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**Abstract:** This paper deals with the problem of consistent discretization of the generalized homogeneous differential inclusions which appear, in particular, as mathematical models of many sliding mode systems. An algorithm of consistent implicit discretization for homogeneous finite-time stable system is developed. It preserves the finite-time stability property and reduces the numerical chattering. The scheme is demonstrated for the so-called "nested" second order sliding mode algorithm.

*Keywords:* Sliding Modes; Digital Implementation; Lyapunov Methods

## 1. INTRODUCTION

Differential inclusions are known to be conventional mathematical models of the so-called Sliding Mode Control (SMC) systems. Indeed, both classical (see Utkin (1992)) and modern (see Levant (2005)) SMC methodologies use discontinuous (with respect to state variables) control laws. According to Filippov (1988), differential inclusions are consistent well-posed regularizations of discontinuous ordinary differential equations (ODEs).

SMC algorithms are known to be difficult in practical realization due to their discontinuous nature (Acary et al. (2012), Kikuuwe et al. (2010)), which may involve chattering (unexpected oscillations) caused by several reasons. One of them is an improper digital implementation (the so-called *numerical chattering*). In contrast to other sources of chattering (e.g. unmodeled dynamics or unknown delays), the numerical chattering can be essentially reduced by means of a proper discretization (Huber et al. (2016)).

Therefore, discretization issues are very important for a digital implementation of SMC algorithms as well as for a computer simulation of controlled processes. Construction of the so-called consistent stable discretization is a non-trivial problem. Consistent discretization (see Polyakov et al. (2019)) means that the discrete-time model approximates solutions and preserves stability/convergence rates of the original continuous-time system.

A symmetry of functions (operators) with respect to a special group of transformations (dilations) is known as homogeneity (see Zubov (1958), Hermes (1986), Rosier (1992), Polyakov et al. (2016)). Nonlinear homogeneous differential equations/inclusions form an important class of control systems Kawski (1990), Perruquetti et al. (2008), Nakamura et al. (2007). They can be local approximations Hermes (1986), Andrieu et al. (2008) or set-valued extensions Orlov (2005), Levant (2005) of nonlinear systems. Asymptotic stability of homogeneous system with negative degree implies its global finite-time stability.

A type of homogeneity is identified by the dilation group, which can be linear as in Zubov (1958), Bhat and Bernstein (2005), Polyakov (2018) or nonlinear as in Khomenuk (1961), Kawski (1990). In this paper, we deal with the linear geometric homogeneity studied originally in Polyakov et al. (2016) for infinite dimensional models. In $\mathbb{R}^n$ the corresponding dilation group (see e.g. Polyakov (2018)) is given by the matrix exponential function $e^{sG_{\mathbf{d}}}$, where $s \in \mathbb{R}$ is the group parameter and $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ is an anti-Hurwitz matrix known as a *generator* of the dilation. Most of the high order sliding mode algorithms Levant (2005) are homogeneous of negative degree with the dilation generator of $G_{\mathbf{d}} = \mathrm{diag}\{r_i\}, r_i > 0, i = 1, 2, ..., n$. Such SMC systems may have finite-time stable manifolds or equilibria.

Recently Polyakov et al. (2019), a problem of consistent discretization of homogeneous finite-time stable ODEs has been studied. In particular, a discretization algorithm preserving stability/convergence rates in the resulting discrete-time model has been developed. It is essentially based on topological equivalence of any stable homogeneous system to a quadratically stable one (see Fig. 1).

It is shown in Polyakov et al. (2019) that an implementation of finite-time controllers using the developed scheme reduces numerical chattering and essentially improves the

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$$\dot{x} = f(x) \quad \overset{y=\Phi(x)}{\Leftrightarrow} \quad \dot{y} = \tilde{f}(y)$$

$\uparrow$ a consistent discrete-time approximation $\uparrow$ $\qquad$ $\downarrow$ a consistent discrete-time approximation $\downarrow$

$$x_{i+1} = \Phi^{-1}(\Phi(x_i) + h\tilde{f}(\Phi(x_{i+1}))) \quad \overset{x=\Phi^{-1}(y)}{\Leftrightarrow} \quad y_{i+1} = y_i + h\tilde{f}(y_{i+1})$$

Fig. 1. The scheme of the consistent implicit discretization of $\dot{x} = f(x)$

control precision. This paper continues this research direction and proposes a consistent discretization scheme for homogeneous differential inclusions satisfying standard assumptions of Filippov (1988).

### *1.1 Notation*

$\mathbb{R}$ is the field of real numbers; $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$; $\|\cdot\|$ denotes a norm in $\mathbb{R}^n$ and

$$\|A\|_{\mathbb{A}} = \sup_{x \in \mathbb{R}^n} \frac{\|Ax\|}{\|x\|} \text{ and } \lfloor A \rfloor_{\mathbb{A}} = \inf_{x \in \mathbb{R}^n} \frac{\|Ax\|}{\|x\|} \text{ if } A \in \mathbb{R}^{n\times n};$$

$C^n(X, Y)$ is the set of continuously differentiable (at least up to the order $n$) maps $X \to Y$, where $X, Y$ are open subsets of finite dimensional spaces; $\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right)$; $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix; $\mathbf{0}$ denotes zero element, e.g., $\mathbf{0} \in \mathbb{R}^n$ is the zero vector but $\mathbf{0} \in \mathbb{R}^{n\times n}$ is the zero matrix; $\text{diag}\{\lambda_1, .., \lambda_n\}$ - diagonal matrix with elements $\lambda_i$; the order relation $P \succ 0$ means positive definiteness of the symmetric matrix $P \in \mathbb{R}^{n\times n}$; $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote maximal and minimal eigenvalues of the symmetric matrix $P \in \mathbb{R}^{n\times n}$; $\Re(\lambda)$ denotes the real part of the complex number $\lambda$; the notation $P^{\frac{1}{2}}$ means that $P^{\frac{1}{2}} = M$ is such that $P = M^2$; a function $c : [0, +\infty) \to [0, +\infty)$ belongs to the class $\mathcal{K}$ if it is continuous, monotone increasing and $c(0) = 0$; $B = \{x \in \mathbb{R}^n : \|x\| < 1\}$ is an open unit ball in $\mathbb{R}^n$; a set-valued mapping is a mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, which associates a subset $F(x)$ of $\mathbb{R}^m$ with $x \in \mathbb{R}^n$.

## 2. PROBLEM STATEMENT

Let us consider the non-linear differential inclusion

$$\dot{x} \in F(x), \quad t > 0, \quad x(0) = x_0 \in \mathbb{R}^n, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system's state and the nonlinear mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is nonempty-, compact-, convex-valued and upper semi-continuous [1].

Recall (see e.g. Filippov (1988)) that under given assumptions (known also as **standard assumptions**) the differential inclusion (1) has a strong solution for any $x_0 \in \mathbb{R}^n$, i.e. there exists an absolutely continuous function $\phi(\cdot, x_0) : [0, +\infty) \to \mathbb{R}^n$ , $\phi(0, x_0) = x_0$ satisfying the differential inclusion (1) almost everywhere on a time interval $(0, t_{max})$, $0 < t_{\max} \le +\infty$.

