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Deciding Feasibility of a Booking in the European Gas Market on a Cycle is in P for the Case of Passive Networks

Martine Labbé, Fränk Plein, Martin Schmidt, and Johannes Thürauf

Abstract. We show that the feasibility of a booking in the European entry-exit gas market can be decided in polynomial time on single-cycle networks that are passive, i.e., do not contain controllable elements. The feasibility of a booking can be characterized by solving polynomially many nonlinear potential-based flow models for computing so-called potential-difference maximizing load flow scenarios. We thus analyze the structure of these models and exploit both the cyclic graph structure as well as specific properties of potential-based flows. This enables us to solve the decision variant of the nonlinear potential-difference maximization by reducing it to a system of polynomials of constant dimension that is independent of the cycle's size. This system of fixed dimension can be handled with tools from real algebraic geometry to derive a polynomial-time algorithm. The characterization in terms of potential-difference maximizing load flow scenarios then leads to a polynomial-time algorithm for deciding the feasibility of a booking. Our theoretical results extend the existing knowledge about the complexity of deciding the feasibility of bookings from trees to single-cycle networks.

1. Introduction

During the last decades, the European gas market has undergone ongoing liberalization [26–28], resulting in the so-called entry-exit market system [19]. The main goal of this market re-organization is the decoupling of trading and actual gas transport. To achieve this goal within the European entry-exit market, gas traders interact with transport system operators (TSOs) via bookings and nominations. A booking is a capacity-right contract in which a trader reserves a maximum injection or withdrawal capacity at an entry or exit node of the TSO's network. On a day-ahead basis, these traders are then allowed to nominate an actual load flow up to the booked capacity. To this end, the traders specify the actual amount of gas to be injected to or withdrawn from the network such that the total injection and withdrawal quantities are balanced. On the other hand, the TSO is responsible for the transport of the nominated amounts of gas. By having signed the booking contract, the TSO guarantees that the nominated amounts can actually be transported through the network. More precisely, the TSO needs to be able to transport every set of nominations that complies with the signed booking contracts. Thus, an infinite number of possible nominations must be anticipated and checked for feasibility when the TSO accepts bookings. As a consequence, the entry-exit market decouples trading and transport. However, it also introduces many new challenges, e.g., the checking of feasibility of bookings or the computation of bookable capacities on the network [10, 23].

A large branch of research considers the feasibility of nominations, as well as the physics and the optimal control of gas networks w.r.t. single nominations. Early works such as [25] or [8] study the physical properties of pipe networks. In particular, it is shown that in connected pressure-based networks the flow corresponding to a given load scenario is unique (given that the pressure at an arbitrary node is fixed). This result holds more generally for a potential-based flow model; see, e.g., [30]. Such a potential-based flow model is also used in [16] as an abstract model that approximates, among others like water or lossless direct current (DC) power flow, the physics of stationary flows in gas networks. More generally, the study of gas transport and the feasibility of nominations has been researched from many different optimization perspectives. For instance, in [9] and [4], the authors study the cost-optimal transport of gas in the Belgian network before and after the market liberalization. An extension of the simplex algorithm is proposed to solve the problem for the case in which gas physics are approximated by piecewise-linear functions, enabling mixed-integer linear programming (MILP) techniques to be used. MILP approaches can also be found, e.g., in [12, 13, 24]. On the other hand, purely continuous and highly accurate nonlinear optimization (NLP) models are discussed, e.g., in [33]. The combination of both worlds leads to challenging mixed-integer nonlinear models that are tackled, e.g., in [14, 20]. For an in-depth overview of optimization problems in gas networks, we also refer to the recent survey [29] as well as the book [21] and the references therein.

In contrast to the very rich literature on nominations, there is much less literature on checking the feasibility of a booking. First mathematical analyses of bookings are presented in the PhD theses [18, 34]. Moreover, the early technical report [11] discusses the reservation-allocation problem, which is highly related to the feasibility of bookings in the European entry-exit gas market. Deciding the feasibility of a booking can also be seen as an adjustable robust feasibility problem [6], where the set of booking-compliant nominations is the uncertainty set. Exploiting this perspective, the authors of [3] propose set containment techniques to decide robust feasibility and infeasibility with an application to the Greek gas transport network. With an application to a tree-shaped hydrogen network, the problem of robust discrete arc sizing is discussed in [31]. In [2], the uncertainty of physical parameters is considered. On the other hand, structural properties of the sets of feasible nominations and bookings such as nonconvexity and star-shapedness are discussed in [32]. For networks consisting of pipes only, a characterization of feasible bookings is given in [22] by conditions on nominations with maximum potential difference in the network. Using a linear potential-based flow model, these nominations can be computed efficiently using linear programming. In the nonlinear case, the authors give a polynomial-time dynamic programming approach for deciding the feasibility of a booking, if the underlying network is a tree. For the general case, i.e., nonlinear potential-based physics and arbitrary network topologies, the complexity of deciding the feasibility of a booking is not yet clear and only exponential upper bounds are given in [18]. However, neither hardness results nor polynomial-time algorithms can be found in the literature for cases where the network is not a tree.

In the light of this literature, our contribution is as follows. We extend the knowledge on the hardness of the problem by showing that deciding the feasibility of a booking on single-cycle networks is in P. We analyze the structure of potential-difference maximizing nominations by exploiting the cyclic structure of the network as well as techniques specific to potential-based flow models. Interestingly, this allows to reduce the task of checking the feasibility of a booking to checking the solvability of a system of polynomial equalities and inequalities in fixed dimension, where the latter does not depend on the size of the cycle. These systems of fixed

dimension can then be tackled with tools from real algebraic geometry to derive a polynomial-time algorithm for deciding the feasibility of a booking.

The remainder of this paper is structured as follows. In Section 2, the problem of checking the feasibility of a booking is formally defined. Section 3 collects notations and known results that are used in this work. In Section 4, we introduce a notion of so-called flow-meeting points in cycle networks and study properties of potential-difference maximizing nominations in Section 5. These results are then combined in Section 6 to derive a polynomial-time algorithm for deciding the feasibility of a booking on a cycle. Finally, we draw a conclusion and pose some open questions for future research in Section 7.

2. Problem Description

Before restricting ourselves to cycles, we first introduce the problem of verifying the feasibility of bookings for general networks. Thus, we model a gas network using a weakly connected directed graph $G = (V, A)$ with node set V and arc set A. The set of nodes is partitioned into the set V_+ of entry nodes, at which gas is supplied, the set V_- of exit nodes, where gas is withdrawn, and the set V_0 of the remaining inner nodes. The node types are encoded in a vector $\sigma = (\sigma_u)_{u \in V}$, given by

$$\sigma_u = \begin{cases} 1, & \text{if } u \in V_+, \\ -1, & \text{if } u \in V_-, \\ 0, & \text{if } u \in V_0. \end{cases}$$

In real-world gas networks, the arc set is typically partitioned into different types of arcs that correspond to different elements of the network; e.g., pipes, compressors, control valves, etc. However, we restrict our analysis to passive networks that consist of pipes only. We follow the notation and definitions of [32], which we briefly introduce in the following.

Definition 2.1. A *load* is a vector $\ell = (\ell_u)_{u \in V} \in \mathbb{R}_{\geq 0}^V$, with $\ell_u = 0$ for all $u \in V_0$. The set of load vectors is denoted by L.

A load vector thus corresponds to an actual situation at a single point in time by specifying the amount of gas ℓ_u that is supplied at $u \in V_+$ or withdrawn at $u \in V_-$. Since we only consider stationary flows, we need to impose that the supplied amount of gas equals the withdrawn amount, which leads to the definition of a nomination.

Definition 2.2. A *nomination* is a balanced load vector ℓ, i.e., $\sigma^\top \ell = 0$. The set of nominations is given by

$$N := \left\{ \ell \in L : \sigma^\top \ell = 0 \right\}.$$

A booking, on the other hand, is a load vector defining bounds on the admissible nomination values. More precisely, we have the following definition.

