A geometric stabilization of planar switched systems

Cyrille Chenavier, Rosane Ushirobira, Giorgio Valmorbida

A geometric stabilization of planar switched systems ⋆

Cyrille Chenavier * **Rosane Ushirobira** * **Giorgio Valmorbida** **

* *Inria, Univ. Lille, CNRS, UMR 9189 - CRIStAL, F-59000 Lille, France. (e-mail: cyrille.chenavier@inria.fr, rosane.ushirobira@inria.fr).*
** *L2S, CentraleSupélec, CNRS, Université Paris-Saclay, Gif-sur- Yvette 91192, France. Inria projet DISCO, giorgio.valmorbida@centralesupelec.fr*

Abstract: In this paper, we investigate a particular class of switching functions between two linear systems in the plan. The considered functions are defined in terms of geometric constructions. More precisely, we introduce two criteria for proving uniform stability of such functions, both criteria are based on the construction of a Lyapunov function. The first criterion is constructed in terms of an algebraic reformulation of the problem and linear matrix inequalities. The second one is purely geometric. Finally, we illustrate these methods with a numerical example.

Keywords: Linear systems, switching functions, stabilization, algebraic approaches, geometric approaches.

1. INTRODUCTION

A switched system is a (continuous or discrete-time) dynamical system composed of a finite number of subsystems together with a rule, called the switching function or the switching law, that orchestrates the switching between subsystems. Such systems have been studied in various areas of control theory. In particular, stability DeCarlo et al. (2000); Liberzon and Morse (1999); Molchanov and Pyatnitskiy (1989), controllability Sun (2006); Sun et al. (2002), observability Bemporad et al. (2000); Egerstedt and Babaali (2005); Hespanha et al. (2005), stabilization Johansson (2003); Pettersson (2003); Sun and Ge (2005), optimal control Bemporad and Morari (1999); Xu and Antsaklis (2004), aperiodic sampling Hetel et al. (2017), or for discrete-time delay systems Hetel et al. (2017); Fridman (2014), for instance.

Stability and stabilization problems generally consist in searching for or proving that a class of switching functions induce, for every initial condition, a convergent trajectory, see Lin and Antsaklis (2005); Sun and Ge (2005). Numerical methods for proving stability results are based on linear matrix inequalities (LMI) and Lyapunov functions or their adaptations, such as, quasi-quadratic Hu and Blanchini (2010), parameter dependent Daafouz et al. (2002), path-dependent Lee and Dullerud (2006), non-monotonic Athanasopoulos and Lazar (2014); Megretski (1997); Bliman and Ferrari-Trecate (2003); Kruszewski et al. (2008); Ahmadi and Parrilo (2008), with an augmented state vector Gomide and Lacerda (2018), composite quadratic Hu and Blanchini (2010); Hetel et al. (2011) Lyapunov functions or using a Gaussian elimination procedure Aleksandrov et al. (2011).

In this work, we are interested in linear planar switched systems with two subsystems. A complete stability analysis of such systems was done using algebraic invariants, namely the traces and the determinants of the two matrices of the two subsystems Balde et al. (2009). In the present paper, we are interested in stabilization problems, where we wish to construct a stable switching function using geometric methods. Indeed, our original goal is to partition the plane into four regions using two distinct lines passing through the origin, and define the switching signal as being the same on two adjacent regions, but alternating while passing from a region to another, see Fig. 1. A different problem is studied here: we assume that two lines passing through the origin are given, and we search for criteria to guarantee that the switching signal defined by these two lines is globally stabilizing, that is, the trajectories induced by this signal converge to the origin. Two such criteria are proposed, both of them involve the design of Lyapunov functions. The first one is based on an algebraic reformulation of the problem in terms of quadratic forms, and the Lyapunov function is constructed in terms of solutions of linear matrix inequalities. The second approach uses purely geometric tools, and the Lyapunov function is constructed in terms of the existence of a parallelogram such that the trajectories are contracted along them. As an illustration of this method we recover an example treated in Lin and Antsaklis (2007).

The paper is organized as follows. In Section 2, we recall general notions on continuous-time switched systems, and classical Lyapunov functions properties to prove global exponential stability of such systems. In Section 3, we formulate our general problem and give two reformulations that are used in the sequel. In Section 4, our two main results are proven: two sufficient conditions, one algebraic and the other geometric, for the existence of a solution to the general problem. In Section 5, we illustrate the two conditions with a complete numerical example.

2. PRELIMINARIES

Let us begin with some notation. For n, m two strictly positive integers, the set of real $n \times m$-matrices is denoted by $\mathbb{R}^{n\times m}$. Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^n$. For a matrix $P \in \mathbb{R}^{n\times n}$, we write $P > 0$ if P is a positive definite matrix and $P < 0$ if P is a negative definite matrix.

⋆ .

In this short section, we recall some general notions of continuous-time switched linear systems that are used in the sequel.

