On the density of sets of the Euclidean plane avoiding distance 1

Arnaud Pêcher

Univ. Bordeaux (LaBRI / INRIA RealOpt)
Joint work with
Thomas Bellitto (University of Southern Denmark, Denmark)
Antoine Sedillot (ENS Paris-Saclay, France)

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Maximum density of a set avoiding distance 1

- Normed space $E = (\mathbb{R}^n, \|\cdot\|)$
- A set $A \in \mathbb{R}^n$ avoids distance 1 iff $\forall x, y \in A, \|x - y\| \neq 1$
- (Upper) density of a measurable set A:

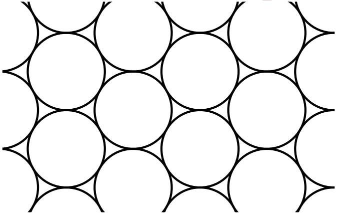
$$\delta = \limsup_{R \to \infty} \frac{\mathrm{Vol}(A \cap [-R, R]^n)}{\mathrm{Vol}([-R, R]^n)}$$

- Maximum density of a set avoiding distance 1:

$$m_1(\mathbb{R}^n, \|\cdot\|) = \sup_{A \text{ avoiding } 1} \delta(A).$$

Lower bounds

- Construction: let Λ be a set of pairwise disjoint balls of radius 1.
  A set avoiding distance 1 of density $\frac{\delta(\Lambda)}{2^n}$:



- $m_1(\mathbb{R}^2, \|\cdot\|_2) \geqslant 0.9069/4 \geqslant 0.2267$

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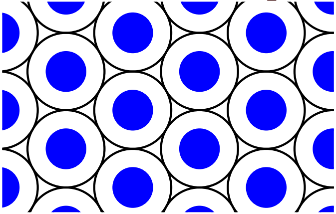

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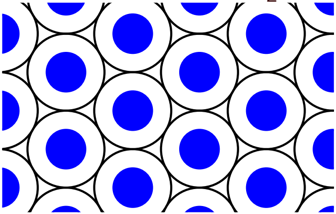
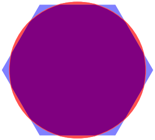


- $m_1(\mathbb{R}^2, \|\cdot\|_2) \geqslant 0.9069/4 \geqslant 0.2267$
- Croft (1967): $m_1(\mathbb{R}^2, \|\cdot\|_2) \geqslant 0.229$



Upper bounds

- Best published upper bound : $m_1(\mathbb{R}^2, \|\cdot\|_2) \leqslant 0.259$ (Keleti, Matolcsi, de Oliveira Filho, Ruzsa, 2015)
- Erdös' conjecture : $m_1(\mathbb{R}^2, \|\cdot\|_2) < 1/4$
- Generalization (Moser, Larman, Rogers): $m_1(\mathbb{R}^n, \|\cdot\|_2) < \frac{1}{2^n}$

Chromatic number of the unit-distance graph

Chromatic number of a metric space

The chromatic number χ of a metric space(X, d) is the smallest number of colours required to colour each point of X so that no two points at distance 1 share the same colour.

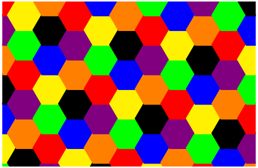
Unit-distance graph

The unit-distance graph associated to a metric space (X, d) is the graph G such that $V(G) = X$ and $E(G) = \{\{x, y\} : d(x, y) = 1\}$.

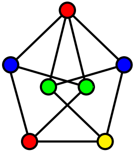
$$\chi(\mathbb{R}^2) = \chi(\text{unit-distance graph of } (\mathbb{R}^2, \| \cdot \|_2))$$

The Euclidean plane

- $\chi(\mathbb{R}^2) \leqslant 7$:



- $\chi(\mathbb{R}^2) \geqslant 4$ (Moser's spindle):



- De Grey (April 2018): $\chi(\mathbb{R}^2) \geqslant 5$ (1581 vertices)
- alternative proof/graph by Exoo and Ismailescu

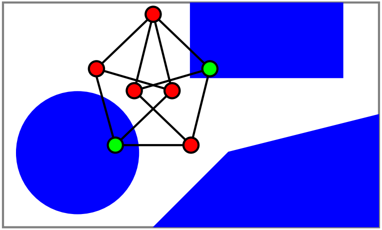
Measurable chromatic number

Measurable chromatic number χ_m of a metric space (X, d): colour classes must be measurable sets.

$$\chi_m(\mathbb{R}^n, \|\cdot\|) \geqslant \frac{1}{m_1(\mathbb{R}^n, \|\cdot\|)}$$

Euclidean plane: $\chi_m(\mathbb{R}^2) \geqslant 5$ (Falconer, 1981)

Discretization Lemma



Set S of density δ. X at random in $\mathbb{R}^n$: $\mathbb{P}(X \in S) = \delta$.

Unit-distance subgraph $G = (V, E)$ in $\mathbb{R}^n$: $\mathbb{E}(|V \cap S|) = |V| \times \delta$.

If S avoids distance 1: $|V \cap S| \leqslant \alpha(G) \to \delta \leqslant \frac{\alpha(G)}{|V|}$.

