Spectral alignment of correlated Gaussian random matrices

Luca Ganassali, Marc Lelarge, Laurent Massoulié

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Abstract

In this paper we analyze a simple method ($EIG1$) for the problem of matrix alignment, consisting in aligning their leading eigenvectors : given A and B, we compute v_1 and v_1' two leading eigenvectors of A and B. The algorithm returns a permutation $\hat{\Pi}$ such that the rank of the coordinate $\hat{\Pi}(i)$ in v_1 is the rank of the coordinate i in v_1' (up to the sign of v_1').

We consider a model where A belongs to the Gaussian Orthogonal Ensemble (GOE), and $B = \Pi^T(A + \sigma H)\Pi$, where Π is a permutation matrix and H is an independent copy of A. We show the following 0-1 law: under the condition $\sigma N^{7/6+\epsilon} \to 0$, the $EIG1$ method recovers all but a vanishing part of the underlying permutation Π. When $\sigma N^{7/6-\epsilon} \to \infty$, this algorithm cannot recover more than $o(N)$ correct matches.

This result gives an understanding of the simplest and fastest spectral method for matrix alignment (or complete weighted graph alignment), and involves proof methods and techniques which could be of independent interest.

1 Introduction

Motivation: graph alignment

The aim of graph alignment (or graph matching, network alignment) is to find a bijective mapping between the nodes of two graphs of same size N, such that connectivity between nodes is preserved. In the general case such a perfect mapping is impossible to recover because of random noise, the aim is thus to find a mapping that minimizes the error (or maximizes the overlap).

Many questions can be phrased as graph alignment problems. They are found in various fields, such as network privacy and data de-anonymization [13, 14], biology and protein-protein interaction networks [18], natural language processing [11], as well as pattern recognition in image processing [5].

A very common way to represent graphs and networks is through adjacency matrices, or more generally affinity matrices. For two graphs of size N with adjacency matrices A and B, the graph matching problem can be formalized as an optimization problem:

$$\max_{\Pi \in \mathcal{S}_N} \langle A, \Pi B \Pi^T \rangle, \tag{1}$$

where the maximum is taken over all $N \times N$ permutation matrices, and $\langle \cdot, \cdot \rangle$ is the classical matrix inner product. This formulation is a special case of the well studied *quadratic assignment problem* (QAP) [17], which is known to be NP-hard in the worst case, as well as some of its approximations [12].

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Related work

Some general spectral methods for random graph alignment are presented in [9], one of which is based on low rank approximation. This method is tested over synthetic graphs and gene regulatory graphs across several species. However, no precise theoretical guarantee (e.g. an error control of the inferred mapping depending on the signal-to-noise ratio) can be found for such techniques. It is important to note that the signs of eigenvectors are ambiguous: in order to optimize $\langle A, \hat{\Pi} B \hat{\Pi}^T \rangle$ in practice, it is necessary to test over all possible signs of eigenvectors. This additional complexity has no consequence when reducing A and B to rank-one matrices, but becomes costly when the reduction made is of rank $k = \omega(1)$. This combinatorial observation makes implementation and analysis of general rank-reduction methods (as the ones proposed in [9]) more difficult. We therefore focus on the rank-one reduction ($EIG1$ hereafter).

Most recently, a spectral method for matrix and graph alignment was proposed in [7, 8] and computes a similarity matrix which takes into account all pairs of eigenvalues (λ_i, μ_j) and eigenvectors (u_i, v_j) of both matrices. meeting the state-of-the-art performances for alignment of sparse Erdös-Rényi graphs in polynomial time, and improving the performances among spectral methods for matrix alignment. This method can tolerate a noise σ up to $O\left((\log N)^{-1}\right)$ to recover the entire underlying vertex correspondence. Since the computations of all eigenvectors is required, the time complexity of this method is at least $O(N^3)$. Algorithm $EIG1$ is the simplest spectral graph alignment method, where only the leading eigenvectors are computed, with complexity $O(N^2)$.

Random weighted graph matching : model and method

We now focus on the case where the two graphs are complete weight-correlated graphs. We can work in a symmetric matrix setup, with correlated Gaussian weights. In the matrix model of our study, A and H are two $N \times N$ independent normalized matrices of the Gaussian Orthogonal Ensemble (GOE) (Gaussian Orthogonal Ensemble), i.e. such that for all $1 \leq i \leq j \leq N$,

$$A_{i,j} = A_{j,i} \sim \begin{cases} \frac{1}{\sqrt{N}} \mathcal{N}(0,1) & \text{if } i \neq j, \\ \frac{\sqrt{2}}{\sqrt{N}} \mathcal{N}(0,1) & \text{if } i = j, \end{cases} \tag{2}$$

and H is an independent copy of A. A permutation matrix Π is a matrix such that $\Pi_{i,j} = \delta_{i,\pi(j)}$ (where δ is the Kronecker delta) with π a permutation of $\{1, \dots, N\}$. Let Π be a permutation matrix of size $N \times N$, and $B = \Pi^T (A + \sigma H) \Pi$, where $\sigma = \sigma(N)$ is the *noise parameter*.

Let us now define the *aligning permutation* of two vectors: given two vectors $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ having all distinct coordinates, the permutation ρ which *aligns* x and y is the permutation such that for all $1 \leq i \leq n$, the rank (for the usual order) of $x_{\rho(i)}$ in x is the rank of y_i in y.

Remark 1.1. *We note that in our case, since all the probability distributions are absolutely continuous with respect to Lebesgue measure, the eigenvectors of A and B all have almost surely (a.s.) pairwise distinct coordinates.*

The aim of this problem is to infer the underlying permutation Π given the information in A and B, using a simple algorithm derived from [9], which we call *EIG1*, that can be thought as the relaxation of QAP (1) when reducing the information in A and B to rank-one matrices. Indeed, is it easy to see that

$$\arg\max_{\Pi \in \mathcal{S}_N} \langle v_1, \Pi v_1' \rangle = \rho,$$

where ρ is the aligning permutation of the two vectors, as soon as v_1 and v_1' have pair wise distinct coordinates.

Computing the two normalized leading eigenvectors (i.e. corresponding to the highest eigenvalues) v_1 and v_1' of A and B, this algorithm returns the permutation $\hat{\Pi}$ that aligns v_1 and v_1'. The main question is: *what are the conditions on the noise σ and the size N that guarantee that $\hat{\Pi}$ is close enough from Π, with high probability?*

2 Notations, main results and proof scheme

In this section we introduce some notations that will be used throughout this paper and we mention the main results and the proof scheme. These notations and results will be recalled in the next parts in more detail.

First, let us describe the *EIG*1 Algorithm, mentioned above, that we will use for our matrix alignment problem.

Algorithm 1: *EIG*1 Algorithm for matrix alignment

Input: A, B two matrices of size $N \times N$.
Output: A permutation $\hat{\Pi}$.
Compute v_1 a normalized leading eigenvector of A and v_1' a normalized leading eigenvector of B;
Compute Π_+ the permutation that aligns v_1 and v_1';
Compute Π_- the permutation that aligns v_1 and $-v_1'$;
If $\langle A, \Pi_+ B \Pi_+^T \rangle \geq \langle A, \Pi_- B \Pi_-^T \rangle$ **then** return Π_+ ;
Else return Π_- ;

2.1 Notations

Spectral and metric notations

Recall that A and H are two $N \times N$ matrices drawn under model (2) here above. The matrix B is equal to $\Pi^T (A + \sigma H) \Pi$, where Π is a $N \times N$ permutation matrix and σ is the noise parameter, depending on N.

In the following, $(v_1, v_2, \dots, v_N)$ (resp. $(v_1', v_2', \dots, v_N')$) denote two orthonormal bases of eigenvectors of A (resp. of B) with respect to the (real) eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ of A (resp. $\lambda_1' \geq \lambda_2' \geq \dots \geq \lambda_N'$ of B). Through all the study, the sign of v_1' is fixed such that $\langle \Pi v_1, v_1' \rangle = \langle v_1, v_1' \rangle > 0$.

Denote by $\|\cdot\|$ the euclidean norm of $\mathbb{R}^N$. Let $\langle \cdot, \cdot \rangle$ denote the corresponding inner product.

For any estimator $\hat{\Pi}$ of Π, define its overlap:

$$\mathcal{L}(\hat{\Pi}, \Pi) := \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\hat{\Pi}(i) = \Pi(i)}. \tag{3}$$

This metric is used to quantify the quality of a given estimator of Π.

Probabilistic notations

The equality $\overset{(d)}{=}$ will refer to equality in distribution. Most of the following results are valid *with high probability* (we will use the abbreviation "w.h.p."), which means that their probabilities converges to 1 when $N \to \infty$.

For two random variables $u = u(N)$ and $v = v(N)$, we will use the notation $u = o_{\mathbb{P}}(v)$ if $\frac{u(N)}{v(N)} \xrightarrow{\mathbb{P}} 0$ when $N \to \infty$. We also use this notation when $X = X(N)$ and $Y = Y(N)$ are $N-$dimensional random vectors: $X = o_{\mathbb{P}}(Y)$ if $\frac{\|X(N)\|}{\|Y(N)\|} \xrightarrow{\mathbb{P}} 0$ when $N \to \infty$.

Finally we introduce a last asymptotic notation. Define

$$\mathcal{F} := \left\{ f : \mathbb{N} \to \mathbb{R} \mid \forall t > 0, N^t f(N) \to \infty, \frac{f(N)}{N^t} \to 0 \right\}. \tag{4}$$

For two random variables $u = u(N)$ and $v = v(N)$, $u \asymp v$ refers to logarithmic equivalence in probability, meaning that there exists a function $f \in \mathcal{F}$ such that

$$\mathbb{P}\left(\frac{v(N)}{f(N)} \le u(N) \le f(N)v(N) \right) \to 1 \tag{5}$$

Throughout the paper, all limits are taken when $N \to \infty$, and the dependency of all quantities in N will most of the time be eluded, as an abuse of notation.

2.2 Main results, proof scheme

The result shown can be stated as follows: there exists a condition on σ and N under which the *EIG1* method enables us to recover Π, in terms of the overlap $\mathcal{L}$ defined in (3). Over this condition, we show that *EIG1* Algorithm cannot recover more than a vanishing part of Π.

Theorem I (0-1 law in the *EIG1* method for Gaussian matrix alignment)**.** *For all N, Π_N denotes an arbitrary permutation of size N, $\hat{\Pi}_N$ is the estimator obtained with Algorithm EIG1, for A and B of model (2), with permutation Π_N and noise parameter σ.*
We have the following 0-1 law:

(*i*) *If there exists $\epsilon > 0$ such that $\sigma = o(N^{-7/6-\epsilon})$ then*

$$\mathcal{L}(\hat{\Pi}_N, \Pi_N) \xrightarrow{L^1} 1.$$

(*ii*) *If there exists $\epsilon > 0$ such that $\sigma = \omega(N^{-7/6+\epsilon})$ then*

$$\mathcal{L}(\hat{\Pi}_N, \Pi_N) \xrightarrow{L^1} 0.$$

This result is shown in section 5. In order to prove this theorem, it is necessary to establish two intermediate results, which could also be of independent interest.

Remark 2.1. *We can now underline that without loss of generality, we can assume that* $\Pi = \mathrm{Id}$. *Indeed, one can return to the general case with the rotations $A \to \Pi A \Pi^T$ and $H \to \Pi H \Pi^T$. From this point, in the rest of the paper, we will thus assume that* $\Pi = \mathrm{Id}$.