Consider n, p, two strictly positive integers, and a finite set of matrices $\mathbf{A} = \{A_i \in \mathbb{R}^{n\times n} \mid 1 \le i \le p\}$. A continuous-time switched linear system is described by:

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad t \in \mathbb{R}_{>0}, \quad x(0) = x_0 \in \mathbb{R}^n \tag{1}$$

where $x : \mathbb{R}_{>0} \to \mathbb{R}^n$ represents the system state, x_0 is the initial condition and $\sigma : \mathbb{R}_{>0} \to \{1, \cdots, p\}$ is a switching function. The flow associated to σ is denoted by $(t, x_0) \mapsto \phi_\sigma(t, x_0)$.

Definition 1. Given a switching function σ, the equilibrium point $x = 0$ of the switched linear system (1) is said to be *globally exponentially stable* if there exist constant $c > 0$ and $\lambda > 0$ such that

$$\|\phi_\sigma(t, x_0)\|^2 \le ce^{-\lambda t}\|x_0\|^2 \tag{2}$$

holds for all initial conditions $x_0 \in \mathbb{R}^n$ and all $t \in \mathbb{R}_{>0}$. In this situation, we also say that the system (1) is globally exponentially stable.

In the sequel, we will work with systems of the form (1) with $n = p = 2$: so $\mathbf{A} = \{A_1, A_2 \in \mathbb{R}^{2\times 2}\}$ and

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad t \in \mathbb{R}_{>0}, \quad x(0) = x_0 \in \mathbb{R}^2 \tag{3}$$

wit $\sigma : \mathbb{R}_{>0} \to \{1, 2\}$.

Our goal is to provide algebraic and geometric conditions on the switching function σ to prove that (3) is globally exponentially stable. Our proofs are based on the design of Lyapunov functions (so, positive functions $V : \mathbb{R}^2 \to \mathbb{R}_{\ge 0}$, with the property that V is decreasing along trajectories of (3)). Indeed, recall that the existence of such a function guarantees global exponential stability.

3. GENERAL PROBLEM

For two angles θ_1, $\theta_2 \in [0, 2\pi)$ such that $\theta_1 \neq \theta_2 \mod (\pi)$, denote by $\mathcal{D}_{\theta_i}$ the unique line passing through the origin making an angle $\theta_i \mod (\pi)$ with the x_1-axis.

Our general problem can be stated as following: given two square matrices $A_1, A_2 \in \mathbb{R}^{2\times 2}$, do there exist angles θ_1 and θ_2 as above such that (3) is globally exponentially stable, with the switching function $\sigma : \mathbb{R}_{>0} \to \{1, 2\}$ defined by $\sigma(t) = i$ if $x(t) \in \mathcal{O}_i$, where $\mathcal{O}_1$, $\mathcal{O}_2$ are the two regions determined respectively by $\mathcal{D}_{\theta_1}$ and $\mathcal{D}_{\theta_2}$, and pictured in Fig. 1, and $\sigma(t) \in \{1, 2\}$ if $x(t) \in \overline{\mathcal{O}}_i$, where $\overline{\mathcal{O}}_i$ ($i = 1$, 2) denote the topological closures. Remark that the lines $\mathcal{D}_{\theta_i}$ form the intersection of the two regions $\overline{\mathcal{O}}_i$, $i = 1, 2$. Moreover, we also point out that the switching function is not well-defined on these two lines. However, in our main results stated in Section 4, we assume that there is no sliding motion along trajectories, which is sufficient for σ to be well-defined. For more details, see Definition 5 and the discussion after this definition.

Next, we will use the following terminology:

Definition 2. We say that a pair of angles (θ_1, θ_2) is *stabilizing* for (A_1, A_2) if it is a solution to the general problem.

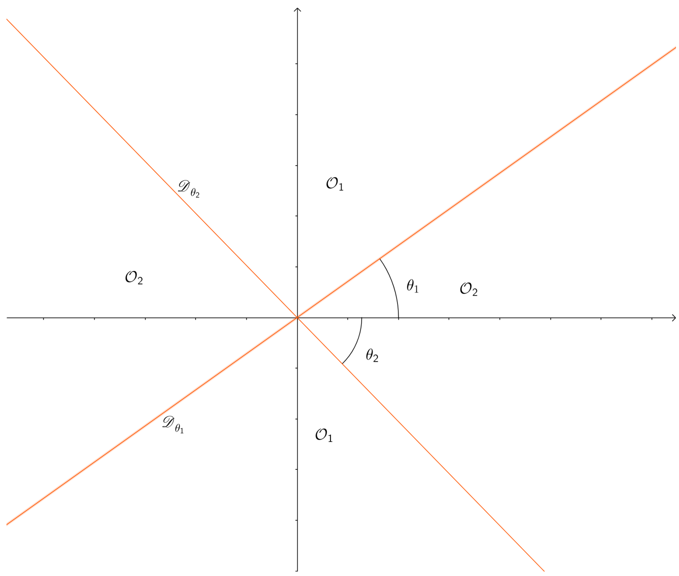

Fig. 1. Definition of regions $\mathcal{O}_1$ and $\mathcal{O}_2$

In the next section, two sufficient conditions are presented such that a given pair of angles (θ_1, θ_2) is stabilizing for (A_1, A_2). The first sufficient condition, given in Section 4.1, is algebraic, and the second one, given in Section 4.2, is geometric. In these sections, we use reformulations of the problem by giving other descriptions of Fig. 1.

