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Boundary stabilization in finite time of one-dimensional linear hyperbolic balance laws with coefficients depending on time and space

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Abstract

In this article we are interested in the boundary stabilization in finite time of one-dimensional linear hyperbolic balance laws with coefficients depending on time and space. We extend the so called "backstepping method" by introducing appropriate time-dependent integral transformations in order to map our initial system to a new one which has desired stability properties. The kernels of the integral transformations involved are solutions to non standard multi-dimensional hyperbolic PDEs, where the time dependence introduces several new difficulties in the treatment of their well-posedness. This work generalizes previous results of the literature, where only time-independent systems were considered.

Keywords: Hyperbolic systems, Boundary stabilization, Non-autonomous systems, Backstepping method.

1 Introduction and main result

In the present paper we are interested in the one-sided boundary stabilization in finite time of one-dimensional linear hyperbolic balance laws when the coupling coefficients of the system depend on both time and space variables. To investigate this stabilization property we use the by now so-called "backstepping method", a method that consists in transforming our initial system into another system - called target system - for which the stabilization properties are simpler to study. In finite dimension it relies on a recursive design procedure, which in the case of partial differential equations leads to Volterra transformations of the second kind.

The idea of the possibility to transform a control system into another one in order to study its controllability or stabilization properties already goes back to the development of the control theory for linear finite-dimensional systems in the late 60's, notably with the celebrated work [Bru70] where the author introduced the so-called "control canonical form". Concerning infinite-dimensional systems, such as systems modeled by partial differential equations (PDEs), this approach is much more complicated. The first attempt in this direction seems to be [Rus78], where the author was interested in the spectral determination (i.e. pole placement) of a particular 2×2 first-order hyperbolic

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system. The difficult task in this approach is, in general, to find an invertible transformation that allows to pass from one system to another and, to the best of our knowledge, there is no general theory for infinite-dimensional systems so far (if possible). In [Rus78], the author proposed to use a Volterra transformation of the second kind to pass from what he called the "control normal form" to the control canonical form of his hyperbolic system and, in this way, easily solved his spectral determination problem. In that paper, the use of such a transformation was justified by the analogy with finite-dimensional systems when using transformations of the simple form $\mathrm{Id} + K$ with K being a triangular matrix (while for Volterra transformations of the second kind, K is an integral operator whose kernel is supported in a triangular domain). The use of a Volterra transformation of the second kind to transform a PDE into another one was also introduced at almost the same time in [Col77]. Therein, the author showed that a one-dimensional perturbed heat equation, with a time and space dependent perturbation, can be transformed into the classical heat equation by means of a Volterra transformation of the second kind whose kernel has to satisfy some PDE posed on a non-standard domain which is triangular. The equation that the kernel has to satisfy is now commonly referred to as the "kernel equation" and the method was then referred by the author of [Col77] to as the "method of integral operators". The result of [Col77] was notably applied in [Sei84] to deduce the boundary null-controllability in one space dimension of the perturbed heat equation from that of the classical heat equation.

In the 90's a method with similar spirit appeared under the name of "backstepping method". This method was primarily designed to transform, thanks to a recursive procedure, finite-dimensional control systems, which may be nonlinear, into control systems which can be stabilized by means of simple feedback laws. This method was later on extended to linear PDEs. The first result in this direction is in [CdN98] for a beam equation; see also[LK00] for a Burgers' equation. However, the main breakthrough for the PDEs case are in [BKL01, BK02, Liu03], which deal with 1-D heat equations and where Volterra transformations of the second kind are introduced or used. In particular in [BK02] the backstepping recursive procedure in finite dimension is applied to the semi-discretized finite difference approximation of these equations and it is proved that, as the spatial step size tends to 0, the backstepping transformation at the finite dimensional level is converging to a Volterra transformation of the second kind. The fact that the transformation which appears with this approach is a Volterra transformation of the second kind comes from the recursive procedure of the backstepping method. With this method the authors, directly inspired by the backstepping in finite dimension, independently arrived at the use of exactly the same transformation as in the two above mentioned pioneering references [Rus78] and [Col77]. This is the reason of the use of the terminology "backstepping" for the construction of stabilizing feedback laws relying on the use of Volterra transformations of the second kind to transform a given control PDE to another control PDE (called the target system) which can be easily stabilized (usually with the null feedback law).

The use of Volterra transformations of the second kind also matches very well with the boundary stabilization of one-dimensional systems since this transformation somehow removes the undesirable terms (or adds desirable ones) of the equation by "bringing" them to the part of the boundary where the feedback is acting (through the kernel equations). This approach rapidly turned out to be very successful in the study of the boundary stabilization of various important PDEs such as heat equations, wave equations, Schrödinger equations, Korteweg-de Vries equations, Kuramoto-Sivashinsky equations, etc. and it eventually leads to the by now reference book [KS08] on this subject. This method is nowadays systematically used as a standard tool to analyze the boundary stabilization for (mainly one-dimensional) PDEs. This method has also received some recent developments. Notably, the use of Volterra transformations of the second kind has started to show some serious limitation for some problems and it has been replaced by more general integral transformations such as Fredholm integral transformations (see e.g. [CL14, CL15, BAK15, CHO16, CHO17, CGM18]) or other kind of integral transformations (see e.g. [SGK09]). In these cases the transformation on the state does not have any special structure and the method is no longer related to the finite dimensional backstepping approach. It is related to the older notion of feedback equivalence, as initiated in [Bru70]; see also

[Kal72], [Won85, Section 5.7], and [Son98, Section 5.2].

Concerning more specifically systems of hyperbolic equations and the finite-time stabilization property, which is the focus of this article, the first result was obtained in [CVKB13]. In this paper, the authors developed the original backstepping method to prove the boundary stabilization of a 2×2 hyperbolic system in finite time, with the best time that can be achieved. The generalization of the result of [CVKB13] to $n \times n$ systems was a non-trivial task which was eventually solved in [HDMVK16, HVDMK19] using the ideas introduced previously in [HDM15] for 3×3 systems. The key point was to add additional constraints on the kernel to obtain a specific structure of the coupling parameter in the target system. The time of stabilization found in [HDMVK16, HVDMK19] was then improved in [ADM16, CHO17], using two different target systems.

The goal of the present article is to extend the results of the previously mentioned references to time-dependent systems. For the finite-time stabilization of non-autonomous hyperbolic systems, the only works that we are aware of are [DJK16] and [AA18] which concerned a single equation with constant speed. Therefore, the non-autonomous case for systems was still left without investigation. The introduction of the time variable in the coupling coefficients obviously complicates the whole situation. As in [Col77, DJK16, AA18] we need to introduce integral transformations with time-dependent kernels, resulting in much more complex kernel equations to solve. Finally, in addition to the previous references, we would also like to mention the work [Wan06] on time-dependent quasilinear hyperbolic systems concerning the related notion of controllability and the works [SK05, KD19], with the references therein, concerning the stabilization of time-dependent parabolic systems (where strong regularity conditions are required to make the backstepping method work, because of the result of [Kan90]).

The rest of this paper is organized as follows. In the remaining part of Section 1 we present in details the class of hyperbolic systems that we consider and we state our main result. In Section 2 we perform several transformations to show that our initial system can be mapped to a target system which is finite-time stable with desired settling time. In Section 3 we prove the existence and regularity of the kernels of the integral transformations that were used in the previous section. Finally, we gathered in Appendices A, B and C some auxiliary results.

1.1 System description

In this article, we focus on the following general $n \times n$ linear hyperbolic systems, which appear for instance in the linearized Saint-Venant equations, plug flow chemical reactors equations, heat exchangers equations and many other physical models of balance laws (see e.g. [BC16, Chapter 1]) around time-varying trajectories:

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,x) + \Lambda(t,x)\dfrac{\partial y}{\partial x}(t,x) = M(t,x)y(t,x), \\ y_-(t,1) = u(t), \quad y_+(t,0) = Q(t)y_-(t,0), \\ y(t^0,x) = y^0(x). \end{cases} \tag{1}$$

In (1), $t > t^0 \geq 0$ and $x \in (0,1)$, $y(t,\cdot)$ is the state at time t, y^0 is the initial data at time t^0 and $u(t)$ is the control at time t. The matrix M couples the equations of the system inside the domain and the matrix Q couples the equations of the system on the boundary $x = 0$. We assume that the matrix Λ is diagonal:

$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, , \lambda_n). \tag{2}$$

We denote by $m \in \{1, \ldots, n-1\}$ the number of equations with negative speeds and by $p = n - m \in \{1, \ldots, n-1\}$ the number of equations with positive speeds (all along this work we assume that $n \geq 2$, see Remark 1.11 below for the case $m = n \geq 1$). We assume that there exists some $\varepsilon > 0$ such that, for every $t \geq 0$ and $x \in [0,1]$, we have

$$\lambda_1(t,x) < \cdots < \lambda_m(t,x) < -\varepsilon < 0 < \varepsilon < \lambda_{m+1}(t,x) < \cdots < \lambda_n(t,x), \tag{3}$$

and, for every $i \in \{1, \dots, n-1\}$,

$$\lambda_{i+1}(t,x) - \lambda_i(t,x) > \varepsilon. \tag{4}$$

Assumptions (3) and (4) will be commented, respectively, in Remarks 1.9 and 1.10 below.

All along this paper, for a vector (or vector-valued function) $v \in \mathbb{R}^n$ and a matrix (or matrix-valued function) $A \in \mathbb{R}^{n \times n}$, we use the notation

$$v = \begin{pmatrix} v_- \\ \\ v_+ \end{pmatrix}, \qquad A = \begin{pmatrix} A_{--} & A_{-+} \\ \\ A_{+-} & A_{++} \end{pmatrix},$$

where $v_- \in \mathbb{R}^m$, $v_+ \in \mathbb{R}^p$ and $A_{--} \in \mathbb{R}^{m \times m}$, $A_{-+} \in \mathbb{R}^{m \times p}$, $A_{+-} \in \mathbb{R}^{p \times m}$, $A_{++} \in \mathbb{R}^{p \times p}$.

We will always assume the following regularities for the parameters involved in the system (1):

$$\begin{gathered} \Lambda \in C^1([0,+\infty) \times [0,1])^{n \times n}, \qquad M \in C^0([0,+\infty) \times [0,1])^{n \times n}, \qquad Q \in C^0([0,+\infty))^{p \times m}, \\ \Lambda, \frac{\partial \Lambda}{\partial x}, M \in L^\infty((0,+\infty) \times (0,1))^{n \times n}, \qquad Q \in L^\infty(0,+\infty)^{p \times m}. \end{gathered} \tag{5}$$

In this article, we use the notion of "solution along the characteristics" or "broad solution" for the system (1). The necessary background on this notion is given in Appendix A (see also [Bre00, Section 3.4] for more information). For the moment we only need to know that, for every $F \in L^\infty((0,+\infty) \times (0,1))^{m \times n}$, $t^0 \geq 0$ and $y^0 \in L^2(0,1)^n$, there exists a unique (broad) solution $y \in C^0([t^0,+\infty); L^2(0,1)^n)$ to the system (1) with

$$u(t) = \int_0^1 F(t,\xi) y(t,\xi) \, d\xi. \tag{6}$$

The relation (6) will be called the feedback law and the function F will be called the state-feedback gain function.

Let us now give the notion of stability that we are interested in this article (see, for example, [BB98, Definition], [BR05, Section 3.2] and [Cor07, Definitions 11.11 and 11.27] for time-varying systems in finite dimension).

Definition 1.1. Let $T > 0$. We say that the system (1) with feedback law (6) is finite-time stable with settling time T if the following two properties hold:

(i) **Finite-time global attractor.** For every $t^0 \geq 0$ and $y^0 \in L^2(0,1)^n$,

$$y(t^0 + T, \cdot) = 0. \tag{7}$$

(ii) **Uniform stability.** For every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $t^0 \geq 0$ and $y^0 \in L^2(0,1)^n$,

$$\left(\|y^0\|_{L^2(0,1)^n} \leq \delta \right) \quad \Longrightarrow \quad \left(\|y(t,\cdot)\|_{L^2(0,1)^n} \leq \varepsilon, \quad \forall t \geq t^0 \right). \tag{8}$$

Remark 1.2. The property (8) guarantees that, inside any time interval of the form $[t^0, t^0 + T]$, the solution is controlled solely by its value at the initial time t^0, even if this time t^0 is very large. For our system (1) this property is in fact a consequence of the first property (7) and that the state-feedback gain function F is in $L^\infty((0,+\infty)\times(0,1))^{m\times n}$ (see Remark A.3). Such an implication is in general not true for time-dependent hyperbolic systems. A simple example is the following transport equation:

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,x) - \dfrac{\partial y}{\partial x}(t,x) = 0, \\ y(t,1) = f(t)\displaystyle\int_0^1 y(t,\xi)d\xi, \\ y(t^0,x) = y^0(x), \end{cases}$$

where $f \in C^\infty([0,+\infty))$ is such that, for every $k \in \mathbb{N}$,

$$\begin{cases} f(t) = 0, \quad \forall t \in [2k, 2k+1], \\ f(t) = t, \quad \forall t \in \left[2k + \dfrac{5}{4}, 2k + \dfrac{7}{4}\right], \end{cases}$$

(note that $f \notin L^\infty(0,+\infty)$). Then the finite-time global attractor property holds (with $T = 3$) but the uniform stability property does not hold (consider the sequences $y^0_\delta(x) = \delta$ for every $x \in (0,1)$ and $t^0_\delta = 2\left\lceil \frac{1}{\delta} \right\rceil + \frac{5}{4}$, where $\lceil\cdot\rceil$ denotes the ceiling function).

Remark 1.3. As we are trying to find a state-feedback gain function F so that (1) with feedback law (6) is finite-time stable, let us first point out that, in general, $F = 0$ does not work. A simple example is provided by the 2×2 system with constant coefficients ($t^0 = 0$ to simplify)

$$\begin{cases} \dfrac{\partial y_-}{\partial t}(t,x) - \dfrac{\partial y_-}{\partial x}(t,x) = -cy_+(t,x), \\ \dfrac{\partial y_+}{\partial t}(t,x) + \dfrac{\partial y_+}{\partial x}(t,x) = -cy_-(t,x), \\ y_-(t,1) = 0, \quad y_+(t,0) = y_-(t,0), \\ y(0,x) = y^0(x), \end{cases}$$

which is exponentially unstable for $c > \pi$ (see e.g. [BC16, Proposition 5.12] with $y_-(t,x) = S_1(t,1-x)$ and $y_+(t,x) = S_2(t,1-x)$), and thus not finite-time stable.

1.2 The characteristics

To state the main result of this paper we need to introduce the characteristic curves associated with system (1). To this end, it is convenient to first extend Λ to a function of $\mathbb{R}^2$ (still denoted by Λ).

Remark 1.4. This extension procedure can be done in such a way that the properties (2), (3), (4) and (5) remain valid on $\mathbb{R}^2$. We can take for instance

$$\bar{\lambda}_i(t,x) = \begin{cases} \lambda_i(t,x) & \text{if } t \geq 0, \\ \lambda_i(0,x) + \delta\left(\lambda_i(0,x) - \lambda_i\left(1 - e^{t/\delta}, x\right)\right) & \text{if } t < 0, \end{cases}$$

where $\delta > 0$ is small enough so that $-\varepsilon + 4\delta \max_i \|\lambda_i\|_{L^\infty((0,1)\times(0,1))} < -\varepsilon/2$ to guarantee the properties (3) and (4) with $\varepsilon/2$ in place of ε. This extends the function to $\mathbb{R}\times[0,1]$. We can use a similar procedure to then extend it to $\mathbb{R}^2$. We can check that the results of this paper do not depend on such a choice of extension (all the important data are uniquely determined on the domain of interest $(0,+\infty)\times(0,1)$).

1.2.1 The flow

For every $i \in \{1, \cdots, n\}$, let χ_i be the flow associated with λ_i, i.e. for every $(t,x) \in \mathbb{R} \times \mathbb{R}$, the function $s \mapsto \chi_i(s;t,x)$ is the solution to the ODE

$$\begin{cases} \dfrac{\partial \chi_i}{\partial s}(s;t,x) = \lambda_i(s, \chi_i(s;t,x)), \quad \forall s \in \mathbb{R}, \\ \chi_i(t;t,x) = x. \end{cases} \tag{9}$$

The existence and uniqueness of the solution to the ODE (9) follows from the (local) Cauchy-Lipschitz theorem and this solution is global since λ_i is bounded (by the finite time blow-up theorem, see e.g. [Har02, Theorem II.3.1]). The uniqueness of the solution to the ODE (9) also yields the group property

$$\chi_i\left(\sigma; s, \chi_i(s;t,x)\right) = \chi_i(\sigma;t,x), \quad \forall \sigma \in \mathbb{R}. \tag{10}$$

By classical regularity results on ODEs (see e.g. [Har02, Theorem V.3.1]), χ_i has the regularity

$$\chi_i \in C^1(\mathbb{R}^3), \tag{11}$$

and, for every $s,t,x \in \mathbb{R}$, we have

$$\frac{\partial \chi_i}{\partial t}(s;t,x) = -\lambda_i(t,x) e^{\int_t^s \frac{\partial \lambda_i}{\partial x}(\theta, \chi_i(\theta;t,x))\, d\theta}, \qquad \frac{\partial \chi_i}{\partial x}(s;t,x) = e^{\int_t^s \frac{\partial \lambda_i}{\partial x}(\theta, \chi_i(\theta;t,x))\, d\theta}. \tag{12}$$

Note in particular that

$$\begin{cases} \dfrac{\partial \chi_i}{\partial t}(s;t,x) > 0 \quad & \text{if } i \in \{1, \ldots, m\}, \\ \dfrac{\partial \chi_i}{\partial t}(s;t,x) < 0 \quad & \text{if } i \in \{m+1, \ldots, n\}, \\ \dfrac{\partial \chi_i}{\partial x}(s;t,x) > 0. \end{cases} \tag{13}$$

1.2.2 The entry and exit times

For every $i \in \{1, \cdots, n\}$, $t \in \mathbb{R}$ and $x \in [0,1]$, let $s_i^{\text{in}}(t,x), s_i^{\text{out}}(t,x) \in \mathbb{R}$ be the entry and exit times of the flow $\chi_i(\cdot;t,x)$ inside the domain $[0,1]$, i.e. the respective unique solutions to

$$\begin{cases} \chi_i(s_i^{\text{in}}(t,x);t,x) = 1, \qquad \chi_i(s_i^{\text{out}}(t,x);t,x) = 0, \qquad \text{if } i \in \{1, \ldots, m\}, \\ \chi_i(s_i^{\text{in}}(t,x);t,x) = 0, \qquad \chi_i(s_i^{\text{out}}(t,x);t,x) = 1, \qquad \text{if } i \in \{m+1, \ldots, n\}. \end{cases} \tag{14}$$

The existence and uniqueness of $s_i^{\text{out}}(t,x)$ and $s_i^{\text{in}}(t,x)$ are guaranteed by the assumption (3). Note that we always have

$$s_i^{\text{in}}(t,x) \leq t \leq s_i^{\text{out}}(t,x) \tag{15}$$

and the cases of equalities are given by

$$\begin{cases} s_i^{\text{in}}(t,x) = t \iff x = 1, \qquad s_i^{\text{out}}(t,x) = t \iff x = 0, \qquad \text{if } i \in \{1, \ldots, m\}, \\ s_i^{\text{in}}(t,x) = t \iff x = 0, \qquad s_i^{\text{out}}(t,x) = t \iff x = 1, \qquad \text{if } i \in \{m+1, \ldots, n\}. \end{cases} \tag{16}$$

It readily follows from (10) and the uniqueness of $s_i^{\text{in}}, s_i^{\text{out}}$ that, for every $s \in [s_i^{\text{in}}(t,x), s_i^{\text{out}}(t,x)]$,

$$s_i^{\text{in}}\left(s, \chi_i(s;t,x)\right) = s_i^{\text{in}}(t,x), \quad s_i^{\text{out}}\left(s, \chi_i(s;t,x)\right) = s_i^{\text{out}}(t,x). \tag{17}$$

From (11) and by the implicit function theorem, we have

$$s_i^{\mathrm{in}}, s_i^{\mathrm{out}} \in C^1(\mathbb{R} \times [0,1]). \tag{18}$$

Moreover, integrating the ODE (9) and using the assumption (3), we have the following bounds, valid for every $t \in \mathbb{R}$ and $x \in [0,1]$,

$$t - s_i^{\mathrm{in}}(t,x) < \frac{1}{\varepsilon}, \quad s_i^{\mathrm{out}}(t,x) - t < \frac{1}{\varepsilon}. \tag{19}$$

On the other hand, differentiating (14) and using (13) with (3), we see that, for every $t \in \mathbb{R}$ and $x \in [0,1]$, we have

$$\begin{cases} \dfrac{\partial s_i^{\mathrm{in}}}{\partial t}(t,x) > 0, & \dfrac{\partial s_i^{\mathrm{out}}}{\partial t}(t,x) > 0, \\ \dfrac{\partial s_i^{\mathrm{in}}}{\partial x}(t,x) > 0, & \dfrac{\partial s_i^{\mathrm{out}}}{\partial x}(t,x) > 0 \quad \text{if } i \in \{1,\dots,m\}, \\ \dfrac{\partial s_i^{\mathrm{in}}}{\partial x}(t,x) < 0, & \dfrac{\partial s_i^{\mathrm{out}}}{\partial x}(t,x) < 0 \quad \text{if } i \in \{m+1,\dots,n\}. \end{cases} \tag{20}$$

Finally, from the assumption (3) and classical results on comparison for ODEs (see e.g. [Har02, Corollary III.4.2]), we have, for every $t \in \mathbb{R}$ and $x \in [0,1]$,

$$\begin{cases} s_m^{\mathrm{in}}(t,x) < \ldots < s_1^{\mathrm{in}}(t,x) \quad \text{if } x \neq 1, & s_1^{\mathrm{out}}(t,x) < \ldots < s_m^{\mathrm{out}}(t,x) \quad \text{if } x \neq 0, \\ s_{m+1}^{\mathrm{in}}(t,x) < \ldots < s_n^{\mathrm{in}}(t,x) \quad \text{if } x \neq 0, & s_n^{\mathrm{out}}(t,x) < \ldots < s_{m+1}^{\mathrm{out}}(t,x) \quad \text{if } x \neq 1. \end{cases} \tag{21}$$

1.3 Main result and comments

We are now in position to state the main result of this paper:

Theorem 1.5. *Let Λ, M and Q satisfy* (2), (3), (4) *and* (5). *Then, there exists a state-feedback gain function $F \in L^\infty((0,+\infty) \times (0,1))^{m \times n}$ such that the system* (1) *with feedback law* (6) *is finite-time stable with settling time $T_{\mathrm{unif}}(\Lambda)$ defined by*

$$T_{\mathrm{unif}}(\Lambda) = \sup_{t^0 \geq 0} s_{m+1}^{\mathrm{out}}\left(s_m^{\mathrm{out}}\left(t^0, 1\right), 0\right) - t^0. \tag{22}$$

Moreover, if for some $\tau > 0$, Λ, M and Q are τ-periodic with respect to time (that is $\Lambda(t+\tau,x) = \Lambda(t,x)$ for every $t \geq 0$ and $x \in [0,1]$, same for M and Q) then one can also impose to F to be τ-periodic with respect to time (almost everywhere).

