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EXISTENCE OF STRONG SOLUTIONS FOR A SYSTEM OF INTERACTION BETWEEN A COMPRESSIBLE VISCOUS FLUID AND A WAVE EQUATION

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Abstract. In this article, we consider a fluid-structure interaction system where the fluid is viscous and compressible and where the structure is a part of the boundary of the fluid domain and is deformable. The fluid is governed by the barotropic compressible Navier-Stokes system whereas the structure displacement is described by a wave equation. We show that the corresponding coupled system admits a unique strong solution for an initial fluid density and an initial fluid velocity in H^3 and for an initial deformation and an initial deformation velocity in H^4 and H^3 respectively. The reference configuration for the fluid domain is a rectangular cuboid with the elastic structure being the top face. We use a modified Lagrangian change of variables to transform the moving fluid domain into the rectangular cuboid and then analyze the corresponding linear system coupling a transport equation (for the density), a heat-type equation and a wave equation. The corresponding results for this linear system and estimations of the coefficients coming from the change of variables allow us to perform a fixed point argument and to prove the existence and uniqueness of strong solutions for the nonlinear system, locally in time.

1. Introduction

The mathematical study of fluid-structure interaction systems has been an active subject of research during the last decades, and this can be explained by the numerous applications in fluid mechanics but also by the challenges associated with such systems that involve free boundaries, coupling between different dynamical systems and nonlinearities. Here, we focus on the case where the fluid is a compressible Navier-Stokes system in a domain where a part of the boundary can deform following a wave equation. More precisely, the reference configuration for the fluid domain is $\mathcal{F}$ defined by

$$\mathcal{F} = \mathcal{S} \times (0,1), \quad \mathcal{S} = (\mathbb{R}/L_1\mathbb{Z}) \times (\mathbb{R}/L_2\mathbb{Z})$$

with $L_1, L_2 > 0$. The elastic structure is located at the upper boundary $\mathcal{S} \times \{1\}$. For an admissible vertical displacement of the structure $\eta : \mathcal{S} \to (-1, \infty)$, the fluid domain is transformed into

$$\mathcal{F}_\eta = \{[x_1, x_2, x_3] \in \mathcal{S} \times \mathbb{R} \; ; \; 0 < x_3 < 1 + \eta(x_1, x_2)\},$$

the position of the elastic structure becomes

$$\Gamma_\eta = \{[x_1, x_2, 1 + \eta(x_1, x_2)] \; ; \; [x_1, x_2] \in \mathcal{S}\},$$

and the other part of the boundary

$$\Gamma_b = \mathcal{S} \times \{0\}$$

remains fixed (see Fig. 1). By working on the torus $\mathcal{S}$, we assume that all the quantities at stake are periodic in the e_1 and e_2 directions, where we have denoted by (e_1, e_2, e_3) the canonical basis of $\mathbb{R}^3$.

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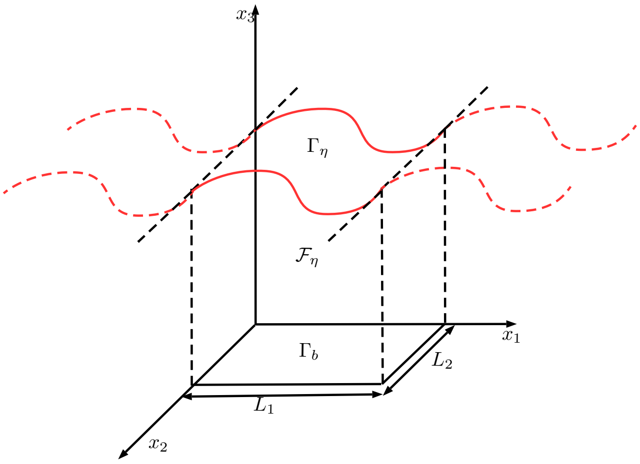

FIGURE 1.

The evolution of the density $\widetilde{\rho}$ and the velocity $\widetilde{u}$ of the fluid and of the elastic displacement η are given by the following system:

$$\begin{cases} \partial_t \widetilde{\rho} + \operatorname{div}(\widetilde{\rho}\widetilde{u}) = 0 & t > 0,\ x \in \mathcal{F}_{\eta(t)}, \\ \widetilde{\rho}\,(\partial_t \widetilde{u} + (\widetilde{u} \cdot \nabla)\widetilde{u}) - \operatorname{div} \mathbb{T}(\widetilde{u}, \widetilde{\pi}) = 0 & t > 0,\ x \in \mathcal{F}_{\eta(t)}, \\ \partial_{tt}\eta - \Delta_s \eta = \mathbb{H}_\eta(\widetilde{u}, \widetilde{\pi}) & t > 0, s \in \mathcal{S}, \end{cases} \tag{1.1}$$

with the boundary conditions

$$\begin{cases} \widetilde{u}(t, s, 1 + \eta(t, s)) = \partial_t \eta(t, s) e_3 & t > 0,\ s \in \mathcal{S}, \\ \widetilde{u} = 0 & t > 0,\ x \in \Gamma_b, \end{cases} \tag{1.2}$$

and the initial conditions

$$\begin{cases} \eta(0, \cdot) = \eta_1^0, \quad \partial_t \eta(0, \cdot) = \eta_2^0 & \text{in } \mathcal{S}, \\ \widetilde{\rho}(0, \cdot) = \widetilde{\rho}^0, \quad \widetilde{u}(0, \cdot) = \widetilde{u}^0 & \text{in } \mathcal{F}_{\eta_1^0}. \end{cases} \tag{1.3}$$

In the above system, the fluid stress tensor $\mathbb{T}(\widetilde{u}, \widetilde{\pi})$ is given by:

$$\mathbb{T}(\widetilde{u}, \widetilde{\pi}) = 2\mu \mathbb{D}\widetilde{u} + (\alpha \operatorname{div} \widetilde{u} - \widetilde{\pi})\mathbb{I}_3, \quad \mathbb{D}\widetilde{u} = \frac{1}{2}(\nabla \widetilde{u} + \nabla \widetilde{u}^\top),$$

and the pressure law is

$$\widetilde{\pi} = a\widetilde{\rho}^\gamma.$$

We assume that

$$\mu > 0,\ \alpha + \mu > 0,\ a > 0,\ \gamma \geqslant 1. \tag{1.4}$$

The force of the fluid acting on the structure is given by

$$\mathbb{H}_\eta(\widetilde{u}, \widetilde{\pi}) = -\sqrt{1 + |\nabla_s \eta|^2} \Big(\mathbb{T}(\widetilde{u}, \widetilde{\pi})\widetilde{n} \Big)|_{\Gamma_{\eta(t)}} \cdot e_3, \tag{1.5}$$

where $\widetilde{n}$ is the unit exterior normal of $\mathcal{F}_{\eta(t)}$. In particular, on $\Gamma_{\eta(t)}$ we have

$$\widetilde{n} = \frac{1}{\sqrt{1 + |\nabla_s \eta|^2}} \left[-\nabla_s \eta, 1 \right].$$

In the whole article, we add the index s in the gradient and in the Laplace operators if they apply to functions defined on $\mathcal{S}$ (and we keep the usual notation for functions defined on a domain of $\mathbb{R}^3$).

Let us give conditions on the initial data that we need to solve (1.1)-(1.3):

$$\eta_1^0 \in H^4(\mathcal{S}), \quad \min_{\overline{\mathcal{S}}} \left(1+\eta_1^0\right) > 0, \tag{1.6}$$

$$\widetilde{\rho}^0 \in H^3(\mathcal{F}_{\eta_1^0}), \quad \min_{\overline{\mathcal{F}_{\eta_1^0}}} \widetilde{\rho}^0 > 0, \tag{1.7}$$

$$\eta_2^0 \in H^3(\mathcal{S}), \quad \widetilde{u}^0 \in H^3(\mathcal{F}_{\eta_1^0})^3, \quad \widetilde{u}^0 = 0 \text{ on } \Gamma_b, \quad \widetilde{u}^0\left(s, 1+\eta_1^0(s)\right) = \eta_2^0(s) e_3 \quad (s \in \mathcal{S}), \tag{1.8}$$

$$\widetilde{\pi}^0 = a(\widetilde{\rho}^0)^\gamma, \quad \frac{1}{\widetilde{\rho}^0} \operatorname{div} \mathbb{T}(\widetilde{u}^0, \widetilde{\pi}^0) = 0 \text{ on } \Gamma_b, \tag{1.9}$$

$$\left[\frac{1}{\widetilde{\rho}^0} \operatorname{div} \mathbb{T}(\widetilde{u}^0, \widetilde{\pi}^0)\right]\left(s, 1+\eta_1^0(s)\right) = \left[\Delta_s \eta_1^0 + \mathbb{H}_{\eta_1^0}(\widetilde{u}^0, \widetilde{\pi}^0)\right](s) e_3 \quad (s \in \mathcal{S}). \tag{1.10}$$

We now state our main result :

Theorem 1.1. *Assume that $\left[\widetilde{\rho}^0, \widetilde{u}^0, \eta_1^0, \eta_2^0\right]$ satisfies (1.6)–(1.10). Then there exists $T>0$ such that the system (1.1)-(1.3) admits a unique strong solution $[\widetilde{\rho}, \widetilde{u}, \eta]$ satisfying*

$$\widetilde{\rho} \in H^1(0,T; H^3(\mathcal{F}_{\eta(\cdot)})) \cap W^{1,\infty}(0,T; H^2(\mathcal{F}_{\eta(\cdot)})),$$

$$\widetilde{u} \in L^2(0,T; H^4(\mathcal{F}_{\eta(\cdot)}))^3 \cap C^0([0,T); H^3(\mathcal{F}_{\eta(\cdot)}))^3 \cap H^1(0,T; H^2(\mathcal{F}_{\eta(\cdot)}))^3$$
$$\cap C^1([0,T); H^1(\mathcal{F}_{\eta(\cdot)}))^3 \cap H^2(0,T; L^2(\mathcal{F}_{\eta(\cdot)}))^3,$$

$$\eta \in L^\infty(0,T; H^4(\mathcal{S})) \cap H^2(0,T; H^2(\mathcal{S})) \cap H^3(0,T; L^2(\mathcal{S})),$$
$$\partial_t \eta \in L^2(0,T; H^{7/2}(\mathcal{S})) \cap L^\infty(0,T; H^3(\mathcal{S})).$$

Moreover,

$$1+\eta(t,s) > 0 \quad (t \in [0,T], s \in \overline{\mathcal{S}}), \quad \widetilde{\rho}(t,x) > 0 \quad \left(t \in [0,T], \ x \in \overline{\mathcal{F}_{\eta(t)}}\right).$$

In the above statement, we are using spaces of the form $H^s(0,T; H^r(\mathcal{F}_{\eta(\cdot)}))$, $W^{s,\infty}(0,T; H^r(\mathcal{F}_{\eta(\cdot)}))$ and $C^s([0,T); H^r(\mathcal{F}_{\eta(\cdot)}))$. They are defined through a diffeomorphism $X(t,\cdot) : \overline{\mathcal{F}} \to \overline{\mathcal{F}_{\eta(t)}}$ (see Section 2 for an example of construction of such a diffeomorphism):

$$f \in H^s(0,T; H^r(\mathcal{F}_{\eta(\cdot)})) \quad \text{if} \quad f \circ X \in H^s(0,T; H^r(\mathcal{F})),$$
$$f \in W^{s,\infty}(0,T; H^r(\mathcal{F}_{\eta(\cdot)})) \quad \text{if} \quad f \circ X \in W^{s,\infty}(0,T; H^r(\mathcal{F})),$$
$$f \in C^s([0,T); H^r(\mathcal{F}_{\eta(\cdot)})) \quad \text{if} \quad f \circ X \in C^s([0,T); H^r(\mathcal{F})).$$

Remark 1.2. *The result in Theorem 1.1 corresponds to the existence of strong solutions for a system coupling the compressible Navier-Stokes system with a wave equation. Note that we keep the regularity of the initial conditions given in (1.6)–(1.8) during the time of existence. More precisely, we have $\eta \in C^0_{\text{weak}}([0,T]; H^4(\mathcal{S}))$, $\partial_t \eta \in C^0_{\text{weak}}([0,T]; H^3(\mathcal{S}))$, $\widetilde{\rho} \in C^0([0,T]; H^3(\mathcal{F}_{\eta(\cdot)}))$ and $\widetilde{u} \in C^0([0,T]; H^3(\mathcal{F}_{\eta(\cdot)}))^3$. With respect to previous results, it is important to notice that we do not add any damping on the wave equation (see below for details on the references on similar systems).*

Remark 1.3. *One can show an analogue of Theorem 1.1 on a corresponding $2D/1D$ system, that is where the fluid reference domain is given by*

$$\mathcal{F} = \mathcal{S} \times (0,1), \quad \mathcal{S} = \mathbb{R}/L_1\mathbb{Z},$$

with $L_1 > 0$ and the elastic structure is located at the upper boundary $\mathcal{S} \times \{1\}$.

As mentioned at the beginning of the introduction, lots of articles have been devoted to the mathematical analysis of fluid-structure systems involving moving interfaces. Broadly speaking, these types of models can be classified into two types: either the structure is immersed inside the fluid, or the structure is located at the boundary of the fluid domain. For the second case, we can mention the survey paper [38], where the authors describe some models for blood flow in arteries as a fluid-structure interaction system. They consider, in particular, the simplified case of a viscous incompressible fluid modeled by the Navier-Stokes system and of a structure governed by a damped beam equation. The corresponding model was mathematically analyzed in [12] (existence of weak solutions) and in [4] (existence of strong solutions). In [19], the author obtains the existence of weak solutions with a similar model but without damping on the plate equation.

The existence of weak solutions was obtained in the case of more complex models: for instance, the case of a linear elastic Koiter shell for the structure was considered in [25, 24]. Let us also mention [33], where the authors deal with the case of dynamic pressure boundaries conditions and for the structure modeled either by a linear viscoelastic beam or by a linear elastic Koiter shell equation. The methodology is different from the earlier ones and it is based on constructing an approximation solution via a semi-discrete, operator splitting Lie scheme (also known as kinematically coupled scheme). Later, these results were extended in [34, 35] to the 3D cylindrical domain where the structure is modeled by a linear (respectively nonlinear in [35]) elastic cylindrical Koiter shell whose displacements are not necessarily radially symmetric.

Concerning strong solutions, let us note that the result [4] was obtained under some restrictions on the physical parameters. In [39], the author considers a monolithic method to study the stabilization of this coupled problem, and using this approach, the existence of local in time strong solutions are obtained in [26] without restrictions on the physical parameters. Similar results have been obtained in [11] with boundary conditions involving the pressure and in [17] with Navier boundary conditions. Finally, let us mention several works in the case where the structure is modeled by a nonlinear shell equation: [13, 14, 28, 35]. The work [20] is devoted to the global in time existence and uniqueness of solutions for the case of a damped beam equation and investigates, in particular, the possible contacts between the structure and the bottom of the domain. We also mention [30, 16] where the authors obtain the existence of strong solutions within an "$L^p - L^q$" framework instead of a "Hilbert" space framework.

Finally, some works have been devoted to the case of a structure modeled by the wave equation instead of the beam equation. First [27] shows the existence and the uniqueness of strong solutions in the case of a damped wave equation. In [21], the authors consider three types of structure: the damped beam equation, the wave equation (without damping), and the beam equation with an inertia term. The case of an undamped beam equation is tackled in [2, 3] by using the notion of Gevrey semigroups. Here our aim is to show a similar result as [21] in the case of a compressible fluid and with an undamped wave equation.

Concerning compressible fluids interacting with plate/beam equations through the boundary of the fluid domain, there are only a few results available in the literature. Global existence of weak solutions until the structure touches the boundary of the fluid domain was proved in [18, 10]. Local in time existence of strong solutions in the corresponding $2D/1D$ case was recently obtained in [32]. Global in time existence of strong solution for small data within an "$L^p - L^q$" framework for heat conducting compressible fluid and a damped plate equation was established in [29]. Well-posedness and stability of linear compressible fluid-structure systems were studied in [15, 1]. Let us mention some works in the case of a viscous compressible fluid but with rigid bodies or elastic structured immersed into the fluid: [9, 8, 7, 6, 23, 31, 22], etc.

