Oceanic fluid dynamics under location uncertainty Part.I : Towards a stochastic modeling for the Quasi-Geostrophic system

Long Li, Etienne Mémin, Bruno Deremble

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Introduction

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- better small-scale representation

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- identify regously subgrid effects

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- fluid flows under location uncertainty
- stochastic QG equations
- multi-layer stochastic shallow water system

Uncertainty formulation (Mémin, 2014)

Principes

- Lagrangian displacement :

$$d\boldsymbol{X}(\boldsymbol{x},t) = \boldsymbol{w}(\boldsymbol{X}(\boldsymbol{x},t),t)dt + \boldsymbol{\sigma}(\boldsymbol{X}(\boldsymbol{x},t),t)d\boldsymbol{B}_t, \ \forall(\boldsymbol{x},t) \in D \times \mathbb{R}^+, D \subset \mathbb{R}^3$$
$$\boldsymbol{X}(\boldsymbol{x},0) = \boldsymbol{x}, \ \forall \boldsymbol{x} \in D$$

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- Eulerian velocity :

$$U(\boldsymbol{x},t) = \underbrace{\boldsymbol{w}(\boldsymbol{x},t)}_{\text{large-scale}} + \underbrace{\boldsymbol{\sigma}(\boldsymbol{x},t)\dot{\boldsymbol{B}}_t}_{\text{small-scale}}, \ \forall \ (\boldsymbol{x},t) \in D \times \mathbb{R}^+$$

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- $(\boldsymbol{B}_t)_{t\in\mathbb{R}^+}$ is a cylindrical Wiener process in $L^2(D,\mathbb{R}^3)$

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- Subgrid tensor :

$$\boldsymbol{a} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T = \frac{1}{dt}\mathbb{E}\Big[(\boldsymbol{\sigma}d\boldsymbol{B}_t)(\boldsymbol{\sigma}d\boldsymbol{B}_t)^T\Big]$$

Stochastic Reynolds Transport Theorem (SRRT)

SRRT for solenoidal turbulence

- Volumetric rate of change of a scalar for $0 = \nabla \cdot \boldsymbol{\sigma} d\boldsymbol{B}_t = \nabla \cdot \boldsymbol{\sigma}$:

$$d \int_{\mathcal{V}(t)} q(\boldsymbol{x}, t) d\boldsymbol{x} = \int_{\mathcal{V}(t)} \left[d_t q + \left(\nabla \cdot (q\boldsymbol{w}) - \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} (q a_{ij}) \right) dt + \nabla q \cdot \boldsymbol{\sigma} d\boldsymbol{B}_t \right] d\boldsymbol{x}$$

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- Conservation of extensive scalar :

energy balance

$$d_t q + \underbrace{\boldsymbol{w}^* dt\cdot\nabla q}_{\text{advection}^*} + \underbrace{\boldsymbol{\sigma}d\boldsymbol{B}_t\cdot\nabla q}_{\text{forcing}} - \underbrace{\nabla\cdot\frac{1}{2}(\boldsymbol{a}\nabla q)dt}_{\text{diffusion}} = -q\nabla\cdot\boldsymbol{w}^* dt$$

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- Effective drift :

$$\boldsymbol{w}^* = \boldsymbol{w} - \frac{1}{2} \nabla \cdot \boldsymbol{a}$$

Stokes drift ?

Stochastic conservation laws

Navier Stokes equations under location uncertainty

- Momentum equation :

$$d_t \boldsymbol{w} + \left(\boldsymbol{w}^* dt + \boldsymbol{\sigma} d\boldsymbol{B}_t\right) \cdot \nabla \boldsymbol{w} - \frac{1}{\rho} \nabla \cdot \left(\frac{1}{2} \rho \boldsymbol{a} \nabla \boldsymbol{w}\right) dt + \boldsymbol{f} \times \left(\boldsymbol{w} dt + \boldsymbol{\sigma} d\boldsymbol{B}_t\right)$$
$$= -\frac{1}{\rho} \nabla \left(p dt + dp'_t\right) - \rho \boldsymbol{k} dt + \nu \nabla^2 \left(\boldsymbol{w} dt + \boldsymbol{\sigma} d\boldsymbol{B}_t\right),$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity and dp'_t is a centered random process.