Let us remark that, thanks to (15), (16) and (19), we always have

$$0 < T_{\mathrm{unif}}(\Lambda) < \frac{2}{\varepsilon}.$$

Note as well, thanks to (21) and the first line in (20), that we have

$$T_{\mathrm{unif}}(\Lambda) = \max_{j \in \{m+1,\dots,n\}} \max_{i \in \{1,\dots,m\}} \sup_{t^0 \geq 0} s_j^{\mathrm{out}}\left(s_i^{\mathrm{out}}\left(t^0, 1\right), 0\right) - t^0.$$

Example 1.6. Theorem 1.5 applies for instance to the following coupled 2×2 system:

$$\begin{cases} \dfrac{\partial y_1}{\partial t}(t,x) - \dfrac{\partial y_1}{\partial x}(t,x) = m_{11}(t,x)y_1(t,x) + m_{12}(t,x)y_2(t,x), \\ \dfrac{\partial y_2}{\partial t}(t,x) + \left(1 + \dfrac{1}{1+t}\right)\dfrac{\partial y_2}{\partial x}(t,x) = m_{21}(t,x)y_1(t,x) + m_{22}(t,x)y_2(t,x), \\ y_1(t,1) = u(t), \quad y_2(t,0) = q(t)y_1(t,0), \\ y(t^0,x) = y^0(x), \end{cases} \tag{23}$$

where $M = (m_{ij})_{1 \leq i,j \leq 2}$ and $Q = (q)$ are any parameters with the regularity (5). Let us show how to compute $T_{\text{unif}}(\Lambda)$ for this example. First of all, it is clear that $\chi_1(s;t,x) = -s + t + x$, so that $s_1^{\text{out}}(t^0,1) = t^0 + 1$. On the other hand, we have $\chi_2(s;t,x) = s + \ln(1+s) - t - \ln(1+t) + x$. Therefore, $h(t^0) = s_2^{\text{out}}(t^0+1,0) - t^0$ solves $\Psi(h(t^0),t^0) = 0$, where

$$\Psi(h,t^0) = h + \ln(1+h+t^0) - 2 - \ln(2+t^0).$$

Taking the derivative of the relation $\Psi(h(t^0),t^0) = 0$ and using the fact that $h \geq 1$ by (15), we see that $h'(t^0) \geq 0$, so that h is non-decreasing. Since $h \leq 2$ by (19), the function h is thus a bounded non-decreasing function and, consequently, $\lim_{t^0 \to +\infty} h(t^0)$ exists and is equal to $\sup_{t^0 \geq 0} h(t^0) = T_{\text{unif}}(\Lambda)$. Writing the relation $\Psi(h(t^0),t^0) = 0$ as follows for $t^0 > 0$

$$h(t^0) + \ln\left(\frac{1}{t^0} + \frac{h(t^0)}{t^0} + 1\right) - 2 - \ln\left(\frac{2}{t^0} + 1\right) = 0,$$

and letting $t^0 \to +\infty$ we obtain the value $T_{\text{unif}}(\Lambda) = 2$.

Remark 1.7. Observe that the time $T_{\text{unif}}(\Lambda)$ does not depend on the parameters M and Q. It depends only on Λ on $[0,+\infty) \times (0,1)$. Moreover, this is the best time one can obtain, uniformly with respect to all the possible choices of M and Q (this explains our notation "$T_{\text{unif}}(\Lambda)$"). More precisely,

$$T_{\text{unif}}(\Lambda) = \min E,$$

where E is the set of $T > 0$ such that, for every M and Q with the regularity (5), there exists a state-feedback gain function $F \in L^\infty((0,+\infty) \times (0,1))^{m \times n}$ so that the system (1) with feedback law (6) is finite-time stable with settling time T. Indeed, Theorem 1.5 establishes that $T_{\text{unif}}(\Lambda) \in E$, so that $E \neq \emptyset$. On the other hand, taking $M = 0$ and the constant matrix

$$Q = \begin{array}{c} \left(\begin{array}{c|c} \quad 0 \quad & 1 \\ \hline \quad 0 \quad & 0 \end{array}\right)\begin{array}{l} \}1 \\ \}p-1 \end{array} \\ \underbrace{\hphantom{\quad 0 \quad}}_{m-1}\underbrace{\hphantom{0}}_{1} \end{array} .$$

we can check from the very definition of broad solution (see Definition A.1) that, if $T < T_{\text{unif}}(\Lambda)$, then there exist $t^0 \geq 0$ and $y^0 \in L^2(0,1)^n$ such that the corresponding solution to (1) satisfies $y(t^0+T,\cdot) \neq 0$, whatever $u \in L^\infty(t^0,t^0+T)^m$ is.

Of course, for particular choices of M and Q one may obtain a better settling time (a trivial example being $M = 0$ and $Q = 0$). In the case of time-independent systems, the minimal time in which one can achieve the stabilization and related controllability properties has been recently discussed in [CN19] and [HO19] (see also the references therein).

Remark 1.8. If the speeds do not depend on time, i.e. $\lambda_\ell(t,x) = \lambda_\ell(x)$ for every $t \geq 0$, then we have a more explicit formula for the time $T_{\text{unif}}(\Lambda)$, namely:

$$T_{\text{unif}}(\Lambda) = \int_0^1 \frac{1}{-\lambda_m(\xi)}\, d\xi + \int_0^1 \frac{1}{\lambda_{m+1}(\xi)}\, d\xi. \tag{24}$$

The value (24) is obtained by integrating over $\xi \in [0,1]$ the differential equation satisfied by the inverse functions $\xi \longmapsto \chi_m^{-1}(\xi; t, 1)$ and $\xi \longmapsto \chi_{m+1}^{-1}(\xi; t, 0)$.

Remark 1.9. The assumption (3) that the negative (resp. positive) speeds are uniformly bounded from above (resp. below), despite not being necessary for the existence of a solution to (1), is to be expected for the system (1) to be finite-time stable. This is an issue that is not specific to systems and that already occurs for a single equation. Indeed, let us consider for instance the equation with speed $\lambda(t) = -e^{-t}$ (and $t^0 = 0$ to simplify):

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,x) - e^{-t}\dfrac{\partial y}{\partial x}(t,x) = 0, \\ y(t,1) = u(t), \\ y(0,x) = y^0(x). \end{cases}$$

Then, whatever $y^0 \in L^2(0,1)$ and $u \in L^\infty(0,+\infty)$ are, if $y^0 \neq 0$ in a neighborhood of 1 we have

$$y(T,\cdot) \neq 0, \quad \forall T > 0.$$

This is easily seen thanks to the explicit representation of the solution (obtained by the characteristic method):

$$y(t,x) = \begin{cases} y^0\left(1 - (e^{-t} - x)\right) & \text{if } 0 < x < e^{-t}, \\ u\left(\ln\left(\dfrac{1}{1 + e^{-t} - x}\right)\right) & \text{if } e^{-t} < x < 1. \end{cases}$$

Remark 1.10. Contrary to (3), the assumption (4) is mainly technical. This assumption is needed because we will have to divide in the sequel by the quantities $\lambda_j - \lambda_i$ (see in particular (67) below) and we will need this inverse function to be bounded. However, this condition is clearly not necessary for some systems (1) to be finite-time stable. Indeed, consider for instance the following 3×3 system:

$$\begin{cases} \dfrac{\partial y_1}{\partial t}(t,x) - \dfrac{\partial y_1}{\partial x}(t,x) = y_2(t,x), \\ \dfrac{\partial y_2}{\partial t}(t,x) - \left(1 - \dfrac{e^{-t}}{2}\right)\dfrac{\partial y_2}{\partial x}(t,x) = 0, \\ \dfrac{\partial y_3}{\partial t}(t,x) + \dfrac{\partial y_3}{\partial x}(t,x) = 0, \\ y_1(t,1) = u_1(t), \quad y_2(t,1) = u_2(t), \quad y_3(t,0) = y_2(t,0), \\ y(t^0,x) = y^0(x). \end{cases} \tag{25}$$

Then, using the characteristic method it is not difficult to see that the system (25) with $u_1 = u_2 = 0$ is finite-time stable with settling time $T+1$, where T is the unique positive solution to the equation $T + \frac{e^{-T}}{2} = \frac{3}{2}$.

Remark 1.11. The case $m = n \geq 1$ (no boundary conditions at $x = 0$) is easier and does not require the techniques presented in this paper. Indeed, it can be checked using for instance the constructive method of [LR03, Wan06] that in this case the system (1) with $u = 0$ is finite-time stable with settling time equal to $\sup_{t^0 \geq 0} s_m^{\text{out}}(t^0, 1) - t^0$.

2 System transformations

The goal of this section is to show that we can use several invertible transformations in order to remove or transform some coupling terms in the initial system (1) and to obtain in the end a system for which we can directly establish that it is finite-time stable with settling time $T_{\text{unif}}(\Lambda)$. The plan of this section is as follows:

1) In Section 2.1, we use a diagonal transformation to remove the diagonal terms in M.

2) Next, in Section 2.2, inspired by the seminal works [Col77, Rus78, BK02] for equations and [CVKB13, HDM15, HDMVK16, HVDMK19] for hyperbolic systems, we use a Volterra transformation of the second kind to transform the system obtained by the previous step into a new system in the so-called "control normal form" and with an additional triangular structure for the couplings.

3) Finally, in Section 2.3, inspired by the work [CHO17] for time-independent systems, we use an invertible Fredholm integral transformation to transform the system obtained by the previous step into a new system with a very simple coupling structure that allows us to readily see that it is finite-time stable with settling time $T_{\text{unif}}(\Lambda)$.

In Section 2 only the properties of the transformations and new systems are discussed. The existence of the transformations is the main technical point of this paper and will be proved in Section 3 below for the sake of the presentation.

Finally, because of the nature of the transformations that we will use in the sequel, we are led to consider a class of systems that is slightly more general than (1). All the systems of this paper will have the following form:

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,x) + \Lambda(t,x)\dfrac{\partial y}{\partial x}(t,x) = M(t,x)y(t,x) + G(t,x)y(t,0), \\ y_-(t,1) = \displaystyle\int_0^1 F(t,\xi)y(t,\xi)\,d\xi, \quad y_+(t,0) = Q(t)y_-(t,0), \\ y(t^0,x) = y^0(x), \end{cases} \tag{26}$$

where M and Q will have at least the regularity (5), $F \in L^\infty((0,+\infty)\times(0,1))^{m\times n}$ and

$$G \in C^0([0,+\infty)\times[0,1])^{n\times n} \cap L^\infty((0,+\infty)\times(0,1))^{n\times n}.$$

Therefore, (26) is similar to (1) but has the extra term with G. In what follows, we will also refer to a system of the form (26) as

$$(M, G, F, Q).$$

Hyperbolic equations similar to $(0, G, F, Q)$ were called in "control normal form" in the pioneering work [Rus78, p. 212] for the similarity with the finite-dimensional setting (see also the earlier paper [Bru70]).

2.1 Removal of the diagonal terms

In this section we just perform a simple preliminary transformation in order to remove the diagonal terms in M. This is only a technical step, which is nevertheless necessary in view of the existence of the transformation that we will use in the next section, see Remark 2.7 below. This step is sometimes called "exponential pre-transformation" in the case of time-independent systems (see Remark 2.3 below). More precisely, the goal of this section is to establish the following result:

Proposition 2.1. *There exists* $M^1 = (m^1_{ij})_{1\leq i,j\leq n} \in C^0([0,+\infty)\times[0,1])^{n\times n} \cap L^\infty((0,+\infty)\times(0,1))^{n\times n}$ *with diagonal terms equal to zero:*

$$m^1_{ii} = 0, \quad \forall i \in \{1,\dots,n\}, \tag{27}$$

and there exists $Q^1 \in C^0([0,+\infty))^{p\times m} \cap L^\infty(0,+\infty)^{p\times m}$ *such that, for every* $F^1 \in L^\infty((0,+\infty)\times(0,1))^{m\times n}$*, there exists* $F \in L^\infty((0,+\infty)\times(0,1))^{m\times n}$ *such that the following property holds for every* $T > 0$*:*

$$\begin{aligned}&(M^1,0,F^1,Q^1) \text{ is finite-time stable with settling time } T\\ &\qquad\qquad \Longrightarrow \quad (M,0,F,Q) \text{ is finite-time stable with settling time } T.\end{aligned} \tag{28}$$

2.1.1 Formal computations

To prove Proposition 2.1, the idea is to show that, for every F^1, there exists F such that we can transform a solution of $(M,0,F,Q)$ into a solution of $(M^1,0,F^1,Q^1)$. Let then y be the solution to the system $(M,0,F,Q)$ with state-feedback gain function F to be determined below and initial data y^0. Let $\Phi : [0,+\infty)\times[0,1] \longrightarrow \mathbb{R}^{n\times n}$ be a smooth matrix-valued function and set

$$w(t,x) = \Phi(t,x)y(t,x). \tag{29}$$

Let us now perform some formal computations in order to see what w can solve. Using the equation satisfied by y, we have

$$\frac{\partial w}{\partial t} + \Lambda\frac{\partial w}{\partial x} = \left(\frac{\partial \Phi}{\partial t} + \Phi M + \Lambda\frac{\partial \Phi}{\partial x}\right) y + (-\Phi\Lambda + \Lambda\Phi)\frac{\partial y}{\partial x}.$$

On the other hand, using the boundary condition satisfied by y at $x = 0$, we have

$$\begin{aligned}&w_+(t,0) - Q^1(t)w_-(t,0)\\ &\qquad = \left(\Phi_{+-}(t,0) + \Phi_{++}(t,0)Q(t) - Q^1(t)\Phi_{--}(t,0) - Q^1(t)\Phi_{-+}(t,0)Q(t)\right) y_-(t,0).\end{aligned}$$

Finally, at $x = 1$, we have

$$w_-(t,1) - \int_0^1 F^1(t,\xi)w(t,\xi)\,d\xi = \int_0^1 \left(\Phi_{--}(t,1)F(t,\xi) - F^1(t,\xi)\Phi(t,\xi)\right) y(t,\xi)\,d\xi + \Phi_{-+}(t,1)y_+(t,1).$$

Thus, we see that w satisfies at $x = 1$ the boundary condition $w_-(t,1) = \int_0^1 F^1(t,\xi)w(t,\xi)\,d\xi$ if $\Phi_{-+}(t,1) = 0$ and

$$F(t,\xi) = \Phi_{--}(t,1)^{-1}F^1(t,\xi)\Phi(t,\xi), \tag{30}$$

provided that $\Phi_{--}(t,1)$ is also invertible. Moreover, note that F belongs to $L^\infty((0,+\infty)\times(0,1))^{m\times n}$ provided that F^1 belongs to this space as well and

$$\exists C > 0, \quad \left\|\Phi_{--}(\cdot,1)^{-1}\right\|_{L^\infty(0,+\infty)^{m\times m}} \leq C. \tag{31}$$

In summary, w defined by (29) is the solution of $(M^1,0,F^1,Q^1)$ with state-feedback gain function F^1 (which is assumed to be known) and initial data $w^0(\cdot) = \Phi(0,\cdot)y^0(\cdot)$ if we have the following four properties:

(i) $\Lambda(t,x)\Phi(t,x) = \Phi(t,x)\Lambda(t,x)$ for every $t \geq 0$ and $x \in [0,1]$.

(ii) The matrices $\Phi(t,x)$ and $\Phi_{--}(t,0) + \Phi_{-+}(t,0)Q(t)$ are invertible for every $t \geq 0$ and $x \in [0,1]$.

(iii) $\Phi_{-+}(t,1) = 0$ for every $t \geq 0$ (it then follows with (ii) that $\Phi_{--}(t,1)$ is invertible).

(iv) M^1 and Q^1 are defined by

$$\begin{cases} M^1(t,x) = \left(\dfrac{\partial \Phi}{\partial t}(t,x) + \Lambda(t,x)\dfrac{\partial \Phi}{\partial x}(t,x) + \Phi(t,x)M(t,x)\right)\Phi(t,x)^{-1}, \\ Q^1(t) = \left(\Phi_{+-}(t,0) + \Phi_{++}(t,0)Q(t)\right)\left(\Phi_{--}(t,0) + \Phi_{-+}(t,0)Q(t)\right)^{-1}. \end{cases} \tag{32}$$

Finally, it is not difficult to check that the stability property (28) is indeed satisfied since the state-feedback gain function F is solely determined by the state-feedback gain function F^1 and, at every fixed $t \geq 0$, the transformation (29) defines an injective (in fact, invertible) map of $L^2(0,1)^n$.

2.1.2 Existence of the transformation

Let us now prove the existence of a function Φ with the properties listed above and which in addition ensures that the condition (27) on M^1 holds.

Proposition 2.2. *There exists* Φ *with* $\Phi, \frac{\partial \Phi}{\partial t} + \Lambda\frac{\partial \Phi}{\partial x} \in C^0([0,+\infty) \times [0,1])^{n\times n} \cap L^\infty((0,+\infty) \times (0,1))^{n\times n}$ *such that the properties (i), (ii), (iii) and* (31) *are satisfied and such that the matrix-valued function* M^1 *defined in* (32) *satisfies* (27)*.*

Proof. Let Φ be the diagonal matrix-valued function defined for every $t \geq 0$ and $x \in [0,1]$ by

$$\Phi(t,x) = \mathrm{diag}(\phi_1(t,x), \ldots, \phi_n(t,x)),$$

where, for every $i \in \{1, \ldots, n\}$,

$$\phi_i(t,x) = e^{-\int_{s_i^{\mathrm{in}}(t,x)}^{t} m_{ii}(\sigma, \chi_i(\sigma;t,x))\, d\sigma}, \tag{33}$$

where m_{ii} is extended to negative times by an arbitrary function that keeps the regularity (5). Clearly, $\phi_i \in C^0([0,+\infty) \times [0,1])$ and it follows from (19) that $\phi_i \in L^\infty((0,+\infty) \times (0,1))$.

It is clear that the first property (i) holds since Λ and Φ are both diagonal matrices. Since $\Phi_{-+} = 0$, the third property (iii) is automatically satisfied. It also follows that, to check the second property (ii), we only need to show that $\Phi(t,x)$ is invertible, which readily follows from the explicit expression of ϕ_i. The estimate (31) is obviously true since $\Phi_{--}(t,1) = \mathrm{Id}_{\mathbb{R}^{m\times m}}$ (recall (16)). Finally, M^1 defined in (32) satisfies (27) since ϕ_i satisfies the following linear hyperbolic equation:

$$\frac{\partial \phi_i}{\partial t}(t,x) + \lambda_i(t,x)\frac{\partial \phi_i}{\partial x}(t,x) + m_{ii}(t,x)\phi_i(t,x) = 0.$$

□

Remark 2.3. There are obviously other possible choices for ϕ_i, for instance in the time-independent case we can take the slightly simpler function $\phi_i(t,x) = e^{-\int_0^x \frac{m_{ii}(\xi)}{\lambda_i(\xi)}\, d\xi}$ (which coincides with (33) only for $i \in \{m+1, \ldots, n\}$).

2.2 Volterra transformation

In this section we perform a second transformation to remove some coupling terms of the system. The system will then have a triangular coupling structure, which is the key point to show later on (Section 2.3 below) that this system is finite-time stable with settling time $T_{\mathrm{unif}}(\Lambda)$. More precisely, the goal of this section is to establish the following result:

Proposition 2.4. *There exists a strictly lower triangular matrix* $G^2_{--} = (g^2_{ij})_{1 \leq i,j \leq m} \in C^0([0,+\infty) \times [0,1])^{m\times m} \cap L^\infty((0,+\infty)\times(0,1))^{m\times m}$*:*

$$g^2_{ij} = 0, \quad \forall\, 1 \leq i \leq j \leq m, \tag{34}$$

and there exists $G^2_{+-} \in C^0([0,+\infty)\times[0,1])^{p\times m} \cap L^\infty((0,+\infty)\times(0,1))^{p\times m}$ *such that, for every* $F^2 \in L^\infty((0,+\infty)\times(0,1))^{m\times n}$*, there exists* $F^1 \in L^\infty((0,+\infty)\times(0,1))^{m\times n}$ *such that the following property holds for every* $T > 0$*:*

$$\begin{aligned} &(0, G^2, F^2, Q^1) \textit{ is finite-time stable with settling time } T \\ &\qquad\qquad \Longrightarrow \quad (M^1, 0, F^1, Q^1) \textit{ is finite-time stable with settling time } T, \end{aligned} \tag{35}$$

where

$$G^2 = \begin{pmatrix} G^2_{--} & 0 \\ & \\ G^2_{+-} & 0 \end{pmatrix}. \tag{36}$$

Remark 2.5. Thanks to the triangular structure (34) and (36) of G^2, we can check from the very definition of broad solution (see Definition A.1) that the system provided by Proposition 2.4 with state-feedback gain function equal to zero, i.e. $(0, G^2, 0, Q^1)$, is finite-time stable with settling time $T(0, G^2, 0, Q^1)$ defined by

$$T(0, G^2, 0, Q^1) = \sup_{t^0 \geq 0} s^{\text{out}}_{m+1}\left(T_m(t^0), 0\right) - t^0,$$

where

$$\begin{cases} T_1(t^0) = s^{\text{out}}_1(t^0, 1), \\ \\ T_i(t^0) = s^{\text{out}}_i(T_{i-1}(t^0), 1), \quad \forall i \in \{2, \ldots, m\}. \end{cases}$$

We do not detail this point here because it is not needed, and we refer to the arguments used in the proof of Proposition 2.12 below for an idea of the proof of this assertion. As a result, the combination of Proposition 2.4 with Proposition 2.1 already shows that our initial system $(M, 0, F, Q)$ is finite-time stable for some F, with settling time $T(0, G^2, 0, Q^1)$. However, this time $T(0, G^2, 0, Q^1)$ is always strictly larger than the time $T_{\text{unif}}(\Lambda)$ given in Theorem 1.5 (as long as $m > 1$). In the case of time-independent systems, the time $T(0, G^2, 0, Q^1)$ is the time obtained in [HDMVK16, HVDMK19] and it has the more explicit expression

$$T(0, G^2, 0, Q^1) = \sum_{i=1}^{m} \int_0^1 \frac{1}{-\lambda_i(\xi)}\, d\xi + \int_0^1 \frac{1}{\lambda_{m+1}(\xi)}\, d\xi.$$

2.2.1 Formal computations

Let us now show how to establish Proposition 2.4. As before, the goal is to show that, for every F^2, there exists F^1 such that we can transform a solution of $(M^1, 0, F^1, Q^1)$ into a solution of $(0, G^2, F^2, Q^1)$. Let then w be the solution to the system $(M^1, 0, F^1, Q^1)$ with state-feedback gain function F^1 to be determined below and initial data w^0. Inspired by the works mentioned at the beginning of Section 2, we use a Volterra transformation of the second kind as follows:

$$\gamma(t,x) = w(t,x) - \int_0^x K(t,x,\xi) w(t,\xi)\, d\xi, \tag{37}$$

where we suppose for the moment that the kernel K is smooth on $\overline{\mathcal{T}}$, where $\mathcal{T}$ is the infinite triangular prism defined by

$$\mathcal{T} = \{(t,x,\xi) \in (0,+\infty) \times (0,1) \times (0,1), \qquad \xi < x\}.$$

Let us now perform some formal computations to see what γ can solve. We have

$$\begin{aligned}\frac{\partial \gamma}{\partial t}(t,x) + \Lambda(t,x)\frac{\partial \gamma}{\partial x}(t,x) = {} & \frac{\partial w}{\partial t}(t,x) + \Lambda(t,x)\frac{\partial w}{\partial x}(t,x) \\ & - \int_0^x \frac{\partial K}{\partial t}(t,x,\xi)w(t,\xi)\,d\xi - \int_0^x K(t,x,\xi)\frac{\partial w}{\partial t}(t,\xi)\,d\xi \\ & - \Lambda(t,x)K(t,x,x)w(t,x) - \Lambda(t,x)\int_0^x \frac{\partial K}{\partial x}(t,x,\xi)w(t,\xi)\,d\xi.\end{aligned}$$

Using the equation satisfied by w, we obtain

$$\begin{aligned}\frac{\partial \gamma}{\partial t}(t,x) + \Lambda(t,x)\frac{\partial \gamma}{\partial x}(t,x) = {} & M^1(t,x)w(t,x) \\ & - \int_0^x \frac{\partial K}{\partial t}(t,x,\xi)w(t,\xi)\,d\xi - \int_0^x K(t,x,\xi)\left(-\Lambda(t,\xi)\frac{\partial w}{\partial \xi}(t,\xi) + M^1(t,\xi)w(t,\xi)\right)d\xi \\ & - \Lambda(t,x)K(t,x,x)w(t,x) - \Lambda(t,x)\int_0^x \frac{\partial K}{\partial x}(t,x,\xi)w(t,\xi)\,d\xi.\end{aligned}$$