Definition 2.3. A *booking* is load vector $b \in L$. A nomination ℓ is called *booking-compliant* w.r.t. the booking b if $\ell \leq b$ holds, where "$\leq$" is meant component-wise throughout this paper. The set of booking-compliant (or b-compliant) nominations is given by

$$N(b) := \left\{ \ell \in N : \ell \leq b \right\}.$$

Next, we introduce the notion of feasibility for nominations and bookings. We model stationary gas flows using an abstract physics model based on the Weymouth pressure drop equation and potential flows; see, e.g., [16] or [22]. It consists of arc flow variables $q = (q_a)_{a \in A} \in \mathbb{R}^A$ and potentials on the nodes $\pi = (\pi_u)_{u \in V} \in \mathbb{R}_{\geq 0}^V$.

We note that, in this context, potentials are linked to gas pressures at the nodes via $\pi_u = p_u^2$ for the case of horizontal pipes. An in-depth explanation for non-horizontal pipes is given in [16].

Definition 2.4. A nomination $\ell \in N$ is *feasible* if a point (q, π) exists that satisfies

$$\sum_{a \in \delta^{\mathrm{out}}(u)} q_a - \sum_{a \in \delta^{\mathrm{in}}(u)} q_a = \sigma_u \ell_u, \qquad u \in V, \qquad (1\mathrm{a})$$

$$\pi_u - \pi_v = \Lambda_a q_a \left| q_a \right|, \qquad a = (u, v) \in A, \qquad (1\mathrm{b})$$

$$\pi_u \in [\pi_u^-, \pi_u^+], \qquad u \in V, \qquad (1\mathrm{c})$$

where $\delta^{\mathrm{out}}(u)$ and $\delta^{\mathrm{in}}(u)$ denote the sets of arcs leaving and entering node $u \in V$, $\Lambda_a > 0$ is an arc-specific constant for any $a \in A$, and $0 < \pi_u^- \leq \pi_u^+$ are potential bounds for any $u \in V$.

Constraints (1a) ensure that flow is conserved at every node w.r.t. the nomination ℓ. For any $a = (u, v) \in A$, Constraint (1b) links the flow q_a to the difference $\pi_u - \pi_v$ of potentials at the endpoints of a. We note that flow can be negative, if it flows in the opposite direction of the orientation of the arc. Finally, due to technical restrictions of the network, the potentials need to satisfy bounds (1c). In a weakly connected network that only consists of pipes, the flow $q = q(\ell)$ corresponding to a given nomination $\ell \in N$ is unique since it is the optimal solution of a strictly convex minimization problem [25]. The potentials $\pi = \pi(\ell)$ are the corresponding dual variables and are unique as soon as a reference potential is fixed; see, e.g., [30]. The potentials are therefore unique up to shifts, which in particular implies that potential differences between nodes are unique for a given nomination ℓ. The feasibility of a given nomination can be checked using the approach described in [15]. In contrast, verifying the feasibility of a booking is less researched and much more difficult.

Definition 2.5. We say that a booking b is *feasible* if all booking-compliant nominations $\ell \in N(b)$ are feasible.

To assess the feasibility of a booking, by definition, a possibly infinite number of nominations need to be checked.

Remark 2.6. Deciding the feasibility of a booking can be seen as very special case of deciding the feasibility of an adjustable robust optimization problem with uncertainty set $N(b)$. Let us briefly highlight this relationship in this remark. In principle, for every booking-compliant nomination, we are allowed to adjust the corresponding flow and the corresponding potentials according to the feasibility system (1). However, the decision rule (in terms of adjustable robust optimization) is very special. Note again that, for a given nomination $\ell \in N(b)$, the resulting flow is uniquely determined and all potentials are uniquely determined if we fix a certain potential π_w at an arbitrarily chosen reference node w, e.g., if we set $\pi_w = \psi$ for a reference potential ψ. Thus, we face the adjustable robust problem in which the uncertainty set consists of all booking-compliant nominations and that can be formalized as

$$\forall \ell \in N(b) : \exists y^\psi \in Y : \pi_u^- \leq y_u^\psi(\ell) \leq \pi_u^+, \quad u \in V. \qquad (2)$$

Here, Y corresponds to the ψ-parameterized set of decision rules, which map given nominations to node potentials, i.e., $y^\psi \in Y$ and $y^\psi : N(b) \to \mathbb{R}^V$. This, in

particular, means that for a given nomination, the only choice is the reference pressure since all flows and potentials are uniquely determined afterward by (1).

Consequently, deciding the feasibility of a booking is equivalent to finding a specific decision rule in the ψ-parameterized family of functions Y. Note that the uncertainty is not given in a constraint-wise way. Additionally, the decision rules must satisfy (1b), which has a nonlinear right-hand side that is nonsmooth in zero. Consequently, the decision rules are also nonlinear and nonsmooth in general. The related adjustable robust problem is thus a very special one that is, in general, not tractable in terms of adjustable robust optimization; see, e.g., [7] or the recent survey [35] as well as the references therein. One particular contribution of this paper is that the problem-specific structure at hand is exploited so that the considered problem (which looks highly intractable at a first glance) can be solved efficiently. The further question on whether the developed techniques may be generalized to general adjustable robust flow problems is beyond the scope of this paper.

In every network, the zero flow associated with the zero nomination is always feasible. It is achieved by having the same potential at every node. This, in particular, leads to the following assumption on the bounds of the potentials.

Assumption 2.7. The potential bound intervals have a non-empty intersection, i.e.,
$$\bigcap_{u \in V} [\pi_u^-, \pi_u^+] \neq \emptyset.$$
Since the zero nomination is always booking compliant, this assumption is required for having a feasible booking at all. Thus, the assumption is also required to allow for a reasonable study of deciding the feasibility of bookings.

It is shown in Theorem 7 of [22] that a feasible booking b can be characterized by constraints on the maximum potential differences between all pairs of nodes. Therefore, the authors introduce, for every fixed pair of nodes $(w_1, w_2) \in V^2$, the following problem

$$\varphi_{w_1 w_2}(b) := \max_{\ell, q, \pi} \quad \pi_{w_1} - \pi_{w_2} \tag{3a}$$
$$\text{s.t.} \quad (1a), (1b),$$
$$0 \leq \ell_u \leq b_u, \quad u \in V, \tag{3b}$$

where $\varphi_{w_1 w_2}$ is the corresponding optimal value function (depending on the booking b). Then, the booking b is feasible if and only if

$$\varphi_{w_1 w_2}(b) \leq \pi_{w_1}^+ - \pi_{w_2}^- \tag{4}$$

holds for every fixed pair of nodes $(w_1, w_2) \in V^2$. Hence, to verify the feasibility of a booking using this approach, it is necessary to solve the nonlinear and nonconvex global optimization problems (3). For tree-shaped networks, the authors give a polynomial-time dynamic programming algorithm solving (3). As a consequence, verifying the feasibility of a booking on trees can be done in polynomial time, which can also be obtained by adapting the results of [32]. In this paper, we show that (4) can still be decided in polynomial time on a single cycle.

3. Notations and Basic Observations

Entry and exit nodes $v \in V_+ \cup V_-$ are called *active* if $\ell_v > 0$ holds. We denote by $V_+^> := \{v \in V_+ : \ell_v > 0\}$ and $V_-^> := \{v \in V_- : \ell_v > 0\}$ the set of active entries and exits, respectively.

Using directed graphs to represent gas networks is a modeling choice that allows to interpret the direction of arc flows. However, the physical flow in a potential-based network is not influenced by the direction of the arcs. Thus, for $u, v \in V$, we introduce the so-called *flow-paths* $P := P(u, v) = (V(P(u, v)), A(P(u, v)))$ in which $V(P(u, v)) \subseteq V$ contains the nodes of the path from u to v in the undirected version of the graph G and $A(P(u, v)) \subseteq A$ contains the corresponding arcs of this path. Note that these flow-paths are not necessarily unique. For another pair of nodes $u', v' \in V$, we say that $P(u', v')$ is a *flow-subpath* of $P(u, v)$ if $P(u', v') \subseteq P(u, v)$, i.e., $V(P(u', v')) \subseteq V(P(u, v))$ and $A(P(u', v')) \subseteq A(P(u, v))$, and if $P(u', v')$ is itself a flow-path. In particular, this allows us to define an order on the nodes of a flow-path. For $P = P(u, v)$ and $u', v' \in P$, we define $u' \preceq_P v'$ if and only if a flow-subpath $P(u, u') \subseteq P(u, v')$.[1] If $u' \neq v'$ holds, we write $u' \prec_P v'$.