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- Mass equation :

$$d_t \rho + \left(\boldsymbol{w}^* dt + \boldsymbol{\sigma} d\boldsymbol{B}_t\right) \cdot \nabla \rho - \nabla \cdot \left(\frac{1}{2} \boldsymbol{a} \nabla \rho\right) dt = -\rho \nabla \cdot \boldsymbol{w}^* dt$$
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- Continuity equation (a sufficient constraint) :

$$0 = \nabla \cdot \boldsymbol{w} = \nabla \cdot (\nabla \cdot \boldsymbol{a}) = \nabla \cdot \boldsymbol{\sigma}$$

Stochastic governing equations for stratified ocean

Simple Boussinesq equations under location uncertainty

- Boussinesq approximations :

$$\rho(\boldsymbol{x},t) = \rho_b + \delta\rho(\boldsymbol{x},t), \text{ with } |\delta\rho| \ll \rho_b$$
$$p(\boldsymbol{x},t) = p_0(z) + \delta p(\boldsymbol{x},t), \text{ with } |\delta p| \ll |p_0|$$

- Hydrostatic balance :

$$\frac{\partial p_0}{\partial z}(z) = -g\rho_b$$

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- Boussinesq thermodynamic equation :

  ▷ Applies SRRT with $\nabla\cdot\boldsymbol{\sigma} d\boldsymbol{B}_t = 0$ for $b(\boldsymbol{x},t) = b_0(z) + b'(\boldsymbol{x},t)$:

$$d_t b' + \Big(\boldsymbol{w}^* dt + \boldsymbol{\sigma} d\boldsymbol{B}_t\Big)\cdot\nabla b' + N^2\Big(w^* dt + (\sigma dB_t)_z\Big) = \nabla\cdot\Big(\frac{1}{2}\boldsymbol{a}\nabla b'\Big)dt + \nabla\cdot\Big(\frac{1}{2}\boldsymbol{a}_{\cdot z} N^2\Big)dt$$

Continuously stratified QG system

Geostrophic scaling assumptions

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   - A small Rossby number : $Ro \ll 1$
   - A small variation of f : $|\beta L| \ll f_0$
   - The scale of motion is not significantly larger than the deformation scale :

$$\frac{Ro}{Bu} = \mathcal{O}(Ro) \Rightarrow \frac{\partial b'}{\partial z} \ll N^2$$

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2. Uncertainties scalings :

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- The vertical uncertainty is small compared with the horizontal uncertainties :

$$\frac{(\sigma dB_t)_z}{\|(\boldsymbol{\sigma} d\boldsymbol{B}_t)_H\|} \sim \frac{Ro}{Bu} D, \; D = \frac{h}{L} \ll 1$$

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- A moderate uncertainty such that the energy dissipated by the horizontal small-scale flow is the same order than the large-scale kinetic energy :

$$\boldsymbol{a}_H \sim UL \Leftarrow MKE \sim U^2, \, TKE \sim A_H/T_\sigma$$

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- Results :

$$(\sigma dB_t)_z \frac{\partial}{\partial z} = \mathcal{O}\Big(\frac{Ro}{Bu}\Big)$$

$$\frac{\boldsymbol{a}_{Hz}}{\boldsymbol{a}_H} \sim \frac{Ro}{Bu} D, \quad \frac{a_{zz}}{\boldsymbol{a}_H} \sim \Big(\frac{Ro}{Bu}\Big)^2 D^2$$

$$\forall \; i \in H, \; \boldsymbol{a}_{Hz} \frac{\partial^2}{\partial x_i \partial z} dt = \mathcal{O}\Big(\frac{Ro}{Bu}\Big), \quad a_{zz} \frac{\partial^2}{\partial z^2} dt = \mathcal{O}\Big(\Big(\frac{Ro}{Bu}\Big)^2\Big)$$

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Non-dimensional primitive equations under location uncertainty

- Momentum :

$$Ro\Big[d_t\boldsymbol{u} + \Big(\boldsymbol{u}^* dt + (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H\Big)\cdot\nabla_H\boldsymbol{u} - \nabla_H\cdot\Big(\frac{1}{2}\boldsymbol{a}_H\nabla_H\boldsymbol{u}\Big)dt + \mathcal{O}\Big(\frac{Ro}{Bu}\Big)\Big]$$
$$+\Big(f_0 + Ro\beta(y-y_0)\Big)\boldsymbol{k}\times\Big(\boldsymbol{u}dt + (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H\Big) = -\nabla_H\Big(\tilde{p}dt + d\tilde{p}_t'\Big)$$

