Classification of edge-critical underlying absolute planar cliques for signed graphs

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Abstract

A simple signed graph (G, Σ) is a simple graph G having two different types of edges, positive edges and negative edges, where Σ denotes the set of negative edges of G. A closed walk of a signed graph is positive (resp., negative) if it has even (resp., odd) number of negative edges, taking repeated edges into account. A homomorphism (resp., colored homomorphism) of a simple signed graph to another simple signed graph is a vertex-mapping that preserves adjacencies and signs of closed walks (resp., signs of edges). A simple signed graph (G, Σ) is a signed absolute clique (resp., $(0, 2)$-absolute clique) if any homomorphism (resp., colored homomorphism) of it is an injective function, in which case G is called an underlying signed absolute clique (resp., underlying $(0, 2)$-absolute clique). Moreover, G is edge-critical if $G - e$ is not an underlying signed absolute clique (resp., underlying $(0, 2)$-absolute clique) for any edge e of G. In this article, we characterize all edge-critical outerplanar underlying $(0, 2)$-absolute cliques and all edge-critical planar underlying signed absolute cliques. We also discuss the motivations and implications of obtaining these exhaustive lists.

Keywords: signed graphs, graph homomorphisms, absolute cliques, planar graphs.

1 Introduction

The work of Naserasr, Rollová and Sopena [19], based on the work of Zaslavsky [30], has generated attention to the topic of homomorphisms of signed graphs in recent times [23, 8, 6, 2, 20, 18]. In particular, the seminal work of Naserasr, Rollová and Sopena and the series of works that followed showed how one can extend a number of classical results and conjectures, including the Four-Color Theorem and Hadwiger's Conjecture, to the signed graph context. In this article, we deal only with simple signed graphs; thus, the definition of homomorphism to be given is the one appropriate for simple signed graphs. In particular, we voluntarily omit more complicated definitions, such as the one from [21].

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Homomorphisms of signed graphs

A *signed graph* (G, Σ) is a graph with set of vertices $V(G)$ and set of edges $E(G)$ where each edge is assigned one of two possible signs, $+$ and $-$. Edges with sign $+$ are called *positive*, while edges with sign $-$ are called *negative*. For any two adjacent vertices u, v of (G, Σ), we call u a *positive neighbor* of v if uv is a positive edge. Analogously, u is a *negative neighbor* of v if uv is a negative edge. We denote by $N(u)$, $N^+(u)$ and $N^-(u)$ the sets of neighbors, positive neighbors and negative neighbors of u in (G, Σ). The *signature* Σ of G is its set of negative edges. A closed walk (in particular, a cycle) C in (G, Σ) is *positive* (respectively, *negative*) if it has an even (respectively, odd) number of negative edges, taking repeated edges into account.

A *homomorphism* f of a signed graph (G, Σ) to a signed graph (H, Π) is a vertex-mapping $f : V(G) \to V(H)$ that preserves adjacencies and the signs of closed walks. We use the notation $(G, \Sigma) \to (H, \Pi)$ to denote that there exists a homomorphism of (G, Σ) to (H, Σ).

To apprehend homomorphisms of signed graphs, some graph parameters and objects such as the notions of chromatic number, relative and absolute clique, relative and absolute clique number and underlying absolute clique, need to be defined. The *chromatic number* of a signed graph (G, Σ) is given by

$$\chi_s(G, \Sigma) = \min \{|V(H)| : (G, \Sigma) \to (H, \Pi)\}.$$

Observe that this definition generalizes the usual notion of chromatic number for simple graphs, as $\chi_s(G, \emptyset) = \chi(G)$ where $\chi(G)$ denote the chromatic number of the simple graph G.

Likewise the notion of clique number was also generalized to the context of signed graphs. However, in this case the generalization ramified into two different parameters, namely the signed relative clique number and the signed absolute clique number [19], the latter being of interest for our investigations in this work. For the sake of completeness and for the sake of providing a big picture, we will also recall the definition of the former.

- On the one hand, a *signed relative clique* of a signed graph (G, Σ) is a vertex subset $R \subseteq V(G)$ such that, for any homomorphism f of (G, Σ), we have $|f(R)| = |R|$. The *signed relative clique number* $\omega_{rs}(G, \Sigma)$ of (G, Σ) is the maximum $|R|$ where R is a signed relative clique of (G, Σ).

- On the other hand, a signed graph (C, Λ) is an *signed absolute clique* if $\chi_s(C, \Lambda) = |V(C)|$. The *signed absolute clique number* $\omega_{as}(G, \Sigma)$ of a signed graph (G, Σ) is the maximum $|C|$ such that C induces a signed absolute clique in (G, Σ).

Observe that $\omega_{rs}(G, \emptyset) = \omega_{as}(G, \emptyset) = \omega(G)$, and also that $(G, \emptyset)$ is a signed absolute clique if and only if G is a complete graph (or, equivalently, a clique). Furthermore, notice that, from the definitions, as noted e.g. in [19] one can directly derive the following inequalities for any signed graph (G, Σ):

$$\omega_{as}(G, \Sigma) \leq \omega_{rs}(G, \Sigma) \leq \chi_s(G, \Sigma) \tag{1}$$

A simple graph G is an *underlying signed absolute clique* if there exists a signature Σ of G such that (G, Σ) is a signed absolute clique. Moreover, an underlying signed absolute clique G is *edge-critical* if for any edge $e \in E(G)$ the signed graph $(G - e, \Sigma \setminus \{e\})$ is not a signed absolute clique. To the best of our knowledge, an analogous notion for signed relative cliques does not exist yet.

Let $p \in \{\chi_s, \omega_{rs}, \omega_{as}\}$ be one of the parameters defined earlier. Each of these three parameters defined for signed graphs can be extended to a family $\mathcal{F}$ of graphs by setting:

$$p(\mathcal{F}) = \max \{p(G, \Sigma) : G \in \mathcal{F} \text{ for all signature } \Sigma \text{ of } G\}.$$

Motivations

Before moving into the specific problems that we address in this article, let us discuss our motivations. In the literature, different types of graphs and their homomorphisms have been studied. Some of them are relevant to present our motivations.

