

Quasi-geostrophic flow under location uncertainty

Long Li, Etienne Mémin, Bertrand Chapron

► **To cite this version:**

Long Li, Etienne Mémin, Bertrand Chapron. Quasi-geostrophic flow under location uncertainty. Seminar of Stochastic Transport in Upper Ocean Dynamics (STUOD) project, May 2020, Rennes, France. pp.1-52. hal-02711026

HAL Id: hal-02711026

<https://hal.inria.fr/hal-02711026>

Submitted on 1 Jun 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Quasi-geostrophic flow under location uncertainty

Long Li¹ Etienne Mémin¹ Bertrand Chapron²

¹Inria/Irmar, Fluminance Group
Campus Universitaire de Beaulieu, 35042 Rennes Cedex, France

²Ifremer, Laboratoire Spatial et Interfaces Air-Mer
Centre Bretagne, ZI de la Pointe du Diable, 29280 Plouzané, France

- 1 Location Uncertainty (LU) Principles
- 2 Barotropic quasi-geostrophic (QG) flow under LU
- 3 Multi-layered QG under LU

Stochastic flow

$$d\mathbf{X}_t = \mathbf{u}(\mathbf{X}_t, t)dt + \underbrace{\sigma(\mathbf{X}_t, t)d\mathbf{B}_t}_{\text{Uncertainty or Noise}} \quad (1)$$

Spatial structure of noise

- Correlation operator

$$\sigma[f](\mathbf{x}, t) = \int_{\Omega} \check{\sigma}(\mathbf{x}, \mathbf{y}, t) f(\mathbf{y}) d\mathbf{y} \quad (2)$$

$\check{\sigma}$ is assumed to be bounded

Spatial structure of noise

- Correlation operator

$$\sigma[f](x, t) = \int_{\Omega} \check{\sigma}(x, y, t) f(y) dy \quad (2)$$

$\check{\sigma}$ is assumed to be bounded

- Covariance and variance

$$\begin{aligned} Q(x, y, t, s) &= \mathbb{E} \left[(\sigma(x, t) dB_t) (\sigma(y, s) dB_s)^T \right] \\ &= \delta(t - s) dt \int_{\Omega} \check{\sigma}(x, z, t) \check{\sigma}^T(y, z, s) dz \end{aligned} \quad (3)$$

$$a \triangleq Q/dt = \sigma \sigma^T \quad (4)$$

Spectral representation

- Noise

$$\sigma(\mathbf{x}, t) d\mathbf{B}_t = \sum_{n \in \mathbb{N}} \phi_n(\mathbf{x}, t) d\beta_t^n \quad (5)$$

ϕ_n : orthogonal eigenfunctions of the covariance

β^n : 1D standard Brownian motions

Spectral representation

- Noise

$$\sigma(\mathbf{x}, t) d\mathbf{B}_t = \sum_{n \in \mathbb{N}} \phi_n(\mathbf{x}, t) d\beta_t^n \quad (5)$$

- Variance

$$\mathbf{a}(\mathbf{x}, t) = \sum_{n \in \mathbb{N}} \phi_n(\mathbf{x}, t) \phi_n^T(\mathbf{x}, t) \quad (6)$$

ϕ_n : orthogonal eigenfunctions of the covariance

β^n : 1D standard Brownian motions

Stochastic Reynolds transport theorem (Mémin, 2014)

$$d \int_{\mathcal{V}(t)} \theta(\mathbf{x}, t) d\mathbf{x} = \int_{\mathcal{V}(t)} (\mathbf{D}_t \theta + \theta \nabla \cdot (\mathbf{u} - \mathbf{u}_s)) d\mathbf{x} \quad (7)$$

- Stochastic transport operator (Resseguier et al., 2017)

$$\mathbf{D}_t \theta \triangleq d_t \theta + (\mathbf{u} - \mathbf{u}_s) \cdot \nabla \theta dt + \sigma d\mathbf{B}_t \cdot \nabla \theta - \frac{1}{2} \nabla \cdot (\mathbf{a} \nabla \theta) dt \quad (8)$$

Stochastic Reynolds transport theorem (Mémin, 2014)

$$d \int_{\mathcal{V}(t)} \theta(\mathbf{x}, t) d\mathbf{x} = \int_{\mathcal{V}(t)} (D_t \theta + \theta \nabla \cdot (\mathbf{u} - \mathbf{u}_s)) d\mathbf{x} \quad (7)$$

