mathbf{I}_{3\times3}+\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_e))\mathrm{vec}(\hat{\boldsymbol{q}}_e^*\otimes(\hat{\boldsymbol{q}}_e-\hat{\boldsymbol{q}}_I)^s)^s\\&+\frac{1}{2}(\hat{\eta}_e\mathbf{I}_{3\times3}+\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_e))\hat{\boldsymbol{\omega}}_e^b\end{aligned}$$

Eq. (54) shows that $\hat{\boldsymbol{\delta}}$ is a bounded signal consisting of the bounded states $\hat{\boldsymbol{\xi}}_e$ and $\hat{\boldsymbol{\omega}}_e^b$. Thus (55) is actually an asymptotically stable linear filter with bounded input. Since $\lim_{t\to\infty}\hat{\boldsymbol{\xi}}_e(t)=\hat{\boldsymbol{0}}_3$, so $\lim_{t\to\infty}\hat{\boldsymbol{\delta}}(t)=\hat{\boldsymbol{0}}_3$, which results in $\lim_{t\to\infty}\hat{\eta}_e\hat{\boldsymbol{\omega}}_e^b(t)=\hat{\boldsymbol{0}}_3$, and $\lim_{t\to\infty}\hat{\eta}_e(t)=\pm1+\varepsilon 0$, so finally we have $\lim_{t\to\infty}\hat{\boldsymbol{\omega}}_e^b(t)=\hat{\boldsymbol{0}}_3$, and the proof is complete.

IV. Observer-based Controller Design and Lyapunov-Like Function Strictification

In this section, based on the proposed observer, an independently designed PD-like controller is introduced to achieve 6-DOF relative tracking objectives, and then the Lyapunov strictification strategy is employed to prove the separation property of the observer-controller construction, and guarantee the almost global asymptotic convergence of tracking errors.

A. Necessity Analysis

The definition of strict Lyapunov function is given firstly.

Definition [24]: For a nonautonomous system

$$\dot{\boldsymbol{x}}=\boldsymbol{f}(t,\boldsymbol{x})\tag{56}$$

A Lyapunov function $V(t,\boldsymbol{x})$ is a strict Lyapunov function for system (56), if there exists a positive definite function $\alpha(\cdot)$ such that

$$\frac{\partial V(t,\boldsymbol{x})}{\partial t}+\frac{\partial V(t,\boldsymbol{x})}{\partial\boldsymbol{x}}\boldsymbol{f}(t,\boldsymbol{x})\leq-\alpha(\|\boldsymbol{x}\|)\tag{57}$$

By this definition, if one could find a strict Lyapunov function for system (56), then the asymptotic convergence result of $\boldsymbol{x}$ is directly guaranteed.

As mentioned in Sec. II. E, when the relative dual angular velocity $\hat{\boldsymbol{\omega}}_{bl}^b$ is available, $\hat{\boldsymbol{\omega}}_{bt}^b = \hat{\boldsymbol{\omega}}_{bl}^b - \hat{\boldsymbol{\omega}}_{tl}^b$ is available, and one can use a PD-like controller given in Proposition 2 to stabilize the closed-loop system. When $\hat{\boldsymbol{\omega}}_{bl}^b$ can't be measured, a direct idea is using the estimation value $\hat{\boldsymbol{\omega}}_{ol}^b$ to replace it, then the new controller is,

$$\begin{aligned}\hat{\boldsymbol{u}}^b =& - k_p \text{vec}(\hat{\boldsymbol{q}}_{bt}^* \otimes (\hat{\boldsymbol{q}}_{bt} - \hat{\boldsymbol{q}}_I)^s) - k_d(\hat{\boldsymbol{\omega}}_{ol}^b - \hat{\boldsymbol{\omega}}_{tl}^b)^s \\ &+ \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{bt})\dot{\hat{\boldsymbol{\omega}}}_{ti}^t) + \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{ti}^b)(\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{ti}^b)\end{aligned} \tag{58}$$

To analysis the stability of the closed-loop system, consider the following Lyapunov-like function candidate,

$$V_c = k_p(\hat{\boldsymbol{q}}_{bt} - \hat{\boldsymbol{q}}_I) \circ (\hat{\boldsymbol{q}}_{bt} - \hat{\boldsymbol{q}}_I) + \frac{1}{2}(\hat{\boldsymbol{\omega}}_{bt}^b)^s \circ (\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{bt}^b) \tag{59}$$

Recall Lemma 4 and the property given in (24), then substitute the dynamics (35) and the control law (58) into the time differential of V_c, one has

$$\begin{aligned}\dot{V}_c =& - k_d(\hat{\boldsymbol{\omega}}_{bt}^b) \circ (\hat{\boldsymbol{\omega}}_{ol}^b - \hat{\boldsymbol{\omega}}_{tl}^b) \\ &- (\hat{\boldsymbol{\omega}}_{bt}^b)^s \circ [-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{ti}^b)(\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{bt}^b) + \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bt}^b)\hat{\boldsymbol{\omega}}_{ti}^b)]\end{aligned} \tag{60}$$

Similar with the analysis given in (51) to (53), it can be proved that

$$(\hat{\boldsymbol{\omega}}_{bt}^b)^s \circ [-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{ti}^b)(\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{bt}^b) + \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bt}^b)\hat{\boldsymbol{\omega}}_{ti}^b)] = 0 \tag{61}$$

Further notice

$$\hat{\boldsymbol{\omega}}_{ol}^b - \hat{\boldsymbol{\omega}}_{tl}^b = \hat{\boldsymbol{\omega}}_{bl}^b - \hat{\boldsymbol{\omega}}_e^b - \hat{\boldsymbol{\omega}}_{tl}^b = \hat{\boldsymbol{\omega}}_{bt}^b - \hat{\boldsymbol{\omega}}_e^b \tag{62}$$

we have

$$\dot{V}_c = -k_d \hat{\boldsymbol{\omega}}_{bt}^b \circ \hat{\boldsymbol{\omega}}_{bt}^b + k_d \hat{\boldsymbol{\omega}}_{bt}^b \circ \hat{\boldsymbol{\omega}}_e^b \tag{63}$$

It can be seen that the cross term $k_d \hat{\boldsymbol{\omega}}_{bt}^b \circ \hat{\boldsymbol{\omega}}_e^b$ is a big obstacle for stability analysis. Furthermore, consider a composite Lyapunov-like function candidate $V_{oc} = \nu V_o + V_c$, where $\nu \in \mathbb{R}$ is a positive constant. By (54) and (63), the time differential of V_{oc} is,

$$\dot{V}_{oc} = -\nu\gamma\lambda \hat{\varsigma}_e \circ \hat{\varsigma}_e - k_d \hat{\boldsymbol{\omega}}_{bt}^b \circ \hat{\boldsymbol{\omega}}_{bt}^b + k_d \hat{\boldsymbol{\omega}}_{bt}^b \circ \hat{\boldsymbol{\omega}}_e^b \tag{64}$$

where $\hat{\varsigma}_e = \text{vec}(\hat{\boldsymbol{q}}_e^* \otimes (\hat{\boldsymbol{q}}_e - \hat{\boldsymbol{q}}_I)^s) = \frac{1}{2}\boldsymbol{r}_e^b + \varepsilon \boldsymbol{\xi}_e$ is defined for ease of notation. Eq. (64) shows that the cross term $k_d \hat{\boldsymbol{\omega}}_{bt}^b \circ \hat{\boldsymbol{\omega}}_e^b$ still can't be canceled or dominated because there is no negative term in $\hat{\boldsymbol{\omega}}_e^b$, which presents itself as a serious obstacle for further stability analysis.

The strictification strategy will be carried out to solve this problem in next subsection. The objective of strictification is judiciously modifying the Lyapunov-like function candidate V_o by introducing a negative term in $\hat{\boldsymbol{\omega}}_e^b$ in the right part of (64), so one can dominate the cross term $k_d \hat{\boldsymbol{\omega}}_{bt}^b \circ \hat{\boldsymbol{\omega}}_e^b$. But to achieve this goal, we need to prove the boundedness of $\hat{\boldsymbol{\omega}}_{bt}^b$ firstly. What's more, if we can also introduce a negative term of $\hat{\boldsymbol{q}}_{bt}$ into the time differential of V_c, then (64) could directly show the result that both the estimation errors and the tracking errors will converge to zero asymptotically, and consequently the separation property can be established.

B. Boundedness Analysis for the Closed-Loop System

To analyze the boundedness of system states under the controller given in (58), introduce a cross term N_c,

$$N_c = \hat{\varsigma}_{bt} \circ (\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{bt}^b) \tag{65}$$

where

$$\hat{\varsigma}_{bt} = \text{vec}(\hat{\boldsymbol{q}}_{bt}^* \otimes (\hat{\boldsymbol{q}}_{bt} - \hat{\boldsymbol{q}}_I)^s) = \frac{1}{2}\boldsymbol{r}_{bt}^b + \varepsilon \boldsymbol{\xi}_{bt} \tag{66}$$

The differential of N_c with respect to time is,

$$\begin{aligned}\dot{N}_c =& [\frac{1}{2}\boldsymbol{v}_{bt}^b + \varepsilon(\eta_{bd}\boldsymbol{\omega}_{bt}^b + \boldsymbol{S}(\boldsymbol{\xi}_{bt})\boldsymbol{\omega}_{bt}^b)] \circ \\ &[m_b(\boldsymbol{v}_{bt}^b + \boldsymbol{\omega}_{bt}^b \times \boldsymbol{r}_{bt}^b) + \varepsilon \boldsymbol{J}_b \boldsymbol{\omega}_{bt}^b] + \hat{\varsigma}_{bt} \circ (\hat{\boldsymbol{M}}_b \dot{\hat{\boldsymbol{\omega}}}_{bt}^b)\end{aligned} \tag{67}$$

Substituting $\hat{\boldsymbol{u}}^b$ given in (58) into (67) yields,

$$\begin{aligned}\dot{N}_c =& \frac{m_b}{2}(\boldsymbol{v}_{bt}^b + \boldsymbol{\omega}_{bt}^b \times \boldsymbol{r}_{bt}^b)^{\mathrm{T}}(\boldsymbol{v}_{bt}^b + \boldsymbol{\omega}_{bt}^b \times \boldsymbol{r}_{bt}^b) \\ &- \frac{m_b}{2}(\boldsymbol{\omega}_{bt}^b \times \boldsymbol{r}_{bt}^b)^{\mathrm{T}}(\boldsymbol{v}_{bt}^b + \boldsymbol{\omega}_{bt}^b \times \boldsymbol{r}_{bt}^b) \\ &+ (\eta_{bt}\boldsymbol{\omega}_{bt}^b)^{\mathrm{T}}(\boldsymbol{J}_b\boldsymbol{\omega}_{bt}^b) + (\boldsymbol{S}(\boldsymbol{\xi}_{bt})\boldsymbol{\omega}_{bt}^b)^{\mathrm{T}}(\boldsymbol{J}_b\boldsymbol{\omega}_{bt}^b) \\ &- k_p \hat{\varsigma}_{bt} \circ \hat{\varsigma}_{bt} + \hat{\varsigma}_{bt} \circ [-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bt}^b)(\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{bt}^b) \\ &- \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{ti}^b)(\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{bt}^b) - \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bt}^b)(\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{ti}^b) \\ &+ \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bt}^b)\hat{\boldsymbol{\omega}}_{ti}^b) - k_d(\hat{\boldsymbol{\omega}}_{bt}^b)^s + k_d(\hat{\boldsymbol{\omega}}_e^b)^s]\end{aligned} \tag{68}$$

For further analysis, consider the dual part (attitude part) of controller (58),

$$\boldsymbol{\tau}^b = -k_p \boldsymbol{\xi}_{bt} - k_d \boldsymbol{\omega}_{bt}^b + k_d \boldsymbol{\omega}_e^b + \boldsymbol{J}_b \boldsymbol{C}(\boldsymbol{q}_{bt})\dot{\boldsymbol{\omega}}_{ti}^b + \boldsymbol{S}(\boldsymbol{\omega}_{ti}^b)(\boldsymbol{J}_b \boldsymbol{\omega}_{ti}^b) \tag{69}$$