An (m,n)-*colored mixed graph* is a graph with m different types of arcs and n different types of edges [22]. A *colored homomorphism* of an (m,n)-colored mixed graph G to an (m,n)-colored mixed graph H is a vertex-mapping $f : V(G) \xrightarrow{(m,n)} V(H)$ such that an arc (resp., edge) uv of G implies that $f(u)f(v)$ is an arc (resp., edge) of H of the same type as uv. It is worth mentioning that (m,n)-colored mixed graphs encapsulate simple graphs[1] [13], oriented graphs[2] [28], 2-edge-colored graphs[3] [16] and k-edge-colored graphs[4] [1]. For each of these particular cases, the corresponding homomorphisms have been well studied independently.

Notice that a signed graph can be thought of as a $(0,2)$-colored mixed graph. However, homomorphisms of $(0,2)$-colored mixed graphs preserve adjacencies and edge signs only. For signed graphs, their homomorphisms preserve adjacencies and signs of their closed walks instead, which is a little more flexible in the sense that colored homomorphisms of signed graphs (treated as $(0,2)$-colored mixed graphs) are homomorphisms, but homomorphisms of signed graphs are not colored homomorphisms. Still, we make the connection between the two notions more explicit in what follows.

To *switch* a vertex v of a signed graph (G,Σ) is to reverse the signs of the edges incident to v. If it is possible to obtain a signed graph (G,Σ') by switching a set of vertices of (G,Σ), then we say that the two signed graphs are in a *switch relation* and denote it by $(G,\Sigma') \sim (G,\Sigma)$. Note that the switch relation is an equivalence relation.

An alternative, but equivalent, definition of homomorphisms of signed graphs is the following [21]: a *homomorphism* f of a signed graph (G,Σ) to a signed graph (H,Π) is a vertex-mapping $f : V(G) \to V(H)$ such that there exists a (G,Σ') with $(G,\Sigma') \sim (G,\Sigma)$ for which $f : (G,\Sigma') \xrightarrow{(0,2)} (H,\Pi)$. This definition is the key for connecting the two notions. This relation was explored in [19, 23].

Another interesting related notion is that of pushable homomorphisms [] of oriented graphs. To *push* a vertex v of an oriented graph is to reverse the direction of all the arcs incident to v. A *pushable homomorphism* of an oriented graph $\overrightarrow{G}$ to another oriented graph $\overrightarrow{H}$ is a vertex-mapping $f : V(\overrightarrow{G}) \to V(\overrightarrow{H})$ such that it is possible to push a set of vertices of $\overrightarrow{G}$ to obtain a $\overrightarrow{G}'$ for which $f : \overrightarrow{G}' \xrightarrow{(1,0)} \overrightarrow{H}$. Note the similarity between that definition and the alternative definition of homomorphisms of signed graphs above. In practice it was noticed that, due to this similarity, several results that hold for either of the two topics, namely homomorphisms of signed graphs and pushable homomorphisms of oriented graphs, can be similarly proved for the other. However, no systematic relation between the two notions has been established yet.

For each of these types of graphs and homomorphisms, several parameters and objects such as the notions of chromatic number, relative and absolute clique, relative and absolute clique number and underlying absolute clique, can be defined similarly as in the case of homomorphisms of signed graphs. Therefore, any question/problem raised regarding the parameters or objects associated to any of the graph types becomes a natural topic of study for the other types of graphs as well.

[1] These are $(0,1)$-colored mixed graphs.
[2] These are $(1,0)$-colored mixed graphs.
[3] These are $(0,2)$-colored mixed graphs.
[4] These are $(0,k)$-colored mixed graphs.

	(m,n)-colored mixed [5]	oriented	2-edge-colored	pushable	signed
χ	$\in [p^3+\epsilon p^2+p+1, 5p^4]$ [12, 22]	$\in [18,80]$ [15, 25]	$\in [20,80]$ [23, 1]	$\in [9,40]$ [27]	$\in [10,40]$ [23]
ω_{rel}	$\in [3p^2+p+1, 42p^2+8]$ [10]	$\in [15,50]$ [9]	$\in [15,50]$ [9]	$=8$ [11]	$=8$ [11]
ω_{abs}	$\in [3p^2+p+1, 9p^2+2p+2]$ [3]	$=15$ [17]	$=15$ [17]	$=8$ [19]	$=8$ [19]

Table 1: Known lower and upper bounds on different parameters for the family of planar graphs. Here χ, ω_{rel} and ω_{abs} denote the chromatic number, relative clique number and absolute clique number for the corresponding graph types and homomorphisms.

For example, in any of the contexts above, the question of finding the chromatic number for the family of planar graphs is a crucial, which is open for all the cases except for that of $(0,1)$-colored mixed graphs. Recall indeed that $(0,1)$-colored mixed graphs are nothing but simple graphs, and that the Four-Color Theorem solves that very question for them.

Question 1.1. *In any of the contexts above, what is the chromatic number for the family of planar graphs?*

In fact, answering Question 1.1 in any other context seems to be difficult. This explains why many efforts have been put into tackling the following seemingly simpler questions.

Question 1.2. *In any of the contexts above, what is the relative clique number for the family of planar graphs?*

Question 1.3. *In any of the contexts above, what is the absolute clique number for the family of planar graphs?*

The answer to Question 1.2 is trivial for the case of $(0,1)$-colored mixed graphs, while an answer for the case of signed graphs was given in [11]. A similar proof yields the similar bound for pushable homomorphisms as well. The answer to Question 1.3 is trivial for the case of $(0,1)$-colored mixed graphs, while an answer was given in the cases of $(1,0)$-colored mixed graphs [17], $(0,2)$-colored mixed graphs [26], pushable graphs [4] and signed graphs [19]. The questions remain open for the remaining cases [28, 9, 3]. Table 1 summarizes what is currently known to date regarding Questions 1.1, 1.2 and 1.3.

A way to get some sort of progress towards the questions above is to consider subclasses of planar graphs. In particular, outerplanar graphs have been attracting some attention. For the family of outerplanar graphs, exact values are known for every parameter and every type of graphs and homomorphisms listed in Table 1, except, in general, for the chromatic number of (m,n)-colored mixed graphs [28, 3]. Some of the values are actually easy to prove. Table 2 summarizes what is currently known to date regarding Questions 1.1, 1.2 and 1.3 for outerplanar graphs.