We now introduce the *characteristic function* of an arc $a = (u, v) \in A$. For any flow-path P, it is given by

$$\chi_a(P) := \begin{cases} 1, & \text{if } u \prec_P v, \\ -1, & \text{if } v \prec_P u, \\ 0, & \text{if } a \notin P. \end{cases}$$

Next, we adapt a classical result from linear flow models to construct a flow decomposition in a gas network.

Lemma 3.1. Given $\ell \in N \setminus \{0\}$, let $\mathcal{P}_\ell := \left\{ P(u, v) : u \in V_+^>, v \in V_-^> \right\}$ be the set of flow-paths in G with an active entry as start node and an active exit as end node. Then, a decomposition of the given flow $q = q(\ell)$ into path flows exists, such that

$$q_a = \sum_{P \in \mathcal{P}_\ell} \chi_a(P) \, q(P), \quad a \in A, \tag{5}$$

where $q(P)$ is the nonnegative flow along the flow-path $P \in \mathcal{P}_\ell$.

Furthermore, we require that if $q_a > 0$ for $a \in A$ and $\chi_a(P) = -1$ for $P \in \mathcal{P}_\ell$, then $q(P) = 0$ holds. Similarly, if $q_a < 0$ for $a \in A$ and $\chi_a(P) = 1$ for $P \in \mathcal{P}_\ell$, then $q(P) = 0$ holds.

Proof. If $q_a < 0$ holds, then we replace arc $a = (u, v)$ by (v, u) and set $q_{(v,u)} = -q_{(u,v)}$. The resulting flow still corresponds to nomination ℓ. We now apply Theorem 3.5 of Chapter 3.5 of the book by Ahuja et al. [1]. Given Constraints (1b), the flow q cannot contain any cycle flows. As a consequence, we obtain a flow decomposition that satisfies all the properties. $\square$

Observe that, by construction, the flow q and the path flows need to traverse arcs in the same direction. A direct consequence of the flow decomposition is that the nomination can be expressed as a function of the path flows.

Corollary 3.2. For any $u \in V_+^>$, the condition

$$\sum_{v \in V_-^>} q(P(u, v)) = \ell_u \tag{6}$$

and for any $v \in V_-^>$, the condition

$$\sum_{u \in V_+^>} q(P(u, v)) = \ell_v \tag{7}$$

is satisfied.

---

[1] For the ease of presentation, we also use the notation $u \in P = P(u, v)$ instead of $u \in V(P(u, v))$ or $a \in P$ instead of $a \in A(P(u, v))$, if it is clear from the context.

Next, we define the potential-difference function along a given flow-path.

Definition 3.3. Let $\ell \in N$ and a flow-path P be given. Then, the *potential-difference function* along P is given by

$$\Pi_P : \mathbb{R}^A \to \mathbb{R}, \quad \Pi_P(q) := \sum_{a \in P} \chi_a(P) \Lambda_a q_a \left| q_a \right|, \tag{8}$$

where $q = q(\ell)$.

As a consequence of Constraint (1b), for any node pair $u, v \in V$ and for any flow-path $P := P(u, v)$, the equation $\pi_u(\ell) - \pi_v(\ell) = \Pi_P(q(\ell))$ holds. We note that if the path P is directed from u to v, the potential-difference function simplifies to

$$\Pi_P(q) = \sum_{a \in P} \Lambda_a q_a \left| q_a \right|.$$

Since, we will mostly use directed paths in what follows, we state some properties that hold in this case.

Lemma 3.4. For $u, v \in V$, let $P := P(u, v)$ be a directed path. Then, the following holds:

(a) Π_P is continuous.
(b) Π_P is strictly increasing w.r.t. every component. That means, for q fixed except for one value q_a, $a \in P$, Π_P is increasing in q_a.
(c) Π_P is unbounded w.r.t. every component, i.e., for $a \in P$,

$$\lim_{q_a \to -\infty} \Pi_P(q) = -\infty \quad \text{and} \quad \lim_{q_a \to \infty} \Pi_P(q) = \infty.$$

(d) Π_P is additive w.r.t. the flow-path, i.e., for all $v' \in P$,

$$\Pi_P = \Pi_{P(u,v')} + \Pi_{P(v',v)}$$

where $P = P(u, v') \cup P(v', v)$.
(e) $\Pi_P \geq 0$ holds if and only if $\pi_u \geq \pi_v$ holds.

4. PROBLEM REDUCTION VIA FLOW-MEETING POINTS

In the remainder of this paper, we restrict ourselves to a network that is a single cycle. A stylized example of a cyclic gas network is shown in Figure 1. A first observation is that in a potential-based flow model, there cannot be any cycling flow. Thus, flow in a cycle has to "meet" in at least one node. In this section, we show that the set of all feasible flows in Problem (3) can be restricted to flow along two different paths without changing direction along the way.

In a cycle, for every pair of nodes $u, v \in V$, exactly two flow-paths exist. We denote by $P^l(u, v) = (V^l(u, v), A^l(u, v))$ the *left path* obtained when v is reached in counter-clockwise direction from u. Similarly, $P^r(u, v) = (V^r(u, v), A^r(u, v))$ is the *right path* obtained by using the clockwise direction. Moreover, $A = A^l(u, v) \cup A^r(u, v)$ holds. If it is clear from the context, we use previously introduced notations indexed by "l" (left) or "r" (right), when they have to be understood w.r.t. P^l or P^r.

It is not hard to observe that, given Constraints (1a) and (1b), the highest potential in G is attained at an entry node.

Lemma 4.1. Let $\ell \in N \setminus \{0\}$ and $o \in V_+$ be an entry with highest potential. Then, $\pi_o(\ell) \geq \pi_v(\ell)$ holds for all $v \in V$.

Given that no cycle flow is possible in a gas network, flow needs to change the direction along a single cycle. We now specify a node as flow-meeting point if arc flows from different directions "meet" at this node.
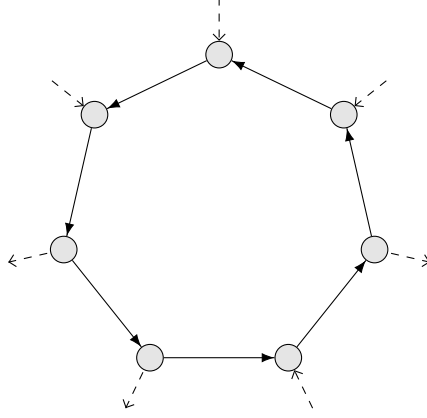
FIGURE 1. Example of a cyclic gas network. Dashed arcs to or from a node indicate entries or exits, respectively.
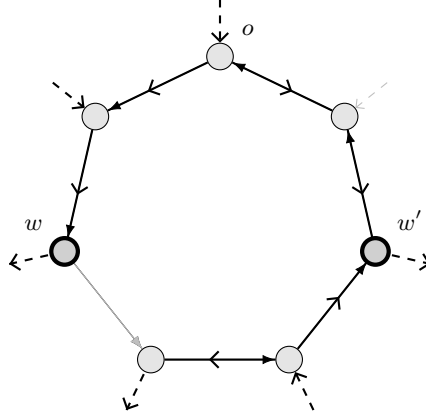

FIGURE 2. Flow directions and resulting flow-meeting points. Bold arcs indicate entry-exit activity and flow directions, whereas gray arcs indicate inactive nodes or zero arc-flow. In this example, exits w and w' are flow-meeting points.

Definition 4.2. Let $\ell \in N \setminus \{0\}$ and $o \in V_+$ be an entry node with highest potential, i.e., $\pi_o(\ell) \geq \pi_v(\ell)$ for all $v \in V$. Then, $w \in V \setminus \{o\}$ is a *flow-meeting point* if there exist $u \in V^{\mathrm{l}}(o,w)$ adjacent to w that satisfies $\pi_u(\ell) > \pi_w(\ell)$ as well as $v \in V^{\mathrm{r}}(o,w)$ such that $\pi_v(\ell) > \pi_w(\ell)$ and $\pi_{v'}(\ell) = \pi_w(\ell)$ holds for all $v' \in V^{\mathrm{r}}(v,w) \setminus \{v\}$.

This definition is illustrated in Figure 2. Note that we choose the node $o \in V_+$ with highest potential to ensure that there is no flow through node o. If multiple entry nodes with highest potential exist, flow-meeting points are still well-defined. In fact, as a direct consequence of Lemma 4.1, the definition of a flow-meeting point is independent of the choice of node o. By definition, the flow-meeting point w has non-zero flow entering on one arc and possibly zero flow on the other arc. Thus, w is necessarily an exit.

