Steps in the Representation of Concept Lattices and Median Graphs

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Abstract. Median semilattices have been shown to be useful for dealing with phylogenetic classification problems since they subsume median graphs, distributive lattices as well as other tree based classification structures. Median semilattices can be thought of as distributive $\vee$-semilattices that satisfy the following property (TRI): for every triple x, y, z, if $x \wedge y$, $y \wedge z$ and $x \wedge z$ exist, then $x \wedge y \wedge z$ also exists. In previous work we provided an algorithm to embed a concept lattice L into a distributive $\vee$-semilattice, regardless of (TRI). In this paper, we take (TRI) into account and we show that it is an invariant of our algorithmic approach. This leads to an extension of the original algorithm that runs in polynomial time while ensuring that the output is a median semilattice.

Keywords: Median graph · Distributive lattice · Order embedding· Formal Concept Analysis.

1 Motivations

Trees constitute noteworthy examples of classification structures that are commonly used in applied sciences [?,?]. However, they do not offer a sufficient representation power for some applications. For instance, in phylogenetic modeling they fail to capture the complexity of evolution in presence of parallel or reverse mutations [?]. Indeed, the latter mutations introduce ambiguity, and it is not possible to design a tree representing such evolution.

Distributive lattices can model phylogenetic evolution by exhibiting in a single structure all potential evolutions, while preserving the principle of parsimony, which states that no unnecessary evolution is introduced. The drawback is that classical tree-like structures are not lattices but remain relevant for several simple cases.

A generalization of both trees and distributive lattices is provided by the so called median algebras [?]. In finite cases, they can be considered in two equivalent ways:

- as median graphs that ensure the existence and uniqueness of a "barycenter" of every triple of vertices (which, in turn, ensures parsimony): for every

triple x, y, z of vertices, there exists a unique vertex at the intersection of the shortest paths between any pair of these vertices, or
– as median semilattices, i.e., distributive $\vee$-semilattices that satisfy the following additional property: for every triple x, y, z of elements, if $x \wedge y$, $y \wedge z$ and $x \wedge z$ exist, then $x \wedge y \wedge z$ also exists.

An example of median graph is shown in Fig. **??** for binary data used by Bandelt [**?**] and coming from [**?**]. The data table shows variations from a consensus value for some sequences in mitochondrial DNA (nucleotide positions, denoted by a, b, ... , j in table) for 8 individuals groups of Khoisan-speaking hunter-gathered populations in southern Africa. Vertices of the median graph are either individuals (numbered from 0 to 7) or latent vertices (spotted as Lx), added such that only one variation exists for nucleotide sequences (principle of parsimony supposes that there is no chance that two variations arise exactly at the same moment for a population in an evolutionary process).

As stated, this graph contains every parsimonious tree as covering tree: For individuals represented by vertex 0, there is not enough data to determine if the evolutionary path from the consensus started with a variant on position k and then j, or started with a variant on j and then k. The strength of median graphs is to display these two possibilities at once, which is not possible with a parsimonious tree. Every finite $\vee$-semilattice with a $\perp$ element constitutes

	a	b	c	d	e	f	g	h	i	j	k
0							×			×	×
1	×				×						
2			×							×	
3	×								×		
4							×	×		×	×
5				×							
6		×						×			×
7			×			×				×	

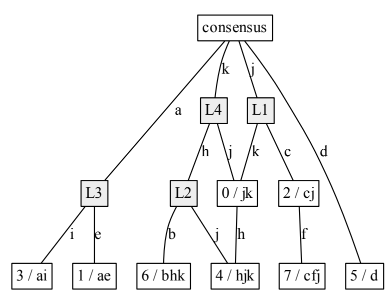


Fig. 1. (Left) A data table of genetic variations for individuals of a particular population. (Right) A median graph for the data table.

a lattice, and thus a concept lattice to some extent. Uta Priss [**?**,**?**] used this observation to establish some facts about median semilattices from a Formal Concept Analysis [**?**] perspective, and sketched an algorithm to compute median semilattices from concept lattices.