The main novelties of our work are the following:

- Regarding the interaction between a purely elastic structure (i.e, no additional damping term) on the boundary and a viscous fluid, the construction of a mathematical theory for strong solutions with "no regularity loss" (i.e. a solution such that the unknowns of the system remain in the same Sobolev spaces as the initial data, at least locally in time) is a challenging and critical issue. We have settled this issue for the compressible fluid-elastic structure case. In the literature the existing results for the compressible fluid-elastic structure interaction are stated with a mismatch between the regularities of the initial data and of the solution: for instance, in [32, Theorem 1.7], there is a loss of order $1/2$ in the space regularity for the fluid velocity at initial time even for the compressible fluid-damped beam interaction in a $2D/1D$ framework.
- Previously, the "no regularity loss" issue for the incompressible fluid-elastic structure case has been obtained in [21] in the $2D/1D$ framework. Here, we have established the same kind of results in the compressible fluid-elastic structure case for the $3D/2D$ framework.
- We do not need initial displacements of the structure to be zero. This is a difference with respect to some of the previous works, for instance [32]. The case of a system coupling the incompressible Navier-Stokes equations and a damped beam (respectively wave) with a non zero initial beam displacement

is addressed in [11, 17] (respectively in [21]). In this article, we have addressed this issue for the compressible fluid-wave equation case.

The strategy to prove Theorem 1.1 is standard: we first use a change of variables (see Section 2) to write the system in a fixed spatial domain. In the context of compressible fluid, it is convenient to use a Lagrangian change of variables, but to take advantage of the geometry we compose it with a geometric transformation. Then we linearize the system and study in Section 3 the corresponding linear system. This is the main part of this article. We follow the approach in [21] and start by approximating this linear system by adding a damping on the wave equation. Then we obtain several a priori estimates and passing to the limit as the damping goes to 0 allows us to prove a well-posedness result on the linear system (see Theorem 3.1). Finally, in Section 4, we prove the main result by using the result on the linear system and a fixed point argument.

2. Change of variables

This section is devoted to the construction of the change of variables to transform the fluid domain $\mathcal{F}_{\eta(t)}$ into $\mathcal{F}$. This change of variables is the composition of a Lagrangian change of variables and of a geometric change of variables. More precisely, we first define the transformation $X^0 : \overline{\mathcal{F}} \to \overline{\mathcal{F}_{\eta_1^0}}$:

$$X^0(y_1, y_2, y_3) := \left[y_1, y_2, (1+\eta_1^0(y_1,y_2))y_3\right] \quad ([y_1,y_2,y_3] \in \overline{\mathcal{F}}). \tag{2.1}$$

We assume that η_1^0 satisfies (1.6), so that X^0 is C^1-diffeomorphism from $\overline{\mathcal{F}}$ onto $\overline{\mathcal{F}_{\eta_1^0}}$.

Then, our change of variables X is defined as the characteristics associated with the fluid velocity $\widetilde{u}$:

$$\begin{cases} \partial_t X(t,y) = \widetilde{u}\,(t, X(t,y))\,, \quad (t>0), \\ X(0,y) = X^0(y), \quad y \in \overline{\mathcal{F}}. \end{cases} \tag{2.2}$$

If $\widetilde{u}$ satisfies some good properties, then X is a C^1-diffeomorphism from $\overline{\mathcal{F}}$ onto $\overline{\mathcal{F}_{\eta(t)}}$ for all $t \geqslant 0$. In that case, we denote by $Y(t,\cdot) = [X(t,\cdot)]^{-1}$ the inverse of $X(t,\cdot)$. Using this transformation, we can write (1.1)–(1.3) using only the fixed domain $\mathcal{F}$. We first write

$$\rho(t,y) = \widetilde{\rho}(t, X(t,y)), \quad u(t,y) = \widetilde{u}(t, X(t,y)), \quad \pi = a\rho^\gamma. \tag{2.3}$$

for $(t,y) \in \mathbb{R}_+ \times \mathcal{F}$. In particular,

$$\widetilde{\rho}(t,x) = \rho(t, Y(t,x)), \quad \widetilde{u}(t,x) = u(t, Y(t,x)). \tag{2.4}$$

for $(t,x) \in \mathbb{R}_+ \times \mathcal{F}_{\eta(t)}$. We also introduce the notation

$$\mathbb{B}_X := \operatorname{Cof} \nabla X, \quad \delta_X := \det \nabla X, \quad \mathbb{A}_X := \frac{1}{\delta_X}\mathbb{B}_X^\top \mathbb{B}_X, \tag{2.5}$$

$$\mathbb{B}^0 := \mathbb{B}_{X^0}, \quad \delta^0 := \delta_{X^0}, \quad \mathbb{A}^0 := \mathbb{A}_{X^0} \tag{2.6}$$

and

$$\mathcal{T}\eta(y) = \left\{ \begin{array}{ll} \eta(y_1,y_2)e_3 & \text{if } y = (y_1,y_2,1) \in \Gamma_0 \\ 0 & \text{if } y = (y_1,y_2,0) \in \Gamma_b \end{array} \right., \quad (\mathcal{Q}H)(s) = H(s,1)e_3 \cdot e_3 \quad (s \in \mathcal{S}). \tag{2.7}$$

Here, we have set

$$\Gamma_0 = \mathcal{S} \times \{1\}\,.$$

Then, the system (1.1)–(1.3) becomes

$$\left\{ \begin{array}{ll} \partial_t \rho + \dfrac{\rho^0}{\delta^0}\nabla u : \mathbb{B}^0 = F_1 & \text{in } (0,T)\times\mathcal{F}, \\ \rho^0\delta^0\partial_t u - \operatorname{div}\mathbb{T}^0(u) = F_2 + \operatorname{div} H & \text{in } (0,T)\times\mathcal{F}, \\ \partial_{tt}\eta - \Delta_s \eta = -\mathcal{Q}(\mathbb{T}^0 + H) & \text{in } (0,T)\times\mathcal{S}, \end{array} \right. \tag{2.8}$$

with the boundary conditions

$$u = \mathcal{T}(\partial_t \eta) \quad \text{on } (0,T)\times\partial\mathcal{F}, \tag{2.9}$$

and the initial conditions

$$\begin{cases} \eta(0,\cdot) = \eta_1^0, \quad \partial_t\eta(0,\cdot) = \eta_2^0 \quad \text{in } \mathcal{S}, \\ \rho(0,\cdot) = \rho^0, \quad u(0,\cdot) = u^0 \quad \text{in } \mathcal{F}. \end{cases} \tag{2.10}$$

where we have used the following notation

$$\rho^0 := \widetilde{\rho}^0 \circ X^0, \quad u^0 := \widetilde{u}^0 \circ X^0, \tag{2.11}$$

$$\mathbb{T}^0(u) := \mu \nabla u \mathbb{A}^0 + \frac{\mu}{\delta^0} \mathbb{B}^0 (\nabla u)^\top \mathbb{B}^0 + \frac{\alpha}{\delta^0} (\mathbb{B}^0 : \nabla u) \mathbb{B}^0, \tag{2.12}$$

$$F_1(\rho, u, \eta) := \frac{\rho^0}{\delta^0} \nabla u : \mathbb{B}^0 - \frac{\rho}{\delta_X} \nabla u : \mathbb{B}_X, \quad F_2(\rho, u, \eta) := \left(\rho^0 \delta^0 - \rho \delta_X\right) \partial_t u, \tag{2.13}$$

$$H(\rho, u, \eta) := \mu \left(\nabla u \mathbb{A}_X - \nabla u \mathbb{A}^0\right) + \mu \left(\frac{1}{\delta_X} \mathbb{B}_X (\nabla u)^\top \mathbb{B}_X - \frac{1}{\delta^0} \mathbb{B}^0 (\nabla u)^\top \mathbb{B}^0 \right) \\ + \alpha \left(\frac{1}{\delta_X} (\mathbb{B}_X : \nabla u) \mathbb{B}_X - \frac{1}{\delta_0} (\mathbb{B}^0 : \nabla u) \mathbb{B}^0 \right) - a \rho^\gamma \mathbb{B}_X. \tag{2.14}$$

The characteristics X defined in (2.2) can now be written as

$$X(t, y) = X^0(y) + \int_0^t u(r, y) \, \mathrm{d}r, \tag{2.15}$$

for every $y \in \mathcal{F}$ and $t \geqslant 0$.

We introduce the following spaces:

$$\mathcal{E}_\rho := H^1(0, T; H^3(\mathcal{F})) \cap W^{1,\infty}(0, T; H^2(\mathcal{F})), \tag{2.16}$$

$$\mathcal{E}_u := L^2(0, T; H^4(\mathcal{F}))^3 \cap H^1(0, T; H^2(\mathcal{F}))^3 \cap H^2(0, T; L^2(\mathcal{F}))^3, \tag{2.17}$$

$$\mathcal{E}_\eta := L^\infty(0, T; H^4(\mathcal{S})) \cap W^{1,\infty}(0, T; H^3(\mathcal{S})) \cap H^2(0, T; H^2(\mathcal{S})) \cap H^3(0, T; L^2(\mathcal{S})). \tag{2.18}$$

with the norms

$$\|\rho\|_{\mathcal{E}_\rho} := \|\rho\|_{H^1(0,T;H^3(\mathcal{F}))} + \|\partial_t \rho\|_{L^\infty(0,T;H^2(\mathcal{F}))}, \tag{2.19}$$

$$\|u\|_{\mathcal{E}_u} := \|u\|_{L^2(0,T;H^4(\mathcal{F}))^3} + \|u\|_{H^1(0,T;H^2(\mathcal{F}))^3} + \|u\|_{H^2(0,T;L^2(\mathcal{F}))^3} \\ + \|u\|_{C^0([0,T];H^3(\mathcal{F}))^3} + \|u\|_{C^1([0,T];H^1(\mathcal{F}))^3}, \tag{2.20}$$

$$\|\eta\|_{\mathcal{E}_\eta} := \|\eta\|_{L^\infty(0,T;H^4(\mathcal{S}))} + \|\eta\|_{H^2(0,T;H^2(\mathcal{S}))} + \|\eta\|_{H^3(0,T;L^2(\mathcal{S}))} \\ + \|\partial_t \eta\|_{L^\infty(0,T;H^3(\mathcal{S}))} + \|\partial_{tt} \eta\|_{C^0([0,T];H^1(\mathcal{S}))}. \tag{2.21}$$

Let us restate our main result. Using the definition of X^0 defined in (2.1), the assumptions (1.6)–(1.10) on the initial data transformed into the following conditions:

$$\eta_1^0 \in H^4(\mathcal{S}), \quad \min_{\overline{\mathcal{S}}} \left(1 + \eta_1^0\right) > 0, \tag{2.22}$$

$$\rho^0 \in H^3(\mathcal{F}), \quad \min_{\overline{\mathcal{F}}} \rho^0 > 0, \tag{2.23}$$

$$\eta_2^0 \in H^3(\mathcal{S}), \quad u^0 \in H^3(\mathcal{F})^3, \quad u^0 = \mathcal{T} \eta_2^0 \text{ on } \partial\mathcal{F}, \tag{2.24}$$

$$\frac{1}{\rho^0 \delta^0} \operatorname{div} \left[\mathbb{T}^0(u^0) - a(\rho^0)^\gamma \mathbb{B}^0\right] = \mathcal{T} \left(\Delta_s \eta_1^0 - \mathcal{Q}\Big(\mathbb{T}^0(u^0) - a(\rho^0)^\gamma \mathbb{B}^0\Big) \right) \text{ on } \partial\mathcal{F}. \tag{2.25}$$

Using the above change of variables, Theorem 1.1 can be rephrased as

Theorem 2.1. *Assume that $\left[\rho^0, u^0, \eta_1^0, \eta_2^0\right]$ satisfies (2.22)–(2.25). Then there exists $T > 0$ such that the system (2.8)-(2.15) admits a unique strong solution $[\rho, u, \eta] \in \mathcal{E}_\rho \times \mathcal{E}_u \times \mathcal{E}_\eta$. Moreover,*

$$\min_{[0,T] \times \overline{\mathcal{S}}} (1 + \eta) > 0, \qquad \min_{[0,T] \times \overline{\mathcal{F}}} \rho > 0,$$

and, for all $t \in [0, T]$, $X(t, \cdot) : \mathcal{F} \to \mathcal{F}_{\eta(t)}$ is a C^1-diffeomorphism.

Remark 2.2. *From the boundary condition (2.9) and the regularity of u, we also get $\partial_t \eta \in L^2(0, T; H^{7/2}(\mathcal{S}))$.*

Let us give some preliminary results on the change of variables described above. First we state the following lemma which proof is skipped since it is standard.

Lemma 2.3. *Assume η_1^0 satisfies (1.6) and let us consider $X^0, \mathbb{A}^0, \mathbb{B}^0$ and δ^0 defined by (2.1), (2.5) and (2.6). Then*

$$\delta^0 = 1 + \eta_1^0, \quad \frac{1}{\delta^0} \in H^4(\mathcal{S}), \quad X^0 \in C^\infty([0,1]; H^4(\mathcal{S}))^3. \quad \mathbb{B}^0, \mathbb{A}^0 \in C^\infty([0,1]; H^3(\mathcal{S}))^9. \tag{2.26}$$

In particular, $X^0 \in W^{2,\infty}(\mathcal{F})^3$, $1/\delta^0 \in W^{2,\infty}(\mathcal{S})$, $\mathbb{B}^0, \mathbb{A}^0 \in W^{1,\infty}(\mathcal{F})^9$.

Note that with the above lemma, if $\widetilde{f} \in H^3(\mathcal{F}_{\eta_1^0})$, then

$$f = \widetilde{f} \circ X^0 \in H^3(\mathcal{F}),$$

which justifies the regularity in (2.23), (2.24).

With the definitions (2.5), (2.6), if we have

$$v = \widetilde{v} \circ X^0, \tag{2.27}$$

then we have the following relations:

$$\nabla \widetilde{v} \circ X^0 = \frac{1}{\delta^0} \nabla v (\mathbb{B}^0)^\top, \quad \operatorname{div} \widetilde{v} \circ X^0 = \frac{1}{\delta^0} \nabla v : \mathbb{B}^0. \tag{2.28}$$

Lemma 2.4. *Assume $v \in H^1(\mathcal{F})^3$ such that $v_1 = v_2 = 0$ on $\partial\mathcal{F}$. Let $\mathbb{T}^0$ be the tensor introduced in (2.12). Then the following relation holds*

$$\int_{\mathcal{F}} \mathbb{T}^0(v) : \nabla v \ dy = \mu \int_{\mathcal{F}} \frac{1}{\delta^0} |\nabla v (\mathbb{B}^0)^\top|^2 \ dy + (\mu + \alpha) \int_{\mathcal{F}} \frac{1}{\delta^0} (\nabla v : \mathbb{B}^0)^2 \ dy. \tag{2.29}$$

In particular, if $\mu + \alpha \geqslant 0$, there exist two constants $\overline{C}, \underline{C} > 0$ such that

$$\underline{C} \|\nabla v\|^2_{L^2(\mathcal{F})^9} \leqslant \int_{\mathcal{F}} \mathbb{T}^0(v) : \nabla v \ dy \leqslant \overline{C} \|\nabla v\|^2_{L^2(\mathcal{F})^9}. \tag{2.30}$$

Proof. Relation (2.30) is a consequence of (2.29) with the properties of $(\delta^0)^{-1} \in L^\infty(\mathcal{F})$ and $\nabla X^0 \in L^\infty(\mathcal{F})^9$.

To prove (2.29), we use a standard density argument by first considering v smooth with $v_1 = v_2 = 0$ on $\partial\mathcal{F}$ and then pass to the limit. We can thus assume that v is smooth in the remaining part of the proof. Using the expression (2.12) of $\mathbb{T}^0$ and (2.28), we obtain

$$\begin{aligned} \int_{\mathcal{F}} \mathbb{T}^0(v) : \nabla v \ dy &= \int_{\mathcal{F}} \frac{1}{\delta^0} \left(\frac{\mu}{2} \left| \nabla v (\mathbb{B}^0)^\top + \mathbb{B}^0 \nabla v^\top \right|^2 + \alpha \left(\mathbb{B}^0 : \nabla v \right)^2 \right) \ dy \\ &= \int_{\mathcal{F}_{\eta_1^0}} \frac{\mu}{2} |\nabla \widetilde{v} + \nabla \widetilde{v}^\top|^2 + \alpha (\operatorname{div} \widetilde{v})^2 \ dx, \end{aligned} \tag{2.31}$$

where $\widetilde{v}$ is given by (2.27).