From Constraints (1a) and (1b), it directly follows that for every non-zero nomination at least one flow-meeting point exists. We note that there can be

multiple flow-meeting points with different potentials. However, since every flow-meeting point is an exit, it is not hard to observe that the following result holds.

Lemma 4.3. Let $\ell \in N \setminus \{0\}$ and w be a flow-meeting point with lowest potential. Then, $\pi_w(\ell) \leq \pi_v(\ell)$ holds for all $v \in V$.

In the remainder of this section, we show that for fixed $(w_1, w_2) \in V^2$ there are optimal solutions of (3) with at most one flow-meeting point. More precisely, we prove that an optimal solution exists that has a special entry node $o \in V_+$, a special exit node $w \in V_-$, and has nonnegative flow from o to w.

Before we prove several auxiliary results, let us first make a notational comment. For $(o, w) \in V_+ \times V_-$, we are interested in the partition of the cycle into two flow-paths $P^l(o, w)$ and $P^r(o, w)$. When discussing the order of nodes along $P^l(o, w)$, we therefore simply write $u \preceq_l v$ instead of $u \preceq_{P^l(o,w)} v$. We use an analogous simplification for $P^r(o, w)$.

A first observation is that nominations can be modified such that the flow from an entry node with highest potential to an exit node with lowest potential is nonnegative, while preserving particular potential differences.

Lemma 4.4. Given $\ell \in N \setminus \{0\}$ with flow $q = q(\ell)$, let $o \in V_+$ be an entry with highest potential and w a flow-meeting point with lowest potential. Furthermore, assume that $P^l(o, w)$ and $P^r(o, w)$ are directed paths. Then, for a given $x \in V^l(o, w)$, a nomination $\ell' \in N$ exists such that the following properties hold (with $q' = q(\ell')$):

$$\ell' \leq \ell, \tag{9a}$$

$$0 \leq q'_a \text{ for all } a \in A^l(o, w), \tag{9b}$$

$$q'_a = q_a \text{ for all } a \in A^r(o, w), \tag{9c}$$

$$\Pi_{P^l(o,x)}(q') = \Pi_{P^l(o,x)}(q) \geq 0, \tag{9d}$$

$$\Pi_{P^l(o,w)}(q') = \Pi_{P^l(o,w)}(q). \tag{9e}$$

Proof. We modify nomination ℓ and $q(\ell)$ such that the required properties are satisfied. To this end, we consider a flow decomposition as in Lemma 3.1.

Since $o \in V_+$ has highest potential and $P^l(o, w)$ and $P^r(o, w)$ are directed paths, it follows $q_a \geq 0$ for all $a \in \delta^{\text{out}}(o)$. In analogy, $q_a \geq 0$ for all $a \in \delta^{\text{in}}(w)$. From Lemma 3.1, we then deduce that $q(P(u,v)) = 0$ if one of the following four conditions holds:

- $u \in V^l_+(o, w) \setminus \{o\}$ and $v \in V^r_-(o, w) \setminus \{w\}$,
- $u \in V^r_+(o, w) \setminus \{o\}$ and $v \in V^l_-(o, w) \setminus \{w\}$,
- $u \in V^l_+(o, w) \setminus \{o\}$, $v \in V^l_-(o, w) \setminus \{w\}$, and $P^r(o, w) \subseteq P(u, v)$, or
- $u \in V^r_+(o, w) \setminus \{o\}$, $v \in V^r_-(o, w) \setminus \{w\}$, and $P^l(o, w) \subseteq P(u, v)$.

In other words, since there is no flow through o or w, there cannot be a flow-path with non-zero flow through them, either. Consequently, for an arc $a \in P^l(o, w)$, we can simplify Equation (5) to

$$q_a = \sum_{P \in \mathcal{P}^l_\ell} \chi_a(P) \, q(P), \tag{10}$$

where $\mathcal{P}^l_\ell$ contains all flow-paths on the left side of the cycle, i.e.,

$$\mathcal{P}^l_\ell := \left\{ P(u, v) \in \mathcal{P}_\ell \colon P(u, v) \subseteq P^l(o, w) \right\}.$$

We note that $\mathcal{P}^l_\ell$ depends on the choice of o and w. However, these are fixed nodes throughout this section. We now modify the flow q such that Property (9b) is

satisfied. To this end, for every arc $a \in P^{l}(o, w)$ and for every $P \in \mathcal{P}^{l}_{\ell}$, we set the flow $q(P) = 0$ if $\chi_a(P) = -1$ holds. We denote the modified flow and the corresponding nomination by q' and ℓ'. Then, for $a \in A^{l}(o, w)$, the modified flow is given by

$$q'_a = \sum_{P \in \mathcal{P}^{l}_{\ell}: \chi_a(P)=1} q(P) \tag{11}$$

and satisfies (9b). Furthermore, by Corollary 3.2, the corresponding modified nomination ℓ' satisfies (9a). Additionally, (9c) is satisfied because we have not modified any arc flows q_a for $a \in A^{r}(o, w)$. Due to Lemma 3.4 (b), the modifications possibly increase the potential difference between o and x, as well as, between o and w. This is the case if and only if the corresponding flow-path contains an arc with negative flow in q, which is now set to zero in the modified flow q'. Next, we need to iteratively adapt nomination ℓ' and flow q' to ensure the remaining properties (9d) and (9e).

Step 1: If an arc $a \in A^{l}(o, x)$ with $q_a < 0$ exists, then, the potential difference between o and x is increased, i.e., $\Pi_{P^{l}(o,x)}(q') > \Pi_{P^{l}(o,x)}(q)$ holds. Let $u \in V^{l}(o, x)$, possibly with $u = x$, such that $q_{a'} \geq 0$ holds for all $a' \in A^{l}(u, x)$ and $|V^{l}(u, x)|$ is maximal. Given the flow decomposition, we then know that we have not modified arc flows on $P^{l}(u, x)$. Consequently, $\Pi_{P^{l}(u,x)}(q') = \Pi_{P^{l}(u,x)}(q)$ holds. Thus, $\Pi_{P^{l}(o,u)}(q') > \Pi_{P^{l}(o,u)}(q)$ must hold. In particular, we have $a \in A^{l}(o, u)$. From Lemma 3.1 and the construction of u it follows that for $v_1 \in V^{l}_+(o, u) \setminus \{u\}$ and $v_2 \in V^{l}_-(u, w)$, we have $q(P^{l}(v_1, v_2)) = 0$. Consequently, the potential difference $\Pi_{P^{l}(o,u)}(q')$ is only determined by positive path flows $q(P^{l}(v_1, v_2))$ with $o \preceq_l v_1 \preceq_l v_2 \preceq_l u$. We further note that $\Pi_{P^{l}(o,u)}(0) = 0$ and, by Lemma 3.4 (e), $\Pi_{P^{l}(o,u)}(q) \geq 0$ holds because $\pi_o \geq \pi_u$. Consequently, due to Lemma 3.4 (a) and 3.4 (b), we can decrease path flows $q(P^{l}(v_1, v_2))$ with $o \preceq_l v_1 \preceq_l v_2 \preceq_l u$ to yield flow q' such that $\Pi_{P^{l}(o,u)}(q') = \Pi_{P^{l}(o,u)}(q)$ holds. These flow modifications only decrease the nomination at entries and exits in $V^{l}(o, u)$. Thus, Lemma 3.4 (d) implies the Properties (9a)–(9d). We note that we have not changed an arc flow of $A^{l}(x, w)$ in the modifications of Step 1.

Now it is left to show that we can modify the flow q' and the corresponding nomination ℓ' such that, additionally, Property (9e) is satisfied. To this end, we assume that an arc $a \in A^{l}(x, w)$ with $q_a < 0$ exists. Otherwise the claim follows directly from Lemma 3.4.