Integrating by parts the third term of the right hand side and using the boundary condition $w_+(t,0) = Q^1(t)w_-(t,0)$, we finally obtain

$$\begin{aligned}\frac{\partial \gamma}{\partial t}(t,x) + \Lambda(t,x)\frac{\partial \gamma}{\partial x}(t,x) = {} & \int_0^x \Bigg(- \frac{\partial K}{\partial t}(t,x,\xi) - \frac{\partial K}{\partial \xi}(t,x,\xi)\Lambda(t,\xi) - K(t,x,\xi)\frac{\partial \Lambda}{\partial \xi}(t,\xi) \\ & - K(t,x,\xi)M^1(t,\xi) - \Lambda(t,x)\frac{\partial K}{\partial x}(t,x,\xi)\Bigg)w(t,\xi)\,d\xi \\ & + \big(M^1(t,x) + K(t,x,x)\Lambda(t,x) - \Lambda(t,x)K(t,x,x)\big)\,w(t,x) - K(t,x,0)\Lambda(t,0)\begin{pmatrix}\mathrm{Id}_{\mathbb{R}^{m\times m}} \\ Q^1(t)\end{pmatrix} w_-(t,0).\end{aligned}$$

On the other hand, since $\gamma(t,0) = w(t,0)$, γ satisfies the same boundary condition as w at $x = 0$:

$$\gamma_+(t,0) - Q^1(t)\gamma_-(t,0) = w_+(t,0) - Q^1(t)w_-(t,0) = 0.$$

Finally, at $x = 1$, we have

$$\begin{aligned}\gamma_-(t,1) - \int_0^1 F^2(t,\xi)\gamma(t,\xi)\,d\xi = {} & \\ & \int_0^1 \left(F^1(t,\xi) - K_-(t,1,\xi) - F^2(t,\xi) + \int_\xi^1 F^2(t,\zeta)K(t,\zeta,\xi)\,d\zeta\right)w(t,\xi)\,d\xi,\end{aligned}$$

where K_- denotes the $m\times n$ sub-matrix of K formed by its first m rows. Thus, we see that γ satisfies at $x = 1$ the boundary condition $\gamma_-(t,1) = \int_0^1 F^2(t,\xi)\gamma(t,\xi)\,d\xi$ if we take

$$F^1(t,\xi) = K_-(t,1,\xi) + F^2(t,\xi) - \int_\xi^1 F^2(t,\zeta)K(t,\zeta,\xi)\,d\zeta. \tag{38}$$

Note that F^1 belongs to $L^\infty((0,+\infty)\times(0,1))^{m\times n}$ provided that F^2 belongs to this space as well and

$$K \in L^\infty(\mathcal{T})^{n\times n}.$$

In summary, γ defined by (37) is the solution to $(0, G^2, F^2, Q^1)$ with initial data $\gamma^0(x) = w^0(x) - \int_0^x K(t^0, x, \xi) w^0(\xi)\, d\xi$ if we have the following two properties:

(i) For every $(t, x, \xi) \in \mathcal{T}$,

$$\begin{cases} \dfrac{\partial K}{\partial t}(t,x,\xi) + \Lambda(t,x)\dfrac{\partial K}{\partial x}(t,x,\xi) + \dfrac{\partial K}{\partial \xi}(t,x,\xi)\Lambda(t,\xi) \\ \qquad\qquad\qquad\qquad + K(t,x,\xi)\left(\dfrac{\partial \Lambda}{\partial \xi}(t,\xi) + M^1(t,\xi)\right) = 0, \\ K(t,x,x)\Lambda(t,x) - \Lambda(t,x)K(t,x,x) = -M^1(t,x). \end{cases} \tag{39}$$

(ii) G^2 is defined by

$$G^2 = \begin{pmatrix} G^2_{--} & 0 \\ G^2_{+-} & 0 \end{pmatrix},$$

with

$$\begin{cases} G^2_{--}(t,x) = -K_{--}(t,x,0)\Lambda_{--}(t,0) - K_{-+}(t,x,0)\Lambda_{++}(t,0)Q^1(t), \\ G^2_{+-}(t,x) = -K_{+-}(t,x,0)\Lambda_{--}(t,0) - K_{++}(t,x,0)\Lambda_{++}(t,0)Q^1(t). \end{cases} \tag{40}$$

Finally, the stability property (35) is clearly satisfied since, at every fixed $t \geq 0$, the Volterra transformation (37) defines an injective map of $L^2(0,1)^n$ (see e.g. [Hoc73, Theorem 2.6]).

2.2.2 The kernel equations

We can prove that there exists $K \in C^0(\overline{\mathcal{T}})^{n\times n} \cap L^\infty(\mathcal{T})^{n\times n}$ that satisfies the so-called "kernel equations" (39) in the sense of broad solutions. However, it is in general not enough to deduce stability results for the initial system $(M, 0, F, Q)$ since the investigation of the stability properties of the system $(0, G^2, F^2, Q^1)$ is not an easier task without knowing any more information about it. The breakthrough idea of the conference paper [HDM15] in the time-independent case (see also [HDMVK16, HVDMK19]) was to construct a solution K to the kernel equations which, in addition, yields a simpler structure for the matrix G^2_{--} defined in (40). This is the key point to prove stability results for the system $(0, G^2, F^2, Q^1)$ (see Remark 2.5 and Section 2.3 below). Such a construction is possible by adding some conditions for K_{--} at $(t, x, 0)$ (see (40)) but the price to pay is that it introduces discontinuities for K_{--}, so that K will not be globally C^0 anymore but only piecewise C^0 in general. We will prove the following result:

Theorem 2.6. *There exists a $n \times n$ matrix-valued function $K = (k_{ij})_{1\leq i,j\leq n}$ such that:*

(i) $K \in L^\infty(\mathcal{T})^{n\times n}$.

(ii) For every $i, j \in \{1, \dots, n\}$ with $j \notin \{i+1, \dots, m\}$, we have $k_{ij} \in C^0(\overline{\mathcal{T}})$.

(iii) For every $i, j \in \{1, \dots, m\}$ with $i < j$, we have $k_{ij} \in C^0(\overline{\mathcal{T}^-_{ij}}) \cap C^0(\overline{\mathcal{T}^+_{ij}})$, where (see Figure 1)

$$\mathcal{T}^-_{ij} = \{(t,x,\xi) \in \mathcal{T}, \quad \xi < \psi_{ij}(t,x)\},$$
$$\mathcal{T}^+_{ij} = \{(t,x,\xi) \in \mathcal{T}, \quad \xi > \psi_{ij}(t,x)\},$$

where $\psi_{ij} \in C^1([0,+\infty) \times [0,1])$ satisfies the following semi-linear hyperbolic equation for every $t \geq 0$ and $x \in [0,1]$:

$$\begin{cases} \dfrac{\partial \psi_{ij}}{\partial t}(t,x) + \lambda_i(t,x)\dfrac{\partial \psi_{ij}}{\partial x}(t,x) - \lambda_j(t,\psi_{ij}(t,x)) = 0, \\ \psi_{ij}(t,0) = 0. \end{cases} \tag{41}$$

(iv) K is a broad solution of (39) *in $\mathcal{T}$ (the exact meaning of this statement will be detailed during the proof of the theorem, in Section 3.2 below).*

(v) For every $t \geq 0$ and $x \in [0,1]$, the matrix $G^2_{--}(t,x)$ defined in (40) *is strictly lower triangular, i.e. it satisfies* (34) *(it then follows from (ii) that $G^2_{--} \in C^0([0,+\infty) \times [0,1])^{m\times m}$).*

The proof of Theorem 2.6 is one of the main technical difficulties of this article and it is postponed to Section 3.2 below for the sake of the presentation. We conclude this section with some important remarks.

Remark 2.7. Let us rewrite the second condition of (39) component-wise:

$$(\lambda_j(t,x) - \lambda_i(t,x))\, k_{ij}(t,x,x) = -m^1_{ij}(t,x). \tag{42}$$

Therefore, we see that for $i = j$ we shall necessarily have $m^1_{ii} = 0$ and it explains why we had to perform a preliminary transformation in Section 2.1 to remove these terms (otherwise the equation (42), and thus the kernel equations (39), have no solution).

Remark 2.8. It is in general not possible to solve (39) with $G^2_{--} = 0$, unless $m = 1$.

Remark 2.9. Observe that, with the regularity stated in Theorem 2.6, we have in particular that, for every $w \in C^0([t^0,+\infty); L^2(0,1)^n)$, $t^0 \geq 0$,

$$(t,x) \mapsto \int_0^x K(t,x,\xi) w(t,\xi)\, d\xi \in C^0([t^0,+\infty) \times [0,1])^n.$$

This follows from Lebesgue's dominated convergence theorem. This shows that γ defined by (37) has the good regularity to be a broad solution (see Definition A.1), if so has w.

Remark 2.10. Observe that the condition (iii) shows that the kernel has possible discontinuities on $\xi = \psi_{ij}(t,x)$ for $i < j \leq m$. Besides, these discontinuities also depend on the component of the kernel that we consider. The appearance of such discontinuities is explained by the requirement of the last condition (v) because we somehow force two boundary conditions at the points $(t,0,0)$, one by the condition already required in (39) (which concerns $i \neq j$, see Remark 2.7) and another one by (v) (which only concerns $i \leq j \leq m$). This results in discontinuities along the characteristics passing through these points. Note that this also complicates the justification of formal computations that we performed above since regularity problems will occur during the computation of the following term (when $i < j \leq m$):

$$\frac{\partial}{\partial t}\left(\int_0^x k_{ij}(t,x,\xi) w_j(t,\xi)\, d\xi\right) + \lambda_i(t,x)\frac{\partial}{\partial x}\left(\int_0^x k_{ij}(t,x,\xi) w_j(t,\xi)\, d\xi\right).$$

More precisely, writing $\int_0^x = \int_0^{\psi_{ij}(t,x)} + \int_{\psi_{ij}(t,x)}^x$ and using integration by parts, we see that the following jump terms notably appear:

$$\left(\frac{\partial \psi_{ij}}{\partial t}(t,x) - \lambda_j(t,\psi_{ij}(t,x)) + \lambda_i(t,x)\frac{\partial \psi_{ij}}{\partial x}(t,x)\right) \\ \times \left(k^-_{ij}(t,x,\psi_{ij}(t,x)) - k^+_{ij}(t,x,\psi_{ij}(t,x))\right) w_j(t,\psi_{ij}(t,x)),$$

where k_{ij}^- (resp. k_{ij}^+) denotes the trace of the restriction of k_{ij} to $\partial\mathcal{T}_{ij}^-$ (resp $\partial\mathcal{T}_{ij}^+$). This is why it is crucial to precise that ψ_{ij} solves the first equation in (41) so that such undesired terms vanish in the end. In the case of time-independent systems, we have in fact

$$\psi_{ij}(t,x) = \phi_j^{-1}\left(\phi_i(x)\right),$$

where we introduced $\phi_\ell(x) = \int_0^x \frac{1}{-\lambda_\ell(\xi)}\, d\xi$ for $\ell \in \{1, \dots, m\}$ (ψ_{ij} is well defined because $i < j$). This is the same function as in [HVDMK19, (A.1)].

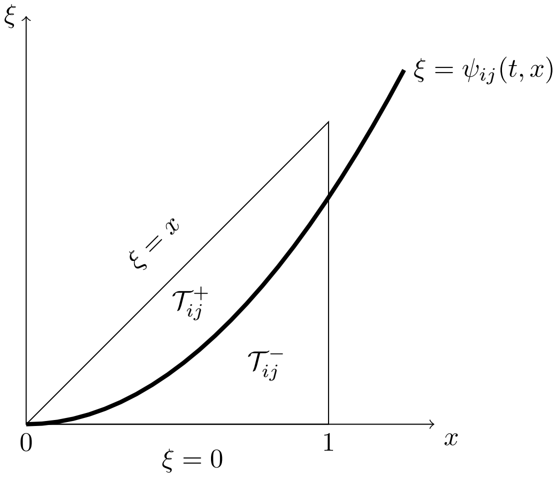

Figure 1: 2D cross-section of the domain $\mathcal{T}$ at a fixed t

2.3 Fredholm integral transformation

We recall that at the moment we already know that the system $(0, G^2, F^2, Q^1)$ of Proposition 2.4 is finite-time stable if we take $F^2 = 0$, but only with a settling time which is strictly larger than $T_{\mathrm{unif}}(\Lambda)$ (unless $m = 1$), see Remark 2.5. In this section, we perform a third and last transformation to remove the coupling term G^2_{--} in the system $(0, G^2, F^2, Q^1)$ and we show that the resulting system has the desired stability properties. More precisely, the goal of this section is to establish the two following results:

Proposition 2.11. *There exists $F^2 \in L^\infty((0,+\infty) \times (0,1))^{m \times n}$ such that the following property holds for every $T > 0$:*

$$\begin{aligned} &(0, G^3, 0, Q^1) \text{ is finite-time stable with settling time } T \\ &\qquad\qquad \Longrightarrow \quad (0, G^2, F^2, Q^1) \text{ is finite-time stable with settling time } T, \end{aligned} \tag{43}$$

where

$$G^3 = \begin{pmatrix} 0 & 0 \\ G^2_{+-} & 0 \end{pmatrix}. \tag{44}$$

Proposition 2.12. *The system $(0, G^3, 0, Q^1)$ is finite-time stable with settling time $T_{\mathrm{unif}}(\Lambda)$ defined by* (22).

Note that the proof of our main result – Theorem 1.5 – will then be complete (recall Propositions 2.1 and 2.4), except for the τ-periodicity statement which will be studied later on in Section 3.3.

2.3.1 Finite-time stability of the system $(0, G^3, 0, Q^1)$

In this section we prove Proposition 2.12 in four steps.

1) Let $t^0 \geq 0$ be fixed. From the very definition of broad solution (see Definition A.1) and the simple structure of G^3, we see that the first m components of the system vanish at time $t^0 + T_{\text{unif}}(\Lambda)$ if (recall that the feedback is equal to zero)

$$s_i^{\text{in}}(t^0 + T_{\text{unif}}(\Lambda), x) > t^0, \quad \forall x \in [0,1], \quad \forall i \in \{1, \dots, m\}, \tag{45}$$

and the remaining p components of the system vanish at time $t^0 + T_{\text{unif}}(\Lambda)$ if

$$\begin{cases} s_i^{\text{in}}(t^0 + T_{\text{unif}}(\Lambda), x) > t^0, \quad \forall x \in [0,1], \quad \forall i \in \{m+1, \dots, n\} \\ s_j^{\text{in}}\left(s_i^{\text{in}}(t^0 + T_{\text{unif}}(\Lambda), x), 0\right) > t^0, \quad \forall x \in [0,1), \quad \forall j \in \{1, \dots, m\}, \quad \forall i \in \{m+1, \dots, n\}. \end{cases} \tag{46}$$

2) First of all, observe that, from (20), (17) and (16) we have the following inverse formula for every $t, \bar{t} \in \mathbb{R}$:

$$\begin{cases} s_i^{\text{in}}(t,0) > \bar{t} \iff t > s_i^{\text{out}}(\bar{t}, 1), & \text{if } i \in \{1, \dots, m\}, \\ s_i^{\text{in}}(t,1) \geq \bar{t} \iff t \geq s_i^{\text{out}}(\bar{t}, 0), & \text{if } i \in \{m+1, \dots, n\}. \end{cases} \tag{47}$$

3) Let us establish (45). Let then $i \in \{1, \dots, m\}$ be fixed. We have:

$$\begin{aligned} s_i^{\text{in}}(t^0 + T_{\text{unif}}(\Lambda), x) > t^0, \quad \forall x \in [0,1] &\iff s_i^{\text{in}}(t^0 + T_{\text{unif}}(\Lambda), 0) > t^0, \quad \text{(by (20))}, \\ &\iff t^0 + T_{\text{unif}}(\Lambda) > s_i^{\text{out}}\left(t^0, 1\right), \quad \text{(by (47))}, \end{aligned}$$

and this last statement holds true since, by definition of $T_{\text{unif}}(\Lambda)$ and (15)-(16), we have, for an arbitrary $j \in \{m+1, \dots, n\}$,

$$t^0 + T_{\text{unif}}(\Lambda) \geq s_j^{\text{out}}(s_i^{\text{out}}(t^0, 1), 0) > s_i^{\text{out}}(t^0, 1).$$

4) Let us now establish (46). We focus on the second inequality since the first one is obtained similarly to (45). Let then $i \in \{m+1, \dots, n\}$ and $j \in \{1, \dots, m\}$ be fixed. We have:

$$\begin{aligned} &s_j^{\text{in}}\left(s_i^{\text{in}}(t^0 + T_{\text{unif}}(\Lambda), x), 0\right) > t^0, \quad \forall x \in [0,1) \\ &\iff s_i^{\text{in}}(t^0 + T_{\text{unif}}(\Lambda), x) > s_j^{\text{out}}\left(t^0, 1\right), \quad \forall x \in [0,1), \quad \text{(by (47))}, \\ &\iff s_i^{\text{in}}(t^0 + T_{\text{unif}}(\Lambda), 1) \geq s_j^{\text{out}}\left(t^0, 1\right), \quad \text{(by (20))}, \\ &\iff t^0 + T_{\text{unif}}(\Lambda) \geq s_i^{\text{out}}\left(s_j^{\text{out}}\left(t^0, 1\right), 0\right), \quad \text{(by (47))}, \end{aligned}$$

and this last statement holds true by definition of $T_{\text{unif}}(\Lambda)$.

□

2.3.2 Proof of Proposition 2.11

We start the proof with some computations. We will show that we can transform a solution of $(0, G^3, 0, Q^1)$ into a solution of $(0, G^2, F^2, Q^1)$ (note the difference in the order of the transformation with respect to the previous sections and see Remark 2.15 below for the reason). Let then z be the solution to the system $(0, G^3, 0, Q^1)$ with initial data z^0. Inspired by the work [CHO17] mentioned before (for time-independent systems), we propose to use a Fredholm integral transformation as follows:

$$\gamma(t,x) = z(t,x) - \int_0^1 H(t,x,\xi) z(t,\xi)\, d\xi, \tag{48}$$

where we suppose for the moment that the kernel H is smooth on $\overline{\mathcal{R}}$, where $\mathcal{R}$ is the infinite rectangular prism defined by

$$\mathcal{R} = (0,+\infty) \times (0,1) \times (0,1).$$

Let us now perform some formal computations and see what γ can solve. We have

$$\begin{aligned} \frac{\partial \gamma}{\partial t}(t,x) + \Lambda(t,x) \frac{\partial \gamma}{\partial x}(t,x) &= \frac{\partial z}{\partial t}(t,x) + \Lambda(t,x) \frac{\partial z}{\partial x}(t,x) \\ &\quad - \int_0^1 \frac{\partial H}{\partial t}(t,x,\xi) z(t,\xi)\, d\xi - \int_0^1 H(t,x,\xi) \frac{\partial z}{\partial t}(t,\xi)\, d\xi - \Lambda(t,x) \int_0^1 \frac{\partial H}{\partial x}(t,x,\xi) z(t,\xi)\, d\xi. \end{aligned}$$

Using the equation satisfied by z, we obtain

$$\begin{aligned} \frac{\partial \gamma}{\partial t}(t,x) + \Lambda(t,x) \frac{\partial \gamma}{\partial x}(t,x) &= G^3(t,x) z(t,0) - \int_0^1 \frac{\partial H}{\partial t}(t,x,\xi) z(t,\xi)\, d\xi \\ &\quad - \int_0^1 H(t,x,\xi) \left(-\Lambda(t,\xi) \frac{\partial z}{\partial \xi}(t,\xi) + G^3(t,\xi) z(t,0) \right) d\xi - \Lambda(t,x) \int_0^1 \frac{\partial H}{\partial x}(t,x,\xi) z(t,\xi)\, d\xi. \end{aligned}$$

Integrating by parts the third term of the right hand side, we obtain

$$\begin{aligned} \frac{\partial \gamma}{\partial t}(t,x) + \Lambda(t,x) \frac{\partial \gamma}{\partial x}(t,x) = \int_0^1 \Bigg(&- \frac{\partial H}{\partial t}(t,x,\xi) - \frac{\partial H}{\partial \xi}(t,x,\xi) \Lambda(t,\xi) - H(t,x,\xi) \frac{\partial \Lambda}{\partial \xi}(t,\xi) \\ &- \Lambda(t,x) \frac{\partial H}{\partial x}(t,x,\xi) \Bigg) z(t,\xi)\, d\xi \\ + H(t,x,1)\Lambda(t,1) z(t,1) &+ \left(G^3(t,x) - H(t,x,0)\Lambda(t,0) - \int_0^1 H(t,x,\xi) G^3(t,\xi)\, d\xi \right) z(t,0). \end{aligned}$$

Using the formula (48) with $x = 0$ we finally obtain

$$\begin{aligned} \frac{\partial \gamma}{\partial t}(t,x) + \Lambda(t,x) \frac{\partial \gamma}{\partial x}(t,x) &= \int_0^1 \Bigg(- \frac{\partial H}{\partial t}(t,x,\xi) - \frac{\partial H}{\partial \xi}(t,x,\xi) \Lambda(t,\xi) - H(t,x,\xi) \frac{\partial \Lambda}{\partial \xi}(t,\xi) \\ - \Lambda(t,x) \frac{\partial H}{\partial x}(t,x,\xi) &+ \left(G^3(t,x) - H(t,x,0)\Lambda(t,0) - \int_0^1 H(t,x,\zeta) G^3(t,\zeta)\, d\zeta \right) H(t,0,\xi) \Bigg) z(t,\xi)\, d\xi \\ + H(t,x,1)\Lambda(t,1) z(t,1) &+ \left(G^3(t,x) - H(t,x,0)\Lambda(t,0) - \int_0^1 H(t,x,\xi) G^3(t,\xi)\, d\xi \right) \gamma(t,0). \end{aligned}$$

Since

$$z_-(t,1) = 0, \tag{49}$$

the boundary term $H(t,x,1)\Lambda(t,1)z(t,1)$ vanishes if we require that H satisfies

$$H_{-+}(t,x,1) = H_{++}(t,x,1) = 0.$$

On the other hand, γ and z satisfy the same boundary condition at $x = 0$ provided that

$$H(t,0,\xi) = 0.$$

Finally, at $x = 1$, we have (recall (49))

$$\gamma_-(t,1) - \int_0^1 F^2(t,\xi)\gamma(t,\xi)\,d\xi = \int_0^1 \left(-H_-(t,1,\xi) - F^2(t,\xi) + \int_0^1 F^2(t,\zeta)H(t,\zeta,\xi)\,d\zeta\right) z(t,\xi)\,d\xi,$$

where H_- denotes again the $m \times n$ sub-matrix of H formed by its first m rows. Thus, we see that γ satisfies at $x = 1$ the boundary condition $\gamma_-(t,1) = \int_0^1 F^2(t,\xi)\gamma(t,\xi)\,d\xi$ if $F^2(t,\cdot)$ satisfies the following Fredholm integral equation (at t fixed):

$$F^2(t,\xi) - \int_0^1 F^2(t,\zeta)H(t,\zeta,\xi)\,d\zeta = -H_-(t,1,\xi). \tag{50}$$

In summary, γ defined by (48) is the solution of $(0, G^2, F^2, Q^1)$ with state-feedback gain function F^2 satisfying (50) (whenever it exists) and initial data $\gamma^0(x) = z^0(x) - \int_0^1 H(t^0,x,\xi)z^0(\xi)\,d\xi$ if we have the following two properties:

(i) For every $(t,x,\xi) \in \mathcal{R}$,

$$\begin{cases} \dfrac{\partial H}{\partial t}(t,x,\xi) + \Lambda(t,x)\dfrac{\partial H}{\partial x}(t,x,\xi) + \dfrac{\partial H}{\partial \xi}(t,x,\xi)\Lambda(t,\xi) + H(t,x,\xi)\dfrac{\partial \Lambda}{\partial \xi}(t,\xi) = 0, \\ H_{-+}(t,x,1) = H_{++}(t,x,1) = H(t,0,\xi) = 0. \end{cases} \tag{51}$$

(ii) G^3 satisfies the Fredholm integral equation

$$G^3(t,x) - \int_0^1 H(t,x,\xi)G^3(t,\xi)\,d\xi = G^2(t,x) + H(t,x,0)\Lambda(t,0). \tag{52}$$

Finally, the stability property (43) is clearly satisfied if, for every $t \geq 0$, the Fredholm transformation (48) defines a surjective map of $L^2(0,1)^n$.