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$$b' dt + \mathcal{O}\Big(RoD^2\Big) = \frac{\partial}{\partial z}\Big(\tilde{p}dt + d\tilde{p}_t'\Big)$$

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- Continuity :

$$\nabla_H\cdot\boldsymbol{u} + \frac{\partial w}{\partial z} = 0$$
$$\nabla_H\cdot(\nabla_H\cdot\boldsymbol{a}_H) + \frac{Ro}{Bu}\frac{\partial}{\partial z}(\nabla_H\cdot\boldsymbol{a}_{Hz}) = \nabla_H\cdot(\boldsymbol{\sigma} d\boldsymbol{B}_t)_H + \frac{Ro}{Bu}\frac{\partial(\sigma dB_t)_z}{\partial z} = 0 \qquad (1)$$

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- Thermodynamic :

$$Ro\Big[d_t b' + \Big(\boldsymbol{u}^* dt + (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H\Big)\boldsymbol{\cdot}\nabla_H b' - \nabla_H\boldsymbol{\cdot}\Big(\frac{1}{2}\boldsymbol{a}_H\nabla_H b'\Big)dt + \frac{\partial b'}{\partial z}wdt\Big]$$
$$+Bu\, wdt + Ro\Big[(\sigma dB_t)_z - \frac{1}{2}\nabla_H\boldsymbol{\cdot}\boldsymbol{a}_{Hz}dt + \mathcal{O}\Big(\frac{Ro}{Bu}\Big)\Big] = 0$$

Zeroth order relations

- Pressure balances rotation :

$$f_0 \boldsymbol{k} \times (\boldsymbol{u}_0 dt + (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H) = -\nabla_H (p_0 dt + d\tilde{p}_t') \Leftrightarrow \begin{cases} f_0 v_0 = \dfrac{\partial p_0}{\partial x}, \ f_0 u_0 = -\dfrac{\partial p_0}{\partial y} \\ f_0 (\sigma dB_t)_y = \dfrac{\partial d\tilde{p}_t'}{\partial x}, \ f_0 (\sigma dB_t)_x = -\dfrac{\partial d\tilde{p}_t'}{\partial y} \end{cases}$$

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- Horizontal incompressibilities :

$$0 = \nabla_H \cdot \boldsymbol{u}_0 = \nabla_H \cdot (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H \xrightarrow{(1)} \frac{\partial (\sigma dB_t)_z}{\partial z} \approx 0, \ \frac{\partial}{\partial z} (\nabla_H \cdot \boldsymbol{a}_{Hz}) \approx 0 \tag{2}$$

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- Relative vorticity :

$$\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} = \frac{\nabla_H^2 p_0}{f_0}$$

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$$0 = \nabla_H \cdot \boldsymbol{u}_0 = \nabla_H \cdot (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H \xrightarrow{(1)} \frac{\partial (\sigma dB_t)_z}{\partial z} \approx 0, \ \frac{\partial}{\partial z} (\nabla_H \cdot \boldsymbol{a}_{Hz}) \approx 0 \tag{2}$$

- Relative vorticity :

$$\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} = \frac{\nabla_H^2 p_0}{f_0}$$

- Hydrostrasy :

$$\frac{\partial p_0}{\partial z} = b_0, \ \frac{\partial \tilde{p}_t'}{\partial z} = \mathcal{O}(RoD^2)$$

Continuously stratified QG system

Zeroth order relations

- Pressure balances rotation :

$$f_0 \boldsymbol{k} \times (\boldsymbol{u}_0 dt + (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H) = -\nabla_H (p_0 dt + d\tilde{p}'_t) \Leftrightarrow \begin{cases} f_0 v_0 = \dfrac{\partial p_0}{\partial x}, \ f_0 u_0 = -\dfrac{\partial p_0}{\partial y} \\ f_0 (\sigma dB_t)_y = \dfrac{\partial d\tilde{p}'_t}{\partial x}, \ f_0 (\sigma dB_t)_x = -\dfrac{\partial d\tilde{p}'_t}{\partial y} \end{cases}$$