First of all, we study the behavior of v_1' with respect to v_1, showing that under some conditions on σ and N, the difference $v_1 - v_1'$ can be approximated by a renormalized Gaussian standard vector, multiplied by a variance term $\mathbf{S}$, where $\mathbf{S}$ is a random variable which behavior is well understood in terms of N and σ when $N \to \infty$. For this we work under the following assumption:

$$\exists\, \alpha > 0,\ \sigma = o\left(N^{-1/2-\alpha}\right), \tag{6}$$

Proposition I. *Under the assumption (6), there exists a standard Gaussian vector $Z \sim \mathcal{N}(0, I_N)$ independent from v_1 and a random variable $\mathbf{S} \asymp \sigma N^{1/6}$, such that*

$$v_1' = (1 + o_{\mathbb{P}}(1)) \left(v_1 + \mathbf{S}\frac{Z}{\|Z\|}\right).$$

This proposition is established in section 3.

Further related work This assumption (6) (or a tighter formulation) arises when studying the diffusion trajectories of eigenvalues and eigenvectors in random matrices, and corresponds to the *microscopic regime* in [2]. Basically, this assumption ensures that all eigenvalues of B are close enough to the eigenvalues of A. This comparison term is justified from the random matrix theory ($N^{-1/2}$ is the typical amplitude of the spectral gaps $\sqrt{N}(\lambda_i - \lambda_{i+1})$ in the bulk, which are the smaller ones). Eigenvectors diffusions in similar models (diffusion processes arise with the scaling $\sigma = \sqrt{t}$) are studied in [2], where the main tool is the Dyson Brownian motion (see e.g. [3]) and its formulation for eigenvectors trajectories, giving stochastic differential equations for the evolutions of $v_j'(t)$ with respect to vectors $v_i = v_i'(0)$. These equations lead to a system of differential equations for the overlaps $\langle v_i, v_j'(t)\rangle$, which is quite difficult to analyze rigorously. We use here an elementary method to get a expansion of v_1' around v_1, where this very condition (6) arises.

This result thus suggests the study of v_1' as a Gaussian perturbation of v_1. The main question is now formulated as follows: *what is the probability that the perturbation on v_1 has an impact on the overlap of the estimator $\hat{\Pi}$ from the $EIG1$ method?*

To answer this question, we introduce a correlated Gaussian vectors model (or *toy model*) of parameters $s > 0$ and $N \geq 1$. In this model, we draw X a standard Gaussian vector of size N. Define $Y = X + sZ$ where Z is an independent copy of X. We will use the notation $(X, Y) \sim \mathcal{J}(N, s)$.

Define r_1 the application that associates to any vector $T = (t_1, \ldots, t_p)$ the rank of t_1 in T (for the usual order). For $(X, Y) \sim \mathcal{J}(N, s)$ we evaluate

$$p(N, s) := \mathbb{P}\left(r_1(X) = r_1(Y)\right).$$

A second result is proved: there is a 0-1 law for the property of rank preservation in the toy model $\mathcal{J}(N, s)$.

Proposition II (0-1 law for $p(N, s)$)**.** *In the correlated Gaussian vectors model we have the following 0-1 law*

(i) If $s = o(1/N)$ then $p(N, s) \underset{N\to\infty}{\to} 1$.

(ii) If $s = \omega(1/N)$ then $p(N, s) \underset{N\to\infty}{\to} 0$.

This proposition is shown in section 4 and illustrated in figure 1, showing the 0-1 law at $s = N^{-1}$.

Finally, in section 5, **Propositions I** and **II** enable us to show **Theorem I**, illustrated on figure 2 showing the 0-1 law at $\sigma = N^{-7/6}$. The convergence to the step function appears to be slow.

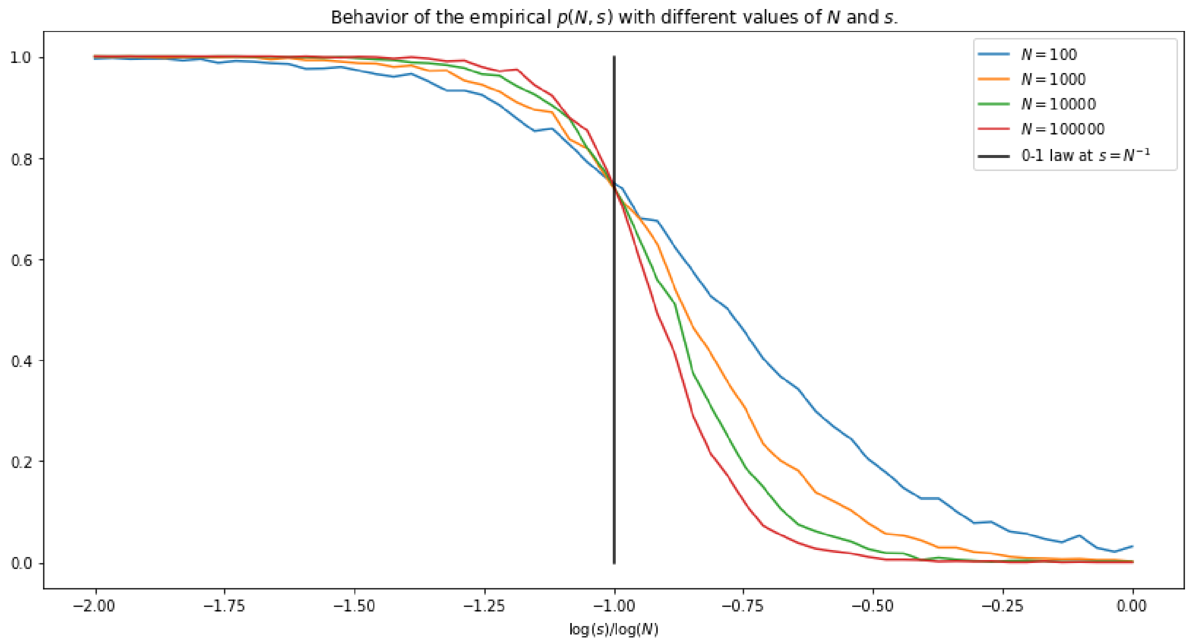

Figure 1: Behavior of the empirical $p(N, s)$ in the toy model (**Proposition II**) with different values of N and s. 1367 iterations per value of s.

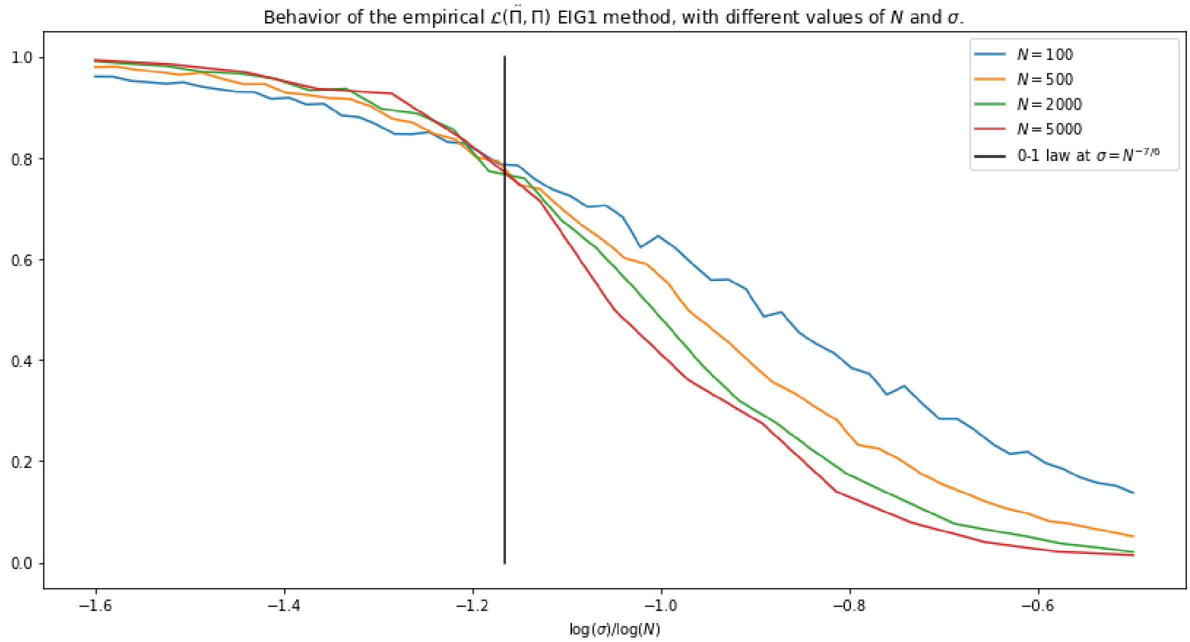

Figure 2: Behavior of the empirical overlap $\mathcal{L}(\hat{\Pi}, \Pi)$ for $\Pi = \mathrm{Id}$ in model (2) (**Theorem I**), for different values of N and σ. 200 iterations per value of σ for $N = 100$, 100 for $N = 500$, 90 for $N = 2000$ and 15 for $N = 5000$.

3 Behavior of the leading eigenvectors in the correlated matrices model

The main idea of this section is to find a first order expansion of v_1' around v_1. Recall that we use the notations $(v_1, v_2, \ldots, v_N)$ for normalized eigenvectors of A, corresponding to the eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$. Similarly, $(v_1', v_2', \ldots, v_N')$ and $\lambda_1' \geq \lambda_2' \geq \ldots \geq \lambda_N'$ will refer to eigenvectors and some eigenvalues of B. We also recall that v_1' is taken such that $\langle v_1, v_1' \rangle > 0$.

3.1 Computation of a leading eigenvector of B

Let w' be an (non normalized) eigenvector of B for the eigenvalue λ_1' of the form

$$w' := \sum_{i=1}^{N} \theta_i v_i,$$

where we assume that $\theta_1 = 1$. Such an assumption can be made a.s. since any hyperplane of $\mathbb{R}^N$ has a null Lebesgue measure in $\mathbb{R}^N$ (see Remark 1.1).

Recall now that we are working under assumption (6):

$$\exists\, \alpha > 0,\ \sigma = o\left(N^{-1/2-\alpha}\right).$$

We first obtain the following expansion.

Proposition 3.1. *Under the assumption (6) one has the following:*

$$w' = v_1 + \sigma \sum_{i=2}^{N} \frac{\langle H v_i, v_1 \rangle}{\lambda_1 - \lambda_i} v_i + o_{\mathbb{P}}\left(\sigma \sum_{i=2}^{N} \frac{\langle H v_i, v_1 \rangle}{\lambda_1 - \lambda_i} v_i\right). \tag{7}$$

Remark 3.1. *Based on the studies of the trajectories of the eigenvalues and eigenvectors in the GUE [2] and the GOE [1], since we only look at the leading eigenvectors, we expect the result of Proposition 3.1 to hold under the weaker assumption $\sigma N^{1/6+\alpha} \to 0$, for $N^{-1/6}$ is the typical spectral gap $\sqrt{N}(\lambda_1 - \lambda_2)$ on the edge. However, our analysis doesn't require this more optimal assumption. We also know that the expansion (7) doesn't hold as soon as $\sigma = \omega(N^{-1/6})$. A result proved by Chatterjee ([4], Theorem 3.8) shows that the eigenvectors corresponding to the highest eigenvalues v_1 of A and v_1' of $B = A + \sigma H$, when A and H are two independent matrices from the GUE, are delocalized (in the sense that $\langle v_1, v_1' \rangle$ converges in probability to 0 as $N \to \infty$), when $\sigma = \omega(N^{-1/6})$.*

We will use the following lemmas:

Lemma 3.2. *W.h.p., for all $\delta > 0$,*

$$\sum_{j=2}^{N} \frac{1}{\lambda_1 - \lambda_j} \leq O\left(N^{1+\delta}\right). \tag{8}$$

Lemma 3.3. *We have*

$$\sum_{j=2}^{N} \frac{1}{(\lambda_1 - \lambda_j)^2} \asymp N^{4/3}. \tag{9}$$

Lemma 3.4. *For any $C > 0$, w.h.p.*

$$\lambda_1 - \lambda_2 \geq N^{-2/3} (\log N)^{-C \log\log N}. \tag{10}$$

Lemma 3.5. *We have the following concentration*

$$\sum_{i=2}^{N} \frac{\langle Hv_i, v_1\rangle^2}{(\lambda_1-\lambda_i)^2} \asymp N^{1/3}. \tag{11}$$

Proof of proposition 3.1. Let us establish a first inequality: since the GOE distribution is invariant by rotation (see e.g. [3]), the random variables $\langle Hv_j, v_i\rangle$ are zero-mean Gaussian, with variance $1/N$ of $i \neq j$ and $2/N$ if $i=j$. Hence, w.h.p.

$$\sup_{1\le i,j\le N} |\langle Hv_j, v_i\rangle| \le C_1\sqrt{\frac{\log N}{N}}. \tag{12}$$

Throughout this proof we will work under the event where inequalities implied by equations (8), (9), (10), (11) and (12) are satisfied. We also use the following short-hand notation for $1 \le i,j \le N$:

$$m_{i,j} := \langle Hv_j, v_i\rangle,$$

The defining eigenvector equations projected on vectors v_i write

$$\begin{cases} \theta_i &= \frac{\sigma}{\lambda_1'-\lambda_i}\sum_{j=1}^N \theta_j m_{i,j}, \\ \lambda_1'-\lambda_1 &= \sigma\sum_{j=1}^N \theta_j m_{1,j}. \end{cases} \tag{13}$$

In order to approximate the θ_i variables, we define the following iterative scheme:

$$\begin{cases} \theta_i^k &= \frac{\sigma}{\lambda_1^{k-1}-\lambda_i}\sum_{j=1}^N \theta_j^{k-1} m_{i,j}, \\ \lambda_1^k-\lambda_1 &= \sigma\sum_{j=1}^N \theta_j^{k-1} m_{1,j}, \end{cases} \tag{14}$$

with initial conditions $(\theta_i^0)_{2\le i\le N} = 0$ and $\lambda_1^0 = \lambda_1$, and setting $\theta_1^k = 1$ for all k.