In [**?**], we tried to formalize these ideas and to implement this algorithm. In our approach, we set an assumption on the family of $\vee$-irreducible elements, exhibiting an invariant between the initial lattice and the lattice produced by the algorithm *i.e.*, the poset of $\vee$-irreducible elements of the two lattices are isomorphic. We showed that the proposed algorithm produces a distributive $\vee$-semilattice but not necessarily a minimal one. In fact, we observed in [**?**] that such a minimum may not exist. Moreover, our algorithm does not take into account the condition:

$$\text{if } x \wedge y, y \wedge z, x \wedge z \text{ exist, then } x \wedge y \wedge z \text{ also exists,} \tag{TRI}$$

which ensures that the semilattice is a median semilattice. In this paper we revisit our algorithm, taking into account the condition (TRI), and we show that it effectively and (to some extent) efficiently computes median semilattices from any formal context. Moreover, it does so in polynomial time.

The paper is organized as follows. After recalling some basic notions, preliminary and results (Section **??**), we show in Section **??** that if the condition (TRI) is not satisfied in an input concept lattice L, then there is only one distributive $\vee$-semilattice with the same poset of $\vee$-irreducible elements as L. We discuss in Section **??** about possible hypotheses other than the isomorphism of a $\vee$-irreducible poset to embed a concept lattice (minus bottom element) in a median semilattice.

2 From Lattices to Median Graphs

In this section we recall basic notions and notations needed throughout the paper. We will mainly adopt the formalism of [**?**], and we refer the reader to [**?**,**?**] for related notions. All sets considered in this paper are assumed to be finite.

2.1 Posets, lattices and semilattices

Throughout the paper $(P, \leq)$ denotes a partially ordered set (or poset, for short) with the (partial) order $\leq$, $(L, \vee, \wedge)$ denotes a lattice, $S^{\vee} = (S, \vee)$ denotes a $\vee$-semilattice, and $S^{\wedge} = (S, \wedge)$ denotes a $\wedge$-semilattice. As usual, $\vee$ is referred to as the supremum or join, whereas $\wedge$ is referred to as the infimum or meet. When clear from the context, we will use P, L and S to respectively denote a poset, a lattice and a semilattice.

Note that every lattice is a semilattice, and that every semilattice $S^{\vee}$ (similarly, $S^{\wedge}$) is a poset with the partial order defined by $x \leq y$ if $y = x \vee y$ ($x = x \wedge y$, respectively). Also, every (finite) poset and every lattice can be visualized thanks to their Hasse diagrams [**?**] that represent the covering relations between elements.

Let (X_1, R_1) and (X_2, R_2) be two relational structures (e.g., posets, semilattices or lattices). A mapping $f \colon X_1 \to X_2$ is said to be a homomorphism between (X_1, R_1) and (X_2, R_2) if for every $x, y \in X_1$, $xR_1y \iff f(x)R_2f(y)$. If f is an injective homomorphism, then it is said to be an embedding, and if the inverse f^{-1} exists and both f and f^{-1} are embeddings, then f is said to be an isomorphism. More specifically,

- if $R_1 = " \leq "$, then f is said to be an order-homomorphism (resp., order embedding, order isomorphism);
- if R_1 is the graph of $\wedge$ (dually of $\vee$), then f is said to be a $\wedge$-homomorphism (resp., $\wedge$-embedding, $\wedge$-isomorphism);

Moreover, f is a lattice homomorphism if it is both a $\wedge$-homomorphism and a $\vee$-homomorphism. The notions of lattice embedding and lattice homomorphism are defined similarly. A poset P_1 is said to be a subposet of a poset P_2 if there exists an order embedding from P_1 into P_2. Similarly, a (semi)lattice L_1 is said to be a sub(semi)lattice of a (semi)lattice L_2 if there exists a (semi)lattice embedding from L_1 into L_2. Note that if L_1 is a sub(semi)lattice of L_2, then it is also a subposet. However the converse is not necessarily true.