- Stochastic transport operator (Resseguier et al., 2017)

$$D_t \theta \triangleq d_t \theta + (\mathbf{u} - \mathbf{u}_s) \cdot \nabla \theta dt + \sigma d\mathbf{B}_t \cdot \nabla \theta - \frac{1}{2} \nabla \cdot (\mathbf{a} \nabla \theta) dt \quad (8)$$

- Itô–Stokes drift (Bauer et al., 2020)

$$\mathbf{u}_s \triangleq \frac{1}{2} \nabla \cdot \mathbf{a} \quad (9)$$

Transport under LU

- Conservation of extensive tracer

$$D_t \theta = 0 \quad (10)$$

Transport under LU

- Conservation of extensive tracer

$$D_t \theta = 0 \quad (10)$$

- Conservation of tracer energy (Resseguier et al., 2017)

$$d_t \int_{\Omega} \frac{1}{2} \theta^2 d\mathbf{x} = 0 \quad (11)$$

- 1 Location Uncertainty (LU) Principles
- 2 Barotropic quasi-geostrophic (QG) flow under LU
- 3 Multi-layered QG under LU

Barotropic quasi-geostrophic (QG) flow under LU

- Potential vorticity (PV)

$$q = \nabla \times \mathbf{u} - \psi/R_d^2 + f \quad (\mathbf{u} = \nabla^\perp \psi) \quad (12)$$

Barotropic quasi-geostrophic (QG) flow under LU

- Potential vorticity (PV)

$$q = \nabla \times \mathbf{u} - \psi/R_d^2 + f \quad (\mathbf{u} = \nabla^\perp \psi) \quad (12)$$

- Evolution of PV under LU

$$D_t q = (\mathcal{D} + \mathcal{F})dt + dS_t^\sigma \quad (13)$$

Barotropic quasi-geostrophic (QG) flow under LU

- Potential vorticity (PV)

$$q = \nabla \times \mathbf{u} - \psi/R_d^2 + f \quad (\mathbf{u} = \nabla^\perp \psi) \quad (12)$$

- Evolution of PV under LU

$$D_t q = (\mathcal{D} + \mathcal{F})dt + d\mathcal{S}_t^\sigma \quad (13)$$

- PV sources and sinks

$$d\mathcal{S}_t^\sigma = \sum_i -J(\sigma dB_t^i - u_s^i dt, u^i) + \frac{1}{2} \nabla \cdot (\partial_i^\perp \mathbf{a} \nabla u^i) dt \quad (14)$$

J: Jacobi operator

Barotropic quasi-geostrophic (QG) flow under LU

- Potential vorticity (PV)

$$q = \nabla \times \mathbf{u} - \psi/R_d^2 + f \quad (\mathbf{u} = \nabla^\perp \psi) \quad (12)$$

- Evolution of PV under LU

$$D_t q = (\mathcal{D} + \mathcal{F})dt + dS_t^\sigma \quad (13)$$

- PV sources and sinks

$$dS_t^\sigma = \sum_i -J(\sigma dB_t^i - u_s^i dt, u^i) + \frac{1}{2} \nabla \cdot (\partial_i^\perp \mathbf{a} \nabla u^i) dt \quad (14)$$

J: Jacobi operator

- Energy conservation ($\mathcal{D} = \mathcal{F} = 0$) (Bauer et al., 2020)

$$d_t \int_\Omega \frac{1}{2} \|\nabla \psi\|^2 + \frac{1}{2} (\psi/R_d)^2 = 0 \quad (15)$$

Test case: monochromatic Rossby wave

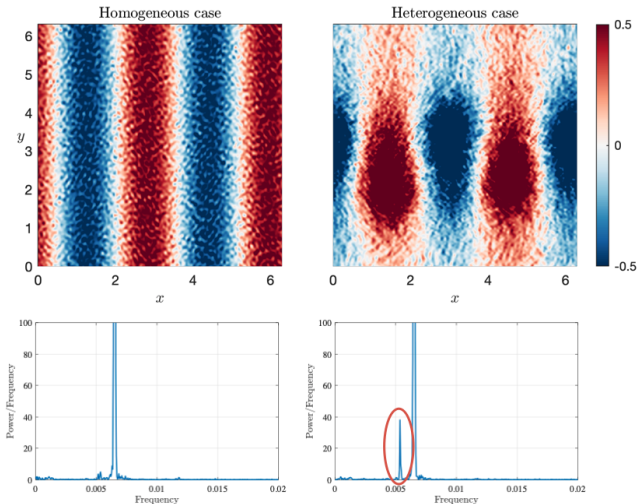
- Aim: study effect of small-scale inhomogeneity on structuration of large-scale flow