- Horizontal incompressibilities :

$$0 = \nabla_H \cdot \boldsymbol{u}_0 = \nabla_H \cdot (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H \xrightarrow{(1)} \frac{\partial (\sigma dB_t)_z}{\partial z} \approx 0, \ \frac{\partial}{\partial z} (\nabla_H \cdot \boldsymbol{a}_{Hz}) \approx 0 \tag{2}$$

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$$\frac{\partial p_0}{\partial z} = b_0, \ \frac{\partial \tilde{p}'_t}{\partial z} = \mathcal{O}(RoD^2)$$

- Thermal wind balance :

$$\frac{\partial \boldsymbol{u}_0}{\partial z} \cdot \nabla_H b_0 = 0, \ \frac{\partial (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H}{\partial z} = \mathcal{O}(RoD^2), \ \frac{\partial \boldsymbol{a}_H}{\partial z} = \mathcal{O}(Ro^2 D^4) \tag{3}$$

Continuously stratified QG system

First order equations

- Momentum equations :

$$d_t u_0 + \left(\boldsymbol{u}_0^* dt + (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H\right) \cdot \nabla_H u_0 - \nabla_H \cdot \left(\frac{\boldsymbol{a}_H}{2} \nabla_H u_0\right) dt - f_0 v_1 dt$$
$$-\beta(y - y_0)\left(v_0 dt + (\sigma dB_t)_y\right) = -\frac{\partial p_1}{\partial x} dt - A_4 \nabla_H^4 u_0 dt \qquad (4)$$

$$d_t v_0 + \left(\boldsymbol{u}_0^* dt + (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H\right) \cdot \nabla_H v_0 - \nabla_H \cdot \left(\frac{\boldsymbol{a}_H}{2} \nabla_H v_0\right) dt + f_0 u_1 dt$$
$$+\beta(y - y_0)\left(u_0 dt + (\sigma dB_t)_x\right) = -\frac{\partial p_1}{\partial y} dt - A_4 \nabla_H^4 v_0 dt \qquad (5)$$

Continuously stratified QG system

First order equations

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$$d_t u_0 + \Big(\boldsymbol{u}_0^* dt + (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H\Big) \cdot \nabla_H u_0 - \nabla_H \cdot \Big(\frac{\boldsymbol{a}_H}{2} \nabla_H u_0\Big) dt - f_0 v_1 dt$$
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- Continuity equation :

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0 \quad (6)$$

Continuously stratified QG system

First order equations

- Momentum equations :

$$d_t u_0 + \Big(\boldsymbol{u}_0^* dt + (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H\Big) \cdot \nabla_H u_0 - \nabla_H \cdot \Big(\frac{\boldsymbol{a}_H}{2} \nabla_H u_0\Big) dt - f_0 v_1 dt$$
$$-\beta(y - y_0)\Big(v_0 dt + (\sigma dB_t)_y\Big) = -\frac{\partial p_1}{\partial x} dt - A_4 \nabla_H^4 u_0 dt \quad (4)$$
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$$+\beta(y - y_0)\Big(u_0 dt + (\sigma dB_t)_x\Big) = -\frac{\partial p_1}{\partial y} dt - A_4 \nabla_H^4 v_0 dt \quad (5)$$

- Continuity equation :

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0 \quad (6)$$

- Cross-differentiating (4)-(5) combining with (6) :

$$d_t\Big[\frac{\nabla_H^2 p_0}{f_0} + \beta(y - y_0)\Big] + \Big(\boldsymbol{u}_0^* dt + (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H\Big) \cdot \nabla_H\Big[\frac{\nabla_H^2 p_0}{f_0} + \beta(y - y_0)\Big]$$
$$-\nabla_H \cdot \Big(\frac{\boldsymbol{a}_H}{2} \nabla_H\Big[\frac{\nabla_H^2 p_0}{f_0} + \beta(y - y_0)\Big]\Big) dt = \Big(f_0 \frac{\partial w_1}{\partial z} - \frac{A_4}{f_0} \nabla_H^6 p_0\Big) dt + R$$