It remains to prove the existence of F^2 and H satisfying the above properties and so that the Fredholm transformation (48) is invertible (let us recall that, unlike Volterra transformations of the second kind, Fredholm transformations are not always invertible). Note that $H = 0$ is a solution of (51). Taking into account the very particular structure (36) of G^2, this motivates our attempt to look for a kernel H with the following simple structure:

$$H = \begin{pmatrix} H_{--} & 0 \\ 0 & 0 \end{pmatrix}. \tag{53}$$

This structure implies that the Fredholm equation (52) is equivalent to

$$\begin{cases} G^3_{--}(t,x) - \displaystyle\int_0^1 H_{--}(t,x,\xi)G^3_{--}(t,\xi)\,d\xi = G^2_{--}(t,x) + H_{--}(t,x,0)\Lambda_{--}(t,0), \\ G^3_{-+}(t,x) - \displaystyle\int_0^1 H_{--}(t,x,\xi)G^3_{-+}(t,\xi)\,d\xi = 0, \\ G^3_{+-}(t,x) = G^2_{+-}(t,x), \\ G^3_{++}(t,x) = 0. \end{cases}$$

These equations are easily solved by taking $G^3_{--} = G^3_{-+} = 0$ (so that G^3 is indeed given by (44)) if we impose the following condition for H_{--} at $(t, x, 0)$:

$$H_{--}(t,x,0) = -G^2_{--}(t,x)\Lambda_{--}(t,0)^{-1}.$$

We point out that this last condition may introduce discontinuities in the kernel, because of possible compatibility conditions at $(t, 0, 0)$ with the previous requirement that $H_{--}(t, 0, \xi) = 0$.

Finally, since G^2_{--} is in fact strictly lower triangular (34), we also look for H_{--} with the same structure. Note that this structure in particular ensures that the Fredholm transformation (48) is invertible and that the Fredholm equation (50) always has a unique solution $F^2 \in L^\infty((0,+\infty) \times (0,1))^{m\times n}$, provided that $H_{--} \in L^\infty(\mathcal{R})^{m\times m}$ (see for instance [CHO17, Appendix] for more details). This property was a priori not guaranteed without additional information (we emphasize again that Fredholm transformations are not always invertible).

The final step is to prove the existence of H_{--} that satisfies all the properties mentioned above. This is the goal of the following theorem, the proof of which is given in Section 3.1 below.

Theorem 2.13. *There exists a $m \times m$ matrix-valued function $H_{--} = (h_{ij})_{1\leq i,j\leq m}$ such that:*

(i) $H_{--} \in L^\infty(\mathcal{R})^{m\times m}$.

(ii) For $i \leq j$, we have $h_{ij} = 0$ (i.e. H_{--} is strictly lower triangular).

(iii) For $i > j$, we have $h_{ij} \in C^0(\overline{\mathcal{R}^-_{ij}}) \cap C^0(\overline{\mathcal{R}^+_{ij}})$, where

$$\mathcal{R}^-_{ij} = \{(t,x,\xi) \in \mathcal{R}, \quad \xi < \psi_{ij}(t,x)\},$$
$$\mathcal{R}^+_{ij} = \{(t,x,\xi) \in \mathcal{R}, \quad \xi > \psi_{ij}(t,x)\},$$

where $\psi_{ij} \in C^1([0,+\infty) \times [0,1])$ satisfies the semi-linear hyperbolic equation (41).

(iv) H_{--} is the unique broad solution in $\mathcal{R}$ of the system

$$\begin{cases} \dfrac{\partial H_{--}}{\partial t}(t,x,\xi) + \Lambda_{--}(t,x)\dfrac{\partial H_{--}}{\partial x}(t,x,\xi) + \dfrac{\partial H_{--}}{\partial \xi}(t,x,\xi)\Lambda_{--}(t,\xi) \\ \qquad\qquad\qquad\qquad\qquad\qquad + H_{--}(t,x,\xi)\dfrac{\partial \Lambda_{--}}{\partial \xi}(t,\xi) = 0, \\ H_{--}(t,0,\xi) = 0, \\ H_{--}(t,x,0) = -G^2_{--}(t,x)\Lambda_{--}(t,0)^{-1}, \end{cases} \tag{54}$$

(once again, the exact meaning of this statement will be detailed during the proof of the theorem, in Section 3.1 below).

This concludes the proof of Proposition 2.11.

□

Remark 2.14. Observe once again that the kernel is discontinuous. This introduces some additional boundary terms along these discontinuities in the formal computations performed above but, as mentioned before in Remark 2.10, these terms cancel each other out thanks to the equation satisfied by ψ_{ij} in (41). As in the time-independent case ([CHO17, Section 3]), the system (54) is easy to solve and its solution is even explicit (see (60)-(58) below).

Remark 2.15. If we prefer to use the inverse transformation

$$z(t,x) = \gamma(t,x) - \int_0^1 L(t,x,\xi)\gamma(t,\xi)\,d\xi, \tag{55}$$

where L has the same structure as H (i.e. only L_{--} is not zero), then the corresponding kernel equations are

$$\begin{cases} \dfrac{\partial L_{--}}{\partial t}(t,x,\xi) + \Lambda_{--}(t,x)\dfrac{\partial L_{--}}{\partial x}(t,x,\xi) + \dfrac{\partial L_{--}}{\partial \xi}(t,x,\xi)\Lambda_{--}(t,\xi) \\ \qquad\qquad\qquad + L_{--}(t,x,\xi)\dfrac{\partial \Lambda_{--}}{\partial \xi}(t,\xi) - L_{--}(t,x,1)\Lambda_{--}(t,1)L_{--}(t,1,\xi) = 0, \\ L_{--}(t,0,\xi) = 0, \\ L_{--}(t,x,0) = \left(G^2_{--}(t,x) - \displaystyle\int_0^1 L_{--}(t,x,\xi)G^2_{--}(t,\xi)d\xi \right) \Lambda_{--}(t,0)^{-1}. \end{cases}$$

We see that these equations are slightly more complicated than (54) since there is a nonlinear and nonlocal term. This explains why we had a preference for the transformation (48) over (55) but there is no obstruction to work with (55).

3 Existence of a solution to the kernel equations

In this section we prove Theorem 2.6 and Theorem 2.13, which are the two key results for the present article, and we describe in Section 3.3 how to obtain a time-periodic feedback. We propose to start with the proof of Theorem 2.13 because it is far more simpler (in particular, no fixed-point argument is needed).

3.1 Kernel for the Fredholm transformation

In this section we prove Theorem 2.13, that is we prove the existence of a suitably smooth matrix-valued function $H_{--} = (h_{ij})_{1 \leq i,j \leq m}$ which is strictly lower triangular and satisfies (54) (in some sense).

Writing (54) component-wise, this gives

$$\begin{cases} \dfrac{\partial h_{ij}}{\partial t}(t,x,\xi) + \lambda_i(t,x)\dfrac{\partial h_{ij}}{\partial x}(t,x,\xi) + \lambda_j(t,\xi)\dfrac{\partial h_{ij}}{\partial \xi}(t,x,\xi) + \dfrac{\partial \lambda_j}{\partial \xi}(t,\xi)h_{ij}(t,x,\xi) = 0, \\ h_{ij}(t,0,\xi) = 0, \\ h_{ij}(t,x,0) = -\dfrac{g^2_{ij}(t,x)}{\lambda_j(t,0)}. \end{cases} \tag{56}$$

Since we see that the equations are uncoupled, we can fix the indices i,j for the remainder of Section 3.1:

$$i,j \in \{1,\dots,m\} \text{ are fixed.}$$

3.1.1 The characteristics of (56)

For each $(t,x,\xi) \in \mathbb{R}^3$ fixed, we introduce the characteristic curve $\chi_{ij}(\cdot;t,x)$ associated with the hyperbolic equation (56) passing through the point (t,x,ξ), i.e.

$$\chi_{ij}(s;t,x,\xi) = (s, \chi_i(s;t,x), \chi_j(s;t,\xi)), \quad \forall s \in \mathbb{R},$$

where we recall that χ_i and χ_j are defined in (9). For every $(t,x,\xi) \in \mathcal{R}$, we have

$$\chi_{ij}(s;t,x,\xi) \in \mathcal{R}, \quad \forall s \in (s_{ij}^{\text{in}}(t,x,\xi), s_{ij}^{\text{out}}(t,x,\xi)),$$

where we introduced

$$s_{ij}^{\text{in}}(t,x,\xi) = \max\left\{0, s_i^{\text{in}}(t,x), s_j^{\text{in}}(t,\xi)\right\} > 0, \qquad s_{ij}^{\text{out}}(t,x,\xi) = \min\left\{s_i^{\text{out}}(t,x), s_j^{\text{out}}(t,\xi)\right\}.$$

Since the speeds λ_i, λ_j are negative $(i, j \leq m)$, when s is increasing, $s \mapsto \chi_i(s;t,x), s \mapsto \chi_j(s;t,\xi)$ are decreasing. Therefore, the associated characteristic $\chi_{ij}(\cdot;t,x,\xi)$ will exit the domain $\mathcal{R}$ through the planes $x = 0$ or $\xi = 0$. This is why we can impose boundary conditions at $(t,0,\xi)$ and $(t,x,0)$ (see (56)) and this is why it is enough to (uniquely) determine a solution on $\mathcal{R}$. To be more precise, we can split $\mathcal{R}$ into three disjoint subsets:

$$\mathcal{R} = \mathcal{R}_{ij}^{+} \cup \mathcal{R}_{ij}^{-} \cup \mathcal{D}_{ij},$$

where

$$\begin{aligned}
\mathcal{R}_{ij}^{+} &= \left\{(t,x,\xi) \in \mathcal{R}, \quad s_i^{\text{out}}(t,x) < s_j^{\text{out}}(t,\xi)\right\}, \\
\mathcal{R}_{ij}^{-} &= \left\{(t,x,\xi) \in \mathcal{R}, \quad s_i^{\text{out}}(t,x) > s_j^{\text{out}}(t,\xi)\right\}, \\
\mathcal{D}_{ij} &= \left\{(t,x,\xi) \in \mathcal{R}, \quad s_i^{\text{out}}(t,x) = s_j^{\text{out}}(t,\xi)\right\}.
\end{aligned}$$

With these notations, the characteristic $\chi_{ij}(\cdot;t,x,\xi)$ will either exit the domain $\mathcal{R}$ through the plane $x = 0$ if $(t,x,\xi) \in \mathcal{R}_{ij}^{+}$ or through the plane $\xi = 0$ if $(t,x,\xi) \in \mathcal{R}_{ij}^{-}$:

Proposition 3.1.

(i) For every $(t,x,\xi) \in \mathcal{R}_{ij}^{+}$, we have $\chi_{ij}(s;t,x,\xi) \in \mathcal{R}_{ij}^{+}$ for every $s \in (t, s_i^{\text{out}}(t,x))$.

(ii) For every $(t,x,\xi) \in \mathcal{R}_{ij}^{-}$, we have $\chi_{ij}(s;t,x,\xi) \in \mathcal{R}_{ij}^{-}$ for every $s \in (t, s_j^{\text{out}}(t,\xi))$.

(iii) For every $(t,x,\xi) \in \mathcal{D}_{ij}$, we have $\chi_{ij}(s;t,x,\xi) \in \mathcal{D}_{ij}$ for every $s \in (t, s_i^{\text{out}}(t,x)) = (t, s_j^{\text{out}}(t,\xi))$.

These three points directly follow from (17).

3.1.2 Existence and regularity of a solution to (56)

Writing the solution of (56) along the characteristic curve $\chi_{ij}(s;t,x,\xi)$ for $s \in [s_{ij}^{\text{in}}(t,x,\xi), s_{ij}^{\text{out}}(t,x,\xi)]$ and using the boundary conditions, we obtain the following ODE:

$$\begin{cases} \dfrac{d}{ds} h_{ij}(\chi_{ij}(s;t,x,\xi)) = -\dfrac{\partial \lambda_j}{\partial \xi}(s, \chi_j(s;t,\xi)) h_{ij}(\chi_{ij}(s;t,x,\xi)), \\ h_{ij}(\chi_{ij}(s_{ij}^{\text{out}}(t,x,\xi);t,x,\xi)) = b_{ij}(t,x,\xi), \end{cases} \tag{57}$$

where

$$b_{ij}(t,x,\xi) = \begin{cases} 0 & \text{if } (t,x,\xi) \in \mathcal{R}_{ij}^{+}, \\ -\dfrac{g_{ij}^2(s_j^{\text{out}}(t,\xi), \chi_i(s_j^{\text{out}}(t,\xi);t,x))}{\lambda_j(s_j^{\text{out}}(t,\xi), 0)} & \text{if } (t,x,\xi) \in \mathcal{R}_{ij}^{-}. \end{cases} \tag{58}$$

Integrating this ODE over $[t, s_{ij}^{\text{out}}(t,x,\xi)]$ yields the integral equation

$$h_{ij}(t,x,\xi) = b_{ij}(t,x,\xi) + \int_t^{s_{ij}^{\text{out}}(t,x,\xi)} \frac{\partial \lambda_j}{\partial \xi}(s, \chi_j(s;t,\xi)) h_{ij}(\chi_{ij}(s;t,x,\xi))\, ds. \tag{59}$$

In this case, this integral equation is very easily solved by taking (as it is in fact directly seen from the ODE (57)):

$$h_{ij}(t,x,\xi) = b_{ij}(t,x,\xi) e^{\int_t^{s_{ij}^{\mathrm{out}}(t,x,\xi)} \frac{\partial \lambda_j}{\partial \xi}(s,\chi_j(s;t,\xi))\,ds}. \tag{60}$$

Clearly, $h_{ij} = 0$ for $i \leq j$ (i.e. H_{--} is indeed strictly lower triangular) since $g_{ij}^2 = 0$ for such indices (see (34)). Obviously, $h_{ij} \in C^0(\overline{\mathcal{R}_{ij}^+}) \cap L^\infty(\mathcal{R}_{ij}^+)$. On the other hand, thanks in particular to the regularities (11), (18), the bounds (19) and the assumption $\frac{\partial \Lambda}{\partial x} \in L^\infty((0,+\infty)\times(0,1))^{n\times n}$, we can check that

$$h_{ij} \in C^0(\overline{\mathcal{R}_{ij}^-}) \cap L^\infty(\mathcal{R}_{ij}^-).$$

3.1.3 Characterization of $\mathcal{R}_{ij}^\pm$ and $\mathcal{D}_{ij}$

Let us now show that

$$\begin{aligned} \mathcal{R}_{ij}^- &= \{(t,x,\xi) \in \mathcal{R}, \quad \xi < \psi_{ij}(t,x)\}, \\ \mathcal{R}_{ij}^+ &= \{(t,x,\xi) \in \mathcal{R}, \quad \xi > \psi_{ij}(t,x)\}, \end{aligned} \tag{61}$$

where $\psi_{ij} \in C^1([0,+\infty)\times[0,1])$ satisfies the semi-linear hyperbolic equation (41). First of all, it follows from (20) and the implicit function theorem that there exists a function $\psi_{ij} \in C^1([0,+\infty)\times[0,1])$, $0 \leq \psi_{ij} \leq 1$, such that

$$s_i^{\mathrm{out}}(t,x) = s_j^{\mathrm{out}}(t,\xi) \qquad \Longleftrightarrow \qquad \xi = \psi_{ij}(t,x). \tag{62}$$

This shows that

$$\mathcal{D}_{ij} = \{(t,x,\xi) \in \mathcal{R}, \quad \xi = \psi_{ij}(t,x)\}. \tag{63}$$

On the other hand, thanks to (20) and (62) we have

$$\xi > \psi_{ij}(t,x) \qquad \Longleftrightarrow \qquad s_j^{\mathrm{out}}(t,\xi) > s_j^{\mathrm{out}}(t,\psi_{ij}(t,x)) = s_i^{\mathrm{out}}(t,x).$$

This shows the equality (61) for $\mathcal{R}_{ij}^+$. The equality for $\mathcal{R}_{ij}^-$ can be proved similarly.

It remains to show that ψ_{ij} satisfies the semi-linear hyperbolic equation (41). This in fact follows from (63) and (iii) of Proposition 3.1. Indeed, thanks to these results, we have

$$\chi_j(s;t,\psi_{ij}(t,x)) = \psi_{ij}(s,\chi_i(s;t,x)), \quad \forall s \in (t, s_i^{\mathrm{out}}(t,x)) = (t, s_j^{\mathrm{out}}(t,\psi_{ij}(t,x))).$$

Taking the derivative of this identity at $s = t^+$, we immediately obtain the equation in (41). On the other hand, letting $s \to s_i^{\mathrm{out}}(t,x)^- = s_j^{\mathrm{out}}(t,\psi_{ij}(t,x))^-$ and then letting $x \to 0^+$, we obtain the second condition $\psi_{ij}(t,0) = 0$.

3.2 Kernel for the Volterra transformation

In this section we prove Theorem 2.6, that is we prove the existence of a suitably smooth matrix-valued function $K = (k_{ij})_{1\leq i,j\leq n}$ such that

$$\begin{cases} \dfrac{\partial K}{\partial t}(t,x,\xi) + \Lambda(t,x)\dfrac{\partial K}{\partial x}(t,x,\xi) + \dfrac{\partial K}{\partial \xi}(t,x,\xi)\Lambda(t,\xi) + K(t,x,\xi)\widetilde{M}^1(t,\xi) = 0, \\ K(t,x,x)\Lambda(t,x) - \Lambda(t,x)K(t,x,x) = -M^1(t,x), \end{cases} \tag{64}$$

where we introduced the notation

$$\widetilde{M}^1(t,\xi) = \frac{\partial \Lambda}{\partial \xi}(t,\xi) + M^1(t,\xi). \tag{65}$$

Note in particular that $\widetilde{M}^1 \in L^\infty((0,+\infty)\times(0,1))^{n\times n}$ thanks to the assumption (5). Besides, noticing (40), we also want the matrix

$$-K_{--}(t,x,0)\Lambda_{--}(t,0) - K_{-+}(t,x,0)\Lambda_{++}(t,0)Q^1(t) \qquad \text{to be strictly lower triangular.} \tag{66}$$

3.2.1 Preliminaries

Let us rewrite (64) by block. It is equivalent to the following four sub-systems:

$$\begin{cases} \dfrac{\partial K_{--}}{\partial t}(t,x,\xi) + \Lambda_{--}(t,x)\dfrac{\partial K_{--}}{\partial x}(t,x,\xi) + \dfrac{\partial K_{--}}{\partial \xi}(t,x,\xi)\Lambda_{--}(t,\xi) \\ \qquad\qquad\qquad + K_{--}(t,x,\xi)\widetilde{M}^1_{--}(t,\xi) + K_{-+}(t,x,\xi)\widetilde{M}^1_{+-}(t,\xi) = 0, \\ K_{--}(t,x,x)\Lambda_{--}(t,x) - \Lambda_{--}(t,x)K_{--}(t,x,x) = -M^1_{--}(t,x), \end{cases}$$

$$\begin{cases} \dfrac{\partial K_{-+}}{\partial t}(t,x,\xi) + \Lambda_{--}(t,x)\dfrac{\partial K_{-+}}{\partial x}(t,x,\xi) + \dfrac{\partial K_{-+}}{\partial \xi}(t,x,\xi)\Lambda_{++}(t,\xi) \\ \qquad\qquad\qquad + K_{--}(t,x,\xi)\widetilde{M}^1_{-+}(t,\xi) + K_{-+}(t,x,\xi)\widetilde{M}^1_{++}(t,\xi) = 0, \\ K_{-+}(t,x,x)\Lambda_{++}(t,x) - \Lambda_{--}(t,x)K_{-+}(t,x,x) = -M^1_{-+}(t,x), \end{cases}$$

$$\begin{cases} \dfrac{\partial K_{+-}}{\partial t}(t,x,\xi) + \Lambda_{++}(t,x)\dfrac{\partial K_{+-}}{\partial x}(t,x,\xi) + \dfrac{\partial K_{+-}}{\partial \xi}(t,x,\xi)\Lambda_{--}(t,\xi) \\ \qquad\qquad\qquad + K_{+-}(t,x,\xi)\widetilde{M}^1_{--}(t,\xi) + K_{++}(t,x,\xi)\widetilde{M}^1_{+-}(t,\xi) = 0, \\ K_{+-}(t,x,x)\Lambda_{--}(t,x) - \Lambda_{++}(t,x)K_{+-}(t,x,x) = -M^1_{+-}(t,x), \end{cases}$$

$$\begin{cases} \dfrac{\partial K_{++}}{\partial t}(t,x,\xi) + \Lambda_{++}(t,x)\dfrac{\partial K_{++}}{\partial x}(t,x,\xi) + \dfrac{\partial K_{++}}{\partial \xi}(t,x,\xi)\Lambda_{++}(t,\xi) \\ \qquad\qquad\qquad + K_{+-}(t,x,\xi)\widetilde{M}^1_{-+}(t,\xi) + K_{++}(t,x,\xi)\widetilde{M}^1_{++}(t,\xi) = 0, \\ K_{++}(t,x,x)\Lambda_{++}(t,x) - \Lambda_{++}(t,x)K_{++}(t,x,x) = -M^1_{++}(t,x), \end{cases}$$

Remark 3.2. We see that K_{--} is coupled only with K_{-+} and that K_{+-} is coupled only with K_{++}. Moreover, the systems satisfied by (K_{--}, K_{-+}) and by (K_{+-}, K_{++}) are similar. Therefore, from now on we only focus on the system satisfied by (K_{--}, K_{-+}) (note that the extra condition (66) only concerns this system). In addition, because of the nature of the coupling terms inside the domain (namely, matrix multiplication by the right), we see that the entries from different rows are not coupled. Therefore, for the rest of Section 3.2, we assume that

$$i \in \{1,\dots,m\} \text{ is fixed.}$$

Let us now rewrite the equations for K_{--} and K_{-+} component-wise. For convenience, we introduce

$$r_{ij}(t,x) = \frac{-m^1_{ij}(t,x)}{\lambda_j(t,x) - \lambda_i(t,x)} \qquad (j \neq i). \tag{67}$$

Note that $r_{ij} \in C^0([0,+\infty)\times[0,1])$. Moreover, $r_{ij} \in L^\infty((0,+\infty)\times(0,1))$ thanks to (4). We have:

1) If $j \neq i$, then

$$\begin{cases} \dfrac{\partial k_{ij}}{\partial t}(t,x,\xi) + \lambda_i(t,x)\dfrac{\partial k_{ij}}{\partial x}(t,x,\xi) + \lambda_j(t,\xi)\dfrac{\partial k_{ij}}{\partial \xi}(t,x,\xi) + \displaystyle\sum_{\ell=1}^{n} k_{i\ell}(t,x,\xi)\widetilde{m}^1_{\ell j}(t,\xi) = 0, \\ k_{ij}(t,x,x) = r_{ij}(t,x). \end{cases} \tag{68}$$

2) If $j = i$, then

$$\frac{\partial k_{ii}}{\partial t}(t,x,\xi) + \lambda_i(t,x)\frac{\partial k_{ii}}{\partial x}(t,x,\xi) + \lambda_i(t,\xi)\frac{\partial k_{ii}}{\partial \xi}(t,x,\xi) + \sum_{\ell=1}^{n} k_{i\ell}(t,x,\xi)\widetilde{m}^1_{\ell i}(t,\xi) = 0. \tag{69}$$

The geometric situation of the characteristics is more complicated than in Section 3.1, it is detailed in Section 3.2.2 below. For the moment, let us just point out that we will have to consider parameters $s < t$ (compare with Section 3.1) and, consequently, we should also add an artificial boundary condition at $t = 0$ (the value of k_{ij} at a point $(t,x,\xi) \in \mathcal{T}$ for sufficiently small t can not be obtained from its values on the planes $\xi = x$ or $x = 1$). To avoid imposing such a condition we can equivalently study (68)-(69) on the domain extended in time

$$\mathcal{P} = \{(t,x,\xi) \in \mathbb{R} \times (0,1) \times (0,1), \quad \xi < x\}.$$

Therefore, we need the values of $\widetilde{m}^1_{\ell j}$ and r_{ij} for negative t. We also need the values of $q^1_{\ell j}$ for negative t since we want to consider the property (66). To this end we extend M to $\mathbb{R} \times [0,1]$ (recall that its diagonal elements were already extended in the proof of Proposition 2.2) and we extend Q to $\mathbb{R}$ in such a way that the property (5) is preserved. This extends $\widetilde{m}^1_{\ell j}$ and r_{ij} to $\mathbb{R} \times [0,1]$ and $q^1_{\ell j}$ to $\mathbb{R}$ through the formula (65), (67) and (32), (33), with

$$\widetilde{m}^1_{\ell j}, r_{ij} \in C^0(\mathbb{R} \times [0,1]) \cap L^\infty(\mathbb{R} \times (0,1)), \qquad q^1_{\ell j} \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}).$$

3.2.2 The characteristics of (68)-(69)

For each $(t,x,\xi) \in \mathbb{R}^3$ fixed, we still denote by $\chi_{ij}(\cdot;t,x,\xi)$ the characteristic curve associated with the hyperbolic system (68)-(69) passing through the point (t,x,ξ), i.e.

$$\chi_{ij}(s;t,x,\xi) = (s, \chi_i(s;t,x), \chi_j(s;t,\xi)), \quad \forall s \in \mathbb{R}.$$

We now need to find for which parameters s the characteristic $\chi_{ij}(s;t,x,\xi)$ stays in the domain $\mathcal{P}$ when $(t,x,\xi) \in \mathcal{P}$. To this end, we introduce the following sets for $j \in \{1,\ldots,m\}$:

$$\begin{aligned} \mathcal{P}^{\text{in},+}_{ij} &= \left\{(t,x,\xi) \in \mathcal{P}, \quad s^{\text{in}}_i(t,x) < s^{\text{in}}_j(t,\xi)\right\}, \\ \mathcal{P}^{\text{in},-}_{ij} &= \left\{(t,x,\xi) \in \mathcal{P}, \quad s^{\text{in}}_i(t,x) > s^{\text{in}}_j(t,\xi)\right\}, \\ \mathcal{D}^{\text{in}}_{ij} &= \left\{(t,x,\xi) \in \mathcal{P}, \quad s^{\text{in}}_i(t,x) = s^{\text{in}}_j(t,\xi)\right\}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{P}^{\text{out},+}_{ij} &= \left\{(t,x,\xi) \in \mathcal{P}, \quad s^{\text{out}}_i(t,x) < s^{\text{out}}_j(t,\xi)\right\}, \\ \mathcal{P}^{\text{out},-}_{ij} &= \left\{(t,x,\xi) \in \mathcal{P}, \quad s^{\text{out}}_i(t,x) > s^{\text{out}}_j(t,\xi)\right\}, \\ \mathcal{D}^{\text{out}}_{ij} &= \left\{(t,x,\xi) \in \mathcal{P}, \quad s^{\text{out}}_i(t,x) = s^{\text{out}}_j(t,\xi)\right\}. \end{aligned}$$

As in Section 3.1.3 we can show that $\mathcal{P}^{\text{out},+}_{ij} \cap \mathcal{T} = \mathcal{T}^+_{ij}$ and $\mathcal{P}^{\text{out},-}_{ij} \cap \mathcal{T} = \mathcal{T}^-_{ij}$ (we recall that $\mathcal{T}^+_{ij}$ and $\mathcal{T}^-_{ij}$ are defined in the statement of Theorem 2.6).