For $k \ge 1$, define

$$\Delta_k := \sum_{i\ge 2} \left|\theta_i^k - \theta_i^{k-1}\right|.$$

and for $k \ge 0$,

$$S_k := \sum_{i\ge 1}\left|\theta_i^k\right|.$$

Recall that under assumption (6), there exists $\alpha > 0$ such that $\sigma = o\left(N^{-1/2-\alpha}\right)$. We define ϵ as follows:

$$\epsilon = \epsilon(N) = \sqrt{\sigma N^{1/2+\alpha}}.$$

We show the following result:

Lemma 3.6. *With the same notations and under the assumption (6) of Proposition 3.1, one has w.h.p.*

(i) $\forall k \ge 1,\ \Delta_k \le \Delta_1 \epsilon^{k-1}$,

(ii) $\forall k \ge 0, \forall\, 2 \le i \le N,\ \left|\lambda_1^k - \lambda_i\right| \ge \frac{1}{2}\left|\lambda_1-\lambda_i\right|\left(1-\epsilon-\ldots-\epsilon^{k-1}\right)$,

(iii) $\forall k \ge 0,\ S_k \le 1 + (1+\ldots+\epsilon^{k-1})\Delta_1$, (*iv*) $\sum_{i=2}^N |\theta_i - \theta_i^1|^2 = o\left(\sum_{i=2}^N |\theta_i^1|^2\right)$.

Equation (*iv*) of Lemma 3.6 yields

$$\begin{aligned} w' &= v_1 + \sum_{i=2}^N \theta_i^1 v_i + \sum_{i=2}^N \left(\theta_i - \theta_i^1\right) v_i \\ &= v_1 + \sigma \sum_{i=2}^N \frac{\langle Hv_i, v_1 \rangle}{\lambda_1 - \lambda_i} v_i + o_{\mathbb{P}}\left(\sigma \sum_{i=2}^N \frac{\langle Hv_i, v_1 \rangle}{\lambda_1 - \lambda_i} v_i\right). \end{aligned}$$

□

3.2 Gaussian representation of $v_1' - v_1$

We still work under assumption (6). After renormalization, we thus have

$$v_1' = \frac{w'}{\|w'\|}.$$

A crucial point is now the computation of the scalar product $\langle v_1', v_1 \rangle$:

$$\langle v_1', v_1 \rangle = \left(1 + \sigma^2(1 + o_{\mathbb{P}}(1)) \sum_{i=2}^N \frac{\langle Hv_i, v_1 \rangle^2}{(\lambda_1 - \lambda_i)^2}\right)^{-1/2}$$

Hence w.h.p.

$$\langle v_1', v_1 \rangle = 1 - \frac{\sigma^2}{2} \sum_{i=2}^N \frac{\langle Hv_i, v_1 \rangle^2}{(\lambda_1 - \lambda_i)^2} + o_{\mathbb{P}}\left(\sigma^2 \sum_{i=2}^N \frac{\langle Hv_i, v_1 \rangle^2}{(\lambda_1 - \lambda_i)^2}\right). \tag{15}$$

With Lemma 3.5 and with the same notations, we have

$$\sigma^2 \sum_{i=2}^N \frac{\langle Hv_i, v_1 \rangle^2}{(\lambda_1 - \lambda_i)^2} \asymp \sigma^2 N^{1/3} \ (= o(1)).$$

Remark 3.2. *The previous result can be seen as a first order expansion in the diffusion process found in the literature mentioned at the beginning of this section (see e.g. [2]). The comparison between σ and $N^{1/6}$ (c.f. Chatterjee, [4]) clearly appears here, as $\sigma^2 N^{1/3}$ is the typical shift of v_1' with respect to v_1.*

Let us now introduce an useful lemma for the proof of **Proposition I**:

Lemma 3.7. *Given v_1, when writing the decomposition*

$$w' = v_1 + \underbrace{\sum_{i=2}^N \theta_i v_i}_{=:w},$$

the distribution of w is invariant by rotation in the orthogonal complement of v_1. This implies in particular that given v_1, $\|w\|$ and $\frac{w}{\|w\|}$ are independent, and that $\frac{w}{\|w\|}$ is uniformly distributed on $\mathbb{S}^{N-2}$, the unit sphere of $v_1^\perp$.

We can now formulate and show the main result of this section:

Proposition I. *Under the assumption (6), there exists a standard Gaussian vector* $Z \sim \mathcal{N}(0, I_N)$ *independent from* v_1 *and a random variable* $\mathbf{S} \asymp \sigma N^{1/6}$*, such that*

$$v_1' = (1 + o_{\mathbb{P}}(1)) \left(v_1 + \mathbf{S} \frac{Z}{\|Z\|} \right).$$

Proof of Proposition I. Recall the decomposition

$$w' = v_1 + \underbrace{\sum_{i=2}^{N} \theta_i v_i}_{=:w}.$$

According to Lemma 3.7, $\frac{w}{\|w\|}$ is uniformly distributed on the unit sphere of $\mathbb{R}^{N-2}$. A classical result (see e.g. [16]) shows that there exist $Z_2, \dots, Z_N$ standard Gaussian independent variables, independent from v_1 and $\|w\|$, such that:

$$w' = v_1 + \frac{\|w\|}{\left(\sum_{i=2}^{N} Z_i^2\right)^{1/2}} \sum_{i=2}^{N} Z_i v_i.$$

Let Z_1 be another standard Gaussian variable, independent from everything else. Then

$$w' = \left(1 - \frac{\|w\| Z_1}{\left(\sum_{i=2}^{N} Z_i^2\right)^{1/2}} \right) v_1 + \frac{\|w\|}{\left(\sum_{i=2}^{N} Z_i^2\right)^{1/2}} \sum_{i=1}^{N} Z_i v_i.$$

Let $Z = \sum_{i=1}^{N} Z_i v_i$, which is a standard Gaussian vector. Since the distribution of Z is invariant by permutation of indices of $(Z_i)_{1 \le i \le N}$, Z and v_1 are independent. We have

$$\begin{aligned} v_1' &= \frac{w'}{\|w'\|} = \frac{w'}{\sqrt{1 + \|w\|^2}} \\ &= \frac{1}{\sqrt{1 + \|w\|^2}} \left(1 - \frac{\|w\| Z_1}{\left(\sum_{i=2}^{N} Z_i^2\right)^{1/2}} \right) v_1 + \frac{\|w\| \|Z\|}{\sqrt{1 + \|w\|^2} \left(\sum_{i=2}^{N} Z_i^2\right)^{1/2}} \frac{Z}{\|Z\|}. \end{aligned}$$

Taking

$$\mathbf{S} = \frac{\|w\| \|Z\|}{\left(\sum_{i=2}^{N} Z_i^2\right)^{1/2} - \|w\| Z_1},$$

we get

$$v_1' = \frac{1}{\sqrt{1 + \|w\|^2}} \left(1 - \frac{\|w\| Z_1}{\left(\sum_{i=2}^{N} Z_i^2\right)^{1/2}} \right) \left(v_1 + \mathbf{S} \frac{Z}{\|Z\|} \right).$$

The law of large numbers, Lemma 3.5 and Proposition 3.1, showing that $\|w\| = o_{\mathbb{P}}(1)$, lead to

$$v_1' = (1 + o_{\mathbb{P}}(1)) \left(v_1 + \mathbf{S} \frac{Z}{\|Z\|} \right).$$

Lemma 3.5 directly shows that $\mathbf{S} = (1 + o_{\mathbb{P}}(1)) \|w\| \asymp \sigma N^{1/6}$ and ends the proof. $\square$

4 Definition and analysis of a toy model

Now that we have established a expansion of v_1' with respect to v_1, the following natural question remains: what is the probability that the perturbation on v_1 has an impact on the overlap (3) of the estimator $\hat{\Pi}$ from the $EIG1$ method? We thus study here the effect of a random Gaussian perturbation of a Gaussian vector in terms of rank of its coordinates: if these ranks are preserved, the permutation that aligns these two vectors will be $\hat{\Pi} = \mathrm{Id}$. Otherwise we want to understand the error made between $\hat{\Pi}$ and $\Pi = \mathrm{Id}$.

4.1 Definitions and notations

We refer to section 2.1 for the definition of the toy model $\mathcal{J}(N,s)$. Recall that we want to compute, when $(X,Y) \sim \mathcal{J}(N,s)$, the probability

$$p(N,s) := \mathbb{P}\left(r_1(X) = r_1(Y)\right).$$

In this section, we denote by E the probability density function function of a standard Gaussian variable, and F its cumulative distribution function.

We explain hereafter the link between this toy model and our first matrix model (2) in section 3. Since v_1 is uniformly distributed on the unit sphere, we have the classical equality in law $v_1 = \frac{X}{\|X\|}$ where X is a standard Gaussian vector of size N, independent of Z. **Proposition I** writes

$$\begin{aligned} v_1 &= \frac{X}{\|X\|}, \\ v_1' &= (1 + o_{\mathbb{P}}(1)) \left(\frac{X}{\|X\|} + \mathbf{S} \frac{Z}{\|Z\|} \right). \end{aligned}$$

Note that for all $\lambda > 0$, $r_1(\lambda T) = r_1(T)$, hence

$$r_1(v_1) = r_1(X), \quad r_1(v_1') = r_1\left(X + \mathbf{s}Z\right), \tag{16}$$

where

$$\mathbf{s} = \frac{\mathbf{S}\|X\|}{\|Z\|} \asymp \sigma N^{1/6}.$$

Equation (16) shows that this toy model is thus relevant for our initial problem, up to the fact that the noise term $\mathbf{s}$ is random in the matrix model (though we know its order of magnitude to be $\asymp \sigma N^{1/6}$).