In the line of the previous three questions, a natural fourth one to ask, connected tightly to Question 1.3, is whether planar absolute cliques admit some nice characterization. This is where the notion of edge-critical underlying absolute cliques comes into play.

Question 1.4. *In any of the contexts above, what are the planar underlying absolute cliques?*

[5] Here, $p = 2m+n$ and $\epsilon = 1$ if $n = 0$ or n is odd, while $\epsilon = 2$ otherwise.

	(m,n)-colored mixed [6]	oriented	2-edge-colored	pushable	signed
χ	$\in [p^3+\epsilon p^2+p+1, 5p^4]$ [12, 22]	$=7$ [28]	$=9$ [1]	$=4$ [14]	$=5$ [19]
ω_{rel}	$=(3p^2+p+1)$ [3]	$=7$ [28]	$=7$ [26]	$=4$ [26]	$=4$ [26]
ω_{abs}	$=(3p^2+p+1)$ [3]	$=7$ [17]	$=7$ [17]	$=4$ [26]	$=4$ [26]

Table 2: Known lower and upper bounds on different parameters for the family of outerplanar graphs. Here χ, ω_{rel} and ω_{abs} denote the chromatic number, relative clique number and absolute clique number for the corresponding graph types and homomorphisms.

In the article [17] where Nandy, Sen and Sopena provided the answer to Question 1.3 for $(1,0)$-colored mixed graphs by showing that the absolute clique number of a planar $(1,0)$-graph can be at most 15, they also provided a list of all edge-critical underlying absolute outerplanar cliques and addressed the problem of finding a similar one for planar graphs. They also showed that there is a unique edge-critical underlying absolute planar clique [17] on 15 vertices for $(1,0)$-colored mixed graphs.

Of course the answer to Question 1.4 is trivial for $(0,1)$-colored mixed graphs. Among the other cases, until now such lists for planar graphs have been found only for pushable graphs [4] and an "almost exhaustive" computer-generated list for $(1,0)$-colored mixed graphs is reported in the Ph.D. thesis of Prabhu [24]. Here "almost exhaustive" means that the exhaustiveness of the list was computer checked and/or theoretically proved for all except the graphs having 14 vertices. Thus, in the list provided, it is possible that we may have to add a few more graphs having 14 vertices. Even such a list for outerplanar graphs is yet to be found for any case other than for $(1,0)$-colored mixed graphs. It is worth mentioning that for signed graphs the problem of deciding whether a given graph is an underlying absolute clique is known to be NP-hard [3].

Our contribution

In this article, we first exhibit, in Section 2, the exhaustive list of all edge-critical underlying absolute outerplanar cliques for $(0,2)$-colored mixed graphs. In other words, we answer Question 1.4 restricted to the family of outerplanar graphs for $(0,2)$-colored mixed graphs. After that, using this list, we exhibit, in Section 3, the exhaustive list of all edge-critical underlying absolute planar cliques for signed graphs. That is, we answer Question 1.4 for signed graphs. Finally, we discuss, in Section 4, the status of Question 1.1 for signed graphs.

2 Underlying absolute outerplanar cliques for $(0,2)$-colored mixed graphs

A signed graph is a $(0,2)$-*absolute clique* if any two non-adjacent vertices of (G,Σ) are connected by a 2-path (path of length 2) whose edges have different signs [3]. A simple graph G is an *underlying* $(0,2)$-*absolute clique* if there exists a signature Σ such that (G,Σ) is a $(0,2)$-absolute clique. Moreover, an underlying $(0,2)$-absolute clique G is *edge-critical* if for any edge $e \in E(G)$ the graph $G-e$ is not an underlying $(0,2)$-absolute clique.

[6] Here, $p=2m+n$ and $\epsilon=1$ if $n=0$ or n is odd, while $\epsilon=2$ otherwise.

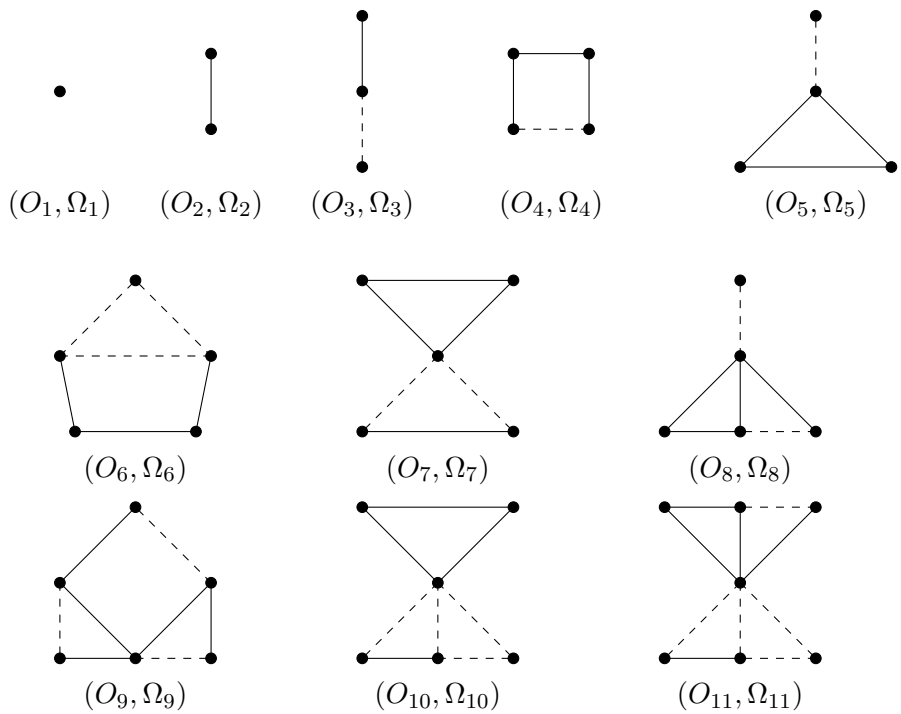

Figure 1: Exhaustive list of all edge-critical underlying $(0, 2)$-absolute outerplanar cliques along with a corresponding signature. Solid edges are positive edges, while dashed edges are negative edges.