Observe that

$$\frac{\mu}{2} |\nabla \widetilde{v} + \nabla \widetilde{v}^\top|^2 + \alpha (\operatorname{div} \widetilde{v})^2 = \mu |\nabla \widetilde{v}|^2 + (\alpha + \mu)(\operatorname{div} \widetilde{v})^2 + \mu \sum_{i,j} \frac{\partial \widetilde{v}_i}{\partial x_j} \frac{\partial \widetilde{v}_j}{\partial x_i} - \frac{\partial \widetilde{v}_i}{\partial x_i} \frac{\partial \widetilde{v}_j}{\partial x_j}.$$

Now,

$$\sum_{i,j} \frac{\partial \widetilde{v}_i}{\partial x_j} \frac{\partial \widetilde{v}_j}{\partial x_i} - \frac{\partial \widetilde{v}_i}{\partial x_i} \frac{\partial \widetilde{v}_j}{\partial x_j} = 2 \sum_{i<j} \frac{\partial \widetilde{v}_i}{\partial x_j} \frac{\partial \widetilde{v}_j}{\partial x_i} - \frac{\partial \widetilde{v}_i}{\partial x_i} \frac{\partial \widetilde{v}_j}{\partial x_j}$$

and using that $\widetilde{v}_i = 0$ on $\partial\mathcal{F}_{\eta_1^0}$ for $i < 3$, we deduce by integration by parts that

$$\int_{\mathcal{F}_{\eta_1^0}} 2 \sum_{i<j} \frac{\partial \widetilde{v}_i}{\partial x_j} \frac{\partial \widetilde{v}_j}{\partial x_i} - \frac{\partial \widetilde{v}_i}{\partial x_i} \frac{\partial \widetilde{v}_j}{\partial x_j} \ dx = 0$$

Combining the above relations together with (2.31) and (2.28), we deduce the result. □

We also need the following integration by parts formulas (see, for instance [21, Lemma 3.6 and Lemma 3.8]):

Lemma 2.5. *Assume $v \in L^2(0,T;H^2(\mathcal{F}))^3 \cap H^1(0,T;L^2(\mathcal{F}))^3$ and $\zeta \in H^1(0,T;L^2(\mathcal{S})) \cap L^2(0,T;H^2(\mathcal{S}))$ with $v = \mathcal{T}\zeta$ on $\partial\mathcal{F}$. Then for all $t \in [0,T]$,*

$$-\int_0^t \int_{\mathcal{F}} \operatorname{div} \mathbb{T}^0(v) \cdot \partial_t v \ \mathrm{d}y\mathrm{d}\tau = \frac{1}{2}\int_{\mathcal{F}} \mathbb{T}^0(v(t,y)) : \nabla v(t,y) \ \mathrm{d}y - \frac{1}{2}\int_{\mathcal{F}} \mathbb{T}^0(v(0,y)) : \nabla v(0,y) \ \mathrm{d}y - \int_0^t \int_{\mathcal{S}} \mathcal{Q}(\mathbb{T}^0(v))\partial_t \zeta \ \mathrm{d}s\mathrm{d}\tau.$$

Assume $v \in H^2(\mathcal{F})^3$, $\zeta \in H^2(\mathcal{S})$ with $v = \mathcal{T}\zeta$ on $\partial\mathcal{F}$, and $\mathbb{H} \in H^1(\mathcal{F})^9$. Then for all $j = 1,2$,

$$\int_{\mathcal{F}} \operatorname{div} \mathbb{H} \cdot \partial_{y_j y_j} v \mathrm{d}y = \int_{\mathcal{F}} \partial_{y_j} \mathbb{H} : \nabla(\partial_{y_j} v)\mathrm{d}y + \int_{\mathcal{S}} (\mathcal{Q}\mathbb{H})\partial_{s_j s_j}\zeta \mathrm{d}s.$$

We end this section with the following result on the product of functions:

Lemma 2.6. *Assume $f \in H^1(\mathcal{S})$ and $g \in L^2(0,1;H^1(\mathcal{S}))$, then $fg \in L^2(\mathcal{F})$ and there exists a constant C such that*

$$\|fg\|_{L^2(\mathcal{F})} \leqslant C \|f\|_{H^1(\mathcal{S}))} \|g\|_{L^2(0,1;H^1(\mathcal{S}))}.$$

In particular, if $f \in H^1(\mathcal{S})$ and $g \in H^1(\mathcal{F})$, then $fg \in L^2(\mathcal{F})$.

3. Linear system

In this section, we consider the following linear problem associated with (2.8)–(2.10):

$$\begin{cases} \partial_t \rho + \dfrac{\rho^0}{\delta^0}\nabla u : \mathbb{B}^0 = f_1 & \text{in } (0,T)\times\mathcal{F}, \\ \rho(0,\cdot) = \widehat{\rho}^0 & \text{in } \mathcal{F}, \end{cases} \tag{3.1}$$

$$\begin{cases} (\rho^0\delta^0)\partial_t u - \operatorname{div} \mathbb{T}^0(u) = f_2 + \operatorname{div} h & \text{in } (0,T)\times\mathcal{F}, \\ u = \mathcal{T}(\partial_t \eta) & \text{on } (0,T)\times\partial\mathcal{F}, \\ u(0,\cdot) = u^0 & \text{in } \mathcal{F}, \end{cases} \tag{3.2}$$

$$\begin{cases} \partial_{tt}\eta - \Delta_s \eta = -\mathcal{Q}(\mathbb{T}^0(u) + h) & \text{in } (0,T)\times\mathcal{S}, \\ \eta(0,\cdot) = \widehat{\eta}_1^0, \quad \partial_t\eta(0,\cdot) = \eta_2^0 & \text{in } \mathcal{S}, \end{cases} \tag{3.3}$$

where $\mathbb{A}^0, \mathbb{B}^0$ and δ^0 are defined in (2.6) and where $\mathbb{T}^0$ is defined by (2.12). We also recall that $\mathcal{T}$ and $\mathcal{Q}$ are defined by (2.7). In the above system, with respect to system (2.8)–(2.10), the nonlinearities F_1, F_2 and H have been replaced by given source terms f_1, f_2 and h. Moreover since η_1^0 and ρ^0 are involved in the coefficients of (3.1)–(3.3), we also replaced the initial conditions of ρ and of η by $\widehat{\rho}^0$ and by $\widehat{\eta}_1^0$.

Throughout this section we assume (2.22)–(2.23). We consider the subset of initial conditions

$$\mathcal{I} = \left\{ [\widehat{\rho}^0, u^0, \widehat{\eta}_1^0, \eta_2^0] \in H^3(\mathcal{F}) \times H^3(\mathcal{F})^3 \times H^4(\mathcal{S}) \times H^3(\mathcal{S}), \quad u^0 = \mathcal{T}\eta_2^0 \text{ on } \partial\mathcal{F} \right\}, \tag{3.4}$$

endowed with the norm

$$\| [\widehat{\rho}^0, u^0, \widehat{\eta}_1^0, \eta_2^0] \|_{\mathcal{I}} := \|\widehat{\rho}^0\|_{H^3(\mathcal{F})} + \|u^0\|_{H^3(\mathcal{F})^3} + \|\widehat{\eta}_1^0\|_{H^4(\mathcal{S})} + \|\eta_2^0\|_{H^3(\mathcal{S})}. \tag{3.5}$$

We also consider $\mathcal{R}_T$, the space of source terms (3.1)-(3.3):

$$\mathcal{R}_T = \Big\{ [f_1, f_2, h] \mid f_1 \in L^2(0,T;H^3(\mathcal{F})) \cap L^\infty(0,T;H^2(\mathcal{F})), \quad f_2 \in H^1(0,T,L^2(\mathcal{F}))^3 \cap L^2(0,T,H^2(\mathcal{F}))^3, \\ h \in H^1(0,T;H^1(\mathcal{F}))^9 \cap L^2(0,T,H^3(\mathcal{F}))^9, \quad f_2(0,\cdot) \in H^1(\mathcal{F})^3, \quad h(0,\cdot) \in H^2(\mathcal{F})^9 \Big\} \tag{3.6}$$

with

$$\|[f_1,f_2,h]\|_{\mathcal{R}_T} := \|f_1\|_{L^2(0,T;H^3(\mathcal{F}))\cap L^\infty(0,T;H^2(\mathcal{F}))} + \|f_2\|_{H^1(0,T,L^2(\mathcal{F}))^3\cap L^2(0,T,H^2(\mathcal{F}))^3} \\ + \|h\|_{L^2(0,T;H^3(\mathcal{F}))^9\cap H^1(0,T;H^1(\mathcal{F}))^9} + \|f_2(0,\cdot)\|_{H^1(\mathcal{F})^3} + \|h(0,\cdot)\|_{H^2(\mathcal{F})^9}. \tag{3.7}$$

We recall that $\mathcal{E}_\rho$, $\mathcal{E}_u$ and $\mathcal{E}_\eta$ are defined by (2.16), (2.17) and (2.18).

We state the main result of this section:

Theorem 3.1. *Assume* (2.22) *and* (2.23)*. Then for any*

$$[\widehat{\rho}^0, u^0, \widehat{\eta}_1^0, \eta_2^0] \in \mathcal{I}, \quad [f_1, f_2, h] \in \mathcal{R}_T, \tag{3.8}$$

satisfying the compatibility conditions

$$\frac{1}{\rho^0\delta^0}\left(\operatorname{div}\mathbb{T}^0(u^0) + f_2(0,\cdot) + \operatorname{div} h(0,\cdot)\right) = \mathcal{T}\left(\Delta_s\widehat{\eta}_1^0 - \mathcal{Q}(\mathbb{T}^0(u^0) + h(0,\cdot))\right) \quad \text{on } \partial\mathcal{F}. \tag{3.9}$$

the system (3.1)-(3.3) *admits a unique solution* $[\rho, u, \eta] \in \mathcal{E}_\rho \times \mathcal{E}_u \times \mathcal{E}_\eta$ *and there exists a constant* $C > 0$ *such that*

$$\|[\rho, u, \eta]\|_{\mathcal{E}_\rho\times\mathcal{E}_u\times\mathcal{E}_\eta} \leqslant Ce^{CT}\left(\left\|[\widehat{\rho}^0, u^0, \widehat{\eta}_1^0, \eta_2^0]\right\|_{\mathcal{I}} + \|[f_1, f_2, h]\|_{\mathcal{R}_T}\right). \tag{3.10}$$

3.1. **A first result for (3.2)–(3.3).** Since the system (3.2)-(3.3) can be solved independently of the system (3.1), we first start by studying it. We also begin by looking for solutions less regular than in the statement of Theorem 3.1. More precisely, let us define the following spaces

$$\mathcal{I}^0 := \left\{ [u^0, \widehat{\eta}_1^0, \eta_2^0] \in H^1(\mathcal{F})^3 \times H^2(\mathcal{S}) \times H^1(\mathcal{S}) \; ; \; u^0 = \mathcal{T}\eta_2^0 \text{ on } \partial\mathcal{F}\right\}, \tag{3.11}$$

$$\mathcal{R}_T^0 = L^2(0,T; L^2(\mathcal{F}))^3 \times L^2(0,T; H^1(\mathcal{F}))^9, \tag{3.12}$$

$$\mathcal{E}_u^0 := L^2(0,T; H^2(\mathcal{F}))^3 \cap C^0([0,T]; H^1(\mathcal{F}))^3 \cap H^1(0,T; L^2(\mathcal{F}))^3, \tag{3.13}$$

$$\mathcal{E}_\eta^0 := L^\infty(0,T; H^2(\mathcal{S})) \cap W^{1,\infty}(0,T; H^1(\mathcal{S})) \cap H^2(0,T; L^2(\mathcal{S})), \tag{3.14}$$

endowed with their canonical norms. With the above notation, our first result for (3.2)-(3.3) states as follows:

Proposition 3.2. *Assume* (2.22)*,* (2.23)*. Then for any*

$$[u^0, \widehat{\eta}_1^0, \eta_2^0] \in \mathcal{I}^0, \quad [f_2, h] \in \mathcal{R}_T^0, \tag{3.15}$$

the system (3.2)–(3.3) *admits a unique solution* $[u, \eta] \in \mathcal{E}_u^0 \times \mathcal{E}_\eta^0$ *and there exists a constant* $C > 0$ *independent of* T *such that*

$$\|[u, \eta]\|_{\mathcal{E}_u^0\times\mathcal{E}_\eta^0} \leqslant Ce^{CT}\left(\left\|[u^0, \widehat{\eta}_1^0, \eta_2^0]\right\|_{\mathcal{I}^0} + \|[f_2, h]\|_{\mathcal{R}_T^0}\right). \tag{3.16}$$

In order to show the above result, we first consider the following regularized system, where we add a damping in the wave equation of size $\varepsilon \in (0,1)$:

$$\begin{cases} (\rho^0\delta^0)\partial_t u_\varepsilon - \operatorname{div}\mathbb{T}^0(u_\varepsilon) = f_2 + \operatorname{div} h & \text{in } (0,T)\times\mathcal{F}, \\ u_\varepsilon = \mathcal{T}(\partial_t\eta_\varepsilon) & \text{on } (0,T)\times\partial\mathcal{F}, \\ u_\varepsilon(0,\cdot) = u^0 & \text{in } \mathcal{F}, \end{cases} \tag{3.17}$$

$$\begin{cases} \partial_{tt}\eta_\varepsilon - \Delta_s\eta_\varepsilon - \varepsilon\Delta_s\partial_t\eta_\varepsilon = -\mathcal{Q}(\mathbb{T}^0(u_\varepsilon) + h) & \text{in } (0,T)\times\mathcal{S}, \\ \eta_\varepsilon(0,\cdot) = \widehat{\eta}_1^0, \quad \partial_t\eta(0,\cdot) = \eta_2^0 & \text{in } \mathcal{S}. \end{cases} \tag{3.18}$$

For the above system, we have a maximal regularity property:

Proposition 3.3. *Assume* (2.22)*,* (2.23) *and* (3.15)*. Then the system* (3.17)–(3.18) *admits a unique solution*

$$u_\varepsilon \in \mathcal{E}_u^0, \quad \eta_\varepsilon \in H^1(0,T; H^2(\mathcal{S})) \cap H^2(0,T; L^2(\mathcal{S})). \tag{3.19}$$

The proof of this proposition is postponed in Appendix A. Using this result, the idea to show Proposition 3.2 is to pass to the limit as $\varepsilon \to 0$ in (3.17)–(3.18). For this purpose, we first obtain some estimates independent on ε for the solutions of (3.17)–(3.18).