Remark 4.1. *The intuition for the 0-1 law for $p(N,s)$ is as follows. If we sort the N coordinates of X on the real axis, they will all be w.h.p. in an interval of length $O(\sqrt{\log N})$. All coordinates being typically perturbed by a factor s, it seems natural to compare s with the typical gap between two coordinates of order $1/N$ to decide whether the rank of the first coordinate of X is preserved in Y.*

Let us show that this intuition is rigorously verified. For every couple (x,y) of real numbers, define

$$\begin{aligned} \mathcal{N}_{N,s}^{+-}(x,y) &:= \sharp \left\{ 1 \le i \le N,\ X_i > x, Y_i < y \right\}, \\ \mathcal{N}_{N,s}^{-+}(x,y) &:= \sharp \left\{ 1 \le i \le N,\ X_i < x, Y_i > y \right\}. \end{aligned}$$

In the following, we omit all dependencies in N and s, with the abuses of notation $\mathcal{N}$, $\mathcal{N}^{+-}$ and $\mathcal{N}^{-+}$. The corresponding regions are shown on figure 3.

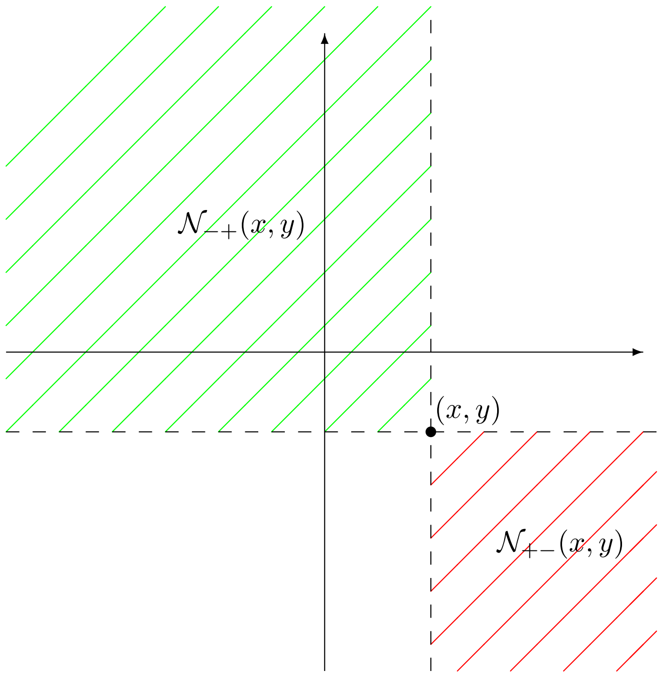

Figure 3: Areas corresponding to $\mathcal{N}^{+-}(x,y)$ and $\mathcal{N}^{-+}(x,y)$.

We will also need the following probabilities

$$S^{+-}(x,y) := \mathbb{P}\left(X_1 > x, Y_1 < y\right), \text{ and}$$
$$S^{-+}(x,y) := \mathbb{P}\left(X_1 < x, Y_1 > y\right) = S^{+-}(-x,-y).$$

We recall that E refers to the probability density function of a standard Gaussian variable, and F its cumulative distribution function. In terms of distribution, the random vector $(\mathcal{N}^{+-}(x,y), \mathcal{N}^{-+}(x,y), N-1-\mathcal{N}^{+-}(x,y)-\mathcal{N}^{+-}(x,y))$ follows a multinomial law of parameters $(N-1, S^{+-}(x,y), S^{-+}(x,y), 1-S^{+-}(x,y)-S^{-+}(x,y))$.

In order to have $r_1(X) = r_1(Y)$, there must be the same number of points on the two domains on figure 3, for $x = X_1$ and $y = Y_1$. We then have the following expression of $p(N,s)$:

$$\begin{aligned} p(N,s) &= \mathbb{E}\left[\mathbb{P}\left(\mathcal{N}^{+-}(X_1,Y_1) = \mathcal{N}^{-+}(X_1,Y_1)\right)\right] \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{P}(dx,dy)\mathbb{P}(\mathcal{N}^{+-}(x,y) = \mathcal{N}^{-+}(x,y)) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} E(x)E(z)\phi_{x,z}(N,s)\,dx\,dz, \end{aligned}$$

with

$$\phi_{x,z}(N,s) := \sum_{k=0}^{\lfloor (N-1)/2 \rfloor} \binom{N-1}{k}\binom{N-1-k}{k}\left(S^{+-}_{x,z}\right)^k\left(S^{-+}_{x,z}\right)^k\left(1-S^{+-}_{x,z}-S^{-+}_{x,z}\right)^{N-1-2k}, \tag{17}$$

using the notations $S^{+-}_{x,z} = S^{+-}(x, x+sz)$ and $S^{-+}_{x,z} = S^{-+}(x, x+sz)$.

A simple computation shows that

$$S^{+-}(x,x+sz) = \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \left(\int_{-\infty}^{z+\frac{x-u}{s}} \frac{1}{\sqrt{2\pi}} e^{-v^2/2}\, dv \right) du$$

$$= \int_x^{+\infty} E(u)\, F\left(z - \frac{u-x}{s}\right) du, \tag{18}$$

$$= s \int_0^{+\infty} E(x+vs)\, F\left(z-v\right) dv. \tag{19}$$

We have the integration result

$$\int_{-\infty}^{z} F(u)du = zF(z) + \frac{1}{\sqrt{2\pi}} e^{-z^2/2}. \tag{20}$$

From (18), (19) and (20) we derive the following easy lemma:

Lemma 4.1. *For all x and z,*

$$S^{+-}(x,x+sz) \underset{s\to 0}{=} s\left[E(x)\left(zF(z)+E(z)\right)\right] + o(s),$$
$$S^{+-}(x,x+sz) \underset{s\to\infty}{\longrightarrow} F(x)\left(1-F(z)\right),$$
$$S^{-+}(x,x+sz) \underset{s\to 0}{=} s\left[E(x)\left(-z+zF(z)+E(z)\right)\right] + o(s),$$
$$S^{-+}(x,x+sz) \underset{s\to\infty}{\longrightarrow} F(z)\left(1-F(x)\right).$$

Moreover, both $s \mapsto S^{+-}(x,x+sz)$ and $s \mapsto S^{-+}(x,x+sz)$ are increasing.

4.2 0-1 law for $p(N,s)$

We now establish the main result of this section.

Proposition II (0-1 law for $p(N,s)$)**.** *In the correlated Gaussian vectors model we have the following 0-1 law*

- *(i) If $s = o(1/N)$ then $p(N,s) \underset{N\to\infty}{\to} 1$.*
- *(ii) If $s = \omega(1/N)$ then $p(N,s) \underset{N\to\infty}{\to} 0$.*

Proof of Proposition II. In the first case *(i)*, if $s = o(1/N)$, we have the following inequality

$$p(N,s) \geq \int_{\mathbb{R}} \int_{\mathbb{R}} dxdz E(x)E(z) \mathbb{P}\left(\mathcal{N}^{+-}(x,x+sz) = \mathcal{N}^{-+}(x,x+sz) = 0\right). \tag{21}$$

According to Lemma 4.1, for all $x, z \in \mathbb{R}$

$$\mathbb{P}\left(\mathcal{N}^{+-}(x,x+sz) = \mathcal{N}^{-+}(x,x+sz) = 0\right) = \left(1 - S^{+-}(x,x+sz) - S^{-+}(x,x+sz)\right)^{N-1}$$
$$\sim \exp\left(-NsE(x)\left[z(2F(z)-1)+2E(z)\right]\right)$$
$$\underset{N\to\infty}{\longrightarrow} 1,$$

By applying the dominated convergence theorem in (21), we conclude that $p(N,s) \to 1$.

In the second case *(ii)*, if $sN \to \infty$, recall that

$$p(N,s) = \int_{\mathbb{R}} \int_{\mathbb{R}} dxdz E(x)E(z) \phi_{x,z}(N,s), \tag{22}$$

with $\phi_{x,z}$ defined in equation (17). In the rest of the proof, we fix x and z two real numbers. Letting

$$b(N,s,k) := \binom{N-1}{k} \left(S^{+-}_{x,z}\right)^k \left(1 - S^{+-}_{x,z}\right)^{N-1-k}$$

and

$$M(N,s) := \max_{0 \le k \le N-1} b(N,s,k),$$

a classical computation shows that there exists $C = C(x,z) > 0$ such that for N large enough,

$$M(N,s) = b(N,s,\lfloor NS^{+-}_{x,z} \rfloor) \le \limsup_N \frac{C}{\left(NS^{+-}_{x,z}(N)\right)^{1/2} + C},$$

and the assumption (ii) $(sN \to \infty)$ together with Lemma 4.1 ensure that $\liminf_N NS^{+-}_{x,z}(N) = \infty$ and that $M(N,s) \underset{N\to\infty}{\longrightarrow} 0$. We obtain the following control

$$\begin{aligned}
\phi_{x,z}(N,s) &\le M(N,s) \times \sum_{k=0}^{\lfloor (N-1)/2 \rfloor} \binom{N-1-k}{k} \left(S^{-+}_{x,z}\right)^k \frac{\left(1 - S^{+-}_{x,z} - S^{-+}_{x,z}\right)^{N-1-2k}}{\left(1 - S^{+-}_{x,z}\right)^{N-1-k}} \\
&\overset{(a)}{=} M(N,s) \times \frac{(1 - S^{+-}_{x,z})\left(1 - \left(\frac{-S^{+-}_{x,z}}{1 - S^{-+}_{x,z}}\right)^N\right)}{1 + S^{-+}_{x,z} - S^{+-}_{x,z}} \\
&\overset{(b)}{=} M(N,s) \times O(1) \underset{N\to\infty}{\longrightarrow} 0.
\end{aligned}$$

We used in (b) the fact that $S^{+-}_{x,z} + S^{-+}_{x,z}$ is increasing in the variable s, and that given x and z, for all $s > 0$, $S^{+-}_{x,z} + S^{-+}_{x,z} < F(x)\,(1 - F(z)) + F(z)\,(1 - F(x)) < 1$, where F is the cumulative distribution function of a standard Gaussian variable (Lemma 4.1). We used in (a) the following combinatorial result:

Lemma 4.2. *For all $\alpha > 0$,*

$$\sum_{k=0}^{\lfloor (N-1)/2 \rfloor} \binom{N-1-k}{k} \alpha^k = \frac{1}{\sqrt{1+4\alpha}} \left[\left(\frac{1+\sqrt{1+4\alpha}}{2}\right)^N - \left(\frac{1-\sqrt{1+4\alpha}}{2}\right)^N \right]. \quad (23)$$

To obtain (a) from Lemma 4.2, we apply (23) to $\alpha = \frac{S^{-+}_{x,z}(1-S^{+-}_{x,z})}{(1-S^{+-}_{x,z}-S^{-+}_{x,z})^2}$, with $\sqrt{1+4\alpha} = \frac{1-S^{+-}_{x,z}+S^{-+}_{x,z}}{1-S^{+-}_{x,z}-S^{-+}_{x,z}}$. The dominated convergence theorem in (22) shows that $p(N,s) \to 0$ and ends the proof. □

Remark 4.2. *The above computations also imply the existence of a non-degenerate limit of $p(N,s)$ in the critical case where $sN \to c > 0$.*

5 Analysis of the *EIG1* method for matrix alignment

By now, we come back to our initial problem, which is to infer the underlying permutation Π with *EIG*1 method. Recall that for any estimator $\hat{\Pi}$ of Π, its overlap is defined as follows

$$\mathcal{L}(\hat{\Pi}, \Pi) := \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\hat{\Pi}(i) = \Pi(i)}.$$

We now show how the **Propositions I** and **II** lead to the main result of our study:

Theorem I (0-1 law in the *EIG1* method for Gaussian matrix alignment)**.** *For all N, Π_N denotes an arbitrary permutation of size N, $\hat{\Pi}_N$ is the estimator obtained with Algorithm EIG1, for A and B of model (2), with permutation Π_N and noise parameter σ.*
We have the following 0-1 law:

(*i*) *If there exists $\epsilon > 0$ such that $\sigma = o(N^{-7/6-\epsilon})$ then*

$$\mathcal{L}(\hat{\Pi}_N, \Pi_N) \xrightarrow{L^1} 1.$$

(*ii*) *If there exists $\epsilon > 0$ such that $\sigma = \omega(N^{-7/6+\epsilon})$ then*

$$\mathcal{L}(\hat{\Pi}_N, \Pi_N) \xrightarrow{L^1} 0.$$

Proof of Theorem I. In the first case (i), condition (6) holds. As detailed in Section 4, for rank preservation, one can identify v_1 and v_1' with the following vectors:

$$v_1 \sim X, \quad v_1' \sim X + \mathbf{s}Z, \tag{24}$$

where X and Z are two independent Gaussian vectors from the toy model, and where $\mathbf{s} \asymp \sigma N^{1/6}$ w.h.p. Recall that we work under the assumptions $\Pi = \mathrm{Id}$ and $\langle v_1, v_1' \rangle > 0$. In this case, we expect Π_+ to be very close to Id.