To *contract* an edge uv of a graph G is to replace the vertices u, v with a new vertex w which is adjacent to all the vertices of $N(u) \cup N(v)$. A graph H is a *minor* of a graph G if it is possible to obtain H from G through a sequence of vertex deletions, edge deletions, and edge contractions. Finally, a graph is outerplanar if and only if it does not contain K_4 or $K_{2,3}$ as a minor [7].

Our main result in this section reads as follows:

Theorem 2.1. *There are exactly* 11 *edge-critical underlying* $(0, 2)$*-absolute outerplanar cliques. These* 11 *graphs are depicted in Figure 1.*

The proof of this theorem is contained in the upcoming lemmas. Before starting, let us present a few definitions and notations. Given a signed graph (G, Σ), two vertices u and v *see* each other if they are adjacent or are endpoints of a 2-path with edges having different signs. Also if u and v are endpoints of a 2-path with edges having different signs and the third vertex of the 2-path is w, then we say that u and v see each other *through* w.

Lemma 2.2. *If O is an underlying $(0, 2)$-absolute outerplanar clique having $|V(O)| = 7$, then O contains the graph O_{11} of Figure 1 as a spanning subgraph.*

Proof. Suppose that (O, Ω) is a $(0, 2)$-absolute outerplanar clique with $|V(O)| = 7$. As O is a connected outerplanar graph, O has a vertex u having degree 1 or 2.

Assume that there exists a degree-1 vertex u in O adjacent to its only neighbor v. Furthermore, let v be an α-neighbor of u for some $\{\alpha, \bar{\alpha}\} = \{+, -\}$. Therefore we must have

$N^{\bar{\alpha}}(v) = V(O) \setminus \{u, v\}$ in order for a vertex $x \in V(O) \setminus \{u, v\}$ to see u. Observe that two vertices from $V(O) \setminus \{u, v\}$ see each other without using u or v. Thus the graph induced by $N^{\bar{\alpha}}(v)$ is an underlying $(0, 2)$-absolute clique. Moreover it is a path as O is an outerplanar graph. We know from [3] that an underlying $(0, 2)$-absolute clique that is a forest can have at most three vertices, a contradiction.

Thus there exists a degree 2 vertex u in O adjacent to its only neighbors v and w. Furthermore, let v be an α-neighbor and w be a β-neighbor of u for some $\{\alpha, \bar{\alpha}\} = \{\beta, \bar{\beta}\} = \{+, -\}$. Therefore we must have $N^{\bar{\alpha}}(v) \cup N^{\bar{\beta}}(w) = V(O) \setminus \{u, v, w\}$ in order for a vertex $x \in V(O) \setminus \{u, v, w\}$ to see u. Note that $|N^{\bar{\alpha}}(v) \cap N^{\bar{\beta}}(w)| \leq 1$, as otherwise it would contain $K_{2,3}$ as a minor contradicting the fact that O is an outerplanar graph.

If $|N^{\bar{\alpha}}(v) \cap N^{\bar{\beta}}(w)| = 0$, then two vertices from $V(O) \setminus \{u, v, w\}$ see each other without using u, v or w. Thus, the vertices of $V(O) \setminus \{u, v, w\}$ induce an underlying $(0, 2)$-absolute clique on four vertices that is also a path. We know from [3] that an underlying $(0, 2)$-absolute clique that is a forest can have at most three vertices, a contradiction.

Hence we must have $|N^{\bar{\alpha}}(v) \cap N^{\bar{\beta}}(w)| = 1$. Moreover, let $N^{\bar{\alpha}}(v) \cap N^{\bar{\beta}}(w) = z$. Assume that $A = N^{\bar{\alpha}}(v) \setminus \{z\}$ and $B = N^{\bar{\beta}}(w) \setminus \{z\}$. Note that any edge between A and B would force $K_{2,3}$ to be a minor, contradicting the fact that O is an outerplanar graph.

Without loss of generality, assume that $|A| \geq |B|$. If $|B| > 0$, then the only way for the vertices of A to see the vertices of B is through z. Then $K_{2,3}$ is a minor because $|A \cup B| = 3$, contradicting the outerplanarity of O. Hence $|B| = 0$.

If $|B| = 0$, then $|A| = 3$. The only two options for a vertex $x \in A$ to see w are through v or z. If all three vertices of A see w through z, then there is $K_{2,3}$ as a minor, a contradiction. Thus at least one vertex from A sees w through v forcing the edge vw in O.

Observe that the vertices of A induce a forest. Furthermore, the vertices of A cannot see each other only through u, v, w. Also, it is not possible that all three vertices of A, see each other through z. Therefore, at least two vertices of A see each other through the third vertex of A forcing a 2-path in the graph induced by A.

However, this forces the graph O_{11} to be a subgraph of O. □

Next we handle the case where $|V(O)| \leq 6$.

Lemma 2.3. *If O is an underlying $(0, 2)$-absolute outerplanar clique having $|V(O)| \leq 6$, then O contains one of $O_1, \ldots, O_{10}$ as a spanning subgraph.*

Proof. Suppose that (O, Ω) is a $(0, 2)$-absolute outerplanar clique with $|V(O)| \leq 6$.

As O is connected, if $|V(O)| = 1, 2$ or 3, then O contains O_1, O_2 or O_3 as a spanning subgraph, respectively.

Next suppose that $|V(O)| \in \{4, 5, 6\}$. It is known that an outerplanar graph either have a cut-vertex or is Hamiltonian [7]. Thus we can consider that O is either Hamiltonian or has a cut-vertex v.

Let us first suppose that O has a cut-vertex v. Assume that $O - v$ has components $C_1, \ldots, C_k$. Then the vertices of C_i must see the vertices of C_j, for all $i \neq j$, through v in (O, Ω). Therefore, $k = 2$ and $V(C_1) \subseteq N^{\alpha}(u)$ and $V(C_2) \subseteq N^{\bar{\alpha}}(u)$ for some $\{\alpha, \bar{\alpha}\} = \{+, -\}$. Thus the graph induced by $V(C_i)$ from (O, Ω) is a $(0, 2)$-absolute clique for each $i \in \{1, 2\}$. Moreover, as O is outerplanar, all C_i's are paths. As we know from [3] that an underlying $(0, 2)$-absolute clique that is a forest can have at most three vertices, $|V(C_1)|, |V(C_2)| \leq 3$. Thus if O has a cut-vertex, then O must contain O_5, O_7, O_8 or O_{10} as a spanning subgraph.

