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Veeraruna Kavitha[1] and Eitan Altman[2*]

[1]IEOR, Indian Institute of Technology Bombay, India, [2]INRIA Sophia Antipolis, LINCS Paris, and CERI/LIA, Avignon University, France

Abstract. Many systems require frequent and regular updates of certain information. These updates have to be transferred regularly from the source(s) to a common destination. We consider scenarios in which an old packet (entire information unit) becomes completely obsolete, in the presence of a new packet. We consider transmission channels with unit storage capacity; upon arrival of a new packet, if another packet is being transmitted then one of the packets is lost. We consider the control problem that consists of deciding which packet to discard so as to maximise the average age of information (AAoI). We derive drop policies that optimize the AAoI. We show that the state independent (static) policies like dropping always the old packets or dropping always the new packets are optimal in many scenarios, among an appropriate set of stationary Markov policies.

Keywords: Age of Information, Freshness of information, Lossy systems, Renewal Processes, Dynamic and static policies.

1 Introduction

The performance measures that have been studied traditionally in queueing systems have been related to delays and losses. Recently, with the advent of applications demanding frequent and regular updates of a certain information, there is also significant focus towards the freshness of information. Timely updates of the information is an important aspect of such systems, e.g, sensor networks, news feed (social network) over mobile networks or remote control/monitoring of autonomous vehicles etc. Many more such applications are mentioned in [3, 7, 8]. Most of the times the regular updates are transferred from the source of information to the destination using wireless communication systems.

To measure the freshness of information, the concept of age of information (AoI), has been introduced. AoI is defined as the difference between the current time and the generation time of the latest available information ([3]). Peak age of information (PAoI) and Average age of Information (AAoI) are the relevant performance measures, introduced recently in [7, 3]. The study of AAoI/PAoI differs significantly from the conventional performance metrics, such as expected transmission delay, expected number of losses etc.

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There has been considerable work in this direction since its recent introduction, we discuss a relevant few of them. In [3] authors discuss the optimal rate of information generation that minimizes the AAoI for various queuing systems. They showed that the smallest age under FCFS can be achieved if a new packet is available exactly when the packet in service finishes service. In [1] the authors consider AoI only for the packets waiting to be transferred/processed. When the queue is empty their AoI is zero, their definition accounts for the oldness of the information waiting at the head of the line. In [2] authors study PAoI and generalize the previously available results to the systems with heterogeneous service time distributions. The authors consider update rates that minimize the maximum PAoI among all the sources. In [8] authors discuss attempt probabilities for slotted aloha system that optimize AAoI.

Most of the work, discussed above, considers lossless systems, where all the packets are transferred (possibly after some delays). However often in systems that require regular updates of the same information, the old packet[1] becomes obsolete once a new one is available. Then the old packet may be dropped. Thus it is appropriate to consider lossy systems, for such scenarios. As an example, in sensor networks the information is consolidated (to generate data packets) from random sets of nodes at random instances of times. Further the transmission of information to the final destination can be over wireless links, which is again random. Further more in some sensor-based applications, the update rates could be significantly high leading to possible availability of a new packet(s) before the old one is completely transferred and then the later becomes obsolete.

If a new packet arrives at source while an old packet is being transferred, it appears upfront that the transfer of the old packet (entire information unit) has to be abandoned. But if the transfer of the old packet is on the verge of getting completed, and since the new packets may require considerable time for transmission, it might be better to discard the new packet and continue the transmission of old packet. Further, the packet transfer times have large fluctuations when the packets are transferred through wireless medium. Thus it is not clear as to which packet is to be discarded. In this work we study the way in which the choice of the packet to be dropped influences the freshness of the information.

We showed that dropping the old packets (always) is optimal for AAoI, when the packet transfer times are distributed according to exponential or hyper exponential distribution. This is a static policy as the drop decision does not depend upon the state of the system, but is optimal among all the stationary Markov and randomized (SMR) policies. The SMR dynamic policies depend upon the age of information at an appropriate decision epoch. We also establish certain conditions under which dropping the new packets (always) is opti-

[1] Throughout we refer an entire information unit as a packet, that could stand for message or a post or a frame, which needs to be updated frequently. Here are few examples: messages describing weather forecast, or cricket score, or the sensed events related to the entire area in a sensor network, or the information from stock exchanges etc.

mal among SMR policies. For transfer time distributions like uniform, Weibull, Poisson, log-normal etc., one of the two static policies is optimal (even among dynamic/Markov policies) based on the parameters. The AAoI cost is the *ratio of two time-average costs resulting in non standard Markov decision processes (MDPs),* and the above conclusions were obtained by solving these. With the aid of numerical computations we showed for almost all cases that, one of the two static policies is near optimal.

2 System with losses and fresh updates

Consider source(s) sending regular updates of a certain information to a destination. The information update packets (entire information units) arrive at any source according to a Poisson process with rate λ. The packets are of constant length or of random lengths, and the transfer times depend upon the (random) medium. In all, the source requires IID (independent and identically distributed) times $\{T_i\}$ to deliver the packets to the destination, which are equivalently the job times in the queue. Our focus is on measures related to the freshness of information available at the destination.

Age/Freshness of information The age of information (AoI), from the given source and at the given destination, at time t is defined

$$G(t) := t - r_t,$$

where r_t is the time at which the last successfully received packet (at destination) before time t, is generated. Our aim is to study the (time) average age of information (AAoI), defined as below[2]:

$$\bar{a} := \lim_{T\to\infty} \frac{\int_0^T G(t)dt}{T}. \tag{1}$$

We consider freshness of information in a lossy system, and our focus is on the packet to be dropped when there are two simultaneous packets. We begin with analysis of the system that drops new packets, when busy.

2.1 Drop the new packets (DNP)

The source does not interrupt transmission of any packet. If a new update packet arrives, during transmission, it is dropped. Once the transfer is complete (after random time T), the source waits for new packet, and starts transmission of the new packet immediately after. And this continues (see Figure 1).

The age of the information $G(t)$ grows linearly with time at unit rate, at all time instances, except for the one at which a packet is just received at the destination. At that time epoch the age drops to T_k, because: a) T_k is the time taken to transfer the (new) packet from source to destination, after its arrival at the source queue; and b) this represents the age of the new packet at destination.

[2] Limit exists almost surely in all our scenarios, as will be shown in respective proofs.