Step 2: If an arc $a \in A^{l}(x, w)$ with $q_a < 0$ exists, then, $\Pi_{P^{l}(x,w)}(q') > \Pi_{P^{l}(x,w)}(q)$ holds. Let $u \in V^{l}(x, w)$ be a node such that $q_{a'} \geq 0$ holds for all $a' \in A^{l}(x, u)$ and $|V^{l}(x, u)|$ is maximal. Given the flow decomposition, we then know that we have not modified arc flows on $A^{l}(x, u)$. Thus, $\Pi_{P^{l}(x,u)}(q') = \Pi_{P^{l}(x,u)}(q)$ and $\Pi_{P^{l}(u,w)}(q') > \Pi_{P^{l}(u,w)}(q)$ hold. Furthermore, $\Pi_{P^{l}(u,w)}(0) = 0$ and $\Pi_{P^{l}(u,w)}(q) \geq 0$ are valid. The latter is satisfied due to $\pi_w \leq \pi_u$ and Lemma 3.4 (e). Similarly to Step 1, the potential difference $\Pi_{P^{l}(u,w)}(q')$ is only determined by positive path flows $q(P^{l}(v_1, v_2))$ with $u \preceq_l v_1 \preceq_l v_2 \preceq_l w$. Due to Lemma 3.4, we can again decrease path flows $q(P^{l}(v_1, v_2))$ for $u \preceq_l v_1 \preceq_l v_2 \preceq_l w$ such that $\Pi_{P^{l}(u,w)}(q') = \Pi_{P^{l}(u,w)}(q)$ holds and Property (9e) is satisfied. Furthermore, this modification does not affect any of Properties (9b)–(9d). Since we only decrease the nomination at entries and exits, Property (9a) is also satisfied.

In total, we can modify nomination ℓ and $q(\ell)$ by repeatedly applying Step 1 and 2 such that ℓ' and the corresponding $q(\ell')$ satisfy Properties (9). $\square$

The same result can be established for the symmetric situation.

Corollary 4.5. *Given $\ell \in N \setminus \{0\}$ with flow $q = q(\ell)$, let $o \in V_+$ be an entry with highest potential and w a flow-meeting point with lowest potential. Furthermore, assume that $P^l(o, w)$ and $P^r(o, w)$ are directed paths. Then, for a given $x \in V^r(o, w)$, a nomination $\ell' \in N$ exists such that the following properties hold (with $q' = q(\ell')$):*

$$\ell' \le \ell, \tag{12a}$$

$$0 \le q'_a \text{ for all } a \in A^r(o, w), \tag{12b}$$

$$q'_a = q_a \text{ for all } a \in A^l(o, w), \tag{12c}$$

$$\Pi_{P^r(o,x)}(q') = \Pi_{P^r(o,x)}(q) \ge 0, \tag{12d}$$

$$\Pi_{P^r(o,w)}(q') = \Pi_{P^r(o,w)}(q). \tag{12e}$$

Lemma 4.6. *Given $\ell \in N \setminus \{0\}$ with flow $q = q(\ell)$, let $o \in V_+$ be an entry with highest potential and w a flow-meeting point with lowest potential. Furthermore, assume that $P^l(o, w)$ and $P^r(o, w)$ are directed paths. Then, for given $o \preceq_l x \preceq_l y \preceq_l w$ with $\Pi_{P^l(x,y)}(q) \ge 0$, a nomination $\ell' \in N$ with $q' = q(\ell')$ exists such that Properties (9a) and (9b) are satisfied and $\Pi_{P^l(x,y)}(q') = \Pi_{P^l(x,y)}(q) \ge 0$ holds.*

Proof. In analogy to the proof of Lemma 4.4, we consider a flow decomposition of Lemma 3.1. Furthermore, for every arc $a \in A^l(o, w)$, we set the flow $q(P^l(v_1, v_2)) = 0$ if $\chi_a(P^l(v_1, v_2)) = -1$ holds. Consequently, the modified flow q', given as in (11), and the corresponding nomination ℓ' satisfy (9a) and (9b). By this modification, we increase the potential difference only if an arc in $P^l(o, w)$ with negative flow in q exists.

If an arc $a \in A^l(o, x)$ with $q_a < 0$ exists, we apply Step 1 of the proof of Lemma 4.4, where we do not change the flow on any arc of $P^l(x, w)$. On the other hand, if an arc $a \in A^l(y, w)$ with $q_a < 0$ exists, we apply Step 2, where we do not change the flow on any arc of $P^l(o, y)$. If an arc $a \in A^l(x, y)$ with $q_a < 0$ exists, then, $\Pi_{P^l(x,y)}(q') > \Pi_{P^l(x,y)}(q) \ge 0$ holds. Due to Lemma 3.4 (a), 3.4 (b), and $\Pi_{P^l(x,y)}(0) = 0$, we can decrease path flows $q(P^l(v_1, v_2))$ such that $\Pi_{P^l(x,y)}(q') = \Pi_{P^l(x,y)}(q)$ and Properties (9a) and (9b) are still satisfied. This modification possibly decreases the potential differences $\Pi_{P^l(o,x)}(q')$ and $\Pi_{P^l(y,w)}(q')$. As a consequence of Lemma 3.4, we deduce that $\Pi_{P^l(o,w)}(q') \le \Pi_{P^r(o,w)}(q')$.

If $\Pi_{P^l(o,w)}(q') < \Pi_{P^r(o,w)}(q')$ is satisfied, q' is not feasible. However, this can be easily fixed. Since the arc flow of any $a \in A^r(o, w)$ stays unchanged, $\Pi_{P^r(o,w)}(q') = \Pi_{P^r(o,w)}(q) \ge 0$ holds as a consequence of Lemma 3.4 (e). Since Property (9b) is satisfied for the modified flow q', we deduce that $\Pi_{P^l(o,w)}(q') \ge 0$. It follows that $0 = \Pi_{P^r(o,w)}(0) \le \Pi_{P^l(o,w)}(q') < \Pi_{P^r(o,w)}(q')$. By Lemma 3.4, we can decrease $q(P^r(v_1, v_2))$ such that $\Pi_{P^l(o,w)}(q') = \Pi_{P^r(o,w)}(q')$ holds. Furthermore, $\Pi_{P^l(x,y)}(q') = \Pi_{P^l(x,y)}(q)$ and Properties (9a) and (9b) still hold. $\square$

Analogously, we derive the symmetric result.

Corollary 4.7. *Given $\ell \in N \setminus \{0\}$ with flow $q = q(\ell)$, let $o \in V_+$ be an entry with highest potential and w a flow-meeting point with lowest potential. Furthermore, assume that $P^l(o, w)$ and $P^r(o, w)$ are directed paths. Then, for given $o \preceq_r x \preceq_r y \preceq_r w$ with $\Pi_{P^r(x,y)}(q) \ge 0$, a nomination $\ell' \in N$ with $q' = q(\ell')$ exists such that Properties (12a) and (12b) are satisfied and $\Pi_{P^l(x,y)}(q') = \Pi_{P^l(x,y)}(q) \ge 0$ holds.*

As a final auxiliary result, we give a sufficient condition for the existence of a unique flow-meeting point.

Lemma 4.8. Given $\ell \in N \setminus \{0\}$ with flow $q = q(\ell)$, let $o \in V_+$ be an entry with highest potential and let $w \in V \setminus \{o\}$ be an arbitrary node. Furthermore, assume that $P^{\mathrm{l}}(o, w)$ and $P^{\mathrm{r}}(o, w)$ are directed paths. If $q_a \geq 0$ for all $a \in A = A^{\mathrm{l}}(o, w) \cup A^{\mathrm{r}}(o, w)$, then there is a unique flow-meeting point x. Furthermore, $x \in V^{\mathrm{l}}(o, w)$ holds.

Proof. Since $q \geq 0$, it holds $\pi_w \leq \pi_v$ for all $v \in V$. Let $x \in V^{\mathrm{l}}(o, w)$ be such that $q_a = 0$ holds for all $a \in A^{\mathrm{l}}(x, w)$ and $|V^{\mathrm{l}}(x, w)|$ is maximal. By construction of x, it is the only flow-meeting point and it may hold $x = w$. $\square$

Recall that it is sufficient to solve Problem (3) for each fixed node pair $(w_1, w_2) \in V^2$ and then check Inequality (4) to decide the feasibility of a booking. We now combine the previous results to show that an optimal solution of Problem (3) with at most one flow-meeting point exists.

Theorem 4.9. Let b be a booking and $(w_1, w_2) \in V^2$ a fixed pair of nodes. Then, there is an optimal solution of Problem (3) that has at most one flow-meeting point w.