Since $\boldsymbol{\omega}_e^b$ is guaranteed to be bounded, by straightforward Lyapunov-based analysis, one can readily prove $\boldsymbol{\omega}_{bt}^b$ is bounded under the control law given in (69) (refer [21] for details), which means there exists a positive constant c_1, such that $\|\boldsymbol{\omega}_{bt}^b\| \leq c_1$. Moreover, because $\hat{\boldsymbol{\omega}}_{li}^l$ is bounded, and $\hat{\boldsymbol{\omega}}_{tl}^t$ is user-designed (which should also be bounded), then $\hat{\boldsymbol{\omega}}_{ti}^t = \hat{\boldsymbol{\omega}}_{tl}^t + \hat{\boldsymbol{\omega}}_{li}^t$ is bounded, and there exists positive constants c_2 and c_3, such that $\boldsymbol{\omega}_{ti}^t \leq c_2$ and $\boldsymbol{v}_{ti}^t + \boldsymbol{\omega}_{ti}^t \times \boldsymbol{r}_{ti}^t \leq c_3$. Then recall (27), one has $\|\hat{\boldsymbol{\omega}}_{ti}^b\| \leq c_2 + \varepsilon(c_3 + c_2\|\boldsymbol{r}_{bt}^b\|)$. Based on these facts and notice $\|\eta_{bt}\|, \|\boldsymbol{\xi}_{bt}\| \leq 1$, by applying the Cauchy-Schwarz inequality, we have,

$$\begin{aligned}&\hat{\varsigma}_{bt} \circ [-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bt}^b)(\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{bt}^b)] \leq \\ &\quad \frac{1}{2}m_b c_1 \|\boldsymbol{r}_{bt}^b\| \|\boldsymbol{v}_{bt}^b + \boldsymbol{\omega}_{bt}^b \times \boldsymbol{r}_{bt}^b\| + J_{b\max}\|\boldsymbol{\omega}_{bt}^b\|^2\end{aligned} \tag{70}$$

$$\begin{aligned}&\hat{\varsigma}_{bt} \circ [-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{ti}^b)(\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{bt}^b)] \leq \\ &\quad \frac{3}{2}m_b c_2 \|\boldsymbol{r}_{bt}^b\| \|\boldsymbol{v}_{bt}^b + \boldsymbol{\omega}_{bt}^b \times \boldsymbol{r}_{bt}^b\| + J_{b\max} c_2 \|\boldsymbol{\xi}_{bt}\| \|\boldsymbol{\omega}_{bt}^b\| \\ &\quad + m_b c_3 \|\boldsymbol{\xi}_{bt}\| \|\boldsymbol{v}_{bt}^b + \boldsymbol{\omega}_{bt}^b \times \boldsymbol{r}_{bt}^b\|\end{aligned} \tag{71}$$

$$\begin{aligned}&\hat{\varsigma}_{bt} \circ [-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bt}^b)(\hat{\boldsymbol{M}}_b \hat{\boldsymbol{\omega}}_{ti}^b) + \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{bt}^b)\hat{\boldsymbol{\omega}}_{ti}^b)] \\ &\leq \frac{3}{2}m_b c_2 \|\boldsymbol{r}_{bt}^b\| \|\boldsymbol{v}_{bt}^b + \boldsymbol{\omega}_{bt}^b \times \boldsymbol{r}_{bt}^b\| + 2J_{b\max} c_2 \|\boldsymbol{\xi}_{bt}\| \|\boldsymbol{\omega}_{bt}^b\| \\ &\quad + m_b c_3 \|\boldsymbol{\xi}_{bt}\| \|\boldsymbol{v}_{bt}^b + \boldsymbol{\omega}_{bt}^b \times \boldsymbol{r}_{bt}^b\|\end{aligned} \tag{72}$$

and here $J_{b\max}$ is the maximum eigenvalue of $\boldsymbol{J}_b$. So

$$\begin{aligned}\dot{N}_c \leq & -k_p\hat{\varsigma}_{bt}\circ\hat{\varsigma}_{bt} - k_d\hat{\varsigma}_{bt}\circ(\hat{\boldsymbol{\omega}}_e^b)^s \\ & + l_1\|\boldsymbol{r}_{bt}^b\|\|\boldsymbol{v}_{bt}^b+\boldsymbol{\omega}_{bt}^b\times\boldsymbol{r}_{bt}^b\|\| + l_2\|\boldsymbol{\omega}_{bt}^b\|^2 \\ & + l_3\|\boldsymbol{\xi}_{bt}\|\|\boldsymbol{v}_{bt}^b+\boldsymbol{\omega}_{bt}^b\times\boldsymbol{r}_{bt}^b\| + l_4\|\boldsymbol{\xi}_{bt}\|\|\boldsymbol{\omega}_{bt}^b\| \\ & + l_5\|(\boldsymbol{v}_{bt}^b+\boldsymbol{\omega}_{bt}^b\times\boldsymbol{r}_{bt}^b)\|^2\end{aligned} \tag{73}$$

where $l_1 = m_bc_1 + 3m_bc_2 + k_d/2$, $l_2 = 3J_{b\max}$, $l_3 = 2m_bc_3$, $l_4 = 3J_{b\max}c_2 + k_d$, $l_5 = m_b/2$.

Proposition 3: Consider the 6-DOF tracking model given in (34-35), and the control input designed as (58), then system states $\hat{\boldsymbol{q}}_{bt}$, $\hat{\boldsymbol{\omega}}_{bt}^b$, $\hat{\boldsymbol{\omega}}_{bi}^b$ and $\hat{\boldsymbol{\omega}}_{oi}^b$ are uniformly bounded.

Proof: Employ the following augmented Lyapunov-like function candidate,

$$\begin{aligned}V_{c1} =& \rho V_c + N_c = \rho k_p(\hat{\boldsymbol{q}}_{bt}-\hat{\boldsymbol{q}}_I)\circ(\hat{\boldsymbol{q}}_{bt}-\hat{\boldsymbol{q}}_I) \\ & + \frac{\rho}{2}(\hat{\boldsymbol{\omega}}_{bt}^b)^s\circ(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_{bt}^b) + \hat{\varsigma}_{bt}\circ(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_{bt}^b)\end{aligned} \tag{74}$$

where $\rho\in\mathbb{R}$ is a constant. By the definition of $\hat{\boldsymbol{q}}_{bt}$, $\hat{\varsigma}_{bt}$, and $\hat{\boldsymbol{\omega}}_{bt}^b$, and then applying the Binet-Cauchy identity of cross product along with the Cauchy-Schwarz inequality, one can get,

$$\begin{aligned}V_{c1} \geq & \rho k_p((\eta_{bt}-1)^2 + \|\boldsymbol{\xi}_{bt}\|^2 + \frac{1}{4}\|\boldsymbol{r}_{bt}\|^2) \\ & + \frac{\rho m_b}{2}\|\boldsymbol{v}_{bt}^b+\boldsymbol{\omega}_{bt}^b\times\boldsymbol{r}_{bt}^b\|^2 + \frac{\rho J_{b\min}}{2}\|\boldsymbol{\omega}_{bt}^b\|^2 \\ & - \frac{m_b}{4}\|\boldsymbol{r}_{bt}^b\|^2 - \frac{m_b}{4}\|\boldsymbol{v}_{bt}^b+\boldsymbol{\omega}_{bt}^b\times\boldsymbol{r}_{bt}^b\|^2 \\ & - \frac{J_{b\max}}{2}\|\boldsymbol{\xi}_{bt}\|^2 - \frac{J_{b\max}}{2}\|\boldsymbol{\omega}_{bt}^b\|^2 \\ \geq & \rho k_p(\eta_{bt}-1)^2 + (\rho k_p - \frac{J_{b\max}}{2})\|\boldsymbol{\xi}_{bt}\|^2 \\ & + (\frac{\rho k_p}{4} - \frac{m_b}{4})\|\boldsymbol{r}_{bt}^b\|^2 + (\frac{\rho J_{b\min}}{2} - \frac{J_{b\max}}{2})\|\boldsymbol{\omega}_{bt}^b\|^2 \\ & + (\frac{\rho m_b}{2} - \frac{m_b}{4})\|\boldsymbol{v}_{bt}^b+\boldsymbol{\omega}_{bt}^b\times\boldsymbol{r}_{bt}^b\|^2\end{aligned} \tag{75}$$

Choosing

$$\rho \geq \max\{\frac{J_{b\max}}{k_p}, \frac{2m_b}{k_p}, \frac{2J_{b\max}}{J_{b\min}}, 1\}$$

one has,

$$\begin{aligned}V_{c1} \geq & \rho k_p(\eta_{bt}-1)^2 + \frac{J_{b\max}}{2}\|\boldsymbol{\xi}_{bt}\|^2 + \frac{m_b}{4}\|\boldsymbol{r}_{bt}^b\|^2 \\ & + \frac{J_{b\max}}{2}\|\boldsymbol{\omega}_{bt}^b\|^2 + \frac{m_b}{4}\|\boldsymbol{v}_{bt}^b+\boldsymbol{\omega}_{bt}^b\times\boldsymbol{r}_{bt}^b\|^2\end{aligned} \tag{76}$$

Eq. (76) guarantees $V_{c1}\geq 0$, and $V_{c1}=0$ only when $\hat{\boldsymbol{q}}_{bt}=\hat{\boldsymbol{q}}_I$ and $\hat{\boldsymbol{\omega}}_{bt}^b=\hat{\boldsymbol{0}}_3$, so V_{c1} is a valid Lyapunov function.

Substituting (63) and (73) into the time differential of V_{c1}, one can get,

$$\begin{aligned}\dot{V}_{c1} \leq & -\kappa_1(\|\boldsymbol{\xi}_{bt}\|^2 + \frac{1}{4}\|\boldsymbol{r}_{bt}^b\|^2) \\ & - \frac{\rho k_d}{2}(\|\boldsymbol{\omega}_{bt}^b\|^2 + \|\boldsymbol{v}_{bt}^b+\boldsymbol{\omega}_{bt}^b\times\boldsymbol{r}_{bt}^b\|^2) + \kappa_2\hat{\boldsymbol{\omega}}_e^b\circ\hat{\boldsymbol{\omega}}_e^b \\ & + l_1\|\boldsymbol{r}_{bt}^b\|\|\boldsymbol{v}_{bt}^b+\boldsymbol{\omega}_{bt}^b\times\boldsymbol{r}_{bt}^b\| + l_2\|\boldsymbol{\omega}_{bt}^b\|^2 \\ & + l_3\|\boldsymbol{\xi}_{bt}\|\|\boldsymbol{v}_{bt}^b+\boldsymbol{\omega}_{bt}^b\times\boldsymbol{r}_{bt}^b\| + l_4\|\boldsymbol{\xi}_{bt}\|\|\boldsymbol{\omega}_{bt}^b\| \\ & + l_5\|\boldsymbol{v}_{bt}^b+\boldsymbol{\omega}_{bt}^b\times\boldsymbol{r}_{bt}^b\|^2\end{aligned} \tag{77}$$

where $\kappa_1 = k_p - k_d\varpi/2$ and $\kappa_2 = k_d/(2\varpi) + \rho k_d/2$, and here ϖ is a positive constant. To guarantee $\kappa_1,\kappa_2>0$, choose $\varpi = k_p/k_d$.

Utilize the Cauchy-Schwarz inequality again and complete squares, we have

$$\begin{aligned}\dot{V}_{c1} \leq & -\frac{\kappa_1}{2}\|\boldsymbol{\xi}_{bt}\|^2 - \frac{\kappa_1}{8}\|\boldsymbol{r}_{bt}^b\|^2 \\ & -(\frac{\rho k_d}{2} - \frac{4l_1^2}{\kappa_1} - \frac{l_3^2}{\kappa_1} - l_5)\|\boldsymbol{v}_{bt}^b+\boldsymbol{\omega}_{bt}^b\times\boldsymbol{r}_{bt}^b\|^2 \\ & -(\frac{\rho k_d}{2} - \frac{l_4^2}{\kappa_1} - l_2)\|\boldsymbol{\omega}_{bt}^b\|^2 \\ & +\kappa_2\hat{\boldsymbol{\omega}}_e^b\circ\hat{\boldsymbol{\omega}}_e^b - (\frac{4l_1^2}{\kappa_1}\|\boldsymbol{v}_{bt}^b+\boldsymbol{\omega}_{bt}^b\times\boldsymbol{r}_{bt}^b\|^2 - ... \\ & l_1\|\boldsymbol{r}_{bt}^b\|\|\boldsymbol{v}_{bt}^b+\boldsymbol{\omega}_{bt}^b\times\boldsymbol{r}_{bt}^b\| + \frac{\kappa_1}{16}\|\boldsymbol{r}_{bt}^b\|^2) \\ & -(\frac{l_3^2}{\kappa_1}\|\boldsymbol{v}_{bt}^b+\boldsymbol{\omega}_{bt}^b\times\boldsymbol{r}_{bt}^b\|^2 - ... \\ & l_3\|\boldsymbol{\xi}_{bt}\|\|\boldsymbol{v}_{bt}^b+\boldsymbol{\omega}_{bt}^b\times\boldsymbol{r}_{bt}^b\| + \frac{\kappa_1}{4}\|\boldsymbol{\xi}_{bt}\|^2) \\ & -(\frac{l_4^2}{\kappa_1}\|\boldsymbol{\omega}_{bt}^b\|^2 - l_4\|\boldsymbol{\xi}_{bt}\|\|\boldsymbol{\omega}_{bt}^b\| + \frac{\kappa_1}{4}\|\boldsymbol{\xi}_{bt}\|^2) \\ \leq & -\frac{\kappa_1}{2}\|\boldsymbol{\xi}_{bt}\|^2 - \frac{\kappa_1}{8}\|\boldsymbol{r}_{bt}^b\|^2 \\ & -(\frac{\rho k_d}{2} - \frac{4l_1^2}{\kappa_1} - \frac{l_3^2}{\kappa_1} - l_5)\|\boldsymbol{v}_{bt}^b+\boldsymbol{\omega}_{bt}^b\times\boldsymbol{r}_{bt}^b\|^2 \\ & -(\frac{\rho k_d}{2} - \frac{l_4^2}{\kappa_1} - l_2)\|\boldsymbol{\omega}_{bt}^b\|^2 + \kappa_2\hat{\boldsymbol{\omega}}_e^b\circ\hat{\boldsymbol{\omega}}_e^b\end{aligned} \tag{78}$$