Lemma 3.4. *Assume* (2.22), (2.23) *and* (3.15)*. Let us consider the solution* $[u_\varepsilon, \eta_\varepsilon]$ *of the system* (3.17)–(3.18) *with regularity* (3.19)*. Then there exist a constant* $C_1 > 0$*, independent of* ε *and* T*, such that, for any* $t \in (0,T)$ *we have*

$$\int_{\mathcal{F}} |u_\varepsilon(t,y)|^2 \ \mathrm{d}y + \int_{\mathcal{S}} |\partial_t \eta_\varepsilon(t,s)|^2 \ \mathrm{d}s + \int_{\mathcal{S}} |\nabla_s \eta_\varepsilon(t,s)|^2 \ \mathrm{d}s \\ + \int_0^t \int_{\mathcal{F}} |\nabla u_\varepsilon|^2 \mathrm{d}y\mathrm{d}\tau + \varepsilon \int_0^t \int_{\mathcal{S}} |\nabla_s \partial_t \eta_\varepsilon|^2 \mathrm{d}s\mathrm{d}\tau \leqslant C_1 \left(\left\| \left[u^0, \widehat{\eta}_1^0, \eta_2^0\right] \right\|_{\mathcal{I}^0}^2 + \|[f_2, h]\|_{\mathcal{R}_T^0}^2 \right). \tag{3.20}$$

Proof. We multiply the first equation of (3.17) by u_ε and the first equation of (3.18) by $\partial_t \eta_\varepsilon$:

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\mathcal{F}} \rho^0 \delta^0 |u_\varepsilon|^2 \ \mathrm{d}y + \int_{\mathcal{S}} \left(|\partial_t \eta_\varepsilon|^2 + |\nabla_s \eta_\varepsilon|^2 \right) \ \mathrm{d}s \right) + \int_{\mathcal{F}} \mathbb{T}^0(u_\varepsilon) : \nabla u_\varepsilon \ \mathrm{d}y + \varepsilon \int_{\mathcal{S}} |\nabla_s \partial_t \eta_\varepsilon|^2 \ \mathrm{d}s \\ = \int_{\mathcal{F}} f_2 \cdot u_\varepsilon \ \mathrm{d}y - \int_{\mathcal{F}} h : \nabla u_\varepsilon \ \mathrm{d}y. \tag{3.21}$$

Combining Lemma 2.4 and Poincaré's inequality, we thus deduce

$$\frac{1}{2}\frac{d}{dt}\Big(\int_{\mathcal{F}} \rho^0 \delta^0 |u_\varepsilon|^2 \ \mathrm{d}y + \int_{\mathcal{S}} |\partial_t \eta_\varepsilon|^2 \ \mathrm{d}s + \int_{\mathcal{S}} |\nabla_s \eta_\varepsilon|^2 \ \mathrm{d}s \Big) + C \int_{\mathcal{F}} |\nabla u_\varepsilon|^2 \ \mathrm{d}y \\ + \varepsilon \int_{\mathcal{S}} |\nabla_s \partial_t \eta_\varepsilon|^2 \ \mathrm{d}s \leqslant C \left(\int_{\mathcal{F}} |f_2|^2 \ \mathrm{d}y + \int_{\mathcal{F}} |h|^2 \ \mathrm{d}y \right). \tag{3.22}$$

Integrating the above relation on $(0,t)$ and using that $\rho^0\delta^0 \in L^\infty(\mathcal{F})$, $1/(\rho^0\delta^0) \in L^\infty(\mathcal{F})$, we deduce (3.20). □

Lemma 3.5. *Assume* (2.22), (2.23) *and* (3.15)*. Let us consider the solution* $[u_\varepsilon, \eta_\varepsilon]$ *of the system* (3.17)–(3.18) *with regularity* (3.19)*. Then there exist a constant* $C_2 > 0$*, independent of* ε *and* T*, such that, for any* $t \in (0,T)$ *we have*

$$\int_0^t \int_{\mathcal{F}} |\partial_t u_\varepsilon|^2 \ \mathrm{d}y\mathrm{d}\tau + \int_{\mathcal{F}} |\nabla u_\varepsilon(t,\cdot)|^2 \ \mathrm{d}y + \int_0^t \int_{\mathcal{S}} |\partial_{tt} \eta_\varepsilon|^2 \ \mathrm{d}s\mathrm{d}\tau + \varepsilon \int_{\mathcal{S}} |\nabla_s \partial_t \eta_\varepsilon(t,\cdot)|^2 \ \mathrm{d}s \\ \leqslant C_2 \left(\left\| \left[u^0, \widehat{\eta}_1^0, \eta_2^0\right] \right\|_{\mathcal{I}^0}^2 + \|[f_2, h]\|_{\mathcal{R}_T^0}^2 + \int_0^t \int_{\mathcal{S}} |\nabla_s \partial_t \eta_\varepsilon|^2 \ \mathrm{d}s\mathrm{d}\tau + \int_{\mathcal{S}} \Delta_s \eta_\varepsilon(t,\cdot) \partial_t \eta_\varepsilon(t,\cdot) \ \mathrm{d}y \right). \tag{3.23}$$

Proof. We multiply the first equation of (3.17) by $\partial_t u_\varepsilon$ and the first equation of (3.18) by $\partial_{tt}\eta_\varepsilon$:

$$\int_0^t \int_{\mathcal{F}} \rho^0 \delta^0 |\partial_t u_\varepsilon|^2 \ \mathrm{d}y\mathrm{d}\tau - \int_0^t \int_{\mathcal{F}} \operatorname{div} \mathbb{T}^0(u_\varepsilon) \cdot \partial_t u_\varepsilon \ \mathrm{d}y\mathrm{d}\tau + \int_0^t \int_{\mathcal{S}} |\partial_{tt} \eta_\varepsilon|^2 \ \mathrm{d}s\mathrm{d}\tau + \int_{\mathcal{S}} \nabla_s \eta_\varepsilon \cdot \nabla_s \partial_t \eta_\varepsilon \ \mathrm{d}s \\ + \frac{\varepsilon}{2} \int_{\mathcal{S}} |\nabla_s \partial_t \eta_\varepsilon|^2 \ \mathrm{d}s = \int_0^t \int_{\mathcal{F}} f_2 \cdot \partial_t u_\varepsilon \ \mathrm{d}y\mathrm{d}\tau + \int_0^t \int_{\mathcal{F}} \operatorname{div} h \cdot \partial_t u_\varepsilon \ \mathrm{d}y\mathrm{d}\tau - \int_0^t \int_{\mathcal{S}} \mathcal{Q}(\mathbb{T}^0(u_\varepsilon) + h) \partial_{tt} \eta_\varepsilon \ \mathrm{d}s\mathrm{d}\tau \\ + \int_0^t \int_{\mathcal{S}} |\nabla_s \partial_t \eta_\varepsilon|^2 \ \mathrm{d}s\mathrm{d}\tau + \int_{\mathcal{S}} \nabla_s \widehat{\eta}_1^0 \cdot \nabla_s \eta_2^0 \ \mathrm{d}s + \frac{\varepsilon}{2} \int_{\mathcal{S}} |\nabla_s \eta_2^0|^2 \ \mathrm{d}s. \tag{3.24}$$

Using Lemma 2.5 and $\int_{\mathcal{S}} \nabla_s \eta_\varepsilon \cdot \nabla_s \partial_t \eta_\varepsilon \ \mathrm{d}s = -\int_{\mathcal{S}} \Delta_s \eta_\varepsilon \partial_t \eta_\varepsilon \ \mathrm{d}y$, the relation (3.24) can be written as

$$\int_0^t \int_{\mathcal{F}} \rho^0 \delta^0 |\partial_t u_\varepsilon|^2 \mathrm{d}y\mathrm{d}\tau + \frac{1}{2} \int_{\mathcal{F}} \mathbb{T}^0(u_\varepsilon) : \nabla u_\varepsilon \ \mathrm{d}y + \int_0^t \int_{\mathcal{S}} |\partial_{tt} \eta_\varepsilon|^2 \ \mathrm{d}s\mathrm{d}\tau + \frac{\varepsilon}{2} \int_{\mathcal{S}} |\nabla_s \partial_t \eta_\varepsilon(t,\cdot)|^2 \ \mathrm{d}s \\ = \int_0^t \int_{\mathcal{F}} f_2 \cdot \partial_t u_\varepsilon \mathrm{d}y\mathrm{d}\tau + \int_0^t \int_{\mathcal{F}} \operatorname{div} h \cdot \partial_t u_\varepsilon \ \mathrm{d}y\mathrm{d}\tau - \int_0^t \int_{\mathcal{S}} (he_3 \cdot e_3) \partial_{tt} \eta_\varepsilon \ \mathrm{d}s\mathrm{d}\tau \\ + \frac{1}{2} \int_{\mathcal{F}} \mathbb{T}^0(u^0) : \nabla u^0 \ \mathrm{d}y + \int_{\mathcal{S}} \Delta_s \eta_\varepsilon \partial_t \eta_\varepsilon \ \mathrm{d}y + \int_{\mathcal{S}} \nabla_s \widehat{\eta}_1^0 \cdot \nabla_s \eta_2^0 \ \mathrm{d}s\mathrm{d}\tau + \frac{\varepsilon}{2} \int_{\mathcal{S}} |\nabla_s \eta_2^0|^2 \ \mathrm{d}s. \tag{3.25}$$

Using Lemma 2.4 and Young's inequality, we deduce the result. □

Lemma 3.6. *Assume* (2.22), (2.23) *and* (3.15). *Let us consider the solution* $[u_\varepsilon, \eta_\varepsilon]$ *of the system* (3.17)–(3.18) *with regularity* (3.19). *Then there exist a constant* $C_3 > 0$, *independent of* ε *and* T, *such that, for any* $t \in (0,T)$ *we have*

$$\sum_{j=1}^{2} \int_0^t \int_{\mathcal{F}} |\nabla(\partial_{y_j} u_\varepsilon)|^2 \, \mathrm{d}y\mathrm{d}\tau + \int_{\mathcal{S}} |\nabla_s \partial_t \eta_\varepsilon|^2 \mathrm{d}s + \int_{\mathcal{S}} |\Delta_s \eta_\varepsilon|^2 \mathrm{d}s + \varepsilon \int_0^t \int_{\mathcal{S}} |\Delta_s \partial_t \eta_\varepsilon|^2 \, \mathrm{d}s\mathrm{d}\tau \\ \leqslant C_3 \left(\left\| [u^0, \widehat{\eta}_1^0, \eta_2^0] \right\|_{\mathcal{I}^0}^2 + \|[f_2, h]\|_{\mathcal{R}_T^0}^2 + \int_0^t \int_{\mathcal{F}} |\partial_t u_\varepsilon|^2 \mathrm{d}y\mathrm{d}\tau \right). \quad (3.26)$$

Proof. We multiply the first equation of (3.17) by

$$-\Delta' u_\varepsilon := -\sum_{j=1}^{2} \partial_{y_j y_j} u_\varepsilon$$

and the first equation of (3.18) by $-\Delta_s \partial_t \eta_\varepsilon$ and we use Lemma 2.5:

$$\sum_{j=1}^{2} \int_0^t \int_{\mathcal{F}} \partial_{y_j} \mathbb{T}^0(u_\varepsilon) : \nabla(\partial_{y_j} u_\varepsilon) \mathrm{d}y\mathrm{d}\tau + \frac{1}{2} \left(\int_{\mathcal{S}} |\nabla_s \partial_t \eta_\varepsilon|^2 \mathrm{d}s + \int_{\mathcal{S}} |\Delta_s \eta_\varepsilon|^2 \mathrm{d}s \right) \\ + \varepsilon \int_0^t \int_{\mathcal{S}} |\Delta_s \partial_t \eta_\varepsilon|^2 \mathrm{d}s\mathrm{d}\tau = \int_0^t \int_{\mathcal{F}} \rho^0 \delta^0 \partial_t u_\varepsilon \cdot \Delta' u_\varepsilon \mathrm{d}y\mathrm{d}\tau - \int_0^t \int_{\mathcal{F}} f_2 \cdot \Delta' u_\varepsilon \mathrm{d}y\mathrm{d}\tau \\ - \sum_{j=1}^{2} \int_0^t \int_{\mathcal{F}} \partial_{y_j} h : \nabla(\partial_{y_j} u_\varepsilon) \mathrm{d}y\mathrm{d}\tau + \frac{1}{2} \left(\int_{\mathcal{S}} |\nabla_s \eta_2^0|^2 \mathrm{d}s + \int_{\mathcal{S}} |\Delta_s \widehat{\eta}_1^0|^2 \mathrm{d}s \right). \quad (3.27)$$

Using the expression (2.12) of $\mathbb{T}^0$, we obtain

$$\partial_{y_j} \mathbb{T}^0(u_\varepsilon) = \mathbb{T}^0(\partial_{y_j} u_\varepsilon) + \mathbb{M}^j$$

with

$$\mathbb{M}^j := \mu \nabla u_\varepsilon (\partial_{y_j} \mathbb{A}^0) + \mu \partial_{y_j} \left(\frac{\mathbb{B}^0}{\delta^0} \right) \nabla u_\varepsilon^\top \mathbb{B}^0 + \mu \frac{\mathbb{B}^0}{\delta^0} \nabla u_\varepsilon^\top (\partial_{y_j} \mathbb{B}^0) + \alpha (\nabla u_\varepsilon : \partial_{y_j} \mathbb{B}^0) \frac{\mathbb{B}^0}{\delta^0} + (\nabla u_\varepsilon : \mathbb{B}^0) \partial_{y_j} \left(\frac{\mathbb{B}^0}{\delta^0} \right).$$

From Lemma 2.3 and Lemma 2.4, we deduce that

$$\int_{\mathcal{F}} \partial_{y_j} \mathbb{T}^0(u_\varepsilon) : \nabla(\partial_{y_j} u_\varepsilon) \mathrm{d}y \geqslant C_0 \left\| \nabla(\partial_{y_j} u_\varepsilon) \right\|_{L^2(\mathcal{F})^9}^2 - C \left\| \nabla u_\varepsilon \right\|_{L^2(\mathcal{F})^9}^2 \quad (3.28)$$

for some constants $C_0, C > 0$. Moreover, there exists a constant $C > 0$ such that

$$\left| \int_0^t \int_{\mathcal{F}} \rho^0 \delta^0 \partial_t u_\varepsilon \cdot \Delta' u_\varepsilon \mathrm{d}y\mathrm{d}\tau - \int_0^t \int_{\mathcal{F}} f_2 \cdot \Delta' u_\varepsilon \mathrm{d}y\mathrm{d}\tau - \sum_{j=1}^{2} \int_0^t \int_{\mathcal{F}} \partial_{y_j} h : \nabla(\partial_{y_j} u_\varepsilon) \mathrm{d}y\mathrm{d}\tau \right| \\ \leqslant \frac{C_0}{2} \sum_{j=1}^{2} \int_0^t \int_{\mathcal{F}} |\nabla(\partial_{y_j} u_\varepsilon)|^2 \, \mathrm{d}y\mathrm{d}\tau + C \left(\int_0^t \int_{\mathcal{F}} |\partial_t u_\varepsilon|^2 \mathrm{d}y\mathrm{d}\tau + \|[f_2, h]\|_{\mathcal{R}_T^0}^2 \right). \quad (3.29)$$

Combining (3.27), (3.28), (3.29) and using (3.20), we deduce (3.26). □

Lemma 3.7. *Assume* (2.22), (2.23) *and* (3.15). *Let us consider the solution* $[u_\varepsilon, \eta_\varepsilon]$ *of the system* (3.17)–(3.18) *with regularity* (3.19). *Then there exist two constants* $C_4, C_5 > 0$, *independent of* ε *and* T, *such that, for any*

$t\in(0,T)$ *we have*

$$\int_0^t\int_{\mathcal{F}}|\partial_t u_\varepsilon|^2\ \mathrm{d}y\mathrm{d}\tau+\int_{\mathcal{F}}|\nabla u_\varepsilon(t,\cdot)|^2\ \mathrm{d}y+\int_0^t\int_{\mathcal{S}}|\partial_{tt}\eta_\varepsilon|^2\ \mathrm{d}s\mathrm{d}\tau+\varepsilon\int_{\mathcal{S}}|\nabla_s\partial_t\eta_\varepsilon(t,\cdot)|^2\ \mathrm{d}s$$
$$+\sum_{j=1}^2\int_0^t\int_{\mathcal{F}}|\nabla(\partial_{y_j}u_\varepsilon)|^2\ \mathrm{d}y\mathrm{d}\tau+\int_{\mathcal{S}}|\nabla_s\partial_t\eta_\varepsilon|^2\mathrm{d}s+\int_{\mathcal{S}}|\Delta_s\eta_\varepsilon|^2\mathrm{d}s+\varepsilon\int_0^t\int_{\mathcal{S}}|\Delta_s\partial_t\eta_\varepsilon|^2\ \mathrm{d}s\mathrm{d}\tau$$
$$\leqslant C_4e^{C_5T}\left(\left\|\left[u^0,\widehat{\eta}_1^0,\eta_2^0\right]\right\|_{\mathcal{I}^0}^2+\|[f_2,h]\|_{\mathcal{R}_T^0}^2\right). \tag{3.30}$$

Proof. We multiply (3.23) and (3.26) by respectively $a=1/(2C_3)$ and by $b=a^2$ and we add them:

$$a\int_0^t\int_{\mathcal{F}}|\partial_t u_\varepsilon|^2\ \mathrm{d}y\mathrm{d}\tau+a\int_{\mathcal{F}}|\nabla u_\varepsilon(t,\cdot)|^2\ \mathrm{d}y+a\int_0^t\int_{\mathcal{S}}|\partial_{tt}\eta_\varepsilon|^2\ \mathrm{d}s\mathrm{d}\tau+a\varepsilon\int_{\mathcal{S}}|\nabla_s\partial_t\eta_\varepsilon(t,\cdot)|^2\ \mathrm{d}s$$
$$+b\sum_{j=1}^2\int_0^t\int_{\mathcal{F}}|\nabla(\partial_{y_j}u_\varepsilon)|^2\ \mathrm{d}y\mathrm{d}\tau+b\int_{\mathcal{S}}|\nabla_s\partial_t\eta_\varepsilon|^2\mathrm{d}s+b\int_{\mathcal{S}}|\Delta_s\eta_\varepsilon|^2\mathrm{d}s+b\varepsilon\int_0^t\int_{\mathcal{S}}|\Delta_s\partial_t\eta_\varepsilon|^2\ \mathrm{d}s\mathrm{d}\tau$$
$$\leqslant(aC_2+bC_3)\left(\left\|\left[u^0,\widehat{\eta}_1^0,\eta_2^0\right]\right\|_{\mathcal{I}^0}^2+\|[f_2,h]\|_{\mathcal{R}_T^0}^2\right)+aC_2\int_0^t\int_{\mathcal{S}}|\nabla_s\partial_t\eta_\varepsilon|^2\ \mathrm{d}s\mathrm{d}\tau$$
$$+\frac{b}{4}\int_{\mathcal{S}}|\Delta_s\eta_\varepsilon(t,\cdot)|^2\ \mathrm{d}y+C_2^2\int_{\mathcal{S}}|\partial_t\eta_\varepsilon(t,\cdot)|^2\ \mathrm{d}s+\frac{a}{2}\int_0^t\int_{\mathcal{F}}|\partial_t u_\varepsilon|^2\mathrm{d}y\mathrm{d}\tau. \tag{3.31}$$

Combining this with (3.20) and with Grönwall's inequality, we deduce (3.30) □

We are now in a position to prove Proposition 3.2.

