Linking Focusing and Resolution with Selection

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Focusing and selection are techniques that shrink the proof-search space for respectively sequent calculi and resolution. To bring out a link between them, we generalize them both: we introduce a sequent calculus where each *occurrence* of an atomic formula can have a positive or a negative polarity; and a resolution method where each literal, whatever its sign, can be selected in input clauses. We prove the equivalence between cut-free proofs in this sequent calculus and derivations of the empty clause in that resolution method. Such a generalization is not semi-complete in general, which allows us to consider complete instances that correspond to theories of any logical strength. We present three complete instances: first, our framework allows us to show that ordinary focusing corresponds to hyperresolution and semantic resolution; the second instance is deduction modulo theory and the related framework called superdeduction; and a new setting, not captured by any existing framework, extends deduction modulo theory with rewriting rules having several left-hand sides, which restricts even more the proof-search space.

CCS Concepts: • **Theory of computation** → **Proof theory**; *Automated reasoning*.

Additional Key Words and Phrases: sequent calculus, refinements of resolution, deduction modulo theory, polarization

1 INTRODUCTION

In addition to clever implementation techniques and data structures, a key point that explains the success of state-of-the-art automated theorem provers is the use of calculi that dramatically reduce proof-search space. In the last decades, the independent developments of two families of techniques can be highlighted. First, in the kind of methods based on resolution, proof-search space can be shrunk using ordering and selection techniques. The intuition is to restrict the application of the resolution rule to only some literals in a clause. If equality is considered, this leads to the superposition calculus [2] which is the base calculus of the currently most efficient automated provers for first-order classical logic. Second, in sequent calculi, Andreoli [1] introduced a technique called focusing to reduce non-determinism in the application of sequent-calculus rules. It works by first applying all invertible rules (those whose conclusion is logically equivalent to their premises) and second by chaining the application of non-invertible rules. Originally developed for linear logic, focusing has been extended to intuitionistic and classical first-order logic [31]. Focusing is mostly used in fields where sequent calculi, and related inverse and tableaux methods, are the

most accurate proving methods. For instance, there exists tools for first-order linear logic [15], for intuitionistic logic [32] and for modal logic [33]. Focusing is also the key ingredient in Miller's ProofCert project aiming at building a universal framework for proof certification [18].

Despite their apparent lack of relation, we show in this paper that selection in refinements of the resolution calculus and focusing in sequent calculus are in fact strongly related, so that ordinary focusing in classical first-order logic corresponds actually to hyperresolution, where all negative literals are selected in a clause and are resolved at once. This connection is obtained by relaxing both techniques: concerning resolution, we allow any literal of the input clauses to be selected, whatever its sign; for the focusing part, we allow polarization not only of connectives, but also of all occurrences of literals. The main theorem of this paper, Theorem 4.2, shows that the sets of clauses whose insatisfiability can be proved by the resolution method with arbitrary input selection are exactly the sequents that have a cut-free proof in the generalized focusing setting.

This generalization allows us to cover a wider spectrum of proof systems. In particular, this permits to consider systems that search for proofs modulo some theory. Indeed, in real-world applications, proof obligations are often verified within one or several theories. This explains the interest in and the success of Satisfiability Modulo Theory tools in recent years. Embedding a theory in our framework amounts to giving an axiomatic presentation of it where some literals are selected.

By relaxing the conditions for selecting literals, our framework is not always refutationaly complete. However, this should not be considered as a drawback, but as an essential point to be able to represent efficiently all kinds of theories. Indeed, let us consider a proof-search method $\mathbb{P}(\mathcal{T})$ parameterized by a theory $\mathcal{T}$. Ideally, $\mathbb{P}(\mathcal{T})$ should be as efficient as a generic proof-search method if it is fed with a formula that is not related to the theory $\mathcal{T}$. In particular, if it tries to refute the true formula $\top$, it should terminate, and with the answer "NO". Let us say that $\mathbb{P}(\mathcal{T})$ is relatively consistent if it is the case. As we pointed out with Dowek [11], we cannot have a generic proof of the completeness of a relatively consistent method $\mathbb{P}(\mathcal{T})$ that would work for all $\mathcal{T}$. Indeed, such a proof would imply the consistency of the theory $\mathcal{T}$, and, according to Gödel, this cannot be performed in $\mathcal{T}$ itself. So either the completeness of the proof system is proved once and for all, but it cannot represent theories that are logically at least as strong as that proof of completeness; or it is not complete in general but it can be proved to be complete for particular theories, whatever their logical strengths. What is interesting therefore is to give proofs of completeness of $\mathbb{P}(\mathcal{T})$ for particular theories $\mathcal{T}$.

Consequently, we give three instances of our framework, where we can have proofs of completeness. First, as stated above, we link ordinary focusing with hyperresolution, and, in the ground case, with semantic resolution. Second, we show that Deduction Modulo Theory [23] is also a particular instance of this framework, knowing that there exists numerous proof techniques to prove the completeness of Deduction Modulo a particular theory, for instance [9, 21, 24, 28]. Third, we show how completeness in our framework can be reduced to completeness of several instances of Deduction Modulo Theory, thus allowing to reuse these techniques.

To give an intuition about our framework, and to illustrate how much the proof-search space can be constrained without losing completeness, let us consider for example the theory defining the powerset:

$$\forall X, \forall Y, (X \in \mathcal{P}(Y)) \Leftrightarrow (\forall Z, (Z \in X) \Rightarrow (Z \in Y))$$

This theory can be put in clausal normal form, using d as a Skolem symbol, and we select (by underlining them) some literals in these clauses[1]:

$$\neg X \in \mathcal{P}(Y) \vee \underline{\neg Z \in X} \vee Z \in Y \tag{1}$$

$$\underline{X \in \mathcal{P}(Y)} \vee d(X, Y) \in X \tag{2}$$

$$\underline{X \in \mathcal{P}(Y)} \vee \underline{\neg d(X, Y) \in Y} \tag{3}$$

Using focusing in general, and in our framework in particular, the decomposition of connectives is so restricted that, given an axiom, a proof derivation decomposing this axiom would necessarily have only certain shapes. Thus, the axiom can be replaced by new inference rules, called synthetic rules, that are used instead of the derivation of those shapes. See Section 2.1, for more details. In our framework, this would lead to the following three synthetic rules, that can be used in place of the axioms (the explanation how these rules are obtained is given in Section 5.4):

$$(1) \vdash \frac{\Delta, u \in \mathcal{P}(v), t \in u, t \in v \vdash}{\Delta, u \in \mathcal{P}(v), t \in u \vdash} \qquad\qquad (2) \vdash \frac{}{\Delta, \neg u \in \mathcal{P}(v), d(u, v) \in v \vdash}$$

$$(3) \vdash \frac{\Delta, \neg u \in \mathcal{P}(v), d(u, v) \in u \vdash}{\Delta, \neg u \in \mathcal{P}(v) \vdash}$$

If one tries to prove transitivity of the membership in the powerset, i.e. to refute the formula $\exists A. \exists B. \exists C. A \in \mathcal{P}(B) \wedge B \in \mathcal{P}(C) \wedge \neg A \in \mathcal{P}(C)$, only one proof can be built, namely:

$$(2) \vdash \frac{}{a \in \mathcal{P}(b), b \in \mathcal{P}(c), \neg a \in \mathcal{P}(c), d(a, c) \in a, d(a, c) \in b, d(a, c) \in c \vdash}$$
$$(1) \vdash \frac{}{a \in \mathcal{P}(b), b \in \mathcal{P}(c), \neg a \in \mathcal{P}(c), d(a, c) \in a, d(a, c) \in b \vdash}$$
$$(1) \vdash \frac{}{a \in \mathcal{P}(b), b \in \mathcal{P}(c), \neg a \in \mathcal{P}(c), d(a, c) \in a \vdash}$$
$$(3) \vdash \frac{}{a \in \mathcal{P}(b), b \in \mathcal{P}(c), \neg a \in \mathcal{P}(c) \vdash}$$
$$\wedge \vdash \frac{}{a \in \mathcal{P}(b) \wedge b \in \mathcal{P}(c) \wedge \neg a \in \mathcal{P}(c) \vdash}$$
$$\exists \vdash \frac{}{\exists A. \exists B. \exists C. A \in \mathcal{P}(B) \wedge B \in \mathcal{P}(C) \wedge \neg A \in \mathcal{P}(C) \vdash}$$

where the active formulas in a sequent are underwaved, and double lines indicate potentially several applications of an inference rule.

On the resolution side, clauses (1) to (3) lead to the following ground derived rules (see also Section 5.4):

$$(1) \frac{u \in \mathcal{P}(v) \veebar C \qquad t \in u \veebar D}{t \in v \veebar C \veebar D} \qquad\qquad (2) \frac{\neg u \in \mathcal{P}(v) \veebar C}{d(u, v) \in u \veebar C}$$

$$(3) \frac{\neg u \in \mathcal{P}(v) \veebar C \qquad d(u, v) \in v \veebar D}{C \veebar D}$$

Once again, there is only one proof of transitivity, i.e. starting from the set of clauses $\{a \in \mathcal{P}(b); b \in \mathcal{P}(c); \neg a \in \mathcal{P}(c)\}$:

---

[1]We use the associative-commutative-idempotent symbol $\veebar$ in clauses to distinguish it from the symbol $\vee$ that is used in formulas.

$$(3) \dfrac{\neg a \in \mathcal{P}(c) \qquad (1)\ \dfrac{b \in \mathcal{P}(c) \qquad (1)\ \dfrac{a \in \mathcal{P}(b) \qquad (2)\ \dfrac{\neg a \in \mathcal{P}(c)}{d(a,c) \in a}}{d(a,c) \in b}}{d(a,c) \in c}}{\Box}$$

and we cannot even infer other clauses than those. We let the reader compare with what happens if we use clauses (1) to (3) in resolution, even using the ordered resolution with selection refinement.

In the next section, we present the sequent calculus LKF$^\perp$, which extends the calculus LKF of Liang and Miller [31]. Section 3 introduces the resolution method with arbitrary input selection. Then, the main theorem of Section 4, and of this paper, namely Theorem 4.2, establishes the link between the two proof systems. In Section 5 are highlighted particular instances where proofs of completeness can be obtained, which also permit to show an *a priori* unsuspected relation between existing proof systems such as for instance hyperresolution and ordinary focusing. We also show how to relate completeness in our setting to completeness in Deduction Modulo Theory.

Related work

As we do here, Goubault-Larrecq [27] proves completeness of several refinements of resolution, including hyperresolution and semantic resolution, using syntactic transformations instead of relying on the construction of a model.

Chaudhuri, Pfenning, and Price [16] show that hyperresolution for Horn clauses can be explained as an instance of a sequent calculus for intuitionistic linear logic with focusing where atomic formulas are given a negative polarity.

Farooque, Graham-Lengrand, and Mahboubi [26] developed a sequent calculus, based on focusing, that is able to simulate DPLL($\mathcal{T}$), the most common calculus used in SMT provers. The main difference with our framework is that in [26], the theory is considered as a black box which is called as an oracle. Here, the theory is considered as a first-class citizen.

Within the ProofCert project, resolution proofs can be checked by a kernel built upon a sequent calculus with focusing [18]. Based on this, the tool Checkers [17] is able to verify proofs coming from automated theorem provers based on resolution such as E-prover. Different from here, they translate resolution derivations using cuts to get smaller proofs.

Hermant [29] proves the correspondance between the cut-free fragment of a sequent calculus and a resolution method, in the setting of Deduction Modulo Theory. Since Deduction Modulo Theory is subsumed by our framework, Theorem 4.2 is a generalization of Hermant's work. Proving it is simpler in our setting because focusing restrains the shape of possible sequent calculus proofs, whereas Hermant had to prove technical lemmas to give proofs a canonical shape.

Notations and conventions

We use standard definitions for terms, predicates, formulas (with connectives $\perp, \top, \neg, \wedge, \vee$ and quantifiers $\forall, \exists$), sequents and substitutions. Since we consider classical logic, implication $A \Rightarrow B$ is syntactic sugar for $\neg A \vee B$, and similarly equivalence $A \Leftrightarrow B$ denotes $(A \Rightarrow B) \wedge (B \Rightarrow A)$. A literal is an atomic formula or its negation. A clause is a set of literals. We will identify a literal with the unit clause containing it. Unless stated otherwise, letters $P, Q, R, P', P_1, \dots$ denote atomic formulas, $L, K, L', L_1, \dots$ denote literals, $A, B, A', A_1, \dots$ denote formulas, $C, D, C', C_1, \dots$ denote clauses, Γ, Δ denote sets of clauses or multisets of formulas (depending on the context). In particular, sequents contain multisets of formulas. $A^\perp$ denotes the negation normal form of $\neg A$.

2 FOCUSING WITH POLARIZED OCCURRENCES OF ATOMIC FORMULAS

Focusing was introduced by Andreoli [1] to restrict the non-determinism in some sequent calculus for linear logic. It relies on the alternation of two phases: During the asynchronous phase (sequents with ⇑), all invertible rules are applied on the formulas of the sequent. Recall that a rule is said invertible if its conclusion implies the conjunction of its premises. During the synchronous phase (sequents with ⇓), a particular formula is selected —the focus is on it— and all possible non-invertible rules are successively applied on it. This idea has been extended to intuitionistic and classical first-order logic by Liang and Miller [31]. In these, connectives may have invertible and non-invertible versions of their sequent calculus rules. Therefore, one considers in that case two versions of a connective, one called positive when the right introduction rule is non-invertible, and one called negative when it is invertible. Some connectives, i.e. ∃ in classical logic, only have a positive version, and dually, others, such as ∀ in classical logic, only have a negative version. Given a usual formula, one can decide which version of a connective one wants to use at a particular occurrence, which is called a polarization of the formula.[2] Note that the polarity of a connective does not affect its semantics, it only alters the shape of the sequent calculus proofs. Similarly, one can decide the polarity of each literal. If a literal with negative polarity L is focused on in a branch, then this branch must necessarily be closed, with $L^\perp$ in the context. (See rule $\widehat{\Downarrow}\vdash$ in Figure 1.)

In the ordinary presentation of focusing, this polarity is chosen globally for all occurrences of each atomic formula, and the polarity of $\neg P$ is defined as the inverse of that of P. In our setting, the polarity can be different for each occurrence of a literal. More precisely, the polarity is attached to the position of the literal in the formula. Let us consider the usual subformula relation: in particular, for all terms t, $\{t/x\}P$ is a subformula of $\forall x.\ P$ and of $\exists x.\ P$. In the same way as what happens for polarities of connectives in ordinary focusing, when considering subformulas in the premises of inference rules, the polarities of connectives and literals remain the same. In particular, if a literal L has a negative polarity in e.g. $\forall x.\ P$, then $\{t/x\}L$ will have a negative polarity too in $\{t/x\}P$. Also, if formulas are copied by an inference rule (e.g. the formulas in Δ in the rule $\Downarrow \lor \vdash$ of Figure 1), both copies in the premises contain the same polarities. Furthermore, when considering the application of a first-order term substitution to a formula, the polarities of the literals in the resulting formula remain the same. The polarity of a formula is defined as the polarity of its top connective.

Liang and Miller [31] introduce the sequent calculus LKF, and prove it to be complete for classical first-order logic. In Figure 1, we present the calculus LKF$^\perp$, which is almost the same with the following differences:

- All formulas are put on the left-hand side of the sequent, instead of the right-hand side. Therefore, one does not try to prove a disjunction of formulas, but one tries to refute a conjunction of formulas. This is the same thanks to the dual nature of classical first-order logic, and this helps to be closer to the resolution derivations. Note that, consequently, the focus is on negative formulas, and invertible rules are applied on positive formulas.
- The polarity of atomic formulas is not chosen globally, but each *occurrence* of a literal can have a positive or a negative polarity. In particular, we can have two literals L and $L^\perp$ which are both negative, or both positive. We denote by $\underline{L}$ the fact that the literal L has a negative polarity. To be able to close branches on which we have two positive opposed literals, we add a rule $\widehat{\Uparrow}\vdash$. Note that in this rule, L and $L^\perp$ are assumed to have a positive polarity.