Continuously stratified QG system

First order equations

- Nonlinear source-sink terms :

$$
\begin{aligned}
R_1 &= \Big(-\frac{\partial u_0^*}{\partial x}\frac{\partial v_0}{\partial x} + \frac{\partial u_0^*}{\partial y}\frac{\partial u_0}{\partial x} - \frac{\partial v_0^*}{\partial x}\frac{\partial v_0}{\partial y} + \frac{\partial v_0^*}{\partial y}\frac{\partial u_0}{\partial y}\Big) dt \\
&= -tr\Big(\mathcal{D}[\boldsymbol{u}]\boldsymbol{J}\mathcal{D}[-\nabla_H \cdot \frac{\boldsymbol{a}_H}{2}]\Big) dt \\
R_2 &= -\frac{\partial(\sigma dB_t)_x}{\partial x}\frac{\partial v_0}{\partial x} + \frac{\partial(\sigma dB_t)_x}{\partial y}\frac{\partial u_0}{\partial x} - \frac{\partial(\sigma dB_t)_y}{\partial x}\frac{\partial v_0}{\partial y} + \frac{\partial(\sigma dB_t)_y}{\partial y}\frac{\partial u_0}{\partial y} \\
&= -tr\Big(\mathcal{D}[\boldsymbol{u}]\boldsymbol{J}\mathcal{D}[(\boldsymbol{\sigma} d\boldsymbol{B}_t)_H]\Big) \\
R_3 &= \nabla_H \cdot \Big(\frac{1}{2}\frac{\partial \boldsymbol{a}_H}{\partial x}\nabla_H v_0\Big) dt - \nabla_H \cdot \Big(\frac{1}{2}\frac{\partial \boldsymbol{a}_H}{\partial y}\nabla_H u_0\Big) dt \\
R_4 &= -\beta \nabla_H \cdot \boldsymbol{a}_{\cdot y}\, dt
\end{aligned}
$$

Continuously stratified QG system

First order equations

- Nonlinear source-sink terms :

$$R_1 = \Big(-\frac{\partial u_0^*}{\partial x}\frac{\partial v_0}{\partial x} + \frac{\partial u_0^*}{\partial y}\frac{\partial u_0}{\partial x} - \frac{\partial v_0^*}{\partial x}\frac{\partial v_0}{\partial y} + \frac{\partial v_0^*}{\partial y}\frac{\partial u_0}{\partial y}\Big)dt$$

$$= -tr\Big(\mathcal{D}[\boldsymbol{u}]\boldsymbol{J}\mathcal{D}[-\nabla_H \cdot \frac{\boldsymbol{a}_H}{2}]\Big)dt$$

$$R_2 = -\frac{\partial(\sigma dB_t)_x}{\partial x}\frac{\partial v_0}{\partial x} + \frac{\partial(\sigma dB_t)_x}{\partial y}\frac{\partial u_0}{\partial x} - \frac{\partial(\sigma dB_t)_y}{\partial x}\frac{\partial v_0}{\partial y} + \frac{\partial(\sigma dB_t)_y}{\partial y}\frac{\partial u_0}{\partial y}$$

$$= -tr\Big(\mathcal{D}[\boldsymbol{u}]\boldsymbol{J}\mathcal{D}[(\boldsymbol{\sigma} d\boldsymbol{B}_t)_H]\Big)$$

$$R_3 = \nabla_H \cdot \Big(\frac{1}{2}\frac{\partial \boldsymbol{a}_H}{\partial x}\nabla_H v_0\Big)dt - \nabla_H \cdot \Big(\frac{1}{2}\frac{\partial \boldsymbol{a}_H}{\partial y}\nabla_H u_0\Big)dt$$

$$R_4 = -\beta \nabla_H \cdot \boldsymbol{a}_{\cdot y}\, dt$$

$\triangleright$ $\boldsymbol{a}$ homogeneous $\Rightarrow$ $R = R_2,\ \mathbb{E}[R] = 0$

N.B. $\mathcal{D}[\boldsymbol{u}] = \frac{1}{2}\Big(\nabla_H \boldsymbol{u} + \nabla_H^T \boldsymbol{u}\Big)$: deformation tensor, $\boldsymbol{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$: 90° rotation matrix

Continuously stratified QG system

First order equations

- Thermodynamic equation :

$$d_t b_0 + \left(\boldsymbol{u}_0^* dt + (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H\right) \cdot \nabla_H b_0 - \nabla_H \cdot \left(\frac{\boldsymbol{a}_H}{2} \nabla_H b_0\right) dt + Bu \left[\left(w_1 - \nabla_H \cdot \frac{\boldsymbol{a}_{Hz}}{2}\right) dt + (\sigma dB_t)_z\right] = 0 \quad (7)$$