An element $x \in L$ such that $x = y \vee z$ implies $x = y$ or $x = z$ is called $\vee$-irreducible. Dually, an element $x \in L$ such that $x = y \wedge z$ implies $x = y$ or $x = z$ is called $\wedge$-irreducible. We will denote the set of $\wedge$-irreducible and $\vee$-irreducible elements of L by $\mathcal{M}(L)$ and $\mathcal{J}(L)$, respectively. Observe that both $\mathcal{M}(L)$ and $\mathcal{J}(L)$ are posets when ordered by $\leq_L$. In this case $(\mathcal{M}(L), \leq_L)$ stands for the poset and $\leq_L$ is understood as the restriction of $\leq_L$, the order relation on the lattice L, to elements in $\mathcal{M}(L)$ (resp. for $(\mathcal{J}(L), \leq_L)$).

We say that two lattices L_1 and L_2 are invariant wrt $\wedge$-irreducible elements (resp. $\vee$-irreducible elements) *iff* there exists an order-isomorphism f between $(\mathcal{M}(L_1), \leq_{L_1})$ and $(\mathcal{M}(L_2), \leq_{L_2})$ (resp. $(\mathcal{J}(L_1), \leq_{L_1})$ and $(\mathcal{J}(L_2), \leq_{L_2})$). Clearly, L_1 and L_2 are invariant wrt $\wedge$-irreducible and $\vee$-irreducible elements *iff* there exists a lattice isomorphism $f : L_1 \to L_2$.

The family of lattices invariant wrt $\vee$-irreducible elements was studied by Bordalo and Monjardet in [**?**]. The family of lattices that order-embed a given poset was studied by Nation and Pogel in [**?**]. Notice that these two families are themselves lattices when ordered by inclusion.

2.2 Formal concept analysis

Formal Concept Analysis [**?**] is a mathematical formalism based on lattice theory and aimed at data analysis and classification tasks. Concept lattices are built from a binary table called a *formal context* via a *Galois connection*.

A formal context is a triple (G, M, I), where G is a set of objects, M is a set of attributes and I is an incidence relation between objects and attributes. For instance, in phylogenetic data, objects are usually species, attributes are mutations, and I is a binary relation where $(g, m) \in I$ (or in infix notation gIm) states that the mutation m is spotted in specie g.

Every formal context (G, M, I) induces a Galois connection between objects and attributes: for $X \subseteq G$ and $Y \subseteq M$, defined by:

$$X' = \{y \in M \mid xIy \text{ for all } x \in X\},$$
$$Y' = \{x \in G \mid xIy \text{ for all } y \in Y\}.$$

A formal concept is then a pair (X, Y) such that $X' = Y$ and $Y' = X$, called respectively the intent and the extent of the formal concept (X, Y). It should be noticed that both X and Y are closed sets, i.e., $X = X''$ and $Y = Y''$. The set of all formal concepts can be ordered by inclusion of the extents or, dually, by the reversed inclusion of the intents. The resulting order, denoted by $\leq$, gives rise to the *concept lattice* of the context (G, M, I). The existence of a supremum and an infimum enables the use of lattices for classification purposes. Concepts can be viewed as classes, indeed a concept (X, Y) is a representation of a maximal set X of objects that share a maximal set Y of attributes. If another concept (X_1, Y_1) is greater than (X, Y), it contains more objects but it is described by less attributes.

A *clarified context* is a context such that for any $x, y \in G$, if $x' = y'$, then $x = y$ and for any $a, b \in M$, if $a' = b'$, then $a = b$. In other words, the sets of attributes (resp. objects) of two distinct objects (resp. attributes) are distinct. Moreover, a clarified context is *reduced iff* it does not contain:

- $x \in G$ such that $x' = X'$ with $X \subseteq G$, $x \notin X$
- $y \in M$ such that $y' = Y'$ with $Y \subseteq M$, $y \notin Y$

The reduced context is also called a *standard context* [**?**]. Note that the standard context of lattice L verifies $G = \mathcal{J}(L)$ and $M = \mathcal{M}(L)$.