Test case: monochromatic Rossby wave

- Aim: study effect of small-scale inhomogeneity on structuration of large-scale flow
- Configurations: initialized by a single Rossby wave; doubly periodic domain; under both homogeneous ($u_s = 0$) and heterogeneous noise with same amplitude

Numerical results: effect of small-scale inhomogeneity

Relative vorticity at $t = 1000$



Click for video

Test case: wind-driven circulation

- Aim: study low-frequency variability of LU coarse models

Test case: wind-driven circulation

- Aim: study low-frequency variability of LU coarse models
- Configurations: double gyre wind forcing $\mathcal{F} = \frac{1}{f_0} \nabla \times \boldsymbol{\tau}$; Biharmonic dissipation of vorticity $\mathcal{D} = -A_4 \nabla^6 \psi$; shallow basin ($R_d = \infty$) without topography; free-slip boundary condition; LU coarse model at 16×32 , 32×64 and 64×128

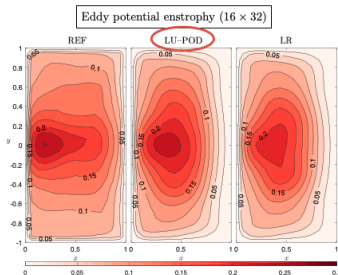
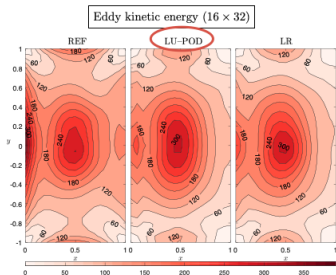
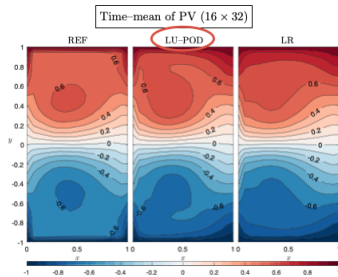
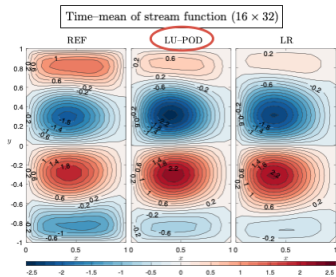
Test case: wind-driven circulation

- Aim: study low-frequency variability of LU coarse models
- Configurations: double gyre wind forcing $\mathcal{F} = \frac{1}{f_0} \nabla \times \boldsymbol{\tau}$; Biharmonic dissipation of vorticity $\mathcal{D} = -A_4 \nabla^6 \psi$; shallow basin ($R_d = \infty$) without topography; free-slip boundary condition; LU coarse model at 16×32 , 32×64 and 64×128
- POD noise

$$\boldsymbol{\sigma} d\mathbf{B}_t = \sum_{n=N_1}^{N_2} \phi_n d\beta_t^n, \quad \mathbf{a} = \sum_{n=N_1}^{N_2} \phi_n \phi_n^T \quad (16)$$

where the EOFs ϕ_n are learned from eddy-resolving (256×512) data.

Numerical results: qualitative accuracy of statistics



Numerical results: quantitative accuracy of statistics

Model	Mean	EKE	Skewness	Kurtosis
LR	0.108	0.073	0.190	0.218
LU	0.094	0.064	0.161	0.182
LU-NS	0.100	0.067	0.185	0.191
LU-CP	0.102	0.068	0.185	0.208

Table: Normalized RMSE of statistics for stream function (32×64)

Model	Mean	EKE	Skewness	Kurtosis
LR	0.061	0.122	0.166	0.155
LU	0.056	0.116	0.146	0.122
LU-NS	0.056	0.116	0.148	0.130
LU-CP	0.056	0.115	0.156	0.138

Table: Normalized RMSE of statistics for PV (32×64)