Let's take $f \in \mathcal{F}$ such that w.h.p., $\sigma N^{1/6} f(N)^{-1} \leq \mathbf{s} \leq \sigma N^{1/6} f(N)$. We have for all $1 \leq i \leq N$,

$$\begin{aligned}
\mathbb{P}\left(\Pi_+(i) = \Pi(i)\right) &= \mathbb{P}\left(\Pi_+(1) = \Pi(1)\right) \\
&= \mathbb{E}\left[\iint dxdz E(x)E(z)\phi_{x,z}\left(N, \mathbf{s}\right) \mathbf{1}_{\sigma N^{1/6} f(N)^{-1} \leq \mathbf{s} \leq \sigma N^{1/6} f(N)}\right] + o(1) \\
&= \iint dxdz E(x)E(z)\mathbb{E}\left[\phi_{x,z}\left(N, \mathbf{s}\right) \mathbf{1}_{\sigma N^{1/6} f(N)^{-1} \leq \mathbf{s} \leq \sigma N^{1/6} f(N)}\right] + o(1).
\end{aligned}$$

When $\sigma N^{1/6} f(N)^{-1} \leq \mathbf{s} \leq \sigma N^{1/6} f(N)$, $\mathbf{s}N \xrightarrow{\text{a.s.}} 0$ by condition (i) and $\phi_{x,z}\left(N, \mathbf{s}\right) \xrightarrow{\text{a.s.}} 1$ by **Proposition I**. Uniform integrability is guaranteed by the obvious domination $|\phi_{x,z}(N, \mathbf{s})| \leq 1$, which gives

$$\phi_{x,z}\left(N, \mathbf{s}\right) \mathbf{1}_{\sigma N^{1/6} f(N)^{-1} \leq \mathbf{s} \leq \sigma N^{1/6} f(N)} \xrightarrow{L^1} 1,$$

which implies with the dominated convergence theorem that

$$\mathbb{E}\left[\mathcal{L}(\Pi_+, \Pi)\right] \to 1$$

and thus

$$\mathcal{L}(\Pi_+, \Pi) \xrightarrow{L^1} 1.$$

Lemma 5.1. *In the case (i), if $\langle v_1, v_1' \rangle > 0$, we have w.h.p.*

$$\langle A, \Pi_+ B \Pi_+^T \rangle > \langle A, \Pi_- B \Pi_-^T \rangle,$$

so the Algorithm EIG1 returns w.h.p. $\hat{\Pi} = \Pi_+$.

With Lemma 5.1 we can conclude that

$$0 \leq \mathbb{E}\left[1 - \mathcal{L}(\hat{\Pi}, \Pi)\mathbf{1}_{\hat{\Pi} = \Pi_+}\right] \leq \mathbb{E}\left[1 - \mathcal{L}(\Pi_+, \Pi)\right] + 1 \times \mathbb{E}\left[1 - \mathbf{1}_{\hat{\Pi} = \Pi_+}\right] \longrightarrow 0.$$

Hence

$$\mathbb{E}\left[\mathcal{L}(\hat{\Pi}, \Pi)\right] \geq \mathbb{E}\left[\mathcal{L}(\hat{\Pi}, \Pi)\mathbf{1}_{\hat{\Pi} = \Pi_+}\right] \to 1$$

and

$$\mathcal{L}(\hat{\Pi}, \Pi) \xrightarrow{L^1} 1.$$

Of course, this convergence also happens in probability, by Markov's inequality.

In the second case (ii), if condition (6) is verified then the identification (24) still holds and the proof of case (i) adapts well. However, if (6) is not verified, we can still make a link with the toy model studied in section 4. Let's use a simple coupling argument: if $\sigma = \omega(N^{-1/2-\alpha})$ for some $\alpha \geq 0$, let's take $\sigma_1, \sigma_2 > 0$ such that

$$\sigma^2 = \sigma_1^2 + \sigma_2^2$$

and fix for instance $\sigma_1 = N^{-1}$. We will use the notation $\widetilde{v_1'}$, now viewed as the leading eigenvector of the matrix

$$\widetilde{B} = A + \sigma_1 H + \sigma_2 \widetilde{H},$$

where $\widetilde{H}$ is an independent copy of H. This has no consequence in terms of distribution : $(A, \widetilde{B})$ is still drawn under model (2). Let's denote v_1' the leading eigenvector of $B_1 = A + \sigma_1 H$, chosen so that $\langle v_1, v_1' \rangle > 0$. It is clear that σ_1 satisfies (6). We have the following result, based on the invariance by rotation of the GOE:

Lemma 5.2. *We still have the following equalities in distribution:*

$$r_1(v_1) \overset{(d)}{=} r_1(X), \tag{25}$$

and

$$r_1(\widetilde{v_1'}) \overset{(d)}{=} r_1(X + \mathbf{s}Z), \tag{26}$$

where X, Z are two Gaussian vectors from the toy model, with w.h.p.

$$\mathbf{s} \geq \mathbf{s^1} \asymp \sigma_1 N^{1/6}.$$

Since w.h.p. $\mathbf{s} \geq \mathbf{s^1}$ and $\mathbf{s^1} N \asymp \sigma_1 N^{7/6} \to \infty$, we have for all $1 \leq i \leq N$,

$$\begin{aligned}
\mathbb{P}\left(\Pi_+(i) = \Pi(i)\right) &= \mathbb{P}\left(\Pi_+(1) = \Pi(1)\right) \\
&= \mathbb{E}\left[\iint dxdz E(x)E(z)\phi_{x,z}(N, \mathbf{s})\mathbf{1}_{\mathbf{s}N\to\infty}\right] + o(1) \\
&= \iint dxdz E(x)E(z)\mathbb{E}\left[\phi_{x,z}(N, \mathbf{s})\mathbf{1}_{\mathbf{s}N\to\infty}\right] + o(1).
\end{aligned}$$

With the same arguments as in the case (i), we show that

$$\phi_{x,z}(N, \mathbf{s})\mathbf{1}_{\mathbf{s}N\to\infty} \xrightarrow{L^1} 0,$$

which implies

$$\mathbb{E}\left[\mathcal{L}(\Pi_+, \Pi)\right] \underset{N\to\infty}{\longrightarrow} 0,$$

and thus

$$\mathcal{L}(\Pi_+, \Pi) \xrightarrow[N\to\infty]{L^1} 0.$$

Lemma 5.3. *In the case (ii), if $\langle v_1, v_1' \rangle > 0$, we also have*

$$\mathcal{L}(\Pi_-, \Pi) \xrightarrow[N\to\infty]{L^1} 0.$$

Lemma 5.3 then gives

$$\mathbb{E}\left[\mathcal{L}(\hat{\Pi},\Pi)\right] \leq \mathbb{E}\left[\mathcal{L}(\Pi_+,\Pi)\right] + \mathbb{E}\left[\mathcal{L}(\Pi_-,\Pi)\right] \underset{N\to\infty}{\longrightarrow} 0,$$

and thus

$$\mathcal{L}(\hat{\Pi},\Pi) \overset{L^1}{\underset{N\to\infty}{\longrightarrow}} 0.$$

Of course, this convergence also happens in probability, by Markov's inequality. □

6 Proof of lemmas

Throughout all these proofs, variables denoted by C_i with i a positive integer are positive universal constants (independent of everything else).

Proof of Lemma 3.4

Proof of Lemma 3.4. This lemma provides a control of the spectral gap $\lambda_1 - \lambda_2$. Given a good rescaling (in $N^{2/3}$), the asymptotic joint law of the eigenvalues in the edge has been investigated in a great amount of research work, for Gaussian ensembles, and for more general Wigner matrices. The GOE case has been mostly studied by Tracy, Widom, and Forrester ; in [10] and [19], the convergence of the joint distribution of the first k eigenvalues towards a density distribution is established:

Proposition 6.1 ([10], [19])**.** *For a given $k \geq 1$, and all $s_1, \dots, s_k$ real numbers,*

$$\mathbb{P}\left(N^{2/3}(\lambda_1 - 2) \leq s_1, \dots, N^{2/3}(\lambda_k - 2) \leq s_k\right) \underset{N\to\infty}{\longrightarrow} \mathcal{F}_{1,k}(s_1,\dots,s_k), \tag{27}$$

where the $\mathcal{F}_{1,k}$ are continuous and can be expressed as solutions of non linear PDEs. Thus the re-scaled spectral gap $N^{2/3}(\lambda_1 - \lambda_2)$ has a limit probability density law supported by $\mathbb{R}_+$, which implies that

$$\mathbb{P}\left(N^{2/3}(\lambda_1 - \lambda_2) \geq (\log N)^{-C\log\log N}\right) \underset{N\to\infty}{\longrightarrow} 1.$$

□

Proof of Lemma 3.3

Proof of Lemma 3.3. This result needs an understanding of the behavior of the spectral gaps of matrix A, in the bulk and in the edges (left and right). The eigenvalues in the *edge* correspond to indices i such that $i = o(N)$ (left) or $i = N - o(N)$ (right). Eigenvalues in the *bulk* are the remaining eigenvalues. For this, we use a result of rigidity of eigenvalues, due to L. Erdös et al. [6], which consists in a control of the probability of the gap between the eigenvales of A and the typical eigenvalues γ_j of the semi-circle law, defined as follows

$$\forall i \in \{1,\dots,n\},\ \frac{1}{2\pi}\int_{-2}^{\gamma_j} \sqrt{4-x^2}\,dx = 1 - \frac{j}{N}. \tag{28}$$

Proposition 6.2 ([6])**.** *For some positive constants $C_5 > 0$ and $C_6 > 0$, for N large enough,*

$$\mathbb{P}\left(\exists j \in \{1,\dots,n\} \mid |\lambda_j - \gamma_j| \geq (\log N)^{C_5 \log\log N} \left(\min\left(j, N+1-j\right)\right)^{-1/3} N^{-2/3}\right) \leq C_5 \exp\left(-(\log N)^{C_6 \log\log N}\right). \tag{29}$$

Remark 6.1. *Another similar result that goes in the same direction for the GOE is already known: it has been shown by O'Rourke in [15] that the variables $\lambda_i - \gamma_i$ behave as Gaussian variables when $N \to \infty$. However, the rigidity result obtained in (29) can apply in more general models. This quantitative probabilistic statement was not previously known even for the GOE case.*

Remark 6.2. *Let us note that one of the assumptions made in [6] is that variances of each column sum to 1, which is not directly the case in our model (2). Nevertheless, one may use (29) for the re-scaled matrix $\tilde{A} := A\left(1+\frac{1}{N}\right)^{-1/2}$, then easily check that there is a possible step back to A: $|\lambda_j - \gamma_j| \leq \left|\lambda_j\left(1+\frac{1}{N}\right)^{-1/2} - \gamma_j\right| + N^{-1} + o(N^{-1})$, and $N^{-1}+o(N^{-1}) \leq 2\left(\min\left(j, N+1-j\right)\right)^{-1/3} N^{-2/3}$ for N big enough. Tolerating a slight increase of the constant C_5, the result (29) is thus valid in the GOE.*