3.1 Algebraic formulation

The algebraic reformulation of Section 4.1 is based on the following proposition.

Proposition 3. There exists a one-to-one correspondence between pairs of angles (θ_1, θ_2) such that $\theta_1 \neq \theta_2 \mod (\pi)$ and real symmetric matrices R with eigenvalues $(\lambda, -1)$, where $\lambda > 0$.

Proof. It is easy to see that pairs of angles that are different modulo π are in bijective correspondence with pairs of distinct lines passing through the origin. Given such a pair of distinct lines, there exists a basis composed of orthonormal unit vectors and a scalar $a \neq 0$ such that the two lines are represented by the two equations

$$\widetilde{x}_2 = a\widetilde{x}_1, \quad \widetilde{x}_2 = -a\widetilde{x}_1. \tag{4}$$

In this coordinate system, these lines are solutions of the equation $\widetilde{x}^T\widetilde{R}\widetilde{x} = 0$, where

$$\widetilde{R} = \begin{pmatrix} a^2 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

The matrix $\widetilde{R}$ has eigenvalues -1 and $a^2 > 0$. In the original coordinate system, the two lines are solutions of $x^TRx = 0$, where $R = (R_\theta)^T\widetilde{R}R_\theta$, with R_θ the rotation matrix of angle θ corresponding to the change of coordinates. So R has the same eigenvalues that $\widetilde{R}$, which proves one implication. Conversely, let R be a real symmetric matrix with eigenvalues $\lambda > 0$ and -1, so that R admits a diagonal form $\widetilde{R}$ such has (5). In the coordinate system corresponding to this diagonal form, the two lines with equations such as in (4) are distinct, which proves the other direction.

Using the previous notation, from Proposition 3, we deduce

$$\mathcal{D}_{\theta_1} \cup \mathcal{D}_{\theta_2} = \left\{x \in \mathbb{R}^2 \mid x^TRx = 0\right\}.$$

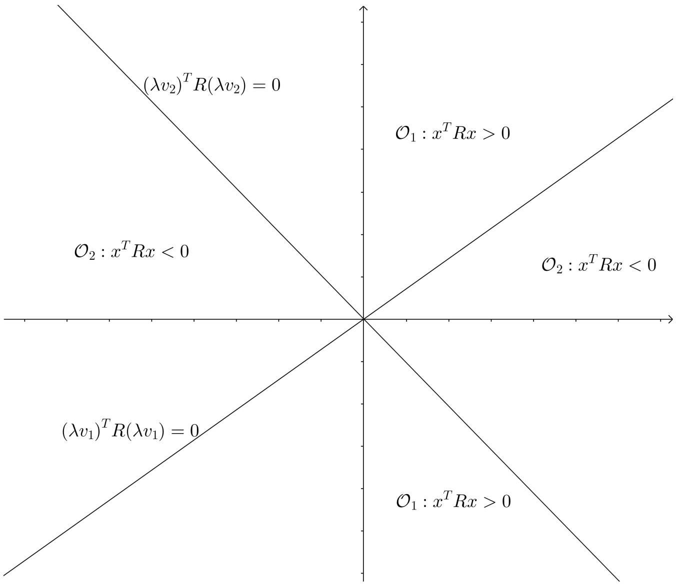

Fig. 2. Definition of matrices dynamics

Notice that if we replace R by the matrix $\tilde{R} = \lambda R$, with $\lambda \neq 0$, then $x^T Rx = 0$ is equivalent to $x^T \tilde{R}x = 0$ and that R has eigenvalues $\lambda > 0$ and -1 if and only if $\tilde{R}$ has a strictly positive and a strictly negative eigenvalue. Moreover, a real symmetric matrix with a strictly positive and a strictly negative eigenvalue is nothing but a real symmetric matrix with a strictly negative determinant. Hence, by the Proposition 3 an algebraic characterization of Fig.1 can be proposed. Given a real symmetric matrix R with a strictly negative determinant, let v_1 and v_2 be two linearly independent vectors [1] that are solutions to the quadratic equation $x^T Rx = 0$. The vector v_i is a direction vector of $\mathcal{D}_{\theta_i}$ $(i = 1,\ 2)$, so if we allow to change R by $-R$, we may assume that $\mathcal{O}_1$ is the region where the quadratic form $x^T Rx$ is strictly positive, and $\mathcal{O}_2$ is the region where it is negative. Hence, Fig. 1 corresponds to the diagram given in Fig. 2.

3.2 Geometric formulation

The geometric reformulation of Fig. 1 consists in starting with the lines $\mathcal{D}_{\theta_1}$ and $\mathcal{D}_{\theta_2}$ instead of the angles, that is, we consider two different lines $\mathcal{D}_1$ and $\mathcal{D}_2$ passing through the origin, and we denote by $\mathcal{O}_1$ and $\mathcal{O}_2$ the induced regions. This is pictured in Fig. 3.

To finish this section, it remains to adapt Definition 2 to our new situations.

Definition 4. (1) We say that a real symmetric matrix with strictly negative determinant is *stabilizing* for (A_1, A_2) if the corresponding pair of angles by Proposition 3 is stabilizing for (A_1, A_2).