Continuously stratified QG system

First order equations

- Thermodynamic equation :

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- Derivating (7) along z :

$$d_t \frac{\partial b_0}{\partial z} + \Big(\boldsymbol{u}_0^* dt + (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H \Big) \cdot \nabla_H \frac{\partial b_0}{\partial z} - \nabla_H \cdot \Big(\frac{\boldsymbol{a}_H}{2} \nabla_H \frac{\partial b_0}{\partial z} \Big) dt$$
$$+ Bu \Big[\Big(\frac{\partial w_1}{\partial z} - \underbrace{\frac{\partial}{\partial z} \nabla_H \cdot \frac{\boldsymbol{a}_{Hz}}{2}}_{0 \Leftarrow (2)} \Big) dt + \underbrace{\frac{\partial (\sigma d B_t)_z}{\partial z}}_{0 \Leftarrow (2)} \Big] + S = 0$$

Continuously stratified QG system

First order equations

- Thermodynamic equation :

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$$S = \Big[\Big(\frac{\partial u_0}{\partial z} - \frac{1}{2}\nabla_H\cdot\frac{\partial \boldsymbol{a}_H}{\partial z}\Big)dt + \frac{\partial(\boldsymbol{\sigma} d\boldsymbol{B}_t)_H}{\partial z}\Big]\cdot\nabla_H b_0 - \nabla_H\cdot\Big(\frac{1}{2}\frac{\partial \boldsymbol{a}_H}{\partial z}\nabla_H b_0\Big)dt = 0 \;\Leftarrow\; (3)$$

Continuously stratified QG system

First order equations

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- Results :

$$-\frac{\partial w_1}{\partial z} = \frac{1}{Bu}\Big[d_t \frac{\partial b_0}{\partial z} + \Big(\boldsymbol{u}_0^* dt + (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H\Big) \cdot \nabla_H \frac{\partial b_0}{\partial z} - \nabla_H \cdot \Big(\frac{\boldsymbol{a}_H}{2} \nabla_H \frac{\partial b_0}{\partial z}\Big) dt\Big]$$

Continuously stratified QG system

QG equations under location uncertainty

- Potential vorticity (PV) :

$$q_0 = \frac{\nabla_H^2 p_0}{f_0} + \beta(y - y_0) + \frac{\partial}{\partial z}\left(\frac{f_0}{Bu} b_0\right), \; b_0 = \frac{\partial p_0}{\partial z}$$

Continuously stratified QG system

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- Evolution of PV :

$$d_t q_0 + \left(\boldsymbol{u}_0^* dt + (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H\right) \cdot \nabla_H q_0 - \nabla_H \cdot \left(\frac{\boldsymbol{a}_H}{2} \nabla_H q_0\right) dt = -\frac{A_4}{f_0} \nabla_H^6 p_0 dt + R$$

Continuously stratified QG system

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- Dimensional version :

$$q = \frac{\nabla_H^2 \tilde{p}}{f_0} + f + \frac{\partial}{\partial z}\Big(\frac{f_0}{N^2} \frac{\partial \tilde{p}}{\partial z}\Big)$$

$$d_t q + \Big(\boldsymbol{u}^* dt + (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H\Big) \boldsymbol{\cdot} \nabla_H q - \nabla_H \boldsymbol{\cdot} \Big(\frac{\boldsymbol{a}_H}{2} \nabla_H q\Big) dt = -\frac{A_4}{f_0} \nabla_H^6 \tilde{p} dt + R$$

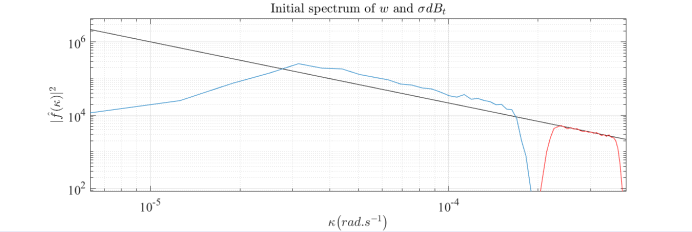
Parametrisation of noise

Some existing approaches

- Homogeneous and isotropic turbulence model :

  a is diagonal and constant

  ▷ Through a pass-band spectral cutoff :