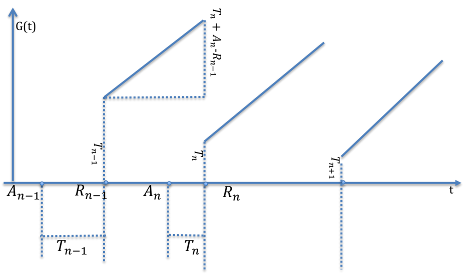

Fig. 1: DNP scheme, Renewal cycles

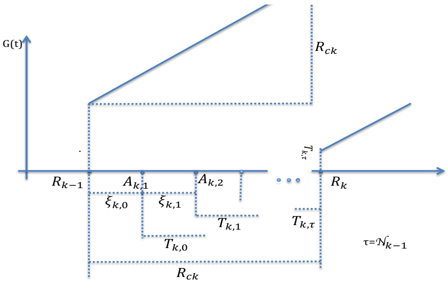

Fig. 2: DOP scheme, a Renewal cycle

Thus we have a process (resembling a renewal process) as in Figure 1. Here $\{R_k\}$ are the epochs at which a message is transferred successfully (these would become the renewal instances in two concatenated processes, see footnote 3), while $\{A_k\}$ are the arrival instances of the packets (at the source and of those transferred) governed by Poisson point process (PPP). Let $\{\xi_k\}$ represent these (residual) inter-arrival times, note $\xi_k = A_k - R_{k-1}$ for each k. As seen from the figure, the age of the information is given by sawtooth waveform. Further more, clearly, the alternate renewal cycles are independent of one another, thus by RRT [3] the long run time average of the age of the information (1) equals:

$$\bar{a} = \frac{E\left[\int_{R_{k-1}}^{R_k} G(s)ds\right]}{E[R_k - R_{k-1}]}, \text{ almost surely (a.s.), and with } G_k := G(R_k).$$

$$= \frac{E[G_{k-1}(R_k - R_{k-1})] + 0.5E[(R_k - R_{k-1})^2]}{E[R_k - R_{k-1}]} = E[G_{k-1}] + \frac{E[(R_k - R_{k-1})^2]}{2E[R_k - R_{k-1}]}. \quad (2)$$

The last line follows by independence (Figure 1) and memoryless property of PPP. For DNP, $G_{k-1} = T_{k-1}$, and

$$\bar{a}_{DNP} = E[T_{k-1}] + \frac{1}{2}\frac{E[(T_k + \xi_k)^2]}{E[T_k + \xi_k]} \text{ a.s.,}$$

where ξ_k, the inter-arrival time, is exponentially distributed with parameter λ and is independent of the transfer times T_k, T_{k-1}. Simplifying

$$\bar{a}_{DNP} = E[T] + \frac{1}{\lambda} + \frac{E[T^2]}{2E[T]}\frac{\rho}{1+\rho} \text{ with } \rho := \lambda E[T]. \quad (3)$$

---

[3] The adjacent cycles (e.g., intervals between R_k, R_{k+1} and R_{k+1}, R_{k+2}) are not independent, because of $G_k = T_k$, however the alternate ones are. Concatenate odd and even cycles to obtain two separate renewal process, observe that $R_k \to \infty$ as $k \to \infty$ and apply (Renewal Reward Theroem) RRT to both the processes to obtain:

$$\bar{a} = \lim_{k\to\infty} \frac{\sum_{l\le k}\int_{R_{l-1}}^{R_l} G(t)dt}{R_k}$$

$$= \lim_{k\to\infty}\left(\frac{\sum_{2l\le k}\int_{R_{2l-1}}^{R_{2l}} G(t)dt}{\sum_{2l\le k} R_{2l} - R_{2l-1}}\frac{\sum_{2l\le k} R_{2l} - R_{2l-1}}{R_k} + \frac{\sum_{2l+1\le k}\int_{R_{2l}}^{R_{2l+1}} G(t)dt}{\sum_{2l\le k} R_{2l+1} - R_{2l}}\frac{\sum_{2l\le k} R_{2l+1} - R_{2l}}{R_k}\right)$$

$$= \frac{E\left[\int_{R_1}^{R_2} G(s)ds\right]}{E[R_2 - R_1]}\frac{1}{2} + \frac{E\left[\int_{R_2}^{R_3} G(s)ds\right]}{E[R_3 - R_1]}\frac{1}{2} = \frac{E\left[\int_{R_1}^{R_2} G(s)ds\right]}{E[R_2 - R_1]} \text{ a.s., as the two processes are identical.}$$

For renewal process with even cycles, the time intervals between two successful packet receptions $\{(R_{2k} - R_{2k-1})\}_k$ form the renewal periods and the time integral of the costs in (1) for each of even renewal periods, $\left\{\int_{R_{2k-1}}^{R_{2k}} G(s)ds\right\}_k$, form the rewards.

2.2 Drop the old packets (DOP)

When source receives a new message, the ongoing transfer (if any) of the old packet is stopped and the old packet is dropped. The source immediately starts transfer of the new packet. The new message would imply a more fresh information, but might also imply longer time (because we now require the transfer of the entire message) before the information at the destination is updated. However the variability in transfer times $\{T_k\}$ might imply interruption is better for average freshness under certain conditions, and we are studying this aspect.

The renewal points (as in footnote 3) will again be the instances at which a message is successfully received. But note that only when a message transfer is not interrupted by a new arrival, we have a successful message reception. Thus the renewal cycles in Figure (1) get prolonged appropriately (see Figure 2). Let $A_{k,1}$ be the first arrival instance after the $(k-1)$-th renewal epoch R_{k-1}. Let $\xi_{k,0}$ be the corresponding inter-arrival time (which is exponentially distributed). Its service (i.e., message transfer) starts immediately and let $T_{k,0}$ be the job size, or the (random) time required to transfer this message. In case a second arrival occurs (after inter-arrival time $\xi_{k,1}$) within this service, we start the service of the new packet by discarding the old one. This happens with probability $1-\gamma$ where $\gamma := P(T_{k,0} \le \xi_{k,1})$. The renewal cycle is completed after second transfer, in case the second message transfer is not interrupted. The second can also get interrupted, independent of previous interruptions and once again with the same probability $1-\gamma$, because of IID nature of the transfer times and the inter arrival times. If second is also interrupted the transfer of the third one starts immediately and this continues till a job is not interrupted (i.e., with probability γ). And then the renewal cycle is completed.