For every unit-distance subgraph $G = (V, E)$ in $\mathbb{R}^n$:

$$m_1(\mathbb{R}^n) \leqslant \frac{\alpha(G)}{|V|}.$$

Weighted version of Discretization Lemma

Weighting of a graph: $w : V \to \mathbb{R}^+$.
Weighted independence number $\alpha_w(G)$ of a weighted graph G: maximum weight of an independent set.
Optimal weighted independence ratio: $\alpha^*(G) = \min_w \frac{\alpha_w(G)}{w(G)}$

Weighted discretization lemma

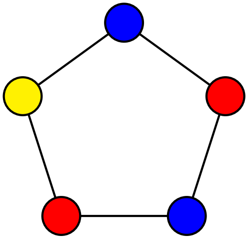
For all unit-distance subgraph G in $\mathbb{R}^n$:

$$m_1(\mathbb{R}^n) \leqslant \alpha^*(G).$$

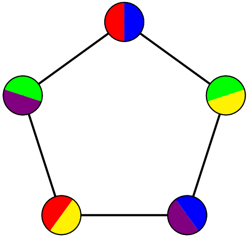
Fractional colouring

Fractional Chromatic number

The fractional chromatic number χ_f of a graph G is the smallest fractional number $\frac{a}{b}$ such that a colours are sufficient to assign b colours to each vertex of G in such a way that no two adjacent vertices share a common colour.



$\chi(C_5) = 3$



$\chi_f(C_5) = \frac{5}{2}$

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Fractional Chromatic number

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$\mathscr{S}$: set of all independent sets in the graph.

$$\begin{cases} \chi = \text{minimize } \sum_{I \in \mathscr{S}} x_I \text{ subject to} \\ \forall v \in V, \sum_{I \in \mathscr{S}: v \in I} x_I = 1 \\ \forall I \in \mathscr{S}, x_I \text{ binary} \end{cases}$$

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Fractional clique number

Fractional clique

A fractional clique is a weight distribution such that no independent set has weight more than 1. The weight of a fractional clique is the total weight of the graph.

The fractional clique number ω_f of a graph is the maximum weight of a fractional clique.

$$\omega_f \begin{cases} \text{maximize } \sum_{v \in V} w_v \\ \forall I \in \mathscr{S}, \sum_{v \in I} w_v \leqslant 1 \\ \forall v \in V, w_v \geq 0 \end{cases}$$

By strong duality,

$$\chi_f = \omega_f$$

Sandwich inequality for m_1

By definition, $\alpha^*(G) = \frac{1}{\omega_f(G)} = \frac{1}{\chi_f(G)}$.

$$\frac{1}{\chi_m(\mathbb{R}^n, \|\cdot\|)} \leqslant m_1(\mathbb{R}^n, \|\cdot\|) \leqslant \frac{1}{\chi_f(\mathbb{R}^n, \|\cdot\|)}.$$

Goals:

- Efficient algorithm for the optimal weighted independence ratio (especially for our geometric instances).
- Method to build graphs of high fractional chromatic number.

Outline of the algorithm

Basic LP for $\alpha^*(G)$
$$\begin{cases} \text{minimize } M \\ \sum_{v \in V} w_v = 1 \\ \forall I \in \mathscr{S}, \sum_{v \in I} w_v \leqslant M \\ \forall v \in V, w_v \geq 0 \end{cases}$$

Outline of our algorithm

Start with $\mathscr{S} = \varnothing$ and a uniform weight distribution W:

Step1 Add to $\mathscr{S}$ a max. weight ind. set for W (gives an upper bound)

Step2 Compute W, the weight distribution that minimizes the maximum weight of sets of $\mathscr{S}$ (gives a lower bound)

Iterate until the two bounds coincide.

Step1

Add to $\mathscr{S}$, a maximum weight independent set for W

Input: $W = (w_1, \ldots, w_p)$ a symmetric weigth distribution
Output: stable set defined by $\mathbf{x}=(x_v)_{v \in V}$.

$$\begin{aligned} \text{maximize} \quad & \sum_{v \in V} w_{\mathrm{orbit}(v)} x_v, \\ \text{subject to} \quad & x_u + x_v \leq 1 \quad \forall uv \in E, \\ \text{and} \quad & x_v \in \{0,1\} \quad \forall v \in V. \end{aligned} \tag{$Step1$}$$

Step1 is slow due to binary variables.

Remarks

- "guided" exploration of stable set polytope
- maximal clique constraints instead of edge constraints
- Moser spindles rank constraints are helpful

Step2

Compute W, the weight distribution that minimizes the maximum weight of sets of $\mathscr{S} = \{S_1, \ldots, S_k\}$.