If we further modify the value of ρ to

$$\begin{aligned}\rho =& \max\{\frac{J_{\max}}{k_p}, \frac{2m_b}{k_p}, \frac{2J_{b\max}}{J_{b\min}}, 1, \frac{16l_1^2}{\kappa_1k_d} + \frac{4l_3^2}{\kappa_1k_d} + \frac{4l_5}{k_d}, ... \\ & \frac{4l_4^2}{\kappa_1k_d} + \frac{4l_2}{k_d}\}\end{aligned}$$

we can further get,

$$\begin{aligned}\dot{V}_{c1} \leq & -\frac{\kappa_1}{2}\|\boldsymbol{\xi}_{bt}\|^2 - \frac{\kappa_1}{8}\|\boldsymbol{r}_{bt}^b\|^2 - \frac{\rho k_d}{4}\|\boldsymbol{\omega}_{bt}^b\|^2 \\ & - \frac{\rho k_d}{4}\|\boldsymbol{v}_{bt}^b+\boldsymbol{\omega}_{bt}^b\times\boldsymbol{r}_{bt}^b\|^2 + \kappa_2\hat{\boldsymbol{\omega}}_e^b\circ\hat{\boldsymbol{\omega}}_e^b\end{aligned} \tag{79}$$

An upper bound of V_{c1} can be obtained from (74),

$$\begin{aligned}V_{c1} \leq & \rho k_p + \frac{J_{b\max}}{2} + \rho k_p\|\boldsymbol{\xi}_{bt}\|^2 + \frac{5\rho k_p + m_b}{4}\|\boldsymbol{r}_{bt}^b\|^2 \\ & + \frac{(2\rho+1)m_b}{4}\|\boldsymbol{v}_{bt}^b+\boldsymbol{\omega}_{bt}^b\times\boldsymbol{r}_{bt}^b\|^2 \\ & + \frac{\rho J_{b\min} + J_{b\max}}{2}\|\boldsymbol{\omega}_{bt}^b\|^2\end{aligned} \tag{80}$$

Then since it has been proved in Theorem 1 that $\hat{\boldsymbol{\omega}}_e^b$ is bounded, there exists positive constant p, such that $\hat{\boldsymbol{\omega}}_e^b\circ\hat{\boldsymbol{\omega}}_e^b\leq p$. And by (79) and (80), we can get,

$$\dot{V}_{c1} \leq -\chi V_{c1} + \sigma \tag{81}$$

where

$$\begin{aligned}\chi =& \min\{\frac{\kappa_1}{2\rho k_p}, \frac{\kappa_1}{2(5\rho k_p + m_b)}, \frac{\rho k_d}{2(\rho J_{b\min} + J_{b\max})}, ... \\ & \frac{\rho k_d}{(2\rho+1)m_b}\},\ \sigma = \chi(\rho k_p + \frac{J_{b\max}}{2}) + \kappa_2 p\end{aligned}$$

So that V_{c1} is uniformly bounded, which guarantees $\hat{\boldsymbol{q}}_{bt}$ and $\hat{\boldsymbol{\omega}}_{bt}^b$ are uniformly bounded, since $\hat{\boldsymbol{\omega}}_{ti}^t$ is bounded, so $\hat{\boldsymbol{\omega}}_{bi}^b$ is bounded. Furthermore, because $\hat{\boldsymbol{\omega}}_e^b$ is bounded, $\hat{\boldsymbol{\omega}}_{oi}^b$ is also bounded. The proof is complete.

C. Toward a Strict Observer Lyapunov-Like Function

To conduct the strictification process, consider a new cross term,

$$N_o = -2(\hat{\eta}_e \hat{\boldsymbol{\xi}}_e) \circ \hat{\boldsymbol{\omega}}_e^b \tag{82}$$

Notice $\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_e)\hat{\boldsymbol{\xi}}_e = \hat{\boldsymbol{0}}_3$ and $\hat{\eta}_e\hat{\eta}_e + \hat{\boldsymbol{\xi}}_e^{\mathrm{T}}\hat{\boldsymbol{\xi}}_e = 1 + \varepsilon 0$, the time differential of N_o can be written as:

$$\begin{aligned}\dot{N}_o =& -\hat{\boldsymbol{\omega}}_e^b \circ \hat{\boldsymbol{\omega}}_e^b + ((\hat{\boldsymbol{\xi}}_e^{\mathrm{T}}\hat{\boldsymbol{\xi}}_e)\hat{\boldsymbol{\omega}}_e^b) \circ \hat{\boldsymbol{\omega}}_e^b \\ &+ [(\hat{\boldsymbol{\xi}}_e^{\mathrm{T}}(\hat{\boldsymbol{\omega}}_e^b - \lambda(\hat{\varsigma}_e)^s))\hat{\boldsymbol{\xi}}_e] \circ \hat{\boldsymbol{\omega}}_e^b + (\lambda \hat{\eta}_e \hat{\eta}_e (\hat{\varsigma}_e)^s) \circ \hat{\boldsymbol{\omega}}_e^b \\ &- (\hat{\eta}_e \hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_e)\hat{\boldsymbol{\omega}}_e^b) \circ \hat{\boldsymbol{\omega}}_e^b - 2(\hat{\eta}_e\hat{\boldsymbol{\xi}}_e) \circ \dot{\hat{\boldsymbol{\omega}}}_e^b\end{aligned} \tag{83}$$

Since $\|\boldsymbol{\xi}_e\| \leq 1$, so $\|\hat{\boldsymbol{\xi}}_e\| \leq \|(\hat{\varsigma}_e)^s\|$. And because $\hat{\boldsymbol{\omega}}_e^b$, $\hat{\boldsymbol{\xi}}_e$, $\hat{\eta}_e$, $\hat{\varsigma}_e$, $\hat{\boldsymbol{\omega}}_{bi}^b$ and $\hat{\boldsymbol{\omega}}_{oi}^b$ are all proven to be bounded, so there exists positive constant dual numbers $\hat{b}_j = b_{jr} + \varepsilon b_{jd}, j = 1, 2, ..., 6$, satisfy $\|\hat{\boldsymbol{\xi}}_e\| \leq \hat{b}_1$, $\|\hat{\boldsymbol{\omega}}_e^b\| \leq \hat{b}_2$, $\|(\hat{\varsigma}_e)^s\| \leq \hat{b}_3$, $\|\hat{\eta}_e\| \leq \hat{b}_4$, $\|\hat{\boldsymbol{\omega}}_{bi}^b\| < \hat{b}_5$ and $\|\hat{\boldsymbol{\omega}}_{oi}^b\| < \hat{b}_6$. Then the following inequations can be established:

$$((\hat{\boldsymbol{\xi}}_e^{\mathrm{T}}\hat{\boldsymbol{\xi}}_e)\hat{\boldsymbol{\omega}}_e^b) \circ \hat{\boldsymbol{\omega}}_e^b \leq (\hat{b}_1\hat{b}_2\|(\hat{\varsigma}_e)^s\|) \circ \|\hat{\boldsymbol{\omega}}_e^b\| \tag{84}$$

$$\begin{aligned}&[(\hat{\boldsymbol{\xi}}_e^{\mathrm{T}}(\hat{\boldsymbol{\omega}}_e^b - \lambda(\hat{\varsigma}_e)^s))\hat{\boldsymbol{\xi}}_e] \circ \hat{\boldsymbol{\omega}}_e^b \leq \\ &\quad (\hat{b}_1\hat{b}_2\|(\hat{\varsigma}_e)^s\|) \circ \|\hat{\boldsymbol{\omega}}_e^b\| + (\lambda \hat{b}_1^2\|(\hat{\varsigma}_e)^s\|) \circ \|\hat{\boldsymbol{\omega}}_e^b\|\end{aligned} \tag{85}$$

$$(\lambda \hat{\eta}_e \hat{\eta}_e (\hat{\varsigma}_e)^s) \circ \hat{\boldsymbol{\omega}}_e^b \leq (\lambda \hat{b}_4^2 \|(\hat{\varsigma}_e)^s\|) \circ \|\hat{\boldsymbol{\omega}}_e^b\| \tag{86}$$

$$-(\hat{\eta}_e \hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_e)\hat{\boldsymbol{\omega}}_e^b) \circ \hat{\boldsymbol{\omega}}_e^b \leq (\hat{b}_2\hat{b}_4\|(\hat{\varsigma}_e)^s\|) \circ \|\hat{\boldsymbol{\omega}}_e^b\| \tag{87}$$

$$-2(\hat{\eta}_e\hat{\boldsymbol{\xi}}_e) \circ \dot{\hat{\boldsymbol{\omega}}}_e^b \leq (2\hat{b}_4\|(\hat{\varsigma}_e)^s\|) \circ \|\dot{\hat{\boldsymbol{\omega}}}_e^b\| \tag{88}$$

Further define $\hat{\ell}_1 = m_b \frac{\mathrm{d}}{\mathrm{d}\varepsilon} + \varepsilon J_{b\max}$ and $\hat{\ell}_2 = \frac{1}{J_{b\min}} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} + \varepsilon \frac{1}{m_b}$, where $J_{b\max}$ and $J_{b\min}$ are the maximum and minimum eigenvalues of $\boldsymbol{J}_b$, respectively. Then according to (49), a upper bound of $\dot{\hat{\boldsymbol{\omega}}}_e^B$ can be obtained,

$$\begin{aligned}\|\dot{\hat{\boldsymbol{\omega}}}_e^b\| =& \|\hat{\boldsymbol{M}}_b^{-1}[-\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^b)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_{bi}^b) - \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{oi}^b)(\hat{\boldsymbol{M}}_b\hat{\boldsymbol{\omega}}_e^b) + ... \\ & \hat{\boldsymbol{M}}_b(\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_e^b)\hat{\boldsymbol{\omega}}_{oi}^b) - \gamma\hat{\varsigma}_e]\| \\ \leq& \hat{\ell}_2(\|\hat{\boldsymbol{\omega}}_e^b\|(\hat{\ell}_1\hat{b}_5)) + \hat{\ell}_2(\hat{b}_6(\hat{\ell}_1\|\hat{\boldsymbol{\omega}}_e^b\|)) \\ & + \hat{b}_6\|\hat{\boldsymbol{\omega}}_e^b\| + \gamma\hat{\ell}_2\|\hat{\varsigma}_e\|\end{aligned} \tag{89}$$