We denote by $\Gamma \Uparrow \Delta \vdash$ (with Γ or Δ, possibly empty, containing polarized formulas) the fact that there exists a proof of the sequent $\Gamma \Uparrow \Delta \vdash$ in LKF$^\perp$, that is, a derivation starting from this sequent

---

[2]Let us note that this notion of polarity is a standard denomination when dealing with focusing, and should not be confused with the more usual but unrelated notion defined by the parity of the negation-depth of a position in a formula.

Asynchronous phase: $\widehat{\Uparrow}\vdash \dfrac{}{\Gamma,L,L^{\perp}\Uparrow\vdash}$ $\qquad$ $\Uparrow\exists\vdash \dfrac{\Gamma\Uparrow\Delta,A\vdash}{\Gamma\Uparrow\Delta,\exists x.\,A\vdash}$ (x not free in Γ,Δ)

$\Uparrow\vee\vdash \dfrac{\Gamma\Uparrow\Delta,A\vdash \qquad \Gamma\Uparrow\Delta,B\vdash}{\Gamma\Uparrow\Delta,A\vee^{+}B\vdash}$ $\qquad$ $\Uparrow\wedge\vdash \dfrac{\Gamma\Uparrow\Delta,A,B\vdash}{\Gamma\Uparrow\Delta,A\wedge^{+}B\vdash}$ $\qquad$ $\Uparrow\top\vdash \dfrac{\Gamma\Uparrow\Delta\vdash}{\Gamma\Uparrow\Delta,\top\vdash}$

Synchronous phase: $\widehat{\Downarrow}\vdash \dfrac{}{\Gamma,L^{\perp}\Downarrow\underline{L}\vdash}$ $\qquad$ $\Downarrow\forall\vdash \dfrac{\Gamma\Downarrow\{t/x\}A\vdash}{\Gamma\Downarrow\forall x.\,A\vdash}$ $\qquad$ $\Downarrow\perp\vdash \dfrac{}{\Gamma\Downarrow\perp\vdash}$

$\Downarrow\vee\vdash \dfrac{\Gamma\Downarrow A\vdash \qquad \Gamma\Downarrow B\vdash}{\Gamma\Downarrow A\vee^{-}B\vdash}$ $\qquad$ $\Downarrow\wedge_{1}\vdash \dfrac{\Gamma\Downarrow A\vdash}{\Gamma\Downarrow A\wedge^{-}B\vdash}$ $\qquad$ $\Downarrow\wedge_{2}\vdash \dfrac{\Gamma\Downarrow B\vdash}{\Gamma\Downarrow A\wedge^{-}B\vdash}$

Focus $\dfrac{\Gamma,A\Downarrow A\vdash}{\Gamma,A\Uparrow\vdash}$ $\qquad$ Release $\dfrac{\Gamma\Uparrow A\vdash}{\Gamma\Downarrow A\vdash}$ $\qquad$ Store $\dfrac{\Gamma,A\Uparrow\Delta\vdash}{\Gamma\Uparrow A,\Delta\vdash}$

(A negative) $\qquad\qquad$ (A positive) $\qquad\qquad$ (A negative or literal)

Fig. 1. The sequent calculus LKF$^{\perp}$

and whose branches are all closed (by $\widehat{\Downarrow}\vdash$, $\widehat{\Uparrow}\vdash$ or $\Downarrow\perp\vdash$). Thanks to focusing, such a proof has the following shape :

- Since one starts in an asynchronous ($\Uparrow$) phase, invertible rules are successively applied to the positive formulas of Δ, until one obtains negative formulas or literals that are put on the left of $\Uparrow$ using Store.
- When no formula appears on the right of $\Uparrow$, then either the branch is closed by $\widehat{\Uparrow}\vdash$; or the focus is put on a negative formula using Focus.
- In the latter case, one is now in synchronous ($\Downarrow$) phase where non-invertible rules are successively applied to the formula upon which the focus is, until either the branch is closed using $\widehat{\Downarrow}\vdash$ or $\Downarrow\perp\vdash$; or one obtains a positive formula and the synchronous phase ends using Release.
- In the latter case, one starts again in the asynchronous phase.

Focusing therefore strongly constrains the shape of possible proofs, and therefore reduces the proof-search space. The rule $\widehat{\Downarrow}\vdash$ in particular imposes to close branches immediately when the focus is on a negative literal, and thus rules out many derivations.

Soundness of LKF$^{\perp}$ comes directly from the soundness of the standard sequent calculus. Indeed, by replacing $\Uparrow$ and $\Downarrow$ by , (comma), one obtains rules that are admissible in the standard sequent calculus.

Example 2.1. Consider the polarized formula $\forall x.\,(\underline{P(x)}\wedge^{-}\exists y.\,\neg P(y))$ (the negation of the drinker paradox). The simplest refutation in LKF$^{\perp}$ is :

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{
          \cfrac{
            \cfrac{
              \cfrac{
                \cfrac{
                  \cfrac{
                    \cfrac{
                      \cfrac{
                        \cfrac{\widehat{\Downarrow}\vdash}{\forall x.\,(\underline{P(x)} \wedge^- \exists y.\,\neg P(y)), \neg P(y) \Downarrow \underline{P(y)} \vdash}
                      }{\forall x.\,(\underline{P(x)} \wedge^- \exists y.\,\neg P(y)), \neg P(y) \Downarrow \underline{P(y)} \wedge^- \exists z.\,\neg P(z) \vdash}\ {\scriptstyle\Downarrow\wedge_1\vdash}
                    }{\forall x.\,(\underline{P(x)} \wedge^- \exists y.\,\neg P(y)), \neg P(y) \Downarrow \forall x.\,(\underline{P(x)} \wedge^- \exists y.\,\neg P(y)) \vdash}\ {\scriptstyle\Downarrow\forall\vdash}
                  }{\forall x.\,(\underline{P(x)} \wedge^- \exists y.\,\neg P(y)), \neg P(y) \Uparrow \vdash}\ {\scriptstyle\text{Focus}}
                }{\forall x.\,(\underline{P(x)} \wedge^- \exists y.\,\neg P(y)) \Uparrow \neg P(y) \vdash}\ {\scriptstyle\text{Store}}
              }{\forall x.\,(\underline{P(x)} \wedge^- \exists y.\,\neg P(y)) \Uparrow \exists y.\,\neg P(y) \vdash}\ {\scriptstyle\Uparrow\exists\vdash}
            }{\forall x.\,(\underline{P(x)} \wedge^- \exists y.\,\neg P(y)) \Downarrow \exists y.\,\neg P(y) \vdash}\ {\scriptstyle\text{Release}}
          }{\forall x.\,(\underline{P(x)} \wedge^- \exists y.\,\neg P(y)) \Downarrow \underline{P(t)} \wedge^- \exists y.\,\neg P(y) \vdash}\ {\scriptstyle\Downarrow\wedge_2\vdash}
        }{\forall x.\,(\underline{P(x)} \wedge^- \exists y.\,\neg P(y)) \Downarrow \forall x.\,(\underline{P(x)} \wedge^- \exists y.\,\neg P(y)) \vdash}\ {\scriptstyle\Downarrow\forall\vdash}
      }{\forall x.\,(\underline{P(x)} \wedge^- \exists y.\,\neg P(y)) \Uparrow \vdash}\ {\scriptstyle\text{Focus}}
    }{\Uparrow \forall x.\,(\underline{P(x)} \wedge^- \exists y.\,\neg P(y)) \vdash}\ {\scriptstyle\text{Store}}
$$

Note that it would not have been possible to apply $\Downarrow\wedge_1\vdash$ where $\Downarrow\wedge_2\vdash$ has been applied, because we would be stuck with the sequent $\forall x.\,(\underline{P(x)} \wedge^- \exists y.\,\neg P(y)) \Downarrow \underline{P(t)} \vdash$ that cannot be closed by $\widehat{\Downarrow}\vdash$.

To switch the polarity of a formula, e.g. to impose a change of phase, one can prefix it by so-called delays: $\delta^- A$ is negative whatever the polarity of A, whereas $\delta^+ A$ is always positive. If a delay appears in the corresponding phase, then it can be silently ignored. For instance, one can have the following derivation:

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{A \wedge^- \delta^- \forall x.\,B \Downarrow \{t/x\}B \vdash}{A \wedge^- \delta^- \forall x.\,B \Downarrow \delta^- \forall x.\,B \vdash}\ {\scriptstyle\Downarrow\forall\vdash}
    }{A \wedge^- \delta^- \forall x.\,B \Downarrow A \wedge^- \delta^- \forall x.\,B \vdash}\ {\scriptstyle\Downarrow\wedge_2\vdash}
  }{A \wedge^- \delta^- \forall x.\,B \Uparrow \vdash}\ {\scriptstyle\text{Focus}}
$$

However, if the delay appears in the opposite phase, then the phase has to end. For instance, this gives the following derivation:

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{A \wedge^- \delta^+ \forall x.\,B, \forall x.\,B \Uparrow \vdash}{A \wedge^- \delta^+ \forall x.\,B \Uparrow \forall x.\,B \vdash}\ {\scriptstyle\text{Store}}
      }{A \wedge^- \delta^+ \forall x.\,B \Downarrow \delta^+ \forall x.\,B \vdash}\ {\scriptstyle\text{Release}}
    }{A \wedge^- \delta^+ \forall x.\,B \Downarrow A \wedge^- \delta^+ \forall x.\,B \vdash}\ {\scriptstyle\Downarrow\wedge_2\vdash}
  }{A \wedge^- \delta^+ \forall x.\,B \Uparrow \vdash}\ {\scriptstyle\text{Focus}}
$$

There is no need to extend the syntax of first-order formulas to include delays, because they can be defined for instance using quantifiers: $\delta^- A = \forall x.\,A$ and $\delta^+ A = \exists x.\,A$, where x is not free in A. We let the reader check that this induces indeed the desired behavior.

Note that proofs can be closed when the polarities of an atomic formula and its negation are both positive (rule $\widehat{\Uparrow}\vdash$), or when one is positive and the other negative (rule $\widehat{\Downarrow}\vdash$), but not when they are both negative. Therefore, this restricts how formulas that contains literals with negative polarities can interact one with the others, and this is the main point of $\text{LKF}^\perp$ to reduce the proof-search space.

The sequent calculus $\text{LKF}^\perp$ is not complete in general. One of the simplest examples of incompleteness is the sequent $\underline{P} \vee^- Q, \underline{\neg P} \vee^- Q, \neg Q \Uparrow\vdash$ which has no proof although $P \vee Q, \neg P \vee Q, \neg Q$ is not satisfiable.

One could choose a global polarization for atomic formulas and use delays to enforce the polarity of occurrences of literals without compromising completeness. However, this would break synchronization phases, and the proof-search space would not be as restricted as in our calculus.

For instance, let A be an arbitrary large tautology, and let P be an atomic formula not appearing in A. Obviously, the set of formulas $P \vee A, \neg P$ is satisfiable and thus cannot be refuted. If one considers the polarization where both P and $\neg P$ are negative in the formulas, one can quickly conclude that no derivation of $\underline{P} \vee^{-} A, \underline{\neg P} \Uparrow \vdash$ exists. However, if one considers a global polarization where for instance P is positive, even if P is enforced to be negative in the first formula using a delay δ^{-}, then a proof search can begin with

$$
\begin{array}{c}
\text{Focus } \dfrac{\widehat{\Uparrow} \vdash \dfrac{}{\Gamma, P \Downarrow \underline{\neg P} \vdash}}{\dfrac{\Gamma, P \Uparrow \vdash}{\Gamma, P \Uparrow \vdash}} \\[2pt]
\text{Store } \dfrac{\Gamma, P \Uparrow \vdash}{\Gamma \Uparrow P \vdash} \\[2pt]
\text{Release } \dfrac{\Gamma \Uparrow P \vdash}{\Gamma \Downarrow \delta^{-} P \vdash} \qquad \dfrac{\text{attempt to refute } A}{\Gamma \Downarrow A \vdash} \\[2pt]
\Downarrow\vee\vdash \dfrac{\Gamma \Downarrow \delta^{-} P \vdash \qquad \Gamma \Downarrow A \vdash}{\Gamma \Downarrow \delta^{-} P \vee^{-} A \vdash} \\[2pt]
\text{Focus } \dfrac{\Gamma \Downarrow \delta^{-} P \vee^{-} A \vdash}{\delta^{-} P \vee^{-} A, \underline{\neg P} \Uparrow \vdash}
\end{array}
$$

where Γ is $\delta^{-} P \vee^{-} A, \underline{\neg P}$. Depending on A, we cannot even guarantee that the attempt to refute A will eventually end with a failure. Given a particular theory, it is therefore possible to carefully choose the polarization of the axioms of the theory so that the proof-search space is dramatically reduced but completeness w.r.t. that theory is not lost. Finding this particular polarization is the interesting part of our framework, since in general completeness can be lost. Section 5 presents some approaches to do this.

2.1 Synthetic rules

Restricting proof search using focusing leads to what are called synthetic rules (see for instance [16, pp.148–150] where they are called derived rules). The idea is to replace some formula A in the context of the sequent by new inference rules. Instead of proving the sequent $A, \Gamma \Uparrow \Delta \vdash$ in LKF$^{\perp}$, one proves $\Gamma \Uparrow \Delta \vdash$ in (LKF$^{\perp}$ + the synthetic rules obtained from A). Indeed, a proof focusing on A can only have certain shapes, and thus instead of having A in the context, it can be replaced by new rules synthesizing those shapes. For instance, the formula $\underline{P} \vee^{-} (Q \wedge^{-} \underline{R})$ in a context Γ can only lead to the following derivations when the focus is put on it:

$$
\begin{array}{c}
\text{Store } \dfrac{\Gamma, Q \Uparrow \vdash}{\Gamma \Uparrow Q \vdash} \\[2pt]
\text{Release } \dfrac{\Gamma \Uparrow Q \vdash}{\Gamma \Downarrow Q \vdash} \\[2pt]
\widehat{\Downarrow} \vdash \dfrac{}{\Gamma \Downarrow \underline{P} \vdash} \qquad \Downarrow\wedge_1\vdash \dfrac{\Gamma \Downarrow Q \vdash}{\Gamma \Downarrow Q \wedge^{-} \underline{R} \vdash} \\[2pt]
\Downarrow\vee^{-}\dfrac{\Gamma \Downarrow \underline{P} \vdash \qquad \Gamma \Downarrow Q \wedge^{-} \underline{R} \vdash}{\Gamma \Downarrow \underline{P} \vee^{-} (Q \wedge^{-} \underline{R}) \vdash} \\[2pt]
\text{Focus } \dfrac{\Gamma \Downarrow \underline{P} \vee^{-} (Q \wedge^{-} \underline{R}) \vdash}{\Gamma \Uparrow \vdash}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\widehat{\Downarrow} \vdash \dfrac{}{\Gamma \Downarrow \underline{P} \vdash} \qquad \Downarrow\wedge_2\vdash \dfrac{\widehat{\Downarrow} \vdash \dfrac{}{\Gamma \Downarrow \underline{R} \vdash}}{\Gamma \Downarrow Q \wedge^{-} \underline{R} \vdash} \\[2pt]
\Downarrow\vee\vdash \dfrac{\Gamma \Downarrow \underline{P} \vdash \qquad \Gamma \Downarrow Q \wedge^{-} \underline{R} \vdash}{\Gamma \Downarrow \underline{P} \vee^{-} (Q \wedge^{-} \underline{R}) \vdash} \\[2pt]
\text{Focus } \dfrac{\Gamma \Downarrow \underline{P} \vee^{-} (Q \wedge^{-} \underline{R}) \vdash}{\Gamma \Uparrow \vdash}
\end{array}
$$

In the left derivation, $P^{\perp}$ must be in Γ to be able to close the left branch, so Γ is in fact of the form $\underline{P} \vee^{-} (Q \wedge^{-} \underline{R}), \Delta, P^{\perp}$. In the right one, Γ must be of the form $\underline{P} \vee^{-} (Q \wedge^{-} \underline{R}), \Delta, P^{\perp}, R^{\perp}$. Instead of searching for a proof with $\underline{P} \vee^{-} (Q \wedge^{-} \underline{R})$ in the context, the following two synthetic rules can therefore be used:

$$
\text{Syn1 } \dfrac{\Delta, P^{\perp}, Q \Uparrow \vdash}{\Delta, P^{\perp} \Uparrow \vdash} \qquad\qquad \text{Syn2 } \dfrac{}{\Delta, P^{\perp}, R^{\perp} \Uparrow \vdash}
$$

Note that in Syn1 and Syn2, Δ is any context. In particular it need not contain $\underline{P} \vee^{-} (Q \wedge^{-} \underline{R})$. This was the point of using synthetic rules instead of having $\underline{P} \vee^{-} (Q \wedge^{-} \underline{R})$ in the context. This is completely equivalent in terms of provability because each application of a synthetic rule can be replaced by applying Focus on $\underline{P} \vee^{-} (Q \wedge^{-} \underline{R})$ and following the derivation leading to the synthetic rule, and vice versa. This is used for instance in provers based on the inverse method and focusing [32].