Input: $O_1, \ldots, O_p$ the orbits of the graph, $n_{i,j} = |S_i \cap O_j|$.
Output: $\mathbf{W}$=$(w_1, \ldots, w_p)$.

$$
\begin{aligned}
\text{minimize} \quad & M, \\
\text{subject to} \quad & \sum_{j=1}^{p} n_{i,j} w_j \leq M \quad \forall i \in \{1, \ldots, k\}\,, \\
& \sum_{j=1}^{p} w_j \, |O_j| = 1, \\
\text{and} \quad & w_1, \ldots, w_p, M \geq 0
\end{aligned}
\tag{$Step2$}
$$

Step2 is fast as k is small in practice

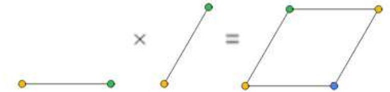
Constructing unit-distance graph with high fractional chromatic number

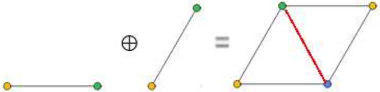
Rough steps to get unit-distance graph with higher fractional chromatic number

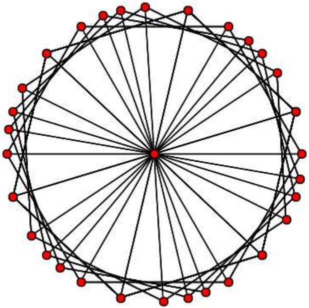
1. Start from a "promising" unit-distance graph
2. Apply graph operations in order to decrease α^* (but makes the graph bigger)
3. Compute α^* and an optimal weight function
4. (cleaning) Remove some vertices with small weight
5. Repeat from step 2.

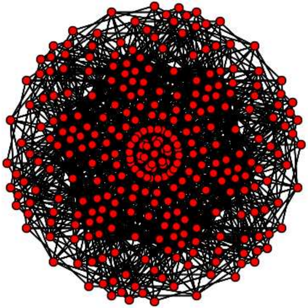
Graph operation : Minkowski sum

Let G and G' be two unit-distance graphs. The Minkowski sum of G and G' is a geometric variant of the Cartesian product of G and G'.



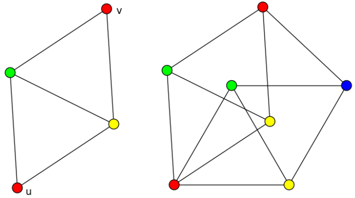






Graph operation : spindling

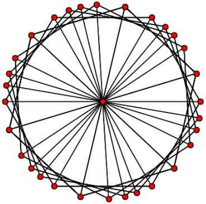
Let $u, v \in V$. The spindling between u and v is the graph obtained by taking the union of G and a rotated copy of G around u such that v and its copy are at distance 1.



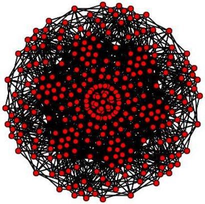
Möser's graph as a spindling of the left graph

Results

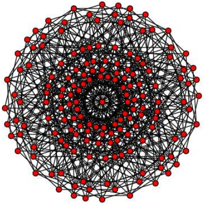
- We started from a 31-vertex unit-distance graph.
- We applied the operations in order to lower the α^* of the obtained graph.



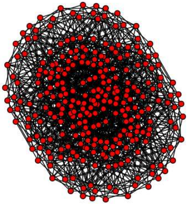
$n = 31$, $\alpha^* = 0.33333...$



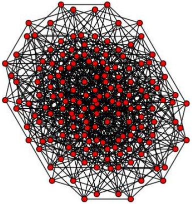
$n = 301$, $\alpha^* = 0,26620...$



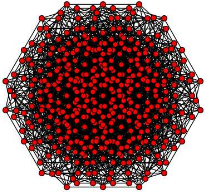
$n = 163$, $\alpha^* = 0,26620...$



$n = 278$, $\alpha^* = 0,26175...$



$n = 166$, $\alpha^* = 0,26225...$



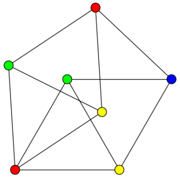
$n = 415$, $\alpha^* = 0,25773...$

Results

Graph	Number of vertices	α^*	χ_f
W415	415	0,25773...	3,8800...
W283	283	0,25800...	3,8758...
W384	384	0,25723...	3,8874...
W282	282	0,25807...	3,8749...
W487	487	0,25682...	3,8936...
W313	313	0,25775...	3,8796...
W420	420	0.257071...	3.89776...
W286	286	0,258127...	3,87405...
W565	565	0.256557...	3.8977...
W607	607	0.25646...	3.8992...

Going on

authors	date	χ_f	m_1	vert.	method
Keleti et al.	2015	-	$\leqslant 0.259$	-	analysis
Cranston, Rabern	2017	$\geqslant 3.619$	$\leqslant 0.277$	-	discharg.
Exoo, Ismailescu	2017	$\geqslant 3.754$	$\leqslant 0.267$	73	graph
Ambrus, Matolsci	2018	-	$\leqslant 0.257$	-	analysis
Bellitto, P., Sedillot	2018	$\geqslant 3.899$	$\leqslant 0.257$	607	our alg.
Jaan Parts	2019	$\geqslant 3.962$	$\leqslant 0.253$	625	our alg.



Thank you!

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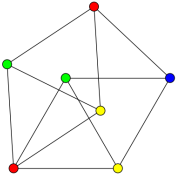

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