Proof of Proposition 3.2. Using (3.20) and (3.30), we can pass to the limit as ε to zero in (3.17)–(3.18). We obtain a solution $[u,\eta]$ of (3.2)–(3.3) with

$$u\in L^\infty(0,T;H^1(\mathcal{F}))^3\cap H^1(0,T;L^2(\mathcal{F}))^3,\quad \partial_{y_iy_j}u\in L^2(0,T;L^2(\mathcal{F}))^3 \text{ if } (i,j)\neq(3,3), \tag{3.32}$$
$$\eta\in L^\infty(0,T;H^2(\mathcal{S}))\cap W^{1,\infty}(0,T;H^1(\mathcal{S}))\cap H^2(0,T;L^2(\mathcal{S})), \tag{3.33}$$

It remains to show that $\partial_{y_3y_3}u\in L^2(0,T;L^2(\mathcal{F}))^3$. From (3.2) and (2.12) we obtain

$$-\mathbb{D}\partial_{y_3y_3}u=f_2+\operatorname{div}h-\rho^0\delta^0\partial_t u+R, \tag{3.34}$$

where

$$\mathbb{D}_{i,j}=\mu\mathbb{A}_{3,3}^0\delta_{i,j}+\frac{\mu+\alpha}{\delta_0}\mathbb{B}_{i,3}^0\mathbb{B}_{j,3}^0=\frac{\mu}{\delta_0}\sum_k\left(\mathbb{B}_{k,3}^0\right)^2\delta_{i,j}+\frac{\mu+\alpha}{\delta_0}\mathbb{B}_{i,3}^0\mathbb{B}_{j,3}^0, \tag{3.35}$$

and where R is a linear combination of terms of the form

$$\frac{\partial^2 u_{i_1}}{\partial y_{i_2}\partial y_{i_3}}\frac{1}{\delta^0}\mathbb{B}_{i_4,i_5}^0\mathbb{B}_{i_6,i_7}^0,\quad (i_2,i_3)\neq(3,3),\quad\text{and}\quad \frac{\partial u_{i_1}}{\partial y_{i_2}}\frac{\partial}{\partial y_{i_3}}\left(\frac{1}{\delta^0}\mathbb{B}_{i_4,i_5}^0\mathbb{B}_{i_6,i_7}^0\right). \tag{3.36}$$

From (1.4) and Lemma 2.3, we deduce that $\mathbb{D}\in L^\infty(\mathcal{F})^9$ is a symmetric positive matrix. Moreover, using that $\mathbb{B}_{3,3}^0=1$, we deduce from (3.35) that $\mathbb{D}^{-1}\in L^\infty(\mathcal{F})^9$ and using (3.32), (3.33), we conclude the proof of Proposition 3.2. □

3.2. **Proof of Theorem 3.1.** In order to prove Theorem 3.1, we first focus on the system (3.2)–(3.3) and extend Proposition 3.2.

Proof of Theorem 3.1. Comparing (3.4), (3.6) with (3.11), (3.12), we can apply Proposition 3.2 to obtain a solution $[u,\eta]\in\mathcal{E}_u^0\times\mathcal{E}_\eta^0$ of (3.2)–(3.3), with the estimate

$$\|[u,\eta]\|_{\mathcal{E}_u^0\times\mathcal{E}_\eta^0}\leqslant Ce^{CT}\left(\left\|\left[\widehat{\rho}^0,u^0,\widehat{\eta}_1^0,\eta_2^0\right]\right\|_{\mathcal{I}}+\|[f_1,f_2,h]\|_{\mathcal{R}_T}\right). \tag{3.37}$$

Now we are going to differentiate (3.2)–(3.3) in time and in space to deduce higher regularity results. To preserve the boundary conditions, we only differentiate in the tangential directions, that with respect to y_j, $j<3$. We then use the system to recover the regularity in the e_3 direction. The last step consists in solving (3.1).

Step 1: Differentiating (3.2)–(3.3) with respect to time, we deduce that

$$\overline{u} = \partial_t u, \qquad \overline{\eta} = \partial_t \eta$$

satisfy

$$\begin{cases} (\rho^0\delta^0)\partial_t \overline{u} - \operatorname{div} \mathbb{T}^0(\overline{u}) = \overline{f}_2 + \operatorname{div} \overline{h} & \text{in } (0,T)\times\mathcal{F}, \\ \overline{u} = \mathcal{T}(\partial_t \overline{\eta}) & \text{on } (0,T)\times\partial\mathcal{F}, \\ \overline{u}(0,\cdot) = \overline{u}^0 & \text{in } \mathcal{F}, \end{cases} \tag{3.38}$$

$$\begin{cases} \partial_{tt}\overline{\eta} - \Delta_s \overline{\eta} = -\mathcal{Q}(\mathbb{T}^0(\overline{u}) + \overline{h}) & \text{in } (0,T)\times\mathcal{S}, \\ \overline{\eta}(0,\cdot) = \overline{\eta}_1^0, \quad \partial_t\overline{\eta}(0,\cdot) = \overline{\eta}_2^0 & \text{in } \mathcal{S}, \end{cases} \tag{3.39}$$

where

$$\overline{f}_2 = \partial_t f_2, \quad \overline{h} = \partial_t h, \quad \overline{u}^0 = \frac{1}{\rho^0\delta^0}\left(\operatorname{div}\mathbb{T}^0(u^0) + f_2(0,\cdot) + \operatorname{div} h(0,\cdot)\right),$$

$$\overline{\eta}_1^0 = \eta_2^0, \quad \overline{\eta}_2^0 = \Delta_s \widehat{\eta}_1^0 - \mathbb{T}^0(u^0)e_3\cdot e_3 - h(0,\cdot)e_3\cdot e_3.$$

From (2.23) and Lemma 2.3, we have $1/(\rho^0\delta^0) \in H^3(\mathcal{F})$. Thus from (3.4), (3.6) and (3.9), we deduce that

$$\left[\overline{f}_2, \overline{h}\right] \in \mathcal{R}_T^0 \qquad \left[\overline{u}^0, \overline{\eta}_1^0, \overline{\eta}_2^0\right] \in \mathcal{I}^0,$$

and

$$\left\| \left[\overline{f}_2, \overline{h}\right] \right\|_{\mathcal{R}_T^0} + \left\| \left[\overline{u}^0, \overline{\eta}_1^0, \overline{\eta}_2^0\right] \right\|_{\mathcal{I}^0} \leqslant Ce^{CT}\left(\left\| \left[\widehat{\rho}^0, u^0, \widehat{\eta}_1^0, \eta_2^0\right] \right\|_{\mathcal{I}} + \|[f_1, f_2, h]\|_{\mathcal{R}_T} \right). \tag{3.40}$$

Therefore, by Proposition 3.2, $[\partial_t u, \partial_t \eta] \in \mathcal{E}_u^0 \times \mathcal{E}_\eta^0$, with

$$\|[\partial_t u, \partial_t \eta]\|_{\mathcal{E}_u^0\times\mathcal{E}_\eta^0} \leqslant Ce^{CT}\left(\left\| \left[\widehat{\rho}^0, u^0, \widehat{\eta}_1^0, \eta_2^0\right] \right\|_{\mathcal{I}} + \|[f_1, f_2, h]\|_{\mathcal{R}_T} \right). \tag{3.41}$$

Step 2: Differentiating (3.2)–(3.3) with respect to y_j, $j = 1, 2$, we deduce that

$$\widetilde{u} = \partial_{y_j} u, \quad \widetilde{\eta} = \partial_{y_j}\eta,$$

satisfy

$$\begin{cases} (\rho^0\delta^0)\partial_t \widetilde{u} - \operatorname{div} \mathbb{T}^0(\widetilde{u}) = \widetilde{f}_2 + \operatorname{div} \widetilde{h} & \text{in } (0,T)\times\mathcal{F}, \\ \widetilde{u} = \mathcal{T}(\partial_t \widetilde{\eta}) & \text{on } (0,T)\times\partial\mathcal{F}, \\ \widetilde{u}(0,\cdot) = \widetilde{u}^0 & \text{in } \mathcal{F}, \end{cases} \tag{3.42}$$

$$\begin{cases} \partial_{tt}\widetilde{\eta} - \Delta_s \widetilde{\eta} = -\mathcal{Q}(\mathbb{T}^0(\widetilde{u}) + \widetilde{h}) & \text{in } (0,T)\times\mathcal{S}, \\ \widetilde{\eta}(0,\cdot) = \widetilde{\eta}_1^0, \quad \partial_t\widetilde{\eta}(0,\cdot) = \widetilde{\eta}_2^0 & \text{in } \mathcal{S}, \end{cases} \tag{3.43}$$

where

$$\widetilde{f}_2 = \partial_{y_j} f_2 - \partial_{y_j}(\rho^0\delta^0)\partial_t u,$$

$$\widetilde{h} = \partial_{y_j} h + \mu\nabla u(\partial_{y_j}\mathbb{A}^0) + \mu\partial_{y_j}\left(\frac{\mathbb{B}^0}{\delta^0}\right)\nabla u^\top \mathbb{B}^0 + \mu\frac{\mathbb{B}^0}{\delta^0}\nabla u^\top(\partial_{y_j}\mathbb{B}^0)$$
$$+ \alpha(\nabla u : \partial_{y_j}\mathbb{B}^0)\frac{\mathbb{B}^0}{\delta^0} + \alpha(\nabla u : \mathbb{B}^0)\partial_{y_j}\left(\frac{\mathbb{B}^0}{\delta^0}\right),$$

$$\widetilde{u}^0 = \partial_{y_j} u^0, \quad \widetilde{\eta}_1^0 = \partial_{y_j}\widehat{\eta}_1^0, \quad \widetilde{\eta}_2^0 = \partial_{y_j}\eta_2^0.$$

From (3.4), we obtain $\left[\widetilde{u}^0, \widetilde{\eta}_1^0, \widetilde{\eta}_2^0\right] \in \mathcal{I}^0$. On the other hand, from (3.6), (2.23), (3.37), Lemma 2.3 and Lemma 2.6, we deduce $\left[\widetilde{f}_2, \widetilde{h}\right] \in \mathcal{R}_T^0$, and there exists a constant $C > 0$ such that

$$\left\| \left[\widetilde{f}_2, \widetilde{h}\right] \right\|_{\mathcal{R}_T^0} + \left\| \left[\widetilde{u}^0, \widetilde{\eta}_1^0, \widetilde{\eta}_2^0\right] \right\|_{\mathcal{I}^0} \leqslant C\left(\left\| \left[\widehat{\rho}^0, u^0, \widehat{\eta}_1^0, \eta_2^0\right] \right\|_{\mathcal{I}} + \|[f_1, f_2, h]\|_{\mathcal{R}_T} \right). \tag{3.44}$$

Therefore, by Proposition 3.2, $\left[\partial_{y_j} u, \partial_{y_j}\eta\right] \in \mathcal{E}_u^0 \times \mathcal{E}_\eta^0$, $j = 1, 2$, and

$$\left\| \left[\partial_{y_j} u, \partial_{y_j}\eta\right] \right\|_{\mathcal{E}_u^0\times\mathcal{E}_\eta^0} \leqslant Ce^{CT}\left(\left\| \left[\widehat{\rho}^0, u^0, \widehat{\eta}_1^0, \eta_2^0\right] \right\|_{\mathcal{I}} + \|[f_1, f_2, h]\|_{\mathcal{R}_T} \right), \quad j = 1, 2. \tag{3.45}$$

Step 3: Differentiating (3.2)–(3.3) with respect to y_j and to y_k, $j,k\in\{1,2\}$, we deduce that

$$\check{u}=\partial_{y_jy_k}u,\quad \check{\eta}=\partial_{y_jy_k}\eta,$$

satisfy

$$\begin{cases}(\rho^0\delta^0)\partial_t\check{u}-\operatorname{div}\mathbb{T}^0(\check{u})=\check{f}_2+\operatorname{div}\check{h} & \text{in } (0,T)\times\mathcal{F},\\ \check{u}=\mathcal{T}(\partial_t\check{\eta}) & \text{on } (0,T)\times\partial\mathcal{F},\\ \check{u}(0,\cdot)=\check{u}^0 & \text{in } \mathcal{F},\end{cases}\tag{3.46}$$

$$\begin{cases}\partial_{tt}\check{\eta}-\Delta_s\check{\eta}=-\mathcal{Q}(\mathbb{T}^0(\check{u})+\check{h}) & \text{in } (0,T)\times\mathcal{S},\\ \check{\eta}(0,\cdot)=\check{\eta}_1^0,\quad \partial_t\check{\eta}(0,\cdot)=\check{\eta}_2^0 & \text{in } \mathcal{S},\end{cases}\tag{3.47}$$

where

$$\check{f}_2=\partial_{y_jy_k}f_2-\partial_{y_jy_k}(\rho^0\delta^0)\partial_tu-\partial_{y_j}(\rho^0\delta^0)\partial_t(\partial_{y_k}u)-\partial_{y_k}(\rho^0\delta^0)\partial_t(\partial_{y_j}u),$$

and where $\check{h}$ is the sum of $\partial_{y_jy_k}h$ and of a linear combination of terms of the form

$$\frac{\partial^2u_{i_1}}{\partial y_{i_2}\partial y_{i_3}}\frac{\partial}{\partial y_{i_4}}\left(\frac{1}{\delta^0}\right)\mathbb{B}^0_{i_5,i_6}\mathbb{B}^0_{i_7,i_8},\quad \frac{\partial^2u_{i_1}}{\partial y_{i_2}\partial y_{i_3}}\frac{1}{\delta^0}\frac{\partial\mathbb{B}^0_{i_5,i_6}}{\partial y_{i_4}}\mathbb{B}^0_{i_7,i_8},\quad (i_2,i_3)\neq(3,3),\tag{3.48}$$

$$\frac{\partial u_{i_1}}{\partial y_{i_2}}\frac{\partial^2}{\partial y_{i_3}\partial y_{i_4}}\left(\frac{1}{\delta^0}\right)\mathbb{B}^0_{i_5,i_6}\mathbb{B}^0_{i_7,i_8},\quad \frac{\partial u_{i_1}}{\partial y_{i_2}}\frac{\partial}{\partial y_{i_3}}\left(\frac{1}{\delta^0}\right)\frac{\partial\mathbb{B}^0_{i_5,i_6}}{\partial y_{i_4}}\mathbb{B}^0_{i_7,i_8},\tag{3.49}$$

and

$$\frac{\partial u_{i_1}}{\partial y_{i_2}}\frac{1}{\delta^0}\frac{\partial^2\mathbb{B}^0_{i_5,i_6}}{\partial y_{i_3}\partial y_{i_4}}\mathbb{B}^0_{i_7,i_8},\quad \frac{\partial u_{i_1}}{\partial y_{i_2}}\frac{1}{\delta^0}\frac{\partial\mathbb{B}^0_{i_5,i_6}}{\partial y_{i_3}}\frac{\partial\mathbb{B}^0_{i_7,i_8}}{\partial y_{i_4}}.\tag{3.50}$$

The initial conditions are given by

$$\check{u}^0=\partial_{y_jy_k}u^0,\quad \check{\eta}_1^0=\partial_{y_jy_k}\widehat{\eta}_1^0,\quad \check{\eta}_2^0=\partial_{y_jy_k}\eta_2^0.$$

From (3.4), we obtain $\left[\check{u}^0,\check{\eta}_1^0,\check{\eta}_2^0\right]\in\mathcal{I}^0$. From (3.6), we have $\partial_{y_jy_k}f_2\in L^2(0,T;L^2(\mathcal{F}))^3$. From (2.22), (2.23), we have $\partial_{y_jy_k}(\rho^0\delta^0)\in H^1(\mathcal{F})$ and using (3.41), we deduce that $\partial_{y_jy_k}(\rho^0\delta^0)\partial_tu\in L^2(0,T;L^2(\mathcal{F}))^3$. Similarly, $\partial_{y_j}(\rho^0\delta^0)\partial_t(\partial_{y_k}u),\partial_{y_k}(\rho^0\delta^0)\partial_t(\partial_{y_j}u)\in L^2(0,T;L^2(\mathcal{F}))^3$ and thus $\check{f}_2\in L^2(0,T;L^2(\mathcal{F}))^3$.