So by (84)-(89), one can get:

$$\begin{aligned}\dot{N}_o \leq& -\hat{\boldsymbol{\omega}}_e^b \circ \hat{\boldsymbol{\omega}}_e^b + (\hat{b}_7\|(\hat{\varsigma}_e)^s\|) \circ \|\hat{\boldsymbol{\omega}}_e^b\| \\ &+ (2\hat{b}_4\|(\hat{\varsigma}_e)^s\|) \circ [\hat{\ell}_2(\hat{b}_5\hat{\ell}_1\|\hat{\boldsymbol{\omega}}_e^b\|) + \hat{\ell}_2(\|\hat{\boldsymbol{\omega}}_e^b\|(\hat{\ell}_1\hat{b}_6)) \\ &+ \hat{b}_6\|\hat{\boldsymbol{\omega}}_e^b\| + \gamma\hat{\ell}_2\|\hat{\varsigma}_e\|]\end{aligned} \tag{90}$$

where $\hat{b}_7 = 2\hat{b}_1\hat{b}_2 + \lambda\hat{b}_1^2 + \lambda\hat{b}_4^2 + \hat{b}_2\hat{b}_4$. Using $\boldsymbol{\omega}_{er}^b$ and $\boldsymbol{\omega}_{ed}^b$ denote the real part and dual part of $\hat{\boldsymbol{\omega}}_e^b$, respectively, then we can further obtain,

$$\begin{aligned}&(\hat{b}_7\|(\hat{\varsigma}_e)^s\|) \circ \|\hat{\boldsymbol{\omega}}_e^b\| = \\ &b_{7r}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{er}^b\| + \frac{1}{2}b_{7r}\|\boldsymbol{r}_e^b\|\|\boldsymbol{\omega}_{ed}^b\| + b_{7d}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{ed}^b\|\end{aligned} \tag{91}$$

$$\begin{aligned}&(2\hat{b}_4\|(\hat{\varsigma}_e)^s\|) \circ [\hat{\ell}_2(\|\hat{\boldsymbol{\omega}}_e^b\|(\hat{\ell}_1\hat{b}_5))] = \\ &\frac{2m_b}{J_{b\min}}b_{4r}b_{5d}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{ed}^b\| + \frac{2J_{b\max}}{J_{b\min}}b_{4r}b_{5r}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{er}^b\| \\ &+ 2b_{4d}b_{5d}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{er}^b\| + b_{4r}b_{5d}\|\boldsymbol{r}_e^b\|\|\boldsymbol{\omega}_{er}^b\|\end{aligned} \tag{92}$$

$$\begin{aligned}&(2\hat{b}_4\|(\hat{\varsigma}_e)^s\|) \circ [\hat{\ell}_2(\hat{b}_6(\hat{\ell}_1\|\hat{\boldsymbol{\omega}}_e^b\|))] = \\ &\frac{2m_b}{J_{b\min}}b_{4r}b_{6d}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{ed}^b\| + \frac{2J_{b\max}}{J_{b\min}}b_{4r}b_{6r}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{er}^b\| \\ &+ 2b_{4d}b_{6r}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{ed}^b\| + b_{4r}b_{6r}\|\boldsymbol{r}_e^b\|\|\boldsymbol{\omega}_{ed}^b\|\end{aligned} \tag{93}$$

$$\begin{aligned}&(2\hat{b}_4\|(\hat{\varsigma}_e)^s\|) \circ [\hat{b}_6\|\hat{\boldsymbol{\omega}}_e^b\|] = \\ &2b_{4r}b_{6r}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{er}^b\| + 2b_{4d}b_{6r}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{ed}^b\| \\ &+ 2b_{4d}b_{6d}\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{er}^b\| + b_{4r}b_{6r}\|\boldsymbol{r}_e^b\|\|\boldsymbol{\omega}_{ed}^b\| \\ &+ b_{4r}b_{6d}\|\boldsymbol{r}_e^b\|\|\boldsymbol{\omega}_{er}^b\|\end{aligned} \tag{94}$$

$$\begin{aligned}&(2\hat{b}_4\|(\hat{\varsigma}_e)^s\|) \circ [\gamma\hat{\ell}_2\|\hat{\varsigma}_e\|] = \\ &\frac{2\gamma}{J_{b\min}}b_{4r}\|\boldsymbol{\xi}_e\|^2 + \frac{\gamma}{m_b}b_{4d}\|\boldsymbol{\xi}_e\|\|\boldsymbol{r}_e^b\| + \frac{\gamma}{2m_b}b_{4r}\|\boldsymbol{r}_e^b\|^2\end{aligned} \tag{95}$$

So

$$\begin{aligned}\dot{N}_o \leq& -\hat{\boldsymbol{\omega}}_e^b \circ \hat{\boldsymbol{\omega}}_e^b + \alpha_1\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{er}^b\| + \alpha_2\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{ed}^b\| \\ &+ \alpha_3\|\boldsymbol{r}_e^b\|\|\boldsymbol{\omega}_{er}^b\| + \alpha_4\|\boldsymbol{r}_e^b\|\|\boldsymbol{\omega}_{ed}^b\| + \beta_1\|\boldsymbol{\xi}_e\|^2 \\ &+ \beta_2\|\boldsymbol{\xi}_e\|\|\boldsymbol{r}_e^b\| + \beta_3\|\boldsymbol{r}_e^b\|^2\end{aligned} \tag{96}$$

where $\alpha_1 = b_{7r} + \frac{2J_{b\max}}{J_{b\min}}b_{4r}(b_{5r} + b_{6r}) + 4b_{4d}b_{5d} + 2b_{4r}b_{5r}$, $\alpha_2 = b_{7d} + \frac{2m_b}{J_{b\min}}b_{4r}(b_{5d} + b_{6d}) + 2b_{4d}b_{6r} + 2b_{4d}b_{5r}$, $\alpha_3 = 2b_{4r}b_{5d}$, $\alpha_4 = \frac{1}{2}b_{7r} + b_{4r}(b_{5r} + b_{6r})$, $\beta_1 = \frac{2\gamma}{J_{b\min}}b_{4r}$, $\beta_2 = \frac{\gamma}{m_b}b_{4d}$ and $\beta_3 = \frac{\gamma}{2m_b}b_{4r}$. The function of cross term N_o is shown in the following proposition.

Proposition 4: For the observer construction given in (39) and (40), the following augmented function V_{o1},

$$V_{o1} = \mu V_o + N_o \tag{97}$$

is a partially strict Lyapunov-like function for the estimation error dynamics. Where $\mu > 0$ is a positive constant, satisfying,

$$\begin{aligned}\mu =& \max\{\frac{2}{\gamma}, \frac{4}{J_{b\min}}, \frac{8}{m_b}, \frac{4(2\alpha_3^2 + 2\alpha_4^2 + \beta_2 + 2\beta_3)}{\gamma\lambda}, ... \\ & \frac{2\alpha_1^2 + 2\alpha_2^2 + 2\beta_1 + \beta_2}{\gamma\lambda}\}\end{aligned} \tag{98}$$

Proof: By the definition of μ, one has

$$\begin{aligned}V_{o1} \geq& (\mu\gamma - 1)(\|\boldsymbol{q}_e - \boldsymbol{q}_I\|^2 + \frac{1}{4}\|\boldsymbol{q}_e \otimes \boldsymbol{r}_e^b\|^2) \\ &+ (\frac{J_{b\min}}{2}\mu - 1)\|\boldsymbol{\omega}_{er}^B\|^2 + (\frac{m_b}{2}\mu - 2)\|\boldsymbol{\omega}_{ed}^B\|^2 \\ \geq& 0\end{aligned} \tag{99}$$

So V_{o1} is a valid Lyapunov-like function candidate. According to (54) and (96), the differential of V_{o1} with respect to time satisfies,

$$\begin{aligned}\dot{V}_{o1} =& \mu\dot{V}_o + \dot{N}_o \\ \leq& -\mu\gamma\lambda(\frac{1}{4}\|\boldsymbol{r}_e^b\|^2 + \|\boldsymbol{\xi}_e\|^2) - \|\boldsymbol{\omega}_{er}^b\|^2 - \|\boldsymbol{\omega}_{ed}^b\|^2 \\ &+ \alpha_1\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{er}^b\| + \alpha_2\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{ed}^b\| + \alpha_3\|\boldsymbol{r}_e^b\|\|\boldsymbol{\omega}_{er}^b\| \\ &+ \alpha_4\|\boldsymbol{r}_e^b\|\|\boldsymbol{\omega}_{ed}^b\| + \beta_1\|\boldsymbol{\xi}_e\|^2 + \beta_2\|\boldsymbol{\xi}_e\|\|\boldsymbol{r}_e^b\| + \beta_3\|\boldsymbol{r}_e^b\|^2\end{aligned} \tag{100}$$

Utilize the Cauchy-Schwarz inequality and complete squares, one has,

$$\begin{aligned}\dot{V}_{o1} \leq & -\frac{\mu\gamma\lambda}{8}(\|\boldsymbol{r}_e^b\|^2 + \frac{1}{2}\|\boldsymbol{\xi}_e\|^2) - \frac{1}{2}(\|\boldsymbol{\omega}_{er}^b\|^2 + \|\boldsymbol{\omega}_{ed}^b\|^2) \\ & - (\frac{\mu\gamma\lambda}{8} - \alpha_3^2 - \alpha_4^2 - \frac{1}{2}\beta_2 - \beta_3)\|\boldsymbol{r}_e^b\|^2 \\ & - (\frac{\mu\gamma\lambda}{2} - \alpha_1^2 - \alpha_2^2 - \beta_1 - \frac{1}{2}\beta_2)\|\boldsymbol{\xi}_e\|^2 \\ & - (\alpha_1^2\|\boldsymbol{\xi}_e\|^2 - \alpha_1\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{er}^b\| + \frac{1}{4}\|\boldsymbol{\omega}_{er}^b\|) \\ & - (\alpha_2^2\|\boldsymbol{\xi}_e\|^2 - \alpha_2\|\boldsymbol{\xi}_e\|\|\boldsymbol{\omega}_{ed}^b\| + \frac{1}{4}\|\boldsymbol{\omega}_{ed}^b\|) \\ & - (\alpha_3^2\|\boldsymbol{r}_e^b\|^2 - \alpha_3\|\boldsymbol{r}_e^b\|\|\boldsymbol{\omega}_{er}^b\| + \frac{1}{4}\|\boldsymbol{\omega}_{er}^b\|) \\ & - (\alpha_4^2\|\boldsymbol{r}_e^b\|^2 - \alpha_4\|\boldsymbol{r}_e^b\|\|\boldsymbol{\omega}_{ed}^b\| + \frac{1}{4}\|\boldsymbol{\omega}_{ed}^b\|) \\ \leq & -\frac{\mu\gamma\lambda}{8}(\|\boldsymbol{r}_e^b\|^2 + 4\|\boldsymbol{\xi}_e\|^2) - \frac{1}{2}(\|\boldsymbol{\omega}_{er}^b\|^2 + \|\boldsymbol{\omega}_{ed}^b\|^2) \\ \leq & -\frac{\mu\gamma\lambda}{2}\hat{\varsigma}_e \circ \hat{\varsigma}_e - \frac{1}{2}\hat{\boldsymbol{\omega}}_e^b \circ \hat{\boldsymbol{\omega}}_e^b\end{aligned} \tag{101}$$

So V_{o1} is now a partially strict Lyapunov-like function, the asymptotic convergence of $\hat{\boldsymbol{\xi}}_e$ and $\hat{\boldsymbol{\omega}}_e^b$ can be directly guaranteed by (101), through the use of the corollary of Barbalat's lemma. The proof is complete.

V. Proof of the Separation Property

After the strictification process, another important contribution of this paper is organized in Theorem 2.

Theorem 2: Consider the dual-quaternion based relative tracking dynamics in (34-35), with the observer proposed in (39-40), and the total control input designed in (58), then the tracking and estimation errors satisfy $\lim_{t\to\infty}[\hat{\boldsymbol{\xi}}_e(t), \hat{\boldsymbol{\omega}}_e^b(t), \hat{\boldsymbol{\xi}}_{bt}(t), \hat{\boldsymbol{\omega}}_{bt}^b(t)] = \hat{\mathbf{0}}_3$.

Proof: Define the following Lyapunov-like function candidate, which is the combination of an observer part and a controller part,

$$V_{oc1} = \underbrace{\nu V_{o1}}_{\text{observer part}} + \underbrace{V_{c1}}_{\text{controller part}} \tag{102}$$

where $\nu = 4\kappa_2$ is a positive constant.