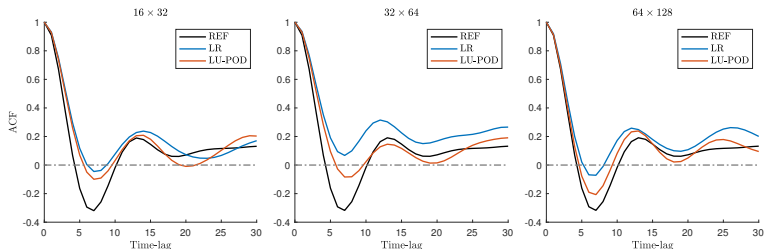
LU-NS: $\overline{u_s}$

LU-CP: $\overline{u_s}$ & dS_t^σ

Autocorrelation function for total circulation

$$\text{ACF}(\tau) = \frac{1}{\sigma_{\Psi}^2} \overline{(\Psi(t) - \bar{\Psi}^t)(\Psi(t + \tau) - \bar{\Psi}^t)}^t, \quad (17)$$

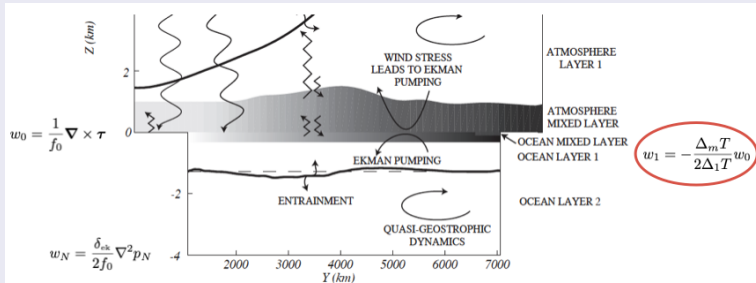
where $\Psi(t) \triangleq \frac{1}{\Omega} \int_{\Omega} \psi(\mathbf{x}, t) d\mathbf{x}$ with its standard derivation σ_{Ψ} .



- 1 Location Uncertainty (LU) Principles
- 2 Barotropic quasi-geostrophic (QG) flow under LU
- 3 Multi-layered QG under LU

Multi-layered QG under LU

Quasi-geostrophic coupled model (Q-GCM) (Hogg et al., 2003)



Governing equations

- Evolution of mixed-layer temperature

$$D_t T_m = (\mathcal{D}_m + \mathcal{F}_m) dt \quad (18)$$

Governing equations

- Evolution of mixed-layer temperature

$$D_t T_m = (\mathcal{D}_m + \mathcal{F}_m) dt \quad (18)$$

- Evolution of k -th layer PV

$$D_t q_k = (\mathcal{D}_k + \mathcal{F}_k) dt + dS_{t,k}^\sigma, \quad \mathcal{F}_k = -\frac{f_0}{H_k} (w_k - w_{k-1}) \quad (19)$$

w : vertical entrainment

Governing equations

- Evolution of mixed-layer temperature

$$D_t T_m = (\mathcal{D}_m + \mathcal{F}_m) dt \quad (18)$$

- Evolution of k -th layer PV

$$D_t q_k = (\mathcal{D}_k + \mathcal{F}_k) dt + dS_{t,k}^\sigma, \quad \mathcal{F}_k = -\frac{f_0}{H_k} (w_k - w_{k-1}) \quad (19)$$

w : vertical entrainment

- Evolution of k -th layer pressure

$$q_k = \frac{1}{f_0} \nabla^2 p_k + \frac{f_0}{H_k} (\eta_k - \eta_{k-1}) + \beta y \quad (20)$$

η : perturbed interface height

LU of type Gent–McWilliams (GM)

- Aim: constraint noise along “isopycnal” surfaces

$$\tilde{\sigma}_k d\mathbf{B}_t \perp \nabla \Theta_k, \quad \Theta_k \triangleq \frac{f_0}{H_k} (\eta_k - \eta_{k-1}) \quad (21)$$

LU of type Gent–McWilliams (GM)

- Aim: constraint noise along “isopycnal” surfaces

$$\tilde{\sigma}_k d\mathbf{B}_t \perp \nabla \Theta_k, \quad \Theta_k \triangleq \frac{f_0}{H_k} (\eta_k - \eta_{k-1}) \quad (21)$$

- Projected noise

$$\tilde{\sigma}_k d\mathbf{B}_t = \mathbf{P}_k (\sigma_k d\mathbf{B}_t), \quad \mathbf{P}_k = \mathbf{I} - \frac{\nabla \Theta_k (\nabla \Theta_k)^T}{\|\nabla \Theta_k\|^2} \quad (22)$$