(2) We say that a pair of lines $(\mathcal{D}_1, \mathcal{D}_2)$ is *stabilizing* for (A_1, A_2) if the pair of angles they define with the x_1-axis is stabilizing for (A_1, A_2).

4. MAIN RESULTS

In this section, we establish our criteria for proving stability. Both of these criteria require that no *sliding motion* occur. We first recall this notion.

[1] That is, they are not pointing in the same direction.

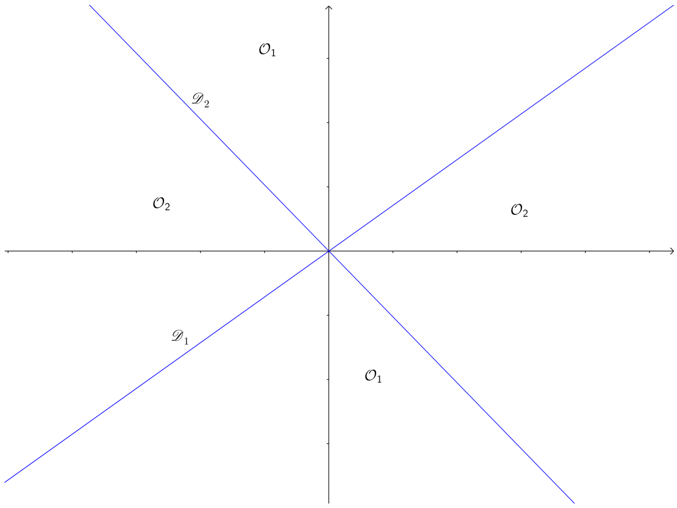

Fig. 3. Regions delimited by lines $\mathcal{D}_1$ and $\mathcal{D}_2$

Definition 5. Let $\mathcal{D}$ be a line passing through the origin and let $p \in \mathcal{D}$. Let d be an orthogonal vector at $\mathcal{D}$ in p. We say that *no sliding motion* occurs at p if

$$(d^\top A_1 d)(d^\top A_2 d) > 0.$$

If no sliding motion occurs at any point of $\mathcal{D}$, we say that $\mathcal{D}$ induces *no sliding motion* for (A_1, A_2).

The previous definition means that both matrices A_1 and A_2 point in the same direction at every point of $\mathcal{D}$, that is, the trajectory defined by the matrices necessarily cross $\mathcal{D}$. Equivalently, that means that no trajectory slides along $\mathcal{D}$.

4.1 A sufficient algebraic condition

In this subsection, we fix a symmetric matrix R with strictly negative determinant as well as two linearly independent vectors v_1 and v_2, solutions to $x^T Rx = 0$, such that the lines passing through the origin and oriented by v_i $(i = 1,\ 2)$ induce no sliding motion. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the regions they define,respectively, and let $\overline{\mathcal{O}}_1$, $\overline{\mathcal{O}}_2$ be their topological closures, see Fig. 2. Let us consider the system (3) defined in the previous section.

Our sufficient condition for the matrix R to be stabilizing for (A_1, A_2) is based on the construction of a piecewise Lyapunov function. The construction of the latter is based on the existence of a solution to a linear matrix inequalities (LMI) problem.

Theorem 6. Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$, let R be a symmetric matrix with strictly negative determinant and let v_1 and v_2 be two linearly independent solutions to $x^T Rx = 0$ such that the lines passing through the origin and oriented by v_i $(i = 1,\ 2)$ induce no sliding motion. Assume that the following LMI problem admits a solution: there exist $\tau_1, \tau_2 \geq 0$ and symmetric matrices P_1, P_2 such that:

$$P_1 - R > 0, \quad P_2 + R > 0, \tag{6a}$$

$$\begin{aligned} &-(A_1^T P_1 + P_1 A_1) - \tau_1 R > 0, \\ &-(A_2^T P_2 + P_2 A_2) + \tau_2 R > 0, \end{aligned} \tag{6b}$$

$$v_1^\top (P_1 - P_2) v_1 = 0, \quad v_2^\top (P_1 - P_2) v_2 = 0. \tag{6c}$$

Then R is stabilizing for (A_1, A_2).

Proof. First, notice that $\overline{\mathcal{O}}_1$ and $\overline{\mathcal{O}}_2$ are the sets of points x where the quadratic form $x^T Rx$ is non-negative and non-positive, respectively. Let τ_i and P_i, $i = 1, 2$, be solutions to the

LMI problem given in the statement of the theorem. Consider the function $V : \mathbb{R}^2 \to \mathbb{R}$ defined by $V(x) = x^T P_i x$, if $x \in \overline{\mathcal{O}}_i$, $i = 1, 2$. The function V is well-defined since equation (6c) means that $x^T P_1 x = x^T P_2 x$ whenever x satisfies $x^T R x = 0$, that is whenever $x \in \overline{\mathcal{O}}_1 \cap \overline{\mathcal{O}}_2$ [2]. Moreover, from (6a), P_1 is positive definite when R is positive definite and P_2 is positive definite when R is negative definite. So by definition, we deduce that V is positive everywhere. An adaptation of this argument shows that (6b) implies that $\dot{V}$ is strictly negative along the trajectories of (3), so that V is strictly decreasing along trajectories. Hence, the trajectories converge exponentially to zero.