For another example, this time with quantifiers, consider the formula $\forall x.\ \underline{P(x)} \vee^- ((\exists y.\ Q(x,y)) \wedge^- \underline{R(f(x))})$ in a context Γ. If the focus is put on this formula, only the following derivations are possible:

$$
\text{Focus}\ \cfrac{\Downarrow\!\forall\vdash\ \cfrac{\Downarrow\vee\vdash\ \cfrac{\widehat{\Downarrow}\vdash\ \cfrac{}{\Gamma \Downarrow \underline{P(t)} \vdash}\qquad \Downarrow\!\wedge_1\!\vdash\ \cfrac{\text{Release}\ \cfrac{\Uparrow\!\exists\vdash\ \cfrac{\text{Store}\ \cfrac{\Gamma, Q(t,y) \Uparrow\ \vdash}{\Gamma \Uparrow Q(t,y) \vdash}}{\Gamma \Uparrow \exists y.\ Q(t,y) \vdash}}{\Gamma \Downarrow \exists y.\ Q(t,y) \vdash}}{\Gamma \Downarrow (\exists y.\ Q(t,y)) \wedge^- \underline{R(f(t))} \vdash}}{\Gamma \Downarrow \underline{P(t)} \vee^- ((\exists y.\ Q(t,y)) \wedge^- \underline{R(f(t))}) \vdash}}{\Gamma \Downarrow \forall x.\ \underline{P(x)} \vee^- ((\exists y.\ Q(x,y)) \wedge^- \underline{R(f(x))}) \vdash}}{\Gamma \Uparrow\ \vdash}
$$

with $P(t)^\perp$ in Γ to be able to apply $\widehat{\Downarrow}\vdash$, and y not free in Γ;

$$
\text{Focus}\ \cfrac{\Downarrow\!\forall\vdash\ \cfrac{\Downarrow\vee\vdash\ \cfrac{\widehat{\Downarrow}\vdash\ \cfrac{}{\Gamma \Downarrow \underline{P(t)} \vdash}\qquad \Downarrow\!\wedge_2\!\vdash\ \cfrac{\widehat{\Downarrow}\vdash\ \cfrac{}{\Gamma \Downarrow \underline{R(f(t))} \vdash}}{\Gamma \Downarrow (\exists y.\ Q(t,y)) \wedge^- \underline{R(f(t))} \vdash}}{\Gamma \Downarrow \underline{P(t)} \vee^- ((\exists y.\ Q(t,y)) \wedge^- \underline{R(f(t))}) \vdash}}{\Gamma \Downarrow \forall x.\ \underline{P(x)} \vee^- ((\exists y.\ Q(x,y)) \wedge^- \underline{R(f(x))}) \vdash}}{\Gamma \Uparrow\ \vdash}
$$

with both $P(t)^\perp$ and $R(f(t))^\perp$ in Γ to be able to apply $\widehat{\Downarrow}\vdash$.

In the first case, Γ is therefore $\forall x.\ \underline{P(x)} \vee^- ((\exists y.\ Q(x,y)) \wedge^- \underline{R(f(x))}), \Delta, P(t)^\perp$; in the second case, it is $\forall x.\ \underline{P(x)} \vee^- ((\exists y.\ Q(x,y)) \wedge^- \underline{R(f(x))}), \Delta, P(t)^\perp, R(f(t))^\perp$. Instead of having the formula in the context, the following two synthetic rules can therefore be used:

$$
\text{Syn1'}\ \cfrac{\Delta, P(t)^\perp, Q(t,y) \Uparrow\ \vdash}{\Delta, P(t)^\perp \Uparrow\ \vdash}\ y \text{ not free in } \Delta, P(t)^\perp \qquad\qquad \text{Syn2'}\ \cfrac{}{\Delta, P(t)^\perp, R(f(t))^\perp \Uparrow\ \vdash}\ .
$$

3 RESOLUTION WITH INPUT SELECTION

Two approaches can be used to reduce the proof-search space of the resolution calculus: first, one can restrict on which pairs of clauses the resolution rule can be applied; this leads for instance to the set-of-support strategy [37], in which clauses are split into two sets, called the theory and the set of support; at least one of the clauses involved in a resolution step must be in the set of support. Second, one can restrict which literals in the clauses can be resolved upon; those literals are said to be selected in the clause. Resolution with free selection is complete for Horn clauses, but incomplete in general. Selecting a subset of the negative literals (if no literal is selected, then any literal of the clause can be used in resolution) is however complete, and combining this with an ordering restriction on clauses with no selected literals leads to Ordered Resolution with Selection, which was introduced by Bachmair and Ganzinger [2] (see also Bachmair and Ganzinger [3]) as a complete refinement of resolution.

Resolution with Input Selection combines these two approaches. We will consider pairs composed of a clause and a subset of its literals (which can be empty). This subset of literals is the set of selected literals in the clause. We represent such a pair by underlining the selected literals in the clause. We will see that they indeed correspond to the literals that have a negative polarization in LKF$^\perp$. We denote by $\mathcal{S}$ the second projection on such a pair. If at least one literal is selected in a clause, then only the selected literals can be used in Resolution. For generated clauses, we impose that $\mathcal{S}(C) = \emptyset$. Note that we can have several times the same clause with different selected literals. The inference rules of Resolution with Input Selection are presented in Fig. 2. As usual, variables

Resolution $\dfrac{L \vee C \qquad L'^\perp \vee D}{\sigma(C \vee D)}$ Factoring $\dfrac{L \vee L' \vee C}{\sigma(L \vee C)}$

- $\mathcal{S}(L \vee C) = \emptyset$
- $\mathcal{S}(L'^\perp \vee D) = \emptyset$
- σ is the most general unifier of $L =^? L'$

- $\mathcal{S}(L \vee L' \vee C) = \emptyset$
- σ is the most general unifier of $L =^? L'$

Resolution with Selection $\dfrac{\underline{K_1} \vee \ldots \vee \underline{K_n} \vee C \qquad K_1'^\perp \vee D_1 \qquad \ldots \qquad K_n'^\perp \vee D_n}{\sigma(C \vee D_1 \vee \ldots \vee D_n)}$

- $\mathcal{S}(\underline{K_1} \vee \ldots \vee \underline{K_n} \vee C) = \{K_1; \ldots; K_n\}$
- $\mathcal{S}(K_i'^\perp \vee D_i) = \emptyset$
- σ is the mgu of the simultaneous unification problem $K_1 =^? K_1', \ldots, K_n =^? K_n'$

Fig. 2. Resolution with Input Selection

are renamed in the clauses to avoid that premises of the inference rules share variables. We have two flavors of the resolution rule: the usual binary resolution, that is applied on two premises that do not select any literal; and Resolution with Selection that is applied on a clause in which n literals are selected and n clauses in which no literal is selected. Consequently, a clause with a non-empty selection cannot be resolved with a clause with a non-empty selection. By considering them as the theory part, and the clauses with an empty selection as the set of support, it is easy to see that Resolution with Input Selection is a generalization of the set-of-support strategy. Notwithstanding, note that neither Resolution with Input Selection is a generalization of Ordered Resolution with Selection nor the converse.

There are two ways to see resolution proofs: either as the successive derivations of new clauses until the empty clause is obtained, or as a tree whose root is the empty clause and whose leaves are included in the set of input clauses.

Definition 3.1 (Resolution derivation). We write $\Gamma \rightsquigarrow C$ if C can be derived from some renaming of the clauses in Γ using the inference rules Resolution with Selection, Resolution, or Factoring presented in Figure 2. We write $\Gamma \rightsquigarrow^* C$ if

- $C \in \Gamma$ or if
- there exists D such that $\Gamma \rightsquigarrow D$ and $\Gamma, D \rightsquigarrow^* C$.

Definition 3.2 (Resolution tree). A *resolution tree* is a tree whose nodes contain clauses, such that C is the father of $C_1, \ldots, C_n$ only if C can be obtained from a renaming of $C_1, \ldots, C_n$ using one of the inference rules Resolution with Selection, Resolution, or Factoring. The multiset of the leaves of the tree is called the axioms of the tree. Its root is called the conclusion of the tree.

Of course, the two views are the same:

LEMMA 3.3. $\Gamma \rightsquigarrow^* C$ *if and only if there exists a resolution tree whose axioms are included (as a set) in Γ, and whose conclusion is C.*

PROOF. For the "only if" direction, we construct the resolution tree by induction on the derivation length. If C is in Γ, then we consider the tree with only one node containing C. The conclusion of the tree is therefore C, and its axioms are the multiset $\{C\}$ which is included as a set in Γ.

For the inductive case, suppose that there exists D such that $\Gamma \rightsquigarrow D$ and $\Gamma, D \rightsquigarrow^* C$. By definition of $\rightsquigarrow$, there exists $C_1, \ldots, C_n$ in Γ such that D is obtained from a renaming of $C_1, \ldots, C_n$ using one of the inference rules Resolution with Selection, Resolution, or Factoring. By induction hypothesis, there exists a resolution tree whose conclusion is C and whose axioms are included as a set in

$\Gamma \cup \{D\}$. We can replace each occurrences of the axiom D in the resolution tree by the subtree $\dfrac{C_1 \quad \cdots \quad C_n}{D}$. The resulting tree is indeed a valid resolution tree, and its conclusion is C. Since all C_i are in Γ, the axioms of this resolution tree are included, as a set, in Γ.

For the "if" direction, we proceed by induction on the size of the resolution tree (number of nodes). If the size is 1, this means that the multiset of axioms is $\{C\}$. By assumption, this multiset is included as a set in Γ, hence $C \in \Gamma$. By definition of $\rightsquigarrow^*$, we have $\Gamma \rightsquigarrow^* C$.

For the inductive case, suppose that the size is $n > 1$. Let us choose any node of the tree, containing a clause D, of which all children are leaves, containing respectively clauses $C_1, \ldots, C_n$. D is therefore derived from a renaming of $C_1, \ldots, C_n$ using one of the inference rules Resolution with Selection, Resolution, or Factoring. By assumption, all C_i belong to Γ. Therefore, by definition of $\rightsquigarrow$, we have

$$\Gamma \rightsquigarrow D. \tag{4}$$

Let us consider the resolution tree where we prune the subtree above D. Its size is strictly smaller than n, its conclusion is C and its axioms are included, as a set, in $\Gamma \cup \{D\}$. By induction hypothesis, we have $\Gamma, D \rightsquigarrow^* C$. Consequently, with (4), $\Gamma \rightsquigarrow^* C$. □

Note that transforming a derivation into a tree and back into a derivation is not the identity, because the derivation corresponds more precisely to a directed acyclic graph that is expanded into a tree by the transformation.

As usual in resolution methods, the goal is to produce the empty clause □ starting from a set of clauses Γ to show, since all rules are sound, that Γ is unsatisfiable. Soundness of Resolution with Input Selection can be shown by remarking that the conclusion of each inference rule is a logical consequence of its premises, as in standard resolution.

Here again, the calculus is not complete in general: from the set of clauses $\underline{P} \vee Q, \underline{\neg P} \vee Q, \neg Q$, no inference rule can be applied: to apply Resolution with Selection, we would need a clause where P, or $\neg P$, is not selected, and Resolution needs two clauses without selection.

4 LKF$^{\perp}$ SIMULATES RESOLUTION WITH INPUT SELECTION

To link LKF$^{\perp}$ with Resolution with Input Selection, we need to indicate how clauses are related to polarized formulas.

Definition 4.1. Given a clause $C = \underline{L_1} \vee \cdots \vee \underline{L_n} \vee K_1 \vee \ldots \vee K_m$ whose free variables are $x_1, \ldots, x_l$ and such that $\mathcal{S}(C) = \{L_1; \ldots; \underline{L_n}\}$, we define the associated formula $\ulcorner C \urcorner$ to be $\forall x_1, \ldots, x_l. \underline{L_1} \vee^- \cdots \vee^- \underline{L_n} \vee^- \delta^-(K_1 \vee^+ \cdots \vee^+ K_m)$. $\ulcorner C \urcorner$ is said to be clausal. By extension, $\ulcorner \Gamma \urcorner$ is the set of formulas associated to the clauses of the set Γ.

Note that the formula associated to the renaming of a clause is the same (modulo α-equivalence) as the formula associated to the clause.

Note that if $\mathcal{S}(C) \neq \emptyset$, or if C is not ground, then the negative delay is useless. Indeed, in such cases, one is already in the synchronous phase when arriving at the subformula at that position, because the surrounding connective is either $\forall$ or $\vee^-$. Adding the delay makes the translation of a clause always negative, even in the case of a ground clause without selection. It makes the statement and the proof of Theorem 4.7 more simple.

The main theorem of this article relates LKF$^{\perp}$ with Resolution with Input Selection:

THEOREM 4.2. *Let Γ be a set of clauses. We have $\ulcorner \Gamma \urcorner \Uparrow \vdash$ iff $\Gamma \rightsquigarrow^* \square$.*

The proof is detailed in the following subsections.

4.1 From Focused Proofs to Resolution Derivations

We need a few lemmas to prove the first direction. Note that these lemmas work on clauses where the selection is empty, which is why we have the restriction that generated clauses do not select literals in Resolution with Input Selection.

LEMMA 4.3. *For all sets of clauses Γ, for all clauses $C_1, \ldots, C_n$ and D such that $\mathcal{S}(C_i) = \emptyset$ for all i and $\mathcal{S}(D) = \emptyset$, if $\Gamma, C_1, \ldots, C_n \rightsquigarrow^* \square$ and $\Gamma, D \rightsquigarrow^* \square$, then $\Gamma, C_1 \vee D, \ldots, C_n \vee D \rightsquigarrow^* \square$ where $\mathcal{S}(C_i \vee D) = \emptyset$ for all i.*

PROOF. By induction on the derivation length of $\Gamma, C_1, \ldots, C_n \rightsquigarrow^* \square$, generalizing on Γ.