*Remark 1. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a measurable discontinuous vector field then, according to regularization procedure of Filippov (1988), solutions of the differential equation*

$$\dot{x} = f(x)$$

*can be defined as strong solutions of the differential inclusion*

$$\dot{x} \in F(x) = \bigcap_{\varepsilon>0} \bigcap_{\mu(N)=0} cof(x + \varepsilon B \backslash \{N\}), \tag{2}$$

*where $B$ denotes a unit ball in $\mathbb{R}^n$, co denotes convex closure and $\mu(N)$ means that the Lebesgue measure of the set $N \subset \mathbb{R}^n$ is zero.*

Recall Orlov (2005), Roxin (1966), Polyakov and Fridman (2014) that the origin of system (1) is said to be globally uniformly **finite-time stable**, if it is Lyapunov stable, and there exists a locally bounded function $T : \mathbb{R}^n \to [0, +\infty)$ such that any solution $\phi(\cdot, x_0)$ to (1) satisfies $\phi(t, x_0) = 0$ for $t \ge T(x_0)$, $x_0 \in U$.

We understand the consistency of a discretization scheme in the sense of the following definition.

*Definition 1.* Polyakov et al. (2019) *A set-valued mapping*

$$Q : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$$

*defines a consistent discrete-time approximation of the globally uniformly* ***finite-time stable system*** *(1) if :*

- ***Existence property:*** *for any $\tilde{x} \in \mathbb{R}^n$ and any $h > 0$, there exists $\tilde{x}_h \in \mathbb{R}^n$ such that*
$$\mathbf{0} \in Q(h, \tilde{x}, \tilde{x}_h), \tag{3}$$
*and $\tilde{x}_h = \mathbf{0}$ is the unique solution to $\mathbf{0} \in Q(h, \mathbf{0}, \tilde{x}_h)$.*
- ***Finite-time convergence property:*** *for any $h \in (0, h_0)$ each sequence*
$$\{x_i\}_{i=0}^{+\infty} \tag{4}$$
*generated by the inclusion*
$$\mathbf{0} \in Q(h, x_i, x_{i+1}), \quad i = 0, 1, 2, .... \tag{5}$$
*converges to zero in a finite number of steps, i.e., for any $x_0 \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ there exists $i^* < +\infty$ such that*
$$x_i = \mathbf{0} \qquad \text{for} \quad i \ge i^*.$$
- ***Approximation property:*** *for any $x_i \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ one has*
$$\sup_{x_{i+1} \text{ from } (5)} \inf_{v \in F(x_i)} \left\| \frac{x_{i+1} - x_i}{h} - v \right\| \to 0 \text{ as } h \to 0 \tag{6}$$

According to Wazewski lemma (see e.g. Filippov (1988)) a function $x : [0, T] \to \mathbb{R}^n$ is a solution of (1) if and only if

$$\inf_{v \in F(x(t))} \left\| \frac{x(t+h) - x(t)}{h} - v \right\| \to 0$$

as $h \to 0^+$. Hence, repeating considerations of Filippov (1988) one can be shown that any step function generated by a discrete-time solution of (4) converge to a solution of (1) provided that (6) holds.

The aim of the paper is to design a consistent (in the sense of Definition 1) discretization for the finite-time stable system (1), under the assumption that the mapping $F$ is generalized homogeneous (see Polyakov (2018)).

## 3. PRELIMINARIES

### *3.1 Generalized Homogeneity*

The generalized homogeneity Polyakov et al. (2016), Polyakov (2018), Polyakov et al. (2018) deals with the group of linear transformations (*linear dilations*).

[1] A mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be upper semi-continuous on $\mathbb{R}^n$ if for any $x \in \mathbb{R}^n$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\Delta x\| < \delta$ implies $F(x + \Delta x) \subseteq F(x) + \varepsilon B$.

*Definition 2.* *A map* $\boldsymbol{d}:\mathbb{R}\to\mathbb{R}^{n\times n}$ *is called* ***dilation*** *if*

- ***Group property:*** $\boldsymbol{d}(0)=I_n$ *and* $\boldsymbol{d}(t+s)=\boldsymbol{d}(t)\boldsymbol{d}(s)=\boldsymbol{d}(s)\boldsymbol{d}(t)$ *for all* $t,s\in\mathbb{R}$*;*
- ***Continuity property:*** $\boldsymbol{d}$ *is a continuous map, i.e.,* $\forall t\in\mathbb{R},\ \forall\varepsilon>0,\ \exists\delta>0: |s-t|<\delta\ \Rightarrow\ \|\boldsymbol{d}(s)-\boldsymbol{d}(t)\|\leq\varepsilon$*;*
- ***Limit property:*** $\lim\limits_{s\to-\infty}\|\boldsymbol{d}(s)x\|=0$, $\lim\limits_{s\to+\infty}\|\boldsymbol{d}(s)x\|=+\infty$ *uniformly on the unit sphere* $S:=\{x\in\mathbb{R}^n:\|x\|=1\}$.

The dilation $\mathbf{d}$ is a continuous group of invertible linear maps $\mathbf{d}(s)\in\mathbb{R}^{n\times n}$, $\mathbf{d}(-s)=[\mathbf{d}(s)]^{-1}$. The matrix

$$G_{\mathbf{d}}=\lim_{s\to 0}\frac{\mathbf{d}(s)-I}{s},\quad G_{\mathbf{d}}\in\mathbb{R}^{n\times n}$$

is known (Pazy, 1983, Chapter 1) as the **generator** of the group $\mathbf{d}$. It satisfies the following properties

$$\frac{d\,\mathbf{d}(s)}{ds}=G_{\mathbf{d}}\mathbf{d}(s)\quad\text{and}\quad \mathbf{d}(s)=e^{G_{\mathbf{d}}s}=\sum_{i=0}^{+\infty}\frac{s^iG_{\mathbf{d}}^i}{i!},\ s\in\mathbb{R}.$$

The most popular dilations in $\mathbb{R}^n$ are Levant (2005), Orlov (2005)

- *uniform (standard) dilation* (L. Euler 17th century) :
$$\mathbf{d}_1(s)=e^sI_n,\quad s\in\mathbb{R},$$
- *weighted dilation* (Zubov 1958, Zubov (1958)):
$$\mathbf{d}_2(s)=\begin{pmatrix} e^{r_1s} & 0 & \dots & 0\\ 0 & e^{r_2s} & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & e^{r_ns}\end{pmatrix},\ s\in\mathbb{R},\ r_i>0,\ i=1,...,n$$

They satisfy Definition 2 with $G_{\mathbf{d}_1}=I_n$ and $G_{\mathbf{d}_2}=\mathrm{diag}\{r_i\}$, respectively. In fact, any anti-Hurwitz [2] matrix $G_{\mathbf{d}}\in\mathbb{R}^{n\times n}$ defines a dilation $\mathbf{d}(s)=e^{G_{\mathbf{d}}s}$ in $\mathbb{R}^n$.