Proof. Let (ℓ, q, π) be an optimal solution of (3). Choose an entry $o \in V_+$ with highest potential and a flow-meeting point w with lowest potential. Due to Lemma 4.3, $\pi_w \leq \pi_v$ holds for all $v \in V$. Without loss of generality, we assume that $P^{\mathrm{l}}(o, w)$ and $P^{\mathrm{r}}(o, w)$ are directed.

The zero nomination corresponds to a feasible point that satisfies the claim and $\pi_{w_1} - \pi_{w_2} = 0$. Thus, we can assume that

$$\pi_{w_1} - \pi_{w_2} > 0 \tag{13}$$

holds. If there is only one flow-meeting point, we are done. Hence, we now additionally assume that ℓ admits at least two different flow-meeting points.

Case 1: $w_1 \in V^{\mathrm{l}}(o, w)$ and $w_2 \in V^{\mathrm{r}}(o, w)$ holds. Thus, we can equivalently reformulate (13) as

$$0 < \pi_{w_1} - \pi_{w_2} = -\Pi_{P^{\mathrm{l}}(o, w_1)}(q) + \Pi_{P^{\mathrm{r}}(o, w_2)}(q).$$

We now apply Lemma 4.4 with $x = w_1$, which possibly decreases $\Pi_{P^{\mathrm{l}}(o, w_1)}(q)$ and does not change $\Pi_{P^{\mathrm{r}}(o, w_2)}(q)$. Then, we apply Corollary 4.5 with $x = w_2$, which does not change $\Pi_{P^{\mathrm{l}}(o, w_1)}(q)$ and possibly decreases $\Pi_{P^{\mathrm{r}}(o, w_2)}(q)$. Consequently, the obtained nomination ℓ' and the corresponding flow $q' = q(\ell')$ are still optimal, since the modifications only possibly increase the objective function value $\pi_{w_1} - \pi_{w_2}$. Thus, (13) is satisfied by $q(\ell') \geq 0$. The claim then follows by Lemma 4.8.

Case 2: $w_1 \in V^{\mathrm{r}}(o, w)$ and $w_2 \in V^{\mathrm{l}}(o, w)$ holds. The claim follows in analogy to Case 1.

Case 3: $w_1, w_2 \in V^{\mathrm{l}}(o, w)$ and $w_1 \preceq_{\mathrm{l}} w_2$. In this case, (13) reads

$$0 < \pi_{w_1} - \pi_{w_2} = \Pi_{P^{\mathrm{l}}(w_1, w_2)}(q).$$

We first apply Corollary 4.5 with $x = w$, which does not change $\Pi_{P^{\mathrm{l}}(w_1, w_2)}(q)$. Thus, (13) is still satisfied and $q'_a \geq 0$ holds for every $a \in P^{\mathrm{r}}(o, w)$. We now apply Lemma 4.6 with $x = w_1$ and $y = w_2$, which does not change the objective value $\pi_{w_1} - \pi_{w_2} = \Pi_{P^{\mathrm{l}}(w_1, w_2)}(q)$. Consequently, $q' \geq 0$ holds and (13) is still satisfied. The claim then again follows from Lemma 4.8.

Case 4: $w_1, w_2 \in V^{\mathrm{r}}(o, w)$ and $w_1 \preceq_{\mathrm{r}} w_2$. The claim follows in analogy to Case 3.

Case 5: $w_1, w_2 \in V^{\mathrm{l}}(o, w)$ and $w_2 \preceq_{\mathrm{l}} w_1$. Inequality (13) then reads

$$0 < \pi_{w_1} - \pi_{w_2} = -\Pi_{P^{\mathrm{l}}(w_2, w_1)}(q).$$

We first apply Corollary 4.5 with $x = w$, which does not change $\Pi_{P^{\mathrm{l}}(w_2, w_1)}(q)$. Thus, (13) is still satisfied and $q'_a \geq 0$ for every $a \in P^{\mathrm{r}}(o, w)$. Now take $u \in V^{\mathrm{l}}(o, w)$

such that $q'_a \geq 0$ for all $a \in A^{\mathrm{l}}(u, w)$ and $|A^{\mathrm{l}}(u, w)|$ is maximal. If $u \in V^{\mathrm{l}}(o, w_2)$, then $q'_a \geq 0$ for all $a \in A^{\mathrm{l}}(w_2, w_1)$. Thus, $\Pi_{P^{\mathrm{l}}(w_2, w_1)}(q) \geq 0$ also holds, which contradicts (13). Hence, we conclude that $u \in V^{\mathrm{l}}(w_2, w) \setminus \{w_2\}$. By Lemma 3.1 and the construction of u, we deduce that for $a \in A^{\mathrm{l}}(u, w)$ the flow is given by

$$q'_a = \sum_{P \in \bar{\mathcal{P}}^{\mathrm{l}}_\ell} q(P), \quad \bar{\mathcal{P}}^{\mathrm{l}}_\ell := \left\{ P \in \mathcal{P}_\ell \colon P \subseteq P^{\mathrm{l}}(u, w), \chi_a(P) = 1 \right\}.$$

We now set the flow $q(P^{\mathrm{l}}) = 0$ for $P^{\mathrm{l}} \subseteq P^{\mathrm{l}}(u, w)$ and $\chi_a(P^{\mathrm{l}}) = 1$. By this modification, we have possibly decreased $\Pi_{P^{\mathrm{l}}(w_2, w_1)}$ and thus also $\Pi_{P^{\mathrm{l}}(o, w)}$. In particular, (13) is still satisfied. Lemma 3.4 (d) implies

$$\Pi_{P^{\mathrm{l}}(o, w)}(q') = \Pi_{P^{\mathrm{l}}(o, u)}(q') + \Pi_{P^{\mathrm{l}}(u, w)}(q').$$

After modification, we have $\Pi_{P^{\mathrm{l}}(u, w)}(q') = \Pi_{P^{\mathrm{l}}(u, w)}(0) = 0$ and $\Pi_{P^{\mathrm{l}}(o, u)}(q') = \Pi_{P^{\mathrm{l}}(o, u)}(q)$. By Lemma 3.4 (e) and $\pi_o \geq \pi_u$, $\Pi_{P^{\mathrm{l}}(o, u)}(q)$ is nonnegative. We deduce that $\Pi_{P^{\mathrm{l}}(o, u)}(q') \geq 0$. Given Lemma 3.4 (a) and 3.4 (b), we can now decrease path flows $q(P^{\mathrm{r}})$ such that $\Pi_{P^{\mathrm{l}}(o, w)}(q') = \Pi_{P^{\mathrm{r}}(o, w)}(q')$ holds. After this modification, (13) is still satisfied and its value is possibly increased, i.e., the objective function value $\pi_{w_1} - \pi_{w_2}$ is possibly increased by the modifications. Consequently, the obtained solution is still optimal. Moreover, w is now connected to a flow-meeting point in $V^{\mathrm{l}}(o, u)$ because $q'_a = 0$ holds for all $a \in P^{\mathrm{l}}(u, w)$. Consequently, for nomination ℓ' a flow-meeting point in $V^{\mathrm{l}}(o, u)$ with lowest potential exists. We now repeat this procedure until either the claim holds or a new flow-meeting point with lowest potential is an element of $V^{\mathrm{l}}(o, w_1)$. Then, we apply the respective case of Cases 1–4.

Case 6: $w_1, w_2 \in V^{\mathrm{r}}(o, w)$ and $w_2 \preceq_{\mathrm{r}} w_1$. The claim follows in analogy to Case 5. $\qquad\square$

As a direct consequence of this result, we deduce the following corollary.

Corollary 4.10. Let b be a booking and $(w_1, w_2) \in V^2$ a fixed pair of nodes. Then, there exist nodes $(o, w) \in V_+ \times V_-$ and an optimal solution (ℓ, q, π) of Problem (3) with $q \geq 0$, if we assume that $P^{\mathrm{l}}(o, w)$ and $P^{\mathrm{r}}(o, w)$ are directed paths.

The previous result implies that when determining potential-difference maximizing nominations solving Problem (3) for fixed $(w_1, w_2) \in V^2$, we can additionally restrict the search space by iteratively considering $(o, w) \in V_+ \times V_-$ and imposing that there is flow from o to w. This is further formalized and exploited in the next section.