2.3 Distributive lattices

Formally, a lattice L is distributive if for every $x, y, z \in L$, one (or, equivalently, both) of the following identities holds:

$$(i)\ x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \quad (ii)\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Distributive lattices naturally appear in any classification task and as computation and semantic models; see, e.g., [**?**,**?**,**?**,**?**]. This is partially due to the fact that *every distributive lattice can be thought of as a sublattice of a power-set lattice, i.e., the set $\mathcal{P}(X)$ of all subsets of a given set X*. It should be noticed that each sublattice of a distributive lattice is also distributive, and thus distributivity can be characterized in terms of forbidden sublattices. In particular, a lattice is distributive *iff* it contains neither N_5 (($\perp$, $\top$ and a, b, c with $b < c$)) nor M_3 (3 non-comparable elements, $\perp$ and $\top$), the two smallest non distributive lattices, as sublattices.

A main result about distributive lattices is *the representation theorem of Birkhoff* [**?**] which makes use of the notion of *order ideals*. Let $(P, \leq)$ be a poset. For $X \subseteq P$, let $\downarrow X = \{y \in P : y \leq x$ for some $x \in X\}$ and $\uparrow X = \{y \in P : x \leq y$ for some $x \in X\}$.

A set $X \subseteq P$ is a (*poset*) *ideal* (resp. *filter*) if $X = \downarrow X$ (resp. $X = \uparrow X$). If $X = \downarrow \{x\}$ (resp. $X = \uparrow \{x\}$) for some $x \in P$, then X is said to be a *principal* ideal (resp. filter) of P. For principal ideals, we omit brackets, so that $\uparrow x$ (resp. $\downarrow x$) stands for $\uparrow \{x\}$ (resp. $\downarrow \{x\}$).

2.4 Birkhoff's representation theorem of distributive lattices.

Let $(P, \leq)$ be a poset and consider the set $\mathcal{O}(P)$ of ideals of P, i.e.,

$$\mathcal{O}(P) = \{ \bigcup_{x \in X} \downarrow x \mid X \subseteq P \}.$$

It is well known that for every poset P, the set $\mathcal{O}(P)$ ordered by inclusion is a distributive lattice, called the *ideal lattice* of P [?]. Furthermore, the poset of $\vee$-irreducible elements of $\mathcal{O}(P)$ is $\mathcal{J}(\mathcal{O}(P)) = \{\downarrow x \mid x \in P\}$ which is order-isomorphic to P.

This representation is used to provide a distributive lattice L_d with the same poset of $\vee$-irreducible elements as a given lattice L [?]. In this case, there is a $\wedge$-embedding from L into L_d. Every lattice invariant wrt $\vee$-irreducible poset can be $\wedge$-embedded in the ideal lattice of its $\vee$-irreducible poset.

From a poset P, it is possible to obtain the context of the ideal lattice as $(P, P, \not\geq)$ [?]. For a standard context $C = (\mathcal{J}(L), \mathcal{M}(L), \leq)$, the standard context of the ideal lattice is $(\mathcal{J}(L), \mathcal{J}(L), \not\geq)$. Also, for every distributive lattice L, the poset $(\mathcal{J}(L), \leq_L)$ is isomorphic to the poset $(\mathcal{M}(L), \geq_L)$ (dual poset of $(\mathcal{M}(L), \leq_L)$). This is why standard contexts of distributive lattices are "squares" ($|\mathcal{J}(L)| = |\mathcal{M}(L)|$), and can be built with the information of only one of these two posets.