Once again the alternate cycles are IID, RRT can be applied to AAoI given by (1) and AAoI is given by equation (2). However the renewal cycles $\{R_k - R_{k-1}\}_k$ are more complex now, and we proceed with deriving their moments. The k-th renewal cycle can be written precisely as below, using the arrival sequence $\{\xi_{k,i}\}_{i\ge 0}$ and transfer times sequence $\{T_{k,i}\}_{i\ge 0}$ belonging to k-th renewal cycle:

$$R_{ck} := R_k - R_{k-1} = \xi_{k,0} + \sum_{i=1}^{\mathcal{N}_k - 1} \xi_{k,i} + T_{k,\mathcal{N}_k - 1} \quad = \xi_{k,0} + \boldsymbol{\Gamma}_k, \tag{4}$$

$$\boldsymbol{\Gamma}_k := \sum_{i=1}^{\mathcal{N}_k - 1} \xi_{k,i} + T_{k,\mathcal{N}_k - 1}, \text{ and} \qquad \mathcal{N}_k \;:=\; \inf\left\{ i \ge 1 : \xi_{k,i} > T_{k,i-1} \right\}. \tag{5}$$

In the above $\mathcal{N}$ is the number of interruptions before successful transfer, and it is geometrically distributed with parameter $1-\gamma$ and $\boldsymbol{\Gamma}$ (given by (5)) is the time taken to complete one packet transfer, in the midst of interruptions by new arrivals. The above random variables are specific to a given renewal cycle, but are also IID across different cycles. Further, $G_k = G(R_k)$ is now a 'special' transfer time (represented by $\underline{T}$): one which is not interrupted. Thus

$$G_k = \underline{T}_k := T_{k,\mathcal{N}_k - 1}, \text{ and, } E[G_k] = E[\underline{T}_k] = E[T | T \le \xi] \text{ for any } k. \tag{6}$$

Hence the AAoI of DOP scheme (again by independence of alternate cycles, exactly as in footnote 3) equals (see (2)):

$$\bar{a}_{DOP} = E[T|T \leq \xi] + \frac{E[(\xi + \boldsymbol{\Gamma})^2]}{2E[\xi + \boldsymbol{\Gamma}]} \text{ almost surely (a.s.).} \tag{7}$$

To complete the analysis we require the first two moments[4] of $\boldsymbol{\Gamma}$ (see equation (4)) and finally the AAoI for DOP scheme is obtained in the following (Proof in Appendix A of [12]):

Lemma 1. *The first two moments of the renewal cycle are (with $\gamma = E[T < \xi]$):*

$$E[R_c] = \frac{1}{\lambda\gamma} \text{ and } E[R_c^2] = \frac{2}{\lambda^2\gamma^2} - \frac{2E[Te^{-\lambda T}]}{\lambda\gamma^2}. \tag{8}$$

Further, the AAoI for DOP scheme equals:

$$\bar{a}_{DOP} = \frac{1}{\lambda\gamma} = E[R_c]. \qquad \blacksquare \tag{9}$$

Thus the AAoI with DOP scheme exactly equals the expected renewal cycle, while that with DNP scheme is strictly bigger than the expected renewal cycle (from (3), $\bar{a}_{DNP} = E[R_k - R_{k-1}] + \frac{E[T^2]}{2E[T]}\frac{\rho}{1+\rho}$). It is not guaranteed that the expected renewal cycle with DOP scheme is smaller than that with DNP scheme. Thus it is not clear upfront as to which scheme is better. But it is equally (or more) important to understand if any scheme with controlled drops can perform better than these two schemes.

3 Controlled drops

In the previous section two 'extreme' and static schemes are considered: in one all the old packets are dropped while in the other all the new packets are dropped. Now we investigate if there exists a better scheme with partial/controlled drops. We also study the conditions under which DOP is better than DNP. With message successful transfer epochs $\{R_k\}_k$ as the decision epochs, we consider a *dynamic decision about the (DOP/DNP) scheme to be used.* The dynamic decision depends upon the state[5], the age of information G_k, at decision epoch R_k.

Threshold policies: We initially restrict ourselves to special type of *dynamic policies, called threshold policies:* DNP scheme is selected if age (G_k) is above a threshold (say $\theta \geq 0$) and DOP is selected other wise. With DNP scheme, new packets are dropped (other than the first one in that renewal cycle) till the message transfer is complete. With DOP decision, old packets are dropped and transmission of new packet starts immediately, whenever the former is interrupted. This continues till a message is transferred completely. Further dropping of (old/new) packets depends upon the decision at the next decision epoch.

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[4] At first glance $\boldsymbol{\Gamma}$ may appear like busy period of $M/G/\infty$ queue, but it is not true.

[5] The source can easily have access to $\{G_k\}$, as it can easily keep track of successful/unsuccessful prior transmissions.

In contrast to the previous subsections, the length of renewal cycles $\{R_{ck}\}_k$ are no more identically distributed. The distribution of R_{ck} depends upon the scheme chosen at the decision epoch R_k. It is easy to observe that the length of the renewal cycle R_{ck} does not depend upon the absolute value of state G_k, but only upon the state dependent binary (DOP/DNP) decision. Thus the distribution of R_{ck} can be one among two types and precisely equals (see (4)):

$$R_{ck+1} = \begin{cases} \xi_{k+1,0} + T_{k+1,0}, & \text{with DNP (i.e., with } G_k > \theta) \\ \xi_{k+1,0} + \boldsymbol{\Gamma}_{k+1} & \text{else, and using (6),} \end{cases} \tag{10}$$

$$G_{k+1} = \begin{cases} T_{k+1,0}, & \text{with DNP } (G_k > \theta) \\ \underline{T}_{k+1}, & \text{other wise.} \end{cases} \tag{11}$$

Observe that $\theta = 0$ implies DNP, while DOP is obtained by considering $\theta \to \infty$. *For ease of notation we say $\theta = \infty$ when DOP is selected for all G_k.*

For every θ, the random variables $\{G_k\}_k$ and $\{R_{ck}\}_k$ constitute a Markov chain. Using these, one can rewrite AAoI (1) as:

$$\bar{a}(\theta) = \lim_{k\to\infty} \frac{\int_0^{R_k} G(t)dt}{R_k} \text{ a.s.,}$$

because $R_k \to \infty$ a.s., as $k \to \infty$, and this is because

$$R_k = \sum_{l\le k} R_{cl} \ge \sum_{l \le k} \xi_{l,0} \text{ for all } k \text{ and } \sum_{l\le k} \xi_{l,0} \stackrel{k\to\infty}{\to} \infty \text{ a.s.}$$

Thus,

$$\bar{a}(\theta) = \frac{\sum_{l\le k}\left(G_{l-1}R_{cl} + 0.5R_{cl}^2\right)}{k} \frac{k}{\sum_{l\le k} R_{cl}}. \tag{12}$$