5. Structure of Potential-Difference Maximizing Nominations

In this section, we fix $(w_1, w_2) \in V^2$ and show that there exist optimal solutions of (3) with additional structure that allows to reduce the dimension of the problem. Based on the results of Section 4, in particular, Corollary 4.10, we next show that (4) can be decided by considering the following variant of Problem (3) for every $(o, w) \in V_+ \times V_-$:

$$\bar{\varphi}^{ow}_{w_1 w_2}(b) := \max_{\ell, q, \pi} \quad \pi_{w_1} - \pi_{w_2} \tag{14a}$$

$$\text{s.t.} \sum_{a \in \delta^{\mathrm{out}}(u)} q_a - \sum_{a \in \delta^{\mathrm{in}}(u)} q_a = \sigma_u \ell_u, \quad u \in V,$$

$$0 \leq \ell_u \leq b_u, \quad u \in V,$$

$$\pi_u - \pi_v = \Lambda_a q_a |q_a|, \quad a \in A', \tag{14b}$$

$$q_a \geq 0, \quad a \in A', \tag{14c}$$

where b is a booking and A' is obtained from A by orienting all arcs from o to w. Note that in addition to the constraints of (3), we now also impose nonnegative flow from o to w, thus effectively reducing the feasible domain of the problem.

Theorem 5.1. Let b be a booking, then

$$\varphi_{w_1 w_2}(b) = \max_{(o,w) \in V_+ \times V_-} \bar{\varphi}^{ow}_{w_1 w_2}(b)$$

holds. Furthermore, the optimal values are finite and attained.

Proof. First, observe that ℓ is bounded in (3). As a consequence of Theorem 7.1 of Chapter 7 in [21], an optimal solution of (3) with finite optimal value exists, i.e., $\varphi_{w_1 w_2}(b) < \infty$.

Let (ℓ, q, π) be an optimal solution corresponding to $\max_{(o,w) \in V_+ \times V_-} \bar{\varphi}^{ow}_{w_1 w_2}(b)$. First, observe that the arc orientation does not play any role in Problem (3). If an arc has a different orientation, we just switch the sign of the corresponding flow variable. Thus, we assume w.l.o.g. that $P^l(o,w)$ and $P^r(o,w)$ are directed paths in the given instance of (3). Consequently, (ℓ, q, π) is feasible for (3). Thus,

$$\varphi_{w_1 w_2}(b) \geq \max_{(o,w) \in V_+ \times V_-} \bar{\varphi}^{ow}_{w_1 w_2}(b).$$

The other inequality follows directly from Corollary 4.10. $\square$

As a consequence, the feasibility of a booking can be characterized using Problem (14) as follows.

Corollary 5.2. A booking b is feasible if and only if for every pair $(w_1, w_2) \in V^2$ and for every $(o,w) \in V_+ \times V_-$,

$$\bar{\varphi}^{ow}_{w_1 w_2}(b) \leq \pi^+_{w_1} - \pi^-_{w_2}. \tag{15}$$

We now further analyze the structure of optimal solutions of (14) for fixed $(o,w) \in V_+ \times V_-$ and given $(w_1, w_2) \in V^2$ w.r.t. their respective position in the cycle. Without loss of generality, we assume that $P^l(o,w)$ and $P^r(o,w)$ are directed paths.

5.1. **Nodes on Different Sides of G.** Assume that $w_1 \in P^l(o,w)$ and $w_2 \in P^r(o,w)$ holds. We show that an optimal solution (ℓ, q, π) of (14) exists that additionally satisfies the following properties:

(a) Two entries $s^l_1, s^l_2 \in V^l_+(o,w)$ with $s^l_1 \preceq_l s^l_2$ exist such that

$$\ell_v = 0, \quad v \in \left(V^l_+(o, s^l_1) \cup V^l_+(s^l_2, w)\right) \setminus \{o, s^l_1, s^l_2\},$$
$$\ell_v = b_v, \quad v \in V^l_+(s^l_1, s^l_2) \setminus \{s^l_1, s^l_2\}.$$

(b) An exit $t^l_1 \in V^l_-(o,w)$ exists such that

$$\ell_v = 0, \quad v \in V^l_-(o, t^l_1) \setminus \{t^l_1\},$$
$$\ell_v = b_v, \quad v \in V^l_-(t^l_1, w) \setminus \{t^l_1\}.$$

(c) An entry $s^r_1 \in V^r_+(o,w)$ exists such that

$$\ell_v = b_v, \quad v \in V^r_+(o, s^r_1) \setminus \{s^r_1\},$$
$$\ell_v = 0, \quad v \in V^r_+(s^r_1, w) \setminus \{s^r_1\}.$$

(d) Two exits $t^r_1, t^r_2 \in V^r_-(o,w)$ with $t^r_1 \preceq_r t^r_2$ exist such that

$$\ell_v = 0, \quad v \in \left(V^r_-(o, t^r_1) \cup V^r_-(t^r_2, w)\right) \setminus \{t^r_1, t^r_2, w\},$$
$$\ell_v = b_v, \quad v \in V^r_-(t^r_1, t^r_2) \setminus \{t^r_1, t^r_2\}.$$

FIGURE 3. Configuration of s and t nodes if $w_1 \in P^{\mathrm{l}}(o,w)$ and $w_2 \in P^{\mathrm{r}}(o,w)$. Boxes qualitatively illustrate the amount of the booking that is nominated.

A possible configuration of nodes $o, w_1, s_1^l, s_2^l, t_1^l, w, w_2, s_1^r, t_1^r, t_2^r$ is given in Figure 3. To show the existence of such a solution, we introduce a bilevel problem, where the lower level is given by (14) and the upper level chooses, among all lower-level optimal solutions, one with the additional structure. It is given by

$$\min_{x,y} \quad f_1(\ell, x^{\preceq_1}, x^{\succeq_1}) + f_2(\ell, y^{\preceq_1}) + f_3(\ell, x^{\succeq_r}) + f_4(\ell, y^{\preceq_r}, y^{\succeq_r}) \tag{16a}$$

$$\text{s.t.} \quad (\ell, q, \pi) \text{ solves } (14),$$

$$Mx_v^{\preceq_1} \geq \sum_{u \in V_+^1(o,v) \setminus \{o\}} \ell_u, \quad v \in V_+^1(o,w) \setminus \{o\}, \tag{16b}$$

$$Mx_v^{\succeq_1} \geq \sum_{u \in V_+^1(v,w)} \ell_u, \quad v \in V_+^1(o,w) \setminus \{o\}, \tag{16c}$$

$$My_v^{\preceq_1} \geq \sum_{u \in V_-^1(o,v)} \ell_u, \quad v \in V_-^1(o,w), \tag{16d}$$

$$Mx_v^{\succeq_r} \geq \sum_{u \in V_+^r(v,w)} \ell_u, \quad v \in V_+^r(o,w), \tag{16e}$$

$$My_v^{\preceq_r} \geq \sum_{u \in V_-^r(o,v)} \ell_u, \quad v \in V_-^r(o,w) \setminus \{w\}, \tag{16f}$$

$$My_v^{\succeq_r} \geq \sum_{u \in V_-^r(v,w) \setminus \{w\}} \ell_u, \quad v \in V_-^r(o,w) \setminus \{w\}, \tag{16g}$$

$$x_v^{\preceq_1}, x_v^{\succeq_1}, x_v^{\succeq_r}, y_v^{\preceq_1}, y_v^{\preceq_r}, y_v^{\succeq_r} \in \{0,1\}, \quad v \in V, \tag{16h}$$

where $M = \sum_{u \in V} b_u$ and $f_1, \ldots, f_4$ are continuous functions that we specify later. By Constraints (16b) and (16c), the variables $x_v^{\preceq_1}$ and $x_v^{\succeq_1}$ model the existence of an active entry before and after v on P^{l}. Similarly, Constraints (16d) ensure that $y_v^{\preceq_1}$

determines the existence of an active exit before v on P^l. An analogous interpretation can be given for Constraints (16e)–(16g) and the variables $x^{\succeq_r}$, $y^{\preceq_r}$, $y^{\succeq_r}$. Then, the optimal value function reformulation of (16) is given by

$$\min_{\ell,q,\pi,x,y} \quad f_1(\ell, x^{\preceq_l}, x^{\succeq_l}) + f_2(\ell, y^{\preceq_l}) + f_3(\ell, x^{\succeq_r}) + f_4(\ell, y^{\preceq_r}, y^{\succeq_r}) \tag{17a}$$

$$\text{s.t.} \quad (1a), (3b), (14b), (14c), \tag{17b}$$

$$(16b)–(16h), \tag{17c}$$

$$\pi_{w_1} - \pi_{w_2} \geq \bar{\varphi}^{ow}_{w_1 w_2}(b). \tag{17d}$$

Here, Constraint (17b) determines the feasible domain of Problem (14) and Constraint (17d) guarantees feasible points with a potential difference of at least $\bar{\varphi}^{ow}_{w_1 w_2}(b)$. Thus, we only consider optimal solutions of (14). We denote by

$$z := (\ell, q, \pi, x^{\preceq_l}, x^{\succeq_l}, x^{\succeq_r}, y^{\preceq_l}, y^{\preceq_r}, y^{\succeq_r})$$

a feasible point of (17). In particular, we have the following result.