Let us now assume that O is Hamiltonian.

- If $|V(O)| = 4$, then O must contain O_4 as a spanning subgraph.

- If $|V(O)| = 5$, then O contains a 5-cycle. However, any signature on a 5-cycle forces two incident edges of the same sign. The endpoints of the 2-path induced by those edges cannot see each other. Thus O must have at least one chord in this case, forcing O_6 as a spanning subgraph.
- If $|V(O)| = 6$, then O contains a 6-cycle. However, a 6-cycle does not have diameter 2. If we add some chords to a 6-cycle in order to construct an outerplanar graph having diameter 2, then we are forced to have O_9 as a spanning subgraph. □

Finally we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. First observe that the signed graphs (O_i, Ω_i) for all $i \in \{1, \ldots, 11\}$ are $(0, 2)$-absolute outerplanar cliques. Moreover, Bensmail, Duffy and Sen [3] have shown that if O is an underlying $(0, 2)$-absolute outerplanar clique, then $|V(O)| \leq 7$. From this, the proof now follows directly from Lemmas 2.2 and 2.3. □

3 Underlying absolute planar cliques for signed graphs

A positive (resp., negative) cycle of a signed graph is popularly refered to as a *balanced* (resp., *unbalanced*) cycle. It is known from [19] that a signed graph is a signed absolute clique if and only if any two non-adjacent vertices of the graph are part of an unbalanced 4-cycle. Furthermore, if a signed graph (G, Σ) is a signed absolute clique, then any $(G, \Sigma') \sim (G, \Sigma)$ is also a signed absolute clique [19].

Theorem 3.1. *There are exactly* 15 *edge-critical underlying signed absolute planar cliques. These* 15 *graphs are depicted in Figure 2.*

The proof of this theorem is contained in a series of observations and lemmas below. Before starting with those, let us present a few definitions and notations.

Let u, v be a pair of vertices of a simple graph G. We say that u *reaches* v if either u and v are adjacent or u and v have at least two common neighbors. Furthermore, when writing that u *reaches* v *through* w, we mean that w is a common neighbor of u and v. Also if any two distinct vertices of G reach each other, then we say that G is *reach-complete*. This motivates our first observation.

Observation 3.2. *An underlying signed absolute clique is reach-complete.*

Recall that, for a simple graph G, the parameter $\delta(G)$ denotes its *minimum degree*.

Observation 3.3. *A reach-complete graph G cannot have a cut-vertex. In particular, if $|V(G)| \geq 3$, then $\delta(G) \geq 2$.*

Proof. If a reach-complete graph G has a cut-vertex v, then the vertices from different components of $G - v$ are neither adjacent nor have at least two common neighbors in G. Now, if $|V(G)| \geq 3$ and $\delta(G) = 1$, then the neighbor of a vertex having degree 1 is a cut-vertex. □

Note that a reach-complete graph is either a complete graph or contains a 4-cycle. Thus the following is immediate.

Lemma 3.4. *If (H, Π) is a signed absolute planar clique having $|V(H)| \leq 4$, then H contains one of A_1, A_2, A_3, A_4 as a spanning subgraph.*

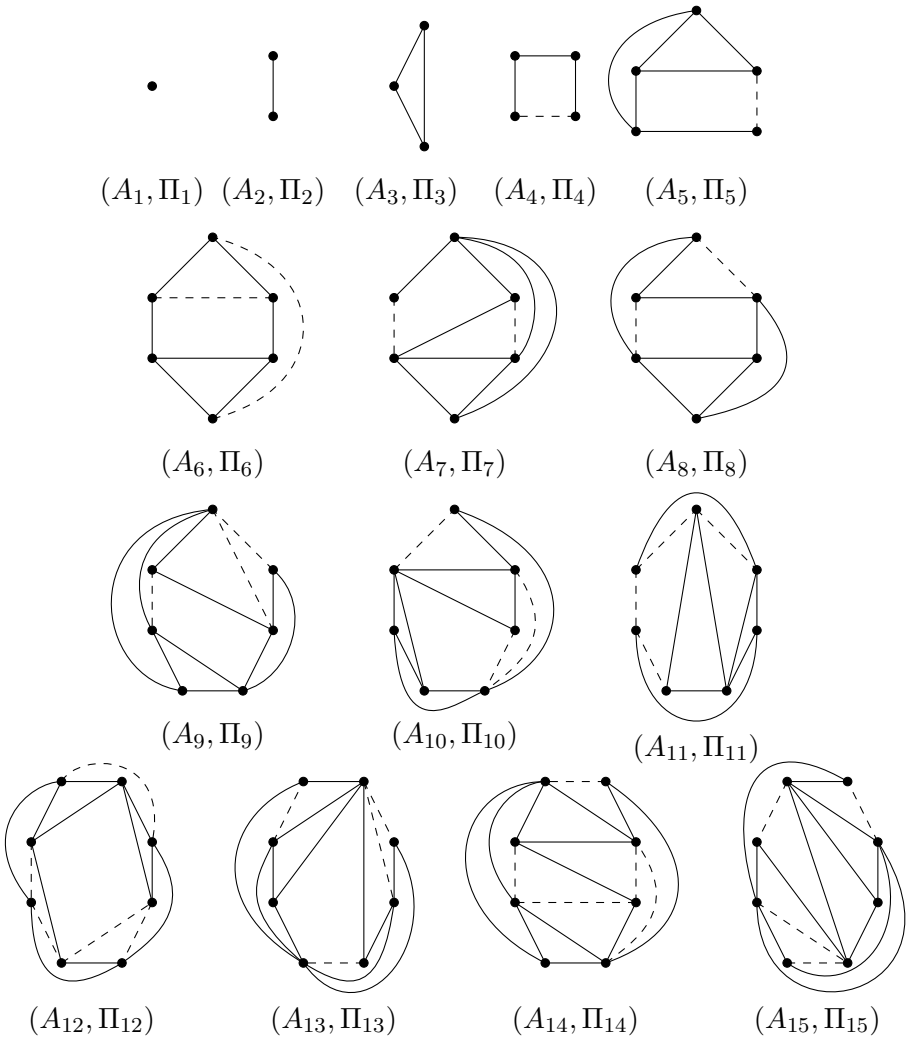

Figure 2: Exhaustive list of all edge-critical underlying signed absolute planar cliques along with a corresponding signature. Solid edges are positive edges, while dashed edges are negative edges.