Let $S^\vee$ be a $\vee$-semilattice and let $x \in S^\vee$. Then $\uparrow x$ is a lattice (in the finite case, every $\vee$-semilattice with $\perp$ is a lattice). A semilattice $S^\vee$ is said to be distributive if $\uparrow x$ is distributive, for all $x \in S^\vee$ [?]. It should be noticed that it is sufficient to check this property for the minimal elements of $S^\vee$. It follows directly from the fact that a sublattice of a distributive lattice is a distributive lattice, and if $x \leq y$ in $S^\vee$, then $\uparrow y$ is a (distributive) sublattice of $\uparrow x$.

2.5 Median algebras, median semilattices and median graphs

As indicated in the introduction, median algebras can encode all parsimonious (with no unnecessary mutations) phylogenetic trees in case of evolution ambiguities. Formally, a *median algebra* is a structure $A = (A, \mathrm{m})$ for a nonempty set A and a ternary symmetric operation $\mathrm{m}\colon A^3 \to A$ such that for every $x, y, z, t, u \in A$, $\mathrm{m}(x, x, y) = x$ and $\mathrm{m}(\mathrm{m}(x, y, z), t, u) = \mathrm{m}(x, \mathrm{m}(y, t, u), \mathrm{m}(z, t, u))$. Such algebras have been studied by several authors and they were shown to be tightly related to several ordered structures such as semilattices and distributive lattices, and to an important class of graphs, the so-called "median graphs" [?,?].

Sholander [?] showed that each element a of a median algebra A gives rise to a *median semilattice* $(A, \vee_a)$ where $x \vee_a y = \mathrm{m}(a, x, y)$. From this definition it follows that their principal filters

$$\uparrow x := \{y \in A \colon y \vee_a x = y\}$$

are indeed distributive lattices. Conversely, if a $\vee$-semilattice $S^\vee$ has the property that for every $a, b, c \in S^\vee$, $a \wedge b \wedge c$ exists whenever $a \wedge b, b \wedge c, c \wedge a$ exist, then $S^\vee$ is a median semilattice.

The following characterization of distributive lattices emphasizes the link between distributive lattices and median algebras.

Property 1. A lattice L is a distributive lattice iff for all $x, y, z \in L$,

$$(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x). \tag{1}$$

Accordingly, we can define a median operation on L by $\mathrm{m}(x, y, z) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$, for every $x, y, z \in L$.

In the case of discrete structures, median algebras can also be considered as a special kind of graphs. A *median graph* is a connected graph (i.e., for every pair of vertices there is a path connecting them) verifying that for any three vertices a, b, c, there is exactly one vertex x that minimizes the sum of the distances to a, b and c. It was shown in [**?**] that every median semilattice whose intervals are finite has a *covering graph* (i.e., undirected Hasse diagram) that is median, and that every median graph is the covering graph of a median semilattice. For further details on the connections between median graphs and median semilattice, see [**?**,**?**].

Summing up, we have the following characterization [**?**].

Property 2. A graph G is a median graph *iff* it is the covering graph of a $\vee$-semilattice $S^{\vee}$ with the following properties:

1. $S^{\vee}$ is distributive, and
2. (TRI) for every $x, y, z \in S^{\vee}$ such that $x \wedge y$, $y \wedge z$ and $z \wedge x$ are defined, $x \wedge y \wedge z$ is also defined.

2.6 Previous Results

In [**?**], we presented an algorithm that takes the reduced context of a lattice L as input, and outputs in polynomial time the context of a lattice L_d such that $S_L = \{L_d \backslash \bot\}$ is a distributive $\vee$-semilattice with the same poset of $\vee$-irreducible elements. This context is obtained by "local" applications of Birkhoff's representation theorem on every subcontext corresponding to a sublattice defined by a filter of atoms (minimal $\vee$-irreducible elements). Due to the fact that some elements of the context may appear in several filters, multiple applications may be necessary until a fix-point is reached. This fix-point is subsumed by the ideal lattice of the poset of $\vee$-irreducible elements derived from the context.