In 2D, $(\boldsymbol{\sigma}(\boldsymbol{x})d\boldsymbol{B}_t)_H = \nabla_H^{\perp}\psi_{\boldsymbol{\sigma}} \star d\boldsymbol{B}_t,\ \hat{\psi}_{\boldsymbol{\sigma}}(\boldsymbol{\kappa}) = A\mathbf{1}_{\boldsymbol{\kappa}_1 \le |\boldsymbol{\kappa}| \le \boldsymbol{\kappa}_2}|\boldsymbol{\kappa}|^{-\alpha}$

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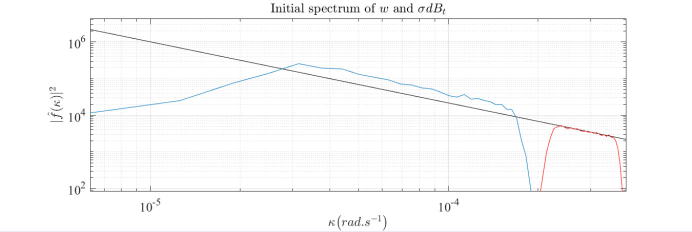

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- Reduced order model :

  ▷ POD approach using observations from velocity field

  $\boldsymbol{a}$ is stationnary

A new proposition

- A type of uncertainty living on the iso-surface of buoyancy, i.e. $\boldsymbol{\sigma} d\boldsymbol{B}_t \cdot \nabla b = 0$:
  - ▷ Iso-surface projector :

$$\mathcal{P}_b = \begin{pmatrix} 1 & 0 & \alpha_x \\ 0 & 1 & \alpha_y \\ \alpha_x & \alpha_y & |\boldsymbol{\alpha}|^2 \end{pmatrix}$$

$$\boldsymbol{\alpha} = (\alpha_x, \alpha_y)^T = -\frac{\nabla_H b}{\partial b/\partial z} = -\frac{\nabla_H b'/N^2}{1 + \mathcal{O}(Ro)} = -\nabla_H \frac{\partial p/\partial z}{N^2}$$

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  - ▷ Divergence-free projector :

$$(\boldsymbol{\sigma} d\boldsymbol{B}_t)_H = \Big(\underbrace{\boldsymbol{I}_2 - \Delta_H^{-1} \nabla_H \nabla_H^T}_{\delta_{ij} - \frac{\kappa_i \kappa_j}{|\kappa|^2} \text{ in Fourier}} \Big) \big(\boldsymbol{\alpha} \xi_t^z \big),$$

where ξ_t^z is the 3rd component of an original noise ξ_t, which may be may be the homogeneous and isotropic one, or Kraichnan turbulent model.

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where ξ_t^z is the 3rd component of an original noise ξ_t, which may be may be the homogeneous and isotropic one, or Kraichnan turbulent model.

$\boldsymbol{a}$ anisotropic, inhomogeneous and non-staionnary

N.B. In a layered model, the derivative along z will be approximated by a finite difference between layer-averaged quantities.

A multi-layer model

Work in progress

- (Hogg et al., 2003) - A QG coupled model

A multi-layer model

Work in progress

- (Hogg et al., 2003) - A QG coupled model
  - ▷ only taken the ocean case in a double gyre basin, without heat flux

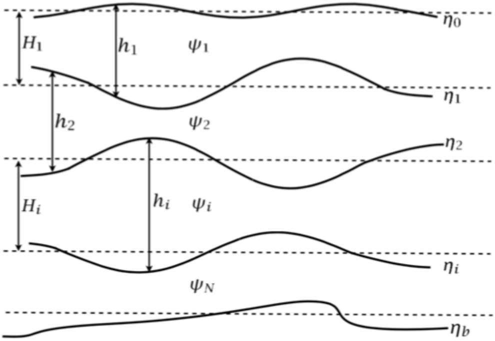

Figure: A multi-layer Shallow Water QG system

A multi-layer model

An N-layers QG shallow water system

- Evolution of $q^{(k)},\ k = 1, \ldots, N$:

$$d_t q^{(k)} + \frac{1}{f_0} J\Big(p^{(k)}, q^{(k)}\Big) dt - \Big(\nabla_H \cdot \frac{\boldsymbol{a}_H^{(k)}}{2}\Big) \cdot \nabla_H q^{(k)} dt + (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H^{(k)} \cdot \nabla_H q^{(k)}$$

$$-\nabla_H \cdot \Big(\frac{\boldsymbol{a}_H^{(k)}}{2} \nabla_H q^{(k)}\Big) dt = -\frac{A_4}{f_0} \nabla_H^6 p^{(k)} dt + R^{(k)} \underbrace{+ \frac{f_0}{H^{(k)}} w_{ek} \delta_{k1}}_{\text{surface pumping}} \underbrace{- \frac{h_{ek}}{2 f_0} \nabla_H^2 p^{(N)} \delta_{kN}}_{\text{bottom drag}}$$