The base case is when $\square \in \Gamma$ or there exists i such that $\square = C_i$. In the former case, trivially $\Gamma, C_1 \vee D, \ldots, C_n \vee D \rightsquigarrow^* \square$. In the latter case, $C_i \vee D = D$, and since $\Gamma, D \rightsquigarrow^* \square$ we have a fortiori $\Gamma, C_1 \vee D, \ldots, C_n \vee D \rightsquigarrow^* \square$.

For the inductive case, suppose that there exists C_{n+1} such that $\Gamma, C_1, \ldots, C_n \rightsquigarrow C_{n+1}$ and $\Gamma, C_1, \ldots, C_n, C_{n+1} \rightsquigarrow^* \square$.

There are two cases:

- C_{n+1} is derived using other clauses than one of the C_i. We therefore have $\Gamma, C_1 \vee D, \ldots, C_n \vee D \rightsquigarrow C_{n+1}$.
  We can apply the induction hypothesis on $\Gamma, C_1, \ldots, C_n, C_{n+1} \rightsquigarrow^* \square$, which can be viewed as $\Gamma, C_{n+1}, C_1, \ldots, C_n \rightsquigarrow^* \square$. Indeed, by weakening we have $\Gamma, C_{n+1}, D \rightsquigarrow^* \square$. We obtain $\Gamma, C_{n+1}, C_1 \vee D, \ldots, C_n \vee D \rightsquigarrow^* \square$. By definition of $\rightsquigarrow^*$ we therefore have $\Gamma, C_1 \vee D, \ldots, C_n \vee D \rightsquigarrow^* \square$.
- At least one of the parents of C_{n+1} is some C_i. Since no literal is selected in the C_i, they can only be side clauses of Resolution with Selection, or any clause in Resolution or Factoring. We can therefore derive $C_{n+1} \vee D$ from $\Gamma, C_1 \vee D, \ldots, C_n \vee D$ with the same inference rule that produced C_{n+1}. We can apply the induction hypothesis on $\Gamma, C_1, \ldots, C_n, C_{n+1} \rightsquigarrow^* \square$ which gives us $\Gamma, C_1 \vee D, \ldots, C_n \vee D, C_{n+1} \vee D \rightsquigarrow^* \square$.
  Hence $\Gamma, C_1 \vee D, \ldots, C_n \vee D \rightsquigarrow^* \square$.   □

COROLLARY 4.4. *For all sets of clauses Γ, for all clauses $C_1, \ldots, C_n$ such that $\mathcal{S}(C_i) = \emptyset$ for all i, if $\Gamma, C_i \rightsquigarrow^* \square$ for all i then $\Gamma, C_1 \vee \cdots \vee C_n \rightsquigarrow^* \square$ where $\mathcal{S}(C_1 \vee \cdots \vee C_n) = \emptyset$.*

LEMMA 4.5. *For all multisets of clauses Γ, for all substitutions θ, for all clauses C, for all resolution trees whose axioms are the multiset $\{\theta C\} \cup \Gamma$ with $\mathcal{S}(\theta C) = \emptyset$, and whose conclusion is $\square$, there exists a resolution tree whose axioms are the multiset $\{C\} \cup \Gamma$ with $\mathcal{S}(C) = \emptyset$, and whose conclusion is $\square$.*

PROOF. By induction on the lexicographic combination of the size (number of nodes) of the tree and the cardinal of C.

If the tree size is 1, this means that $\Gamma = \emptyset$ and $\theta C = \square$. Thus, $C = \square$ too and we have a trivial tree of size 1 whose conclusion is $\square$ and whose multiset of axioms is $\{\square\}$.

If the tree size is $n > 1$, let us consider the inference rule of which θC is a premise.

If it is Factoring, then the conclusion of the inference rule is $\sigma \theta C$, where σ is the most general unifier of two literals of θC. Let us consider the tree where we prune $\text{Factoring } \dfrac{\theta C}{\sigma \theta C}$ to have just $\sigma \theta C$ as an axiom. The axioms of this tree are $\{\sigma \theta C\} \cup \Gamma$ and its conclusion is $\square$. Its size is $n - 1$. By induction hypothesis, using $\sigma \theta$ instead of θ, we obtain a resolution tree whose axioms are $\{C\} \cup \Gamma$ and whose conclusion is $\square$, as was expected.

Let us consider the case when the inference rule is Resolution with Selection or Resolution. Since $\mathcal{S}(\theta C) = \emptyset$, only one literal of θC is used in the inference rule. Let it be denoted by L, then θC is $L \vee \theta C'$, and C is $L_1 \vee \ldots \vee L_m \vee C'$ where the L_i are exactly the literals of C such that

$$\theta L_i = L. \tag{5}$$

If $m > 1$, since $\theta L_1 = \theta L_2 = L$, the literals L_1 and L_2 can be unified. Let ω be their most general unifier. By definition, there exists θ' such that $\theta = \theta'\omega$. The axioms of the resolution tree are therefore the multiset $\{\theta'\omega C\} \cup \Gamma$. The cardinal of ωC is smaller than the cardinal of C, since at least ωL_1 and ωL_2 are identical. By induction hypothesis, using θ' instead of θ, we get a resolution tree whose axioms are $\{\omega C\} \cup \Gamma$ and whose conclusion is $\square$. We can extend this resolution tree by replacing ωC with the subtree Factoring $\dfrac{C}{\omega C}$, thus obtaining a resolution tree whose axioms are $\{C\} \cup \Gamma$ and whose conclusion is $\square$, as was expected.

If $m = 1$, since $\theta C = L \vee \theta C'$ where L is the literal used in the inference rule, the latter is therefore of the form $\dfrac{\theta C \quad C_1 \vee D_1 \quad \ldots \quad C_k \vee D_k}{\sigma(\theta C' \vee D_1 \vee \ldots D_k)}$ where σ is the most general solution of a unification problem of the form $L =^? L', Prob$ for some set of equations $Prob$ (which may be non-empty in the Resolution with Selection case). In particular, this means that

$$\sigma L = \sigma L'. \tag{6}$$

Since clauses are renamed before applying inference rules, we can assume that the variables of all $C_i \vee D_i$ are not in the domain of θ. In particular, L' and all literals in $Prob$ are not affected by θ. Therefore,

$$L' = \theta L' \tag{7}$$

and

$$D_i = \theta D_i \qquad \text{for all } 1 \le i \le k. \tag{8}$$

Combining (5), (6), and (7), we obtain
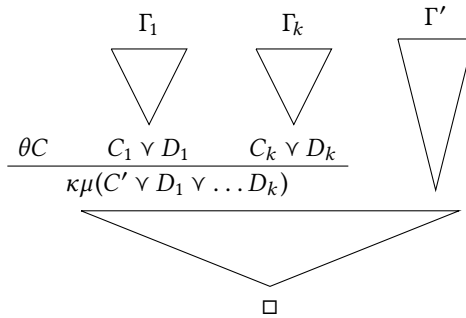
$$\sigma \theta L_1 = \sigma \theta L' \tag{9}$$

$\sigma \theta$ is therefore a solution of the unification problem $L_1 =^? L', Prob$ (recall that variables in $Prob$ are unaffected by θ). Let μ be its most general solution. By definition, there exists κ such that

$$\sigma \theta = \kappa \mu. \tag{10}$$

The conclusion of the inference rule is

$$
\begin{aligned}
&\sigma(\theta C' \vee D_1 \vee \ldots D_k) \\
&= \sigma(\theta C' \vee \theta D_1 \vee \ldots \theta D_k) && \text{by (8)} \\
&= \sigma \theta (C' \vee D_1 \vee \ldots D_k) \\
&= \kappa \mu (C' \vee D_1 \vee \ldots D_k) && \text{by (10).}
\end{aligned}
$$

The resolution tree has the form

where $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_k \cup \Gamma'$.

Let us consider the resolution tree where we prune the subtree above $\kappa \mu(C' \vee D_1 \vee \ldots D_k)$. Its size is strictly smaller than n, its axioms are $\{\kappa \mu(C' \vee D_1 \vee \ldots D_k)\} \cup \Gamma'$, and its conclusion is $\square$.

Since, by definition of Resolution with Input Selection, $\mathcal{S}(\kappa\mu(C' \vee D_1 \vee \ldots D_k)) = \emptyset$, we can apply the induction hypothesis to obtain a resolution tree whose axioms are $\{\mu(C' \vee D_1 \vee \ldots D_k)\} \cup \Gamma'$ and whose conclusion is $\square$. We can extend this tree by replacing the leaf $\mu(C' \vee D_1 \vee \ldots D_k)$ by the tree

$$\frac{C \qquad \overset{\displaystyle\Gamma_1}{\underset{C_1 \vee D_1}{\bigvee}} \qquad \overset{\displaystyle\Gamma_k}{\underset{C_k \vee D_k}{\bigvee}}}{\mu(C' \vee D_1 \vee \ldots D_k)}$$

which is valid because μ is the most general solution to the unification problem $L_1 =^? L', Prob$ which is the one that needs to be solved for the inference rule to be applied. We therefore obtain a resolution tree whose axioms are $\{C\} \cup \Gamma_1 \cup \cdots \cup \Gamma_k \cup \Gamma' = \{C\} \cup \Gamma$ and whose conclusion is $\square$, as was expected. □

Lemma 4.6. *For all sets of clauses Γ, for all substitutions θ, for all clauses C such that $\mathcal{S}(\theta C) = \emptyset$, if $\Gamma, \theta C \rightsquigarrow^* \square$ then $\Gamma, C \rightsquigarrow^* \square$ where $\mathcal{S}(C) = \emptyset$.*

Proof. Using Lemma 3.3, there exists a resolution tree whose axioms are included, as a set, in $\Gamma \cup \{\theta C\}$, and whose conclusion is $\square$. Repetitively applying Lemma 4.5, one can replace all axioms θC by C, thus obtaining a resolution tree whose axioms are included, as a set, in $\Gamma \cup \{C\}$. Using Lemma 3.3 in the converse direction, we obtain $\Gamma, C \rightsquigarrow^* \square$. □

Theorem 4.7. *If $\ulcorner\Gamma\urcorner \Uparrow\vdash$, then $\Gamma \rightsquigarrow^* \square$.*

Proof. By induction on the proof $\ulcorner\Gamma\urcorner \Uparrow\vdash$. We generalize the statement a little by allowing the sequent to contain not only translations of clauses but also literals with a positive polarity, which are associated to the corresponding singleton clauses. On such a sequent, only two rules can be applied, namely $\widehat{\Uparrow}\vdash$ and Focus. Since $\ulcorner\Gamma\urcorner$ contains only clausal formulas and positive literals, there are only four cases:

- $$\widehat{\Uparrow}\vdash \frac{}{\ulcorner\Gamma'\urcorner, L, L^\perp \Uparrow \vdash}$$
  In that case, we can simply apply Resolution on L and $L^\perp$ to derive $\square$, hence $\Gamma', L, L^\perp \rightsquigarrow \square$.
- The proof focuses on a formula corresponding to the empty clause:
  $$\text{Focus} \frac{\Downarrow\perp\vdash \dfrac{}{\ulcorner\Gamma\urcorner \Downarrow \perp \vdash}}{\ulcorner\Gamma\urcorner \Uparrow \vdash}$$
  In that case, $\square$ already belongs to Γ.
- The proof focuses on a formula corresponding to a clause without selection. Because of focusing constraints, the proof is necessarily of the form:

  $$\text{Focus} \frac{\Downarrow\forall\vdash \dfrac{\text{Release} \dfrac{\Uparrow\vee\vdash \dfrac{\text{Store} \dfrac{\ulcorner\Gamma\urcorner, \theta L_1 \Uparrow \vdash}{\ulcorner\Gamma\urcorner \Uparrow \theta L_1 \vdash} \quad \cdots \quad \text{Store} \dfrac{\ulcorner\Gamma\urcorner, \theta L_m \Uparrow \vdash}{\ulcorner\Gamma\urcorner \Uparrow \theta L_m \vdash}}{\ulcorner\Gamma\urcorner \Uparrow \theta(L_1 \vee^+ \cdots \vee^+ L_m) \vdash}}{\ulcorner\Gamma\urcorner \Downarrow \theta(L_1 \vee^+ \cdots \vee^+ L_m) \vdash}}{\ulcorner\Gamma\urcorner \Downarrow \overline{\forall x}.\delta^-(L_1 \vee^+ \cdots \vee^+ L_m) \vdash}}{\ulcorner\Gamma\urcorner \Uparrow \vdash}$$

  By induction hypothesis, we have derivations of $\Gamma, \theta L_k \rightsquigarrow^* \square$ for all $1 \le k \le m$. By Corollary 4.4, we have a derivation $\Gamma, \theta(L_1 \vee \cdots \vee L_k) \rightsquigarrow^* \square$ with nothing selected in $\theta(L_1 \vee \cdots \vee L_k)$. By Lemma 4.6, we have a derivation of $\Gamma \rightsquigarrow^* \square$, with $L_1 \vee \cdots \vee L_k$ in Γ.

- The proof focuses on a formula corresponding to a clause with selection. Because of focusing constraints, the proof is necessarily of the form:

$$
\text{Focus} \dfrac{
  \Downarrow\!\vee\vdash \dfrac{
    \Downarrow\!\vee\vdash \dfrac{
      \cdots \quad
      \widehat{\Downarrow}\vdash \dfrac{}{\ulcorner\Gamma\urcorner \Downarrow \underline{\theta K_j} \vdash} \quad \cdots \quad
      \Uparrow\!\vee\vdash \dfrac{
        \cdots \quad
        \text{Store} \dfrac{\ulcorner\Gamma\urcorner, \theta L_k \Uparrow \vdash}{\ulcorner\Gamma\urcorner \Uparrow \theta L_k \vdash} \quad \cdots
      }{
        \text{Release} \dfrac{\ulcorner\Gamma\urcorner \Uparrow \theta(L_1 \vee^+ \cdots \vee^+ L_m) \vdash}{\ulcorner\Gamma\urcorner \Downarrow \delta^- \theta(L_1 \vee^+ \cdots \vee^+ L_m) \vdash}
      }
    }{\ulcorner\Gamma\urcorner \Downarrow \theta(\underline{K_1} \vee^- \cdots \vee^- \underline{K_n} \vee^- \delta^-(L_1 \vee^+ \cdots \vee^+ L_m)) \vdash}
  }{\ulcorner\Gamma\urcorner \Downarrow \overline{\forall x}.\underline{K_1} \vee^- \cdots \vee^- \underline{K_n} \vee^- \delta^-(L_1 \vee^+ \cdots \vee^+ L_m) \vdash}
}{\ulcorner\Gamma\urcorner \Uparrow \vdash}
$$

where $\overline{\forall x}.\underline{K_1} \vee^- \cdots \vee^- \underline{K_n} \vee^- (L_1 \vee^+ \cdots \vee^+ L_m)$ and $\theta K_j^\perp$ for all $1 \le j \le n$ are members of $\ulcorner\Gamma\urcorner$.

By induction hypothesis, we have derivations of $\Gamma, \theta L_k \rightsquigarrow^* \square$ for all $1 \le k \le m$. By Corollary 4.4, we have a derivation $\Gamma, \theta(L_1 \vee \cdots \vee L_m) \rightsquigarrow^* \square$.