According to (79) and (101), the time derivative of V_{oc1} satisfies

$$\begin{aligned}\dot{V}_{oc1} \leq & -\frac{\nu\mu\gamma\lambda}{2}\hat{\varsigma}_e \circ \hat{\varsigma}_e - \frac{\nu}{2}\hat{\boldsymbol{\omega}}_e^b \circ \hat{\boldsymbol{\omega}}_e^b \\ & - \frac{\kappa_1}{2}\|\boldsymbol{\xi}_{bt}\|^2 - \frac{\kappa_1}{8}\|\boldsymbol{r}_{bt}^b\|^2 - \frac{\rho k_d}{4}\|\boldsymbol{\omega}_{bt}^b\|^2 \\ & - \frac{\rho k_d}{4}\|\boldsymbol{v}_{bt}^b + \boldsymbol{\omega}_{bt}^b \times \boldsymbol{r}_{bt}^b\|^2 + \kappa_2\hat{\boldsymbol{\omega}}_e^b \circ \hat{\boldsymbol{\omega}}_e^b \\ \leq & -\frac{\nu\mu\gamma\lambda}{2}\hat{\varsigma}_e \circ \hat{\varsigma}_e - \frac{\nu}{4}\hat{\boldsymbol{\omega}}_e^b \circ \hat{\boldsymbol{\omega}}_e^b \\ & - \frac{\kappa_1}{2}\hat{\varsigma}_{bt} \circ \hat{\varsigma}_{bt} - \frac{\rho k_d}{4}\hat{\boldsymbol{\omega}}_{bt}^b \circ \hat{\boldsymbol{\omega}}_{bt}^b\end{aligned} \tag{103}$$

Because $V_{oc1} \geq 0$ and $\dot{V}_{oc1} \leq 0$, so $V_{oc1} \in \mathcal{L}_\infty$, and by the definition of V_{oc1}, we have $\hat{\varsigma}_e, \hat{\boldsymbol{\omega}}_e^b, \hat{\varsigma}_{bt}, \hat{\boldsymbol{\omega}}_{bt}^b \in \mathcal{L}_\infty$. Additionally, since $\int_0^\infty \dot{V}_{oc1}(\sigma)\mathrm{d}\sigma$ exists, $\hat{\varsigma}_e, \hat{\boldsymbol{\omega}}_e^b, \hat{\varsigma}_{bt}, \hat{\boldsymbol{\omega}}_{bt}^b \in \mathcal{L}_2$. Furthermore, recall (31), (32), (46) and (49), we have $\dot{\hat{\varsigma}}_e, \dot{\hat{\boldsymbol{\omega}}}_e^b, \dot{\hat{\varsigma}}_{bt}, \dot{\hat{\boldsymbol{\omega}}}_{bt}^b \in \mathcal{L}_\infty$. By the corollary of Barbalat's Lemma, these results guarantee

$$\lim_{t\to\infty}[\hat{\varsigma}_e(t), \hat{\boldsymbol{\omega}}_e^b(t), \hat{\varsigma}_{bt}(t), \hat{\boldsymbol{\omega}}_{bt}^b(t)] = \hat{\mathbf{0}}_3 \tag{104}$$

And it can be readily proved that $\hat{\varsigma}_e(t) = \hat{\mathbf{0}}_3$ and $\hat{\varsigma}_{bt}(t) = \hat{\mathbf{0}}_3$ guarantee $\hat{\boldsymbol{\xi}}_e(t) = \hat{\mathbf{0}}_3$ and $\hat{\boldsymbol{\xi}}_{bt}(t) = \hat{\mathbf{0}}_3$, respectively. So finally we get

$$\lim_{t\to\infty}[\hat{\boldsymbol{\xi}}_e(t), \hat{\boldsymbol{\omega}}_e^b(t), \hat{\boldsymbol{\xi}}_{bt}(t), \hat{\boldsymbol{\omega}}_{bt}^b(t)] = \hat{\mathbf{0}}_3 \tag{105}$$

The proof is completed.

Remark 4: It's noteworthy that, except the controller and observer parameters k_p, k_d, λ and γ, all the other introduced parameters (like ν, ρ, χ, ϖ, and so on) are employed purely for analysis. Which means these parameters put no restriction on the controller or the observer, and won't influence the performance of the closed-loop system.

Remark 5: The logic procedure in this section is crucial and rigorously ordered. Firstly, Theorem 1 guarantees the boundedness of the estimation error $\hat{\boldsymbol{\omega}}_e^b$. And then, as shown in Proposition 3, this result is combined with a strictification-like process, and enables $\hat{\boldsymbol{\omega}}_{bi}^b$ and $\hat{\boldsymbol{\omega}}_{oi}^b$ to be uniformly bounded. Subsequently, in Proposition 4, the boundedness of $\hat{\boldsymbol{\omega}}_{bi}^b$ and $\hat{\boldsymbol{\omega}}_{oi}^b$ are utilized to help us get a partially strict observer Lyapunov-like function, and finally the separation property is proved in Theorem 2.

VI. Numerical Simulations

In this section, based on the observer in given (39-40) and the controller presented in (58), typical simulation examples are illustrated and analyzed. The performance of the proposed observer-controller construction is compared with the PD-like full-state-feedback controller and also the passivity-based output-feedback method introduced in Ref. [22]. And then, the effectiveness of the proposed approach is further demonstrated by Monte Carlo simulations.

To optimize the comparison results, a similar simulation scenario with Ref. [22] is employed, in which the follower spacecraft is required to track an elliptical motion around the leader while the X_B-axis of it precisely pointing to the leader. In the simulation, the inertia frame $\mathcal{I}$ is chosen as the Earth-centered inertial (ECI) frame, and the leader spacecraft is running on a Molniya orbit, the orbital elements of the leader are given in Table I.

TABLE I: Orbital Elements of the Leader Spacecraft

Orbital Elements	Quantity
Semimajor axis, km	26553.937
Eccentricity, -	0.729677
Inclination, deg	63.4
Argument of perigee, deg	-90
RAAN, deg	0
True anomaly, deg	180

The body-fixed frame of the leader (the frame $\mathcal{L}$) is always coincide with it's Local-Vertical-Local Horizontal

(LVLH) frame (the definition of the LVLH frame is: x-axis is along the radius vector of the leader, z-axis is along the angular momentum vector, and y-axis completes the right-handed system). The target frame $\mathcal{T}$ is designed as: $\boldsymbol{q}_{tl}(0) = [0,0,0,1]^{\mathrm{T}}$, $\boldsymbol{r}^l_{tl} = [a_e,0,0]^{\mathrm{T}}$m, $\boldsymbol{\omega}^l_{tl} = [0,0,a_e b_e \omega_0/(a_e^2\cos^2(\omega_0 t) + b_e^2 \sin^2(\omega_0 t))]^{\mathrm{T}}$rad/s, $\boldsymbol{v}^l_{tl} = [-a_e\omega_0\sin(\omega_0 t), b_e\omega_0\cos(\omega_0 t), 0]^{\mathrm{T}}$ m/s, $a_e = 15$m, $b_e = 25$m. And ω_0 denotes the (unperturbed) mean orbital angular velocity of the leader, so $\omega_0 = \sqrt{\mu/a^3}$ rad/s, where $\mu = 398600.4405$ kg^3/m^2 is the geocentric gravitational constant, and here a is the semimajor axis of the leader's orbit. Under this design, the target frame $\mathcal{T}$ conducts an elliptical motion around the leader, while it x-axis always pointing to the leader.

As mentioned in the paper, the control objective is to superimpose the body-fixed frame of the follower onto the target frame. The nominal mass and the inertia of the follower are assumed to be [9]

$$m = 400 \text{ kg},\ \boldsymbol{J}_b = \begin{bmatrix} 55 & 1.5 & -3 \\ 1.5 & 65 & -0.5 \\ -3 & -0.5 & 58 \end{bmatrix} \text{kg}\cdot\text{m}^2$$

To test the robustness of the proposed method, the gravity gradient torque $\boldsymbol{\tau}_{gg}$ and the J2 perturbation force $\boldsymbol{f}_{J2}$ are employed in the simulation, given by Ref. [7],

$$\boldsymbol{\tau}_{gg} = 3\mu \frac{\boldsymbol{r}^b_{bi} \times (\boldsymbol{J}_b \boldsymbol{r}^b_{bi})}{\|\boldsymbol{r}^b_{bi}\|^5}$$

$$\boldsymbol{f}_{J2} = m_b \boldsymbol{a}^b_{J2},\ \boldsymbol{a}^i_{J2} = -\frac{3\mu J_2 R_e^2}{2\|\boldsymbol{r}^i_{bi}\|^4} \begin{bmatrix} (1-5(\frac{z^i_{bi}}{\|\boldsymbol{r}^i_{bi}\|})^2)\frac{x^i_{bi}}{\|\boldsymbol{r}^i_{bi}\|} \\ (1-5(\frac{z^i_{bi}}{\|\boldsymbol{r}^i_{bi}\|})^2)\frac{y^i_{bi}}{\|\boldsymbol{r}^i_{bi}\|} \\ (3-5(\frac{z^i_{bi}}{\|\boldsymbol{r}^i_{bi}\|})^2)\frac{z^i_{bi}}{\|\boldsymbol{r}^i_{bi}\|} \end{bmatrix}$$

where x^i_{bi}, y^i_{bi}, z^i_{bi} are the corresponding components of $\boldsymbol{r}^i_{bi}$, $J_2 = 0.0010826267$, and $R_e = 6378.137$ km is the mean equatorial radius of the Earth.

Furthermore, during the proximity process, the mass ejected by the actuators will introduce changes to the follower's nominal mass and inertia, and lead to parameter uncertainties, this issue is also considered in the simulation. Assume the actuators of the follower are thrusters, and they are able to generate variable thrust amplitudes (for example, the cold-gas thrusters [36]). The follower has a total of 12 thrusters to implement 6-DOF control, and the configuration of them is given in Fig. 2. The specific impulse of thrusters is assumed to be [36]: SI $= 9.80665\times 70$ N·s/kg, and the maximum output of thrusters is 30N. So the mass change induced by every thruster is $\Delta m_j = (\int_0^t |F_j(t)|dt)/\text{SI}$, where $F_j(\cdot)$ is the output of the corresponding thruster j, and $j = 1,2,...,12$. A reasonable assumption is that every Δm_j can be regarded as a mass point removed from the position of the corresponding thruster, and then we can further calculate the change of inertia.

The initial conditions are set to: $\boldsymbol{q}_{bl}(0) = [0,0.8,0.6,0]^{\mathrm{T}}$, $\boldsymbol{r}^l_{bl}(0) = [10,-20,-15]^{\mathrm{T}}$m, $\boldsymbol{v}^l_{bl}(0) = [1,0.5,0.4]^{\mathrm{T}}$ m/s, $\boldsymbol{\omega}^l_{bl}(0) = [0.05,-0.1,0.2]^{\mathrm{T}}$ rad/s, $\hat{\boldsymbol{q}}_{ol}(0) = \hat{\boldsymbol{q}}_{bl}(0)$ and $\hat{\boldsymbol{\omega}}^o_{ol}(0) = [0,0,0,0,0,0,0,0]^{\mathrm{T}}$ rad/s. Parameters of the proposed method are $k_p = 1$, $k_d = 20$, $\lambda = 0.5$ and $\gamma = 10$. For the controller given in Ref. [22], the linear time-invariant subsystem is set to $\boldsymbol{A} = \boldsymbol{I}_{8\times8}$, $\boldsymbol{B} = 5\boldsymbol{I}_{8\times8}$ and $\boldsymbol{C} = k_d\boldsymbol{I}_{8\times8}$.

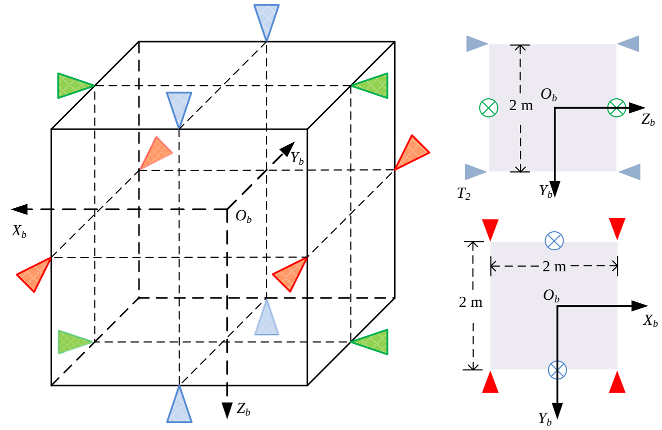

Fig. 2: Configuration of thrusters

A. Noise-Free Simulations

Follow the aforementioned conditions, this subsection demonstrates the simulation results when there is no measurement noises.

Figure 3 shows the estimation errors of angular and linear velocities, it can be seen that the designed observer has a good performance, all estimation errors can converge within 100s, with smooth trajectories and high precisions. The tracking errors of relative positions and quaternions are shown in Fig. 4 and Fig. 5, respectively. And Figure 6 further demonstrates the tracking errors (norms) of linear and angular velocities. These results show that all the three methods can achieve control objectives. For the two velocity-free methods, one can see that the observer-based method introduced in this paper has faster convergence processes and higher precisions with respect to the passivity-based output feedback controller given in Ref. [22]. The reason why the controller in Ref. [22] needs a longer convergence time maybe become this method is based on the output of a special linear time-invariant system which can be regarded as a low-pass filter, this structure could result in extended convergence processes. Compared with the full-state feedback case, the proposed method in this paper leads to small fluctuations due to the indispensable procedure to eliminate estimation errors. The changes of masses and principle inertias are illustrated in Fig. 7, compared to the full-state feedback controller, the extra mass consumption of the proposed method also comes from the estimation process. The steady-state errors of this simulation case are summarized in Table II. Overall, the performance of the proposed observer-controller method is just slight worse than the nominal full-state-feedback PD-like controller.