The following proposition gives precise information about the exit of the characteristics from the domain $\mathcal{P}$ (the proof is postponed to Appendix B for the sake of the presentation; we refer to Figures 2, 3, 4 and 5 for a clarification of the geometric situation at a fixed t):

Proposition 3.3.

(i) For every $j \in \{1, \dots, i-1\}$, there exists a unique $s_{ij}^{\text{in}} \in C^0(\overline{\mathcal{P}})$ with $(t,x,\xi) \mapsto t - s_{ij}^{\text{in}}(t,x,\xi) \in L^\infty(\mathcal{P})$ such that, for every $t \in \mathbb{R}$ and $0 \leq \xi < x < 1$, we have $s_{ij}^{\text{in}}(t,x,\xi) < t$ (and $s_{ij}^{\text{in}}(t,x,\xi) = t$ otherwise) with

$$\chi_{ij}(s;t,x,\xi) \in \mathcal{P}, \quad \forall s \in \left(s_{ij}^{\text{in}}(t,x,\xi), t\right),$$

and

$$\begin{cases} \chi_j(s_{ij}^{\text{in}}(t,x,\xi);t,\xi) = \chi_i(s_{ij}^{\text{in}}(t,x,\xi);t,x) & \textit{if } (t,x,\xi) \in \overline{\mathcal{P}_{ij}^{\text{in},+}}, \\ \chi_i(s_{ij}^{\text{in}}(t,x,\xi);t,x) = 1 & \textit{if } (t,x,\xi) \in \overline{\mathcal{P}_{ij}^{\text{in},-}}. \end{cases}$$

(ii) For $j = i$, there exists a unique $s_{ii}^{\text{out}} \in C^0(\overline{\mathcal{P}})$ with $(t,x,\xi) \mapsto s_{ii}^{\text{out}}(t,x,\xi) - t \in L^\infty(\mathcal{P})$ such that, for every $t \in \mathbb{R}$ and $0 < \xi \leq x \leq 1$, we have $s_{ii}^{\text{out}}(t,x,\xi) > t$ (and $s_{ii}^{\text{out}}(t,x,\xi) = t$ otherwise) and, if in addition $\xi < x$, then we have

$$\chi_{ii}(s;t,x,\xi) \in \mathcal{P}, \quad \forall s \in \left(t, s_{ii}^{\text{out}}(t,x,\xi)\right),$$

and

$$\chi_i(s_{ii}^{\text{out}}(t,x,\xi);t,\xi) = 0.$$

(iii) For every $j \in \{i+1, \dots, m\}$, there exists a unique $s_{ij}^{\text{out}} \in C^0(\overline{\mathcal{P}})$ with $(t,x,\xi) \mapsto s_{ij}^{\text{out}}(t,x,\xi) - t \in L^\infty(\mathcal{P})$ such that, for every $t \in \mathbb{R}$ and $0 < \xi < x \leq 1$, we have $s_{ij}^{\text{out}}(t,x,\xi) > t$ (and $s_{ij}^{\text{out}}(t,x,\xi) = t$ otherwise) with

$$\chi_{ij}(s;t,x,\xi) \in \mathcal{P}, \quad \forall s \in \left(t, s_{ij}^{\text{out}}(t,x,\xi)\right),$$

and

$$\begin{cases} \chi_j(s_{ij}^{\text{out}}(t,x,\xi);t,\xi) = \chi_i(s_{ij}^{\text{out}}(t,x,\xi);t,x) & \textit{if } (t,x,\xi) \in \overline{\mathcal{P}_{ij}^{\text{out},+}}, \\ \chi_j(s_{ij}^{\text{out}}(t,x,\xi);t,\xi) = 0 & \textit{if } (t,x,\xi) \in \overline{\mathcal{P}_{ij}^{\text{out},-}}. \end{cases}$$

(iv) For every $j \in \{m+1, \dots, n\}$, there exists a unique $s_{ij}^{\text{out}} \in C^0(\overline{\mathcal{P}})$ with $(t,x,\xi) \mapsto s_{ij}^{\text{out}}(t,x,\xi) - t \in L^\infty(\mathcal{P})$ such that, for every $t \in \mathbb{R}$ and $0 \leq \xi < x \leq 1$, we have $s_{ij}^{\text{out}}(t,x,\xi) > t$ (and $s_{ij}^{\text{out}}(t,x,\xi) = t$ otherwise) with

$$\chi_{ij}(s;t,x,\xi) \in \mathcal{P}, \quad \forall s \in \left(t, s_{ij}^{\text{out}}(t,x,\xi)\right),$$

and

$$\chi_j(s_{ij}^{\text{out}}(t,x,\xi);t,\xi) = \chi_i(s_{ij}^{\text{out}}(t,x,\xi);t,x).$$

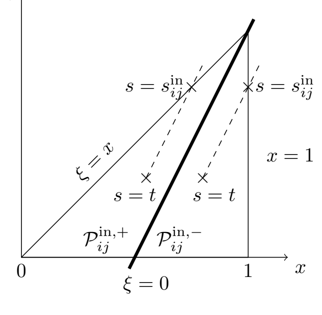

Figure 2: Definition of s_{ij}^{in}

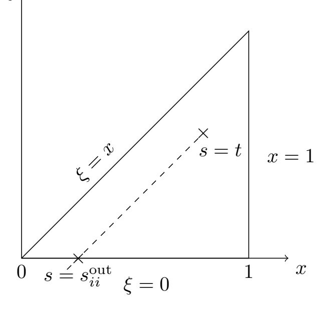

Figure 3: Definition of s_{ii}^{out}

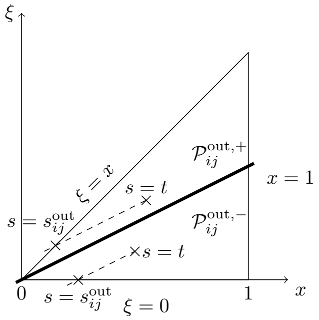

Figure 4: Definition of s_{ij}^{out} when $i < j \leq m$

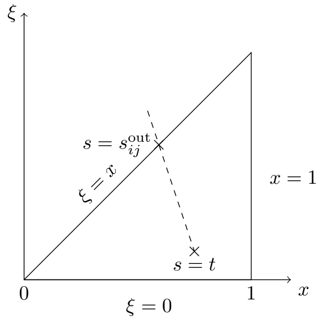

Figure 5: Definition of s_{ij}^{out} when $j > m$

In order to show that the system (68)-(69) is well-posed, we see from Proposition 3.3 that we need to add some conditions:

1) when $j \in \{1, \dots, i-1\}$, we will consider the following artificial boundary condition at $x = 1$:

$$k_{ij}(t, 1, \xi) = a_{ij}(t, \xi), \quad \forall j \in \{1, \dots, i-1\},$$

where $a_{ij} \in C^0(\mathbb{R} \times [0,1]) \cap L^\infty(\mathbb{R} \times (0,1))$ is any function that satisfies the corresponding C^0-compatibility conditions at $(t, x, \xi) = (t, 1, 1)$, namely:

$$a_{ij}(t, 1) = r_{ij}(t, 1), \quad \forall t \in \mathbb{R}. \tag{70}$$

2) when $j \in \{i, \dots, m\}$, we have some freedom for the boundary condition. We choose to consider the following one in order to obtain (66):

$$k_{ij}(t, x, 0) = \sum_{\ell=1}^{p} k_{i,m+\ell}(t, x, 0)\widetilde{q}^{1}_{\ell j}(t), \quad \forall j \in \{i, \dots, m\},$$

where we set

$$\widetilde{q}^{1}_{\ell j}(t) = -\frac{1}{\lambda_j(t, 0)}\lambda_{m+\ell}(t, 0)q^{1}_{\ell j}(t).$$

Note that $\widetilde{q}^{1}_{\ell j} \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

In summary, we are going to solve the following coupled hyperbolic system:

1) If $j \in \{1, \dots, i-1\}$, then

$$\begin{cases} \dfrac{\partial k_{ij}}{\partial t}(t, x, \xi) + \lambda_i(t, x)\dfrac{\partial k_{ij}}{\partial x}(t, x, \xi) + \lambda_j(t, \xi)\dfrac{\partial k_{ij}}{\partial \xi}(t, x, \xi) + \displaystyle\sum_{\ell=1}^{n} k_{i\ell}(t, x, \xi)\widetilde{m}^{1}_{\ell j}(t, \xi) = 0, \\ k_{ij}(t, x, x) = r_{ij}(t, x), \\ k_{ij}(t, 1, \xi) = a_{ij}(t, \xi). \end{cases} \tag{71}$$

2) If $j = i$, then

$$\begin{cases} \dfrac{\partial k_{ii}}{\partial t}(t, x, \xi) + \lambda_i(t, x)\dfrac{\partial k_{ii}}{\partial x}(t, x, \xi) + \lambda_i(t, \xi)\dfrac{\partial k_{ii}}{\partial \xi}(t, x, \xi) + \displaystyle\sum_{\ell=1}^{n} k_{i\ell}(t, x, \xi)\widetilde{m}^{1}_{\ell i}(t, \xi) = 0, \\ k_{ii}(t, x, 0) = \displaystyle\sum_{\ell=1}^{p} k_{i,m+\ell}(t, x, 0)\widetilde{q}^{1}_{\ell i}(t). \end{cases} \tag{72}$$

3) If $j \in \{i+1, \dots, m\}$, then

$$\begin{cases} \dfrac{\partial k_{ij}}{\partial t}(t,x,\xi) + \lambda_i(t,x)\dfrac{\partial k_{ij}}{\partial x}(t,x,\xi) + \lambda_j(t,\xi)\dfrac{\partial k_{ij}}{\partial \xi}(t,x,\xi) + \displaystyle\sum_{\ell=1}^{n} k_{i\ell}(t,x,\xi)\widetilde{m}^1_{\ell j}(t,\xi) = 0, \\ k_{ij}(t,x,x) = r_{ij}(t,x), \\ k_{ij}(t,x,0) = \displaystyle\sum_{\ell=1}^{p} k_{i,m+\ell}(t,x,0)\widetilde{q}^1_{\ell j}(t). \end{cases} \tag{73}$$

4) If $j \in \{m+1, \dots, n\}$, then

$$\begin{cases} \dfrac{\partial k_{ij}}{\partial t}(t,x,\xi) + \lambda_i(t,x)\dfrac{\partial k_{ij}}{\partial x}(t,x,\xi) + \lambda_j(t,\xi)\dfrac{\partial k_{ij}}{\partial \xi}(t,x,\xi) + \displaystyle\sum_{\ell=1}^{n} k_{i\ell}(t,x,\xi)\widetilde{m}^1_{\ell j}(t,\xi) = 0, \\ k_{ij}(t,x,x) = r_{ij}(t,x). \end{cases} \tag{74}$$

3.2.3 Transformation into integral equations

To prove the existence and uniqueness of the solution to the kernel equations (71)-(74) on $\mathcal{P}$, we use the classical strategy that consists in transforming these hyperbolic equations into integral equations. Then, in the next subsection, we will prove that this system of integral equations has a unique solution by using a fixed-point argument and appropriate estimates.

Let us introduce

$$\widetilde{k}^0_{ij}(t,x,\xi) = \begin{cases} r_{ij}\left(s^{\text{in}}_{ij}(t,x,\xi), \chi_i\left(s^{\text{in}}_{ij}(t,x,\xi); t, x\right)\right) & \text{if } j \in \{1,\dots,i-1\} \text{ and } (t,x,\xi) \in \overline{\mathcal{P}^{\text{in},+}_{ij}}, \\ a_{ij}\left(s^{\text{in}}_{ij}(t,x,\xi), \chi_j\left(s^{\text{in}}_{ij}(t,x,\xi); t, \xi\right)\right) & \text{if } j \in \{1,\dots,i-1\} \text{ and } (t,x,\xi) \in \overline{\mathcal{P}^{\text{in},-}_{ij}}, \\ r_{ij}\left(s^{\text{out}}_{ij}(t,x,\xi), \chi_i\left(s^{\text{out}}_{ij}(t,x,\xi); t, x\right)\right) & \text{if } j \in \{i+1,\dots,m\} \text{ and } (t,x,\xi) \in \overline{\mathcal{P}^{\text{out},+}_{ij}}, \\ r_{ij}\left(s^{\text{out}}_{ij}(t,x,\xi), \chi_i\left(s^{\text{out}}_{ij}(t,x,\xi); t, x\right)\right) & \text{if } j \in \{m+1,\dots,n\}. \end{cases}$$

Thanks to the C^0-compatibility condition (70), note in particular that

$$\widetilde{k}^0_{ij} \in C^0(\overline{\mathcal{P}}), \quad \forall j \in \{1,\dots,i-1\}. \tag{75}$$

Using now Proposition 3.3, we can obtain that

1) For $j \in \{1,\dots,i-1\}$, integrating (71) along the characteristic curve $\chi_{ij}(s;t,x,\xi)$ for $s \in (s^{\text{in}}_{ij}(t,x,\xi), t)$ yields the following integral equation:

$$k_{ij}(t,x,\xi) = \widetilde{k}^0_{ij}(t,x,\xi) - \sum_{\ell=1}^{n} \int_{s^{\text{in}}_{ij}(t,x,\xi)}^{t} k_{i\ell}(\chi_{ij}(s;t,x,\xi))\widetilde{m}^1_{\ell j}(s, \chi_j(s;t,\xi))\, ds.$$

2) For $j = i$, integrating (72) along the characteristic curve $\chi_{ii}(s;t,x,\xi)$ for $s \in (t, s^{\text{out}}_{ii}(t,x,\xi))$ yields the following integral equation:

$$\begin{aligned} k_{ii}(t,x,\xi) = {} & \sum_{\ell=1}^{p} k_{i,m+\ell}\left(s^{\text{out}}_{ii}(t,x,\xi), \chi_i\left(s^{\text{out}}_{ii}(t,x,\xi); t, x\right), 0\right) \widetilde{q}^1_{\ell i}\left(s^{\text{out}}_{ii}(t,x,\xi)\right) \\ & + \sum_{\ell=1}^{n} \int_{t}^{s^{\text{out}}_{ii}(t,x,\xi)} k_{i\ell}(\chi_{ii}(s;t,x,\xi))\widetilde{m}^1_{\ell i}(s, \chi_i(s;t,\xi))\, ds. \end{aligned} \tag{76}$$

3) For $j \in \{i+1, \dots, m\}$, integrating (73) along the characteristic curve $\chi_{ij}(s; t, x, \xi)$ for $s \in (t, s_{ij}^{\mathrm{out}}(t, x, \xi))$ yields the following integral equations:

$$
\begin{aligned}
k_{ij}(t,x,\xi) &= \widetilde{k}^0_{ij}(t,x,\xi) \\
&\qquad + \sum_{\ell=1}^{n} \int_t^{s_{ij}^{\mathrm{out}}(t,x,\xi)} k_{i\ell}(\chi_{ij}(s;t,x,\xi)) \widetilde{m}^1_{\ell j}(s, \chi_j(s;t,\xi))\, ds, \quad (t,x,\xi) \in \overline{\mathcal{P}_{ij}^{\mathrm{out},+}},
\end{aligned}
$$

and

$$
\begin{aligned}
k_{ij}(t,x,\xi) &= \sum_{\ell=1}^{p} k_{i,m+\ell}\left(s_{ij}^{\mathrm{out}}(t,x,\xi), \chi_i\left(s_{ij}^{\mathrm{out}}(t,x,\xi); t, x\right), 0\right) \widetilde{q}^1_{\ell j}\left(s_{ij}^{\mathrm{out}}(t,x,\xi)\right) \\
&\quad + \sum_{\ell=1}^{n} \int_t^{s_{ij}^{\mathrm{out}}(t,x,\xi)} k_{i\ell}(\chi_{ij}(s;t,x,\xi)) \widetilde{m}^1_{\ell j}(s, \chi_j(s;t,\xi))\, ds, \quad (t,x,\xi) \in \overline{\mathcal{P}_{ij}^{\mathrm{out},-}}. \quad (77)
\end{aligned}
$$

4) For $j \in \{m+1, \dots, n\}$, integrating (74) along the characteristic curve $\chi_{ij}(s; t, x, \xi)$ for $s \in (t, s_{ij}^{\mathrm{out}}(t, x, \xi))$ yields the following integral equation:

$$
k_{ij}(t,x,\xi) = \widetilde{k}^0_{ij}(t,x,\xi) + \sum_{\ell=1}^{n} \int_t^{s_{ij}^{\mathrm{out}}(t,x,\xi)} k_{i\ell}(\chi_{ij}(s;t,x,\xi)) \widetilde{m}^1_{\ell j}(s, \chi_j(s;t,\xi))\, ds. \quad (78)
$$

5) We now want to plug (78) into (76) and (77), respectively. From (78) we have

$$
\begin{aligned}
&k_{i,m+\ell}\left(s_{ij}^{\mathrm{out}}(t,x,\xi), \chi_i\left(s_{ij}^{\mathrm{out}}(t,x,\xi); t, x\right), 0\right) = \widetilde{k}^0_{i,m+\ell}\left(s_{ij}^{\mathrm{out}}(t,x,\xi), \chi_i\left(s_{ij}^{\mathrm{out}}(t,x,\xi); t, x\right), 0\right) \\
&+ \sum_{q=1}^{n} \int_{s_{ij}^{\mathrm{out}}(t,x,\xi)}^{s_{i,m+\ell}^{\mathrm{out}}\left(s_{ij}^{\mathrm{out}}(t,x,\xi), \chi_i\left(s_{ij}^{\mathrm{out}}(t,x,\xi);t,x\right), 0\right)} k_{iq}\left(\chi_{i,m+\ell}\left(s; s_{ij}^{\mathrm{out}}(t,x,\xi), \chi_i\left(s_{ij}^{\mathrm{out}}(t,x,\xi); t, x\right), 0\right)\right) \\
&\qquad\qquad \times \widetilde{m}^1_{q,m+\ell}\left(s, \chi_{m+\ell}\left(s; s_{ij}^{\mathrm{out}}(t,x,\xi), 0\right)\right) ds. \quad (79)
\end{aligned}
$$

Plugging (79) into (76) and (77), we obtain, for every $j \in \{i, \dots, m\}$ and $(t,x,\xi) \in \overline{\mathcal{P}_{ij}^{\mathrm{out},-}}$,

$$
\begin{aligned}
&k_{ij}(t,x,\xi) = \widehat{k}^0_{ij}(t,x,\xi) \\
&+ \sum_{\ell=1}^{p} \left(\sum_{q=1}^{n} \int_{s_{ij}^{\mathrm{out}}(t,x,\xi)}^{s_{i,m+\ell}^{\mathrm{out}}\left(s_{ij}^{\mathrm{out}}(t,x,\xi), \chi_i\left(s_{ij}^{\mathrm{out}}(t,x,\xi);t,x\right), 0\right)} k_{iq}\left(\chi_{i,m+\ell}\left(s; s_{ij}^{\mathrm{out}}(t,x,\xi), \chi_i\left(s_{ij}^{\mathrm{out}}(t,x,\xi); t, x\right), 0\right)\right) \right. \\
&\qquad \left. \times \widetilde{m}^1_{q,m+\ell}\left(s, \chi_{m+\ell}\left(s; s_{ij}^{\mathrm{out}}(t,x,\xi), 0\right)\right) ds \right) \widetilde{q}^1_{\ell j}\left(s_{ij}^{\mathrm{out}}(t,x,\xi)\right) \\
&\qquad\qquad + \sum_{\ell=1}^{n} \int_t^{s_{ij}^{\mathrm{out}}(t,x,\xi)} k_{i\ell}(\chi_{ij}(s;t,x,\xi)) \widetilde{m}^1_{\ell j}(s, \chi_j(s;t,\xi))\, ds,
\end{aligned}
$$

where we introduced

$$
\widehat{k}^0_{ij}(t,x,\xi) = \sum_{\ell=1}^{p} \widetilde{k}^0_{i,m+\ell}\left(s_{ij}^{\mathrm{out}}(t,x,\xi), \chi_i\left(s_{ij}^{\mathrm{out}}(t,x,\xi); t, x\right), 0\right) \widetilde{q}^1_{\ell j}\left(s_{ij}^{\mathrm{out}}(t,x,\xi)\right).
$$

Note that

$$
\widehat{k}^0_{ij} \in C^0(\overline{\mathcal{P}_{ij}^{\mathrm{out},-}}) \cap L^\infty(\mathcal{P}_{ij}^{\mathrm{out},-}), \quad \forall j \in \{i+1, \dots, m\}, \quad (80)
$$

and, since $\mathcal{P}_{ii}^{\mathrm{out},-} = \mathcal{P}$ (because of (20)),

$$
\widehat{k}^0_{ii} \in C^0(\overline{\mathcal{P}}) \cap L^\infty(\mathcal{P}). \quad (81)
$$

Remark 3.4. Observe that, in general, for $j \in \{i+1, \ldots, m\}$, we have

$$\widehat{k}_{ij}^0 \neq \widetilde{k}_{ij}^0 \text{ on } \mathcal{D}_{ij}^{\text{out}}.$$

This is the reason why we have to consider discontinuous kernels.

3.2.4 Solution to the integral equations

In this subsection we show that there exists a unique solution to the system of integral equations of the previous section. This will conclude the proof of Theorem 2.6.