*Definition 3.* Polyakov (2018) *The dilation* $\boldsymbol{d}$ *is* ***strictly monotone*** *if* $\exists\beta>0$ *such that* $\|\boldsymbol{d}(s)\|\leq e^{\beta s}$ *as* $s<0$.

Monotonicity of dilation may depend on the norm $\|\cdot\|$.

*Theorem 1.* (Polyakov (2018)). *If* $\boldsymbol{d}$ *is a dilation, then*

1) *all eigenvalues* $\lambda_i$ *of the matrix* $G_{\boldsymbol{d}}$ *are placed in the right complex half-plane:* $\Re(\lambda_i)>0, i=1,2,...,n$*;*
2) *there exists a matrix* $P\in\mathbb{R}^{n\times n}$ *such that*
$$PG_{\boldsymbol{d}}+G_{\boldsymbol{d}}^\top P\succ 0,\quad P=P^\top\succ 0;\tag{7}$$
3) *the dilation* $\boldsymbol{d}$ *is strictly monotone with respect to the weighted Euclidean norm* $\|\cdot\|=\sqrt{\langle\cdot,\cdot\rangle}$ *induced by the inner product* $\langle x,z\rangle=x^\top Pz$ *with* $P$ *satisfying* (7).

*Moreover,*

$$\begin{array}{l} e^{\alpha s}\leq\lfloor\boldsymbol{d}(s)\rfloor\leq\|\boldsymbol{d}(s)\|\leq e^{\beta s}\quad if\ \ s\leq 0,\\ e^{\beta s}\leq\lfloor\boldsymbol{d}(s)\rfloor\leq\|\boldsymbol{d}(s)\|\leq e^{\alpha s}\quad if\ \ s\geq 0,\end{array}\tag{8}$$

*where* $\alpha=\frac{1}{2}\lambda_{\max}\left(P^{\frac{1}{2}}G_{\boldsymbol{d}}P^{-\frac{1}{2}}+P^{-\frac{1}{2}}G_{\boldsymbol{d}}^\top P^{\frac{1}{2}}\right)$ *and* $\beta=\frac{1}{2}\lambda_{\min}\left(P^{\frac{1}{2}}G_{\boldsymbol{d}}P^{-\frac{1}{2}}+P^{-\frac{1}{2}}G_{\boldsymbol{d}}^\top P^{\frac{1}{2}}\right)$.

The latter theorem proves that any dilation $\mathbf{d}$ is strictly monotone if $\mathbb{R}^n$ is equipped with the norm $\|x\|=\sqrt{x^\top Px}$, provided that the matrix $P\succ 0$ satisfies (7).

*Definition 4.* (Polyakov (2018)). *A continuous function* $p:\mathbb{R}^n\to\mathbb{R}_+$ *is said to be a* $\boldsymbol{d}$*-homogeneous norm if* $p(x)\to 0$ *as* $x\to\mathbf{0}$ *and* $p(\boldsymbol{d}(s)x)=e^sp(x)>0$ *for* $x\in\mathbb{R}^n\backslash\{\mathbf{0}\}$, $s\in\mathbb{R}$.

The $\mathbf{d}$-homogeneous norm is neither a norm nor semi-norm in the general case, since the triangle inequality may not hold. However, many authors (see Andrieu et al. (2008), Grune (2000) and references therein) use this notion for such homogeneous functions. We follow this tradition.

*The* ***canonical homogeneous norm*** $\|\cdot\|_{\boldsymbol{d}}:\mathbb{R}^n\to\mathbb{R}_+$ *is defined as*

$$\|x\|_{\boldsymbol{d}}=e^{s_x}\ \ where\ s_x\in\mathbb{R}:\|\boldsymbol{d}(-s_x)x\|=1.\tag{9}$$

The map $\|\cdot\|_{\mathbf{d}}:\mathbb{R}^n\to[0,+\infty)$ is well defined and single-valued for monotone dilations. In Polyakov et al. (2018) such a homogeneous norm was called canonical because it is induced by a (canonical) norm in $\mathbb{R}^n$. Notice that

$$\lfloor\mathbf{d}(\ln\|x\|_{\mathbf{d}})\rfloor\leq\|x\|\leq\|\mathbf{d}(\ln\|x\|_{\mathbf{d}})\|\quad\text{for}\quad x\in\mathbb{R}^n,$$

and, due to (8), $\|\cdot\|_{\mathbf{d}}$ is continuous at zero.

*Proposition 1.* (Polyakov (2018)) *If* $\boldsymbol{d}$ *is a strictly monotone dilation then*

- *the canonical homogeneous norm* $\|\cdot\|_{\boldsymbol{d}}$ *is Lipschitz continuous on* $\mathbb{R}^n\backslash\{\mathbf{0}\}$*;*
- *if the norm* $\|\cdot\|$ *is smooth outside the origin then the homogeneous norm* $\|\cdot\|_{\boldsymbol{d}}$ *is also smooth outside the origin,* $\frac{d\|\boldsymbol{d}(-s)x\|}{ds}<0$ *if* $s\in\mathbb{R}$, $x\in\mathbb{R}^n\backslash\{\mathbf{0}\}$ *and*
$$\frac{\partial\|x\|_{\boldsymbol{d}}}{\partial x}=\left.\frac{\|x\|_{\boldsymbol{d}}\left.\frac{\partial\|z\|}{\partial z}\right|_{z=\boldsymbol{d}(-s)x}}{\left.\frac{\partial\|z\|}{\partial z}\right|_{z=\boldsymbol{d}(-s)x}G_{\boldsymbol{d}}\boldsymbol{d}(-s)x}\right|_{s=\ln\|x\|_{\boldsymbol{d}}}\tag{10}$$

***Below we use the notation*** $\|\cdot\|_{\boldsymbol{d}}$ ***only for the canonical homogeneous norm induced by the weighted Euclidean norm*** $\|x\|=\sqrt{x^\top Px}$ ***with a matrix*** $P\succ 0$ ***satisfying*** (7)***. The unit sphere*** $S$ ***is defined using the same norm.***

Vector fields, which are homogeneous with respect to dilation $\mathbf{d}$, have many properties useful for control design and state estimation of linear and nonlinear plants as well as for analysis of convergence rates Rosier (1993), Bhat and Bernstein (2005), Perruquetti et al. (2008).

*Definition 5.* Polyakov et al. (2018) *A mapping* $f:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ *is said to be* $\boldsymbol{d}$*-homogeneous if there exists* $\nu\in\mathbb{R}$

$$F(\mathbf{d}(s)x)=e^{\nu s}\mathbf{d}(s)F(x),\ \forall x\in\mathbb{R}^n\backslash\{\mathbf{0}\},\ \forall s\in\mathbb{R}.\tag{11}$$

*The number* $\nu\in\mathbb{R}$ *is called the homogeneity degree.*

*Let* $\mathbb{F}_{\boldsymbol{d}}(\mathbb{R}^n)$ *be the set of mappings* $\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ *satisfying the identity* (11)*, which are* ***nonempty-, compact-, convex-valued and upper semi-continuous*** *on* $\mathbb{R}^n$.