TABLE II: Steady Tracking and Estimation Errors

	PD-like	Observer-based	Output feedback
$\boldsymbol{\xi}_{bt}$, -	2.5×10^{-6}	5.0×10^{-6}	1.2×10^{-3}
$\boldsymbol{r}^b_{bt}$, m	1.1×10^{-4}	1.8×10^{-4}	9×10^{-3}
$\boldsymbol{\omega}^b_{bt}$, rad/s	1.2×10^{-7}	5.2×10^{-6}	3.6×10^{-4}
$\boldsymbol{v}^b_{bt}$, m/s	4.7×10^{-6}	1.2×10^{-5}	1.2×10^{-4}
$\boldsymbol{\omega}^b_e$, rad/s	-	1.1×10^{-7}	-
$\boldsymbol{v}^b_e$, m/s	-	1.3×10^{-6}	-

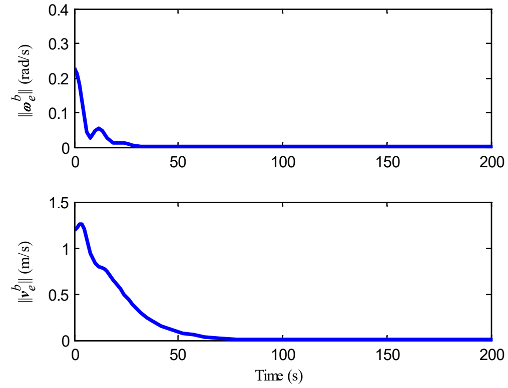

Fig. 3: Estimation errors of dual angular velocites

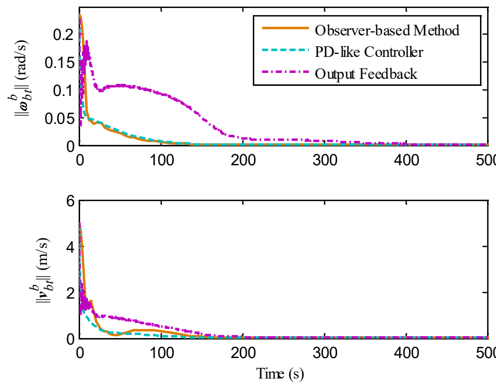

Fig. 6: Tracking errors of dual angular velocites

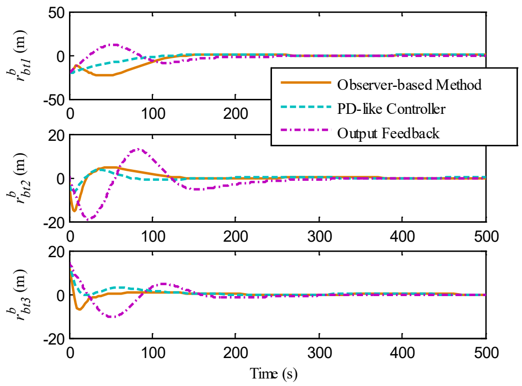

Fig. 4: Tracking errors of relative positions

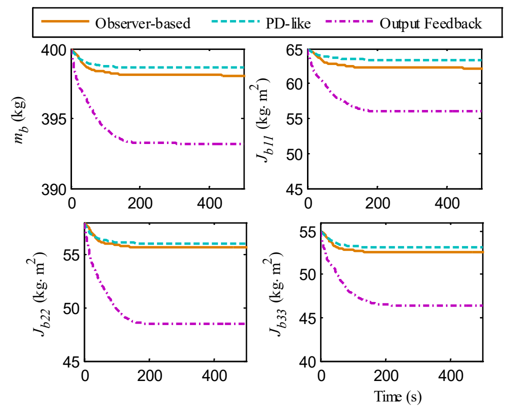

Fig. 7: Changes of masses and principal inertias

To further demonstrate the performance of the proposed method, a 3D illustration is provided in Fig. 8. To clearly show the relative motion between the follower and the leader, the figure is drawn upon the frame $\mathcal{L}$, so the leader is motionless in the figure (but in simulations, both the leader and follower are doing orbital motion around ECI). The leader and the follower are represented by spheres, the mutually perpendicular lines pointing from the follower are employed to denote the instantaneous axis directions of the frame $\mathcal{B}$, and the directions of X_B-axis are further indicated by spherical cones. The instantaneous positions of the follower (denoted by light orange spheres) at $t = 0$s, 35.06s, 213.3s, 17108s and 25972s are given in Fig. 8. It is shown that under the proposed method, the follower could track the predetermined elliptical trajectory around the leader, while it's X_B-axis always pointing to the leader. To sum up, the method proposed in this paper successfully completes the estimation and tracking task with reasonable convergence time and good precisions.

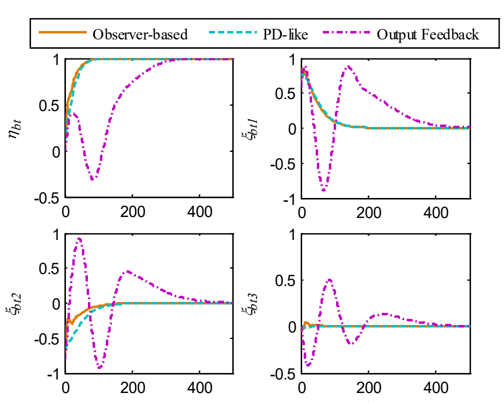

Fig. 5: Tracking errors of relative quaternions

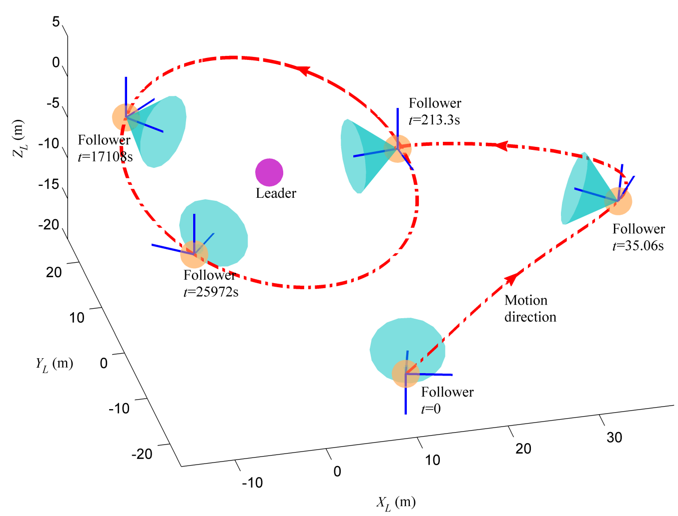

Fig. 8: 3D illustration under the proposed method

B. Simulations under Measurement Noises

In this subsection, to further illustrate the robustness of the proposed observer-controller construction, measurement noises of relative quaternions and positions are considered. Given the definition of quaternions [31]:

$$\boldsymbol{\xi}_{bl} = \sin(\frac{\theta_{bl}}{2})\boldsymbol{e}_{bl},\ \eta_{bl} = \cos(\frac{\theta_{bl}}{2}) \tag{106}$$

where θ_{bl} and $\boldsymbol{e}_{bl}$ are the eigenangle and the eigenaxis associated with $\boldsymbol{q}_{bl}$, respectively. Measurement noises are generated through the random perturbations of $\boldsymbol{e}_{bl}$ within a spherical cone of a prescribed cone half-angle and uniform distribution centered around the true eigenaxis. The cone half-angle is specified as 0.05 deg. For relative position, zero-mean additive white Gaussian noises with standard deviation of 0.01 m are considered as process noises. After adding these noises to the model, the simulation case given in Sec. VI. A is repeated. The time responses of $\|\boldsymbol{\xi}_{bt}\|$ and $\|\boldsymbol{r}_{bt}\|$ are given in Figs. 9 and 10, respectively, and semi-logarithmic scales are employed in these two figures to clearly show the performance differences among different methods. One can see that the overall performance of all the three control methods suffer from the presence of measurement noises, and tracking errors can only converge to small residual sets around the origin. The proposed method has an almost same steady-state performance with the full-state PD-like controller, which means it is not adversely affected any worse than PD-like controller under noisy measurements. The estimation errors of the observer are shown in Fig. 11, though the steady estimate errors are also influenced by noisy measurements, but the proposed observer can still render the estimation errors rapidly converge to a small bounded region.

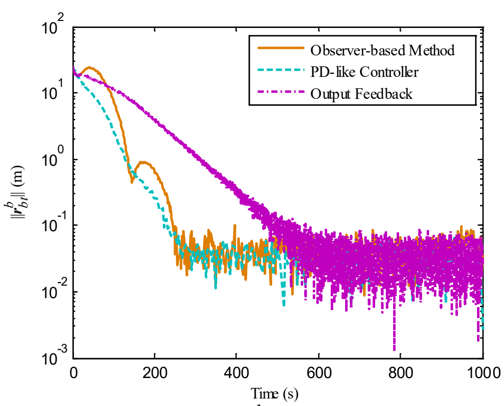

Fig. 9: Time response of $\|\boldsymbol{r}_{bt}^b\|$ under noisy measurements

C. Monte Carlo Simulations

Finally, Monte Carlo simulations are employed to further demonstrate the performance and robustness of the proposed method under external disturbances and parameter uncertainties with variable initial conditions and control parameters.

Table III presents the randomized parameters and initial conditions, and also their random ranges. Notice that the follower can only know the nominal mass and inertia values, and can't get any information about external disturbances.

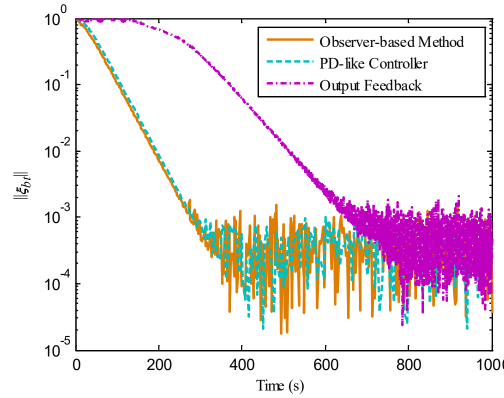

Fig. 10: Time response of $\|\boldsymbol{\xi}_{bt}\|$ under noisy measurements

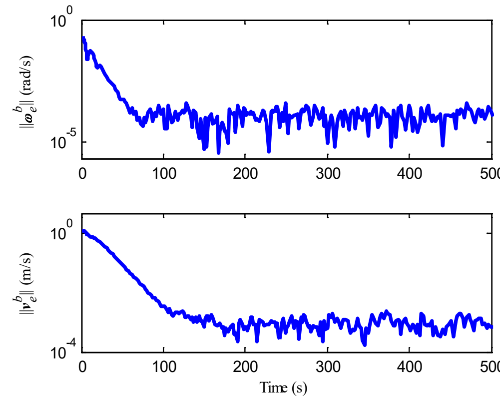

Fig. 11: Estimation errors under noisy measurements

The process of a single simulation is that: First, by Monte Carlo techniques, randomly generate a set of new simulation parameters and initial conditions; then, under the proposed method, the simulation case given in Sec. VI. A is repeated with those new generated random values. Finally, the steady tracking errors and estimation errors are recorded. Under all these conditions, 1000 Monte Carlo runs are conducted. The distributions of tracking errors and estimation errors are given in Figs. 12-14, which shows that the overwhelming majority of simulations have good performance, even in the presence of unknown external disturbances and parameter uncertainties,

TABLE III: Randomized Parameters and Initial Values

Parameters	Ranges
m_b, kg	[350, 450]
$\boldsymbol{J}_b$, kg·m^2	$\begin{bmatrix} 55\pm10, & 1.5\pm1, & -3\pm1 \\ 1.5\pm1, & 65\pm5, & -0.5\pm0.5 \\ -3\pm1, & -0.5\pm0.5, & 50\pm6 \end{bmatrix}$
k_p, -	[0.25, 2]
k_d, -	[5, 25]
λ, -	[0.2, 2]
γ, -	[2, 15]
$\boldsymbol{q}_{bl}(0)$, -	$\mathbb{H}$
$\boldsymbol{r}_{bl}^l(0)$, m	$[-200, -200, -200]^{\mathrm{T}}$ to $[200, 200, 200]^{\mathrm{T}}$
$\boldsymbol{\omega}_{bl}^l(0)$, rad/s	$[-0.2, -0.2, -0.2]^{\mathrm{T}}$ to $[0.2, 0.2, 0.2]^{\mathrm{T}}$
$\boldsymbol{r}_{bl}^l(0)$, m/s	$[-2, -2, -2]^{\mathrm{T}}$ to $[2, 2, 2]^{\mathrm{T}}$

while the tracking and estimation errors of the worst cases are less than 2×10^{-2} m ($\|\boldsymbol{r}_{bt}^b\|$), 4×10^{-4} ($\|\boldsymbol{\xi}_{bt}\|$), 4×10^{-4} m/s ($\|\boldsymbol{v}_{bt}^b\|$), 3×10^{-4} rad/s ($\|\boldsymbol{\omega}_{bt}^b\|$), 3×10^{-4} m/s ($\|\boldsymbol{v}_e^b\|$), and 2×10^{-4} rad/s ($\|\boldsymbol{\omega}_e^b\|$). By further analyzing the simulation conditions of these worst cases, it is found out that their relatively worse performance are caused by the improper observer and controller parameters. For example, for the point on the top of Fig. 12 (the corresponding simulation has the worst position tracking error), the k_p of this simulation case is 0.2723, which is too small to guarantee diminutive tracking errors. Simulation results suggest that $k_p\in[0.5,2]$, $k_d\in[5k_p,20k_p]$, $\lambda\in[0.2,1]$ and $\gamma\in[5\lambda,20\lambda]$ would be reasonable choices, while the final decisions should also base on the initial conditions of the corresponding simulation cases.