In some cases, our algorithm outputs a solution smaller than this ideal lattice. This is indeed the case for M_3 or N_5, for which the algorithm outputs a non modified context. Indeed, $M_3 \backslash \bot$ is a tree and $N_5 \backslash \bot$ is a path, and both are median graphs. Nevertheless, our algorithm does not guarantee the minimality of the solution. In fact, we have shown in [**?**] that several minimal non isomorphic solutions may exist.

Also, being able to build a distributive $\vee$-semilattice is a necessary condition to build a median graph, but it is not sufficient. In [**?**,**?**] the second condition of Property **??** was not considered. Here, we take it into account and provide an improved algorithm that always outputs a median graph for any input lattice.

3 Main results

In this section we show, in particular, that if L is a lattice containing a triple x, y, z such that $x \wedge y, x \wedge z, y \wedge z > \perp$ and $x \wedge y \wedge z = \perp$ then the only median semilattice invariant for the poset of $\vee$-irreducible elements is a lattice, and it is the ideal lattice of the poset of $\vee$-irreducible elements of L.

Proposition 1. *Let L be a lattice and suppose that there exist $x, y, z \in L$ with $x \wedge y, x \wedge z, y \wedge z > \perp$ and $x \wedge y \wedge z = \perp$. Let denote $S_L = \{L_d \backslash \perp\}$, with L_d a distributive $\vee$-semilattice with the same poset of $\vee$-irreducible elements as L. Then the semilattice S_L does not satisfy the second condition of Property* **??**.

Proof. By construction (applications of Birkhoff's representation theorem), L_d is a lattice such that there is an $\wedge$-embedding $f\colon L \to L_d$ i.e., $x \wedge y = z \Rightarrow f(x) \wedge f(y) = f(z)$. Hence, if there are $x, y, z \in L$ such that $x \wedge y, x \wedge z, y \wedge z > \perp$ and $x \wedge y \wedge z = \perp$, then $f(x) \wedge f(y)$, $f(x) \wedge f(z)$, $f(y) \wedge f(z)$ exist, but $f(x) \wedge f(y) \wedge f(z) = \perp \notin S_L$. □

As an immediate consequence, we obtain the following corollary.

Corollary 1. *If L is a lattice for which there exist $x, y, z \in L$ with $x \wedge y, x \wedge z, y \wedge z > \perp$ and $x \wedge y \wedge z = \perp$, then there is exactly one median semilattice S_L such that $(\mathcal{J}(L), \leq_L)$ is isomorphic to $(\mathcal{J}(S_L), \leq_{S_L})$. This semilattice is the ideal lattice of $(\mathcal{J}(L), \leq_L)$.*

Proof. To be a median semilattice, a new element $a = x \wedge y \wedge z$ must be added to the semilattice S_L. There are two possibilities:

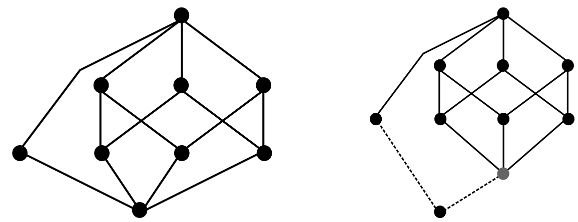
- $(i.)$ $a = \perp$ and in this case S_L is a lattice. For being a median semilattice, it must be a distributive lattice, and thus the ideal lattice of $(\mathcal{J}(L), \leq)$, or
- $(ii.)$ $a \neq \perp$ and in this case a is a new atom of S_L, and thus a new $\vee$-irreducible element. It is not difficult to check that there is no isomorphism between $(\mathcal{J}(L), \leq)$ and $(\mathcal{J}(L_d), \leq)$ □

If one requires invariance of the poset of $\vee$-irreducible elements, then the following lemma provides an algorithm that decides in polynomial time whether the second condition is satisfied. In other words, whether it must output the ideal lattice of $\vee$-irreducible elements or a distributive $\vee$-semilattice.