*Let* $\deg_{\mathbb{F}_{\boldsymbol{d}}}(f)$ *denote homogeneity degree of* $F\in\mathbb{F}_{\boldsymbol{d}}(\mathbb{R}^n)$

### *3.2 Quadratic Stability of Nonlinear Homogeneous Systems*

Homogeneity may simplify the analysis of differential equations. The most important property of $\mathbf{d}$-homogeneous systems is the symmetry of solutions Zubov (1958), Kawski (1995), Rosier (1992),Grune (2000), Bhat and Bernstein (2005). Namely, *if* $\varphi_{\xi_0}:[0,T)\to\mathbb{R}^n$ *is a solution to*

$$\dot{\xi}\in F(\xi),\quad F\in\mathbb{F}_{\boldsymbol{d}}(\mathbb{R}^n)\tag{12}$$

*with the initial condition* $\xi(0)=\xi_0\in\mathbb{R}^n$, *then* $\varphi_{\boldsymbol{d}(s)\xi_0}:[0,e^{-\nu s}T)\to\mathbb{R}^n$ *defined as*

$$\varphi_{\boldsymbol{d}(s)\xi_0}(t)=\boldsymbol{d}(s)\varphi_{\xi_0}(te^{\nu s}),\quad s\in\mathbb{R}$$

*is a solution to* (12) *with the initial condition* $\xi(0)=\boldsymbol{d}(s)\xi_0$, *where* $\nu=\deg_{\mathbb{F}_{\boldsymbol{d}}}(f)$.

The latter property implies many corollaries. In this paper we use the next one.

[2] The matrix $G_{\mathbf{d}}\in\mathbb{R}^n$ is anti-Hurwitz if $-G_{\mathbf{d}}$ is Hurwitz.

*Theorem 2.* Zimenko et al. (2018) *The following five claims are equivalent:*

1) *The origin of the system* (12) *is asymptotically stable.*
2) *The origin of the system*
$$\dot z\in\tilde F(z):=\|z\|^{1+\deg_{\mathbb{F}_{\mathbf{d}}}(f)}\left(\tfrac{(I_n-G_{\mathbf{d}})zz^\top P}{z^\top PG_{\mathbf{d}}z}+I_n\right)F\left(\tfrac{z}{\|z\|}\right) \tag{13}$$
*is asymptotically stable, where* $z=\sqrt{z^\top Pz}$, *and the positive definite matrix* $P\in\mathbb{R}^{n\times n}$ *satisfies* (7).
3) *For any positive definite matrix* $P\in\mathbb{R}^{n\times n}$ *satisfying,* (7) *there exists a homogeneous mapping* $\Psi:\mathbb{R}^n\to\mathbb{R}^n$, $\deg_{\mathbb{F}_{\mathbf{d}}}(\Psi)=0$, *such that* $\Psi\in C(\mathbb{R}^n), C^\infty(\mathbb{R}^n\backslash\{\mathbf{0}\})$ *is a diffeomorphism on* $\mathbb{R}^n\backslash\{\mathbf{0}\}$, *a homeomorphism on* $\mathbb{R}^n$, $\Psi(\mathbf{0})=\mathbf{0}$ *and*
$$\tfrac{\partial\left(\Psi^\top(\xi)P\Psi(\xi)\right)}{\partial\xi}f(\xi)<0 \quad \text{if} \quad \Psi^\top(\xi)P\Psi(\xi)=1. \tag{14}$$
*Moreover,* $\|\Psi\|_{\mathbf{d}}\in C(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n\backslash\{\mathbf{0}\})$ *is a Lyapunov function for the system* (12).
4) *For any matrix* $P\in\mathbb{R}^{n\times n}$ *satisfying* (7), *there exists a smooth mapping* $\Xi\in C^\infty(\mathbb{R}^n\backslash\{\mathbf{0}\},\mathbb{R}^{n\times n})$ *such that*
$$\begin{array}{c}\det(\Xi(z))\neq 0,\ \ \tfrac{\partial\Xi(z)}{\partial z_i}z=0,\ \ \Xi(e^sz)=\Xi(z)\\ \text{for}\ \ z=(z_1,...,z_n)^\top\in\mathbb{R}^n\backslash\{\mathbf{0}\}, s\in\mathbb{R}, i=1,...,n\end{array}\ \ \text{and}$$
$$\sup_{v\in\tilde F(z)} z^\top\Xi^\top(z)P\Xi(z)v<0, \quad z\neq\mathbf{0}. \tag{15}$$

This theorem proves two important facts:

- Any generalized homogeneous system (12) is homeomorphic on $\mathbb{R}^n$ and diffeomorphic on $\mathbb{R}^n\backslash\{\mathbf{0}\}$ to a standard homogeneous one (13). The corresponding change of coordinates is given by
$$z=\Phi(\xi):=\|\xi\|_{\mathbf{d}}\mathbf{d}(-\ln\|\xi\|_{\mathbf{d}})\xi \tag{16}$$
while the inverse transformation is as follows:
$$\xi=\Phi^{-1}(z):=\mathbf{d}(\ln\|z\|)\tfrac{z}{\|z\|}$$
- Any asymptotically stable generalized homogeneous system is homeomorphic on $\mathbb{R}^n$ and diffeomorphic on $\mathbb{R}^n\backslash\{\mathbf{0}\}$ to a **quadratically stable** one. Indeed, making the change of variables $z=\Psi(\xi)$ we derive
$$\dot z=\hat f(z)=\left.\tfrac{\partial\Psi(\xi)}{\partial\xi}f(\xi)\right|_{\xi=\Psi^{-1}(z)},$$
but the criterion (14) implies that $z^\top P\dot z<0$ if $z^\top Pz=1$, so the homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is the Lyapunov function to the latter system. Finally, the change of variable $x=\|z\|_{\mathbf{d}}\mathbf{d}(-\ln\|z\|_{\mathbf{d}})z$ gives $\|z\|_{\mathbf{d}}=\|x\|$, so the transformed system $\dot x=\tilde f(x)$ is quadratically stable with the Lyapunov function $V$ defined as $V(x)=\|x\|^2=x^\top Px$, where
$$\tilde f(x)=\|x\|^{1+\deg_{\mathbb{F}_{\mathbf{d}}}(f)}\left(\tfrac{(I_n-G_{\mathbf{d}})xx^\top P}{x^\top PG_{\mathbf{d}}x}+I_n\right)\hat f\left(\tfrac{x}{\|x\|}\right).$$

Below we show that the transformations $\Phi$ and $\Psi$ can be utilized for the design of a consistent discretization scheme for locally homogeneous systems.