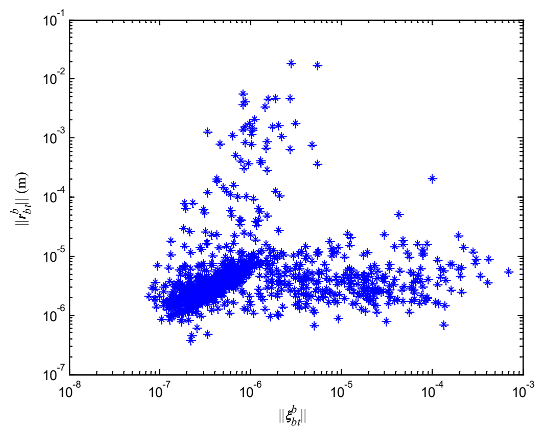

Fig. 12: Error distributions of relative dual quaternions

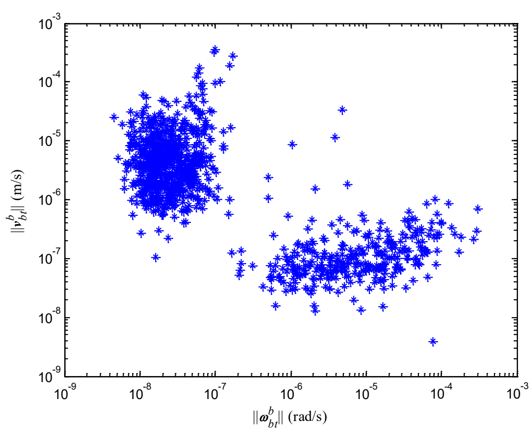

Fig. 13: Error distributions of relative dual angular velocities

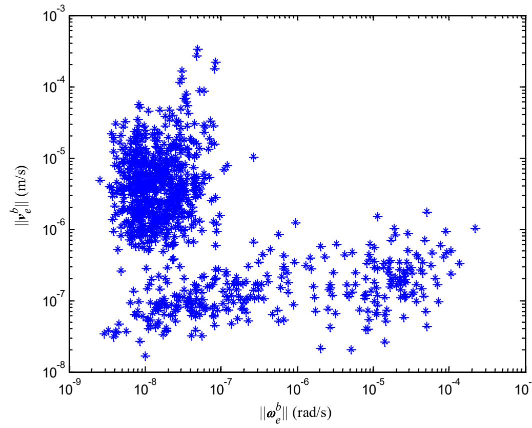

Fig. 14: Distributions of estimation errors

VII. Conclusion

This paper addresses the dual angular velocity observer design problem for the 6-DOF tracking control of a leader-follower spacecraft formation, in the absence of both linear and angular velocity measurements. Some important properties of dual vectors and dual quaternions are studied, the kinematics of dual transformation matrices is derived, and the transition relationship between dual quaternions and dual transformation matrices is subsequently analyzed. Based on these mathematical foundations, a smooth observer is proposed to estimate the dual relative angular velocity, and the estimation errors are guaranteed to be global asymptotic convergence irrespective of control inputs. Subsequently, the observer is combined with a separately designed PD-like controller. A partial Lyapunov strictification process is employed to transform nonstrict Lyapunov functions into strict ones whose derivatives contain additional non-positive terms, then the separation property is proved and the almost global asymptotic convergence of the closed-loop system is guaranteed accordingly. Numerical simulations for a prototypical 6-DOF spacecraft pose tracking mission are employed, which shows that the proposed observer-controller construction has an good performance even in the presence of external disturbances, parameter uncertainties and noisy measurements.

Appendix A
Proof of Lemmas 1-4

For $\hat{\boldsymbol{a}},\hat{\boldsymbol{b}},\hat{\boldsymbol{c}}\in\hat{\mathbb{R}}^3$, from Eq. (13), one has:

$$\begin{aligned}\hat{\boldsymbol{a}}\times(\hat{\boldsymbol{b}}\times\hat{\boldsymbol{c}})=&\boldsymbol{a}_r\times(\boldsymbol{b}_r\times\boldsymbol{c}_r)+\varepsilon[\boldsymbol{a}_r\times(\boldsymbol{b}_r\times\boldsymbol{c}_d)+\ldots\\&\boldsymbol{a}_r\times(\boldsymbol{b}_d\times\boldsymbol{c}_r)+\boldsymbol{a}_d\times(\boldsymbol{b}_r\times\boldsymbol{c}_r)]\end{aligned}\tag{107}$$

Similarly:

$$\begin{aligned}\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{c}}\times\hat{\boldsymbol{a}})=&\boldsymbol{b}_r\times(\boldsymbol{c}_r\times\boldsymbol{a}_r)+\varepsilon[\boldsymbol{b}_r\times(\boldsymbol{c}_r\times\boldsymbol{a}_d)+\ldots\\&\boldsymbol{b}_r\times(\boldsymbol{c}_d\times\boldsymbol{a}_r)+\boldsymbol{b}_d\times(\boldsymbol{c}_r\times\boldsymbol{a}_r)]\end{aligned}\tag{108}$$

and

$$\begin{aligned}\hat{\boldsymbol{c}}\times(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{b}})=&\boldsymbol{c}_r\times(\boldsymbol{a}_r\times\boldsymbol{b}_r)+\varepsilon[\boldsymbol{c}_r\times(\boldsymbol{a}_r\times\boldsymbol{b}_d)+\ldots\\&\boldsymbol{c}_r\times(\boldsymbol{a}_d\times\boldsymbol{b}_r)+\boldsymbol{c}_d\times(\boldsymbol{a}_r\times\boldsymbol{b}_r)]\end{aligned}\tag{109}$$

Consider the Jacobi identity of real 3-dimensional vectors:

$$\boldsymbol{a}_*\times(\boldsymbol{b}_\star\times\boldsymbol{c}_\circ)+\boldsymbol{b}_\star\times(\boldsymbol{c}_\circ\times\boldsymbol{a}_*)+\boldsymbol{c}_\circ\times(\boldsymbol{a}_*\times\boldsymbol{b}_\star)=\boldsymbol{0}_3\tag{110}$$

where $*,\star,\circ$ denote r or d, so by iteratively using this identity, finally one can get:

$$\hat{\boldsymbol{a}}\times(\hat{\boldsymbol{b}}\times\hat{\boldsymbol{c}})+\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{c}}\times\hat{\boldsymbol{a}})+\hat{\boldsymbol{c}}\times(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{b}})=\hat{\boldsymbol{0}}_3\tag{111}$$

The Lemma 1 is proved.

By Eqs. (6) and (10), one has:

$$\hat{\boldsymbol{b}}(\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{c}}) = \boldsymbol{b}_r(\boldsymbol{a}_r\cdot\boldsymbol{c}_r)+\varepsilon[\boldsymbol{b}_r(\boldsymbol{a}_r\cdot\boldsymbol{c}_d)+ ... \boldsymbol{b}_d(\boldsymbol{a}_r\cdot\boldsymbol{c}_r)+\boldsymbol{b}_r(\boldsymbol{a}_d\cdot\boldsymbol{c}_r)] \tag{112}$$

and

$$\hat{\boldsymbol{c}}(\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{b}}) = \boldsymbol{c}_r(\boldsymbol{a}_r\cdot\boldsymbol{b}_r)+\varepsilon[\boldsymbol{c}_d(\boldsymbol{a}_r\cdot\boldsymbol{b}_r)+\boldsymbol{c}_r(\boldsymbol{a}_r\cdot\boldsymbol{b}_d)+\boldsymbol{c}_r(\boldsymbol{a}_d\cdot\boldsymbol{b}_r)] \tag{113}$$

The real 3-dimensional vectors follows the property:

$$\boldsymbol{a}_*\times(\boldsymbol{b}_\star\times\boldsymbol{c}_\circ) = \boldsymbol{b}_\star(\boldsymbol{a}_*\cdot\boldsymbol{c}_\circ)-\boldsymbol{c}_\circ(\boldsymbol{a}_*\cdot\boldsymbol{b}_\star) \tag{114}$$

By iteratively using the above property, one can obtain:

$$\hat{\boldsymbol{a}}\times(\hat{\boldsymbol{b}}\times\hat{\boldsymbol{c}}) = \hat{\boldsymbol{b}}(\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{c}})-\hat{\boldsymbol{c}}(\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{b}}) \tag{115}$$

The Lemma 2 is proved.

By Eqs. (10) and (13), and consider the properties of the cross product of real vectors, we have:

$$\begin{aligned}(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{b}})\cdot\hat{\boldsymbol{a}} =& (\boldsymbol{a}_r\times\boldsymbol{b}_r)\cdot\boldsymbol{a}_r+\varepsilon[(\boldsymbol{a}_r\times\boldsymbol{b}_r)\cdot\boldsymbol{a}_d+ ...\\ &(\boldsymbol{a}_r\times\boldsymbol{b}_d)\cdot\boldsymbol{a}_r+(\boldsymbol{a}_d\times\boldsymbol{b}_r)\cdot\boldsymbol{a}_r)]\\ =& \boldsymbol{0}+\varepsilon[(\boldsymbol{a}_r\times\boldsymbol{b}_r)\cdot\boldsymbol{a}_d+(\boldsymbol{a}_d\times\boldsymbol{b}_r)\cdot\boldsymbol{a}_r]\\ =& \hat{\boldsymbol{0}}_3\end{aligned} \tag{116}$$

Furthermore:

$$\hat{\boldsymbol{a}}\cdot(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{b}}) = (\hat{\boldsymbol{a}}\times\hat{\boldsymbol{b}})\cdot\hat{\boldsymbol{a}} = \hat{\boldsymbol{0}}_3 \tag{117}$$

The Lemma 3 is proved.

By (7), (12) and (13),

$$\begin{aligned}\hat{\boldsymbol{a}}^s\circ(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{b}}) =& \boldsymbol{a}_d^{\mathrm{T}}(\boldsymbol{a}_r\times\boldsymbol{b}_r)+\boldsymbol{a}_r^{\mathrm{T}}(\boldsymbol{a}_d\times\boldsymbol{b}_r)+\varepsilon\boldsymbol{0}_3\\ =& \hat{\boldsymbol{0}}_3\end{aligned} \tag{118}$$

The Lemma 4 is proved.