From $\underline{K_1} \vee \cdots \vee \underline{K_n} \vee L_1 \vee \cdots \vee L_m$, since the application of $\widehat{\Downarrow}\vdash$ above imposes that all $\theta K_j^\perp$ are in $\overline{\Gamma}$, we can apply Resolution with Selection to obtain $\theta(L_1 \vee \cdots \vee L_m)$, hence a derivation $\Gamma \rightsquigarrow^* \square$. $\qquad\square$

4.2 From Resolution Derivations to Focused Proofs

We prove that all inference rules of Resolution with Input Selection are admissible in $\text{LKF}^\perp$: if $\Gamma \rightsquigarrow C$ then $\text{LKF}^\perp$ proofs of $\ulcorner\Gamma\urcorner, \ulcorner C\urcorner \Uparrow\vdash$ can be turned into proofs of $\ulcorner\Gamma\urcorner \Uparrow\vdash$. For this direction, we do not need to assume that generated clauses do not select any literal: if a derivation is found in Resolution with selection, it would be translated into $\text{LKF}^\perp$ even if some literals are selected in generated clauses, as long as selection is compatible with instantiation, i.e. $\sigma \mathcal{S}(C) \subseteq \mathcal{S}(\sigma C)$ for all clauses C and substitutions σ. (Note that we do not need the converse inclusion.)

LEMMA 4.8. *For all sets of formulas Γ, for all clauses C, for all substitutions σ, assuming $\sigma \mathcal{S}(C) \subseteq \mathcal{S}(\sigma C)$, if $\Gamma, \ulcorner C\urcorner, \ulcorner \sigma C\urcorner \Uparrow\vdash$, then $\Gamma, \ulcorner C\urcorner \Uparrow\vdash$.*

PROOF. By induction on the proof $\Gamma, \ulcorner C\urcorner, \ulcorner \sigma C\urcorner \Uparrow\vdash$. Note that to be able to apply the induction hypothesis directly, one needs to have a sequent of the form $\Gamma' \Uparrow\vdash$, so that one considers in general coarse grain proof steps consisting of an alternation of a synchronous and an asynchronous phases.

If the proof does not begin by acting on $\ulcorner \sigma C\urcorner$, this is a simple application of the induction hypothesis. Otherwise, note that the only rule that can act on $\ulcorner \sigma C\urcorner$ is Focus: even in the case when σC consists only of a single literal, $\ulcorner \sigma C\urcorner$ is $\delta^-(\sigma C)$ so that $\widehat{\Uparrow}\vdash$ cannot be applied. Let σC be $\underline{K_1} \vee \cdots \vee \underline{K_n} \vee L_1 \vee \cdots \vee L_m$. The proof begins with

$$
\text{Focus} \dfrac{
  \Downarrow\!\vee\vdash \dfrac{
    \Downarrow\!\vee\vdash \dfrac{
      \cdots \quad
      \widehat{\Downarrow}\vdash \dfrac{}{\Gamma' \Downarrow \underline{\theta K_j} \vdash} \quad \cdots \quad
      \Uparrow\!\vee\vdash \dfrac{
        \cdots \quad
        \text{Store} \dfrac{\Gamma', \theta L_k \Uparrow \vdash}{\Gamma' \Uparrow \theta L_k \vdash} \quad \cdots
      }{
        \text{Release} \dfrac{\Gamma' \Uparrow \theta(L_1 \vee^+ \cdots \vee^+ L_m) \vdash}{\Gamma' \Downarrow \theta\delta^-(L_1 \vee^+ \cdots \vee^+ L_m) \vdash}
      }
    }{\Gamma' \Downarrow \theta(\underline{K_1} \vee^- \cdots \vee^- \underline{K_n} \vee^- \delta^-(L_1 \vee^+ \cdots \vee^+ L_m)) \vdash}
  }{\Gamma' \Downarrow \overline{\forall x}.\underline{K_1} \vee^- \cdots \vee^- \underline{K_n} \vee^- \delta^-(L_1 \vee^+ \cdots \vee^+ L_m) \vdash}
}{\Gamma' \Uparrow \vdash}
$$

where $\Gamma' = \Gamma, \ulcorner C\urcorner, \ulcorner \sigma C\urcorner$ and for all $1 \le j \le n$ the literal $\theta K_j^\perp$ is in Γ.

Let C be $\underline{K_1^1} \vee \cdots \underline{K_1^{k_1}} \vee \cdots \underline{K_n^1} \vee \cdots \underline{K_n^{k_n}} \vee L_1^1 \vee \cdots L_1^{l_1} \vee \cdots L_m^1 \vee \cdots L_m^{l_m}$ where for all i,j we have $\sigma K_i^j = K_i$ and $\sigma L_i^j = L_i$. By hypothesis, the literals selected in C are among the K_i^j (hence the dashed underline). For all i,j, one can either build a proof $\dfrac{\overset{\Downarrow\vdash}{}}{\Gamma, \ulcorner C \urcorner \Downarrow \underline{\theta\sigma K_i^j} \vdash}$ if it is selected,

or $\text{Store} \dfrac{\overset{\widehat{\Uparrow\vdash}}{\Gamma, \ulcorner C \urcorner, \theta\sigma K_i^j \Uparrow \vdash}}{\Gamma, \ulcorner C \urcorner \Uparrow \theta\sigma K_i^j \vdash}$ if it is not.

We apply the induction hypothesis on $\Gamma', \theta L_k \Uparrow\vdash$, hence we have proofs $\Gamma, \ulcorner C \urcorner, \theta L_k \Uparrow\vdash$ for all k. We can therefore build the proof

$$
\text{Focus} \dfrac{
\dfrac{
\dfrac{
\overset{\Downarrow\vdash}{\Gamma,\ulcorner C\urcorner \Downarrow \underline{\theta\sigma K_i^j}\vdash} \quad \cdots \quad \text{Release} \dfrac{
\Uparrow\vee\vdash \dfrac{
\text{Store}\dfrac{\overset{\widehat{\Uparrow\vdash}}{\Gamma,\ulcorner C\urcorner,\theta\sigma K_i^j \Uparrow\vdash}}{\Gamma,\ulcorner C\urcorner \Uparrow \theta\sigma K_i^j\vdash} \quad \cdots \quad \Gamma,\ulcorner C\urcorner,\theta L_i \Uparrow\vdash
}{\Gamma,\ulcorner C\urcorner \Uparrow \cdots \vee^+ \theta\sigma K_l^m \vee^+ \cdots \vee^+ \theta\sigma L_i^j \vee^+ \cdots \vdash}
}{\Gamma,\ulcorner C\urcorner \Downarrow \delta^-(\cdots \vee^+ \theta\sigma K_l^m \vee^+ \cdots \vee^+ \theta\sigma L_i^j \vee^+ \cdots)\vdash}
}{\Downarrow\vee\vdash \quad \Gamma,\ulcorner C\urcorner \Downarrow \cdots \vee^- \underline{\theta\sigma K_i^j} \vee^- \cdots \vee^- \delta^-(\cdots \vee^+ \theta\sigma K_l^m \vee^+ \cdots \vee^+ \theta\sigma L_i^j \vee^+ \cdots)\vdash}
}{\Downarrow\forall\vdash \quad \Gamma,\ulcorner C\urcorner \Downarrow \overline{\forall x.}\cdots \vee^- \underline{K_i^j} \vee^- \cdots \vee^- \delta^-(\cdots \vee^+ K_l^m \vee^+ \cdots \vee^+ L_i^j \vee^+ \cdots)\vdash}
}{\Gamma,\ulcorner C\urcorner \Uparrow\vdash} \qquad \square
$$

COROLLARY 4.9. *Factoring is admissible in* $LKF^\perp$.

LEMMA 4.10. *Resolution is admissible in* $LKF^\perp$:
For all sets of formulas Γ, *for all clauses* $L \vee C$ *and* $L'^\perp \vee D$ *without selection, if* $\sigma = mgu(L, L')$, *if* $\Gamma, \ulcorner L \vee C \urcorner, \ulcorner L'^\perp \vee D \urcorner, \ulcorner \sigma(C \vee D) \urcorner \Uparrow\vdash$ *then* $\Gamma, \ulcorner L \vee C \urcorner, \ulcorner L'^\perp \vee D \urcorner \Uparrow\vdash$.

PROOF. By induction on the proof $\Gamma, \ulcorner L \vee C \urcorner, \ulcorner L'^\perp \vee D \urcorner, \ulcorner \sigma(C \vee D) \urcorner \Uparrow\vdash$.

Let Γ' be $\Gamma, \ulcorner L \vee C \urcorner, \ulcorner L'^\perp \vee D \urcorner$ and Γ'' be $\Gamma', \ulcorner \sigma(C \vee D) \urcorner$. If the proof does not begin by focusing on $\ulcorner \sigma(C \vee D) \urcorner$, this is a simple application of the induction hypothesis. Otherwise, let $\sigma(C \vee D)$ be $\underline{I_1} \vee \cdots \vee \underline{I_n} \vee J_1 \vee \cdots \vee J_m$. The proof begins with

$$
\text{Focus}\dfrac{
\dfrac{
\dfrac{
\cdots \quad \overset{\Downarrow\vdash}{\Gamma''\Downarrow \underline{\theta I_j}\vdash} \quad \cdots \quad \text{Release}\dfrac{
\Uparrow\vee\vdash\dfrac{
\cdots \quad \text{Store}\dfrac{\Gamma'',\theta J_k \Uparrow\vdash}{\Gamma''\Uparrow \theta J_k\vdash} \quad \cdots
}{\Gamma''\Uparrow \theta(J_1 \vee^+ \cdots \vee^+ J_m)\vdash}
}{\Gamma''\Downarrow \theta\delta^-(J_1 \vee^+ \cdots \vee^+ J_m)\vdash}
}{\Downarrow\vee\vdash \quad \Gamma''\Downarrow \theta(\underline{I_1}\vee^- \cdots \vee^- \underline{I_n}\vee^- \delta^-(J_1 \vee^+ \cdots \vee^+ J_m))\vdash}
}{\Downarrow\forall\vdash \quad \Gamma''\Downarrow \overline{\forall x.}\underline{I_1}\vee^- \cdots \vee^- \underline{I_n}\vee^- \delta^-(J_1 \vee^+ \cdots \vee^+ J_m)\vdash}
}{\Gamma''\Uparrow\vdash}
$$

where, to be able to close the left branches, for all $1 \le j \le n$ the literal $\theta I_j^\perp$ is in Γ'.

We know that C is $\cdots \vee I_i^1 \vee \cdots I_i^{k_i} \vee \cdots \vee J_j^1 \vee \cdots J_j^{l_j} \vee \cdots$ where i ranges over a subset of $\{1, \ldots, n\}$ and j over a subset of $\{1, \ldots, m\}$, and $\sigma I_i^x = I_i$ and $\sigma J_j^y = J_j$ for all x, y. Likewise, D is $\cdots \vee I_i'^1 \vee \cdots I_i'^{k_i} \vee \cdots \vee J_j'^1 \vee \cdots J_j'^{l_j} \vee \cdots$ where i ranges over a subset of $\{1, \ldots, n\}$ and j over a subset of $\{1, \ldots, m\}$, and $\sigma I_i'^x = I_i$ and $\sigma J_j'^y = J_j$ for all x, y.

We apply the induction hypothesis on $\Gamma'', \theta J_k \Uparrow\vdash$, hence we have a proof of $\Gamma, \ulcorner L \vee C \urcorner, \ulcorner L'^\perp \vee D \urcorner, \theta J_k \Uparrow\vdash$.

We can build the following proof of $\Gamma' \Uparrow\vdash$: first, we focus on $\ulcorner L \vee C \urcorner$ to get the derivation

$$
\text{Focus} \frac{\Large \begin{array}{c} \text{Release} \frac{\Uparrow\vee\vdash \dfrac{\text{Store} \dfrac{\Gamma', \theta\sigma L \Uparrow\vdash}{\Gamma' \Uparrow \theta\sigma L \vdash} \quad \cdots \quad \text{Store} \dfrac{\widehat{\Uparrow}\vdash \overline{\Gamma', \theta I_i \Uparrow\vdash}}{\Gamma' \Uparrow \theta I_i \vdash} \quad \cdots \quad \text{Store} \dfrac{\Gamma', \theta J_j \Uparrow\vdash}{\Gamma' \Uparrow \theta J_j \vdash} \quad \cdots}{\Gamma' \Uparrow \theta\sigma(L \vee^+ \cdots \vee^+ I_i^1 \vee^+ \cdots I_i^{k_i} \vee^+ \cdots \vee^+ J_j^1 \vee^+ \cdots J_j^{l_j} \vee^+ \cdots) \vdash}}{\Gamma' \Downarrow \theta\sigma\delta^-(L \vee^+ \cdots \vee^+ I_i^1 \vee^+ \cdots I_i^{k_i} \vee^+ \cdots \vee^+ J_j^1 \vee^+ \cdots J_j^{l_j} \vee^+ \cdots) \vdash}} \\ \Downarrow\forall\vdash \overline{\Gamma' \Downarrow \overline{\forall x}.\, \delta^-(L \vee^+ \cdots \vee^+ I_i^1 \vee^+ \cdots I_i^{k_i} \vee^+ \cdots \vee^+ J_j^1 \vee^+ \cdots J_j^{l_j} \vee^+ \cdots) \vdash} \end{array}}{\Gamma' \Uparrow \vdash}
$$

In this derivation, all right branches are closed by induction hypothesis. On the left branch, we focus on $\ulcorner L'^\perp \vee D \urcorner$. Let Γ''' be $\Gamma', \theta\sigma L$. We get the derivation

$$
\text{Focus} \frac{\Large \begin{array}{c} \text{Release} \frac{\Uparrow\vee\vdash \dfrac{\text{Store} \dfrac{\widehat{\Uparrow}\vdash \overline{\Gamma''', \theta\sigma L'^\perp \Uparrow\vdash}}{\Gamma''' \Uparrow \theta\sigma L'^\perp \vdash} \quad \cdots \quad \text{Store} \dfrac{\widehat{\Uparrow}\vdash \overline{\Gamma''', \theta I_i \Uparrow\vdash}}{\Gamma''' \Uparrow \theta I_i \vdash} \quad \cdots \quad \text{Store} \dfrac{\Gamma''', \theta J_j \Uparrow\vdash}{\Gamma''' \Uparrow \theta J_j \vdash} \quad \cdots}{\Gamma''' \Uparrow \theta\sigma(L'^\perp \vee^+ \cdots \vee^+ I_i'^1 \vee^+ \cdots I_i'^{k_i} \vee^+ \cdots \vee^+ J_j'^1 \vee^+ \cdots J_j'^{l_j} \vee^+ \cdots) \vdash}}{\Gamma''' \Downarrow \theta\sigma\delta^-(L'^\perp \vee^+ \cdots \vee^+ I_i'^1 \vee^+ \cdots I_i'^{k_i} \vee^+ \cdots \vee^+ J_j'^1 \vee^+ \cdots J_j'^{l_j} \vee^+ \cdots) \vdash}} \\ \Downarrow\forall\vdash \overline{\Gamma''' \Downarrow \overline{\forall x}.\, \delta^-(L'^\perp \vee^+ \cdots \vee^+ I_i'^1 \vee^+ \cdots I_i'^{k_i} \vee^+ \cdots \vee^+ J_j'^1 \vee^+ \cdots J_j'^{l_j} \vee^+ \cdots) \vdash} \end{array}}{\Gamma''' \Uparrow \vdash}
$$

The left branch can be closed because $\theta\sigma L = \theta\sigma L'$. The right branches are closed by induction hypothesis. □

LEMMA 4.11. *Resolution with Selection is admissible in $LKF^\perp$:*
For all sets of formulas Γ, for all clauses $\underline{K_1} \vee \ldots \vee \underline{K_n} \vee C$, $K_1'^\perp \vee D_1$, ..., and $K_n'^\perp \vee D_n$, where
$\mathcal{S}(\underline{K_1} \vee \ldots \vee \underline{K_n} \vee C) = \{K_1; \ldots; K_n\}$, $\mathcal{S}(\overline{K_i'^\perp \vee D_i}) = \emptyset$ *and σ is the most general unifier of the*
simultaneous unification problem $K_1 =^? K_1', \ldots, K_n =^? K_n'$,
if $\Gamma, \ulcorner \underline{K_1} \vee \ldots \vee \underline{K_n} \vee C \urcorner, \ldots \ulcorner K_i'^\perp \vee D_i \urcorner, \ldots, \ulcorner \sigma(C \vee D_1 \vee \cdots \vee D_n) \urcorner \Uparrow\vdash$
then $\Gamma, \ulcorner \underline{K_1} \vee \ldots \vee \underline{K_n} \vee C \urcorner, \ldots \ulcorner K_i'^\perp \vee D_i \urcorner, \ldots \Uparrow\vdash$.