Now let us consider the case of graphs having five vertices.

Lemma 3.5. *If (H, Π) is a signed absolute planar clique having $|V(H)| = 5$, then H contains A_5 as a spanning subgraph.*

Proof. As H does not have a cut-vertex, it contains either a 5-cycle or $K_{2,3}$ as a subgraph.

- We know that a 5-cycle is not an underlying $(0, 2)$-absolute clique, and thus obviously not an underlying signed absolute clique. Therefore, we need to add at least one chord to make the graph an underlying signed absolute clique. Without loss of generality assume that we have added the chord be. Note that the vertex a is non-adjacent to both c and d. Moreover, both a, c and a, d have at exactly one common neighbor. Thus we need to add some more chords. If we add either ac or ad, then the subgraph A_5 is created. The other way is to add both ce and bd, in which case also the subgraph A_5 is created.

- $K_{2,3}$ is not an underlying signed absolute clique as there exist two vertices from the partite set of cardinality 3 that are neither adjacent nor part of an unbalanced 4-cycle. Thus we need to add at least one edge in the concerned partite set. This creates the subgraph A_5. □

We already know that if (H, Π) is a signed absolute planar clique having $|V(H)| \geq 6$, then $\delta(H) \geq 2$. First we analyse the case when $\delta(H) = 2$.

Lemma 3.6. *If (H, Π) is a signed absolute planar clique having $|V(H)| \geq 6$ and $\delta(H) = 2$, then H contains A_7 as a spanning subgraph.*

Proof. Let u be a degree-2 vertex of H having exactly two neighbors v and w. Then each non-neighbor x of u must be adjacent to both v and w, resulting in a $K_{2,|V(G)|-2}$. Assume that $V(H) \setminus \{u, v, w\} = \{x_1, \ldots, x_t\}$. For being a signed absolute clique, uvx_iwu must be an unbalanced 4-cycle for each $i \in \{1, \ldots, t\}$. Furthermore, x_k and x_l, for some $i \neq j$, have exactly two common neighbors v and w. As both uvx_kwu and uvx_lwu are unbalanced 4-cycles, the 4-cycle vx_kwx_lv is balanced. Thus any two non-neighbors of v are not part of any common unbalanced 4-cycle in (H, Π).

Note that $t \geq 3$ since $|V(H)| \geq 6$. Furthermore, if $t \geq 4$, then x_1 and x_4 cannot have any common neighbor other than v and w. But, as observed earlier, the 4-cycle induced by vx_1wx_4v is balanced. Thus $t = 3$.

Now note that the only common neighbor of x_1 and x_3, other than v, w, is x_2. As we know that the 4-cycle vx_1wx_3v is balanced, the only way for (H, Π) to be a signed absolute planar clique is to have the 2-path $x_1x_2x_3$ with exactly one negative edge. Thus H contains A_7 as a subgraph. □

Recall that Theorem 2.1 provided the list of all edge-critical underlying $(0, 2)$-absolute outerplanar cliques (in Figure 1). If an signed absolute planar clique (H, Π) has a dominating vertex v, then we can switch the negative neighbors of v to obtain a new signature Π^* of H. Observe that signed absolute cliques are invariant under switching. Thus (H, Π^*) is also a signed absolute planar clique. Observe that the signed graph obtained by deleting v from (H, Π^*) is a $(0, 2)$-absolute outerplanar clique. We denote by (H^+, Π^+) be the signed graph obtained by adding a new vertex v to (H, Π) such that v is adjacent to every other vertex through a positive edge.

We are now ready to present our next observation.

Observation 3.7. *A signed graph (H, Π) is a $(0,2)$-absolute outerplanar clique if and only if (H^+, Π^+) is a signed absolute planar clique.*

Recall that (O_k, Ω_k) denotes the kth graph in Figure 1. Then:

Observation 3.8. *If $\sqsubseteq$ denotes spanning subgraph inclusion, then $A_2 \sqsubseteq O_1^+$, $A_3 \sqsubseteq O_2^+$, $A_4 \sqsubseteq O_3^+$, $A_5 \sqsubseteq O_4^+$, $A_5 \sqsubseteq O_5^+$, $A_6 \sqsubseteq O_6^+$, $A_8 \sqsubseteq O_7^+$, $A_7 \sqsubseteq O_8^+$, $A_9 \sqsubseteq O_9^+$, $A_{10} \sqsubseteq O_{10}^+$, $A_{13} \sqsubseteq O_{11}^+$.*

A *dominating set* S of a graph G is a vertex subset such that any vertex from $V(G) \setminus S$ has a neighbor in S. The *domination number* of G is the minimum $|S|$ where S is a dominating set. Due to the previous observations, we need to only focus on edge-critical underlying signed absolute planar cliques having at least six vertices, minimum degree 3, and domination number at least 2.

Now let us introduce a few graphs. Let P_2 and P_4 denote the paths on two and four vertices, respectively. Let A be the graph obtained by taking two vertices ∞ and $-\infty$ along with the disjoint union of P_2 and P_4, and making each of $\infty, -\infty$ adjacent to each vertex of P_2 and P_4. Also let B be the graph obtained from A by adding an edge between the vertices ∞ and $-\infty$. Furthermore, let C denote the 5-*wheel graph* obtained by taking a 5-cycle and a vertex ∞ and adding edges between ∞ and each of the vertices of the 5-cycle.

Lemma 3.9. *The graphs A, B and C are not underlying signed absolute cliques.*

Proof. Observe that A is a subgraph of B. Thus if we prove that B is not an underlying signed absolute clique, then it will imply that A is also not an underlying signed absolute clique.