In order to show that $\check{h}\in L^2(0,T;H^1(\mathcal{F}))^9$, we already notice that from (3.6), $\partial_{y_jy_k}h\in L^2(0,T;H^1(\mathcal{F}))^9$. Then we need to show that the gradients of the terms of the form (3.48)–(3.50) are in $L^2(0,T;L^2(\mathcal{F}))^9$. This can be done in a systematic way, we only point out here the more complicated terms.

First, from (3.45), we deduce that

$$\nabla u\in L^2(0,T;L^2(0,1;H^2(\mathcal{S})))^9\cap L^2(0,T;H^1(0,1;H^1(\mathcal{S})))^9.$$

Moreover, from Lemma 2.3, we have that

$$\nabla^2\mathbb{B}^0\in C^\infty([0,1];H^1(\mathcal{S}))^{81}.$$

Thus using Lemma 2.6, we deduce that

$$\frac{\partial^2u_{i_1}}{\partial y_{i_2}\partial y_{i_3}}\frac{1}{\delta^0}\frac{\partial^2\mathbb{B}^0_{i_5,i_6}}{\partial y_{i_4}\partial y_{i_9}}\mathbb{B}^0_{i_7,i_8},\quad \frac{\partial u_{i_1}}{\partial y_{i_2}}\frac{1}{\delta^0}\frac{\partial^3\mathbb{B}^0_{i_5,i_6}}{\partial y_{i_3}\partial y_{i_4}\partial y_{i_9}}\mathbb{B}^0_{i_7,i_8}\in L^2(0,T;L^2(\mathcal{F})).$$

The other terms involved in the derivatives of $\check{h}$ are of the form

$$\frac{\partial^3u_{i_1}}{\partial y_{i_2}\partial y_{i_3}\partial y_{i_9}}\frac{\partial}{\partial y_{i_4}}\left(\frac{1}{\delta^0}\right)\mathbb{B}^0_{i_5,i_6}\mathbb{B}^0_{i_7,i_8},\quad \frac{\partial^3u_{i_1}}{\partial y_{i_2}\partial y_{i_3}\partial y_{i_9}}\frac{1}{\delta^0}\frac{\partial\mathbb{B}^0_{i_5,i_6}}{\partial y_{i_4}}\mathbb{B}^0_{i_7,i_8},\quad (i_2,i_3)\neq(3,3),\tag{3.51}$$

$$\frac{\partial^2u_{i_1}}{\partial y_{i_2}\partial y_{i_3}}\frac{\partial^2}{\partial y_{i_4}\partial y_{i_9}}\left(\frac{1}{\delta^0}\right)\mathbb{B}^0_{i_5,i_6}\mathbb{B}^0_{i_7,i_8},\quad \frac{\partial^2u_{i_1}}{\partial y_{i_2}\partial y_{i_3}}\frac{\partial}{\partial y_{i_9}}\left(\frac{1}{\delta^0}\right)\frac{\partial\mathbb{B}^0_{i_5,i_6}}{\partial y_{i_4}}\mathbb{B}^0_{i_7,i_8},\tag{3.52}$$

$$\frac{\partial^2u_{i_1}}{\partial y_{i_2}\partial y_{i_3}}\frac{1}{\delta^0}\frac{\partial\mathbb{B}^0_{i_5,i_6}}{\partial y_{i_4}}\frac{\partial\mathbb{B}^0_{i_7,i_8}}{\partial y_{i_9}},\quad \frac{\partial^2u_{i_1}}{\partial y_{i_2}\partial y_{i_3}}\frac{1}{\delta^0}\frac{\partial^2\mathbb{B}^0_{i_5,i_6}}{\partial y_{i_4}\partial y_{i_9}}\mathbb{B}^0_{i_7,i_8},\tag{3.53}$$

$$\frac{\partial u_{i_1}}{\partial y_{i_2}}\frac{\partial^3}{\partial y_{i_3}\partial y_{i_4}\partial y_{i_9}}\left(\frac{1}{\delta^0}\right)\mathbb{B}^0_{i_5,i_6}\mathbb{B}^0_{i_7,i_8},\quad \frac{\partial u_{i_1}}{\partial y_{i_2}}\frac{\partial^2}{\partial y_{i_3}\partial y_{i_4}}\left(\frac{1}{\delta^0}\right)\frac{\partial\mathbb{B}^0_{i_5,i_6}}{\partial y_{i_9}}\mathbb{B}^0_{i_7,i_8},\tag{3.54}$$

$$\frac{\partial u_{i_1}}{\partial y_{i_2}}\frac{\partial}{\partial y_{i_3}}\left(\frac{1}{\delta^0}\right)\frac{\partial^2 \mathbb{B}^0_{i_5,i_6}}{\partial y_{i_4}\partial y_{i_9}}\mathbb{B}^0_{i_7,i_8},\quad \frac{\partial u_{i_1}}{\partial y_{i_2}}\frac{\partial}{\partial y_{i_3}}\left(\frac{1}{\delta^0}\right)\frac{\partial \mathbb{B}^0_{i_5,i_6}}{\partial y_{i_4}}\frac{\partial \mathbb{B}^0_{i_7,i_8}}{\partial y_{i_9}}, \tag{3.55}$$

and

$$\frac{\partial u_{i_1}}{\partial y_{i_2}}\frac{1}{\delta^0}\frac{\partial^3 \mathbb{B}^0_{i_5,i_6}}{\partial y_{i_3}\partial y_{i_4}\partial y_{i_9}}\mathbb{B}^0_{i_7,i_8},\quad \frac{\partial u_{i_1}}{\partial y_{i_2}}\frac{1}{\delta^0}\frac{\partial^2 \mathbb{B}^0_{i_5,i_6}}{\partial y_{i_3}\partial y_{i_4}}\frac{\partial \mathbb{B}^0_{i_7,i_8}}{\partial y_{i_9}}. \tag{3.56}$$

Using (3.37), (3.45), Lemma 2.3 and Lemma 2.6, one can check that they belong to $L^2(0,T;L^2(\mathcal{F}))$ and

$$\left\|[\check{f}_2,\check{h}]\right\|_{\mathcal{R}^0_T}+\left\|[\check{u}^0,\check{\eta}^0_1,\check{\eta}^0_2]\right\|_{\mathcal{I}^0}\leqslant C\left(\left\|[\widehat{\rho}^0,u^0,\widehat{\eta}^0_1,\eta^0_2]\right\|_{\mathcal{I}}+\|[f_1,f_2,h]\|_{\mathcal{R}_T}\right). \tag{3.57}$$

Therefore, by Proposition 3.2, $[\partial_{y_jy_k}u,\partial_{y_jy_k}\eta]\in\mathcal{E}^0_u\times\mathcal{E}^0_\eta$, $j,k\in\{1,2\}$ with the estimate

$$\left\|[\partial_{y_jy_k}u,\partial_{y_jy_k}\eta]\right\|_{\mathcal{E}^0_u\times\mathcal{E}^0_\eta}\leqslant C\left(\left\|[\widehat{\rho}^0,u^0,\widehat{\eta}^0_1,\eta^0_2]\right\|_{\mathcal{I}}+\|[f_1,f_2,h]\|_{\mathcal{R}_T}\right),\quad (j,k\in\{1,2\}). \tag{3.58}$$

<u>*Step 4:*</u> regularity in the e_3 direction. From (3.37), (3.41), (3.45) and (3.58), it only remains (see (2.17), (2.18)) to show that

$$\partial_{y_3y_3y_3}u,\quad \partial_{y_1y_3y_3y_3}u,\quad \partial_{y_2y_3y_3y_3}u,\quad \partial_{y_3y_3y_3y_3}u\in L^2(0,T;L^2(\mathcal{F}))^3.$$

We are going to use (3.34), (3.35) and (3.36) and combine it with the fact that we have already (using Step 1 and (2.22), (2.23))

$$f_2+\operatorname{div}h-\rho^0\delta^0\partial_t u\in L^2(0,T;H^2(\mathcal{F}))^9.$$

Differentiating (3.34) with respect to y_3 yields

$$-\mathbb{D}\partial_{y_3y_3y_3}u=\partial_{y_3}\left(f_2+\operatorname{div}h-\rho^0u^0\partial_t u\right)+\partial_{y_3}R+\left(\partial_{y_3}\mathbb{D}\right)\partial_{y_3y_3}u, \tag{3.59}$$

We can check that $\partial_{y_3}R+\left(\partial_{y_3}\mathbb{D}\right)\partial_{y_3y_3}u$ is a linear combination of terms of the form

$$\frac{\partial^3 u_{i_1}}{\partial y_{i_2}\partial y_{i_3}\partial y_{i_8}}\frac{1}{\delta^0}\mathbb{B}^0_{i_4,i_5}\mathbb{B}^0_{i_6,i_7},\quad (i_2,i_3)\neq(3,3), \tag{3.60}$$

$$\frac{\partial^2 u_{i_1}}{\partial y_{i_2}\partial y_{i_3}}\frac{\partial}{\partial y_{i_8}}\left(\frac{1}{\delta^0}\right)\mathbb{B}^0_{i_4,i_5}\mathbb{B}^0_{i_6,i_7},\quad \frac{\partial^2 u_{i_1}}{\partial y_{i_2}\partial y_{i_3}}\frac{1}{\delta^0}\frac{\partial \mathbb{B}^0_{i_4,i_5}}{\partial y_{i_8}}\mathbb{B}^0_{i_6,i_7}. \tag{3.61}$$

$$\frac{\partial u_{i_1}}{\partial y_{i_2}}\frac{\partial}{\partial y_{i_3}\partial y_{i_8}}\left(\frac{1}{\delta^0}\right)\mathbb{B}^0_{i_4,i_5}\mathbb{B}^0_{i_6,i_7},\quad \frac{\partial u_{i_1}}{\partial y_{i_2}}\frac{\partial}{\partial y_{i_8}}\left(\frac{1}{\delta^0}\right)\frac{\partial \mathbb{B}^0_{i_4,i_5}}{\partial y_{i_3}}\mathbb{B}^0_{i_6,i_7}, \tag{3.62}$$

$$\frac{\partial u_{i_1}}{\partial y_{i_2}}\frac{1}{\delta^0}\frac{\partial^2 \mathbb{B}^0_{i_4,i_5}}{\partial y_{i_3}\partial y_{i_8}}\mathbb{B}^0_{i_6,i_7},\quad \frac{\partial u_{i_1}}{\partial y_{i_2}}\frac{1}{\delta^0}\frac{\partial \mathbb{B}^0_{i_4,i_5}}{\partial y_{i_3}}\frac{\partial \mathbb{B}^0_{i_6,i_7}}{\partial y_{i_8}}. \tag{3.63}$$

As in Step 3, one can check that all the above terms are in $L^2(0,T;L^2(\mathcal{F}))$ and thus, since $\mathbb{D}^{-1}\in L^\infty(\mathcal{F})^9$, we deduce that $\partial_{y_3y_3y_3}u\in L^2(0,T;L^2(\mathcal{F}))^3$. Then we differentiate (3.59) with respect to y_k, $k=1,2$:

$$-\mathbb{D}\partial_{y_3y_3y_3y_k}u=\partial_{y_3y_k}\left(f_2+\operatorname{div}h-\rho^0u^0\partial_t u\right)+\partial_{y_k}\left(\partial_{y_3}R+\left(\partial_{y_3}\mathbb{D}\right)\partial_{y_3y_3}u\right)+\left(\partial_{y_k}\mathbb{D}\right)\partial_{y_3y_3y_3}u. \tag{3.64}$$

The last three terms of (3.64) are the linear combination of terms in (3.51)–(3.56) and of terms of the form

$$\frac{\partial^4 u_{i_1}}{\partial y_{i_2}\partial y_{i_3}\partial y_{i_8}\partial y_k}\frac{1}{\delta^0}\mathbb{B}^0_{i_4,i_5}\mathbb{B}^0_{i_6,i_7},\quad (i_2,i_3)\neq(3,3),k\neq 3, \tag{3.65}$$

$$\frac{\partial^3 u_{i_1}}{\partial y_3\partial y_3\partial y_{i_9}}\frac{\partial}{\partial y_{i_4}}\left(\frac{1}{\delta^0}\right)\mathbb{B}^0_{i_5,i_6}\mathbb{B}^0_{i_7,i_8},\quad \frac{\partial^3 u_{i_1}}{\partial y_3\partial y_3\partial y_{i_9}}\frac{1}{\delta^0}\frac{\partial \mathbb{B}^0_{i_5,i_6}}{\partial y_{i_4}}\mathbb{B}^0_{i_7,i_8}. \tag{3.66}$$

Using (3.45) and that $\partial_{y_3y_3y_3}u\in L^2(0,T;L^2(\mathcal{F}))^3$, we deduce that the above terms are also in $L^2(0,T;L^2(\mathcal{F}))$. Thus, $\partial_{y_3y_3y_3y_k}u\in L^2(0,T;L^2(\mathcal{F}))^3$, for $k=1,2$.

Finally, we differentiate (3.59) with respect to y_3:

$$-\mathbb{D}\partial_{y_3y_3y_3y_3}u=\partial_{y_3y_3}\left(f_2+\operatorname{div}h-\rho^0u^0\partial_t u\right)+\partial_{y_3}\left(\partial_{y_3}R+\left(\partial_{y_3}\mathbb{D}\right)\partial_{y_3y_3}u\right)+\left(\partial_{y_3}\mathbb{D}\right)\partial_{y_3y_3y_3}u. \tag{3.67}$$

The last three terms of (3.67) are the linear combination of terms in (3.51)–(3.56), (3.65)–(3.66) and of terms of the form

$$\frac{\partial^4 u_{i_1}}{\partial y_3\partial y_3\partial y_3\partial y_k}\frac{1}{\delta^0}\mathbb{B}^0_{i_4,i_5}\mathbb{B}^0_{i_6,i_7},\quad k\neq 3, \tag{3.68}$$

that is in $L^2(0,T;L^2(\mathcal{F}))$ since $\partial_{y_3y_3y_3y_k}u \in L^2(0,T;L^2(\mathcal{F}))^3$, for $k=1,2$. We deduce that $\partial_{y_3y_3y_3y_3}u \in L^2(0,T;L^2(\mathcal{F}))^3$.

Step 5: solving (3.1). From (2.23), Lemma 2.3 and the above steps, we find that $\dfrac{\rho^0}{\delta^0}\nabla u : \mathbb{B}^0 \in L^2(0,T;H^3(\mathcal{F}))\cap L^\infty(0,T;H^2(\mathcal{F}))$. This yields that $\rho \in \mathcal{E}_\rho$ and this ends the proof of Theorem 3.1. □

4. Proof of Theorem 2.1

In this section, we prove Theorem 2.1, that is the local existence and uniqueness of strong solutions for the system (2.8)-(2.15). The method is standard, we use a fixed-point argument by noticing that a solution of the system (2.8)-(2.15) is a solution of (3.1)-(3.3) with the source terms

$$[f_1,f_2,h]=[F_1,F_2,H],$$

where the nonlinearities F_1,F_2,H are given by (2.13)-(2.14).