LU of type Gent–McWilliams (GM)

- Aim: constraint noise along “isopycnal” surfaces

$$\tilde{\sigma}_k d\mathbf{B}_t \perp \nabla \Theta_k, \quad \Theta_k \triangleq \frac{f_0}{H_k} (\eta_k - \eta_{k-1}) \quad (21)$$

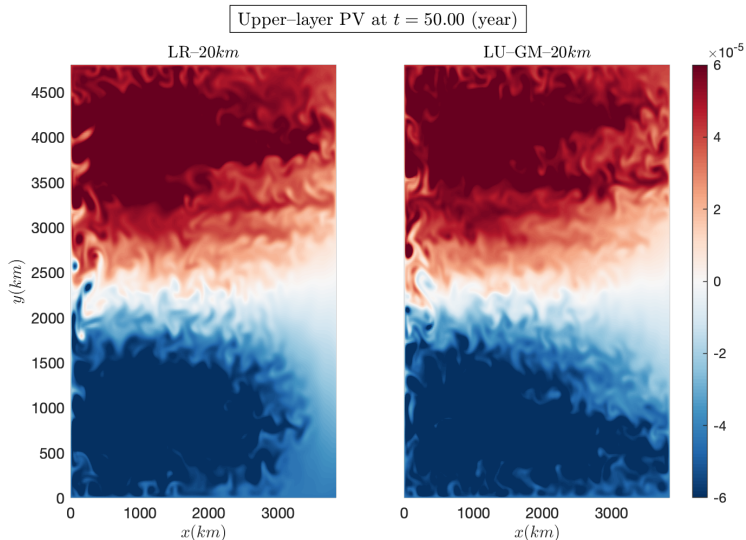
- Projected noise

$$\tilde{\sigma}_k d\mathbf{B}_t = \mathbf{P}_k (\sigma_k d\mathbf{B}_t), \quad \mathbf{P}_k = \mathbf{I} - \frac{\nabla \Theta_k (\nabla \Theta_k)^T}{\|\nabla \Theta_k\|^2} \quad (22)$$

- LU–GM diffusion tensor

$$\tilde{\mathbf{a}}_k = \mathbf{P}_k \mathbf{a}_k \mathbf{P}_k^T \quad (23)$$

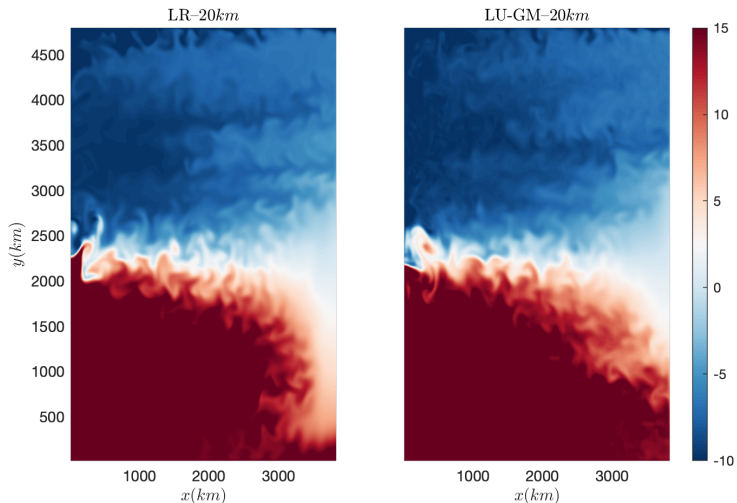
Numerical results: instantaneous variability



Click for video

Numerical results: instantaneous variability

Mixed-layer temperature anomaly at $t = 50.00$ (year)



[Click for video](#)

Measures of statistics (Grooms et al., 2015)

- Pattern correlation (PC)

$$PC = \frac{\int_{\Omega} \sigma_f^2 \sigma_{f_{\text{ref}}}^2 d\mathbf{x}}{(\int_{\Omega} \sigma_f^4 d\mathbf{x})(\int_{\Omega} \sigma_{f_{\text{ref}}}^4 d\mathbf{x})} \quad \nearrow \quad (24)$$

Measures of statistics (Grooms et al., 2015)