Lemma 1. *Let L be a lattice. There exist $x, y, z \in L$ such that $x, y, z \in L$ with $x \wedge y, x \wedge z, y \wedge z > \perp$ and $x \wedge y \wedge z = \perp$* iff *there exist $a_x, a_y, a_z \in Atoms(L)$ and $m_x, m_y, m_z \in \mathcal{M}(L)$ such that*

- $a_x \not\leq m_x$, $a_x \leq m_y$, $a_x \leq m_z$,
- $a_y \not\leq m_y$, $a_y \leq m_x$, $a_y \leq m_z$,
- $a_z \not\leq m_z$, $a_z \leq m_x$, $a_z \leq m_y$.

Proof. Suppose that there exist $x, y, z \in L$ such that $x \wedge y, x \wedge z, y \wedge z > \perp$ and $x \wedge y \wedge z = \perp$. Then there exist 3 atoms a_x, a_y, a_z such that



(*i.*) (*ii.*)

Fig. 2. Illustration of the proof of Corollary **??**. For the lattice in (i), each filter is distributive and so $L\backslash\bot$ is a distributive $\vee$-semilattice. However, the second condition is not satisfied. One possibility is to keep $\bot$ but the whole lattice should be then distributive (which is not the case). The other possibility is to add a new infimum distinct from $\bot$. In this case, the poset of $\vee$-irreducible elements is not invariant from (i) to (ii).

- $a_z \leq x \wedge y$, $a_z \not\leq z$ (otherwise $x \wedge y \wedge z \geq a_z > \bot$),
- $a_y \leq x \wedge z$, $a_y \not\leq y$ (otherwise $x \wedge y \wedge z \geq a_y > \bot$), and
- $a_x \leq y \wedge z$, $a_x \not\leq x$ (otherwise $x \wedge y \wedge z \geq a_x > \bot$).

Take $m_i \in Max(\mathcal{M}(L) \backslash \uparrow a_i)$ for $i \in \{x, y, z\}$. Then,

- m_x is greater than a_y and a_z and not greater than a_x,
- m_y is greater than a_x and a_z and not greater than a_y, and
- m_z is greater than a_x and a_y and not greater than a_z,

showing that the conditions are necessary.

To show that the conditions are also sufficient, suppose that there are $a_1, a_2, a_3 \in Atoms(L)$ and $m_1, m_2, m_3 \in \mathcal{M}(L)$ such that

- $a_1 \not\leq m_1$, $a_1 \leq m_2$, $a_1 \leq m_3$
- $a_2 \not\leq m_2$, $a_2 \leq m_1$, $a_2 \leq m_3$
- $a_3 \not\leq m_3$, $a_3 \leq m_1$, $a_3 \leq m_1$

Let $x = a_2 \vee a_3$, $y = a_1 \vee a_3$, $z = a_1 \vee a_2$. It should be noticed that there cannot be any comparable pair from $\{x, y, z\}$ as it would contradict the existence of $m_1, m_2, m_3 \in \mathcal{M}(L)$ satisfying the displayed conditions. Hence, x, y and z are pairwise incomparable w.r.t. $\leq$. Moreover, $x \wedge y > \bot$, $x \wedge z > \bot$, $y \wedge z > \bot$ and $x \wedge y \wedge z = \bot$. This completes the proof of the lemma. □

This lemma ensures that it is possible to check whether the second condition is satisfied in polynomial time from the reduced context of the lattice. Note that it is a variation (on atoms, i.e., minimal $\vee$-irreducible elements) of a result on totally balanced matrices (see for example [**?**,**?**])

Based on Lemma **??**, we can modify our algorithm to Algorithm **??**, which now returns a context and a Boolean. If the Boolean is true, then it is possible to consider the semilattice obtained by the deletion of $\bot$. If the Boolean is false, then the output context is the one associated with the ideal lattice of $(\mathcal{J}(L), \leq)$ that, together with $\perp$, constitutes a median semilattice (which is, in this case, a lattice).