Fixed-point argument. As it is classical, we reformulate the existence of such a solution into the existence of a fixed-point of the mapping defined by the right-hand sides of these equations. Let us first introduce $K^0 = (k_{ij}^0)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ defined by

$$k_{ij}^0(t,x,\xi) = \begin{cases} \widetilde{k}_{ij}^0(t,x,\xi) & \text{if } j \in \{1, \ldots, i-1\}, \\ \widehat{k}_{ii}^0(t,x,\xi) & \text{if } j = i, \\ \widetilde{k}_{ij}^0(t,x,\xi) & \text{if } j \in \{i+1, \ldots, m\} \text{ and } (t,x,\xi) \in \mathcal{P}_{ij}^{\text{out},+}, \\ \widehat{k}_{ij}^0(t,x,\xi) & \text{if } j \in \{i+1, \ldots, m\} \text{ and } (t,x,\xi) \in \mathcal{P}_{ij}^{\text{out},-}, \\ \widetilde{k}_{ij}^0(t,x,\xi) & \text{if } j \in \{m+1, \ldots, n\}. \end{cases}$$

Thanks in particular to (75), (80) and (81), we see that

$$\begin{aligned} & k_{ij}^0 \in C^0(\overline{\mathcal{P}_{ij}^{\text{out},+}}) \cap C^0(\overline{\mathcal{P}_{ij}^{\text{out},-}}) \cap L^\infty(\mathcal{P}) \quad \text{if } j \in \{i+1, \ldots, m\}, \\ & k_{ij}^0 \in C^0(\overline{\mathcal{P}}) \cap L^\infty(\mathcal{P}) \quad \text{otherwise.} \end{aligned}$$

It is this regularity that dictates the space in which we can work. More precisely, let us introduce the vector space B defined by

$$B = \left\{ K = (k_{ij})_{\substack{1 \leq i \leq m, \\ 1 \leq j \leq n}} \begin{array}{l} k_{ij} \in C^0(\overline{\mathcal{P}_{ij}^{\text{out},+}}) \cap C^0(\overline{\mathcal{P}_{ij}^{\text{out},-}}) \cap L^\infty(\mathcal{P}) \text{ if } j \in \{i+1, \ldots, m\}, \\ k_{ij} \in C^0(\overline{\mathcal{P}}) \cap L^\infty(\mathcal{P}) \text{ otherwise.} \end{array} \right\}. \tag{82}$$

We can check that B is a Banach space when equipped with the L^∞ norm. Let us now introduce the mapping

$$\Phi : B \longrightarrow B,$$

defined, for every $K \in B$, by

$$\Phi(K) = K^0 + \Phi_1(K) + \Phi_2(K),$$

where, for every $(t,x,\xi) \in \overline{\mathcal{P}}$,

$$(\Phi_1(K))_{ij}(t,x,\xi) = \\ \begin{cases} -\displaystyle\sum_{\ell=1}^n \int_{s_{ij}^{\text{in}}(t,x,\xi)}^t k_{i\ell}(\chi_{ij}(s;t,x,\xi)) \widetilde{m}_{\ell j}^1(s, \chi_j(s;t,\xi)) \, ds, & \text{if } j \in \{1, \ldots, i-1\}, \\ \displaystyle\sum_{\ell=1}^n \int_t^{s_{ij}^{\text{out}}(t,x,\xi)} k_{i\ell}(\chi_{ij}(s;t,x,\xi)) \widetilde{m}_{\ell j}^1(s, \chi_j(s;t,\xi)) \, ds, & \text{if } j \in \{i, \ldots, n\}, \end{cases} \tag{83}$$

and

$$\begin{aligned}(\Phi_2(K))_{ij}(t,x,\xi) &= \\ \sum_{\ell=1}^{p}\left(\sum_{q=1}^{n}\int_{s_{ij}^{\mathrm{out}}(t,x,\xi)}^{s_{i,m+\ell}^{\mathrm{out}}\left(s_{ij}^{\mathrm{out}}(t,x,\xi),\chi_i\left(s_{ij}^{\mathrm{out}}(t,x,\xi);t,x\right),0\right)}\right.& k_{iq}\left(\chi_{i,m+\ell}\left(s;s_{ij}^{\mathrm{out}}(t,x,\xi),\chi_i\left(s_{ij}^{\mathrm{out}}(t,x,\xi);t,x\right),0\right)\right) \\ &\left.\times\,\widetilde{m}^1_{q,m+\ell}\left(s,\chi_{m+\ell}\left(s;s_{ij}^{\mathrm{out}}(t,x,\xi),0\right)\right)\,ds\right)\widetilde{q}^1_{\ell j}\left(s_{ij}^{\mathrm{out}}(t,x,\xi)\right),\end{aligned}\tag{84}$$

if $j \in \{i,\dots,m\}$ and $(t,x,\xi) \in \overline{\mathcal{P}_{ij}^{\mathrm{out},-}}$, and $(\Phi_2(K))_{ij}(t,x,\xi) = 0$ otherwise (recall that $\mathcal{P}_{ii}^{\mathrm{out},-} = \mathcal{P}$).

Regularity of the mapping. First of all, we have to show that Φ is well defined, i.e. that for every $K \in B$, we have indeed

$$\Phi_1(K) \in B, \qquad \Phi_2(K) \in B. \tag{85}$$

This is not obvious since the function $s \mapsto \chi_{ij}(s;t,x,\xi)$ may take values in the set $\mathcal{D}_{i\ell}^{\mathrm{out}}$, where $k_{i\ell}$ is discontinuous (even for $j \notin \{i+1,\dots,m\}$, where we expect $(\Phi_1(K))_{ij}$ to be continuous by definition of B). The following result, close to Proposition 3.3, shows that this may happen only at one point:

Proposition 3.5. *Let $\ell \in \{i+1,\dots,m\}$ be fixed.*

(i) For every $j \in \{1,\dots,i-1\}$, for every $(t,x,\xi) \in \overline{\mathcal{P}}$, there is at most one $s_{ij\ell}^{\mathrm{disc}} \in (s_{ij}^{\mathrm{in}}(t,x,\xi),t)$ such that $\chi_{ij}(s_{ij\ell}^{\mathrm{disc}};t,x,\xi) \in \mathcal{D}_{i\ell}^{\mathrm{out}}$.

(ii) For every $j \in \{i,\dots,n\}$ with $j \neq \ell$, for every $(t,x,\xi) \in \overline{\mathcal{P}}$, there is at most one $s_{ij\ell}^{\mathrm{disc}} \in (t,s_{ij}^{\mathrm{out}}(t,x,\xi))$ such that $\chi_{ij}(s_{ij\ell}^{\mathrm{disc}};t,x,\xi) \in \mathcal{D}_{i\ell}^{\mathrm{out}}$.

This result shows in fact a stronger regularity than (85), namely,

$$\Phi_1(K), \Phi_2(K) \in C^0(\overline{\mathcal{P}})^{m\times n} \cap L^\infty(\mathcal{P})^{m\times n}.$$

The proof of Proposition 3.5 is postponed to Appendix B for the sake of the presentation.

Contraction of the mapping. We will now prove that Φ^N is a contraction for $N \in \mathbb{N}^*$ large enough. Therefore, the Banach fixed-point theorem can be applied, giving the existence (and uniqueness) of $K \in B$ such that

$$K = \Phi(K).$$

This will conclude the proof of Theorem 2.6. Now, to show that Φ^N is a contraction when N is large, it is sufficient to prove the following estimate:

Proposition 3.6. *There exists $C > 0$ such that, for every $N \in \mathbb{N}^*$ and $K, H \in B$,*

$$\left\|\Phi^N(K) - \Phi^N(H)\right\|_{L^\infty(\mathcal{P})^{m\times n}} \leq \frac{C^N}{N!}\left\|K - H\right\|_{L^\infty(\mathcal{P})^{m\times n}}. \tag{86}$$

To establish (86) we will use the following key lemma:

Lemma 3.7. *For every $i \in \{1,\dots,m\}$, there exist a function $\Omega_i \in C^1(\overline{\mathcal{P}}) \cap L^\infty(\mathcal{P})$ and $\varepsilon_0 > 0$ such that, for every $(t,x,\xi) \in \overline{\mathcal{P}}$, we have $\Omega_i(t,x,\xi) \geq 0$ with*

$$\frac{\partial \Omega_i}{\partial t}(t,x,\xi) + \lambda_i(t,x)\frac{\partial \Omega_i}{\partial x}(t,x,\xi) + \lambda_j(t,\xi)\frac{\partial \Omega_i}{\partial \xi}(t,x,\xi) \geq \varepsilon_0, \qquad \forall j \in \{1,\dots,i-1\}, \tag{87}$$

and

$$\frac{\partial \Omega_i}{\partial t}(t,x,\xi) + \lambda_i(t,x)\frac{\partial \Omega_i}{\partial x}(t,x,\xi) + \lambda_j(t,\xi)\frac{\partial \Omega_i}{\partial \xi}(t,x,\xi) \leq -\varepsilon_0, \qquad \forall j \in \{i,\dots,n\}. \tag{88}$$

The proof of Lemma 3.7 is postponed to Appendix C for the sake of the presentation.

Remark 3.8. In the time-independent case, we can take $\Omega_i(x,\xi) = \phi_i(x) - \nu\phi_i(\xi)$ (we recall that $\phi_i(x) = \int_0^x \frac{1}{-\lambda_i(y)}\,dy$) where $\nu \in [0,1)$ is any number such that $\nu > \max_{1\leq j<i\leq m} \max_{\xi\in[0,1]} \lambda_i(\xi)/\lambda_j(\xi)$. This function appeared for instance in [HDMVK16, Lemma 6.2] for systems with constant coefficients and in [HVDMK19, (A.32)] for systems with time-independent coefficients (see also [CVKB13, Lemma A.4] for 2×2 systems, where it is enough to take $\nu = 0$ since (87) becomes void).

Remark 3.9. Observe that it follows from the estimate (88) that, for every $j \in \{i,\ldots,n\}$,

$$s \mapsto \Omega_i\left(\chi_{ij}\left(s;t,x,\xi\right)\right) \text{ is strictly decreasing.}$$

This is the analogue to [HDMVK16, Remark 10].

We can now prove Proposition 3.6:

Proof of Proposition 3.6. Let us denote by

$$R = \max\left\{\left\|\widetilde{M}^1\right\|_{L^\infty(\mathbb{R}\times(0,1))^{n\times n}}, \left\|\widetilde{Q}^1\right\|_{L^\infty(\mathbb{R})^{p\times m}}\right\}.$$

1) We start with the estimate of $\|\Phi_1(K)-\Phi_1(H)\|_{L^\infty(\mathcal{P})^{m\times n}}$. Set

$$C_1 = \frac{n}{\varepsilon_0}R.$$

Let $j \in \{1,\ldots,i-1\}$. From the definition (83) of Φ_1 we see that

$$\left|\left(\Phi_1(K)-\Phi_1(H)\right)_{ij}(t,x,\xi)\right| \leq nR\left(\int_{s_{ij}^{\text{in}}(t,x,\xi)}^t 1\,ds\right)\|K-H\|_{L^\infty(\mathcal{P})^{m\times n}}.$$

Thanks to the estimate (87) we can perform the change of variable $s\mapsto\theta(s)=\Omega_i(\chi_{ij}(s;t,x,\xi))$ and obtain

$$\varepsilon_0\left(\int_{s_{ij}^{\text{in}}(t,x,\xi)}^t 1\,ds\right) \leq \int_{s_{ij}^{\text{in}}(t,x,\xi)}^t \frac{d\theta}{ds}(s)\,ds = \theta(t)-\theta(s_{ij}^{\text{in}}(t,x,\xi)) \leq \theta(t) = \Omega_i(t,x,\xi).$$

This gives the estimate

$$\left|\left(\Phi_1(K)-\Phi_1(H)\right)_{ij}(t,x,\xi)\right| \leq C_1\Omega_i(t,x,\xi)\,\|K-H\|_{L^\infty(\mathcal{P})^{m\times n}}.$$

It is important to point out that the right-hand side does not depend on the second index j. Computing $\Phi_1^2(H)-\Phi_1^2(K) = \Phi_1(\Phi_1(H))-\Phi_1(\Phi_1(K))$ and using the previous estimate, we obtain

$$\begin{aligned}&\left|\left(\Phi_1^2(K)-\Phi_1^2(H)\right)_{ij}(t,x,\xi)\right|\\&\qquad\leq nRC_1\left(\int_{s_{ij}^{\text{in}}(t,x,\xi)}^t \Omega_i(\chi_{ij}(s;t,x,\xi))\,ds\right)\|K-H\|_{L^\infty(\mathcal{P})^{m\times n}}.\end{aligned}$$

Using again the change of variable $s\mapsto\theta(s)$ and (87), we obtain

$$\begin{aligned}\varepsilon_0\left(\int_{s_{ij}^{\text{in}}(t,x,\xi)}^t \Omega_i(\chi_{ij}(s;t,x,\xi))\,ds\right) &= \varepsilon_0\int_{s_{ij}^{\text{in}}(t,x,\xi)}^t \theta(s)\,ds \leq \int_{s_{ij}^{\text{in}}(t,x,\xi)}^t \theta(s)\frac{d\theta}{ds}(s)\,ds\\&= \frac{\theta(t)^2}{2}-\frac{\theta(s_{ij}^{\text{in}}(t,x,\xi))^2}{2} \leq \frac{\theta(t)^2}{2} = \frac{\Omega_i(t,x,\xi)^2}{2}.\end{aligned}$$

This gives the estimate

$$\left|\left(\Phi_1^2(K)-\Phi_1^2(H)\right)_{ij}(t,x,\xi)\right| \leq \frac{C_1^2\Omega_i(t,x,\xi)^2}{2}\left\|K-H\right\|_{L^\infty(\mathcal{P})^{m\times n}}.$$

By induction, we easily obtain that, for every $N\in\mathbb{N}^*$,

$$\left|\left(\Phi_1^N(K)-\Phi_1^N(H)\right)_{ij}(t,x,\xi)\right| \leq \frac{C_1^N\Omega_i(t,x,\xi)^N}{N!}\left\|K-H\right\|_{L^\infty(\mathcal{P})^{m\times n}}. \tag{89}$$

Using the estimate (88) instead of (87), we can obtain exactly the same estimate as (89) for $j\in\{i,\dots,n\}$. Since Ω_i is bounded, it follows that

$$\left\|\Phi_1^N(K)-\Phi_1^N(H)\right\|_{L^\infty(\mathcal{P})^{m\times n}} \leq \frac{C^N}{N!}\left\|K-H\right\|_{L^\infty(\mathcal{P})^{m\times n}},$$

for some C independent of N and K,H.

2) Let us now take care of $\Phi_2(K)-\Phi_2(H)$. The idea to estimate this term is essentially the same as before, with the extra use of the decreasing property stated in Remark 3.9. Set

$$C_2=\frac{n}{\varepsilon_0}R^2p.$$

From the definition (84) of Φ_2 we see that, for $j\in\{i,\dots,m\}$ and $(t,x,\xi)\in\overline{\mathcal{P}_{ij}^{\text{out},-}}$,

$$\begin{aligned}&\left|\left(\Phi_2(K)-\Phi_2(H)\right)_{ij}(t,x,\xi)\right|\\ &\qquad\leq nR^2\sum_{\ell=1}^{p}\left(\int_{s_{ij}^{\text{out}}(t,x,\xi)}^{s_{i,m+\ell}^{\text{out}}\left(s_{ij}^{\text{out}}(t,x,\xi),\chi_i\left(s_{ij}^{\text{out}}(t,x,\xi);t,x\right),0\right)}1\,ds\right)\left\|K-H\right\|_{L^\infty(\mathcal{P})^{m\times n}}.\end{aligned}$$

Thanks to the estimate (88) we can perform again the change of variable

$$s\mapsto\theta(s)=\Omega_i(\chi_{ij}(s;t,x,\xi)),$$

which is decreasing since $j\geq i$ (see Remark 3.9), and obtain

$$\begin{aligned}&\varepsilon_0\left(\int_{s_{ij}^{\text{out}}(t,x,\xi)}^{s_{i,m+\ell}^{\text{out}}\left(s_{ij}^{\text{out}}(t,x,\xi),\chi_i\left(s_{ij}^{\text{out}}(t,x,\xi);t,x\right),0\right)}1\,ds\right)\\ &\qquad\leq\int_{s_{ij}^{\text{out}}(t,x,\xi)}^{s_{i,m+\ell}^{\text{out}}\left(s_{ij}^{\text{out}}(t,x,\xi),\chi_i\left(s_{ij}^{\text{out}}(t,x,\xi);t,x\right),0\right)}-\frac{d\theta}{ds}(s)\,ds\\ &=-\theta\left(s_{i,m+\ell}^{\text{out}}\left(s_{ij}^{\text{out}}(t,x,\xi),\chi_i\left(s_{ij}^{\text{out}}(t,x,\xi);t,x\right),0\right)\right)+\theta\left(s_{ij}^{\text{out}}(t,x,\xi)\right)\leq\theta(t)=\Omega_i(t,x,\xi).\end{aligned}$$

This gives the estimate

$$\left|\left(\Phi_2(K)-\Phi_2(H)\right)_{ij}(t,x,\xi)\right|\leq C_2\Omega_i(t,x,\xi)\left\|K-H\right\|_{L^\infty(\mathcal{P})^{m\times n}}.$$

Note that this estimate is also valid if $j\notin\{i,\dots,m\}$ or $(t,x,\xi)\notin\overline{\mathcal{P}_{ij}^{\text{out},-}}$ since $(\Phi_2(\cdot))_{ij}=0$ in this case. Reasoning by induction as before, it is now not difficult to obtain the estimate

$$\left\|\Phi_2^N(K)-\Phi_2^N(H)\right\|_{L^\infty(\mathcal{P})^{m\times n}}\leq\frac{C^N}{N!}\left\|K-H\right\|_{L^\infty(\mathcal{P})^{m\times n}},$$

for some C independent of N and K,H.

□

3.3 On the time-periodicity of F

In this section we assume that, for some $\tau > 0$, Λ, M and Q are τ-periodic with respect to time and show that the above construction of F leads, with minor modifications, to a F which is also τ-periodic with respect to time.

First of all, concerning the extension of Λ to a function of $\mathbb{R}^2$ (and of M and Q whenever needed), it is clear that one can extend Λ to $\mathbb{R} \times [0,1]$ by just requiring the τ-periodicity with respect to time of this extension. This extension is still denoted by Λ. Then one extends Λ to $\mathbb{R}^2$ so that this extension, still denoted by Λ, is τ-periodic with respect to time and so that the properties (2), (3), (4) and (5) remain valid on $\mathbb{R}^2$ (see e.g. Remark 1.4).

From the construction of F (see (30), (38) and (50)) it is clear that F is τ-periodic with respect to time if so are all the matrix-valued functions involved in the several transformations of this article. Now, in order to obtain the τ-periodicity of these transformations, the minor modifications/comments are essentially the following ones.

1) Concerning the diagonal transformation to remove the diagonal terms in M (see Section 2.1), one simply observes that the function (33) is τ-periodic with respect to time if so is m_{ii}, thanks to the properties

$$\chi_i(s+\tau; t+\tau, \xi) = \chi_i(s; t, \xi), \qquad s_i^{\text{in}}(t+\tau, \xi) = s_i^{\text{in}}(t, \xi) + \tau. \tag{90}$$

2) Concerning the kernel H of the Fredholm transformation (see Section 3.1) one easily checks that it is indeed τ-periodic with respect to time. This follows from the uniqueness of the solution to (59) and similar properties to (90).

3) Concerning the kernel K of the Volterra transformation of the second kind (see Section 3.2), to construct it in such a way that it is τ-periodic with respect to time, it suffices to observe that $\widetilde{m}^1_{\ell j}$, r_{ij} and $q^1_{\ell j}$ become τ-periodic with respect to time once M and Q are, and to modify the definition of the space B given in (82) by adding the condition that the k_{ij}, $1 \le i \le m$, $1 \le j \le n$, are τ-periodic with respect to time (alternatively, one can keep (82) and deduce from the uniqueness of the fixed point of Φ that it has to be τ-periodic with respect to time).

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A Background on broad solutions

We recall that all the systems of this paper have the following form:

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,x) + \Lambda(t,x)\dfrac{\partial y}{\partial x}(t,x) = M(t,x)y(t,x) + G(t,x)y(t,0), \\ y_-(t,1) = \displaystyle\int_0^1 F(t,\xi)y(t,\xi)\,d\xi, \quad y_+(t,0) = Q(t)y_-(t,0), \\ y(t^0,x) = y^0(x), \end{cases} \tag{91}$$

where M and Q have at least the regularity (5), $F \in L^\infty((0,+\infty)\times(0,1))^{m\times n}$ and

$$G \in C^0([0,+\infty)\times[0,1])^{n\times n} \cap L^\infty((0,+\infty)\times(0,1))^{n\times n}.$$

A.1 Definition of broad solution

Let us now introduce the notion of solution for such systems. To this end, we have to restrict our discussion to the domain where the system (91) evolves, i.e. on $(t^0,+\infty)\times(0,1)$. For every $(t,x)\in(t^0,+\infty)\times(0,1)$, we have

$$(s,\chi_i(s;t,x)) \in (t^0,+\infty)\times(0,1), \quad \forall s \in (\bar{s}_i^{\text{in}}(t^0;t,x), s_i^{\text{out}}(t,x)),$$

where we introduced

$$\bar{s}_i^{\text{in}}(t^0;t,x) = \max\left\{t^0, s_i^{\text{in}}(t,x)\right\} < t.$$

Formally, writing the i-th equation of the system (91) along the characteristic $\chi_i(s;t,x)$ for $s \in [\bar{s}_i^{\text{in}}(t^0;t,x), s_i^{\text{out}}(t,x)]$, and using the chain rules yields the ODE

$$\begin{cases} \dfrac{d}{ds} y_i\left(s,\chi_i(s;t,x)\right) = \displaystyle\sum_{j=1}^n m_{ij}\left(s,\chi_i(s;t,x)\right) y_j\left(s,\chi_i(s;t,x)\right) + \sum_{j=1}^n g_{ij}\left(s,\chi_i(s;t,x)\right) y_j\left(s,0\right), \\ y_i\left(\bar{s}_i^{\text{in}}(t^0;t,x), \chi_i(\bar{s}_i^{\text{in}}(t^0;t,x);t,x)\right) = b_i(y)(t,x), \end{cases} \tag{92}$$

where the initial condition $b_i(y)(t,x)$ is given by the appropriate boundary or initial conditions of the system (91):

$$b_i(y)(t,x) = \begin{cases} \displaystyle\sum_{j=1}^n \int_0^1 f_{ij}(s_i^{\text{in}}(t,x),\xi) y_j(s_i^{\text{in}}(t,x),\xi)\,d\xi & \text{if } s_i^{\text{in}}(t,x) > t^0 \text{ and } i \in \{1,\dots,m\}, \\ \displaystyle\sum_{j=1}^m q_{i-m,j}(s_i^{\text{in}}(t,x)) y_j(s_i^{\text{in}}(t,x),0) & \text{if } s_i^{\text{in}}(t,x) > t^0 \text{ and } i \in \{m+1,\dots,n\}, \\ y_i^0(\chi_i(t^0;t,x)) & \text{if } s_i^{\text{in}}(t,x) < t^0. \end{cases} \tag{93}$$

Integrating the ODE (92) over $s \in [\bar{s}_i^{\text{in}}(t^0;t,x), t]$, we obtain the following system of integral equations:

$$\begin{aligned} y_i(t,x) = b_i(y)(t,x) &+ \sum_{j=1}^n \int_{\bar{s}_i^{\text{in}}(t^0;t,x)}^t m_{ij}(s,\chi_i(s;t,x)) y_j(s,\chi_i(s;t,x))\,ds \\ &+ \sum_{j=1}^n \int_{\bar{s}_i^{\text{in}}(t^0;t,x)}^t g_{ij}\left(s,\chi_i(s;t,x)\right) y_j\left(s,0\right)\,ds. \end{aligned} \tag{94}$$

This leads to the following notion of "solution along the characteristics" or "broad solution":

Definition A.1. Let $t^0 \geq 0$ and $y^0 \in L^2(0,1)^n$ be fixed. We say that a function $y : (t^0,+\infty)\times(0,1) \longrightarrow \mathbb{R}^n$ is a broad solution to the system (91) if

$$y \in C^0([t^0,t^0+T]; L^2(0,1)^n) \cap C^0([0,1]; L^2(t^0,t^0+T)^n), \quad \forall T > 0, \tag{95}$$

and if the integral equation (94) is satisfied for every $i \in \{1,\dots,n\}$, for a.e. $t > t^0$ and a.e. $x \in (0,1)$.