Let us now compute an asymptotic expansion of γ_j in the right edge, which is for $j = o(N)$. Define

$$G(x) := \frac{1}{2\pi}\int_{-2}^{x}\sqrt{4-t^2}dt = \frac{x\sqrt{4-x^2}+4\arcsin(x/2)}{4\pi} + \frac{1}{2}, \tag{30}$$

for all $x \in [-2, 2]$. We have $\gamma_j = G^{-1}(1-j/N) = -G^{-1}(j/N)$, observing that the integrand in (30) is an even function. We get the following expansion when $x \to -2$,

$$G(x) \underset{x\to-2}{=} \frac{2(x+2)^{3/2}}{3\pi} + o\left((x+2)^{3/2}\right)$$

which implies that

$$G^{-1}(y) \underset{y\to0}{=} -2 + \left(\frac{3\pi y}{2}\right)^{2/3} + o\left(y^{2/3}\right),$$

hence

$$\gamma_j \underset{j/N\to0}{=} 2 - \left(\frac{3\pi j}{2N}\right)^{2/3} + o\left((j/N)^{2/3}\right). \tag{31}$$

Remark 6.3. *One can observe the coherence of this result that arises naturally in [15] as the expectation of the eigenvalues in the edge.*

Let $\epsilon > 0$, to be specified later. To establish our result we will split the variables j in three sets:

$$A_1 := \left\{2 \leq j \leq (\log N)^{(C_5+1)\log\log N}\right\} \text{ (a small part of the right edge)},$$
$$A_2 := \left\{(\log N)^{(C_5+1)\log\log N} < j \leq N^{1-\epsilon}\right\} \text{ (a larger part of the right edge)},$$
$$A_3 := \left\{N^{1-\epsilon} < j \leq N\right\} \text{ (everything else)}.$$

We show that the sum over A_1 is the major contribution in (9). the split in the right edge in A_1 and A_2 is driven by the error term of (29): this term is small compared to γ_j if and only if $(\log N)^{C_5 \log\log N} = o(j)$.

Step 1: estimation of the sum over A_1. According to (29) and Lemma 3.4, w.h.p.

$$N^{-4/3}(\log N)^{-C_6\log\log N} \leq (\lambda_1-\lambda_2)^2 \leq C_7 N^{-4/3}(\log N)^{C_6\log\log N},$$

where C_6, C_7 are positive constants. Hence, w.h.p.

$$\begin{aligned}\frac{N^{4/3}}{C_7 (\log N)^{C_6 \log\log N}} &\leq \sum_{j\in A_1} \frac{1}{(\lambda_1-\lambda_j)^2} \\ &\leq \sum_{j\in A_1} \frac{1}{(\lambda_1-\lambda_2)^2} \\ &\leq N^{4/3} (\log N)^{(C_5+C_6+1)\log\log N}.\end{aligned}$$

Step 2: estimation of the sum over A_2. Let us show that the sum over A_2 is asymptotically small compared to the sum over A_1: using (29) and (31), we know that there exists $C_8 > 0$ such that for all $j \in A_2$, w.h.p.

$$\lambda_j = 2 - C_8 \left(\frac{j}{N}\right)^{2/3} + o\left((j/N)^{2/3}\right),$$

and we know furthermore (se e.g. [3]) that w.h.p.

$$\lambda_1 = 2 + o\left((j/N)^{2/3}\right), \forall j \in A_2 \tag{32}$$

hence w.h.p.

$$\begin{aligned}\sum_{j\in A_2} \frac{1}{(\lambda_1-\lambda_j)^2} &= N^{4/3} \sum_{j\in A_2} \frac{1}{C_9 j^{4/3}(1+o(1))} \\ &= N^{4/3}(1+o(1)) \sum_{j\in A_2} \frac{1}{C_9 j^{4/3}} = o\left(N^{4/3}\right),\end{aligned}$$

using in the last line the fact that the Riemann's series $\sum j^{-4/3}$ converges.

Step 3: estimation of the sum under A_3. With the previous results (29), (31) and (32), assuming that $\epsilon < 1$, we get w.h.p.

$$\lambda_1 - \lambda_{N^{1-\epsilon}} = C_8 N^{-2\epsilon/3} + O\left(N^{-2\epsilon/3}\right),$$

which gives w.h.p. the following control

$$\begin{aligned}\sum_{j\in A_3} \frac{1}{(\lambda_1-\lambda_j)^2} &\leq \left(N - N^{1-\epsilon}\right) \frac{1}{(\lambda_1-\lambda_{N^{1-\epsilon}})^2} \\ &= \left(N - N^{1-\epsilon}\right) \frac{N^{4\epsilon/3}}{C_9(1+o(1))} = O\left(N^{1+4\epsilon/3}\right) = o\left(N^{4/3}\right),\end{aligned}$$

as long as $\epsilon < 1/4$. Taking such a ϵ, these three controls end the proof. □

Proof of Lemma 3.2

Proof of Lemma 3.2. We follow the same steps as in the proof of Lemma 3.3. Let's take $\delta > 0$. We split the j variables in three sets:

$$\begin{aligned}A_1 &:= \left\{2 \leq j \leq N^{1/3}\right\}, \\ A_2 &:= \left\{N^{1/3} < j \leq N^{1-\delta}\right\}, \\ A_3 &:= \left\{N^{1-\delta} < j \leq N\right\}.\end{aligned}$$

We use Lemma 3.4 to obtain the following control w.h.p.

$$\sum_{j\in A_1} \frac{1}{\lambda_1-\lambda_j} \leq N^{1/3}N^{2/3}\left(\log N\right)^{C_5 \log\log N} = O(N^{1+\delta}).$$

Similarly, for A_2

$$\sum_{j\in A_2} \frac{1}{\lambda_1-\lambda_j} \leq \sum_{j\in A_2} \frac{1}{o(N^{-2/3}) + C_8(j/N)^{2/3} + O\left(\left(\log N\right)^{C_5\log\log N} N^{-2/3}j^{-1/3}\right)}$$

$$= N^{2/3}\sum_{j\in A_2} \frac{1}{o(1)+C_8 j^{2/3}} \leq C_{10}N^{2/3}N^{(1-\delta)/3} \leq O(N^{1+\delta}).$$

Finally, using Cauchy–Schwarz inequality

$$\sum_{j\in A_3} \frac{1}{\lambda_1-\lambda_j} \leq \sqrt{N}\left(\sum_{j\in A_3}\frac{1}{(\lambda_1-\lambda_j)^2}\right)^{1/2} \leq \sqrt{N}O(N^{1/2+2\delta/3}) = O(N^{1+\delta}).$$

□

Proof of Lemma 3.6

Proof of Lemma 3.6. In this proof we will use the same notations, as defined in the proof of Proposition 3.1, and we make the assumption (6). Recall that we work under the event (that occurs w.h.p.) on which the equations (8), (9), (10), (11) and (12) are satisfied. We show the following inequalities:

(i) $\forall k \geq 1,\ \Delta_k \leq \Delta_1 \epsilon^{k-1}$,

(ii) $\forall k \geq 0, \forall\, 2 \leq i \leq N,\ \left|\lambda_1^k - \lambda_i\right| \geq \frac{1}{2}\left|\lambda_1-\lambda_i\right|\left(1-\epsilon-\ldots-\epsilon^{k-1}\right)$,

(iii) $\forall k \geq 0,\ S_k \leq 1 + (1+\ldots+\epsilon^{k-1})\Delta_1$.

Recall that ϵ is given by

$$\epsilon = \epsilon(N) = \sqrt{\sigma N^{1/2+\alpha}}.$$

We will denote by $f_i(N)$, with i an integer, functions as defined in Lemma 3.3. All the following inequality will be valid for N large enough (uniformly in i and in k).

Step 1: propagation of the first equation. Let $k \geq 3$.

$$\left|\theta_i^k - \theta_i^{k-1}\right| \leq \left|\frac{\sigma}{\lambda_1^{k-1}-\lambda_i}\sum_{j=2}^{N}\left(\theta_j^{k-1}-\theta_j^{k-2}\right)m_{i,j}\right| + \left|\frac{\sigma\left(\lambda_1^{k-2}-\lambda_1^{k-1}\right)}{\left(\lambda_1^{k-1}-\lambda_i\right)\left(\lambda_1^{k-2}-\lambda_i\right)}\sum_{j=1}^{N}\theta_j^{k-2}m_{i,j}\right|$$

$$\leq \frac{\sigma}{\left|\lambda_1^{k-1}-\lambda_i\right|}C_1\sqrt{\frac{\log N}{N}}\Delta_{k-1} + \sigma C_1\sqrt{\frac{\log N}{N}}S_{k-2}\frac{\left|\lambda_1^{k-2}-\lambda_1^{k-1}\right|}{\left|\lambda_1^{k-1}-\lambda_i\right|\left|\lambda_1^{k-2}-\lambda_i\right|}$$

$$\leq \sigma\underbrace{\frac{3}{\left|\lambda_1-\lambda_i\right|}}_{(ii)\text{ to } k-1}C_1\sqrt{\frac{\log N}{N}}\Delta_{k-1} + \sigma C_1\sqrt{\frac{\log N}{N}}S_{k-2}\underbrace{\frac{9\left|\lambda_1^{k-2}-\lambda_1^{k-1}\right|}{\left|\lambda_1-\lambda_i\right|^2}}_{(ii)\text{ to } k-1,\ k-2}$$

$$\leq \sigma\frac{3}{\left|\lambda_1-\lambda_i\right|}C_1\sqrt{\frac{\log N}{N}}\Delta_{k-1} + \sigma C_1\sqrt{\frac{\log N}{N}}\underbrace{2}_{(iii)\text{ to } k-2}\frac{9\left|\lambda_1^{k-2}-\lambda_1^{k-1}\right|}{\left|\lambda_1-\lambda_i\right|^2}.$$

Note that

$$\left|\lambda_1^{k-2} - \lambda_1^{k-1}\right| = \left|\sigma \sum_{j=1}^{N} \left(\theta_j^{k-2} - \theta_j^{k-3}\right) m_{i,j}\right|$$
$$\leq \sigma C_1 \sqrt{\frac{\log N}{N}} \Delta_{k-2},$$

which yields the inequality:

$$\left|\theta_i^k - \theta_i^{k-1}\right| \leq \frac{\sigma}{|\lambda_1 - \lambda_i|} f_1(N) N^{-1/2} \Delta_{k-1} + \frac{\sigma^2}{|\lambda_1 - \lambda_i|^2} f_2(N) N^{-1} \Delta_{k-2}.$$

We choose δ such that $0 < \delta < \alpha$ (where α is fixed by (6)), and we sum from $i = 2$ to N:

$$\Delta_k \leq \underbrace{\sigma f_1(N) N^{1/2+\delta}}_{=o(\epsilon)} \Delta_{k-1} + \underbrace{\sigma^2 f_3(N) N^{1/3}}_{=o(\epsilon^2)} \Delta_{k-2}$$
$$\leq o(\epsilon) \underbrace{\epsilon^{k-2} \Delta_1}_{(i) \text{ to } k-1} + o(\epsilon^2) \underbrace{\epsilon^{k-3} \Delta_1}_{(i) \text{ to } k-2}$$
$$\leq \epsilon^{k-1} \Delta_1.$$

Step 2: propagation of the second equation. Let $k \geq 2$, and $0 < \delta < \alpha$.

$$\left|\lambda_1^k - \lambda_1^{k-1}\right| \leq \sigma f_1(N) N^{-1/2} \Delta_{k-1}$$
$$\leq \sigma f_1(N) N^{-1/2} \underbrace{\epsilon^{k-2} \Delta_1}_{(i) \text{ to } k-1}$$
$$\leq N^{-2/3} (\log N)^{-C \log \log N} \epsilon^{k-2} \Delta_1$$
$$\leq \frac{\lambda_1 - \lambda_2}{2} \epsilon^{k-2} \Delta_1$$
$$\leq \frac{\lambda_1 - \lambda_i}{2} \epsilon^{k-2} \Delta_1.$$