As already discussed, the distribution of R_{ck} (for any k) can be of two types depending only upon the event $\{G_k < \theta\}$ (see (10). Let G_* and R_{c*} represent the random quantities corresponding to stationary distributions of G_k and R_{ck} respectively. As before, the stationary distribution R_{c*} depends only upon the stationary event $\{G_* < \theta\}$. Thus it suffices to obtain the stationary distribution of $\{G_k\}_k$. In fact the transitions of $\{G_k\}$ given by (11) also depend only upon the events $\{G_{k-1} < \theta\}$. Thus it further suffices to study the two state Markov chain $X_k := 1_{\{G_k<\theta\}}$ (1_A is the indicator of the event A) and the rest of the random quantities can be studied using this two state chain. The Markov chain has the following evolution

$$X_{k+1} = \begin{cases} 1_{\{T_{k+1,0}<\theta\}} & \text{if } X_k = 0, \\ 1_{\{\underline{T}_{k+1}<\theta\}} & \text{else.} \end{cases} \tag{13}$$

When $\theta = \infty$, $X_k \equiv 1$ for all k. The transition probabilities (with $\theta \ne \infty$) are:

$$P(X_{k+1} = x'|X_k = x) = \begin{cases} p_\theta & \text{if } x = 0,\ x' = 1 \\ q_\theta & \text{if } x = 1,\ x' = 0 \end{cases} \text{ where} \tag{14}$$

$$p_\theta := P(T < \theta) \text{ and } q_\theta := P(\underline{T} > \theta) = P(T > \theta \big| T \le \xi).$$

This chain has unique stationary distribution given by

$$\pi_\theta(0) = \frac{q_\theta}{q_\theta + p_\theta} 1_{\{\theta \neq \infty\}} = 1 - \pi_\theta(1), \text{ and } P(X_* = 0) = \pi_\theta(0), \tag{15}$$

where X_* is the random quantity corresponding to stationary distribution of $\{X_k\}$ (see [12]). The stationary distribution of the remaining quantities is dictated by that of $\{X_k\}$: for example the stationary distribution of G_* is the same as that of T, a typical transfer time when $X_* = 0$ and equals that of $\underline{T} = T | T \leq \xi$ (conditional distribution) when $X_* = 1$.

The Markov chain $\{X_k\}$ is clearly ergodic, the rest of the stationary random quantities R_{c*}, G_* depend just upon X_*, hence strong law of large numbers (SLLN) (e.g., [9]) can be applied[6] separately to the numerator and denominator of (12) to obtain:

$$\bar{a}(\theta) = \frac{E_{\pi_\theta}[G_{l-1} R_{cl}] + 0.5 E_{\pi_\theta}[R_{cl}^2]}{E_{\pi_\theta}[R_{cl}]} \text{ a.s. },$$

where $E_{\pi_\theta}[\cdot]$ is the stationary expectation. It is easy to verify (see (10)) by appropriate conditioning that:

$$\begin{aligned}
E_{\pi_\theta}[G_{l-1} R_{cl}] &= E_{\pi_\theta}[G_{l-1} R_{cl}; X_{l-1} = 1] + E_{\pi_\theta}[G_{l-1} R_{cl}; X_{l-1} = 0] \\
&= \left(\frac{1}{\lambda} + E[T]\right) E[G_*; G_* > \theta] + E[G_* \ ; \ G_* < \theta] \left(\frac{1}{\lambda} + E[\boldsymbol{\Gamma}]\right) \text{ and} \\
E[G_*; G_* > \theta] &= E[T; T > \theta] \pi_\theta(0) + E[T; T > \theta | T \leq \xi] \pi_\theta(1).
\end{aligned}$$

Using similar logic,

$$\begin{aligned}
E_{\pi_\theta}[G_{l-1} R_{cl}] + \frac{1}{2} E_{\pi_\theta}[R_{cl}^2] &= d_n E[G_*; G_* > \theta] + d_o E[G_* \ ; \ G_* < \theta] + 0.5 E[R_{c*}^2] \\
&= \beta_\theta(0) \pi_\theta(0) + \beta_\theta(1) \pi_\theta(1), \qquad (16) \\
E_{\pi_\theta}[R_{cl}] &= d_n \pi_\theta(0) + d_o \pi_\theta(1),
\end{aligned}$$

with the following definitions:

$$\begin{aligned}
\beta_\theta(0) &:= d_n E[T; T > \theta] + d_o E[T; T \leq \theta] + 0.5 c_n, \\
\beta_\theta(1) &:= d_n E[T; T > \theta | T \leq \xi] + d_o E[T; T \leq \theta | T \leq \xi] + 0.5 c_o, \\
c_n &:= E[(\xi + T)^2], \qquad c_o := E\left[(\xi + \boldsymbol{\Gamma})^2\right] \text{ and} \\
d_n &:= E[T + \xi], \qquad d_o := E[\xi + \boldsymbol{\Gamma}].
\end{aligned}$$

Thus the AAoI equals

$$\bar{a}(\theta) = \frac{\beta_\theta(0) \pi_\theta(0) + \beta_\theta(1) \pi_\theta(1)}{d_n \pi_\theta(0) + d_o \pi_\theta(1)} \text{ a.s.} \tag{17}$$

Optimal threshold θ: We are interested in optimal threshold, θ^* and hence consider:

$$\min_{\theta \geq 0} \bar{a}(\theta). \tag{18}$$

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[6] One can not apply the usual renewal theory based analysis, as the process is (the odd/even cycles are also) Markovian and <u>can not be modelled as a Renewal process</u>, with IID renewal cycles.

The objective function depends upon θ in a complicated manner, further the dependence is influenced by the distribution of the transfer times. However one can derive the optimal policies by using an appropriate lower bound function.