*Remark 2. Recall Orlov (2005), Levant (2005), Bhat and Bernstein (2005), Polyakov (2018) that if the homogeneous system* (12) *is asymptotically stable and* $\deg_{\mathbb{F}_{\mathbf{d}}}(f)<0$, *then it is globally uniformly* ***finite-time stable****.*

*Remark 3. If* $\mathbf{d}$ *is a dilation with the generator* $G_{\mathbf{d}}$, *then for any fixed* $\alpha>0$, *the group* $\mathbf{d}^\alpha$ *defined as* $\mathbf{d}^\alpha(s):=\mathbf{d}(\alpha s), s\in\mathbb{R}^n$, *is the dilation with the generator* $G_{\mathbf{d}^\alpha}=\alpha G_{\mathbf{d}}$. *If* $f\in\mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$, *then* $f\in\mathbb{F}_{\mathbf{d}^\alpha}(\mathbb{R}^n)$ *and* $\deg_{\mathbb{F}_{\mathbf{d}^\alpha}}(f)=\alpha\deg_{\mathbb{F}_{\mathbf{d}}}(f)$. *In other words, if* $\deg_{\mathbb{F}_{\mathbf{d}}}(f)<0$, *then a new dilation* $\mathbf{d}^\alpha$ *can be selected such that* $\deg_{\mathbb{F}_{\mathbf{d}^\alpha}}(f)=-1$.

## 4. FINITE-TIME STABLE IMPLICIT DISCRETIZATION

The main idea of the design of a finite-time stable discretization for homogeneous systems, is to use the coordinate transformation (16).

*Theorem 3. Let* $F\in\mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ *be a* $\mathbf{d}$*-homogeneous mapping satisfying standard assumption,* $\deg_{\mathbb{F}_{\mathbf{d}}}(F)=-1$ *and the condition* (15) *holds with* $\Xi=I_n$.

*Then the mapping* $Q:\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ *defined as*
$$Q(h,x_i,x_{i+1})=\tilde Q(h,\Phi(x_i),\Phi(x_{i+1})), \tag{17}$$
$$\Phi(x)=\|x\|_{\mathbf{d}}\,\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x,$$
$$\tilde Q(h,y_i,y_{i+1})=y_{i+1}-y_i-h\tilde F(y_{i+1}),$$
$$\tilde F(y)=\bigcap_{\varepsilon>0} co\tilde F_0\left(\{y+\varepsilon B\}\backslash\{\mathbf{0}\}\right), \tag{18}$$
$$\tilde F_0(y)=\left(\tfrac{(I_n-G_{\mathbf{d}})yy^\top P}{y^\top PG_{\mathbf{d}}y}+I_n\right)F\left(\tfrac{y}{\|y\|}\right)$$
*is a consistent discrete-time approximation of the globally uniformly* ***finite-time stable system*** (1)

**Proof.**

First of all, since $F_0(\lambda y)=F_0(y)$ for all $\lambda>0$ and all $y\neq\mathbf{0}$ then
$$\tilde F(y)=\begin{cases}F_0(y) & \text{if } y\neq 0,\\ co\,\tilde F_0\left(B\backslash\{\mathbf{0}\}\right) & \text{if } y=0\end{cases}$$
then the system
$$\dot y\in\tilde F(y), \tag{19}$$
is topologically equivalent to the system (1) with the change of coordinates
$$y=\Phi(x), \qquad x=\Phi^{-1}(y)=\mathbf{d}(\ln\|y\|)\frac{y}{\|y\|}$$
and $\|y\|=\|x\|_{\mathbf{d}}$. For more details about the coordinate transformation $\Phi$ we refer the reader to Polyakov (2018), Zimenko et al. (2018) and/or to explanations presented after Theorem 2 given above.

Notice if $A\in\mathbb{R}^{n\times n}$ is an arbitrary matrix and $\Omega$ is a convex and compact set then the set $A\Omega$ is also compact and convex. This implies that the mapping $\tilde F$ is convex-valued and compact-valued. Moreover, by construction, $\tilde F$ is upper-semi continuous. The system (19) is globally uniformly finite-time stable due to (15) and Remark 2, so $y=\mathbf{0}$ is the strong equilibrium of the system and $\mathbf{0}\in\tilde F(\mathbf{0})$.

If we show that $\tilde Q$ is a consistent discrete-time approximation of (19) then, due to continuity of $\Phi$ on $\mathbb{R}^n$ and its smoothness on $\mathbb{R}^n\backslash\{\mathbf{0}\}$, we derive that $Q$ is a consistent discrete-time approximation of (1) in $\mathbb{R}^n$.

1) **Existence property.** Let us show that for any $\tilde y\in\mathbb{R}^n$ and any $h>0$, there exists $\tilde y_h\in\mathbb{R}^n$ such that $\mathbf{0}\in\tilde Q(h,\tilde y,\tilde y_h)$.

**Case** $\tilde y\in\Omega:=h\tilde F(\mathbf{0})$. Obviously, if $\tilde y\in h\tilde F(\mathbf{0})$ then $\tilde y_h=\mathbf{0}$ is a solution to the inclusion $\mathbf{0}\in\tilde Q(h,\tilde y,\tilde y_h)$.

Let us show now that the inclusion $\mathbf{0} \in \tilde{Q}(h, \mathbf{0}, \tilde{y}_h)$ has the unique solution $\tilde{y}_h = \mathbf{0}$. Suppose the contrary, i.e., $\mathbf{0} \neq \tilde{y}_h \in h\tilde{F}(\tilde{y}_h)$. Since, in our case, (15) is fulfilled with $\Xi = I_n$ we derive

$$\tilde{y}_h^\top P\tilde{v} < 0, \quad \forall \tilde{v} \in h\tilde{F}(y_h) < 0,$$

Hence, for $\tilde{v} = y_h \in h\tilde{F}(y_h)$ we obtain a contradiction because the positive definiteness of the matrix $P$ implies the inequality $\tilde{y}_h^\top P y_h > 0$.

**Case** $\tilde{y} \notin \Omega = h\tilde{F}(\mathbf{0})$. By construction the set-valued mapping $\tilde{F}$ is nonempty-valued, compact-valued and upper semi-continuous. Notice that $\tilde{F}_0(y) \subset \tilde{F}(\mathbf{0})$ for all $y \in \mathbb{R}^n$ (also by construction). This means that the set-valued mapping $h\tilde{F}(\tilde{y} + \cdot) : \Omega \to \Omega$ has a closed-graph. Since $\Omega$ is a convex and compact set, then according Kakutani fixed-point theorem (see Kakutani (1941)) $\exists \Delta^* \in \Omega$ such that $\Delta^* \in h\tilde{F}(\tilde{y} + \Delta^*)$, and taking $\tilde{y}_h = \tilde{y} + \Delta^*$ we derive $\mathbf{0} \in \tilde{Q}(h, \tilde{y}, \tilde{y}_h)$.