Note that

$$\Delta_1 = \sum_{j=2}^{N} \frac{\sigma}{\lambda_1 - \lambda_i} |m_{i,1}|$$
$$\leq \sigma f_1(N) N^{1/2+\delta}.$$
$$\leq o(\epsilon).$$

Applying (ii) to $k-1$, we get

$$\left|\lambda_1^k - \lambda_i\right| \geq \left|\lambda_1 - \lambda_1^{k-1}\right| - \left|\lambda_1^k - \lambda_1^{k-1}\right|$$
$$\geq \frac{\lambda_1 - \lambda_i}{2} \left(1 - \epsilon - \ldots - \epsilon^{k-2}\right) - \frac{\lambda_1 - \lambda_i}{2} \epsilon^{k-1}$$
$$\geq \frac{\lambda_1 - \lambda_i}{2} \left(1 - \epsilon - \ldots - \epsilon^{k-1}\right).$$

Step 3: propagation of the third equation. Let $k \geq 1$.

$$\begin{aligned} S_k &= 1 + \sum_{j=2}^{N} \left|\theta_j^k\right| \\ &\leq 1 + \Delta_k + S_{k-1} - 1 \\ &\leq \underbrace{\epsilon^{k-1}\Delta_1}_{(i) \text{ to } k} + \underbrace{1 + \left(1 + \ldots + \epsilon^{k-2}\right)\Delta_1}_{(iii) \text{ to } k-1} \\ &\leq 1 + \left(1 + \epsilon + \ldots + \epsilon^{k-1}\right)\Delta_1. \end{aligned}$$

Step 4: Proof of (i) **for** $k = 1, 2$**,** (ii) **for** $k = 0, 1$ **and** (iii) **for** $k = 0, 1$**.** The equation (i) for $k = 1$ is obvious. For $k = 2$:

$$\left|\theta_i^2 - \theta_i^1\right| \leq \left|\frac{\sigma}{\lambda_1^1 - \lambda_i} \sum_{j=2}^{N} \left(\theta_j^1 - \theta_j^0\right) m_{i,j}\right| + \left|\frac{\sigma\left(\lambda_1^0 - \lambda_1^1\right)}{\left(\lambda_1^1 - \lambda_i\right)\left(\lambda_1^0 - \lambda_i\right)} \sum_{j=1}^{N} \theta_j^0 m_{i,j}\right|.$$

We have

$$\begin{aligned} \left|\lambda_1^1 - \lambda_i\right| &\geq \left|\lambda_1 - \lambda_i\right| - \left|\lambda_1 - \lambda_1^1\right| \\ &\geq \left|\lambda_1 - \lambda_i\right| - \sigma\left|m_{1,1}\right| \\ &\geq \left|\lambda_1 - \lambda_i\right| - \frac{1}{2}\left|\lambda_1 - \lambda_2\right| \\ &\geq \frac{1}{2}\left|\lambda_1 - \lambda_i\right|, \end{aligned}$$

which shows (ii) for $k = 0, 1$. Thus, for $0 < \delta < \alpha$:

$$\left|\theta_i^2 - \theta_i^1\right| \leq \frac{2\sigma}{\lambda_1 - \lambda_i} C_1 \sqrt{\frac{\log N}{N}} \Delta_1 + \frac{4\sigma}{\left(\lambda_1 - \lambda_i\right)^2} \sigma\left|m_{1,1}\right|\left|m_{i,1}\right|.$$

$$\begin{aligned} \Delta_2 &\leq \sigma f_1(N) N^{1/2+\delta}\Delta_1 + 4\sigma\left|m_{1,1}\right| \sum_{i=2}^{N} \frac{\sigma\left|m_{i,1}\right|}{\left(\lambda_1 - \lambda_i\right)^2} \\ &\leq \sigma f_1(N) N^{1/2+\delta}\Delta_1 + 4\sigma f_4(N) N^{-1/2} N^{2/3} \sum_{i=2}^{N} \frac{\sigma\left|m_{i,1}\right|}{\left(\lambda_1 - \lambda_i\right)} \\ &\leq \sigma f_1(N) N^{1/2+\delta}\Delta_1 + 4\sigma f_4(N) N^{1/6}\Delta_1 \\ &\leq \epsilon\Delta_1. \end{aligned}$$

The proof of (iii) for $k = 0, 1$ is obvious.

Step 5: Proof of equation (iv)**.** Let $k \geq 2$ and $2 \leq i \leq N$. In the same way than in Step 1, we have

$$\left|\theta_i^k - \theta_i^{k-1}\right| \leq \frac{2\sigma C_1}{\lambda_1 - \lambda_i} \sqrt{\frac{\log N}{N}} \epsilon^{k-2}\Delta_1 + \frac{8\sigma^2 C_1^2}{\left(\lambda_1 - \lambda_i\right)^2} \frac{\log N}{N} \epsilon^{(k-3)_+}\Delta_1.$$

In the right-hand term, the ratio of the second term on the first one is smaller that

$$\frac{4\sigma C_1}{\lambda_1 - \lambda_i} \sqrt{\frac{\log N}{N}} \epsilon^{-1} \leq \sigma N^{1/6} f(N) \epsilon^{-1} \leq \epsilon \to 0,$$

using Lemma 3.4, with $f \in \mathcal{F}$. It follows that for N big enough (uniformly in k and i) one has

$$\left|\theta_i^k - \theta_i^{k-1}\right| \leq \frac{\sigma f(N)}{\lambda_1 - \lambda_i} N^{-1/2} \epsilon^{k-2}\Delta_1. \tag{33}$$

From equation (33) and proof of Lemma 3.6, we deduce that the scheme (14) converges, and that the limits are the solutions of the fixed-point equations, $\theta_1 = 1, \theta_2, \ldots, \theta_N$. By a simple summation of (33) over $k \geq 2$ we have

$$\left|\theta_i - \theta_i^1\right| \leq \frac{2\sigma f(N)}{\lambda_1 - \lambda_i} N^{-1/2} \leq \frac{2\sigma^2 f(N)}{\lambda_1 - \lambda_i} N^{\delta},$$

where δ is a positive quantity of Lemma 3.2 specified later. Using Lemma 3.3 one has the following control

$$\sum_{i=2}^{N} \left|\theta_i - \theta_i^1\right|^2 \leq 4\sigma^4 N^{2\delta} f(N) N^{4/3}.$$

Moreover, Lemma 3.5 shows that

$$\sum_{i=2}^{N} \left|\theta_i^1\right|^2 \asymp \sigma^2 N^{1/3} \geq g(N)^{-1} \sigma^2 N^{1/3},$$

where g is another function in $\mathcal{F}$. This yields

$$\sum_{i=2}^{N} \left|\theta_i - \theta_i^1\right|^2 \leq \sum_{i=2}^{N} \left|\theta_i^1\right|^2 4\sigma^2 N^{2\delta+1} f(N) g(N).$$

The proof is completed by taking $\delta = \alpha/2$ and applying (6). □

Proof of Lemma 3.5

Proof of Lemma 3.5. We show that w.h.p.

$$\sum_{i=2}^{N} \frac{\langle Hv_i, v_1\rangle^2}{(\lambda_1 - \lambda_i)^2} - \frac{1}{N} \sum_{i=2}^{N} \frac{1}{(\lambda_1 - \lambda_i)^2} = o\left(\frac{1}{N} \sum_{i=2}^{N} \frac{1}{(\lambda_1 - \lambda_i)^2}\right) \tag{34}$$

Let us recall that H is drawn according to the GOE, hence its law is invaraiant by rotation. This implies that the $\langle Hv_i, v_1\rangle$ are independent variables with variance $1/N$, independent of $\lambda_1, \ldots, \lambda_N$. Define

$$M_N := \sum_{i=2}^{N} \frac{\langle Hv_i, v_1\rangle^2 - 1/N}{(\lambda_1 - \lambda_i)^2}.$$

Computing the second moment of M_N, we get

$$\begin{aligned}\mathbb{E}\left[M_N^2 | \lambda_1, \ldots, \lambda_N\right] &= \mathrm{Var}(M_N | \lambda_1, \ldots, \lambda_N) \\ &= \frac{1}{N^4} \sum_{i=2}^{N} \frac{2}{(\lambda_1 - \lambda_i)^4}.\end{aligned}$$

Adapting the proof of Lemma 3.3, following the same steps, one can also show that w.h.p.

$$\sum_{i=2}^{N} \frac{1}{(\lambda_1 - \lambda_i)^4} \asymp N^{8/3}. \tag{35}$$

Let $\epsilon = \epsilon(N) > 0$ to be specified later. By Markov's inequality

$$\begin{aligned}\mathbb{P}\left(|M_N| \geq \frac{\epsilon}{N} \sum_{i=2}^{N} \frac{1}{(\lambda_1 - \lambda_i)^2} | \lambda_1, \ldots, \lambda_N\right) &\leq \frac{N^2}{\epsilon^2} \frac{\mathbb{E}\left[M_N^2 | \lambda_1, \ldots, \lambda_N\right]}{\left(\sum_{i=2}^{N} \frac{1}{(\lambda_1 - \lambda_i)^2}\right)^2} \\ &\asymp \frac{1}{\epsilon^2 N^2},\end{aligned}$$

by 3.3 and equation (35). Taking e.g. $\epsilon(N) = N^{-1/2}$ concludes the proof. □

Proof of Lemma 3.7

Proof of Lemma 3.7. Let O be an orthogonal transformation of the hyperplane $v_1^\perp$ (such that $Ov_1 = v_1$). Since the GOE's law is invariant by rotation and A and H are independent, $\tilde{B} := O^T AO + \sigma O^T HO$ has he same distribution than $B = A + \sigma H$.
$Ow' = v_1 + Ow$ is an eigenvector of $\tilde{B}$ for the eigenvalue λ_1. Since the distribution of the matrix of eigenvectors $(v_2, \dots, v_n)$ is the Haar measure on the orthogonal group $\mathcal{O}_{n-1}\left(v_1^\perp\right)$ denoted sy $d\mathcal{H}(O)$, the distribution of w is also invariant by rotation in the orthogonal complement of v_1.

Furthermore, for any f, g bounded continuous functions

$$\begin{aligned}
&\forall O \in \mathcal{O}_{n-1}\left(v_1^\perp\right), \\
\mathbb{E}\left[f(\|w\|)g\left(\frac{w}{\|w\|}\right)\right] &= \mathbb{E}\left[f(\|w\|)g\left(\frac{Ow}{\|Ow\|}\right)\right] \\
&= \mathbb{E}\left[f(\|w\|)\int_{\mathcal{O}_{n-1}\left(v_1^\perp\right)} d\mathcal{H}(O)g\left(\frac{Ow}{\|Ow\|}\right)\right] \\
&= \mathbb{E}\left[f(\|w\|)\int_{\mathbb{S}^{n-2}} \frac{g(u)du}{\mathrm{Vol}\left(\mathbb{S}^{n-2}\right)}\right] \\
&= \mathbb{E}\left[f(\|w\|)\right]\mathbb{E}\left[g\left(\frac{w}{\|w\|}\right)\right].
\end{aligned}$$

This completes the proof of Lemma 3.7. □

Proof of Lemma 4.2

Proof of Lemma 4.2. We fix $\alpha > 0$ and we want to prove

$$\sum_{k=0}^{\lfloor (N-1)/2 \rfloor} \binom{N-1-k}{k}\alpha^k = \frac{1}{\sqrt{1+4\alpha}}\left[\left(\frac{1+\sqrt{1+4\alpha}}{2}\right)^N - \left(\frac{1-\sqrt{1+4\alpha}}{2}\right)^N\right]. \quad (36)$$

We denote in the following $\phi_+ := \frac{1+\sqrt{1+4\alpha}}{2}$ and $\phi_- := \frac{1-\sqrt{1+4\alpha}}{2}$, and for all $N \geq 1$:

$$u_N = u_N(\alpha) := \sum_{k=0}^{\lfloor (N-1)/2 \rfloor} \binom{N-1-k}{k}\alpha^k.$$

We clearly have $u_N(\alpha) \leq (1+\alpha)^N$. For all $t > 0$ small enough (e.g. $t < \frac{1}{1+\alpha}$), define

$$f(t) := \sum_{N=1}^{\infty} u_N t^N.$$

On one hand,

$$\begin{aligned}
\frac{t}{1-t-\alpha t^2} &= t\sum_{m=0}^{\infty}(t+\alpha t^2)^m \\
&= \sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{m}{l}\alpha^l t^{l+m+1} \\
&= \sum_{N=1}^{\infty}\left(\sum_{\substack{0\leq l\leq m \\ l+m=N-1}}\binom{m}{l}\alpha^l\right)t^N = \sum_{N=1}^{\infty} u_N t^N = f(t).
\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{t}{1-t-\alpha t^2} &= \frac{t}{(1-\phi_- t)(1-\phi_+ t)} \\ &= \frac{1}{\phi_+ - \phi_-}\left(\frac{1}{1-\phi_+ t} - \frac{1}{1-\phi_- t}\right) \\ &= \frac{1}{\sqrt{1+4\alpha}} \sum_{N=1}^{\infty} \left(\phi_+^N - \phi_-^N\right) t^N.\end{aligned}$$

This proves (36). □

Proof of Lemma 5.2

Proof of Lemma 5.2. Let us represent the situation in the plane spanned by v_1 and v_1', as shown on figure 4.