A.2 Well-posedness

This section is devoted to the following well-posedness result regarding system (91):

Theorem A.2. *For every $t^0 \geq 0$ and $y^0 \in L^2(0,1)^n$, there exists a unique broad solution to* (91)*. Moreover, there exists $C > 0$ such that, for every $T > 0$, $t^0 \geq 0$ and $y^0 \in L^2(0,1)^n$, the corresponding broad solution y satisfies*

$$\|y\|_{C^0([t^0,t^0+T];L^2(0,1)^n)} + \|y\|_{C^0([0,1];L^2(t^0,t^0+T)^n)} \leq Ce^{CT} \left\|y^0\right\|_{L^2(0,1)^n}. \tag{96}$$

Remark A.3. It follows from the uniformity of the constant Ce^{CT} with respect to the initial time t^0 in the estimate (96) that, for systems of the form (91), the uniform stability property (8) is a consequence of the finite-time global attractor property (7) (simply take $\delta > 0$ such that $Ce^{CT}\delta \leq \varepsilon$).

Let us first point out that this well-posedness result for our initial system (1) for the particular F that we have constructed in Section 2 follows in fact from the well-posedness result for the final target system of Proposition 2.11 (easier to establish), since we have shown that both systems are equivalent by means of several invertible transformations. However, it is still important to have such a well-posedness result for any F within the class studied, which is a result that also has its own interest. We will provide a complete proof since, to the best of our knowledge, there are no references that show the well-posedness for the initial-boundary value problem (91) with non-local terms $G(t,x)y(t,0)$, with weak regularity (95) and with uniform estimate (96).

Proof of Theorem A.2. We first remark that it is enough to prove the theorem for $\|Q\|_{L^\infty}$ small enough, say

$$\|Q\|_{L^\infty} \leq \alpha, \tag{97}$$

where $\alpha > 0$ does not depend on T, t^0, y^0 nor on M, G, F. This follows from the following change of variable:

$$y = D\tilde{y}, \qquad D = \begin{pmatrix} \frac{\alpha}{\|Q\|_{L^\infty}+\alpha}\mathrm{Id}_{\mathbb{R}^m} & 0 \\ 0 & \mathrm{Id}_{\mathbb{R}^p} \end{pmatrix},$$

where $\tilde{y}$ is the solution to the system $(\tilde{M}, \tilde{G}, \tilde{F}, \tilde{Q})$ with

$$\tilde{M} = D^{-1}MD, \quad \tilde{G} = D^{-1}GD, \quad \tilde{F} = \left(\frac{\alpha}{\|Q\|_{L^\infty}+\alpha}\right)^{-1} FD, \quad \tilde{Q} = \frac{\alpha}{\|Q\|_{L^\infty}+\alpha}Q.$$

Let us now show how to prove the theorem under the smallness condition (97) with the Banach fixed point theorem ($\alpha > 0$ will be fixed adequately below). Let $T > 0$, $t^0 \geq 0$ and $y^0 \in L^2(0,1)^n$ be fixed for the remainder of the proof. It is clear that a function $y : (t^0, +\infty) \times (0,1) \longrightarrow \mathbb{R}^n$ satisfies the integral equation (94) if, and only if, it is a fixed point of the map $\mathcal{F} : B \longrightarrow B$, where

$$B = C^0([t^0, t^0+T]; L^2(0,1)^n) \cap C^0([0,1]; L^2(t^0, t^0+T)^n),$$

and $(\mathcal{F}(y))_i(t,x)$ is given by the expression on the right-hand side of (94). It can be checked that $\mathcal{F}$ indeed maps B into itself (actually, by computations similar to the upcoming ones). Let us now make B a Banach space by equipping it with the following weighted norm:

$$\|y\|_B = \|y\|_{B_1} + \|y\|_{B_2},$$

where

$$\|y\|_{B_1} = \max_{t\in[t^0,t^0+T]} e^{-\frac{L_1}{2}(t-t^0)} \sqrt{\int_0^1 \sum_{i=1}^n |y_i(t,x)|^2 e^{-L_2 x}\, dx},$$

and

$$\|y\|_{B_2} = \max_{x\in[0,1]} e^{\frac{L_2}{2}(1-x)} \sqrt{\int_{t^0}^{t^0+T} \sum_{i=1}^{n} |y_i(t,x)|^2 \, e^{-L_1(t-t^0)}\, dt},$$

where $L_1, L_2 > 0$ are constants independent of T, t^0 and y^0 that will be fixed below. Our goal is to show that, for $L_1, L_2 > 0$ large enough,

$$\left\| \mathcal{F}(y^1) - \mathcal{F}(y^2) \right\|_B \leq \frac{1}{2} \left\| y^1 - y^2 \right\|_B, \quad \forall y^1, y^2 \in B. \tag{98}$$

It is then not difficult to check that the fixed point of $\mathcal{F}$ satisfies the estimate (96). Indeed, using (98), we easily see that the fixed point y of $\mathcal{F}$ will satisfy

$$\frac{1}{2} \|y\|_B \leq \|\mathcal{F}(0)\|_B ,$$

and some straightforward computations show that

$$\|y\|^2_{C^0([t^0,t^0+T];L^2(0,1)^n)} \leq e^{L_2} e^{L_1 T} \|y\|^2_{B_1} , \qquad \|y\|^2_{C^0([0,1];L^2(t^0,t^0+T)^n)} \leq e^{L_1 T} \|y\|^2_{B_2} ,$$

$$\|\mathcal{F}(0)\|^2_B \leq 2\left(1 + \frac{e^{L_2}}{\varepsilon}\right) e^{\left\|\frac{\partial \Lambda}{\partial x}\right\|_{L^\infty} T} \left\|y^0\right\|^2_{L^2(0,1)^n} .$$

Let us now establish (98). We introduce

$$y = y^1 - y^2,$$

so that $\mathcal{F}(y^1) - \mathcal{F}(y^2)$ is equal to the right-hand side of (94) with $y^0 = 0$. We have to estimate four types of terms in each $\|\cdot\|_{B_i}$-norm $(i = 1, 2)$. For convenience, we denote by

$$R_1 = \max\left\{ \|\Lambda\|_{L^\infty} , \left\| \frac{\partial \Lambda}{\partial x} \right\|_{L^\infty} \right\}, \quad R_2 = \max\left\{ \|M\|_{L^\infty} , \|G\|_{L^\infty} , \|F\|_{L^\infty} \right\}.$$

We recall that it is crucial that α does not depend on R_2.

Estimate of the $\|\cdot\|_{B_1}$-norm. Let $t \in [t^0, t^0+T]$ be fixed. Let $I = \left\{ x \in (0,1), \quad s_i^{\text{in}}(t,x) > t^0 \right\}$.

1) Let $i \in \{1, \ldots, m\}$. For a.e. $x \in I$, using Cauchy-Schwarz inequality, we have

$$\left| \sum_{j=1}^{n} \int_0^1 f_{ij}(s_i^{\text{in}}(t,x), \xi) y_j(s_i^{\text{in}}(t,x), \xi)\, d\xi \right|^2 \leq n R_2^2 e^{L_2} \|y\|^2_{B_1} e^{L_1(s_i^{\text{in}}(t,x) - t^0)}.$$

Using a finer version of (15), namely,

$$\frac{1-x}{R_1} \leq t - s_i^{\text{in}}(t,x), \tag{99}$$

(obtained similarly to (19)) we obtain the estimate

$$\int_I \left| \sum_{j=1}^{n} \int_0^1 f_{ij}(s_i^{\text{in}}(t,x), \xi) y_j(s_i^{\text{in}}(t,x), \xi)\, d\xi \right|^2 e^{-L_2 x}\, dx \leq \left(n R_2^2 \frac{1}{\frac{L_1}{R_1} - L_2} \right) e^{L_1(t-t^0)} \|y\|^2_{B_1} ,$$

provided that

$$\frac{L_1}{R_1} - L_2 > 0. \tag{100}$$

2) Let $i \in \{m+1, \dots, n\}$. Using (15), we have

$$\int_I \left| \sum_{j=1}^{m} q_{i-m,j}(s_i^{\mathrm{in}}(t,x)) y_j(s_i^{\mathrm{in}}(t,x),0) \right|^2 e^{-L_2 x}\, dx$$

$$\leq m\alpha^2 e^{L_1(t-t^0)} \sum_{j=1}^{m} \int_I \left| y_j(s_i^{\mathrm{in}}(t,x),0) \right|^2 e^{-L_1(s_i^{\mathrm{in}}(t,x)-t^0)}\, dx.$$

Doing the change of variables $\sigma = s_i^{\mathrm{in}}(t,x)$ and using the estimate (see (14), (12) and (19))

$$\frac{\partial s_i^{\mathrm{in}}(t,x)}{\partial x} = \frac{-e^{-\int_{s_i^{\mathrm{in}}(t,x)}^{t} \frac{\partial \lambda_i}{\partial x}(\theta, \chi_i(\theta;t,x))\, d\theta}}{\lambda_i(s_i^{\mathrm{in}}(t,x),0)} \leq -\frac{1}{R_1} e^{-\frac{R_1}{\varepsilon}},$$

we obtain

$$\int_I \left| \sum_{j=1}^{m} q_{i-m,j}(s_i^{\mathrm{in}}(t,x)) y_j(s_i^{\mathrm{in}}(t,x),0) \right|^2 e^{-L_2 x}\, dx \leq \left(m\alpha^2 R_1 e^{\frac{R_1}{\varepsilon}} e^{-L_2} \right) e^{L_1(t-t^0)} \left\| y \right\|_{B_2}^2 .$$

3) For the next term, we have

$$\int_0^1 \sum_{i=1}^{n} \left| \sum_{j=1}^{n} \int_{\bar{s}_i^{\mathrm{in}}(t^0;t,x)}^{t} m_{ij}(s,\chi_i(s;t,x)) y_j(s,\chi_i(s;t,x))\, ds \right|^2 e^{-L_2 x}\, dx$$

$$\leq n R_2^2 \frac{1}{\varepsilon} \sum_{i=1}^{n} \int_0^1 \int_{\bar{s}_i^{\mathrm{in}}(t^0;t,x)}^{t} \sum_{j=1}^{n} \left| y_j(s,\chi_i(s;t,x)) \right|^2 \, ds\, dx.$$

Using the change of variable $(\sigma,\xi) = (s,\chi_i(s;t,x))$, whose Jacobian determinant is (see (12))

$$\det \begin{pmatrix} 1 & 0 \\ \lambda_i(s,\chi_i(s;t,x)) & \frac{\partial \chi_i}{\partial x}(s;t,x) \end{pmatrix} = e^{-\int_s^t \frac{\partial \lambda_i}{\partial x}(\theta,\chi_i(\theta;t,x))\, d\theta} \geq e^{-\frac{R_1}{\varepsilon}}, \quad \forall s \in (s_i^{\mathrm{in}}(t,x),t),$$

we obtain

$$\int_0^1 \sum_{i=1}^{n} \left| \sum_{j=1}^{n} \int_{\bar{s}_i^{\mathrm{in}}(t^0;t,x)}^{t} m_{ij}(s,\chi_i(s;t,x)) y_j(s,\chi_i(s;t,x))\, ds \right|^2 e^{-L_2 x}\, dx$$

$$\leq \left(n^2 R_2^2 \frac{1}{\varepsilon} e^{\frac{R_1}{\varepsilon}} e^{L_2} \frac{1}{L_1} \right) e^{L_1(t-t^0)} \left\| y \right\|_{B_1}^2 .$$

4) Finally, the estimate of the remaining term is easy:

$$\int_0^1 \sum_{i=1}^{n} \left| \sum_{j=1}^{n} \int_{\bar{s}_i^{\mathrm{in}}(t^0;t,x)}^{t} g_{ij}\left(s,\chi_i(s;t,x)\right) y_j\left(s,0\right)\, ds \right|^2 e^{-L_2 x}\, dx$$

$$\leq \left(n^2 R_2^2 \frac{1}{\varepsilon} e^{-L_2} \right) e^{L_1(t-t^0)} \left\| y \right\|_{B_2}^2 .$$

In summary, we have established the following estimate (provided that (100) holds):

$$\left\|\mathcal{F}(y^1) - \mathcal{F}(y^2)\right\|_{B_1}^2 \leq 3\left(mnR_2^2 \frac{1}{\frac{L_1}{R_1} - L_2} + n^2 R_2^2 \frac{1}{\varepsilon} e^{\frac{R_1}{\varepsilon}} e^{L_2} \frac{1}{L_1}\right) \|y\|_{B_1}^2 \\ + 3\left((n-m)m\alpha^2 R_1 e^{\frac{R_1}{\varepsilon}} e^{-L_2} + n^2 R_2^2 \frac{1}{\varepsilon} e^{-L_2}\right) \|y\|_{B_2}^2. \quad (101)$$

Estimate of the $\|\cdot\|_{B_2}$-norm. Let $x \in [0,1]$ be fixed. Let $J = \left\{t \in (t^0, t^0 + T), \quad s_i^{\text{in}}(t,x) > t^0\right\}$.

1) Let $i \in \{1, \ldots, m\}$. We have

$$\int_J \left|\sum_{j=1}^n \int_0^1 f_{ij}(s_i^{\text{in}}(t,x), \xi) y_j(s_i^{\text{in}}(t,x), \xi)\, d\xi\right|^2 e^{-L_1(t-t^0)}\, dt \\ \leq nR_2^2 \int_0^1 \left(\int_J \sum_{j=1}^n \left|y_j(s_i^{\text{in}}(t,x), \xi)\right|^2 e^{-L_1(s_i^{\text{in}}(t,x) - t^0)} e^{L_1(s_i^{\text{in}}(t,x) - t)}\, dt\right) d\xi.$$

Using once again (99), (100), performing the change of variable $\sigma = s_i^{\text{in}}(t,x)$, and using the estimate (see (14), (12) and (19))

$$\frac{\partial s_i^{\text{in}}(t,x)}{\partial t} = \frac{\lambda_i(t,x) e^{-\int_{s_i^{\text{in}}(t,x)}^t \frac{\partial \lambda_i}{\partial x}(\theta, \chi_i(\theta;t,x))\, d\theta}}{\lambda_i(s_i^{\text{in}}(t,x), 1)} \geq \frac{\varepsilon}{R_1} e^{-\frac{R_1}{\varepsilon}}, \quad (102)$$

we obtain the estimate

$$\int_J \left|\sum_{j=1}^n \int_0^1 f_{ij}(s_i^{\text{in}}(t,x), \xi) y_j(s_i^{\text{in}}(t,x), \xi)\, d\xi\right|^2 e^{-L_1(t-t^0)}\, dt \leq \left(n \frac{R_1}{\varepsilon} R_2^2 e^{\frac{R_1}{\varepsilon}} \frac{1}{L_2}\right) e^{-L_2(1-x)} \|y\|_{B_2}^2.$$

2) The next estimate is where we will need the smallness assumption on Q. Let $i \in \{m+1, \ldots, n\}$. Using (15) and the change of variables $\sigma = s_i^{\text{in}}(t,x)$ (recall the estimate (102)), we obtain

$$\int_J \left|\sum_{j=1}^m q_{i-m,j}(s_i^{\text{in}}(t,x)) y_j(s_i^{\text{in}}(t,x), 0)\right|^2 e^{-L_1(t-t^0)}\, dt \leq \left(m\alpha^2 \frac{R_1}{\varepsilon} e^{\frac{R_1}{\varepsilon}}\right) e^{-L_2(1-x)} \|y\|_{B_1}^2.$$

3) For $i \in \{1, \ldots, m\}$, using the estimate

$$-R_1(t-s) \leq x - \chi_i(s;t,x),$$

we have

$$\left|\sum_{j=1}^n \int_{\bar{s}_i^{\text{in}}(t^0;t,x)}^t m_{ij}(s, \chi_i(s;t,x)) y_j(s, \chi_i(s;t,x))\, ds\right|^2 e^{-L_1(t-t^0)} \\ \leq nR_2^2 \frac{1}{\varepsilon} \int_{\bar{s}_i^{\text{in}}(t^0;t,x)}^t \sum_{j=1}^n |y_j(s, \chi_i(s;t,x))|^2 e^{-L_1(s-t^0)} e^{-\frac{L_1}{R_1}(\chi_i(s;t,x) - x)}\, ds.$$

Integrating and using the change of variable $(\sigma, \xi) = (s, \chi_i(s;t,x))$, whose Jacobian determinant is uniformly estimated for $s \in (s_i^{\text{in}}(t,x), t)$ by (see (12) and (19))

$$\left| \det \begin{pmatrix} 1 & 0 \\ \lambda_i(s, \chi_i(s;t,x)) & \frac{\partial \chi_i}{\partial t}(s;t,x) \end{pmatrix} \right| = |\lambda_i(t,x)| \, e^{-\int_s^t \frac{\partial \lambda_i}{\partial x}(\theta, \chi_i(\theta;t,x)) \, d\theta} \geq \varepsilon e^{-\frac{R_1}{\varepsilon}},$$

we obtain (using also that $x \leq \chi_i(s;t,x) \leq 1$)

$$\begin{aligned} &\int_{t^0}^{t^0+T} \sum_{i=1}^{m} \left| \sum_{j=1}^{n} \int_{\bar{s}_i^{\text{in}}(t^0;t,x)}^{t} m_{ij}(s, \chi_i(s;t,x)) y_j(s, \chi_i(s;t,x)) \, ds \right|^2 e^{-L_1(t-t^0)} \, dt \\ &\qquad \leq mnR_2^2 \frac{1}{\varepsilon^2} e^{\frac{R_1}{\varepsilon}} \int_x^1 \left(\int_{t^0}^{t^0+T} \sum_{j=1}^{n} |y_j(\sigma, \xi)|^2 \, e^{-L_1(\sigma - t^0)} \, d\sigma \right) e^{-\frac{L_1}{R_1}(\xi - x)} \, d\xi \\ &\qquad\qquad \leq \left(mnR_2^2 \frac{1}{\varepsilon^2} e^{\frac{R_1}{\varepsilon}} \frac{1}{\frac{L_1}{R_1} - L_2} \right) e^{-L_2(1-x)} \, \|y\|_{B_2}^2, \end{aligned}$$

provided that (100) holds. A similar reasoning shows that

$$\begin{aligned} &\int_{t^0}^{t^0+T} \sum_{i=m+1}^{n} \left| \sum_{j=1}^{n} \int_{\bar{s}_i^{\text{in}}(t^0;t,x)}^{t} m_{ij}(s, \chi_i(s;t,x)) y_j(s, \chi_i(s;t,x)) \, ds \right|^2 e^{-L_1(t-t^0)} \, dt \\ &\qquad\qquad \leq \left((n-m)nR_2^2 \frac{1}{\varepsilon^2} e^{\frac{R_1}{\varepsilon}} \frac{1}{L_2 + \frac{L_1}{R_1}} \right) e^{-L_2(1-x)} \, \|y\|_{B_2}^2. \end{aligned}$$

4) For the remaining term, using a similar reasoning to the one used in the previous step, we obtain

$$\begin{aligned} &\int_{t^0}^{t^0+T} \sum_{i=1}^{n} \left| \sum_{j=1}^{n} \int_{\bar{s}_i^{\text{in}}(t^0;t,x)}^{t} g_{ij}(s, \chi_i(s;t,x)) y_j(s, 0) \, ds \right|^2 e^{-L_1(t-t^0)} \, dt \\ &\qquad\qquad \leq \left(n^2 R_1 R_2^2 \frac{1}{\varepsilon^2} e^{\frac{R_1}{\varepsilon}} \frac{1}{L_1} \right) e^{-L_2(1-x)} \, \|y\|_{B_2}^2. \end{aligned}$$

In summary, we have established the following estimate (provided that (100) holds):

$$\begin{aligned} \left\| \mathcal{F}(y^1) - \mathcal{F}(y^2) \right\|_{B_2}^2 &\leq 3 \left((n-m)m\alpha^2 \frac{R_1}{\varepsilon} e^{\frac{R_1}{\varepsilon}} \right) \|y\|_{B_1}^2 \\ &+ 3 \Bigg(mn \frac{R_1}{\varepsilon} R_2^2 e^{\frac{R_1}{\varepsilon}} \frac{1}{L_2} + mnR_2^2 \frac{1}{\varepsilon^2} e^{\frac{R_1}{\varepsilon}} \frac{1}{\frac{L_1}{R_1} - L_2} + (n-m)nR_2^2 \frac{1}{\varepsilon^2} e^{\frac{R_1}{\varepsilon}} \frac{1}{L_2 + \frac{L_1}{R_1}} \\ &\qquad + n^2 R_1 R_2^2 \frac{1}{\varepsilon^2} e^{\frac{R_1}{\varepsilon}} \frac{1}{L_1} \Bigg) \|y\|_{B_2}^2. \end{aligned} \tag{103}$$

Consequently, we see from (101) and (103) that $\mathcal{F}$ indeed satisfies the contraction property (98) if α is small enough (depending only on $n-m, m, R_1$ and ε) and if we fix $L_2 > 0$ and then $L_1 > 0$ large enough. This concludes the proof of Theorem A.2.

□

Remark A.4. It can be shown that the broad solution is also the classical solution if the data of the system are smooth enough. It then follows by standard approximation arguments that the broad solution is also the so-called weak solution. We recall that the notion of weak solution for (91) is obtained by multiplying (91) by a smooth function and integrating by parts, that is, a function $y : (t^0, +\infty) \times (0,1) \longrightarrow \mathbb{R}^n$ is a weak solution to (91) if $y \in C^0([t^0, +\infty); L^2(0,1)^n)$ and if it satisfies:

$$\begin{aligned}\int_0^1 y(t^0+T,x)\cdot\varphi(t^0+T,x)\,dx - \int_0^1 y^0(x)\cdot\varphi(t^0,x)\,dx \\ + \int_{t^0}^{t^0+T}\int_0^1 y(t,x)\cdot\left(-\frac{\partial\varphi}{\partial t}(t,x) - \Lambda(t,x)\frac{\partial\varphi}{\partial x}(t,x) - \left(\frac{\partial\Lambda}{\partial x}(t,x) + M(t,x)^{\mathrm{Tr}}\right)\varphi(t,x)\right)dxdt \\ + \int_{t^0}^{t^0+T}\int_0^1 y(t,\xi)\cdot F(t,\xi)^{\mathrm{Tr}}\Lambda_{--}(t,1)\varphi_-(t,1)\,d\xi dt = 0, \quad (104)\end{aligned}$$

for every $T > 0$ and every $\varphi \in C^1([t^0, t^0+T]\times[0,1])^n$ such that, for every $t \in [t^0, t^0+T]$,

$$\begin{aligned}\varphi_+(t,1) = 0, \\ \varphi_-(t,0) = -\Lambda_{--}(t,0)^{-1}\left(Q(t)^{\mathrm{Tr}}\Lambda_{++}(t,0)\varphi_+(t,0) + \begin{pmatrix}\mathrm{Id}_{\mathbb{R}^m} & Q(t)^{\mathrm{Tr}}\end{pmatrix}\int_0^1 G(t,x)^{\mathrm{Tr}}\varphi(t,x)\,dx\right).\end{aligned}$$

In (104), we denoted by A^{Tr} the transpose of a matrix A and $v_1 \cdot v_2$ denotes the canonical scalar product between two vectors v_1, v_2 of $\mathbb{R}^n$.