2) **Finite-time convergence property.** Let us show that for any $y_0 \in \bar{\mathbb{R}}^n \backslash \{\mathbf{0}\}$, the sequence $\{y_i\}_{i=0}^{+\infty}$ generated by $\mathbf{0} \in \tilde{Q}(h, y_i, y_{i+1})$ converges to zero in a finite number of steps. Since the stability criterion (15) holds for $\Xi = I_n$, then $V(y) = y^\top P y = \|y\|^2$ is a Lyapunov function for the system (19).

Let us show that there exists $i^* < +\infty$ such that $y_{i^*} = \mathbf{0}$. Suppose the contrary, i.e., $y_i \neq \mathbf{0}$ for all $i > 0$. In this case, using (18) one has $y_{i-1} = y_i - h\tilde{v}$ with $\tilde{v} \in \tilde{F}(y_i)$, and

$$V(y_{i-1}) = (y_i - h\tilde{v})^\top P(y_i - h\tilde{v})) =$$
$$V(y_i) - 2hy_i^\top P\tilde{v} + h^2\tilde{v}^\top\tilde{v} \geq V(y_i) + h^2 c.$$

where $c = \inf_{v\in\tilde{F}(y), y\in S} \|\tilde{v}\|^2$. Due to (15) we conclude $\mathbf{0} \notin \tilde{F}(y), y \neq \mathbf{0}$ then $c > 0$.

This means that in a finite number of steps, we will have $V(y_i) < 0$, but this is impossible due to positive definiteness of the Lyapunov function $V$. Therefore, there exists $i^* < +\infty$ such that $y_{i^*} = 0$.

3) **Approximation property**. By construction for all $y_{i+1}$ and $y_i$ satisfying $\mathbf{0} \in \tilde{Q}(h, y_i, y_{i+1})$ we have

$$\frac{y_{i+1} - y_i}{h} \in F(y_{i+1}).$$

This means that

$$\inf_{v\in F(y_i)} \left\| \frac{y_{i+1} - y_i}{h} - v \right\| \leq \sup_{z\in F(y_{i+1})} \inf_{v\in F(y_i)} \|z - v\|$$

for all $y_{i+1}$ satisfying $\mathbf{0} \in \tilde{Q}(h, y_i, y_{i+1})$. On the other hand, obviously, that $\|y_{i+1} - y_i\| \to 0$ as $h \to 0$ but upper semi-continuity of $F$ implies

$$\sup_{z\in F(y_{i+1})} \inf_{v\in F(y_i)} \|z - v\| \to 0 \quad \text{as} \quad h \to 0.$$

■

It is well known (according Heine-Cantor Theorem) that any function continuous on a compact is uniformly continuous on it. A similar results does not hold for upper semi-continuous functions, e.g. the upper semi-continuous set-valued function

$$\mathrm{sgn}(\rho) := \begin{cases} 1 & \text{if } \rho>0, \\ [-1,1] & \text{if } \rho=0, \\ -1 & \text{if } \rho<0. \end{cases} \tag{20}$$

is *not uniformly* upper semi-continuous on $[-\delta, \delta]$, $\delta > 0$.

*Lemma 1. If all condition of Theorem 3 hold and the mapping $\tilde{F}$ is uniformly continuous then the inequality* (**??**) *holds on any compact from $\mathbb{R}^n \backslash \{\mathbf{0}\}$.*

**Proof.** We have

$$\|\tilde{\phi}(h, y_i) - y_{i+1}\| = \left\| \int_0^h \frac{d}{d\tau}\tilde{\phi}(\tau, y_i) - v_{i+1} d\tau \right\|$$

where $\tilde{\phi}(\cdot, y_i)$ is a solution to (19) such that $y(0) = y_i$ and $\frac{d}{d\tau}\tilde{\phi}(\tau, y_i) \in \tilde{F}(\tilde{\phi}(\tau, y_i))$ almost everywhere, $v_{i+1} \in \tilde{F}(y_{i+1})$.

Let $0 < \varepsilon < R < +\infty$ be such that $y_i, y_{i+1} \in K(\varepsilon, R)$ and $\phi(\tau, y_i) \in K(\varepsilon, R)$ for all $\tau \in [0, h]$, where

$$K(r_1, r_2) = \{x \in \mathbb{R}^n : r_1 \leq \|x\| \leq r_2\},$$
$$0 < r_1 < r_2 < +\infty.$$

Since $\mathbf{d}(-\ln \|\cdot\|_{\mathbf{d}}) : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{d}(\ln \|\cdot\|_{\mathbf{d}}) : \mathbb{R}^n \to \mathbb{R}^n$ are smooth outside the origin, the mapping $\tilde{F}$ is uniformly upper semi-contionous on $K(\varepsilon,R)$, with a modulus of continuity $\omega_{\varepsilon,R} \in \mathcal{K}$. Hence,

$$\|\tilde{\phi}(h, y_i) - y_{i+1}\| \leq \int_0^h \omega_{\varepsilon,R}(\|\tilde{\phi}(\tau, y_i) - y_{i+1}\|) d\tau \leq$$
$$h \sup_{\tau\in[0,h]} \omega_{\varepsilon,R}\left(\|\tilde{\phi}(\tau, y_i) - y_{i+1}\|\right) \leq h\omega_{\varepsilon,R}(f^{\max} h),$$

where $f^{\max} = \sup_{v\in\tilde{F}(y), y\in K(\varepsilon,R)} \|v\|$.

Since $\Phi^{-1}$ is continuously differentiable on $\mathbb{R}^n \backslash \{\mathbf{0}\}$, it satisfies a Lipschitz condition on $K(\epsilon, R)$, with a Lipschitz constant $L = L(\varepsilon, R)$ for any $0 < \varepsilon < R < +\infty$. If $\phi(h, x_i)$ is a solution to (1) with $x(0) = x_i$, then

$$\|\phi(h, x_i) - x_{i+1}\| = \|\Phi^{-1}(\tilde{\phi}(h, y_i)) - \Phi^{-1}(y_{i+1})\| \leq$$
$$L\|\tilde{\phi}(h, y_i)) - y_{i+1}\| \leq hL\omega_{\varepsilon,R}(2f^{\max} h).$$

■

Theorem 3 is based on the fact that the system $\dot{y} = \tilde{F}_0(y)$ admits a quadratic Lyapunov function (the condition (15) with $\Xi$ =const). However, as it was shown in Theorem 2, any stable homogeneous system is equivalent to a quadratically stable one. If $F$ in Theorem 3 is replaced with the equivalent one:

$$F^{new}(x) = \frac{\partial\Psi(\xi)}{\partial\xi} F(\xi)\Big|_{\xi=\Psi^{-1}(x)}, \quad x \in \mathbb{R}^n, \quad \xi \in \mathbb{R}^n,$$

where $\Psi \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ is a diffeomorphism on $\mathbb{R}^n \backslash \{\mathbf{0}\}$ given in Theorem 2, then the condition $\Xi = I_n$ required for Theorem 3 is fulfilled.