Let us assume that the initial condition $\left[\rho^0,u^0,\eta_1^0,\eta_2^0\right]$ satisfies (2.22)-(2.25). Note that this implies in particular that $\left[\rho^0,u^0,\eta_1^0,\eta_2^0\right]\in\mathcal{I}$ (see (3.4)). For all $T>0$ and $R>0$, let us consider the following closed subset of $\mathcal{R}_T$ (see (3.6)):

$$\mathcal{B}_{T,R}=\Big\{[f_1,f_2,h]\in\mathcal{R}_T\ ;\ f_2(0,\cdot)=0,\quad h(0,\cdot)=-a(\rho^0)^\gamma\mathbb{B}^0,\quad \|[f_1,f_2,h]\|_{\mathcal{R}_T}\leqslant R\Big\}. \tag{4.1}$$

From Lemma 2.3 and (2.23), there exists $R>0$ such that

$$\left\|a(\rho^0)^\gamma\mathbb{B}^0\right\|_{H^3(\mathcal{F})^9}\leqslant R, \tag{4.2}$$

and thus if $T\leqslant 1$, then $\mathcal{B}_{T,R}$ is non empty. We also take R such that

$$\left\|\left[\rho^0,u^0,\eta_1^0,\eta_2^0\right]\right\|_{\mathcal{I}}\leqslant R. \tag{4.3}$$

With R satisfying (4.2)-(4.3), we define the map

$$\mathcal{N}:\mathcal{B}_{T,R}\to\mathcal{B}_{T,R},\quad [f_1,f_2,h]\to[F_1,F_2,H],$$

where $[\rho,u,\eta]$ is the solution to the system (3.1)-(3.3) associated with the source term $[f_1,f_2,h]$ and with the initial condition $\left[\rho^0,u^0,\eta_1^0,\eta_2^0\right]$ and where F_1,F_2,H are given by (2.13)-(2.14). In order to prove Theorem 2.1, we show below that, for T small enough, $\mathcal{N}$ is well-defined, $\mathcal{N}(\mathcal{B}_{T,R})\subset\mathcal{B}_{T,R}$ and $\mathcal{N}|_{\mathcal{B}_{T,R}}$ is a strict contraction.

First, note that (2.25) and (4.1) yield the compatibility condition (3.9) with $\widehat{\rho}^0=\rho^0$, $\widehat{\eta}_1^0=\eta_1^0$. Thus, we can apply Theorem 3.1 to conclude that the system (3.1)-(3.3) admits a unique solution $[\rho,u,\eta]\in\mathcal{E}_\rho\times\mathcal{E}_u\times\mathcal{E}_\eta$ and there exists a constant $C>0$ independent of T such that

$$\|[\rho,u,\eta]\|_{\mathcal{E}_\rho\times\mathcal{E}_u\times\mathcal{E}_\eta}\leqslant Ce^{CT}\left(\left\|\left[\rho^0,u^0,\eta_1^0,\eta_2^0\right]\right\|_{\mathcal{I}}+\|[f_1,f_2,h]\|_{\mathcal{R}_T}\right).$$

In all what follows, we assume to simplify that $T\leqslant 1$ and the constants C used in the estimate can depend on R and on the initial conditions. For instance using the above estimate, and (4.1), (4.2), (4.3) we deduce

$$\begin{aligned}\|\rho\|_{H^1(0,T;H^3(\mathcal{F}))}+\|\partial_t\rho\|_{L^\infty(0,T;H^2(\mathcal{F}))}+\|u\|_{L^2(0,T;H^4(\mathcal{F}))^3}+\|u\|_{H^1(0,T;H^2(\mathcal{F}))^3}+\|u\|_{H^2(0,T;L^2(\mathcal{F}))^3}\\+\|u\|_{C^0([0,T];H^3(\mathcal{F}))^3}+\|u\|_{C^1([0,T];H^1(\mathcal{F}))^3}+\|\eta\|_{L^\infty(0,T;H^4(\mathcal{S}))}+\|\eta\|_{H^2(0,T;H^2(\mathcal{S}))}+\|\eta\|_{H^3(0,T;L^2(\mathcal{S}))}\\+\|\partial_t\eta\|_{L^\infty(0,T;H^3(\mathcal{S}))}+\|\partial_{tt}\eta\|_{C^0([0,T];H^1(\mathcal{S}))}\leqslant C.\end{aligned} \tag{4.4}$$

Since $\rho(0,\cdot)=\rho^0$, the above estimate implies

$$\|\rho-\rho^0\|_{L^\infty(0,T;H^3(\mathcal{F}))}\leqslant CT^{1/2}. \tag{4.5}$$

In particular, for T small enough, combining the above relation with condition (2.23), we deduce for any $\alpha\in\mathbb{R}$;

$$\|\rho^\alpha\|_{L^\infty(0,T;H^3(\mathcal{F}))}\leqslant C. \tag{4.6}$$

Combining (2.15), (4.4), (2.17) and Lemma 2.3 yields

$$\|X-X^0\|_{L^\infty(0,T;H^4(\mathcal{F}))^3}\leqslant CT^{1/2},\quad \|X\|_{L^\infty(0,T;H^4(\mathcal{F}))^3}+\|X\|_{H^1(0,T;H^4(\mathcal{F}))^3}\leqslant C. \tag{4.7}$$

We deduce from the above estimates that X is a C^1-diffeomorphism for T small enough. Similarly, $\mathbb{B}_X$ and δ_X defined by (2.5) satisfy

$$\|\mathbb{B}_X - \mathbb{B}^0\|_{L^\infty(0,T;H^3(\mathcal{F}))^9} \leqslant CT^{1/2}, \quad \|\mathbb{B}_X\|_{L^\infty(0,T;H^3(\mathcal{F}))^9} + \|\mathbb{B}_X\|_{H^1(0,T;H^3(\mathcal{F}))^9} \leqslant C, \tag{4.8}$$

$$\|\delta_X - \delta^0\|_{L^\infty(0,T;H^3(\mathcal{F}))} \leqslant CT^{1/2}, \quad \|\delta_X\|_{L^\infty(0,T;H^3(\mathcal{F}))} + \|\delta_X\|_{H^1(0,T;H^3(\mathcal{F}))} \leqslant C, \tag{4.9}$$

and, in particular, there exists c_0 depending on η_1^0 such that for T small enough

$$\delta_X \geqslant c_0 > 0.$$

Thus, we also have

$$\left\|\frac{1}{\delta_X} - \frac{1}{\delta^0}\right\|_{L^\infty(0,T;H^3(\mathcal{F}))} \leqslant CT^{1/2}, \quad \left\|\frac{1}{\delta_X}\right\|_{L^\infty(0,T;H^3(\mathcal{F}))} + \left\|\frac{1}{\delta_X}\right\|_{H^1(0,T;H^3(\mathcal{F}))} \leqslant C. \tag{4.10}$$

Using the above estimates and (2.5)-(2.6), we obtain

$$\|\mathbb{A}_X - \mathbb{A}^0\|_{L^\infty(0,T;H^3(\mathcal{F}))^9} \leqslant CT^{1/2}, \quad \|\mathbb{A}_X\|_{L^\infty(0,T;H^3(\mathcal{F}))^9} + \|\mathbb{A}_X\|_{H^1(0,T;H^3(\mathcal{F}))^9} \leqslant C. \tag{4.11}$$

Regarding the time derivatives, we deduce from (4.4), (2.17) the following estimates

$$\|\partial_t X\|_{L^\infty(0,T;H^3(\mathcal{F}))^3} \leqslant C, \quad \|\partial_t \mathbb{B}_X\|_{L^\infty(0,T;H^2(\mathcal{F}))^9} \leqslant C, \quad \|\partial_t \delta_X\|_{L^\infty(0,T;H^2(\mathcal{F}))} \leqslant C,$$
$$\|\partial_t \mathbb{A}_X\|_{L^\infty(0,T;H^2(\mathcal{F}))^9} \leqslant C, \quad \left\|\partial_t\left(\frac{1}{\delta_X}\right)\right\|_{L^\infty(0,T;H^2(\mathcal{F}))} \leqslant C. \tag{4.12}$$

Using the above results, we can now estimate $[F_1, F_2, H]$ given by (2.13)-(2.14) in the norm (3.7) (see, for instance [40, Proposition 2.4] for a similar computation):

$$\|F_1(\rho,u,\eta)\|_{L^2(0,T;H^3(\mathcal{F}))} \leqslant \left\|\frac{\rho^0}{\delta^0}\right\|_{L^\infty(0,T;H^3(\mathcal{F}))} \|\nabla u\|_{L^2(0,T;H^3(\mathcal{F}))^9} \|\mathbb{B}^0 - \mathbb{B}_X\|_{L^\infty(0,T;H^3(\mathcal{F}))^9}$$
$$+ \left\|\frac{\rho^0}{\delta^0} - \frac{\rho}{\delta_X}\right\|_{L^\infty(0,T;H^3(\mathcal{F}))} \|\nabla u\|_{L^2(0,T;H^3(\mathcal{F}))^9} \|\mathbb{B}_X\|_{L^\infty(0,T;H^3(\mathcal{F}))^9} \leqslant CT^{1/2}, \tag{4.13}$$

$$\|F_1(\rho,u,\eta)\|_{L^\infty(0,T;H^2(\mathcal{F}))} \leqslant \left\|\frac{\rho^0}{\delta^0}\right\|_{L^\infty(0,T;H^3(\mathcal{F}))} \|\nabla u\|_{L^\infty(0,T;H^2(\mathcal{F}))^9} \|\mathbb{B}^0 - \mathbb{B}_X\|_{L^\infty(0,T;H^3(\mathcal{F}))^9}$$
$$+ \left\|\frac{\rho^0}{\delta^0} - \frac{\rho}{\delta_X}\right\|_{L^\infty(0,T;H^3(\mathcal{F}))} \|\nabla u\|_{L^\infty(0,T;H^2(\mathcal{F}))^9} \|\mathbb{B}_X\|_{L^\infty(0,T;H^3(\mathcal{F}))^9} \leqslant CT^{1/2}, \tag{4.14}$$

$$\|F_2(\rho,u,\eta)\|_{L^2(0,T;H^2(\mathcal{F}))^3} \leqslant \left\|\rho^0\delta^0 - \rho\delta_X\right\|_{L^\infty(0,T;H^3(\mathcal{F}))} \|\partial_t u\|_{L^2(0,T;H^2(\mathcal{F}))^3} \leqslant CT^{1/2}, \tag{4.15}$$

$$\|\partial_t F_2\|_{L^2(0,T;L^2(\mathcal{F}))^3} \leqslant T^{1/2} \|\partial_t \rho\|_{L^\infty(0,T;H^2(\mathcal{F}))} \|\delta_X\|_{L^\infty(0,T;H^3(\mathcal{F}))} \|\partial_t u\|_{L^\infty(0,T;H^1(\mathcal{F}))^3}$$
$$+ T^{1/2} \|\rho\|_{L^\infty(0,T;H^3(\mathcal{F}))} \|\partial_t(\delta_X)\|_{L^\infty(0,T;H^2(\mathcal{F}))} \|\partial_t u\|_{L^\infty(0,T;H^1(\mathcal{F}))^3}$$
$$+ \|\rho^0\delta^0 - \rho\delta_X\|_{L^\infty(0,T;H^3(\mathcal{F}))} \|\partial_{tt} u\|_{L^2(0,T;L^2(\mathcal{F}))^3} \leqslant CT^{1/2}, \tag{4.16}$$

$$\|H\|_{L^2(0,T;H^3(\mathcal{F}))^9} \leqslant C\Big(\|\mathbb{A}_X - \mathbb{A}^0\|_{L^\infty(0,T;H^3(\mathcal{F}))^9} \|\nabla u\|_{L^2(0,T;H^3(\mathcal{F}))^9}$$
$$+ \left\|\frac{\mathbb{B}^0}{\delta^0} - \frac{\mathbb{B}_X}{\delta_X}\right\|_{L^\infty(0,T;H^3(\mathcal{F}))^9} \|\nabla u\|_{L^2(0,T;H^3(\mathcal{F}))^9} \|\mathbb{B}_X\|_{L^\infty(0,T;H^3(\mathcal{F}))^9}$$
$$+ \left\|\frac{\mathbb{B}^0}{\delta^0}\right\|_{L^\infty(0,T;H^3(\mathcal{F}))^9} \|\nabla u\|_{L^2(0,T;H^3(\mathcal{F}))^9} \|\mathbb{B}^0 - \mathbb{B}_X\|_{L^\infty(0,T;H^3(\mathcal{F}))^9}$$
$$+ T^{1/2} \|\rho^\gamma\|_{L^\infty(0,T;H^3(\mathcal{F}))} \|\mathbb{B}_X\|_{L^\infty(0,T;H^3(\mathcal{F}))^9}\Big) \leqslant CT^{1/2}, \tag{4.17}$$

$$
\begin{aligned}
&\|\partial_t H\|_{L^2(0,T;H^1(\mathcal{F}))^9} \\
&\quad \leqslant C\Big(\|\mathbb{A}_X - \mathbb{A}^0\|_{L^\infty(0,T;H^3(\mathcal{F}))^9} \|\partial_t \nabla u\|_{L^2(0,T;H^1(\mathcal{F}))^9} + T^{1/2} \|\nabla u\|_{L^\infty(0,T;H^2(\mathcal{F}))^9} \|\partial_t(\mathbb{A}_X)\|_{L^\infty(0,T;H^2(\mathcal{F}))^9} \\
&\qquad + T^{1/2}\|\nabla u\|_{L^\infty(0,T;H^2(\mathcal{F}))^9} \left\| \partial_t \left(\frac{\mathbb{B}_X}{\delta_X} \right) \right\|_{L^\infty(0,T;H^2(\mathcal{F}))^9} \|\mathbb{B}_X\|_{L^\infty(0,T;H^3(\mathcal{F}))^9} \\
&\qquad + \left\| \frac{\mathbb{B}^0}{\delta^0} - \frac{\mathbb{B}_X}{\delta_X} \right\|_{L^\infty(0,T;H^3(\mathcal{F}))^9} \|\partial_t \nabla u\|_{L^2(0,T;H^1(\mathcal{F}))^9} \|\mathbb{B}_X\|_{L^\infty(0,T;H^3(\mathcal{F}))^9} \\
&\qquad + \left\| \frac{\mathbb{B}_X}{\delta_X} \right\|_{L^\infty(0,T;H^3(\mathcal{F}))^9} \|\nabla u\|_{L^2(0,T;H^3(\mathcal{F}))^9} \|\partial_t \mathbb{B}_X\|_{L^\infty(0,T;H^2(\mathcal{F}))^9} \\
&\qquad + \left\| \frac{\mathbb{B}^0}{\delta^0} \right\|_{L^\infty(0,T;H^3(\mathcal{F}))^9} \|\partial_t \nabla u\|_{L^2(0,T;H^1(\mathcal{F}))^9} \|\mathbb{B}^0 - \mathbb{B}_X\|_{L^\infty(0,T;H^3(\mathcal{F}))^9} \\
&\qquad + T^{1/2} \|\rho^{\gamma-1}\|_{L^\infty(0,T;H^3(\mathcal{F}))} \|\partial_t \rho\|_{L^\infty(0,T;H^2(\mathcal{F}))} \|\mathbb{B}_X\|_{L^\infty(0,T;H^3(\mathcal{F}))^9} \\
&\qquad + T^{1/2} \|\rho^{\gamma}\|_{L^\infty(0,T;H^3(\mathcal{F}))} \|\partial_t \mathbb{B}_X\|_{L^\infty(0,T;H^2(\mathcal{F}))^9} \Big) \leqslant C T^{1/2}. \qquad (4.18)
\end{aligned}
$$

Combining the above estimates, we deduce

$$\|\mathcal{N}(f_1, f_2, h)\|_{\mathcal{R}_T} \leqslant C T^{1/2}.$$

Thus for T small enough $\mathcal{N}(\mathcal{B}_{T,R}) \subset \mathcal{B}_{T,R}$.