Algorithm 1: Construction of context of a median ($\vee$-semi) lattice.

```
Data: A context (J(L), M(L), I) of a lattice L
Result: a context (J(L_med), M(L_med), I) and a boolean tri
tri ← check_TRI_condition((J(L), M(L), I))
if tri == false then
    return C(J(L), J(L), ≱), false
foreach j ∈ J(L), minimal do
    (P_j, ≤) ← ∅
repeat
    stability ← true;
    foreach j ∈ J(L), minimal do
        compute P_j the poset of ∨-irreducible elements in ↑j
        compute C_j = (P_j, P_j, ≱)
        if P_j modified since last iteration then
            stability ← false;
    Merge all C_j = (P_j, P_j, ≱) in a unique context
    Reduce this context
until stability
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4 Final remarks and open problems

In previous works ([**?**,**?**]), we investigated how to obtain a median semilattice S_L from a concept lattice L with the following properties:

1. $L \backslash \perp$ can be order-embedded in S_L,
2. there is an isomorphism between $(\mathcal{J}(L), \leq)$ and $(\mathcal{J}(S_L \cup \{\perp\}), \leq)$, and
3. $|S_L \backslash L|$ is minimal.

Our algorithm, based on local applications of Birkhoff's representation theorem, verifies the first two properties, but may fail to satisfy the third (we have shown in [**?**] that there can exist several minimal solutions).

The first property seems to be natural in applications. In phylogeny problems, we want to be able to add latent vertices, representing the species not yet observed, but we do not want to change the relative order of the observed species. The second property is an extension of the first one. In a context, all elements can be inferred from irreducible ones. Thus, we decided to keep as invariant the poset of $\vee$-irreducible elements, so that the 'skeleton' of data does not change. The third property is motivated by the parsimony principle: a solution with less additional information (concepts) is preferred.

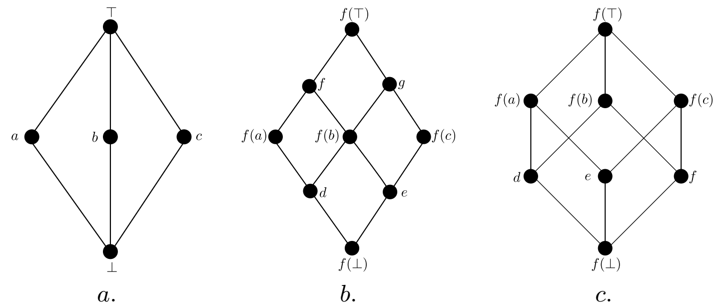

Fig. 3. M_3 (a) and two possible embeddings (b and c)

Fig. **??** proposes two embeddings for the lattice M_3. The first one (Fig. **??**.b) is based on a product of two chains of length 3. The second one (Fig. **??**.c) is based on the ideal lattice of $\mathcal{J}(M_3)$, which is here a Boolean lattice. In this example, there are more concepts in the product of chains than in the Boolean lattice (9 versus 8). Nevertheless, in a similar way, we can embed M_x, a lattice composed by an antichain of x elements, $\top$ and $\bot$ elements into a Boolean lattice of dimension x, as well as a product of two chains of length $x-1$. The product of two chains for length $x-1$ is a distributive lattice with $(x-1)^2$ concepts. The Boolean lattice of dimension x has 2^x concepts. If $x > 4$, there are less concepts in the product of two chains of length $x-1$ than in the Boolean lattice of dimension x. Nevertheless, the number of irreducible elements is greater in the product of chains than in the Boolean lattice (and so, the reduced context of the first has more elements than the context of the last). Without the hypothesis of the invariance of $\vee$-irreducible posets, if follows that there may exist order embeddings with less elements. Then some questions are remaining open:

1. What is the relevance of isomorphism between $\vee$-irreducible elements for real-world problems?
2. What is the parameter to minimize: the number of new concepts in the lattice or the number of elements in the reduced context?
3. How to compute a *minimal* median semilattice from a concept lattice?