Note that (6a) does not impose the matrices P_1 or P_2 to be positive definite. This relaxes conditions in the literatures where composed quadratic functions admit only positive definite functions.

4.2 A sufficient geometric condition

In this subsection, we fix two different lines $\mathcal{D}_1$ and $\mathcal{D}_2$ passing through the origin. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the regions they define and let $\overline{\mathcal{O}}_i$ be the topological closures of these regions, see Fig. 3. Let us consider the system (3) defined in the previous section.

In this section, we are looking for a sufficient condition for $(\mathcal{D}_1, \mathcal{D}_2)$ to be stabilizing for (A_1, A_2) in the case where the lines $\mathcal{D}_1$ and $\mathcal{D}_2$ induce no sliding motion. To establish this sufficient condition, it is required the existence of a *contractive pair* for $(\mathcal{D}_1, \mathcal{D}_2)$. Then that will allow us to construct a piecewise Lyapunov function.

Definition 7. Let $L_1, L_2 \in \mathbb{R}^{1\times 2}$. Given two lines $\mathcal{D}_1$ and $\mathcal{D}_2$, if there exist $\lambda_1, \lambda_2 > 0$ such that

- (i) $\mathcal{D}_1$ and $\mathcal{D}_2$ are the diagonals of the parallelogram bounded by the equations $| L_i x | = \lambda_i$, $i = 1, 2$,
- (ii) for $i = 1, 2$, $L_i A_i x < 0$ for every $x \in \mathbb{R}^2$ satisfying $L_i x = \lambda_i$ and $| L_j x | \leq \lambda_j$, $j \neq i$,

then we say that (L_1, L_2) is *contractive* for $(\mathcal{D}_1, \mathcal{D}_2)$.

Before giving the main result of this section, let us relate the previous notion to the existence of auxiliary stable systems. This approach consisting in using asymptotically stable auxiliary systems for proving stabilization is the one developed in Lin and Antsaklis (2007).

Proposition 8. The pair (L_1, L_2) is contractive for (A_1, A_2) if and only if for every $R_i \in \mathbb{R}^{2\times 1}$ such that $L_i R_i = 1$ and $| L_i R_i | \leq \frac{\lambda_j}{\lambda_i}$, the following auxiliary system is asymptotically stable

$$\dot{\xi}(t) = L_i A_i R_i \xi(t). \quad (7)$$

Proof. The auxiliary system (7) is one-dimensional, so that it is asymptotically stable if and only if $L_i A_i R_i < 0$. Moreover, if x is such that $L_i x = \lambda_i$, with $\lambda_i > 0$, as in Definition 7, then $x = \lambda_i R_i$. Thus, $L_i A_i R_i < 0$ is equivalent to $L_i A_i x < 0$, which shows the proposition.

Now, we may introduce the main result of the section.

Theorem 9. Consider the system (3). Let $\mathcal{D}_1, \mathcal{D}_2$ be two lines passing through the origin inducing no sliding motion for (A_1, A_2). If there exists a contractive pair (L_1, L_2) for $(\mathcal{D}_1, \mathcal{D}_2)$, then $(\mathcal{D}_1, \mathcal{D}_2)$ is stabilizing for (A_1, A_2).

[2] With the previous notations, from Proposition 3, this intersection is $\mathcal{D}_{\theta_1} \cup \mathcal{D}_{\theta_2}$.

Proof. Consider the notations of Definition 7. Set Ω_i to be the region defined by points $p \in \mathbb{R}^2$ such that the line passing through p and the origin meets one of the lines $L_i p = \pm\lambda_i$. Let x be the trajectory defined by $x(0) = x_0 \in \mathbb{R}^2$ and a switching law $\delta(t) = i$ if $x(t) \in \Omega_i$, $\forall t \in \mathbb{R}_+$ Since $\mathcal{D}_1$ and $\mathcal{D}_2$ are the diagonals of the parallelogram defined such as in Definition 7, the region Ω_i is equal to $\overline{\mathcal{O}_i}$ and δ is equal to σ. Hence, it is sufficient to show that the trajectory x converges exponentially to 0.

Consider the positive definite function $V : \mathbb{R}^2 \to \mathbb{R}$ defined by $V(p) = | L_i p |$ if $p \in \Omega_i$. Remark that V is well defined since the regions Ω_1 and Ω_2 determine a partition of $\mathbb{R}^2$. We show that the Dini derivative D^+V of V is strictly negative along x. At time t_0, $x(t_0)$ belongs to Ω_i. First, assume that no switching occurs at t_0, that is $x(t_0)$ belongs to the interior of Ω_i. Recall from Blanchini (1995) that the Dini derivative along x at t_0 satisfies

$$D^+V(x(t_0)) = \lim_{t\to 0, t>0} \frac{V(x(t_0) + tA_i x(t_0)) - V(x(t_0))}{t}. \quad (8)$$

If $L_i x(t_0) > 0$, then we have $L_i A_i x(t_0) < 0$ since, in this case, we have $x(t_0) = \mu x$, where x satisfies $L_i x = \lambda_i$ and $\mu > 0$. Hence, for $t > 0$ sufficiently small,

$$V(x(t_0) + tA_i x(t_0)) = L_i x(t_0) + tL_i A_i x(t_0)$$

is strictly smaller than $L_i x(t_0) = V(x(t_0))$, so that

$$D^+V(x(t_0)) < 0.$$

If $L_i x(t_0) > 0$, by adapting the previous arguments, we show $D^+V(x(t_0)) < 0$.