- Pattern correlation (PC)

$$\text{PC} = \frac{\int_{\Omega} \sigma_f^2 \sigma_{f_{\text{ref}}}^2 d\mathbf{x}}{(\int_{\Omega} \sigma_f^4 d\mathbf{x})(\int_{\Omega} \sigma_{f_{\text{ref}}}^4 d\mathbf{x})} \quad \nearrow \quad (24)$$

- Gaussian approximation of relative entropy

$$\text{Dispersion} = \frac{1}{|\Omega|} \int_{\Omega} \left(\frac{\sigma_{f_{\text{ref}}}^2}{\sigma_f^2} - 1 - \log \left(\frac{\sigma_{f_{\text{ref}}}^2}{\sigma_f^2} \right) \right) d\mathbf{x} \quad \searrow \quad (25)$$

$$\text{Entropy} = \frac{1}{2} \left(\frac{1}{|\Omega|} \int_{\Omega} \frac{(\bar{f}^t - \bar{f}_{\text{ref}}^t)^2}{\sigma_f^2} d\mathbf{x} + \text{Dispersion} \right) \quad \searrow \quad (26)$$

Numerical results: comparison of measures

Model	RMSE of mean	RMSE of std	PC of std	Dispersion	Entropy
LR	0.41	0.71	0.51	79	41
LU-POD	0.38	0.69	0.52	38	19
LU-GM	0.36	0.66	0.53	16	8

Table: Measures of skill for upper-layer pressure (40km).

Model	RMSE of mean	RMSE of std	PC of std	Dispersion	Entropy
LR	0.23	0.64	0.51	194	98
LU-POD	0.22	0.62	0.56	72	36
LU-GM	0.22	0.58	0.64	24	12

Table: Measures of skill for middle-layer pressure (40km).

Model	RMSE of mean	RMSE of std	PC of std	Dispersion	Entropy
LR	0.12	0.67	0.57	1322	661
LU-POD	0.12	0.65	0.65	222	111
LU-GM	0.12	0.60	0.83	41	21

Table: Measures of skill for bottom-layer pressure (40km).

Numerical results: comparison of measures

Model	RMSE of mean	RMSE of std	PC of std	Dispersion	Entropy
LR	0.44	0.71	0.44	851	26
LU-POD	0.44	0.69	0.46	36	18
LU-GM	0.41	0.66	0.50	15	8

Table: Measures of skill for upper-layer pressure (80km).

Model	RMSE of mean	RMSE of std	PC of std	Dispersion	Entropy
LR	0.26	0.60	0.58	188	94
LU-POD	0.26	0.59	0.61	67	34
LU-GM	0.25	0.55	0.70	18	9

Table: Measures of skill for middle-layer pressure (80km).

Model	RMSE of mean	RMSE of std	PC of std	Dispersion	Entropy
LR	0.12	0.67	0.56	1701	851
LU-POD	0.12	0.65	0.65	248	124
LU-GM	0.12	0.59	0.80	40	20

Table: Measures of skill for bottom-layer pressure (80km).

Energy pathways (Hogg and Blundell, 2006)

- KE decomposition

$$\text{KE}_k = \frac{\rho H_k}{2} \left(\underbrace{\langle u_k \rangle^2}_{\text{Zonal KE}} + \underbrace{(\overline{u_k^2} + \overline{v_k^2})}_{\text{Standing EKE}} + \underbrace{(u_k'^2 + v_k'^2)}_{\text{Transient EKE}} \right) \quad (27)$$

Numerical results: (eddy) kinetic energy budget

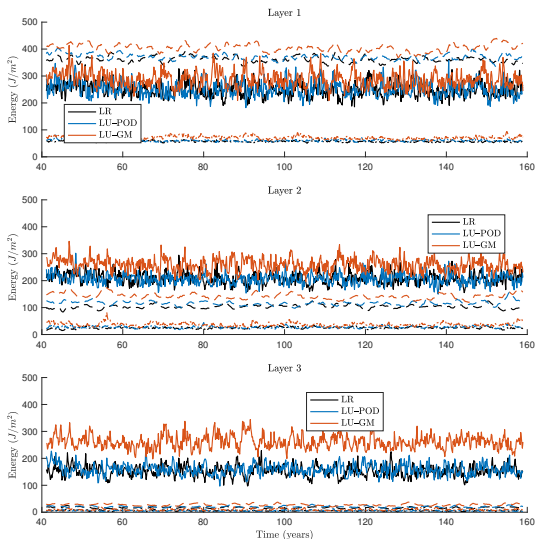


Figure: Comparison of zonal (dotted lines), standing eddy (dashed lines), and transient eddy kinetic energy (solid lines), for different coarse ($40km$) models, in (a) layer 1, (b) layer 2, and (c) layer 3.