PROOF. By induction on the proof
$\Gamma, \ulcorner \underline{K_1} \vee \ldots \vee \underline{K_n} \vee C \urcorner, \ldots, \ulcorner K_i'^\perp \vee D_i \urcorner, \ldots, \ulcorner \sigma(C \vee D_1 \vee \ldots \vee D_n) \urcorner \Uparrow\vdash$. We follow the same idea as in the proofs of the two precedent lemmas.

If the proof does not begin by focusing on $\ulcorner \sigma(C \vee D_1 \vee \cdots \vee D_n) \urcorner$, this is a simple application of the induction hypothesis. Otherwise, let Γ' be $\Gamma, \ulcorner \underline{K_1} \vee \ldots \vee \underline{K_n} \vee C \urcorner, \ldots \ulcorner K_i'^\perp \vee D_i \urcorner, \ldots$ and Γ'' be $\Gamma', \ulcorner \sigma(C \vee D_1 \vee \cdots \vee D_n) \urcorner$. Focusing leads us either to sequents $\Gamma'' \Downarrow \theta I_j \vdash$, with θI_j in Γ, or to sequents $\Gamma'', \theta J_k \Uparrow\vdash$ upon which one can apply the induction hypothesis. Let us remark that for each literal L of C or D_i, $\theta\sigma L$ is either one of θI_j or one of θJ_k. Therefore, we know how to close proofs of $\Gamma' \Uparrow \theta\sigma L \vdash$ for each, either by induction hypothesis or using $\theta I_j^\perp$ in Γ.

To build the proof of $\Gamma' \Uparrow\vdash$, we first focus on $\ulcorner K_1'^\perp \vee D_1 \urcorner$, instantiating the variables using the substitution $\theta\sigma$. As explained above, we know how to close the branches coming from D_1, it remains the branch $\Gamma', \theta\sigma K_1'^\perp \Uparrow\vdash$.

We do the same, focusing on $\ulcorner K_2'^\perp \vee D_2 \urcorner$ then ... then $\ulcorner K_n'^\perp \vee D_n \urcorner$ and the remaining branch is $\Gamma', \theta\sigma K_1'^\perp, \ldots \theta\sigma K_n'^\perp \Uparrow\vdash$.

We can close the proof by focusing on $\ulcorner \underline{K_1} \vee \ldots \vee \underline{K_n} \vee C \urcorner$. Branches coming from C can be closed as before, and the other branches are closed by

$$\widehat{\Downarrow} \vdash \frac{}{\Gamma', \theta\sigma K_1'^{\perp}, \dots \theta\sigma K_n'^{\perp} \Downarrow \underline{\theta\sigma K_i} \vdash}$$

since $\theta\sigma K_i' = \theta\sigma K_i$ for all i. ☐

THEOREM 4.12. *For all sets of clauses* Γ, *if* $\Gamma \rightsquigarrow^* \square$, *then* $\ulcorner\Gamma\urcorner \Uparrow\vdash$.

PROOF. By induction on the length of the derivation $\Gamma \rightsquigarrow^* \square$. If $\square$ is in Γ, then we focus on $\ulcorner\square\urcorner = \bot$ and apply $\Downarrow\bot\vdash$. If the first step is Factoring, we apply Lemma 4.8. If it is Resolution, we apply Lemma 4.10. If it is Resolution with Selection, we apply Lemma 4.11. ☐

Note that the rules of resolution with input selection are admissible, but they are not derivable. In particular, the size of the proof in LKF$^{\perp}$ can be much larger than the resolution derivation, as expected in a cut-free sequent calculus. Using cuts would lead to a closer correspondence between resolution derivations and sequent-calculus proofs, as in [18]. However, we chose to stay in the cut-free fragment to prove that, even in the incomplete case, resolution coincides with cut-free proofs, as in [29].

5 COMPLETE INSTANCES

5.1 Ordinary Focusing and Semantic Hyperresolution

As said earlier, in standard LKF, not all occurrences of literals can have an arbitrary polarity. Instead, each atomic formula P is given globally a polarity, and $P^{\perp}$ has the opposite polarity.

Let us first look at the simple case where atomic formulas are given a positive polarity. We recall the completeness theorem of LKF:

THEOREM 5.1 (COROLLARY OF [31, THEOREM 17]). *If the literals with a positive polarity are exactly the atomic formulas, LKF$^{\perp}$ is (sound and) complete.*

If we look at the corresponding resolution calculus, Resolution with Selection for this particular instance becomes:

$$\text{R.w.S.} \frac{\neg P_1 \vee \dots \vee \neg P_n \vee C \qquad P_1' \vee D_1 \qquad \dots \qquad P_n' \vee D_n}{\sigma(C \vee D_1 \vee \dots \vee D_n)}$$

where C and D_i for all i contain only positive literals, and σ is the most general unifier of $P_1 =^? P_1', \dots, P_n =^? P_n'$. Note that the clause $\sigma(C \vee D_1 \vee \dots \vee D_n)$ contains only atomic formulas, and in agreement with the definition of Resolution with Selection, they have a positive polarity; therefore, no literal would be selected in it even if it was an input clause. Besides, Resolution cannot be applied, since there exists no clause $\neg P \vee C$ with $\mathcal{S}(\neg P \vee C) = \emptyset$.

This corresponding resolution calculus is therefore exactly hyperresolution of Robinson [34]: premises of an inference contain only positive literals, except one clause whose all negative literals are resolved at once. Theorem 4.2 therefore links ordinary focusing with hyperresolution. Consequently, Theorem 5.1 implies the completeness of hyperresolution.

Chaudhuri et al. [16, Theorem 16] prove a similar result by establishing a correspondence between hyperresolution derivations and proofs in a focused sequent calculus for intuitionistic linear logic, but only considering Horn clauses. In their setting, choosing a negative polarity for atomic formulas leads to SLD resolution, which is the reasoning mechanism of Prolog. Note that in our case, choosing a negative polarity for atomic formulas leads to what is called negative hyperresolution, which is the same as hyperresolution but with the role of positive and negative literals inversed. The difference with [16] is that it is concerned with intuitionistic logic, so that positive and negative literals are not dual.

Let us now look at the general case, where atomic formulas are given an arbitrary polarity. Let us first stick to the ground case. We recall a refinement of resolution called Semantic hyperresolution [36][14, Sect. 1.3.5.3]. Let I be an arbitrary Herbrand interpretation, i.e. a model whose domain is the set of terms interpreted as themselves. Note that I is not assumed to be a model of the input set of clauses (which is fortunate, since one is trying to show that it is unsatisfiable). Given a clause C, the idea of semantic hyperresolution is to resolve all literals of C that are valid in I at once, with clauses in which all literals are not valid in I. This gives the rule:

$$\text{SHR} \ \frac{K_1 \vee \ldots \vee K_n \vee C \qquad K_1^\perp \vee D_1 \quad \ldots \quad K_n^\perp \vee D_n}{C \vee D_1 \vee \ldots \vee D_n}$$

where for all i, $I \models K_i$ (and thus $I \not\models K_i^\perp$), $I \not\models C$ and $I \not\models D_i$. Note that $I \not\models C \vee D_1 \vee \ldots \vee D_n$.

Semantic hyperresolution for a Herbrand interpretation I can be seen as an instance of Resolution with Input Selection by using the following polarization of atomic formulas: a literal L has a negative polarity iff $I \models L$. In that case, SHR corresponds exactly to Resolution with Selection, and Resolution cannot be applied since we cannot have clauses $P \vee C$ and $\neg P \vee D$ where both P and $\neg P$ are not valid in I.

This particular instance of polarization is in fact the ordinary version of focusing. Indeed, once a global polarity is assigned to each atomic formula, the set of literals whose polarity is negative defines a Herbrand interpretation, and we saw reciprocally how to design a global polarization from the Herbrand interpretation. Theorem 4.2 therefore links ordinary focusing in the ground case with semantic hyperresolution. They are both complete, thanks to this theorem:

THEOREM 5.2 (COROLLARY OF [31, THEOREM 17]). *Given a global polarization of atomic formulas, where the polarity of $P^\perp$ is the opposite of that of P, LKF$^\perp$ is (sound and) complete.*

Completeness of LKF therefore leads to proofs of completeness of semantic hyperresolution in the ground case, and hyperresolution in the first-order case, that do not rely on the construction of a model. To our knowledge, in addition to Chaudhuri et al. [16], only Goubault-Larrecq [27] showed a similar result. We also provided a proof similar to the one in this paper, but restricted to Ordered Resolution, without selection of literals [6].

Note that we cannot extend this link between semantic hyperresolution and focusing to the non-ground case. Indeed, in the non-ground case, $I \models L$ iff I is a model for all ground instances of L. Hence, given a literal L such that $I \not\models L$, we can nonetheless have an instance σL such that $I \models \sigma L$; σL should therefore be selected in the generated clause, but our resolution calculus cannot take this into account. However, remark that if $I \models L$ then $I \models \sigma L$, so that $\sigma(\mathcal{S}(C)) \subseteq \mathcal{S}(\sigma C)$. Thus, the condition of Lemma 4.8 is fulfilled. Therefore, even in the non-ground case, Theorem 4.12 can be used to show that the completeness of semantic hyperresolution implies the completeness of LKF.

5.2 Deduction Modulo Theory

Deduction Modulo Theory [23] is a framework that consists in applying the inference rules of an existing proof system modulo some congruence over formulas. This congruence represents the theory, and it is in general defined by means of rewriting rules. To be expressive enough, these rules are defined not only at the term level, but also for formulas. To get simpler presentations of theories, we distinguish between rewrite rules that can be applied at positive and at negative positions by giving them a polarity[3], where by negative position we mean under an odd number of $\neg$. We therefore have positive rules $P \rightarrow^+ A$ and negative rules $P \rightarrow^- A$ where P is an atomic formula and A an arbitrary formula whose free variables appear in P. The rewrite relation $B_1 \xrightarrow{+} B_2$ is defined by saying that there exists either a positive position $\mathfrak{p}$ and a positive rule $P \rightarrow^+ A$, or

---

[3]This polarity must not be confused with the other notions of polarity mentioned in the paper.

$$\widehat{\vdash} \ \frac{}{\Gamma, L, L^\perp \vdash} \qquad \top\vdash \frac{\Gamma \vdash}{\Gamma, \top \vdash} \qquad \perp\vdash \frac{}{\Gamma, \perp \vdash} \qquad \vee\vdash \frac{\Gamma, A \vdash \qquad \Gamma, B \vdash}{\Gamma, A \vee B \vdash}$$

$$\wedge\vdash \frac{\Gamma, A, B \vdash}{\Gamma, A \wedge B \vdash} \qquad \exists\vdash \frac{\Gamma, A \vdash}{\Gamma, \exists x.\ A \vdash} \ x \text{ not free in } \Gamma \qquad \forall\vdash \frac{\Gamma, \forall x.\ A, \{t/x\}A \vdash}{\Gamma, \forall x.\ A \vdash}$$

$$\uparrow^-\vdash \frac{\Gamma, P, A \vdash}{\Gamma, P \vdash} \ P \xrightarrow{-} A \qquad\qquad \uparrow^+\vdash \frac{\Gamma, \neg P, A^\perp \vdash}{\Gamma, \neg P \vdash} \ P \xrightarrow{+} A$$

Fig. 3. The sequent calculus PUSC$^\perp$

a negative position $\mathfrak{p}$ and a negative rule $P \to^- A$; and a substitution σ such that the subformula of B_1 at position $\mathfrak{p}$ is σP and B_2 equals B_1 where the subformula at position $\mathfrak{p}$ is replaced by σA. $\xrightarrow{-}$ is defined dually (negative position and positive rule, or positive position and negative rule). In Polarized Sequent Calculus Modulo theory [20], the inference rules of the sequent calculus are applied modulo such a polarized rewriting system, as for instance in

$$\vdash\wedge \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash C, \Delta} \ C \xrightarrow{+}{}^* A \wedge B \cdot$$

Note that the implicit semantics of a negative rule $P \to^- A$ is therefore $\overline{\forall x}.\ (P \Rightarrow A)$, whereas the semantics of $P \to^+ A$ is $\overline{\forall x}.\ (A \Rightarrow P)$, where $\overline{x}$ are the free variables of P.

With Kirchner [12], we proved the equivalence of Polarized Sequent Calculus Modulo theory to a sequent calculus where polarized rewriting rules are applied only on literals, using explicit rules. This calculus, Polarized Unfolding Sequent Calculus, is almost the calculus PUSC$^\perp$ presented in Figure 3. The only difference is that all formulas are put on the left of the sequent in PUSC$^\perp$. We denote by $\Gamma \vdash_{\mathcal{R}}$ the fact that $\Gamma \vdash$ can be proved in PUSC$^\perp$ using the polarized rewriting system $\mathcal{R}$. Note that the rule for the universal quantifier $\forall\vdash$ as well as the unfolding rules $\uparrow^-\vdash$ and $\uparrow^+\vdash$ contain an implicit contraction rule, as in the sequent calculus G4 of Kleene, in order to ensure that all rules of PUSC$^\perp$ are invertible.

We can translate polarized rewriting rules as formulas with selection, and see PUSC$^\perp$ as an instance of LKF$^\perp$. We first consider how to translate formulas of the right-hand side of polarized rewriting rules. We polarize them by choosing positive connectives for $\vee$ and $\wedge$ and, to unchain the introduction of the universal quantifier, we introduce delays. (Let us recall that a delay δ^+ allows to force a formula to be positive, and it can be encoded using an existential quantifier.) This gives the translation:

$$|L| = L \quad \text{when } L \text{ is } \top, \perp \text{ or a literal} \qquad |A \wedge B| = |A| \wedge^+ |B|$$
$$|A \vee B| = |A| \vee^+ |B| \qquad |\exists x.\ A| = \exists x.\ |A| \qquad |\forall x.\ A| = \forall x.\ \delta^+|A|$$

Definition 5.3. Given a negative rewriting rule $P \to^- A$ where the free variables of P are $x_1, \ldots, x_n$, its translation as a formula with selection is $[\![P \to^- A]\!] = \forall x_1 \ldots x_n.\ \underline{\neg P} \vee^- \delta^+|A|$.

Given a positive rewriting rule $P \to^+ A$ where the free variables of P are $x_1, \ldots, x_n$, its translation as a formula with selection is $[\![P \to^+ A]\!] = \forall x_1.\ \ldots \forall x_n.\ \underline{P} \vee^- \delta^+|A^\perp|$.

The translation $[\![\mathcal{R}]\!]$ of a polarized rewriting system $\mathcal{R}$ is the multiset of the translation of its rules.

Definition 5.4. Let A be a non-polarized formula. A is called contractible if its top connective is $\forall$ or $\perp$, or if it is a literal.