Assume that B is an underlying signed absolute clique and Σ is a signature of B such that (B, Σ) is a signed absolute clique. Now switch the negative neighbors of ∞ in (B, Σ) to obtain the signed graph (B, Σ^*). If both edges between $-\infty$ and the vertices of P_2 are negative, then switch $-\infty$ as well. Note that the edges incident to ∞ are all positive and one edge between $-\infty$ and a vertex x of P_2 is also positive. We know that (B, Σ^*) is also a signed absolute clique. Therefore, in order for x to reach the vertices of P_4, the edges between $-\infty$ and the vertices of P_4 must be negative. In this case, the endpoints of P_4 cannot be part of a common unbalanced 4-cycle.

Observe that C is a planar graph. If C is an underlying signed absolute planar clique, then by Observation 3.7 a 5-cycle is an underlying $(0,2)$-absolute outerplanar clique. This is a contradiction due to Theorem 2.1. □

As A, B and C are not underlying signed absolute planar cliques, one may wonder what happens if we add some edges to them. If we add any edge to A to obtain a planar graph A^*, then A^* contains A_{13} as a spanning subgraph. Similarly, if we add any edge to C to obtain a graph C^*, then C^* contains A_6 as a spanning subgraph. As A is a subgraph of B, we have the following lemma.

Lemma 3.10. *If an underlying signed absolute planar clique H contains A, B or C as a spanning subgraph, then H must contain either A_6 or A_{13} as a spanning subgraph.*

Proof. Suppose that (H, Π) an underlying signed absolute planar clique and H contains C as a spanning subgraph. Note that H contains a dominating vertex ∞. Let (H', Π') be the signed graph obtained by deleting ∞ from (H, Π). Notice that H' is an outerplanar graph. Due to Observation 3.7, we know that (H', Π') must be an underlying $(0,2)$-absolute outerplanar clique. Therefore H' must contain O_6 (from Figure 1) as a spanning subgraph due to Theorem 2.1. Hence H contains C as a planar subgraph.

Suppose that (H, Π) an underlying signed absolute planar clique and H contains A as a spanning subgraph. Note that, as A is a subgraph of B, it is enough to show that H contains A_{13}.

Let us first understand the graph A_{13}. In fact, let us understand the difference between A_{13} and A.

Take the bipartite graph $K_{2,6}$ with smaller partite set $\{u, v\}$ and bigger partite set $\{x_1, x_2, \ldots, x_6\}$. Now if we add the edges $x_1x_2, x_2x_3, x_4x_5, x_5x_6$, then we obtain A_{13}. Instead, if we add the edges $x_1x_2, x_2x_3, x_3x_4, x_5x_6$, then we obtain A.

We already know that A is not an underlying signed absolute clique. Thus H has some edges which are not in A. If we add the edge uv, then we obtain the graph B. We know that B is also not an underlying signed absolute clique. Therefore, H must be having some additional edge of the form x_ix_j. However, as H is planar, the only additional edge of the form x_ix_j must have, assuming $i < j$ without loss of generality, $i \in \{1, 4\}$ and $j \in \{5, 6\}$. Hence, without loss of generality, assume that the additional edge is x_4x_5. This graph already contains A_{13}. In fact, if we delete the edges x_3x_4 and uv (if present) from this graph, what we obtain is exactly A_{13}. □

Finally, we present the final lemma needed for the proof of Theorem 3.1 which was implicitly proved by Bensmail, Nandi and Sen [4] (mind the erratum [5] as well).

Lemma 3.11. *If H is a planar edge-critical reach-complete graph having six, seven or eight vertices, minimum degree at least 3 and domination number at least 2, then H contains one of $A_6, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A, B, C$ as a spanning subgraph.*

Proof. This proof is implicitly present in Section 5 of [4]. Let H be as in the statement.

- If $|V(H)| = 6$, then Lemma 5.6 of [4] proves that H contains one of A_6, A_7, A_8, C (which are the graphs H_7, H_6, H_9, H_8 in [4], respectively).
- If $|V(H)| = 7$, then Lemma 5.9 of [4] proves that H contains one of A_9, A_{10}, A_{11} (which are the graphs H_{10}, H_{11}, H_{12} in [4], respectively).
- If $|V(H)| = 8$, then Lemma 5.9 of [4] proves that H contains one of $A_{12}, A_{13}, A_{14}, A_{15}, A$ (which are the graphs $H_{13}, H_{14}, H_{15}, H_{16}, A$ in [4], respectively). □

Finally we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. First observe that the signed graphs (A_i, Π_i) for all $i \in \{1, \ldots, 15\}$ are signed absolute planar cliques. Moreover, Naserasr, Rollová and Sopena [19] have shown that if (H, Π) is a signed absolute planar clique, then $|V(H)| \leq 8$. Thus the proof follows directly from the observations and lemmas proved in this section. □

4 Concluding discussion and remarks

Using our results, we managed to exhibit a computer-generated list of all underlying absolute planar cliques for signed graphs. We report that there are, in total, 47 non-isomorphic underlying signed absolute planar cliques (1 on 1 vertex, 1 on 2 vertices, 1 on 3 vertices, 3 on 4 vertices, 4 on 5 vertices, 10 on 6 vertices, 14 on 7 vertices and 13 on 8 vertices). See the lists in the webpage http://jbensmai.fr/code/signed/ for details.

An interesting aspect of Figure 2 is that its first four graphs are outerplanar as well. As there is no other outerplanar graph in the list, this implies the following corollary.

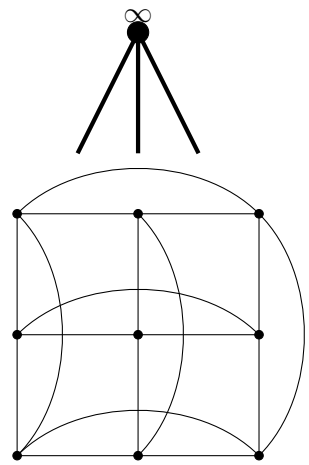

Figure 3: The positive edges of the complete signed graph (P_9^+, Γ^+) are depicted here. The vertex ∞ is adjacent to all the other vertices with a positive edge. The thick edges from it refers to that. All the non-edges in the picture should be replaced by negative edges in order to obtain the signed graph (P_9^+, Γ^+).