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- Evolution of $p^{(k)},\ k = 1, \ldots, N$:

$$q^{(k)} = \frac{\nabla_H^2 p^{(k)}}{f_0} + \beta (y - y_0) + \frac{f_0}{H^{(k)}} \left(\eta^{(k)} - \eta^{(k-1)}\right)$$

A multi-layer model

An N-layers QG shallow water system

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$$d_t q^{(k)} + \frac{1}{f_0} J\left(p^{(k)}, q^{(k)}\right) dt - \left(\nabla_H \cdot \frac{\boldsymbol{a}_H^{(k)}}{2}\right) \cdot \nabla_H q^{(k)} dt + (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H^{(k)} \cdot \nabla_H q^{(k)}$$

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- Evolution of $p^{(k)},\ k=1,\dots,N$:

$$q^{(k)} = \frac{\nabla_H^2 p^{(k)}}{f_0} + \beta(y - y_0) + \frac{f_0}{H^{(k)}} \left(\eta^{(k)} - \eta^{(k-1)}\right)$$

- Perturbation interface height :

$$\eta^{(0)} = 0;\ \eta^{(k)} = \frac{p^{(k+1)} - p^{(k)}}{g^{(k)}}, k = 1, \dots, N-1;\ \eta^{(N)} = D(x,y)$$

$$g^{(k)} = \frac{g}{\rho_b} \left(\rho^{(k+1)} - \rho^{(k)}\right),\ k = 1, \dots, N-1$$

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An N-layers QG shallow water system

- Evolution of $q^{(k)},\ k = 1,\dots,N$:

$$d_t q^{(k)} + \frac{1}{f_0} J\Big(p^{(k)}, q^{(k)}\Big)dt - \Big(\nabla_H \cdot \frac{\boldsymbol{a}_H^{(k)}}{2}\Big) \cdot \nabla_H q^{(k)} dt + (\boldsymbol{\sigma} d\boldsymbol{B}_t)_H^{(k)} \cdot \nabla_H q^{(k)}$$

$$-\nabla_H \cdot \Big(\frac{\boldsymbol{a}_H^{(k)}}{2} \nabla_H q^{(k)}\Big)dt = -\frac{A_4}{f_0}\nabla_H^6 p^{(k)} dt + R^{(k)} \underbrace{+ \frac{f_0}{H^{(k)}} w_{ek}\delta_{k1}}_{\text{surface pumping}} \underbrace{- \frac{h_{ek}}{2 f_0}\nabla_H^2 p^{(N)} \delta_{kN}}_{\text{bottom drag}}$$

- Evolution of $p^{(k)},\ k = 1,\dots,N$:

$$q^{(k)} = \frac{\nabla_H^2 p^{(k)}}{f_0} + \beta(y - y_0) + \frac{f_0}{H^{(k)}}(\eta^{(k)} - \eta^{(k-1)})$$

- Perturbation interface height :

$$\eta^{(0)} = 0;\ \eta^{(k)} = \frac{p^{(k+1)} - p^{(k)}}{g^{(k)}}, k = 1,\dots,N-1;\ \eta^{(N)} = D(x,y)$$

$$g^{(k)} = \frac{g}{\rho_b}(\rho^{(k+1)} - \rho^{(k)}),\ k = 1,\dots,N-1$$

- Horizonal uncertainties :

$$(\boldsymbol{\sigma} d\boldsymbol{B}_t)_H^{(k)} = \Big(\boldsymbol{I}_2 - \Delta_H^{-1}\nabla_H \nabla_H^T\Big)\Big(\Big(\nabla_H \eta^{(k-1)} - \nabla_H \eta^{(k)}\Big)\xi_t^{(k)}\Big),\ k = 1,\dots,N$$

Thanks for your attentions !

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