According to Theorem 2, a homogeneous Lyapunov function $V \in \mathbb{H}_{\mathbf{d}}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ with degree $\deg_{\mathbb{H}}(V) = \mu > 0$ can always be found for any asymptotically stable system with a homogeneous vector field $F \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$. In this case, the required transformation $\Psi$ can be defined as follows

$$\Psi(\xi) = \mathbf{d}\left(\ln \frac{V^{1/\mu}(\xi)}{\|\xi\|_{\mathbf{d}}}\right) \xi.$$

(see Polyakov (2018) for more details).

In other words, if we know a homogeneous Lyapunov function for a finite-time stable homogeneous system then, we can easily design a consistent implicit discretization.

Therefore, *any finite-time stable homogeneous differential inclusion (satisfying standard assumptions) admits a consistent implicit approximation.*

## 5. EXAMPLE: CONSISTENT DISCRETIZATION OF "NESTED" ALGORITHM

Let us consider the control system

$$\begin{array}{c} \dot{x} = Ax + bu, \quad x \in \mathbb{R}^2, \quad |u| \leq 1, \\ x(0) = x_0, \end{array} \tag{21}$$

where $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $b = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. It is well known Levant (2005) that it can be stabilized by means of the so-called "nested" second order sliding mode controller

$$u(x) = -\mathrm{sgn}(x_2|x_2| + 0.5x_1), \tag{22}$$

where $\mathrm{sgn} : \mathbb{R} \rightrightarrows \mathbb{R}$ is defined by (20)

The closed-loop system (21), (22)

$$\dot{x} \in F(x) := Ax + bu(x)$$

is homogenous of degree $-1$ with respect to dilation group given by $\mathbf{d}(s) = e^{G_{\mathbf{d}}s}$, $\quad G_{\mathbf{d}} = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Indeed, simple calculations show that $F(\mathbf{d}(s)x) = e^{-s}\mathbf{d}(s)F(x)$, $\quad s \in \mathbb{R}$.

Let us consider now the coordinate transformation

$$y = \|x\|_{\mathbf{d}}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x, \qquad x = \mathbf{d}(\ln\|y\|)\frac{y}{\|y\|},$$

where, as before, $\|y\| = \sqrt{y^\top P y}$ and $P$ satisfies (7). It can be shown

$$y^\top P\tilde{F}_0(y) < 0, \quad \forall y \neq \mathbf{0}$$

$$\tilde{F}_0(y) := \left(\frac{(I_n - G_{\mathbf{d}})yy^\top P}{y^\top P G_{\mathbf{d}} y} + I_n\right) F\left(\frac{y}{\|y\|}\right)$$

provided that $P = \left(\begin{smallmatrix} 1 & 0.5 \\ 0.5 & 1 \end{smallmatrix}\right)$. Notice that $P$ satisfies (7).

To design a consistent discretization scheme let us apply Theorem 3 using the following notations

$$q_i = \|x_i\|_{\mathbf{d}} = \|y_i\| \quad z_i = \mathbf{d}(-\ln\|x_i\|_{\mathbf{d}})x_i = \frac{y_i}{\|y_i\|}$$

and

$$\Omega := h \cdot co\,\tilde{F}_0(B\backslash\{\mathbf{0}\}), \quad h > 0.$$

**The case $y_i \notin \Omega$.** From Theorem (3) we conclude that the inclusion $\mathbf{0} \in \tilde{Q}(h, y_i, y_{i+1})$ , in our case, is equivalent to the following system

$$q_{i+1}z_{i+1} - h\left(\frac{(I_n - G_{\mathbf{d}})z_{i+1}z_{i+1}^\top P}{z_{i+1}^\top P G_{\mathbf{d}} z_{i+1}} + I_n\right)(Az_{i+1} + bu^*) = y_i,$$

$$z_{i+1}^\top P z_{i+1} = 1, \quad u^* \in u(z_{i+1}), \quad q_{i+1} > 0.$$

Its solution $(q_{i+1}, z_{i+1}, u^*)$ provides a consistent discrete-time approximation of a trajectory of the continuous-time closed-loop system (21), (22) at the time instant $(i+1)h$ as follows

$$x_{i+1} = \mathbf{d}(\ln q_i)z_{i+1}.$$

The value $u^*$ can be treated as *a discrete time approximation of the control input.* **The case $y_i \in \Omega$.** In this case, the inclusion $\mathbf{0} \in \tilde{Q}(h, y_i, y_{i+1})$ has the zero solution (see the proof of Theorem (3)) $y_{i+1} = \mathbf{0}$. Since, $b^\top \tilde{F}_0(y) = u(y)$ then $y_i \in \Omega \quad \Rightarrow \quad -b^\top y_i/h = u(\mathbf{0}) = [-1, 1]$ and the discrete-time approximation of the control input is given by $u^* = -b^\top y_i/h$.

Simulation results for the consistent discretization are shown in Figures 2 and 3 for $h = 0.07$. Their confirm finite-time convergence of the system trajectories to the origin is a finite number of steps. In addition, consistent discretization removes the numerical chattering (in both input and output) in contract to the explicit Euler scheme (Figures 4 and 5).