Now, if a switching occurs at time t_0, then $x(t_0)$ belongs to $\mathcal{D}_1 \cup \mathcal{D}_2$. Since no sliding motion occurs on these lines, $x(t)$ belongs to Ω_j, $j \neq i$, for $t > t_0$ sufficiently small. By replacing i by j in (8) and by adapting the reasoning of the previous paragraph, we show that $D^+V(x(t_0)) < 0$.

Let us finish this section by showing that the existence of a contracting pair for $(\mathcal{D}_1, \mathcal{D}_2)$ is not necessary for the latter to be stabilizing. The proof of Theorem 9 is based on the construction of a piecewise Lyapunov function V. In Theorem 10, we show that there exist systems with a stabilizing switch induced by a stabilizing pair of matrices but without any Lyapunov function such as V.

Before that, we recall from Balde et al. (2009) the notion of *worst trajectory* for $(A_1, A_2) \in \mathbb{R}^{2\times 2}$: this is the trajectory x such that at each t, $\dot{x}(t)$ forms the smallest angle in clockwise sense with the exiting radial direction. In other words, it is the trajectory which moves away in the fastest from the origin. On the other hand, the *best trajectory* is the worst trajectory for $(-A_1, -A_2)$: this is the trajectory which goes the fastest to the origin. From Balde et al. (2009), the switching signal corresponding to this trajectory is orchestrated by a pair of lines passing through the origin.

Theorem 10. There exist matrices $A_1, A_2 \in \mathbb{R}^{2\times 2}$ and a stabilizing pair $(\mathcal{D}_1, \mathcal{D}_2)$ for (A_1, A_2) such that no contractive pair exists for $(\mathcal{D}_1, \mathcal{D}_2)$.

Proof. Consider two *anti-Hurwitz* matrices, that is the real parts of their eigenvalues are non-negative, and such that the best trajectory x is periodic. Let $(\mathcal{D}_1, \mathcal{D}_2)$ be the lines inducing the switching signal corresponding to x. Now, consider a pair of matrices such that their best trajectory goes to zero and is close to x. By continuity, this trajectory is orchestrated by a feedback defined by a pair of lines $(\mathcal{D}_1^\varepsilon, \mathcal{D}_2^\varepsilon)$ closed to $(\mathcal{D}_1, \mathcal{D}_2)$. If the condition of Theorem 9 is necessary, then there exists V^ε such as in the proof. When ε goes to zero, V^ε goes to a nonzero function V. Moreover, the derivative of V along x vanishes since the latter is periodic, that is we have $\dot{x}(t)\dot{V}(x(t)) = 0$. That implies that $\dot{x}(t) = 0$, that is x is constant, which is a contradiction.

5. EXAMPLE

In this section, we illustrate Theorems 6 and 9 with a numerical example coming from Lin and Antsaklis (2007). Consider the two matrices

$$A_1 := \begin{pmatrix} 0 & 10 \\ 0 & 0 \end{pmatrix}, \; A_2 := \begin{pmatrix} 1.5 & 2 \\ -2 & -0.5 \end{pmatrix}.$$

5.1 Algebraic approach

From Lin and Antsaklis (2007), a stabilizing switching is obtained by the lines oriented by the vectors $v_1 := (1 \; 0.3)$ and $v_2 := (1 \; 0.11)$. With notations of Theorem 6, these vectors correspond to the following matrix

$$R := \begin{pmatrix} -0.033 & -0.095 \\ -0.095 & 1 \end{pmatrix}.$$

Then, from Theorem 6, we obtain another proof that this matrix is stabilizing for (A_1, A_2) since there is a solution to the LMIs given by

$$\tau_1 = 1.8890, \quad \tau_2 = 1.3550$$

and matrices P_1 and P_2

$$P_1 := \begin{pmatrix} 0.0229 & -0.1376 \\ -0.1376 & 3.9156 \end{pmatrix}, \quad P_2 := \begin{pmatrix} 0.1424 & 0.2065 \\ 0.2065 & 0.2940 \end{pmatrix}.$$

We note that the matrix P_2 is not sign definite. The obtained level set of the function $V(x) = \max(x^T P_1 x, x^T P_2 x)$ is depicted in Figure 4.

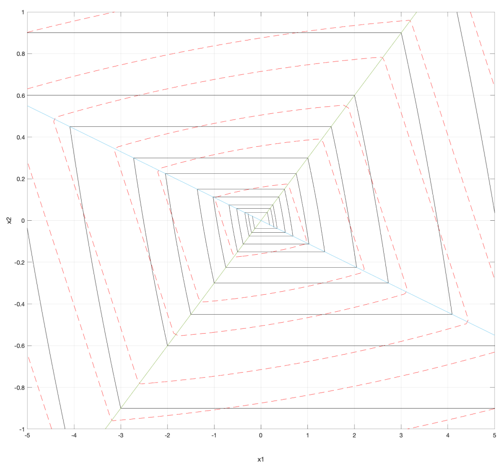

Fig. 4. The level sets of the Lyapunov function are depicted in dashed red curves and trajectories converging to the origin in solid black lines. The straight lines going through the origin correspond to the set verifying $x^\top R x = 0$.