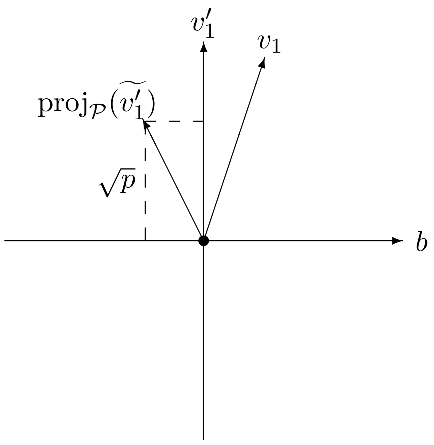

Figure 4: Orthogonal projection of $\widetilde{v_1'}$ on $\mathcal{P} := \text{span}(v_1', v_1)$.

Since $\widetilde{v_1'}$ is taken such that $\langle v_1, \widetilde{v_1'}\rangle > 0$ and σ_1 satisfies (6), we have $\langle \widetilde{v_1'}, v_1'\rangle > 0$ for N large enough. Let $p := \langle \widetilde{v_1'}, v_1'\rangle^2$ and $\widetilde{w} := \widetilde{v_1'} - \sqrt{p} v_1' \in (v_1')^\perp$. By invariance by rotation we can obtain that $\frac{\widetilde{w}}{\|\widetilde{w}\|} = \frac{\widetilde{w}}{\sqrt{1-p}}$ is uniformly distributed on the unit sphere $\mathbb{S}^{N-2}$ of $(v_1')^\perp$, and independent of p, v_1 and v_1'. Hence

$$\langle b, \widetilde{v_1'}\rangle = \langle b, \widetilde{w}\rangle \overset{(d)}{=} \sqrt{1-p} \cdot \frac{\widetilde{Z}_1}{\sqrt{\sum_{i=1}^{N-1} \left(\widetilde{Z}_i\right)^2}},$$

where the $\widetilde{Z}_i$ are independent Gaussian standard variables, independent from everything else. By section 3 we know that $1 - \langle v_1, v_1'\rangle \asymp \sigma_1^2 N^{1/3}$ and thus $\langle v_1, b\rangle \asymp \sigma_1 N^{1/6}$. This yields, for N large enough, w.h.p,

$$\begin{aligned}0 < \langle \widetilde{v_1'}, v_1\rangle &\leq \sqrt{p}\langle v_1, v_1'\rangle + \sqrt{\frac{1-p}{N}} \widetilde{Z}_1 \sigma_1 N^{1/6} f(N) \\ &\leq \sqrt{p}\langle v_1, v_1'\rangle + \sqrt{1-p} N^{-4/3} g(N) \\ &\leq \max\left(\sqrt{p}, \sqrt{1-p}\right) \langle v_1, v_1'\rangle \\ &\leq \langle v_1, v_1'\rangle,\end{aligned}$$

where f and g are two functions as defined in Lemma 3.3.
From this point one can still make the link with the toy model, as done in the beginning

of section 4. By invariance by rotation, letting $t := \widetilde{v_1'} - \langle \widetilde{v_1'}, v_1 \rangle v_1$, we know that $\|t\|$ and $\frac{t}{\|t\|}$ are independent, and that $\frac{t}{\|t\|}$ is uniformly distributed on the unit sphere in $v_1^{\perp}$. We have the following equalities in distribution:

$$r_1(v_1) \overset{(d)}{=} r_1(X),$$

and

$$r_1(\widetilde{v_1'}) \overset{(d)}{=} r_1(X + \mathbf{s}Z),$$

with w.h.p.

$$\mathbf{s} \geq \mathbf{s^1} = \frac{\|w\|\|X\|}{\left(\sum_{i=2}^N Z_i^2\right)^{1/2} \left(1 - \frac{\|w\| Z_1}{\left(\sum_{i=2}^N Z_i^2\right)^{1/2}}\right)} \asymp \sigma_1 N^{1/6},$$

where the X_i, Z_i and w are defined in section 4, for $\sigma = \sigma_1$. □

Proof of Lemma 5.1

Proof of Lemma 5.1. Recall that we work in the case (i) ($\sigma = o(N^{-7/6-\epsilon})$ for some $\epsilon > 0$), with $\langle v_1, v_1' \rangle > 0$ and $\Pi = \mathrm{Id}$. We want to show that w.h.p.

$$\langle A, \Pi_+ B \Pi_+^T \rangle > \langle A, \Pi_- B \Pi_-^T \rangle. \tag{37}$$

Define

$$\mathcal{G} := \{i, \Pi_+(i) = \Pi(i) = i\}.$$

and

$$\mathcal{A} := \left\{\sigma N^{1/6} f(N)^{-1} \leq \mathbf{s} \leq \sigma N^{1/6} f(N)\right\},$$

with $f \in \mathcal{F}$ such that $\mathbb{P}(\mathcal{A}) \to 1$. For N large enough, on the event $\mathcal{A}$, we have $0 \leq \mathbf{s}N \leq N^{-\epsilon} f(N)$. Hence, retaking the proof of **Proposition II**, we have

$$\begin{aligned}\phi_{x,z}(N, \mathbf{s}) &\geq \mathbb{P}\left(\mathcal{N}^{+-}(x, x + \mathbf{s}z) = \mathcal{N}^{-+}(x, x + \mathbf{s}z) = 0\right) \\ &\sim \exp\left(-\mathbf{s}NE(x)\left[z(2F(z) - 1) + 2E(z)\right]\right) = 1 - O(N^{-\epsilon} f(N)).\end{aligned}$$

Thus, with dominated convergence, for N large enough,

$$\mathbb{P}(\Pi_+(i) = \Pi(i) | \mathcal{A}) = \iint dx dz E(x) E(z) \mathbb{E}\left[\phi_{x,z}(N, \mathbf{s}) | \mathcal{A}\right] \geq 1 - O(N^{-\epsilon} f(N)). \tag{38}$$

We use Markov's inequality with (38) to show that $\mathbb{P}\left(\sharp\mathcal{G} \leq N - N^{1-\epsilon/2} \mid \mathcal{A}\right) \leq O\left(N^{-\epsilon/2} f(N)\right)$, hence w.h.p.

$$\sharp\mathcal{G} \geq N - N^{1-\epsilon/2}. \tag{39}$$

Splitting the sum

$$\langle A, \Pi_+ B \Pi_+^T \rangle = \sum_{i,j} A_{i,j} B_{\Pi_+(i), \Pi_+(j)} = \sum_{(i,j) \in \mathcal{G}^2} A_{i,j} B_{i,j} + \sum_{(i,j) \notin \mathcal{G}^2} A_{i,j} B_{\Pi_+(i), \Pi_+(j)},$$

one has, w.h.p.,

$$\begin{aligned}\langle A, \Pi_+ B \Pi_+^T \rangle = &\sum_{(i,j) \in \mathcal{G}^2} A_{i,j}^2 + \sum_{\substack{(i,j) \notin \mathcal{G}^2 \\ (\Pi_+(i), \Pi_+(j)) \neq (j,i)}} A_{i,j} A_{\Pi_+(i), \Pi_+(j)} \\ &+ \sum_{\substack{(i,j) \notin \mathcal{G}^2 \\ (\Pi_+(i), \Pi_+(j)) = (j,i)}} A_{i,j}^2 + \sigma \sum_{1 \leq i,j \leq N} A_{i,j} H_{\Pi_+(i), \Pi_+(j)} \\ &\geq C_1 \frac{(\sharp\mathcal{G})^2}{N} - C_2 \left(N^2 - (\sharp\mathcal{G})^2\right) \frac{\log N}{N} - C_2 \sigma N^2 \frac{\log N}{N}.\end{aligned}$$

We applied the law of large numbers for the first sum, lower-bounded the third sum by zero, and the classical inequality $\max_{i,j}\{A_{i,j}, H_{i,j}\} \leq C_2 \frac{\log N}{N}$ (which holds w.h.p.) for the two others.
Inequality (39) and condition (i) lead to, w.h.p.

$$\langle A, \Pi_+ B \Pi_+^T \rangle \geq C_1 N - 2C_1 N^{1-\epsilon/2} - 2C_2 N^{1-\epsilon/2} \log N - C_2 N^{-1/6-\epsilon} \log N \geq C_3 N.$$

On the other hand, since by definition $\Pi_-(i) = \Pi_+(N+1-i)$, w.h.p.,

$$\begin{aligned}\langle A, \Pi_- B \Pi_-^T \rangle = \sum_{(i,j)\in\mathcal{G}^2} A_{i,j} B_{N+1-i,N+1-j} + \sum_{(i,j)\notin\mathcal{G}^2} A_{i,j} B_{\Pi_-(i),\Pi_-(j)} \\ \leq O(\log N) + \frac{(\sharp\mathcal{G})^2}{N} o(1) + C_2 \left(N^2 - (\sharp\mathcal{G})^2\right) \frac{\log N}{N}.\end{aligned}$$

For the first sum, we used the law of large numbers: the variables $A_{i,j}$ and $B_{N+1-i,N+1-j}$ are independent in all cases but at most $N+1$, and this part of the sum is bounded by $O(\log N)$. We used the same control on Gaussian variables as above.
This gives

$$\left(\langle A, \Pi_- B \Pi_-^T \rangle\right)_+ = o_{\mathbb{P}}(N),$$

where $(x)_+ := \max(0, x)$, which proves (37). □

Proof of Lemma 5.3

Proof of Lemma 5.3. Recall that we work in the case (ii) ($\sigma = \omega(N^{-7/6+\epsilon})$ for some $\epsilon > 0$), with $\langle v_1, v_1' \rangle > 0$ and $\Pi = \mathrm{Id}$. We want to show that the aligning permutation between v_1 and $-v_1'$ has a very bad overlap. Taking the couple $(X, -Y)$ where $(X, Y) \sim \mathcal{J}(N, s)$, one can adapt the proof of **Proposition II**, with the new definitions

$$\begin{aligned}\widetilde{S^{+-}}(x,y) &:= \mathbb{P}\left(X_1 > x, -Y_1 < -y\right), \text{ and} \\ \widetilde{S^{-+}}(x,y) &:= \mathbb{P}\left(X_1 < x, -Y_1 > -y\right).\end{aligned}$$

The analysis is even easier since for all x, z, there exist two constants c, C such that

$$0 < c \leq \widetilde{S^{+-}}(x, x+sz),\ \widetilde{S^{-+}}(x, x+sz) \leq C < 1.$$

It is then easy to check that the proof of **Proposition II**, case (ii) adapts well. □

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