Numerical results: energy exchange

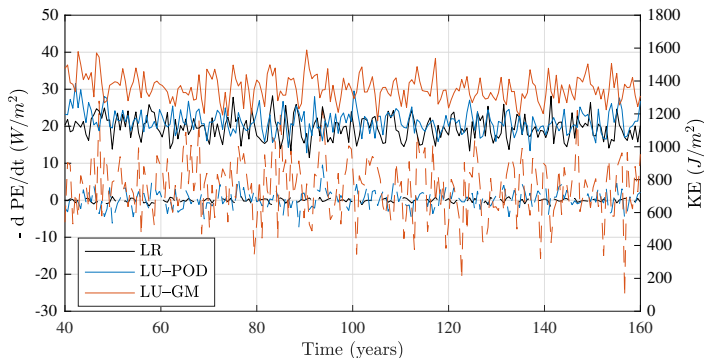


Figure: Time series of the negative rate of change of potential energy (dashed lines) and total kinetic energy (solid lines), for different coarse (40km) models.

- Development of jets

- Development of jets
- LU eddy-resolving simulation ($\sim 10km$)

Future works

- Development of jets
- LU eddy-resolving simulation ($\sim 10km$)
- Southern channel ocean (Hogg and Blundell, 2006)

- Development of jets
- LU eddy-resolving simulation ($\sim 10km$)
- Southern channel ocean (Hogg and Blundell, 2006)
- Atmosphere-ocean coupled mode (Hogg et al., 2003)

- Development of jets
- LU eddy-resolving simulation ($\sim 10km$)
- Southern channel ocean (Hogg and Blundell, 2006)
- Atmosphere-ocean coupled mode (Hogg et al., 2003)
- Multiple scale ocean model (combines planetary and quasi-geostrophic physics) (eg. MSQG)

- Development of jets
- LU eddy-resolving simulation ($\sim 10km$)
- Southern channel ocean (Hogg and Blundell, 2006)
- Atmosphere-ocean coupled mode (Hogg et al., 2003)
- Multiple scale ocean model (combines planetary and quasi-geostrophic physics) (eg. MSQG)
- **Implement LU-GM in primitive model (eg. NEMO)**

- Development of jets
- LU eddy-resolving simulation ($\sim 10km$)
- Southern channel ocean (Hogg and Blundell, 2006)
- Atmosphere-ocean coupled mode (Hogg et al., 2003)
- Multiple scale ocean model (combines planetary and quasi-geostrophic physics) (eg. MSQG)
- Implement LU-GM in primitive model (eg. NEMO)
- Application to ensemble data assimilation (Cotter et al., 2020)

Thank for Your Attention!

- W. Bauer, P. Chandramouli, B. Chapron, L. Li, and E. Mémin. Deciphering the role of small-scale inhomogeneity on geophysical flow structuration: a stochastic approach. *Journal of Physical Oceanography*, 50(4):983–1003, 2020.
- C. Cotter, D. Crisan, D. D. Holm, W. Pan, and I. Shevchenko. Data assimilation for a quasi-geostrophic model with circulation-preserving stochastic transport noise. *Journal of Statistical Physics*, 2020.
- I. Grooms, A. J. Majda, and K. S. Smith. Stochastic superparameterization in a quasigeostrophic model of the Antarctic Circumpolar Current. *Ocean Modelling*, 85: 1–15, 2015.
- A. McC. Hogg and J. R. Blundell. Interdecadal variability of the southern ocean. *Journal of Physical Oceanography*, 36:1626–1645, 2006.
- A. McC. Hogg, W. K. Dewar, P. D. Killworth, and J. R. Blundell. A quasi-geostrophic coupled model (Q-GCM). *Monthly Weather Review*, 131(10):2261–2278, 2003.
- E. Mémin. Fluid flow dynamics under location uncertainty. *Geophysical & Astrophysical Fluid Dynamics*, 108(2):119–146, 2014.
- V. Resseguier, E. Mémin, and B. Chapron. Geophysical flows under location uncertainty, part I: Random transport and general models. *Geophysical & Astrophysical Fluid Dynamics*, 111(3):149–176, 2017.