Corollary 4.1. *There are exactly* 4 *edge-critical underlying signed absolute outerplanar cliques. These* 4 *graphs are the first* 4 *graphs depicted in Figure 2.*

Furthermore, it is worth mentioning that the list of all edge-critical underlying signed absolute planar cliques for pushable graphs have exactly one more graph than the ones depicted in Figure 2, namely the 5-wheel graph. This shows yet another difference between the two contexts.

Now let us discuss the status of the analogue of the Four-Color Theorem for signed graphs, that is, what is $\chi_s(\mathcal{P})$ for the family $\mathcal{P}$ of planar graphs. It is known that $10 \leq \chi_s(\mathcal{P}) \leq 40$ to date [23]. Furthermore, it is also known [23] that $\chi_s(\mathcal{P}) = 10$ if and only if every signed planar graph admits a homomorphism to a particular signed graph called *signed Paley plus graph* (P_9^+, Γ^+) on 10 vertices, which is depicted in Figure 3.

Let us mention which planar graphs are known to admit a homomorphism to (P_9^+, Γ^+). It is known that all planar signed graphs having no cycle of length at most 4 admit a homomorphism to (P_9^+, Γ^+) [23].

Let (H, Π') be a signed graph obtained by switching a set of vertices of (H, Π). We know that $(G, \Sigma) \to (H, \Pi)$ if and only if $(G, \Sigma) \to (H, \Pi')$ [19]. It is known that $(K_4, \emptyset)$ is a subgraph of (P_9^+, Γ^+) and it is possible to switch a set of vertices of (P_9^+, Γ^+) so that the so-obtained signed graph contains $(K_4, E(K_4))$ as a subgraph [23]. Therefore, the Four-Color Theorem implies that for any planar graph P, the signed graphs $(P, \emptyset)$ and $(P, E(P))$ admit a homomorphism to (P_9^+, Γ^+).

A bijective homomorphism of a signed graph to itself whose inverse is also a homomorphism is an *automorphism*. Given any two balanced (respectively, unbalanced) 3-cycles C_1 and C_2 of (G, Σ) if any bijection from $V(C_1)$ to $V(C_2)$ can be extended to an automorphism of (G, Σ), then (G, Σ) is *triangle-transitive*. It is known that the signed graph (P_9^+, Γ^+) is triangle-transitive.

Moreover Ochem, Pinlou and Sen [23] have observed (through computer) that all planar signed graphs having at most 15 vertices admit a homomorphism to (P_9^+, Γ^+). In particular, all signed absolute planar cliques also admit a homomorphism to it. Moreover, due to the triangle-transitive property of (P_9^+, Γ^+), any planar graph obtained by (repeated) *k-clique-sums* (gluing k-cliques of two graphs to obtain a new graph) of planar graphs having 15 or less vertices also

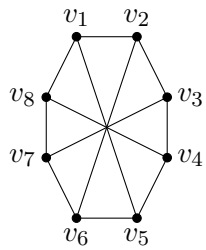

Figure 4: The Wagner graph W_8.

admits a homomorphism (P_9^+, Γ^+) for $k \leq 3$.

These accumulated observations make us confident enough to make the following conjecture:

Conjecture 4.2. *Every planar signed graph admits a homomorphism to* (P_9^+, Γ^+).

In particular, the conjecture, if true, would imply $\chi_s(\mathcal{P}) = 10$.

Recall that long before the Four-Color Theorem was proved, Wagner showed in [29] that if all planar graphs admit a 4-coloring, then so do all K_5-minor-free graphs. He used the following characterization to prove this: G is a K_5-minor-free graph if and only if G is a subgraph of a graph obtained by (repeated) k-clique-sums of graphs belonging to $\mathcal{P} \cup \{W_8\}$ for $k \leq 3$, where W_8 is the Wagner graph depicted in Figure 4. Due to the triangle-transitive property of (P_9^+, Γ^+) we are able to show something similar. In fact, the only non-trivial part is to show that $(W_8, \Sigma) \to (P_9^+, \Gamma^+)$ for any signature Σ of W_8.

Lemma 4.3. *For any signature* Σ *of* W_8*, we have* $(W_8, \Sigma) \to (P_9^+, \Gamma^+)$.

Proof. Let (X, Λ) be the signed graph obtained by adding the following edges to (W_8, Σ): v_1v_3, v_1v_4, v_1v_6 and v_4v_7. Moreover, the new edges are assigned signs in such a way that the cycle $v_1v_3v_7v_6v_1$ is an unbalanced 4-cycle in (X, Λ).

Let (X', Λ') be the signed graph obtained from (X, Λ) by deleting the vertex v_8. Note that it is a planar graph having 8 vertices. Thus there exists a homomorphism $f : (X, \Lambda) \to (P_9^+, \Gamma^+)$. As $v_1v_3v_7v_6v_1$ is an unbalanced 4-cycle, the images $f(v_1), f(v_4)$ and $f(v_7)$ are distinct vertices of (P_9^+, Γ^+). Thus the homomorphism f can be extended to a homomorphism of (X, Λ) to (P_9^+, Γ^+) due the following property (see Lemma 2.10(2) of [23]): given any three distinct vertices u, v, w of (P_9^+, Γ^+) either $N^\alpha(u) \cap N^\beta(v) \cap N^\gamma(w) \neq \emptyset$ or $N^{\bar{\alpha}}(u) \cap N^{\bar{\beta}}(v) \cap N^{\bar{\gamma}}(w) \neq \emptyset$ where $\{\alpha, \bar{\alpha}\} = \{\beta, \bar{\beta}\} = \{\gamma, \bar{\gamma}\} = \{+, -\}$.

As (W_8, Σ) is a subgraph of (X, Λ), we are done. □

Therefore, we indeed have the analogue of Wagner's result.

Proposition 4.4. *Every* K_5*-minor-free signed graph admits a homomorphism to* (P_9^+, Γ^+) *if and only if* $\chi_s(\mathcal{P}) = 10$.

We hope that the above Wagner-like proposition may help formulating a Hadwiger-like conjecture for signed graphs in the future.

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