Definition 5.5. Let $N_1, \ldots, N_n$ be a multiset of contractible formulas, and let $P_1, \ldots, P_m$ be a multiset of non-contractible formulas, then the translation of the PUSC$^\perp$ sequent $N_1, \ldots, N_n, P_1, \ldots, P_m \vdash$ modulo the rewriting system $\mathcal{R}$ is the LKF$^\perp$ sequent $[\![\mathcal{R}]\!], |N_1|, \ldots, |N_n| \Uparrow |P_1|, \ldots, |P_m| \vdash$.

THEOREM 5.6. $N_1, \ldots, N_n, P_1, \ldots, P_m \vdash_\mathcal{R}$ in $PUSC^\perp$ iff $[\![\mathcal{R}]\!], |N_1|, \ldots, |N_n| \Uparrow |P_1|, \ldots, |P_m| \vdash$ in $LKF^\perp$.

PROOF. First, one proves that, in $PUSC^\perp$, the rules $\perp\vdash, \forall\vdash$ and $\uparrow\vdash$ can be delayed until the other rules are no longer applicable. This can be done by showing that these rules permute with the other ones, as done by Hermant [29]. Note that this fact can be related with the strategy used in Tamed [4], a tableaux method based on Deduction Modulo Theory, where rules for universal quantifiers and for rewriting are applied when no other rules can be.

With this proviso, proofs in both calculi correspond almost exactly. $\top\vdash, \wedge\vdash, \vee\vdash$ and $\exists\vdash$ in $PUSC^\perp$ correspond exactly to $\Uparrow\top\vdash, \Uparrow\wedge\vdash, \Uparrow\vee\vdash$ and $\Uparrow\exists\vdash$ in $LKF^\perp$, except that if the subformulas in the premise(s) are contractible, they have to be put on the left hand side of $\Uparrow$ using Store.

The translation of $\perp\vdash$ corresponds to $\Downarrow\perp\vdash$, except that the latter can only be applied when there are only contractible formulas, which is the case if $\perp\vdash$ has been delayed as mentioned above:

$$\perp\vdash \frac{}{N_1, \ldots, \perp, \ldots, N_n \vdash} \quad \text{becomes} \quad \text{Focus} \frac{\Downarrow\perp\vdash \dfrac{}{[\![\mathcal{R}]\!], |N_1|, \ldots, \perp, \ldots, |N_n| \Downarrow \perp \vdash}}{[\![\mathcal{R}]\!], |N_1|, \ldots, \perp, \ldots, |N_n| \Uparrow \vdash} \quad . \text{ Conversely, if a}$$

proof in $LKF^\perp$ starts by focusing on $\perp$, then necessarily it is immediately closed by $\Downarrow\perp\vdash$, which is translated to $\perp\vdash$ in $PUSC^\perp$.

Similarly, $\forall\vdash$ corresponds to $\Downarrow\forall\vdash$, with the same condition that there are only contractible formulas:

$$\forall\vdash \frac{N_1, \ldots, N_n, \forall x.\ A, \{t/x\}A \vdash}{N_1, \ldots, N_n, \forall x.\ A \vdash} \quad \text{becomes}$$

$$\text{Focus} \frac{\Downarrow\forall\vdash \dfrac{\text{Release} \dfrac{[\![\mathcal{R}]\!], |N_1|, \ldots, |N_n|, \forall x.\ \delta^+|A| \Uparrow |\{t/x\}A| \vdash}{[\![\mathcal{R}]\!], |N_1|, \ldots, |N_n|, \forall x.\ \delta^+|A| \Downarrow \delta^+|\{t/x\}A| \vdash}}{[\![\mathcal{R}]\!], |N_1|, \ldots, |N_n|, \forall x.\ \delta^+|A| \Downarrow \forall x.\ \delta^+|A| \vdash}}{[\![\mathcal{R}]\!], |N_1|, \ldots, |N_n|, \forall x.\ \delta^+|A| \Uparrow \vdash}$$

with an extra Store step if $\{t/x\}A$ is contractible. Here again, conversely, if the proof in $LKF^\perp$ starts by focusing on the translation of a formula whose top connective is $\forall$, then it is necessarily of the shape above, which corresponds to $\forall\vdash$ in $PUSC^\perp$.

For the unfolding rules, if P rewrites positively to A, then there exists a rule $Q \to^+ B$ and a substitution θ such that $P = \theta Q$ and $A = \theta B$. This rule corresponds to a formula $[\![Q \to^+ B]\!] = \overline{\forall x.\ \underline{Q} \vee^- \delta^+|B^\perp|}$. Always with the proviso that there are only contractible formulas, let $\Gamma = [\![\mathcal{R}']\!], \overline{\forall x.\ \underline{Q} \vee^- \delta^+|B^\perp|}, |N_1|, \ldots, |N_n|, \neg P$, then $\uparrow^+\vdash$ therefore corresponds to

$$\text{Focus} \frac{\Downarrow\vee\vdash \dfrac{\widehat{\Downarrow}\vdash \dfrac{}{\Gamma \Downarrow \underline{P} \vdash} \quad \text{Release} \dfrac{\Gamma \Uparrow |A^\perp| \vdash}{\Gamma \Downarrow \delta^+|A^\perp| \vdash}}{\Downarrow\vee\vdash \dfrac{\Gamma \Downarrow \theta\underline{Q} \vee^- \delta^+|\theta B^\perp| \vdash}{\Gamma \Downarrow \overline{\forall x.\ \underline{Q} \vee^- \delta^+|B^\perp|} \vdash}}}{\Gamma \Uparrow \vdash}$$

with an extra Store step if $|A^\perp|$ is contractible. Conversely, let us look how the encoding of a rewriting rule $Q \to^+ B$, namely $\overline{\forall x.\ \underline{Q} \vee^- \delta^+|B^\perp|}$, can be involved in a $LKF^\perp$ proof. Given a sequent $[\![\mathcal{R}']\!], \overline{\forall x.\ \underline{Q} \vee^- \delta^+|B^\perp|}, \Gamma \Uparrow\vdash$, the only way to use $\overline{\forall x.\ \underline{Q} \vee^- \delta^+|B^\perp|}$ is to apply a Focus on it. In that case, the derivation is necessarily of the same shape as above, so, in order to close it, we know that there must be a literal $\neg\theta Q$ in Γ. Therefore, the derivation corresponds to an unfolding of $\neg\theta Q$ into $|\theta B^\perp|$ in $PUSC^\perp$.

The case of a negative rewriting is dual. □

$$\text{Resolution } \frac{P \vee C \qquad \neg Q \vee D}{\sigma(C \vee D)} \; {}^a \qquad\qquad \text{Factoring } \frac{L \vee K \vee C}{\sigma(L \vee C)} \; \sigma = mgu(L, K)$$

$$\text{Ext. Narr.}^- \frac{P \vee C}{\sigma(D \vee C)} \; {}^a, Q \to^- D \qquad\qquad \text{Ext. Narr.}^+ \frac{\neg Q \vee D}{\sigma(C \vee D)} \; {}^a, P \to^+ \neg C$$

$${}^a \sigma = mgu(P, Q)$$

Fig. 4. Inference rules of Polarized Resolution Modulo theory

Let us now consider the subcase where the rewriting rules are clausal, according to the terminology of Dowek [22], e.g. they are of the form $P \to^- C$ or $P \to^+ \neg C$ for some clausal formula C. In that case, the resolution method based on Deduction Modulo Theory [23] can be refined into what is called Polarized Resolution Modulo theory [22], whose rules are given in Fig. 4. (A refinement of) Polarized Resolution Modulo theory is actually implemented in the automated theorem prover iProverModulo [7].

By noting that the translation of the rule $Q \to^- D$ is $[\![Q \to^- D]\!] = \forall x_1. \ldots \forall x_n. \neg Q \vee^- \delta^+ |D|$ whereas $\ulcorner \neg Q \vee D \urcorner = \forall x_1. \ldots \forall x_n. \neg Q \vee^- \delta^- |D|$, we can relate the rule $Q \to^- D$ with the clause with selection $\underline{\neg Q} \vee D$, which is called a one-way clause by Dowek [22]. Indeed, they only differ by the polarity of the delays. However, because of the presence of the $\vee^-$ connective, the delay δ^- is useless in $\ulcorner \underline{\neg Q} \vee D \urcorner$. Similarly, since the top connective of $|D|$ is $\vee^+$, the delay δ^+ is useless in $[\![Q \to^- D]\!]$. Ext. Narr.$^-$ can therefore be seen as an instance of the Resolution with Selection rule:

$$\text{Resolution with Selection } \frac{\underline{\neg Q} \vee D \qquad P \vee C}{\sigma(D \vee C)} \; \sigma = mgu(P, Q) \; .$$

Similarly, $P \to^+ \neg C$ is related to $\underline{P} \vee C$.

Consequently, since PUSC$^\perp$ corresponds to LKF$^\perp$, and Resolution with Input Selection corresponds to Polarized Resolution Modulo theory, Theorem 4.2 leads to a new and more generic proof of the correspondence between PUSC$^\perp$ and Polarized Resolution Modulo theory.

Deduction Modulo Theory is not always complete. This is the case only if the cut rule is admissible in Polarized Sequent Calculus Modulo theory. It holds for some particular theories, e.g. Church's Simple Type Theory [23] and arithmetic [25]. There are more or less powerful techniques that ensures this property [9, 21, 24, 28]. We even proved that any consistent first-order theory can be presented by a rewriting system admitting the cut rule [8]. As presented with Dowek [11] and discussed in the introduction, the fact that completeness is not proved once for all, but needs to be proved for each particular theory, is essential. Indeed, if a theory is presented entirely by rewriting rules, completeness implies the consistency of the theory, since no rule can be applied on the empty set of clauses. Consequently, the proof of the completeness cannot be easier than the proof of consistency of the theory, and, according to Gödel, cannot be proven in the theory itself.

5.3 Superdeduction

We can go a step further than what is done concerning Deduction Modulo Theory and benefit from focusing to decompose the right-hand side formula after an unfolding has occurred. This leads to what Brauner, Houtmann, and Kirchner [5] called Superdeduction. Houtmann [30] studied the links between Superdeduction and focusing, but not with the idea that the rules themselves should be considered as polarized formulas. To link Superdeduction with LKF$^\perp$, we just need to change the translation of rewriting rules in order to ensure that the right-hand side is decomposed as much as possible. This is done by suppressing the positive delay δ^+ and trying to stay in synchronous (i.e. focused) phase by using negative connectives, until we reach a $\exists$ quantifier, after which we try to

stay in the asynchronous phase. Note, however, that literals are always given a positive polarization. We introduce the negative translation of a formula:

$$\lfloor L \rfloor = L \quad \text{when } L \text{ is } \top, \bot \text{ or a literal}$$
$$\lfloor A \wedge B \rfloor = \lfloor A \rfloor \wedge^- \lfloor B \rfloor \qquad \lfloor A \vee B \rfloor = \lfloor A \rfloor \vee^- \lfloor B \rfloor$$
$$\lfloor \exists x.\, A \rfloor = \exists x.\, |A| \qquad \lfloor \forall x.\, A \rfloor = \forall x.\, \lfloor A \rfloor$$

and the translation of rewrite rules becomes:

$$\llbracket P \to^- A \rrbracket = \overline{\forall x.\ \underline{\neg P} \vee^- \lfloor A \rfloor}$$
$$\llbracket P \to^+ A \rrbracket = \overline{\forall x.\ \underline{P} \vee^- \lfloor A^\perp \rfloor}$$

The synthetic rules given by the translation of rewriting rules correspond exactly to the superrules of Superdeduction.

Example 5.7. Let us consider the proposition rewrite rules defining the natural numbers as the set of terms verifying the inductive predicates, as in [5].

$$n \in \mathbb{N} \to \forall p.\ 0 \in p \Rightarrow H(p) \Rightarrow n \in p$$
$$H(p) \to \forall m.\ m \in p \Rightarrow s(m) \in p$$

Here, p is a term representing a class of natural number, as presented in [25]. Each rule is considered with both positive and negative polarities.

The translation above gives the following formulas with polarization:

$$\forall n.\ \underline{\neg n \in \mathbb{N}} \vee^- (\forall p.\ \neg 0 \in p \vee^- \neg H(p) \vee^- n \in p) \tag{11}$$
$$\forall n.\ \underline{n \in \mathbb{N}} \vee^- (\exists p.\ 0 \in p \wedge^+ H(p) \wedge^+ \neg n \in p) \tag{12}$$
$$\forall p.\ \underline{\neg H(p)} \vee^- (\forall m.\ \neg m \in p \vee^- s(m) \in p) \tag{13}$$
$$\forall p.\ \underline{H(p)} \vee^- (\exists m.\ m \in p \wedge^+ s(m) \in p) \tag{14}$$

This leads to the following synthetic rules:

$$(11)\vdash \frac{\Gamma, n \in \mathbb{N}, \neg 0 \in t \Uparrow \vdash \qquad \Gamma, n \in \mathbb{N}, \neg H(t) \Uparrow \vdash \qquad \Gamma, n \in \mathbb{N}, n \in t \Uparrow \vdash}{\Gamma, n \in \mathbb{N} \Uparrow \vdash}$$

$$(12)\vdash \frac{\Gamma, \neg n \in \mathbb{N}, 0 \in p, H(p), \neg n \in p \Uparrow \vdash}{\Gamma, \neg n \in \mathbb{N} \Uparrow \vdash} \ p \text{ not free in } \Gamma$$

$$(13)\vdash \frac{\Gamma, H(p), \neg u \in p \Uparrow \vdash \qquad \Gamma, H(p), s(u) \in p \Uparrow \vdash}{\Gamma, H(p) \Uparrow \vdash}$$

$$(14)\vdash \frac{\Gamma, \neg H(p), m \in p, \neg s(m) \in p \Uparrow \vdash}{\Gamma, \neg H(p) \Uparrow \vdash} \ m \text{ not free in } \Gamma$$

For instance $(12)\vdash$ corresponds to the following focused derivation:

$$\text{Store} \cfrac{\Delta, \neg n \in \mathbb{N}, 0 \in p, H(p), \neg n \in p \Uparrow \vdash}{\Delta, \neg n \in \mathbb{N}, \Uparrow 0 \in p, H(p), \neg n \in p \vdash}$$

$$\widehat{\Downarrow} \vdash \cfrac{}{\Delta, \neg n \in \mathbb{N} \Downarrow \underline{n \in \mathbb{N}} \vdash} \qquad \text{Release} \cfrac{\Uparrow\wedge\vdash \cfrac{\Uparrow\exists\vdash \cfrac{\Delta, \neg n \in \mathbb{N} \Uparrow 0 \in p \wedge^+ H(p) \wedge^+ \neg n \in p \vdash}{\Delta, \neg n \in \mathbb{N} \Uparrow \exists p.\, 0 \in p \wedge^+ H(p) \wedge^+ \neg n \in p \vdash}}{\Delta, \neg n \in \mathbb{N} \Downarrow \exists p.\, 0 \in p \wedge^+ H(p) \wedge^+ \neg n \in p \vdash}}{} \; a$$

$$\Downarrow\vee\vdash \cfrac{}{\Downarrow\vee\vdash \cfrac{\Delta, \neg n \in \mathbb{N} \Downarrow \underline{n \in \mathbb{N}} \vee^- (\exists p.\, 0 \in p \wedge^+ H(p) \wedge^+ \neg n \in p) \vdash}{\text{Focus} \cfrac{\Delta, \neg n \in \mathbb{N} \Downarrow \forall n.\, \underline{n \in \mathbb{N}} \vee^- (\exists p.\, 0 \in p \wedge^+ H(p) \wedge^+ \neg n \in p) \vdash}{\Delta, \neg n \in \mathbb{N} \Uparrow \vdash}}}$$

[a] p not free in Δ

where Δ is $\Gamma, \forall n.\, \underline{n \in \mathbb{N}} \vee^- (\exists p.\, 0 \in p \wedge^+ H(p) \wedge^+ \neg n \in p)$.