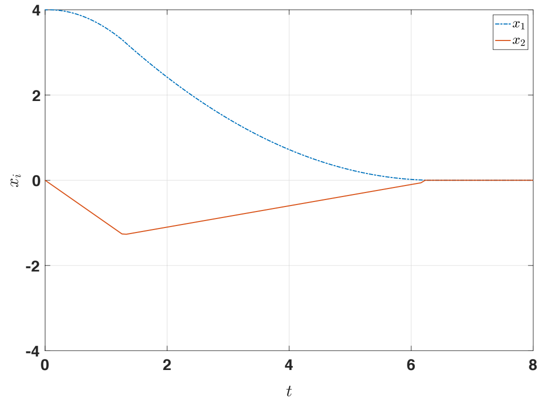


Fig. 2. System states for consistent discretization method

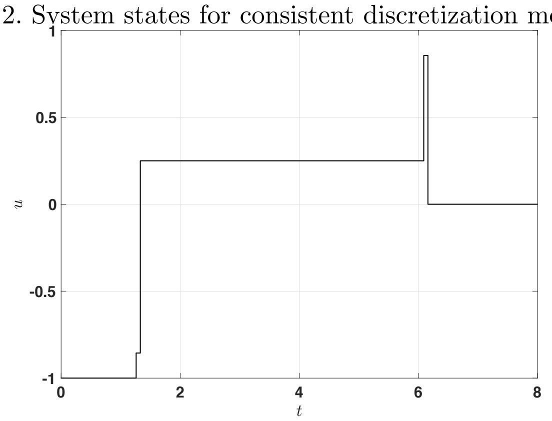


Fig. 3. Control for consistent discretization method

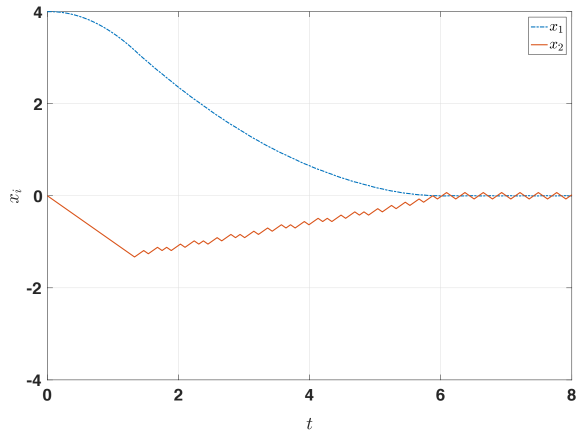


Fig. 4. System states for explicit scheme $x_{i+1} = x_i + hF(x_i)$

## 6. DISCUSSIONS AND CONCLUSIONS

The problem of consistent discrete-time approximation of finite-time stable differential inclusions is studied. It is shown that any homogeneous finite-time stable differential inclusions always admits an implicit discretization scheme preserving finite-time convergence globally. The topological equivalence Polyakov (2018) of homogeneous stable system to a quadratically stable one is utilized for the design of this scheme. Theoretical results are supported with numerical simulations.

## REFERENCES


Acary, V., Brogliato, B., and Orlov, Y. (2012). Chattering-free digital sliding-mode control with state observer and disturbance rejection. *IEEE Transactions on Automatic Control*, 57(5), 1087–1101.

Andrieu, V., Praly, L., and Astolfi, A. (2008). Homogeneous Approximation, Recursive Observer Design, and

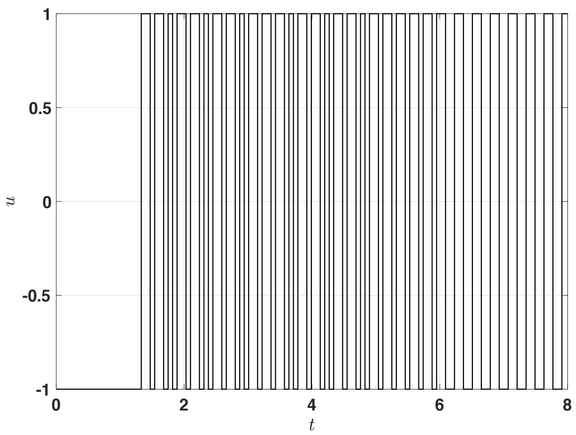


Fig. 5. Control input for explicit scheme $x_{i+1} = x_i + hF(x_i)$

Output Feedback. *SIAM Journal of Control and Optimization*, 47(4), 1814–1850.

Bhat, S.P. and Bernstein, D.S. (2005). Geometric homogeneity with applications to finite-time stability. *Mathematics of Control, Signals and Systems*, 17, 101–127.

Butcher, J.C. (2016). *Numerical Methods for Ordinary Differential Equations.* Wiley.

Filippov, A.F. (1988). *Differential Equations with Discontinuous Right-hand Sides.* Kluwer Academic Publishers.

Grune, L. (2000). Homogeneous state feedback stabilization of homogeneous systems. *SIAM Journal of Control and Optimization*, 38(4), 1288–1308.

Hermes, H. (1986). Nilpotent approximations of control systems and distributions. *SIAM Journal of Control and Optimization*, 24(4), 731.

Huber, O., Acary, V., and Brogliato, B. (2016). Lyapunov stability and performance analysis of the implicit discrete sliding mode control. *IEEE Transactions on Automatic Control*, 61(10), 3016–3030.

Kakutani, S. (1941). A generalization of Brouwer's fixed point theorem. *Duke Mathematical Journal.*, 8(3), 457–459.

Kawski, M. (1990). Homogeneous stabilizing feedback laws. *Control Theory and Advanced Technology*, 6(4), 497–516.

Kawski, M. (1995). Geometric homogeneity and stabilization. In A. Krener and D. Mayne (eds.), *Proc. IFAC Nonlinear Control Symposium*, 164–169. Lake Tahoe, CA.

Khomenuk, V.V. (1961). On systems of ordinary differential equations with generalized homogenous right-hand sides. *Izvestia vuzov. Mathematica (in Russian).*, 3(22), 157–164.

Kikuuwe, R., Yasukouchi, S., Fujimoto, H., and Yamamoto, M. (2010). Proxy-based sliding mode control: A safer extension of pid position control. *IEEE Transactions on Robotics*, 26(4), 670–683.

Levant, A. (2005). Homogeneity approach to high-order sliding mode design. *Automatica*, 41(5), 823–830.

Nakamura, N., Nakamura, H., and Yamashita, Y. (2007). Homogeneous stabilization for input-affine homogeneous systems. In *Conference on Decision and Control*, 80–85.

Orlov, Y. (2005). Finite time stability and robust control synthesis of uncertain switched systems. *SIAM Journal of Control and Optimization*, 43(4), 1253–1271.

Pazy, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer.

Perruquetti, W., Floquet, T., and Moulay, E. (2008). Finite-time observers: application to secure communication. *IEEE Transactions on Automatic Control*, 53(1), 356–360.

Polyakov, A. (2018). Sliding mode control design using canonical homogeneous norm. *International Journal of Robust and Nonlinear Control*, 29(3), 682–701. doi: 10.1002/rnc.4058.

Polyakov, A., Coron, J.M., and Rosier, L. (2018). On homogeneous finite-time control for linear evolution equation in hilbert space. *IEEE Transactions on Automatic Control*, 63(9), 3143–3150.

Polyakov, A., Efimov, D., and Brogliato, B. (2019). Consistent discretization of finite-time and fixed-time stable systems. *SIAM Journal of Control and Optimization*, 57(1), 78–103.

Polyakov, A., Efimov, D., Fridman, E., and Perruquetti, W. (2016). On homogeneous distributed parameters equations. *IEEE Transactions on Automatic Control*, 61(11), 3657–3662.

Polyakov, A. and Fridman, L. (2014). Stability notions and Lyapunov functions for sliding mode control systems. *Journal of The Franklin Institute*, 351(4), 1831–1865.

Rosier, L. (1992). Homogeneous Lyapunov function for homogeneous continuous vector field. *Systems & Control Letters*, 19, 467–473.

Rosier, L. (1993). Etude de quelques problèmes de stabilization. *PhD Thesis, Ecole Normale Superieure de Cachan (France).*

Roxin, E. (1966). On finite stability in control systems. *Rendiconti del Circolo Matematico di Palermo*, 15, 273–283.

Utkin, V.I. (1992). *Sliding Modes in Control Optimization.* Springer-Verlag, Berlin.

Zimenko, K., Polyakov, A., and Efimov, D. (2018). On dynamical feedback control design for generalized homogeneous differential inclusions. In *57th IEEE Conference on Decision and Control.*

Zubov, V. (1958). On systems of ordinary differential equations with generalized homogenous right-hand sides. *Izvestia vuzov. Mathematica (in Russian).*, 1, 80–88.