We first consider the case: $\underline{E[T] > E[\boldsymbol{\Gamma}], \text{ or } d_n > d_o}$. From (16) and the definitions following (16) and because of positivity of the terms:

$$
\begin{aligned}
\bar{a}(\theta) \ &\geq \ f_o(\theta) \text{ for any } \theta \geq 0, \text{ with function,} \\
f_o(\theta) &:= \frac{d_o\big(b_n\pi_\theta(0) + b_o\pi_\theta(1)\big) + 0.5(c_n\pi_\theta(0) + c_o\pi_\theta(1))}{d_n\pi_\theta(0) + d_o\pi_\theta(1)} \\
&= \frac{d_o\big((b_n - b_o)\pi_\theta(0) + b_o\big) + 0.5(c_n - c_o)\pi_\theta(0) + 0.5c_o}{(d_n - d_o)\pi_\theta(0) + d_o}, \text{ with} \\
b_n &:= E(T), \quad b_o := E(T|T < \xi) = \frac{E[Te^{-\lambda T}]}{E[e^{-\lambda T}]}.
\end{aligned} \tag{19}
$$

Further using (7) we have[7] (e.g., $\pi_\theta(0) \to 0$ as $\theta \to \infty$):

$$\lim_{\theta\to\infty} f_o(\theta) = \lim_{\theta\to\infty} \bar{a}(\theta) = \bar{a}_{DOP}. \tag{20}$$

If the DOP scheme is optimal for the lower bound function $f_o(\theta)$, i.e., if

$$\min_\theta f_o(\theta) = \lim_{\theta\to\infty} f_o(\theta), \tag{21}$$

then DOP would be optimal for AAoI, because then using (20):

$$\bar{a}_{DOP} \geq \min_\theta \bar{a}(\theta) \geq \min_\theta f_o(\theta) = \lim_{\theta\to\infty} f_o(\theta) = \bar{a}_{DOP}.$$

We prove that (21) is true when DOP renewal cycle is smaller, and hence show the optimality of DOP (proof in Appendix A and in [12]):

Theorem 1 *If $d_n \geq d_o$ then DOP is optimal, i.e.,*

$$\min_{\theta\geq 0} \bar{a}(\theta) = \lim_{\theta\to\infty} \bar{a}(\theta) = \bar{a}_{DOP}. \qquad \blacksquare$$

It is clear from (3) and (7) that the DOP scheme is better than the DNP scheme when its expected renewal cycle is smaller, i.e., when $d_n \geq d_o$. Theorem 1 proves much more under the same condition, the DOP scheme is better than any other threshold scheme.

We now study the reverse case, i.e., when $d_n < d_o$ or equivalently when $\underline{E[T] < E[\boldsymbol{\Gamma}]}$. In this case $\bar{a}(\theta) > f_n(\theta)$ where

$$f_n(\theta) := \frac{d_n(b_n\pi(0) + b_o\pi(1)) + 0.5c_n\pi(0) + 0.5c_o\pi(1)}{d_n\pi(0) + d_o\pi(1)}.$$

As in the previous case, if DNP is proved optimal for this lower bound function, then DNP is optimal for controlled AAoI, and this is proved in the following (proof in Appendix A and in [12]):

[7] It is not difficult to establish the continuity of the relevant functions as $\theta \to \infty$ and it is not difficult to show that the limit equals that with DOP scheme.

Theorem 2 *If $d_n < d_o$ and (note that*[8] $1 - \lambda E\left[Te^{-\lambda T}\right] > 0$*)*

$$\rho \frac{E[T^2]}{2E[T]} - (1+\rho)(d_o - d_n)\left(1 - \lambda E\left[Te^{-\lambda T}\right]\right) \leq 0, \tag{22}$$

then DNP is optimal, $\min_{\theta \geq 0} \bar{a}(\theta) = \bar{a}(0) = \bar{a}_{DNP}$. ■

Stationary Markov Randomized policies: We now generalize the results to Stationary Markov Randomized (SMR) policies. As seen from (12) the *objective function AAoI is the ratio of two average costs and hence the usual techniques of Markov decision processes may not be applicable. Nevertheless we could use exactly the same techniques as in previous subsection to show the optimality of DNP/DOP policy even under SMR policies.* This is true under the assumptions of Theorems 1-2.

Let α^∞ be any Stationary Markov Randomized policy: $\alpha(G)$ represents the probability with which DNP scheme is selected when the state $G_k = G$, and this is true for all decision epochs k ($^\infty$ implies same state-dependent decision for all decision epochs). Re define $X_k = 1$ if DOP scheme is selected (i.e., if old packet is dropped), else $X_k = 0$. Like before, the random variables X_k, R_{ck} and G_k depend mainly upon X_{k-1}, and same is the case with their stationary distributions. Let π_α represent the stationary probability that $\{X^* = 0\}$, when policy α^∞ is used and note that:

$$\pi_\alpha := \pi_\alpha(0) = \frac{q_\alpha}{q_\alpha + p_\alpha} \text{ with} \tag{23}$$
$$q_\alpha := E[\alpha(\underline{T})] = E[\alpha(T)|T \leq \xi] \text{ and } p_\alpha = E[1 - \alpha(T)].$$

As before, the stationary expectation (see (12))

$$\begin{aligned}
E_{\pi_\alpha}[G_{l-1}R_{cl}] &= E_{\pi_\alpha}[G_{l-1}R_{cl}; X_{l-1} = 1] + E_{\pi_\alpha}[G_{l-1}R_{cl}; X_{l-1} = 0] \\
&= d_n E[G_* E[X^* = 1|G*]] + d_o E[G_* E[X^* = 0|G_*]] \\
&= d_n E[G_* \alpha(G*)] + d_o E[G_*(1 - \alpha(G*))] \\
&= d_n \Big(E[T\alpha(T)]\pi_\alpha(1) + E[T\alpha(T)|T \leq \xi]\pi_\alpha(0) \Big) \\
&\quad + d_o \Big(E[T(1 - \alpha(T))]\pi_\alpha(1) + E[T(1 - \alpha(T))|T \leq \xi]\pi_\alpha(0) \Big).
\end{aligned}$$

Similarly

$$E_{\pi_\alpha}[R_{ck}^2] = E_{\pi_\alpha}[R_{ck}^2; X_{k-1} = 0] + E_{\pi_\alpha}[R_{ck}^2; X_{k-1} = 1] \quad = \quad c_o \pi_\alpha(1) + c_n \pi_\alpha(0),$$

Proceeding exactly as in the case of threshold policies:

$$\begin{aligned}
\bar{a}(\alpha) &= \frac{\beta_\alpha(0)\pi_\alpha(0) + \beta_\alpha(1)\pi_\alpha(1)}{d_n \pi_\alpha(0) + d_o \pi_\alpha(1)} \text{ with} \\
\beta_\alpha(0) &:= d_n E[T\alpha(T)] + d_o E[T(1 - \alpha(T))] + 0.5c_n \text{ and} \\
\beta_\alpha(1) &:= d_n E[T\alpha(T)|T \leq \xi] + d_o E[T(1 - \alpha(T))|T \leq \xi] + 0.5c_o.
\end{aligned}$$