Appendix B
Proof of Proposition 1

Considering Eqs. (10), (13) and (22), and also Lemma 3, one can get,

$$\begin{aligned}&\hat{\boldsymbol{q}}_{yx}^*\otimes[\hat{0},(\hat{\boldsymbol{a}}^x)^{\mathrm{T}}]^{\mathrm{T}}\otimes\hat{\boldsymbol{q}}_{yx}\\ &=[\hat{\eta}_{yx},-\hat{\boldsymbol{\xi}}_{yx}^{\mathrm{T}}]^{\mathrm{T}}\otimes[\hat{0},(\hat{\boldsymbol{a}}^x)^{\mathrm{T}}]^{\mathrm{T}}\otimes[\hat{\eta}_{yx},\hat{\boldsymbol{\xi}}_{yx}^{\mathrm{T}}]^{\mathrm{T}}\\ &=[\hat{0},((\hat{\boldsymbol{\xi}}_{yx}^{\mathrm{T}}\hat{\boldsymbol{a}}^x)\hat{\boldsymbol{\xi}}_{yx}-\hat{\boldsymbol{\xi}}_{yx}\times(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{a}}^x)+\hat{\eta}_{yx}\hat{\eta}_{yx}\hat{\boldsymbol{a}}^x- ...\\ &\quad 2\hat{\eta}_{yx}(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{a}}^x)+2\hat{\boldsymbol{\xi}}_{yx}\times(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{a}}^x))^{\mathrm{T}}]^{\mathrm{T}}\end{aligned} \tag{119}$$

By Lemma 2, and rewrite (119) to the 3-dimensional form, one has

$$\begin{aligned}&\mathrm{vec}(\hat{\boldsymbol{q}}_{yx}^*\otimes[\hat{0},(\hat{\boldsymbol{a}}^x)^{\mathrm{T}}]^{\mathrm{T}}\otimes\hat{\boldsymbol{q}}_{yx}) =\\ &[(\hat{\eta}_{yx}\hat{\eta}_{yx}+\hat{\boldsymbol{\xi}}_{yx}^{\mathrm{T}}\hat{\boldsymbol{\xi}}_{yx})\mathbf{I}_{3\times3}-2\hat{\eta}_{yx}\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_{yx})+2\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_{yx})\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_{yx})]\hat{\boldsymbol{a}}^x\end{aligned} \tag{120}$$

Since $\eta_{yx}\eta_{yx}+\boldsymbol{\xi}_{yx}^{\mathrm{T}}\boldsymbol{\xi}_{yx}=1$, and by (19), it can be proved that

$$\hat{\eta}_{yx}\hat{\eta}_{yx}+\hat{\boldsymbol{\xi}}_{yx}^{\mathrm{T}}\hat{\boldsymbol{\xi}}_{yx} = \hat{\boldsymbol{q}}_{yx}^{\mathrm{T}}\hat{\boldsymbol{q}}_{yx} = 1+\varepsilon 0 \tag{121}$$

Substituting (121) into (120) yields

$$\hat{\boldsymbol{q}}_{yx}^*\otimes\hat{\boldsymbol{a}}^x\otimes\hat{\boldsymbol{q}}_{yx} = \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx})\hat{\boldsymbol{a}}^x \tag{122}$$

and the definition of $\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx})$ is

$$\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx}) = \mathbf{I}_{3\times3}-2\hat{\eta}_{yx}\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_{yx})+2\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_{yx})\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_{yx}) \tag{123}$$

Properties 1 and 2 in Proposition 1 can be readily proved. By (123), we have

$$\begin{aligned}\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx}^*) &= \mathbf{I}_{3\times3}+2\hat{\eta}_{yx}\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_{yx})+2\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_{yx})\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_{yx})\\ &= \mathbf{I}_{3\times3}^{\mathrm{T}}-2\hat{\eta}_{yx}\hat{\boldsymbol{S}}^{\mathrm{T}}(\hat{\boldsymbol{\xi}}_{yx})+2\hat{\boldsymbol{S}}^{\mathrm{T}}(\hat{\boldsymbol{\xi}}_{yx})\hat{\boldsymbol{S}}^{\mathrm{T}}(\hat{\boldsymbol{\xi}}_{yx})\\ &= \hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_{yx})\end{aligned} \tag{124}$$

Eq. (124) shows that, the converse transformation, from $\mathcal{Y}$ to $\mathcal{X}$, can be described by $\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_{yx})$, and so

$$\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_{yx})\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx}) = \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx})\hat{\boldsymbol{C}}^{\mathrm{T}}(\hat{\boldsymbol{q}}_{yx}) = \mathbf{I}_{3\times3}+\varepsilon\boldsymbol{0}_{3\times3} \tag{125}$$

Furthermore, by (27),

$$\begin{aligned}&\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx})(\hat{\boldsymbol{a}}_1^x\times\hat{\boldsymbol{a}}_2^x)-[\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx})\hat{\boldsymbol{a}}_1^x]\times[\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx})\hat{\boldsymbol{a}}_2^x]\\ &=\boldsymbol{0}_3+\varepsilon(\boldsymbol{a}_{1r}^y\times\boldsymbol{a}_{2r}^y)\times\boldsymbol{r}_{yx}^y-\varepsilon(\boldsymbol{a}_{1r}^y\times\boldsymbol{r}_{yx}^y)\times\boldsymbol{a}_{2r}^y\\ &\quad-\varepsilon(\boldsymbol{a}_{1r}^y)\times(\boldsymbol{a}_{2r}^y\times\boldsymbol{r}_{yx}^y)=\hat{\boldsymbol{0}}_3\end{aligned} \tag{126}$$

which guarantees the Property 3.

Finally, to analyze the kinematics of the dual transformation matrix, consider a fixed dual vector $\hat{\boldsymbol{n}}^x$ in frame $\mathcal{X}$, for ease of notation, using $\hat{\boldsymbol{C}}^{yx}$ denote $\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx})$, and then we have,

$$\begin{aligned}\frac{\mathrm{d}(\hat{\boldsymbol{C}}^{yx}\hat{\boldsymbol{n}}^x)}{\mathrm{d}t} &= \frac{\mathrm{d}\hat{\boldsymbol{C}}^{yx}}{\mathrm{d}t}\hat{\boldsymbol{n}}^x+\hat{\boldsymbol{C}}^{yx}\frac{\mathrm{d}\hat{\boldsymbol{n}}^x}{\mathrm{d}t} = \frac{\mathrm{d}\hat{\boldsymbol{C}}^{yx}}{\mathrm{d}t}\hat{\boldsymbol{n}}^x\\ &= \frac{\mathrm{d}[\hat{\boldsymbol{n}}^x-2\hat{\eta}_{yx}\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{n}}^x+2\hat{\boldsymbol{\xi}}_{yx}\times(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{n}}^x)]}{\mathrm{d}t}\end{aligned} \tag{127}$$

Substitute (25) into (127), and consider Lemma 1, one has

$$\begin{aligned}\frac{\mathrm{d}\hat{\boldsymbol{C}}^{yx}}{\mathrm{d}t}\hat{\boldsymbol{n}}^x =& (\hat{\boldsymbol{\xi}}_{yx}^{\mathrm{T}}\hat{\boldsymbol{\omega}}_{yx}^y)\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{n}}^x-\hat{\eta}_{yx}^2\hat{\boldsymbol{\omega}}_{yx}^y\times\hat{\boldsymbol{n}}^x\\ &+2\hat{\eta}_{yx}\hat{\boldsymbol{\omega}}_{yx}^y\times(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{n}}^x)\\ &+(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{\omega}}_{yx}^y)\times(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{n}}^x)\\ &+\hat{\boldsymbol{\xi}}_{yx}\times[(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{\omega}}_{yx}^y)\times\hat{\boldsymbol{n}}^x]\end{aligned} \tag{128}$$

Furthermore,

$$\begin{aligned}&\hat{\boldsymbol{\xi}}_{yx}\times[(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{\omega}}_{yx}^y)\times\hat{\boldsymbol{n}}^x]\\ &=(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{\omega}}_{yx}^y)\times(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{n}}^x)-\hat{\boldsymbol{n}}_{yx}\times[\hat{\boldsymbol{\xi}}_{yx}\times(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{\omega}}_{yx}^y)]\end{aligned} \tag{129}$$

Similarly

$$\begin{aligned}&(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{\omega}}_{yx}^y)\times(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{n}}^x)\\ &=\hat{\boldsymbol{\xi}}_{yx}\times[\hat{\boldsymbol{\omega}}_{yx}^y\times(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{n}}^x)]-\hat{\boldsymbol{\omega}}_{yx}^y\times[\hat{\boldsymbol{\xi}}_{yx}\times(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{n}}^x)]\end{aligned} \tag{130}$$

Substituting (129) and (130) into (128) yields

$$\begin{aligned}\frac{\mathrm{d}\hat{\boldsymbol{C}}^{yx}}{\mathrm{d}t}\hat{\boldsymbol{n}}^x =& (\hat{\boldsymbol{\xi}}_{yx}^{\mathrm{T}}\hat{\boldsymbol{\omega}}_{yx}^y)\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{n}}^x-\hat{\eta}_{yx}^2\hat{\boldsymbol{\omega}}_{yx}^y\times\hat{\boldsymbol{n}}^x\\ &+2\hat{\eta}_{yx}\hat{\boldsymbol{\omega}}_{yx}^y\times(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{n}}^x)\\ &-2\hat{\boldsymbol{\omega}}_{yx}^y\times[\hat{\boldsymbol{\xi}}_{yx}\times(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{n}}^x)]\\ &+2\hat{\boldsymbol{\xi}}_{yx}\times[\hat{\boldsymbol{\omega}}_{yx}^y\times(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{n}}^x)]\\ &-\hat{\boldsymbol{n}}^x\times[\hat{\boldsymbol{\xi}}_{yx}\times(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{\omega}}_{yx}^y)]\end{aligned} \tag{131}$$

then by Lemma 2 and Lemma 3, one has,

$$\begin{aligned}&\hat{\boldsymbol{n}}^x\times[\hat{\boldsymbol{\xi}}_{yx}\times(\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{\omega}}_{yx}^y)]\\ &=\hat{\boldsymbol{n}}^x\times[(\hat{\boldsymbol{\xi}}_{yx}^{\mathrm{T}}\hat{\boldsymbol{\omega}}_{yx}^y)\hat{\boldsymbol{\xi}}_{yx}-(\hat{\boldsymbol{\xi}}_{yx}^{\mathrm{T}}\hat{\boldsymbol{\xi}}_{yx})\hat{\boldsymbol{\omega}}_{yx}^y]\\ &=-(\hat{\boldsymbol{\xi}}_{yx}^{T}\hat{\boldsymbol{\omega}}_{yx}^y)\hat{\boldsymbol{\xi}}_{yx}\times\hat{\boldsymbol{n}}^x+(\hat{\boldsymbol{\xi}}_{yx}^{\mathrm{T}}\hat{\boldsymbol{\xi}}_{yx})\hat{\boldsymbol{\omega}}_{yx}^y\times\hat{\boldsymbol{n}}^x\end{aligned} \tag{132}$$

$$\begin{aligned}&\hat{\boldsymbol{\xi}}_{yx} \times [\hat{\boldsymbol{\omega}}_{yx}^{y} \times (\hat{\boldsymbol{\xi}}_{yx} \times \hat{\boldsymbol{n}}^{x})] \\ &= [\hat{\boldsymbol{\xi}}_{yx}^{\mathrm{T}}(\hat{\boldsymbol{\xi}}_{yx} \times \hat{\boldsymbol{n}}^{x})]\hat{\boldsymbol{\omega}}_{yx}^{y} - (\hat{\boldsymbol{\xi}}_{yx}^{\mathrm{T}}\hat{\boldsymbol{\omega}}_{yx}^{y})(\hat{\boldsymbol{\xi}}_{yx} \times \hat{\boldsymbol{n}}^{x}) \\ &= -(\hat{\boldsymbol{\xi}}_{yx}^{\mathrm{T}}\hat{\boldsymbol{\omega}}_{yx}^{y})\hat{\boldsymbol{\xi}}_{yx} \times \hat{\boldsymbol{n}}^{x}\end{aligned} \tag{133}$$

Substituting Eqs. (132) and (133) into Eq. (131) yields,

$$\begin{aligned}\frac{\mathrm{d}\hat{\boldsymbol{C}}^{yx}}{\mathrm{d}t}\hat{\boldsymbol{n}}^{x} &= -\hat{\boldsymbol{\omega}}_{yx}^{y} \times \hat{\boldsymbol{n}}^{x} + 2\hat{\eta}_{yx}\hat{\boldsymbol{\omega}}_{yx}^{y} \times (\hat{\boldsymbol{\xi}}_{yx} \times \hat{\boldsymbol{n}}^{x}) \\ &\quad - 2\hat{\boldsymbol{\omega}}_{yx}^{y} \times [\hat{\boldsymbol{\xi}}_{yx} \times (\hat{\boldsymbol{\xi}}_{yx} \times \hat{\boldsymbol{n}}^{x})] \\ &= -\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{yx}^{y})[\mathbf{I}_3 - 2\hat{\eta}_{yx}\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_{yx}) + 2\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_{yx})\hat{\boldsymbol{S}}(\hat{\boldsymbol{\xi}}_{yx})]\hat{\boldsymbol{n}}^{x} \\ &= -\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{yx}^{y})\hat{\boldsymbol{C}}\hat{\boldsymbol{n}}^{x}\end{aligned} \tag{134}$$

So that,

$$[\frac{\mathrm{d}\hat{\boldsymbol{C}}^{yx}}{\mathrm{d}t} + \hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{yx}^{y})\hat{\boldsymbol{C}}^{yx}]\hat{\boldsymbol{n}}^{x} = \hat{\mathbf{0}}_3 \tag{135}$$

Due to the arbitrariness of $\hat{\boldsymbol{n}}^{x}$, finally we can get,

$$\frac{\mathrm{d}[\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx})]}{\mathrm{d}t} = -\hat{\boldsymbol{S}}(\hat{\boldsymbol{\omega}}_{yx}^{y})\hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}_{yx}) \tag{136}$$

The proof is complete.

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