A.3 Justification of the formal computations

In this section, we finally rigorously prove that the transformations that we used all along this paper are preserving broad solutions. We show how it works only for the Fredholm transformation of Section 2.3 (because it is simpler to present) but the reasoning is general and can be used for the Volterra transformation of Section 2.2 as well. More precisely, the goal of this section is to prove the following result:

Proposition A.5. *Let $H_{--} = (h_{ij})_{1\leq i,j\leq m}$, where h_{ij} is the solution to the differential equation* (57). *Let $t^0 \geq 0$ be fixed. Let $z^0 \in L^2(0,1)^n$ and let z be the broad solution to*

$$\begin{cases}\dfrac{\partial z}{\partial t}(t,x) + \Lambda(t,x)\dfrac{\partial z}{\partial x}(t,x) = G^3(t,x)z(t,0), \\ z_-(t,1) = 0, \quad z_+(t,0) = Q^1(t)z_-(t,0), \\ z(t^0,x) = z^0(x).\end{cases}$$

Then, the function γ defined by the Fredholm transformation (48) *is the broad solution to*

$$\begin{cases}\dfrac{\partial \gamma}{\partial t}(t,x) + \Lambda(t,x)\dfrac{\partial \gamma}{\partial x}(t,x) = G^2(t,x)\gamma(t,0), \\ \gamma_-(t,1) = \displaystyle\int_0^1 F^2(t,\xi)\gamma(t,\xi)\,d\xi, \quad \gamma_+(t,0) = Q^1(t)\gamma_-(t,0), \\ \gamma(t^0,x) = \gamma^0(x),\end{cases}$$

where $\gamma^0(x) = z^0(x) - \int_0^1 H(t^0,x,\xi)z^0(\xi)\,d\xi$.

We recall that H is given by (53), F^2 is the solution of (50), Q^1 is provided by Proposition 2.1, G^2 is provided by Proposition 2.4 and, finally, G^3 is given by (44).

Remark A.6. Obviously, we could use the explicit expression (60)-(58) of the solution H to simplify the forthcoming arguments but we choose not to do so and to only use the differential equation (57) in order to give a general procedure that can also be used to justify the formal computations of Section 2.2 as well.

A similar result to Proposition A.5 can be found in [CN19, Proposition 3.5]. Here we propose a different and self-contained proof, based on the following characterization of broad solutions:

Lemma A.7. *A function $y : (t^0, +\infty) \times (0,1) \longrightarrow \mathbb{R}^n$ is the broad solution to* (91) *if, and only if, y has the regularity* (95) *and, for every $i \in \{1, \dots, n\}$, for a.e. $t > t^0$ and a.e. $x \in (0,1)$, the function $s \mapsto y_i(s, \chi_i(s;t,x))$ belongs to $H^1(\bar{s}_i^{\text{in}}(t^0;t,x), s_i^{\text{out}}(t,x))$ and it satisfies the ODE* (92).

The proof of Lemma A.7 is not difficult, it simply relies on the properties (10) and (17).

Proof of Proposition A.5.

1) The required regularity

$$\gamma \in C^0([t^0, t^0+T]; L^2(0,1)^n) \cap C^0([0,1]; L^2(t^0, t^0+T)^n), \quad \forall T > 0,$$

is clear since z also has this regularity and $(t,x) \mapsto \int_0^1 H_{--}(t,x,\xi) z_-(t,\xi)\, d\xi$ is continuous (see e.g. Remark 2.9).

2) The initial condition in the ODE formulation

$$\gamma_i\left(\bar{s}_i^{\text{in}}(t^0;t,x), \chi_i(\bar{s}_i^{\text{in}}(t^0;t,x);t,x)\right) = b_i(\gamma)(t,x)$$

is not difficult to check by using the boundary condition $z_-(t,1) = 0$ with the definition (50) of F^2 and Fubini's theorem (case $s_i^{\text{in}}(t,x) > t^0$ and $i \in \{1, \dots, m\}$), the condition $H_{--}(t,0,\xi) = 0$ (case $s_i^{\text{in}}(t,x) > t^0$ and $i \in \{m+1, \dots, n\}$) and the definition of γ^0 (case $s_i^{\text{in}}(t,x) < t^0$).

3) It remains to check that, for every $i \in \{1, \dots, n\}$, for a.e. $t > t^0$ and $x \in (0,1)$, the function $s \mapsto \gamma_i(s, \chi_i(s;t,x))$ belongs to $H^1(\bar{s}_i^{\text{in}}(t^0;t,x), s_i^{\text{out}}(t,x))$ with

$$\frac{d}{ds}\gamma_i(s, \chi_i(s;t,x)) = \sum_{j=1}^{m} g_{ij}^2(s, \chi_i(s;t,x))\, \gamma_j(s,0). \tag{105}$$

By definition (48) of γ, we have

$$\gamma_i(s, \chi_i(s;t,x)) = z_i(s, \chi_i(s;t,x)) - \sum_{j=1}^{m} \int_0^1 h_{ij}(s, \chi_i(s;t,x), \xi) z_j(s,\xi)\, d\xi.$$

For $i \in \{m+1, \dots, n\}$, the identity (105) easily follows from the equation satisfied by z_i, the relation $z(\cdot,0) = \gamma(\cdot,0)$, and the fact that $h_{ij} = 0$ for such indices (recall (53)).

Let us now assume that $i \in \{1, \dots, m\}$. The equation satisfied by z_i then gives

$$\frac{d}{ds} z_i(s, \chi_i(s;t,x)) = 0, \quad \forall i \in \{1, \dots, m\}.$$

On the other hand, since we know some information of h_{ij} along the characteristic curve $s \mapsto \chi_{ij}(s;t,x,\theta) = (s, \chi_i(s;t,x), \chi_j(s;t,\theta))$, we would like to perform the change of variable

$$\xi = \chi_j(s;t,\theta).$$

Thanks to (13) and the implicit function theorem there exists $\theta_j \in C^1(\mathbb{R}^3)$ such that, for every $(s,t,\xi) \in \mathbb{R}^3$, we have

$$\xi = \chi_j(s;t,\theta_j(s;t,\xi)), \qquad \frac{\partial \theta_j}{\partial \xi}(s;t,\xi) > 0. \tag{106}$$

Using this change of variable, we have

$$\gamma_i\left(s,\chi_i(s;t,x)\right) = z_i\left(s,\chi_i(s;t,x)\right) - \sum_{j=1}^{m} \int_{a_j(s)}^{b_j(s)} \eta_{ij}(s,\theta)\, d\theta,$$

where

$$a_j(s) = \theta_j(s;t,0), \qquad b_j(s) = \theta_j(s;t,1),$$

$$\eta_{ij}(s,\theta) = h_{ij}(\chi_{ij}(s;t,x,\theta)) z_j(s,\chi_j(s;t,\theta)) \frac{\partial \chi_j}{\partial x}(s;t,\theta).$$

We would like to use the formula

$$\frac{d}{ds}\left(\int_{a_j(s)}^{b_j(s)} \eta_{ij}(s,\theta)\, d\theta\right) = b_j'(s)\eta_{ij}(s,b_j(s)) - a_j'(s)\eta_{ij}(s,a_j(s)) + \int_{a_j(s)}^{b_j(s)} \frac{\partial \eta_{ij}}{\partial s}(s,\theta)\, d\theta.$$

Clearly, $a_j, b_j \in C^1(\mathbb{R})$. Differentiating the relation $\xi = \chi_j(s;t,\theta_j(s;t,\xi))$ with respect to s we obtain

$$a_j'(s) = \frac{-\lambda_j(s,0)}{\frac{\partial \chi_j}{\partial x}(s;t,\theta_j(s;t,0))}.$$

On the other hand, using (57) with $\xi = 0$, (106) and the boundary condition $z_-(\cdot,1) = 0$, we have

$$\eta_{ij}(s,a_j(s)) = -\frac{g_{ij}^2(s,\chi_i(s;t,x))}{\lambda_j(s,0)} z_j(s,0) \frac{\partial \chi_j}{\partial x}(s;t,\theta_j(s;t,0)), \qquad \eta_{ij}(s,b_j(s)) = 0.$$

Using the ODEs satisfied along the characteristics by h_{ij} (see (57)) and z_j, and using the relation (see (12))

$$\frac{\partial^2 \chi_j}{\partial s \partial x}(s;t,\theta) = \frac{\partial \lambda_j}{\partial x}(s,\chi_j(s;t,\theta)) \frac{\partial \chi_j}{\partial x}(s;t,\theta),$$

we can check that η_{ij} has weak derivative with respect to s which is equal to zero:

$$\frac{\partial \eta_{ij}}{\partial s}(s,\theta) = 0.$$

It follows from all the previous computations and the relation $z(\cdot,0) = \gamma(\cdot,0)$ that

$$\frac{d}{ds}\gamma_i\left(s,\chi_i(s;t,x)\right) = \sum_{j=1}^{m} a_j'(s)\eta_{ij}(s,a_j(s)) = \sum_{j=1}^{m} g_{ij}^2(s,\chi_i(s;t,x))\gamma_j(s,0).$$

□

B Constructions of $s_{ij}^{\text{in}}, s_{ij}^{\text{out}}$ and $s_{ij\ell}^{\text{disc}}$

In this appendix, we give a proof of the existence $s_{ij}^{\text{in}}, s_{ij}^{\text{out}}$ and $s_{ij\ell}^{\text{disc}}$ satisfying the properties stated in Proposition 3.3 and Proposition 3.5. We will make use of the following simple lemma:

Lemma B.1. *Let $f \in C^1([a,b])$ $(a < b)$ satisfy the following property:*

$$\forall s \in [a,b], \quad f(s) = 0 \quad \Longrightarrow \quad f'(s) < 0. \tag{107}$$

Then, there exists a unique $c \in [a,b]$ such that

$$f(s) > 0, \quad \forall s \in (a,c), \quad f(s) < 0, \quad \forall s \in (c,b).$$

Moreover, c has the properties listed in Table 1 (an $\emptyset$ means that such a situation can not occur).

	$f(b) > 0$	$f(b) = 0$	$f(b) < 0$
$f(a) > 0$	$c = b$	$c = b$	$f(c) = 0$
$f(a) = 0$	$\emptyset$	$\emptyset$	$c = a$
$f(a) < 0$	$\emptyset$	$\emptyset$	$c = a$

Table 1: Properties of c

Proof of Proposition 3.3. We recall that $i \in \{1, \dots, m\}$ and we refer to Figures 2, 3, 4 and 5 for a clarification of the geometric situation (at a fixed t). We only focus on the existence part since the uniqueness readily follows from the properties that have to be satisfied.

1) Assume that $j \in \{1, \dots, i-1\}$. For every $(t,x,\xi) \in \overline{\mathcal{P}}$ such that $x < 1$, we introduce the C^1 function

$$f : s \in \left[\max\left\{s_i^{\text{in}}(t,x), s_j^{\text{in}}(t,\xi)\right\}, t\right] \mapsto \chi_j(s;t,\xi) - \chi_i(s;t,x).$$

Note that the interval has a non empty interior since $x < 1$ and $\xi \leq x < 1$ (see (15)-(16)). This function clearly satisfies the property (107) thanks to the ODE (9) and the assumption (3) since $j < i$. Consequently, Lemma B.1 applies and gives the existence of $s_{ij}^{\text{in}}(t,x,\xi)$ with

$$\max\left\{s_i^{\text{in}}(t,x), s_j^{\text{in}}(t,\xi)\right\} \leq s_{ij}^{\text{in}}(t,x,\xi) \leq t,$$

and such that

$$\chi_j(s;t,\xi) < \chi_i(s;t,x), \quad \forall s \in \left(s_{ij}^{\text{in}}(t,x,\xi), t\right).$$

Clearly, $(t,x,\xi) \mapsto t - s_{ij}^{\text{in}}(t,x,\xi) \in L^\infty(\mathcal{P})$ thanks to (19). Moreover, it follows from Table 1 that

$$\begin{cases} s_{ij}^{\text{in}}(t,x,\xi) = t & \text{if } s_i^{\text{in}}(t,x) < s_j^{\text{in}}(t,\xi) \text{ and } \xi = x, \\ f(s_{ij}^{\text{in}}(t,x,\xi)) = 0 & \text{if } s_i^{\text{in}}(t,x) < s_j^{\text{in}}(t,\xi) \text{ and } \xi < x, \\ s_{ij}^{\text{in}}(t,x,\xi) = s_i^{\text{in}}(t,x) & \text{if } s_i^{\text{in}}(t,x) = s_j^{\text{in}}(t,\xi) \text{ and } \xi < x, \\ s_{ij}^{\text{in}}(t,x,\xi) = s_i^{\text{in}}(t,x) & \text{if } s_i^{\text{in}}(t,x) > s_j^{\text{in}}(t,\xi) \text{ and } \xi < x. \end{cases} \tag{108}$$

Let us now complete the definition of s_{ij}^{in} on the remaining parts of $\overline{\mathcal{P}}$. The missing case in (108) is when $s_i^{\text{in}}(t,x) \geq s_j^{\text{in}}(t,\xi)$ and $\xi = x$. However, unless $x = 1$, these conditions are not compatible since $s_i^{\text{in}}(t,x) < s_j^{\text{in}}(t,x)$ for $j < i \leq m$ (see (21)). Consequently, it only remains to define s_{ij}^{in} in the part where $x = 1$, which we do now by setting

$$s_{ij}^{\text{in}}(t,1,\xi) = t. \tag{109}$$

We can check that s_{ij}^{in} defined by (108)-(109) belongs to $C^0(\overline{\mathcal{P}})$ (for the second case in (108) this follows from the implicit function theorem). Therefore, such a s_{ij}^{in} clearly satisfies all the properties claimed in the statement of item (i) of Proposition 3.3.

2) Assume that $j = i$. We will show that, in this case, we can simply take

$$s_{ii}^{\text{out}}(t,x,\xi) = s_i^{\text{out}}(t,\xi).$$

Clearly, $s_{ii}^{\text{out}} \in C^0(\overline{\mathcal{P}})$ with $(t,x,\xi) \mapsto s_{ii}^{\text{out}}(t,x,\xi) - t \in L^\infty(\mathcal{P})$ thanks to (19) and $s_{ii}^{\text{out}}(t,x,\xi) > t$ as long as $\xi > 0$ (see (15)-(16)). Let us now observe that, for $\xi < x$, we have from (13):

$$\chi_i(s;t,\xi) < \chi_i(s;t,x), \quad \forall s \in \mathbb{R},$$

and $\chi_i(s;t,\xi) > 0$ for $s \in (t, s_{ii}^{\text{out}}(t,x,\xi))$ since $s_i^{\text{out}}(t,\xi) < s_i^{\text{out}}(t,x)$ by (20) (recall that $j = i \in \{1,\ldots,m\}$).

3) The proof for the case $j \in \{i+1,\ldots,m\}$ is similar to the proof of part 1) by considering, for each $(t,x,\xi) \in \overline{\mathcal{P}}$ such that $\xi, x > 0$, the function

$$f : s \in \left[t, \min\left\{s_i^{\text{out}}(t,x), s_j^{\text{out}}(t,\xi)\right\}\right] \mapsto \chi_i(s;t,x) - \chi_j(s;t,\xi). \tag{110}$$

4) The proof for the case $j \in \{m+1,\ldots,n\}$ is also similar to the proof of part 1) by considering, for each $(t,x,\xi) \in \overline{\mathcal{P}}$ such that $0 \leq \xi < x \leq 1$, the function f defined again by (110).

□

Proof of Proposition 3.5. The difference with the proof of Proposition 3.3 is that we do not need to neither track the regularity of the point where the function f vanishes nor its sign on the left and right of this zero. It is a straightforward consequence of Lemma B.1 applied to the following functions (it is enough to consider non empty intervals):

1) For $j \in \{1,\ldots,i-1\}$, we use

$$f : s \in \left[s_{ij}^{\text{in}}(t,x,\xi), t\right] \mapsto \chi_j(s;t,\xi) - \psi_{i\ell}\left(s, \chi_i(s;t,x)\right).$$

Using the ODE (9) satisfied by χ_j and using the equation (41) satisfied by $\psi_{i\ell}$, we have

$$f'(s) = \lambda_j\left(s, \chi_j(s;t,\xi)\right) - \lambda_\ell\left(s, \psi_{i\ell}\left(s, \chi_i(s;t,x)\right)\right).$$

Since $j < \ell$, this shows that such a f satisfies the property (107) of Lemma B.1.

2) For $j \in \{i,\ldots,\ell-1\}$, we use

$$f : s \in \left[t, s_{ij}^{\text{out}}(t,x,\xi)\right] \mapsto \chi_j(s;t,\xi) - \psi_{i\ell}\left(s, \chi_i(s;t,x)\right). \tag{111}$$

3) For $j \in \{\ell+1,\ldots,m\}$, we use the function $-f$, where f is given by (111).

4) For $j \in \{m+1,\ldots,n\}$, we use the same function f given by (111) (in fact, the result then directly follows from the intermediate value theorem).

□

C Construction of Ω_i

This appendix is devoted to the proof of Lemma 3.7, that is to the existence of the key change of variable needed in the proof of Proposition 3.6. We recall that $i \in \{1, \dots, m\}$.

1) Inspired by the time-independent case (see Remark 3.6), we look for Ω_i in the following form:

$$\Omega_i(t,x,\xi) = \omega_i^1(t,x) - \omega_i^\nu(t,\xi),$$

where, at each fixed $\nu \in (0,1]$, $\omega_i^\nu(\cdot,\cdot)$ is the solution to the following linear hyperbolic equation:

$$\begin{cases} \dfrac{\partial \omega_i^\nu}{\partial t}(t,x) + \dfrac{\lambda_i(t,x)}{\nu}\dfrac{\partial \omega_i^\nu}{\partial x}(t,x) = 0, \\ \omega_i^\nu(t,0) = t, \end{cases} \qquad t \in \mathbb{R},\ x \in [0,1]. \tag{112}$$

The solution of (112) is explicit:

$$\omega_i^\nu(t,x) = \omega_i^\nu(s_i^{\text{out},\nu}(t,x), 0) = s_i^{\text{out},\nu}(t,x), \tag{113}$$

where $s_i^{\text{out},\nu}(t,x) \geq t$ (with $s_i^{\text{out},\nu}(t,x) = t \iff x = 0$) is the unique number such that

$$\chi_i^\nu(s_i^{\text{out},\nu}(t,x); t, x) = 0, \tag{114}$$

where $s \mapsto \chi_i^\nu(s;t,x)$ is the solution to the ODE

$$\begin{cases} \dfrac{\partial \chi_i^\nu}{\partial s}(s;t,x) = \dfrac{1}{\nu}\lambda_i(s, \chi_i^\nu(s;t,x)), \quad \forall s \in \mathbb{R}, \\ \chi_i^\nu(t;t,x) = x. \end{cases} \tag{115}$$

We can check that the map $(t,x,\nu) \mapsto \omega_i^\nu(t,x)$ belongs to $C^1(\mathbb{R} \times [0,1] \times (0,1])$.

2) We now prove that there exists $\delta > 0$ such that, for every $t \in \mathbb{R}$, $x \in [0,1]$ and $\nu \in (0,1]$,

$$\frac{\partial \omega_i^\nu}{\partial t}(t,x) \geq \varepsilon\delta, \qquad \frac{\partial \omega_i^\nu}{\partial x}(t,x) \geq \nu\delta, \qquad \frac{\partial \omega_i^\nu}{\partial \nu}(t,x) \geq 0. \tag{116}$$

Using the equation (112) and the assumption (3), it is clear that the estimate for $\partial\omega_i^\nu/\partial t$ follows from the estimate of $\partial\omega_i^\nu/\partial x$. Note from (113) that $\partial\omega_i^\nu/\partial x = \partial s_i^{\text{out},\nu}/\partial x$. Taking the derivative of (114) with respect to x, we obtain

$$\frac{1}{\nu}\lambda_i(s_i^{\text{out},\nu}(t,x), \chi_i^\nu(s_i^{\text{out},\nu}(t,x);t,x))\frac{\partial s_i^{\text{out},\nu}}{\partial x}(t,x) + \frac{\partial \chi_i^\nu}{\partial x}(s_i^{\text{out},\nu}(t,x);t,x) = 0.$$

Since $\lambda_i \in L^\infty(\mathbb{R} \times (0,1))$, we have to bound $\frac{\partial \chi_i^\nu}{\partial x}(s_i^{\text{out},\nu}(t,x);t,x)$ from below by a positive constant that does not depend on t, x and ν. From (115) we can show that

$$\frac{\partial \chi_i^\nu}{\partial x}(s;t,x) = e^{\frac{1}{\nu}\int_t^s \frac{\partial \lambda_i}{\partial x}(\theta, \chi_i^\nu(\theta;t,x))\, d\theta},$$

so that

$$\frac{\partial \chi_i^\nu}{\partial x}(s_i^{\text{out},\nu}(t,x);t,x) \geq e^{\frac{1}{\nu}(s_i^{\text{out},\nu}(t,x)-t)\inf_{\mathbb{R}\times[0,1]}\frac{\partial \lambda_i}{\partial x}}.$$

This establishes the desired lower bound since $\frac{\partial \lambda_i}{\partial x} \in L^\infty(\mathbb{R}\times(0,1))$ and $0 \leq s_i^{\text{out},\nu}(t,x) - t \leq \frac{x}{\varepsilon}\nu$ (the proof is similar to the one of (19)). Note that it follows as well from this estimate that $\Omega_i \in L^\infty(\mathbb{R} \times (0,1) \times (0,1))$.

To prove the remaining estimate in (116), we denote by $\gamma^\nu = \partial\omega_i^\nu/\partial\nu$ and observe that it satisfies

$$\begin{cases} \dfrac{\partial\gamma^\nu}{\partial t}(t,x) + \dfrac{\lambda_i(t,x)}{\nu}\dfrac{\partial\gamma^\nu}{\partial x}(t,x) = \dfrac{\lambda_i(t,x)}{\nu^2}\dfrac{\partial\omega_i^\nu}{\partial x}(t,x) \leq 0, \\ \gamma^\nu(t,0) = 0, \end{cases} \qquad t \in \mathbb{R},\ x \in [0,1].$$

It immediately follows that $\gamma^\nu \geq 0$.

3) Let us now check the estimates (87) and (88). We have

$$\begin{aligned} &\frac{\partial\Omega_i}{\partial t}(t,x,\nu) + \lambda_i(t,x)\frac{\partial\Omega_i}{\partial x}(t,x,\nu) + \lambda_j(t,\xi)\frac{\partial\Omega_i}{\partial\xi}(t,x,\nu) \\ &\quad = \frac{\partial\omega_i^1}{\partial t}(t,x) + \lambda_i(t,x)\frac{\partial\omega_i^1}{\partial x}(t,x) - \frac{\partial\omega_i^\nu}{\partial t}(t,\xi) - \lambda_j(t,\xi)\frac{\partial\omega_i^\nu}{\partial x}(t,\xi) \\ &\qquad = -\frac{\partial\omega_i^\nu}{\partial t}(t,\xi)\left(1 - \nu\frac{\lambda_j(t,\xi)}{\lambda_i(t,\xi)}\right). \end{aligned}$$

Since $\lambda_j/\lambda_i \leq 1$ for $i \leq j$ and $i \leq m$, we see that the estimate (88) is obtained by simply taking

$$0 < \varepsilon_0 \leq \varepsilon\delta(1-\nu), \qquad 0 < \nu < 1. \tag{117}$$

On the other hand, let us introduce

$$r = \max_{1 \leq j < i} \sup_{\substack{t\in\mathbb{R} \\ \xi\in[0,1]}} \frac{\lambda_i(t,\xi)}{\lambda_j(t,\xi)}.$$

Clearly, $0 < r \leq 1$. In fact, $r < 1$ since from (4) we have, for $j < i \leq m$,

$$\sup_{\substack{t\in\mathbb{R} \\ \xi\in[0,1]}} \frac{\lambda_i(t,\xi)}{\lambda_j(t,\xi)} \leq 1 - \frac{\varepsilon}{\|\lambda_j\|_{L^\infty}}.$$

The estimate (87) now follows from (116) by taking

$$0 < \varepsilon_0 \leq \varepsilon\delta\left(\frac{\nu}{r} - 1\right), \qquad r < \nu \leq 1. \tag{118}$$

Note that the conditions (117) and (118) are compatible by taking ν close enough to 1.

4) It remains to check that $\Omega_i \geq 0$ on $\overline{\mathcal{P}}$. Since both functions $\nu \mapsto \omega_i^\nu(t,x)$ and $x \mapsto \omega_i^\nu(t,x)$ are nondecreasing by (116) and $\xi \leq x$, we have

$$\omega_i^1(t,x) \geq \omega_i^\nu(t,x) \geq \omega_i^\nu(t,\xi).$$

□

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