[8] because ξ is exponential,

$$1 - \lambda E\left[Te^{-\lambda T}\right] = \lambda(E[\xi] - E[T; T \leq \xi]) = \lambda(E[\xi; T > \xi] + E[\xi - T; T \leq \xi]) > 0.$$

Using the lower bound functions, $f_o(\cdot)$ and $f_n(\cdot)$, and following exactly the same logic one can extend Theorems 1-2:

Theorem 3 *a) If $d_n \geq d_o$ then DOP is optimal among SMR policies, i.e.,*

$$\min_{\alpha^\infty \in SMR} \bar{a}(\alpha) = \bar{a}_{DOP}.$$

b) If $d_n < d_o$ and (22) of Theorem 2 is true then DNP is optimal,

$$\min_{\alpha^\infty \in SMR} \bar{a}(\alpha) = \bar{a}_{DNP}. \qquad \blacksquare$$

We thus have that the static policies DOP/DNP are optimal among stationary Markov (dynamic) policies for all the conditions, except when $d_n < d_o$ and (22) of Theorem 2 is not true. Using numerically aided study of the next section, we will show that these 'exception conditions' are 'rare'.

3.1 Numerically aided study

DOP optimal among SMR policies We considered several distributions for transfer times and tested the conditions required for DOP/DNP optimality. The results are summarized in Table 1. By direct substitution one can show that $d_n = d_o$ for exponential and $d_n > d_o$ for hyper exponential distribution. Thus by Theorem 3, DOP is optimal for these transfer times.

DNP/DOP is almost optimal When $d_n < d_o$, but (22) is not satisfied, we do not have theoretical understanding of the optimal policy. We study such test cases by numerically optimizing (17) over threshold policies. One such example is plotted in Figure 3, which considers Erlang distributed transfer times. The AAoI is plotted as a function of θ, it decreases as $\theta \to \infty$, hence confirming that the AAoI is minimized by DOP scheme.

A second example is considered in Figure 4 with uniformly distributed transfer times, distributed between $(0, \phi)$. Here again AAoI $\bar{a}(\theta)$ is plotted as a function of θ for two different parameters. An intermediate $\theta^* \in (0, \infty)$ is optimal in both the examples of this figure, however DOP and DNP perform almost similar. Further AAoI at θ^* is close to that at DNP/DOP (Figure 4). We considered many more such case studies and observed similar pattern: DOP/DNP scheme is (almost) optimal. These examples include truncated exponential, Log normal, Poisson distributed and Erlang transfer times etc.

Best among DNP/DOP Thus either DNP or DOP scheme is (almost) optimal among the threshold policies. Hence it is important to derive the conditions that suggest the best among the two. One can find the best among DNP/DOP schemes by directly using (3) and (9), i.e., DNP is better than DOP iff (recall $\rho = \lambda E[T]$)

$$E[T] - \frac{1-\gamma}{\lambda\gamma} + \frac{E[T^2]}{2E[T]}\frac{\rho}{1+\rho} < 0 \text{ or iff } 1 > \left(\frac{E[T^2]}{2(E[T])^2}\frac{\rho^2}{1+\rho} + 1 + \rho\right)\gamma. \qquad (24)$$

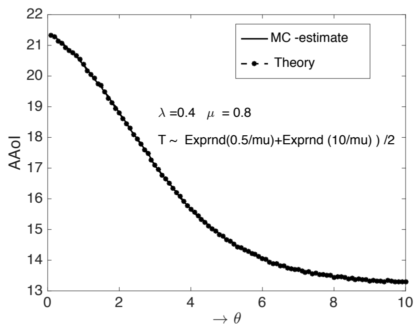

Fig. 3: When $d_o = 13.19 > d_n = 12.97$ and condition (22) negated: optimizer is DOP

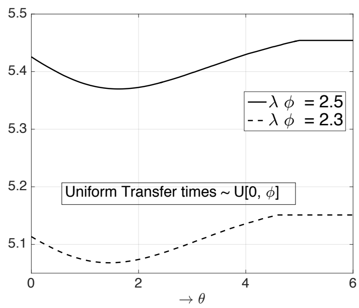

Fig. 4: AAoI versus θ for uniform transfer times: Intermediate θ optimal but DNP/DOP almost optimal

Table 1: Criterion for $\bar{a}_{DOP} \leq \bar{a}_{DNP}$, for different types of T

Distribution	CDF $(P(T \leqslant x))$	$\bar{a}_{DOP} \leq \bar{a}_{DNP}$ when
Uniform $(0,\phi)$	$\left[\frac{x}{\phi}\right]1_{x>0}$	$\frac{1}{1-e^{-\lambda\phi}} < \left(\frac{1}{3}\frac{\lambda\phi}{2+\lambda\phi} + \frac{1}{\lambda\phi} + \frac{1}{2}\right)$, approx. when $\lambda\phi < 2.356$
Weibull $(\mu$, k)	$\left[1 - e^{-(x/\mu)^k}\right]1_{x>0}$	$\frac{\rho^2 w_k^2}{2(1+\rho w_k^1)} + (1 + \rho w_k^1) > \frac{1}{w_\rho^k}$ $w_k^i = \Gamma(1 + \frac{i}{k}) \quad \rho = \lambda\mu \quad w_\rho^k = E_k[e^{-\rho T}]$
Exponential (μ)	$\left[1 - e^{-\mu x}\right]1_{x>0}$	all μ
Hyperexpo$(\{\mu_i, p_i\})$	$\left[1 - \sum_{i=1}^{n} p_i e^{-\frac{x}{\mu_i}}\right]1_{x>0}$	all $\{\mu_i, p_i\}_i$

Note that $E[T^2] = \text{Var}(T) + (E[T])^2$ and we have the following important conclusions:

- DNP is the best for large update rates: as the update rate $\lambda \to \infty$, with distribution of T fixed and with $E[1/T] < \infty$, the above condition is satisfied (RHS converges to 0). Note $\lambda\gamma = E[\lambda e^{-\lambda T}] \to 0$ using L'Hopital's rule (applied point-wise) and dominated convergence theorem.
- DOP is the best for small update rates: as the update rate $\lambda \to 0$, with the distribution of T fixed, the above condition is negated (RHS is approximately $1 + \rho$).
- The range of λ for which DNP is optimal is influenced by the variance. *'DOP scheme becomes optimal as the variance of the transfer times increases, for bigger range of λ'.*

For uniform transfer times we derived the conditions under which DOP performs better than DNP, using (24), and the condition is tabulated in the first row of Table 1. Approximately, DOP is optimal if $\lambda\phi < 2.35$. Weibull is also tabulated in the second row.