5.2 Geometric approach

Regarding Theorem 9, our objective is to construct a stabilizing pair of lines for (A_1, A_2). We proceed in several steps. In particular, we are looking for two matrices $L_i := (a_i \; b_i)$, $i = 1, 2$, which are contractible for the pair of lines we wish to construct. We havee $L_1 A_1 = (0 \; 10a_1)$ and $L_2 A_2 = (1.5a_2 - 2b_2 \;\; 2a_2 - 0.5b_2)$. We will use Ω_1 and Ω_2 as parameters. They are submitted to the restrictions that they form a partition of $\mathbb{R}^2$ and that $L_i A_i$ is strictly negative on one of the half-regions of Ω_i.

Step 1. Letting $x := (1, 0)^\top$, we have $L_1 A_1 x = 0$. Hence, x belongs to Ω_2, so that we must have $L_2 A_2 x < 0$. That imposes the following restriction:

$$\frac{3a_2}{4} < b_2. \tag{9}$$

Moreover, Ω_i being cones, there exists $x' := (x_1, x_2)^\top \in \Omega_1$ such that $x_2 > 0$. The inequality $L_1 A_1 x' = 10a_1 x_2 < 0$ gives the following restriction:

$$a_1 < 0. \tag{10}$$

Step 2. We formalize L_1 and L_2 by taking into account (9) and (10), that is we search for these matrices as follows: $L_1 = (-1 \; b_1)$ and $L_2 = (1 \; b_2)$, with $b_2 > \frac{3}{4}$.

Step 3. We search the top right corner of the parallelogram as in Definition 7 at a point p_1 with coordinates $(1, y_0)$. For that, we select y_0 such that $L_2 A_2 y_0 = 0$, that is

$$y_0 = \frac{2b_2 - 1.5}{2 - 0.5b_2}.$$

We freely select $b_2 = 1$, so that $y_0 = \frac{1}{3}$, and

$$p_1 = \left(1, \frac{1}{3}\right). \tag{11}$$

In particular, for every $y' < y_0$, the point $(1, y')$ is in the region $L_2 A_2 < 0$. Moreover, we already have

$$L_2 = (1 \;\; 1). \tag{12}$$

Step 4. We search the bottom right corner p_2 of the parallelogram. Applying the criterion of Theorem 9, the segments $[p_1, p_2]$ and $[p_1, -p_2]$ have to be in the regions $L_2 A_2 < 0$ and $L_1 A_1 < 0$, respectively. From the previous step, for the first condition, we may select p_2 in the right half-plan with the second coordinate strictly smaller than $\frac{1}{3}$. For the second condition, $-p_2$ must be in the upper half-plan. Finally, p_2 must belong to the line directed by L_2 and passing through p_1, hence we select

$$p_2 = \left(\frac{19}{12}, -\frac{1}{4}\right). \tag{13}$$

Now, we obtain b_1 since the line passing through $-p_2$ and p_1 is directed by L_1:

$$L_1 = (1 \;\; -31). \tag{14}$$

Step 5. We obtain the two lines $\mathcal{D}_1$ and $\mathcal{D}_2$ that are the diagonals passing through p_1 and $-p_1$ and by p_2 and $-p_2$, respectively: $\mathcal{D}_1$ and $\mathcal{D}_2$ have equations $x_2 = \frac{1}{3}x_1$ and $x_2 = \frac{-3}{19}x_1$, respectively. Finally, the switching function defined by these two lines induces a globally exponentially stable system

(3) since, by construction, (L_1, L_2) is contractive for $(\mathcal{D}_1, \mathcal{D}_2)$, so $(\mathcal{D}_1, \mathcal{D}_2)$ is stabilizing for (A_1, A_2).

6. CONCLUSION

We presented two approaches for proving stability of a continuous-time switched system defined by a pair of 2-dimensional square matrices. These proofs were based on the design of Lyapunov functions: by using LMIs and algebraic tools for the first one and by using a geometric method for the second. This work could be extended into the following two directions. The first one will consist in achieving algorithmic constructions from our results. The second one will be to provide a full characterization of the pair of matrices having a solution to the general problem we introduced at the beginning of Section 3.

REFERENCES

Ahmadi, A. and Parrilo, P. (2008). Non–monotonic Lyapunov functions for stability of discrete time nonlinear and switched systems. In *47th IEEE Conference on Decision and Control*, 614–621. Cancun, Mexico.

Aleksandrov, A., Chen, Y., Platonov, A., and Zhang, L. (2011). Stability analysis for a class of switched nonlinear systems. *Automatica*, 47(10), 2286 – 2291.

Athanasopoulos, N. and Lazar, M. (2014). Alternative stability conditions for switched discrete time linear systems. *IFAC Proceedings Volumes*, 47(3), 6007–6012.

Balde, M., Boscain, U., and Mason, P. (2009). A note on stability conditions for planar switched systems. *International journal of control*, 82(10), 1882–1888.