To show that $\mathcal{N}|_{\mathcal{B}_{T,R}}$ is a strict contraction, we proceed in a similar way: we consider

$$\left[f_1^i, f_2^i, h^i\right] \in \mathcal{B}_{T,R}, \quad i = 1, 2$$

and we denote by $\left[\rho^i, u^i, \eta^i\right]$ the solution of the system (3.1)-(3.3) associated with

$$\left[f_1^i, f_2^i, h^i\right] \in \mathcal{R}_T \text{ and } \left[\rho^0, u^0, \eta_1^0, \eta_2^0\right] \in \mathcal{I}.$$

We define

$$[f_1, f_2, h] = \left[f_1^1, f_2^1, h^1\right] - \left[f_1^2, f_2^2, h^2\right], \quad [\rho, u, \eta] = \left[\rho^1, u^1, \eta^1\right] - \left[\rho^2, u^2, \eta^2\right].$$

We can apply Theorem 3.1 and deduce that

$$\|[\rho, u, \eta]\|_{\mathcal{E}_\rho \times \mathcal{E}_u \times \mathcal{E}_\eta} \leqslant C \, \|[f_1, f_2, h]\|_{\mathcal{R}_T}.$$

Since the initial conditions of $[\rho, u, \eta]$ are zero, we have

$$\|\rho\|_{L^\infty(0,T;H^3(\mathcal{F}))} \leqslant C T^{1/2} \, \|[f_1, f_2, h]\|_{\mathcal{R}_T}.$$

We obtain similarly

$$\|X^1 - X^2\|_{L^\infty(0,T;H^4(\mathcal{F}))^3} \leqslant C T^{1/2} \, \|[f_1, f_2, h]\|_{\mathcal{R}_T},$$

and we deduce similar estimates for $\mathbb{B}_{X^1} - \mathbb{B}_{X^2}$, $\mathbb{A}_{X^1} - \mathbb{A}_{X^2}$, $\delta_{X^1} - \delta_{X^2}$. Proceeding as before, we obtain that F_1, F_2, H given by (2.13)-(2.14) satisfy

$$
\begin{aligned}
&\|F_1(\rho^1, u^1, \eta^1) - F_1(\rho^2, u^2, \eta^2)\|_{L^2(0,T;H^3(\mathcal{F}))} + \|F_1(\rho^1, u^1, \eta^1) - F_1(\rho^2, u^2, \eta^2)\|_{L^\infty(0,T;H^2(\mathcal{F}))} \\
&\quad + \|F_2(\rho^1, u^1, \eta^1) - F_2(\rho^2, u^2, \eta^2)\|_{L^2(0,T;H^2(\mathcal{F}))^3} + \|F_2(\rho^1, u^1, \eta^1) - F_2(\rho^2, u^2, \eta^2)\|_{H^1(0,T;L^2(\mathcal{F}))^3} \\
&\quad + \|H(\rho^1, u^1, \eta^1) - H(\rho^2, u^2, \eta^2)\|_{L^2(0,T;H^3(\mathcal{F}))^9} + \|H(\rho^1, u^1, \eta^1) - H(\rho^2, u^2, \eta^2)\|_{H^1(0,T;H^1(\mathcal{F}))^9} \\
&\qquad \leqslant C T^{1/2} \, \|[f_1, f_2, h]\|_{\mathcal{R}_T}.
\end{aligned}
$$

Hence, for T small enough, we have $\mathcal{N}|_{\mathcal{B}_{T,R}}$ is a strict contraction. □

Appendix A. Proof of Proposition 3.3

In order to prove Proposition 3.3, we show that the linear operator corresponding to the system (3.17)–(3.18) is the generator of an analytic semigroup in a suitable function space. More precisely, we first consider

$$\mathcal{X}_F := \left[L^2(\mathcal{F})\right]^3, \quad \mathcal{D}(A_F) = \left(H^2(\mathcal{F}) \cap H^1_0(\mathcal{F})\right)^3, \quad A_F = \frac{1}{\rho^0\delta^0} \operatorname{div} \mathbb{T}^0 : \mathcal{D}(A_F) \to \mathcal{X}_F. \tag{A.1}$$

Due to (2.22)–(2.23) (see Lemma 2.3), the above operator is well-defined. By using the measure $\rho^0\delta^0 \, dy$ for $\mathcal{X}_F$ instead of the Lebesgue measure dy (which gives an equivalent norm), we deduce by integrations by parts that A_F is symmetric. Moreover, using Lemma 2.3 and Lemma 2.4, we also deduce that $-\frac{1}{\rho^0\delta^0} \operatorname{div} \mathbb{T}^0$ is a strongly elliptic operator of order 2. Therefore by [36, Lemma 3.2, p.263], A_F is an isomorphism from $\mathcal{D}(A_F)$ onto $\mathcal{X}_F$. In particular, A_F is a self-adjoint operator and generates an analytic semigroup on $\mathcal{X}_F$ (see for instance [5, Proposition 2.11, p. 122]).

Using again [36, Lemma 3.2, p.263], we introduce the operator $D_F \in \mathcal{L}(H^2(\mathcal{S}), H^2(\mathcal{F})^3)$, defined as follows: $w = D_F g$ is the solution of the system

$$-\operatorname{div} \mathbb{T}^0 w = 0 \text{ in } \mathcal{F}, \quad w = \mathcal{T} g \text{ on } \partial\mathcal{F}. \tag{A.2}$$

By a standard transposition method, the operator D_F can be extended as $D_F \in \mathcal{L}(L^2(\mathcal{S}), L^2(\mathcal{F})^3)$.

Now, we introduce the operator

$$\mathcal{X}_S := H^2(\mathcal{S}) \times L^2(\mathcal{S}), \quad \mathcal{D}(A_S) = H^2(\mathcal{S}) \times H^2(\mathcal{S}), \quad A_S = \begin{bmatrix} 0 & I \\ \Delta_s & \varepsilon\Delta_s \end{bmatrix} : \mathcal{D}(A_S) \to \mathcal{X}_S. \tag{A.3}$$

From [41, Proposition 2.2], we know that A_S generates an analytic semigroup on $\mathcal{X}_S$.

Using the above operators, and by setting

$$\eta_{1,\varepsilon} = \eta_\varepsilon, \quad \eta_{2,\varepsilon} = \partial_t \eta_\varepsilon,$$

we can write the system (3.17)–(3.18) as follows:

$$\frac{d}{dt} \begin{bmatrix} u_\varepsilon \\ \eta_{1,\varepsilon} \\ \eta_{2,\varepsilon} \end{bmatrix} = \mathcal{A}_{FS} \begin{bmatrix} u_\varepsilon \\ \eta_{1,\varepsilon} \\ \eta_{2,\varepsilon} \end{bmatrix} + \begin{bmatrix} f_2 + \operatorname{div} h \\ 0 \\ -\mathcal{Q} h \end{bmatrix}, \qquad \begin{bmatrix} u_\varepsilon \\ \eta_{1,\varepsilon} \\ \eta_{2,\varepsilon} \end{bmatrix}(0) = \begin{bmatrix} u^0 \\ \widehat{\eta}_1^0 \\ \eta_2^0 \end{bmatrix}, \tag{A.4}$$

where $\mathcal{A}_{FS} : \mathcal{D}(\mathcal{A}_{FS}) \to \mathcal{X}$ is defined by

$$\mathcal{X} = \mathcal{X}_F \times \mathcal{X}_S, \tag{A.5}$$

$$\mathcal{D}(\mathcal{A}_{FS}) = \left\{ [u, \eta_1, \eta_2]^\top \in H^2(\mathcal{F})^3 \times \mathcal{D}(A_S) \; ; \; u - D_F \eta_2 \in \mathcal{D}(A_F) \right\}, \tag{A.6}$$

$$\mathcal{A}_{FS} = \mathcal{A}^0_{FS} + \mathcal{B}_{FS},$$

with

$$\mathcal{A}^0_{FS} \begin{bmatrix} u_\varepsilon \\ \eta_{1,\varepsilon} \\ \eta_{2,\varepsilon} \end{bmatrix} = \begin{bmatrix} A_F(u_\varepsilon - D_F \eta_{2,\varepsilon}) \\ A_S \begin{bmatrix} \eta_{1,\varepsilon} \\ \eta_{2,\varepsilon} \end{bmatrix} \end{bmatrix} \quad \text{and} \quad \mathcal{B}_{FS} \begin{bmatrix} u_\varepsilon \\ \eta_{1,\varepsilon} \\ \eta_{2,\varepsilon} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\mathcal{Q}\mathbb{T}^0(u_\varepsilon) \end{bmatrix}. \tag{A.7}$$

Using the analyticity of A_F and A_S, we can show as in the proof of [29, Theorem 4.2] that $\mathcal{A}^0_{FS}$ generates an analytic semigroup on $\mathcal{X}$. Moreover, using Lemma 2.3, standard trace results and compact embeddings, we infer that, for any $\delta > 0$ there exists $C(\delta) > 0$ such that

$$\left\| \mathcal{B}_{FS} \left[u_\varepsilon, \eta_{1,\varepsilon}, \eta_{2,\varepsilon}\right] \right\|_{\mathcal{X}} \leqslant \delta \left\| \mathcal{A}^0_{FS} \left[u_\varepsilon, \eta_{1,\varepsilon}, \eta_{2,\varepsilon}\right] \right\|_{\mathcal{X}} + C(\delta) \left\| \left[u_\varepsilon, \eta_{1,\varepsilon}, \eta_{2,\varepsilon}\right] \right\|_{\mathcal{X}}.$$

In particular, $\mathcal{B}_{FS}$ is a $\mathcal{A}^0_{FS}$-bounded perturbation, and hence, $\mathcal{A}_{FS}$ generates an analytic semigroup on $\mathcal{X}$ (see for instance [37, Chapter 3, Theorem 2.1]). Finally, the conclusion of the theorem follows from [5, Part II, Chapter 1, Theorem 3.1]. □

References

[1] G. Avalos, P. G. Geredeli, and J. T. Webster, *Semigroup well-posedness of a linearized, compressible fluid with an elastic boundary*, Discrete Contin. Dyn. Syst. Ser. B, 23 (2018), pp. 1267–1295.

[2] M. Badra and T. Takahashi, *Gevrey Regularity for a System Coupling the Navier–Stokes System with a Beam Equation*, SIAM J. Math. Anal., 51 (2019), pp. 4776–4814.

[3] M. Badra and T. Takahashi, *Gevrey regularity for a system coupling the Navier-Stokes system with a beam: the non-flat case*, https://hal.archives-ouvertes.fr/hal-02303258, (2019).

[4] H. Beirão da Veiga, *On the existence of strong solutions to a coupled fluid-structure evolution problem*, J. Math. Fluid Mech., 6 (2004), pp. 21–52.

[5] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter, *Representation and control of infinite dimensional systems*, Systems & Control: Foundations & Applications, Birkhäuser Boston, Inc., Boston, MA, second ed., 2007.

[6] M. Boulakia, *Existence of weak solutions for an interaction problem between an elastic structure and a compressible viscous fluid*, J. Math. Pures Appl. (9), 84 (2005), pp. 1515–1554.

[7] ———, *Existence of weak solutions for the three-dimensional motion of an elastic structure in an incompressible fluid*, J. Math. Fluid Mech., 9 (2007), pp. 262–294.

[8] M. Boulakia and S. Guerrero, *A regularity result for a solid-fluid system associated to the compressible Navier-Stokes equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 26 (2009), pp. 777–813.

[9] ———, *On the interaction problem between a compressible fluid and a Saint-Venant Kirchhoff elastic structure*, Adv. Differential Equations, 22 (2017), pp. 1–48.

[10] D. Breit and S. Schwarzacher, *Compressible fluids interacting with a linear-elastic shell*, Arch. Ration. Mech. Anal., 228 (2018), pp. 495–562.

[11] J.-J. Casanova, *Fluid–structure system with boundary conditions involving the pressure*, Journal of Evolution Equations, (2020).

[12] A. Chambolle, B. Desjardins, M. J. Esteban, and C. Grandmont, *Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate*, J. Math. Fluid Mech., 7 (2005), pp. 368–404.

[13] C. H. A. Cheng, D. Coutand, and S. Shkoller, *Navier-Stokes equations interacting with a nonlinear elastic biofluid shell*, SIAM J. Math. Anal., 39 (2007), pp. 742–800.

[14] C. H. A. Cheng and S. Shkoller, *The interaction of the 3D Navier-Stokes equations with a moving nonlinear Koiter elastic shell*, SIAM J. Math. Anal., 42 (2010), pp. 1094–1155.

[15] I. Chueshov, *Dynamics of a nonlinear elastic plate interacting with a linearized compressible viscous fluid*, Nonlinear Anal., 95 (2014), pp. 650–665.

[16] R. Denk and J. Saal, *L^p-theory for a fluid-structure interaction model*, arXiv e-prints, (2019), p. arXiv:1909.09344.

[17] I. A. Djebour and T. Takahashi, *On the existence of strong solutions to a fluid structure interaction problem with Navier boundary conditions*, J. Math. Fluid Mech., 21 (2019).

[18] F. Flori and P. Orenga, *Fluid-structure interaction: analysis of a 3-D compressible model*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 17 (2000), pp. 753–777.

[19] C. Grandmont, *Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate*, SIAM J. Math. Anal., 40 (2008), pp. 716–737.

[20] C. Grandmont and M. Hillairet, *Existence of global strong solutions to a beam-fluid interaction system*, Arch. Ration. Mech. Anal., 220 (2016), pp. 1283–1333.

[21] C. Grandmont, M. Hillairet, and J. Lequeurre, *Existence of local strong solutions to fluid–beam and fluid–rod interaction systems*, in Annales de l'Institut Henri Poincaré C, Analyse non linéaire, vol. 36, Elsevier, 2019, pp. 1105–1149.

[22] B. H. Haak, D. Maity, T. Takahashi, and M. Tucsnak, *Mathematical analysis of the motion of a rigid body in a compressible Navier-Stokes-Fourier fluid*, Math. Nachr., 292 (2019), pp. 1972–2017.

[23] M. Hieber and M. Murata, *The L^p-approach to the fluid-rigid body interaction problem for compressible fluids*, Evol. Equ. Control Theory, 4 (2015), pp. 69–87.

[24] D. Lengeler, *Weak solutions for an incompressible, generalized Newtonian fluid interacting with a linearly elastic Koiter type shell*, SIAM J. Math. Anal., 46 (2014), pp. 2614–2649.

[25] D. Lengeler and M. Růžička, *Weak solutions for an incompressible Newtonian fluid interacting with a Koiter type shell*, Arch. Ration. Mech. Anal., 211 (2014), pp. 205–255.

[26] J. Lequeurre, *Existence of strong solutions to a fluid-structure system*, SIAM J. Math. Anal., 43 (2011), pp. 389–410.

[27] ———, *Existence of strong solutions for a system coupling the Navier-Stokes equations and a damped wave equation*, J. Math. Fluid Mech., 15 (2013), pp. 249–271.

[28] D. Maity, J.-P. Raymond, and A. Roy, *Maximal-in-time existence and uniqueness of strong solution of a 3D fluid-structure interaction model*. working paper or preprint, Dec. 2019.

[29] D. Maity and T. Takahashi, *Existence and uniqueness of strong solutions for the system of interaction between a compressible Navier-Stokes-Fourier fluid and a damped plate equation*, https://hal.archives-ouvertes.fr/hal-02668248, (2020).

[30] ———, *Lp theory for the interaction between the incompressible Navier-Stokes system and a damped beam*, https://hal.archives-ouvertes.fr/hal-02294097, (2020).

[31] D. Maity and M. Tucsnak, *A maximal regularity approach to the analysis of some particulate flows*, in Particles in flows, Adv. Math. Fluid Mech., Birkhäuser/Springer, Cham, 2017, pp. 1–75.

[32] S. Mitra, *Local existence of Strong solutions for a fluid-structure interaction model*, arXiv e-prints, (2018), p. arXiv:1808.06716.
[33] B. Muha and S. Čanić, *Existence of a weak solution to a nonlinear fluid-structure interaction problem modeling the flow of an incompressible, viscous fluid in a cylinder with deformable walls*, Arch. Ration. Mech. Anal., 207 (2013), pp. 919–968.
[34] ———, *A nonlinear, 3D fluid-structure interaction problem driven by the time-dependent dynamic pressure data: a constructive existence proof*, Commun. Inf. Syst., 13 (2013), pp. 357–397.
[35] ———, *Fluid-structure interaction between an incompressible, viscous 3D fluid and an elastic shell with nonlinear Koiter membrane energy*, Interfaces Free Bound., 17 (2015), pp. 465–495.
[36] J. Nečas, *Direct methods in the theory of elliptic equations*, Springer Monographs in Mathematics, Springer, Heidelberg, 2012. Translated from the 1967 French original by Gerard Tronel and Alois Kufner, Editorial coordination and preface by Šárka Nečasová and a contribution by Christian G. Simader.
[37] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, vol. 44 of Applied Mathematical Sciences, Springer-Verlag, New York, 1983.
[38] A. Quarteroni, M. Tuveri, and A. Veneziani, *Computational vascular fluid dynamics: problems, models and methods*, Computing and Visualization in Science, 2 (2000), pp. 163–197.
[39] J.-P. Raymond, *Feedback stabilization of a fluid-structure model*, SIAM J. Control Optim., 48 (2010), pp. 5398–5443.
[40] A. Roy and T. Takahashi, *Stabilization of a rigid body moving in a compressible viscous fluid*, Journal of Evolution Equations, (2020).
[41] G. F. Webb, *Existence and asymptotic behavior for a strongly damped nonlinear wave equation*, Canadian J. Math., 32 (1980), pp. 631–643.

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