Based on this theoretical and numerical case studies we have the following:
• AAoI is (almost) optimized either by DOP scheme or by DNP scheme. No other threshold policy performs significantly better than the best among these two static policies.
• If expected renewal cycle with DOP is smaller than that with DNP, DOP

scheme optimizes AAoI over all SMR policies.
• When DNP has smaller renewal cycle, DOP may still be the optimal (in some test cases).

The Figure 3 also plots the Monte-Carlo estimates of AAoI along with formula (17). For Monte-Carlo estimates we generate several random sample paths and compute the time average of AAoI. As anticipated, the formula well matches the estimates (see Figure 3).

Future Directions: We consider multiple sources in [12] and have some initial results; with multiple sources the drop decision should also include the differential priority that needs to be given to different sources. We also consider the case of multiple resources transferring information to single destination using ALOHA type protocol ([12]).

As of now our analysis considers memory less packet arrivals, it would be interesting to consider more general arrival processes (e.g., renewal processes). It would be interesting to investigate if the static policies (dropping always the new/old packets) are again optimal.

One can also think of more general decision epochs, one can think of dropping at maximum K packets and K can also be controlled etc. One can consider one storage option along with DNP protocol.

4 Conclusions

We considered problems related to freshness of information, for scenarios in which the destination is regularly updated with a certain information. In such cases, old information can become completely obsolete once a new update is available. The systems naturally become lossy, in the sense that, some packets would be discarded. We developed a methodology to study the freshness of information, using average age of information (AAoI) as performance metric, for lossy systems. A packet at destination can automatically be discarded once a new update is available. However a new packet at source, while the source is transferring an older packet, demands an important decision: which packet to be discarded. Older packets can be transferred faster to the destination, while the new packet may have fresh information but may require more time to reach the destination. It may be better to base these decisions on the state of the system, the age of the previous update of the same information at destination. However two static policies, drop always the new packets (DNP) or drop always the old packets (DOP), are optimal among a class of stationary Markov policies, for many scenarios.

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Appendix A: Proofs

Proof of Lemma 1: By conditioning on $\xi_{k,1}$, $T_{k,0}$:

$$\begin{aligned} E[\boldsymbol{\Gamma}_k] &= E\left[\boldsymbol{\Gamma}_k \ ; \ \xi_{k,1} > T_{k,0}\right] + E\left[\boldsymbol{\Gamma}_k \ ; \ \xi_{k,1} \le T_{k,0}\right] \\ &= E\left[T_{k,0} \ ; \ \xi_{k,1} > T_{k,0}\right] + E\left[\xi_{k,1} + \tilde{\boldsymbol{\Gamma}} \ ; \ \xi_{k,1} \le T_{k,0}\right] \\ &= E\left[T_{k,0}; \xi_{k,1} > T_{k,0} + \xi_{k,1} \ ; \ \xi_{k,1} \le T_{k,0}\right] \ + E\left[\tilde{\boldsymbol{\Gamma}} \ ; \ \xi_{k,1} \le T_{k,0}\right], \end{aligned}$$

where $\tilde{\boldsymbol{\Gamma}}$ is an IID copy of $\boldsymbol{\Gamma}_k$, which is independent of $T_{k,0}$ and $\xi_{k,1}$. By independence, $E[\boldsymbol{\Gamma}_k] - E\left[\tilde{\boldsymbol{\Gamma}} \ ; \ \xi_{k,1} \le T_{k,0}\right] = E\left[\boldsymbol{\Gamma}\right]\left(1 - P(\xi \le T)\right)$, and thus by further conditioning on T we have the following:

$$E(\boldsymbol{\Gamma}) = \frac{E\left[T; \xi > T + \xi \ ; \ \xi \le T\right]}{P(T \le \xi)} = \frac{E[Te^{-\lambda T}] + (1 - E[e^{-\lambda T}])/\lambda - E[Te^{-\lambda T}]}{P(T \le \xi)} = \frac{1-\gamma}{\lambda \gamma}. \tag{25}$$

Using exactly similar logic:

$$E[\boldsymbol{\Gamma}^2] = E[\min\{T_0, \xi_1\}^2] + E[\tilde{\boldsymbol{\Gamma}}^2](1-\gamma) + 2E[\tilde{\boldsymbol{\Gamma}}]E[\xi_{k,1} \ ; \ T_{k,0} > \xi_{k,1}).$$

Using (25),

$$E[\boldsymbol{\Gamma}^2] = \frac{E[\min\{T_0, \xi_1\}^2] + 2E[\boldsymbol{\Gamma}]E[\xi_{k,1} \ ; \ T_{k,0} > \xi_{k,1}]}{\gamma}$$

$$= \frac{2(1-\gamma)}{\lambda^2\gamma} - \frac{2E[Te^{-\lambda T}]}{\lambda\gamma} + \frac{2E[\boldsymbol{\Gamma}]}{\gamma}\left(\frac{1-\gamma}{\lambda} - E[Te^{-\lambda T}]\right)$$

$$= \frac{2(1-\gamma)}{\lambda^2\gamma} - \frac{2E[Te^{-\lambda T}]}{\lambda\gamma^2} + \frac{2(1-\gamma)^2}{\lambda^2\gamma^2} = \frac{2(1-\gamma)}{\lambda^2\gamma^2} - \frac{2E[Te^{-\lambda T}]}{\lambda\gamma^2}. \quad (26)$$

Using (25) and (26), the first two moments of the renewal cycle are:

$$E[R_c] = \frac{1}{\lambda} + \frac{1-\gamma}{\lambda\gamma} = \frac{1}{\lambda\gamma} \text{ and}$$

$$E[R_c^2] = E\left[\xi_{k,0}^2 + 2\xi_{k,0}\boldsymbol{\Gamma}_k + \boldsymbol{\Gamma}_k^2\right] = \frac{2}{\lambda^2\gamma^2} - \frac{2E[Te^{-\lambda T}]}{\lambda\gamma^2}. \quad \blacksquare$$

Proof of Theorem 1: As a first step, one can easily observe that the coefficients of the lower bound function f_o depend upon θ only via the stationary distribution π_θ, in particular only via $\pi_\theta(0)$, i.e., $f_o(\theta) = f_o(\pi_\theta(0))$. Further the function $\theta \mapsto \pi_\theta(0)$ is ONTO (see (15)) and hence one can equivalently optimize f_o using $\pi := \pi_\theta(0)$:

$$f_o(\theta) = f_o(\pi) = \frac{d_o\Big((b_n - b_o)\pi + b_o\Big) + 0.5(c_n - c_o)\pi + 0.5c_o}{(d_n - d_o)\pi + d_o}.$$