Bemporad, A., Ferrari-Trecate, G., and Morari, M. (2000). Observability and controllability of piecewise affine and hybrid systems. *IEEE transactions on automatic control*, 45(10), 1864–1876.

Bemporad, A. and Morari, M. (1999). Control of systems integrating logic, dynamics, and constraints. *Automatica*, 35(3), 407–427.

Blanchini, F. (1995). Nonquadratic Lyapunov functions for robust control. *Automatica J. IFAC*, 31(3), 451–461.

Bliman, P.A. and Ferrari-Trecate, G. (2003). Stability analysis of discrete-time switched systems through Lyapunov functions with nonminimal state. *IFAC Proceedings Volumes*, 36(6), 325 – 329. IFAC Conference on Analysis and Design of Hybrid Systems 2003, St Malo, Brittany, France, 16-18 June 2003.

Daafouz, J., Riedinger, P., and Iung, C. (2002). Stability analysis and control synthesis for switched systems: a switched Lyapunov function approach. *IEEE Transactions on Automatic Control*, 47(11), 1883–1887.

DeCarlo, R.A., Branicky, M.S., Pettersson, S., and Lennartson, B. (2000). Perspectives and results on the stability and stabilizability of hybrid systems. *Proceedings of the IEEE*, 88(7), 1069–1082.

Egerstedt, M. and Babaali, M. (2005). On observability and reachability in a class of discrete-time switched linear systems. In *Proceedings of the 2005, American Control Conference, 2005.*, 1179–1180. IEEE.

Fridman, E. (2014). *Introduction to time-delay systems: Analysis and control*. Springer.

Gomide, T.S. and Lacerda, M.J. (2018). Stability analysis of discrete-time switched systems under arbitrary switching. *IFAC-PapersOnLine*, 51(25), 371 – 376. 9th IFAC Symposium on Robust Control Design ROCOND 2018.

Hespanha, J.P., Liberzon, D., Angeli, D., and Sontag, E.D. (2005). Nonlinear norm-observability notions and stability of switched systems. *IEEE Transactions on Automatic Control*, 50(2), 154–168.

Hetel, L., Kruszewski, A., Perruquetti, W., and Richard, J.P. (2011). Discrete-time switched systems, set-theoretic analysis and quasi-quadratic Lyapunov functions. In *2011 19th Mediterranean Conference on Control Automation (MED)*, 1325–1330.

Hetel, L., Fiter, C., Omran, H., Seuret, A., Fridman, E., Richard, J.P., and Niculescu, S.I. (2017). Recent developments on the stability of systems with aperiodic sampling: An overview. *Automatica*, 76, 309–335.

Hu, T. and Blanchini, F. (2010). Non-conservative matrix inequality conditions for stability/stabilizability of linear differential inclusions. *Automatica*, 46(1), 190–196.

Johansson, M.K.J. (2003). *Piecewise linear control systems: a computational approach*, volume 284. Springer.

Kruszewski, A., Wang, R., and Guerra, T.M. (2008). Non-quadratic stabilization conditions for a class of uncertain nonlinear discrete time ts fuzzy models: A new approach. *IEEE Transactions on Automatic Control*, 53(2), 606–611.

Lee, J.W. and Dullerud, G.E. (2006). Uniform stabilization of discrete-time switched and Markovian jump linear systems. *Automatica*, 42(2), 205 – 218.

Liberzon, D. and Morse, A.S. (1999). Basic problems in stability and design of switched systems. *IEEE control systems magazine*, 19(5), 59–70.

Lin, H. and Antsaklis, P.J. (2005). Stability and stabilizability of switched linear systems: A short survey of recent results. In *Proceedings of the 2005 IEEE International Symposium on, Mediterrean Conference on Control and Automation Intelligent Control, 2005.*, 24–29. IEEE.

Lin, H. and Antsaklis, P.J. (2007). Switching stabilizability for continuous-time uncertain switched linear systems. *IEEE Transactions on Automatic Control*, 52(4), 633–646.

Megretski, A. (1997). Integral quadratic constraints derived from the set-theoretic analysis of difference inclusions. In *35th IEEE Conference on Decision and Control*, volume 3, 2389 – 2394 vol.3.

Molchanov, A.P. and Pyatnitskiy, Y.S. (1989). Criteria of asymptotic stability of differential and difference inclusions encountered in control theory. *Systems & Control Letters*, 13(1), 59–64.

Pettersson, S. (2003). Synthesis of switched linear systems. In *42nd IEEE International Conference on Decision and Control (IEEE Cat. No. 03CH37475)*, volume 5, 5283–5288. IEEE.

Sun, Z. (2006). *Switched linear systems: control and design*. Springer Science & Business Media.

Sun, Z. and Ge, S.S. (2005). Analysis and synthesis of switched linear control systems. *Automatica*, 41(2), 181–195.

Sun, Z., Ge, S.S., and Lee, T.H. (2002). Controllability and reachability criteria for switched linear systems. *Automatica*, 38(5), 775–786.

Xu, X. and Antsaklis, P.J. (2004). Optimal control of switched systems based on parameterization of the switching instants. *IEEE transactions on automatic control*, 49(1), 2–16.