These synthetic rules are exactly the rules $\in_{\mathbb{N}L}$, $\in_{\mathbb{N}R}$, $hered_L$, and $hered_R$ obtained by Brauner et al. [5], except that in our presentation all formulas are put on the left of the sequent.

Note that the same kind of encodings can be used to show that Definitional reflection, as defined by Schroeder-Heister [35], can be seen as an instance of LKF$^\perp$.

5.4 Beyond Deduction Modulo Theory

Example 5.8. Let us recall the set of clauses from the Introduction:

$$\neg X \in \mathcal{P}(Y) \vee \underline{\neg Z \in X} \vee Z \in Y \quad (1) \qquad\qquad \underline{X \in \mathcal{P}(Y)} \vee d(X, Y) \in X \quad (2)$$

$$\underline{X \in \mathcal{P}(Y)} \vee \neg d(X, Y) \in Y \quad (3)$$

Note that this example is not covered by Ordered Resolution with Selection, at least not if a simplification ordering is used, because we cannot have $X \in \mathcal{P}(Y) > d(X, Y) \in X$ since with $\theta = \{X \mapsto \mathcal{P}(Z); Y \mapsto Z\}$ their instances are ordered in the wrong direction: $\mathcal{P}(Z) \in \mathcal{P}(Z) \prec d(\mathcal{P}(Z), Z) \in \mathcal{P}(Z)$.

The synthetic rules of the example from the Introduction correspond to the derivations when one of the clauses is focused on. For instance, if we consider the clause (1), in a context Γ containing this clause, a proof putting the focus on $\ulcorner (1) \urcorner$ is necessarily of the following shape:

$$\widehat{\Downarrow} \vdash \cfrac{}{\Gamma \Downarrow \underline{\neg u \in \mathcal{P}(v)} \vdash} \qquad \widehat{\Downarrow} \vdash \cfrac{}{\Gamma \Downarrow \underline{\neg t \in u} \vdash} \qquad \text{Release} \cfrac{\text{Store} \cfrac{\Gamma, t \in v \Uparrow \vdash}{\Gamma \Uparrow t \in v \vdash}}{\Gamma \Downarrow t \in v \vdash}$$

$$\Downarrow\vee\vdash \cfrac{}{\Downarrow\vee\vdash \cfrac{\Gamma \Downarrow \underline{\neg u \in \mathcal{P}(v)} \vee^- \underline{\neg t \in u} \vee^- t \in v \vdash}{\text{Focus} \cfrac{\Gamma \Downarrow \forall X\, Y\, Z.\, \neg X \in \mathcal{P}(Y) \vee^- \underline{\neg Z \in X} \vee^- Z \in Y \vdash}{\Gamma \Uparrow \vdash}}}$$

where t, u, v are arbitrary terms, and where, to be able to close the left and middle branches, $u \in \mathcal{P}(v)$ and $t \in u$ must belong to Γ.

So Γ is in fact of the form $\forall X\, Y\, Z.\, \neg X \in \mathcal{P}(Y) \vee^+ \underline{\neg Z \in X} \vee^+ Z \in Y, \Delta, u \in \mathcal{P}(v), t \in u$ for some Δ, and the axiom $\forall X\, Y\, Z.\, \neg X \in \mathcal{P}(Y) \vee^+ \underline{\neg Z \in X} \vee^+ Z \in Y$ can be replaced by the synthetic rule:

$$(1) \quad \cfrac{\Delta, u \in \mathcal{P}(v), t \in u, t \in v \Uparrow \vdash}{\Delta, u \in \mathcal{P}(v), t \in u \Uparrow \vdash} \quad .$$

The computation of the other synthetic rules is left as an exercise for the reader.

Note, by analogy of this synthetic rule with the unfolding rules in PUSC$^\perp$ (see Section 5.2), how (1) can be seen as a rewriting rule that rewrites simultaneously $x \in \mathcal{P}(y)$ and $z \in x$ into $z \in y$.

On the resolution side, if we consider the ground instances of Resolution with Selection, we have:

$$\text{R.w.S. } \frac{\neg X \in \mathcal{P}(Y) \vee \underline{\neg Z \in X} \vee Z \in Y \quad u \in \mathcal{P}(v) \vee C \quad t \in u \vee D}{t \in v \vee C \vee D}$$

$$\text{R.w.S. } \frac{\underline{X \in \mathcal{P}(Y)} \vee d(X, Y) \in X \qquad \neg u \in \mathcal{P}(v) \vee C}{d(u, v) \in u \vee C}$$

$$\text{R.w.S. } \frac{\underline{X \in \mathcal{P}(Y)} \vee \underline{\neg d(X, Y) \in Y} \quad \neg u \in \mathcal{P}(v) \vee C \quad d(u, v) \in v \vee D}{C \vee D}$$

hence the derived rules given in the introduction, where the clauses of the theory are not mentioned.

The question that remains is how we can prove the completeness of such a selection. We consider the case when there is a fixed selection on a finite set of clauses, and arbitrary other clauses without selection. The intuition is that the clauses with selection constitute the theory in which the proof is searched for, whereas the other clauses represent some goal to be proven in that theory. In that case, one can consider the second projection $\mathcal{S}$ attached to the clauses of the theory, which we call a selection.

We can in fact consider only subselections.

Definition 5.9 (Singleton subselection). Given a selection $\mathcal{S}$, the selection $\mathcal{S}_1$ is a singleton subselection of $\mathcal{S}$ if

- $\mathcal{S}_1(C) \subseteq \mathcal{S}(C)$ for all $C \in \Gamma$
- if $\mathcal{S}(C) \neq \emptyset$ then $\mathrm{card}(\mathcal{S}_1(C)) = 1$.

Example 5.10. A singleton subselection of Example 5.8 can be

$$\neg X \in \mathcal{P}(Y) \vee \underline{\neg Z \in X} \vee Z \in Y \qquad \underline{X \in \mathcal{P}(Y)} \vee d(X, Y) \in X \qquad \underline{X \in \mathcal{P}(Y)} \vee \neg d(X, Y) \in Y$$

In the case of propositional logic, it is enough to consider only singleton subselections. To show this, given a selection $\mathcal{S}$, let us arbitrarily order the occurrences of literals selected by $\mathcal{S}$ in each clause of the theory. Singleton subselections can be partially ordered by the transitive closure of the following relation : $\mathcal{S}_1$ is greater than $\mathcal{S}_1'$ if in some clause, $\mathcal{S}_1$ selects a greater literal than $\mathcal{S}_1'$. Since there are only finitely many subselections, this ordering is well-founded.

We prove the following lemma:

LEMMA 5.11. *Given a selection $\mathcal{S}$ and a singleton subselection $\mathcal{S}_1$, assume that for all literals L that are selected by $\mathcal{S}$ and that are greater than the literal selected by $\mathcal{S}_1$, $L^\perp$ is in Γ.*

If for all singleton subselections $\mathcal{S}_1'$ smaller than or equal to $\mathcal{S}_1$, the sequent $\ulcorner \Gamma \urcorner \Uparrow \vdash$ has a proof using $\mathcal{S}_1'$, then this sequent has a proof using $\mathcal{S}$.

PROOF. By lexicographic induction on $\mathcal{S}_1$ and on the proof of $\ulcorner \Gamma \urcorner \Uparrow \vdash$ using it.

If the last proof step is valid using $\mathcal{S}$, we apply it and proceed by induction on the subproofs.

If it is not, this means that the negation of some literal that is not selected in $\mathcal{S}_1$ but that is selected in $\mathcal{S}$ is not in the context $\ulcorner \Gamma \urcorner$, thus preventing to close a branch using $\widehat{\Downarrow} \vdash$. In other words, the proof ends by focusing on a formula corresponding to a clause $K_1 \vee \cdots \vee K_n \vee C$ where the K_is are in decreasing order, $\mathcal{S}(K_1 \vee \ldots \vee K_n \vee C) = \{K_1, \ldots, K_n\}$ and $\mathcal{S}_1(K_1 \vee \ldots \vee K_n \vee C) = \{K_j\}$, and we have a derivation of the form:

$$\Downarrow\vdash \cfrac{\overset{\widehat{\Downarrow}\vdash}{\ulcorner\Gamma\urcorner \Downarrow \underline{K_j} \vdash} \qquad \ulcorner\Gamma\urcorner \Downarrow \delta^-(K_1 \vee^+ \cdots \vee^+ K_n \vee^+ \theta C) \vdash}{\ulcorner\Gamma\urcorner \Downarrow \theta(\underline{K_j}\vee^-\delta^-(K_1\vee^+\cdots\vee^+K_n\vee^+C))\vdash}$$

$$\Downarrow\forall\vdash \cfrac{}{\ulcorner\Gamma\urcorner \Downarrow \overline{\forall x}.\underline{K_j}\vee^-\delta^-(K_1\vee^+\cdots\vee^+K_n\vee^+C)\vdash}$$

$$\text{Focus} \cfrac{}{\ulcorner\Gamma\urcorner \Uparrow \vdash}$$

using $\mathcal{S}_1$.

By assumption, all $K_i{}^\perp$ for $i < j$ are in $\ulcorner\Gamma\urcorner$. To close the left branch, we also have $K_j{}^\perp$ in $\ulcorner\Gamma\urcorner$. Let us now consider the singleton subselection equal to $\mathcal{S}_1$ except that it selects K_{j+1} instead of K_j in $K_1 \vee \cdots \vee K_n \vee C$. This subselection is smaller than $\mathcal{S}_1$, so we can apply the induction hypothesis to obtain a proof using $\mathcal{S}$.   □

THEOREM 5.12. *Assume that $\mathcal{S}$ selects only ground literals.*
Resolution with input selection $\mathcal{S}$ is complete iff for all singleton subselections $\mathcal{S}_1$ of $\mathcal{S}$, Resolution with input selection $\mathcal{S}_1$ is complete.

PROOF. For the "only if" part, just note that a Resolution with Selection step using $\mathcal{S}$ can easily be simulated by a Resolution with Selection step using $\mathcal{S}_1$ plus possibly one or several applications of Resolution.

Assume Γ to be unsatisfiable. Since Resolution with input selection is complete for all singleton subselection $\mathcal{S}_1$, it means that for each singleton subselection $\mathcal{S}_1$ the sequent $\ulcorner\Gamma\urcorner \Uparrow\vdash$ is provable using $\mathcal{S}_1$, according to Theorem 4.12. Let us consider $\mathcal{S}_1$ the singleton subselection selecting the greatest selected literal in each clause, if it exists. The hypothesis of Lemma 5.11 are fulfilled, hence $\ulcorner\Gamma\urcorner \Uparrow\vdash$ has a proof using $\mathcal{S}$. Using Theorem 4.7, the empty clause can be derived from Γ in Resolution with input selection $\mathcal{S}$. It is therefore complete.   □

We conjecture that the same theorem holds even if non-ground literals are selected.

Since singleton subselections can be linked with rewriting systems in Deduction Modulo Theory according to Section 5.2, we could reduce the problem of completeness in our framework to several problems of completeness in Deduction Modulo Theory.

In particular, to prove completeness if we have a clause $\underline{L_1} \vee \ldots \vee \underline{L_n} \vee K_1 \vee \ldots \vee K_m$, one could prove completeness of the n singleton subselections

$$\underline{L_1} \vee \ldots \vee L_n \vee K_1 \vee \ldots \vee K_m$$
$$\vdots$$
$$L_1 \vee \ldots \vee \underline{L_n} \vee K_1 \vee \ldots \vee K_m$$

which correspond, according to Section 5.2, to the n rewriting systems

$$L_1 \rightarrow^\pm (\neg)\overline{\forall x}.\, L_2 \vee \cdots \vee L_n \vee K_1 \vee \cdots \vee K_m$$
$$\vdots$$
$$L_n \rightarrow^\pm (\neg)\overline{\forall x}.\, L_1 \vee \cdots \vee L_{n-1} \vee K_1 \vee \cdots \vee K_m$$

in Deduction Modulo Theory.

Example 5.13. Following Example 5.8, proving the completeness of our selection would amount to proving cut admissibility in Deduction Modulo Theory for the four following rewriting systems:

$$X \in \mathcal{P}(Y) \to^- \forall Z.\, \neg Z \in X \vee Z \in Y$$
$$X \in \mathcal{P}(Y) \to^+ \neg d(X,Y) \in X$$
$$X \in \mathcal{P}(Y) \to^+ d(X,Y) \in Y$$

$$Z \in X \to^- \forall Y.\, \neg X \in \mathcal{P}(Y) \vee Z \in Y$$
$$X \in \mathcal{P}(Y) \to^+ \neg d(X,Y) \in X$$
$$X \in \mathcal{P}(Y) \to^+ d(X,Y) \in Y$$

$$X \in \mathcal{P}(Y) \to^- \forall Z.\, \neg Z \in X \vee Z \in Y$$
$$X \in \mathcal{P}(Y) \to^+ \neg d(X,Y) \in X$$
$$d(X,Y) \in Y \to^- X \in \mathcal{P}(Y)$$

$$Z \in X \to^- \forall Y.\, \neg X \in \mathcal{P}(Y) \vee Z \in Y$$
$$X \in \mathcal{P}(Y) \to^+ \neg d(X,Y) \in X$$
$$d(X,Y) \in Y \to^- X \in \mathcal{P}(Y)$$

Cut admissibility of these systems can be proved by analyzing so-called critical proofs as defined by Burel and Kirchner [12].

CONCLUSION AND FURTHER WORK

We generalized focusing and resolution with selection, proved that they correspond, and showed how known calculi are instances of this framework, namely ordinary focusing, hyperresolution, Deduction Modulo Theory and Superdeduction. We also showed how to reduce completeness of this framework to several completeness proofs in Deduction Modulo Theory. We can therefore reuse the various techniques for proving completeness in Deduction Modulo Theory [9, 21, 24, 28] in our framework. As Deduction Modulo Theory already gives significant results in industrial applications when the theory is a variant of set theory (more precisely, set theory of the B method) [13], we can expect our framework to lead to even better outcomes. The notable results presented here raise the following new areas of investigations.

First, we need to study how to apply selection also in the generated clauses. This should allow us to cover the cases of Ordered Resolution with Selection and of Semantic Resolution in the first-order case. Dually, in the sequent calculus part, this would correspond to the possibility to dynamically add selection in formulas of subderivations. This could probably be linked with the work of Deplagne [19] where rewrite rules corresponding to induction hypotheses are dynamically added in the rewriting system of a sequent calculus for Deduction Modulo Theory. Note that we already have one direction, namely from Resolution with Input Selection to LKF$^\perp$, since Lemmas 4.8, 4.10, and 4.11 do not assume anything on the generated clauses; except, for Factoring, that it selects at least the instances of literals that were already selected. The converse direction would require a meta-theorem of completeness, since it is obviously not complete for all possible dynamic choices of selection.

Since focusing is defined not only for classical first-order logic but also for linear, intuitionistic, modal logics, the work in this paper could serve as a starting point to study how to get automated proof-search methods for these logics with a selection mechanism.

Another worthwhile point is how equality should be handled in our framework. In particular, it would be interesting to see how paramodulation calculi, in particular superposition, can be embedded into a sequent calculus.

Finally, it would be worth investigating whether completeness proofs based on model construction, such as semantic completeness proofs of tableaux (related to sequent calculus), and completeness proof of superposition [2], can be related one to the other in our framework.

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