The first derivative for the lower bound function is:

$$f_o'(\pi) = \frac{0.5(c_n d_o - c_o d_n) + d_o(b_n d_o - b_o d_n)}{(\pi(d_n - d_o) + d_o)^2}. \quad (27)$$

From (28) of Appendix B:

$$c_n d_o - c_o d_n = \frac{E[T^2]}{\lambda\gamma} + \left(\frac{1}{\lambda} + E[T]\right)\left(E[Te^{-\lambda T}] - \frac{(1-\gamma)}{\lambda}\right)\frac{2}{\lambda\gamma^2}.$$

Thus the numerator of the derivative (27) is proportional to,

$$c_n d_o - c_o d_n + 2d_o(b_n d_o - b_o d_n) = \frac{E[T^2]}{\lambda\gamma} + \left(\frac{1}{\lambda} + E[T]\right)E[Te^{-\lambda T}]\left(\frac{2}{\lambda\gamma^2} - 2\frac{1}{\lambda\gamma^2}\right)$$

$$-\frac{1}{\lambda\gamma}\left(\left(\frac{1}{\lambda} + E[T]\right)\frac{2(1-\gamma)}{\lambda\gamma} - E[T]\frac{2}{\lambda\gamma}\right)$$

$$= \frac{E[T^2]}{\lambda\gamma} + \frac{2}{\lambda^2\gamma}(E[T] - E[\boldsymbol{\Gamma}]) > 0, \text{ when } d_n \geq d_o.$$

Thus the derivative $f_o'(\theta) > 0$ for all θ, hence the lower bound f_o is increasing with π, and thus the unique minimizer of f_o is at $\pi^* = 0$. This implies the DOP scheme (see (15)) is optimal for AAoI $\bar{a}(\centerdot)$. ■

Proof of Theorem 2: As before it suffices to show that the numerator of derivative of f_n (with respect to π) is negative. Recall the following:

$$c_n d_o - c_o d_n = \frac{E[T^2]}{\lambda\gamma} + \left(\frac{1}{\lambda} + E[T]\right)\left(E[Te^{-\lambda T}] - \frac{(1-\gamma)}{\lambda}\right)\frac{2}{\lambda\gamma^2},$$

$$d_o = \frac{1}{\lambda\gamma}, \quad b_o = \frac{E[Te^{-\lambda T}]}{\gamma}, \quad d_n = \frac{1}{\lambda} + E[T]$$

The numerator of derivative of f_n is proportional to,

$$
\begin{aligned}
& c_n d_o - c_o d_n + 2 d_n (b_n d_o - b_o d_n) \\
& \quad = \frac{E[T^2]}{\lambda\gamma} + \left(\frac{1}{\lambda} + E[T]\right) E[Te^{-\lambda T}] \left(\frac{2}{\lambda\gamma^2} - \frac{2}{\gamma}\left(\frac{1}{\lambda} + E[T]\right)\right) \\
& \quad\quad - \frac{2}{\lambda\gamma}\left(\frac{1}{\lambda} + E[T]\right)\left(\frac{1-\gamma}{\lambda\gamma} - E[T]\right) \\
& \quad = \frac{E[T^2]}{\lambda\gamma} - \frac{2}{\lambda\gamma}\left(\frac{1}{\lambda} + E[T]\right)\left(1 - \lambda E[Te^{-\lambda T}]\right)(d_o - d_n).
\end{aligned}
$$

Thus the theorem follows from hypothesis. ∎

Appendix B: Some useful terms used in the proofs

The estimate of the term $c_n d_o - d_n c_o$:

$$
\begin{aligned}
c_n d_o - d_n c_o &= \left(\frac{2}{\lambda^2} + E[T^2] + \frac{2E[T]}{\lambda}\right)\left(\frac{1}{\lambda} + \frac{1-\gamma}{\lambda\gamma}\right) \\
&- \left(\frac{1}{\lambda} + E[T]\right)\left(\frac{2}{\lambda^2} + \frac{2(1-\gamma)}{\lambda^2\gamma} - \frac{2E[Te^{-\lambda T}]}{\lambda\gamma^2} + 2\frac{(1-\gamma)^2}{\lambda^2\gamma^2} + \frac{2(1-\gamma)}{\lambda^2\gamma}\right) \\
&= \left(\frac{2}{\lambda^2} + E[T^2] + \frac{2E[T]}{\lambda}\right)\left(\frac{1}{\lambda\gamma}\right) \\
&\quad - \left(\frac{1}{\lambda} + E[T]\right)\left(\frac{2}{\lambda^2\gamma} - \frac{2E[Te^{-\lambda T}]}{\lambda\gamma^2} + 2\frac{(1-\gamma)^2}{\lambda^2\gamma^2} + \frac{2(1-\gamma)}{\lambda^2\gamma}\right) \\
&= E[T^2]\left(\frac{1}{\lambda\gamma}\right) - \left(\frac{1}{\lambda} + E[T]\right)\left(-\frac{2E[Te^{-\lambda T}]}{\lambda\gamma^2} + 2\frac{(1-\gamma)^2}{\lambda^2\gamma^2} + \frac{2(1-\gamma)}{\lambda^2\gamma}\right) \\
&= \frac{1}{\gamma}\left(\frac{E[T^2]}{\lambda} + \left(\frac{1}{\lambda} + E[T]\right)\left(\frac{2E[Te^{-\lambda T}]}{\lambda\gamma}\right)\right) \\
&\quad - \frac{1}{\gamma}\left(\left(\frac{1}{\lambda} + E[T]\right)\left(\frac{2(1-\gamma)}{\lambda^2\gamma}((1-\gamma)+\gamma)\right)\right) \\
&= \frac{1}{\gamma}\left(\frac{E[T^2]}{\lambda} + \left(\frac{1}{\lambda} + E[T]\right)\left(\frac{2E[Te^{-\lambda T}]}{\lambda\gamma}\right) - \left(\frac{1}{\lambda} + E[T]\right)\frac{2(1-\gamma)}{\lambda^2\gamma}\right) \\
&= \frac{E[T^2]}{\lambda\gamma} + \left(\frac{1}{\lambda} + E[T]\right)\left(E[Te^{-\lambda T}] - \frac{(1-\gamma)}{\lambda}\right)\frac{2}{\lambda\gamma^2}. \qquad (28)
\end{aligned}
$$