Lemma 5.3. Let z be feasible for (17), then (ℓ, q, π) is an optimal solution of (14). Conversely, every optimal solution of (14) can be extended to a feasible point of (17).

Proof. The first statement follows from the previous discussion. For the converse, let an optimal solution (ℓ, q, π) of (14) be given. We construct a solution z as follows:

$$x^{\preceq_l}_v = 1, \quad \text{if and only if an active } u \in V^l_+(o, v) \setminus \{o\} \text{ exists,}$$

$$x^{\succeq_l}_v = 1, \quad \text{if and only if an active } u \in V^l_+(v, w) \setminus \{o\} \text{ exists,}$$

$$y^{\preceq_l}_v = 1, \quad \text{if and only if an active } u \in V^l_-(o, v) \text{ exists,}$$

$$x^{\succeq_r}_v = 1, \quad \text{if and only if an active } u \in V^r_+(v, w) \text{ exists,}$$

$$y^{\preceq_r}_v = 1, \quad \text{if and only if an active } u \in V^r_-(o, v) \setminus \{w\} \text{ exists,}$$

$$y^{\succeq_r}_v = 1, \quad \text{if and only if an active } u \in V^r_-(v, w) \setminus \{w\} \text{ exists.} \qquad \square$$

We now specify the parts of the objective function of (17) and prove connections between these functions and the stated Properties (a)–(d). We discuss and prove the results for f_1 and f_2 in detail, whereas we only state the results for f_3 and f_4, since they are very similar. The proofs for the results concerning f_3 and f_4 can be found in Appendix A.

For the following proofs, we make use of structures resulting from the negation of Properties (a)–(d) on Page 14. More precisely, we observe that

- if Property (a) does not hold, then there are $u_1, u_2, u_3 \in V^l_+(o, w) \setminus \{o\}$ with $u_1 \prec_l u_2 \prec_l u_3$ such that $\ell_{u_1} > 0, \ell_{u_2} < b_{u_2}$, and $\ell_{u_3} > 0$,
- if Property (b) does not hold, then there are $u_1, u_2 \in V^l_-(o, w)$ with $u_1 \prec_l u_2$ such that $\ell_{u_1} > 0$ and $\ell_{u_2} < b_{u_2}$,
- if Property (c) does not hold, then there are $u_1, u_2 \in V^r_+(o, w)$ with $u_1 \prec_l u_2$ such that $\ell_{u_1} < b_{u_1}$ and $\ell_{u_2} > 0$, and
- if Property (d) does not hold, then there are $u_1, u_2, u_3 \in V^r_-(o, w) \setminus \{o\}$ with $u_1 \prec_l u_2 \prec_l u_3$ such that $\ell_{u_1} > 0, \ell_{u_2} < b_{u_2}$, and $\ell_{u_3} > 0$.

Consider, for instance, the negation of Property (a). It is always possible to satisfy the first part of the property, i.e., there exist two entries $s^l_1, s^l_2 \in V^l_+(o, w)$ with $s^l_1 \preceq_l s^l_2$ such that

$$\ell_v = 0, \quad v \in \left(V^l_+(o, s^l_1) \cup V^l_+(s^l_2, w)\right) \setminus \{o, s^l_1, s^l_2\}.$$

To achieve this, we simply choose the first and the last active entry node on the left side of the cycle, i.e., $s^l_1 \preceq_l s^l_2 \in V^l_+(o, w)$ such that $\ell_{s^l_1} > 0$, $\ell_{s^l_2} > 0$, and $\ell_u = 0$

for all $u \in V^1_+(o, s^l_1) \cup V^1_+(s^l_2, w) \setminus \{o, s^l_1, s^l_2\}$. Now, if Property (a) is not satisfied, there has to exist another node $u \in V^1_+(s^l_1, s^l_2) \setminus \{s^l_1, s^l_2\}$ with $\ell_u < b_u$, which shows the claim. Analogously, we can obtain the remaining statements.

Lemma 5.4. Let z be feasible for (17) and

$$f_1(\ell, x^{\preceq_1}, x^{\succeq_1}) := \sum_{\substack{i,j \in V^1_+(o,w) \setminus \{o\}: \\ i \preceq_1 j}} x^{\preceq_1}_i x^{\succeq_1}_j \sum_{v \in V^1_+(i,j) \setminus \{i,j\}} (b_v - \ell_v). \tag{18}$$

Then, there exists $(x^{\preceq_1}, x^{\succeq_1})$ such that $f_1(\ell, x^{\preceq_1}, x^{\succeq_1}) = 0$ holds if and only if ℓ satisfies Property (a).

Proof. Let z be feasible for (17). For $i, j \in V^1_+(o, w) \setminus \{o\}$ where $i \preceq_1 j$,

$$x^{\preceq_1}_i x^{\succeq_1}_j \sum_{v \in V^1_+(i,j) \setminus \{i,j\}} (b_v - \ell_v) \geq 0$$

holds. Assume now that Property (a) does not hold. Consequently, there are $u_1, u_2, u_3 \in V^1_+(o, w) \setminus \{o\}$ with $u_1 \prec_1 u_2 \prec_1 u_3$ such that $\ell_{u_1} > 0, \ell_{u_2} < b_{u_2}$, and $\ell_{u_3} > 0$ hold. Thus, $x^{\preceq_1}_{u_1} = 1 = x^{\succeq_1}_{u_3}$ and $\sum_{v \in V^1_+(u_1,u_3) \setminus \{u_1,u_3\}}(b_v - \ell_v) > 0$, therefore

$$x^{\preceq_1}_{u_1} x^{\succeq_1}_{u_3} \sum_{v \in V^1_+(u_1,u_3) \setminus \{u_1,u_3\}} (b_v - \ell_v) > 0$$

holds. Consequently, $f_1(\ell, x^{\preceq_1}, x^{\succeq_1}) > 0$.

If ℓ satisfies Property (a), then we set $x^{\preceq_1}_u = 0$ for all $u \in P^1(o, s^l_1) \setminus \{s^l_1\}$. Otherwise, we set $x^{\preceq_1}_u = 1$. Additionally, we set $x^{\succeq_1}_u = 1$ for all $u \in P^1(o, s^l_2)$ and otherwise we set $x^{\succeq_1}_u = 0$. Consequently, for $i \in V^1_+(o, s^l_1) \setminus \{s^l_1\}$ or $j \in V^1_+(s^l_2, w) \setminus \{s^l_2\}$, we have $x^{\preceq_1}_i x^{\succeq_1}_j = 0$ and for $i, j \in V^1_+(s^l_1, s^l_2)$,

$$\sum_{v \in V^1_+(i,j) \setminus \{i,j\}} (b_v - \ell_v) = 0$$

holds due to Property (a). Consequently, $f_1(\ell, x^{\preceq_1}, x^{\succeq_1}) = 0$ holds. □

Lemma 5.5. Let z be feasible for (17) and

$$f_2(\ell, y^{\preceq_1}) := \sum_{i \in V^1_-(o,w)} y^{\preceq_1}_i \sum_{v \in V^1_-(i,w) \setminus \{i\}} (b_v - \ell_v). \tag{19}$$

Then, there exists $y^{\preceq_1}$ such that $f_2(\ell, y^{\preceq_1}) = 0$ holds if and only if ℓ satisfies Property (b).

Proof. Let z be feasible for (17). For $i \in V^1_-(o, w)$,

$$y^{\preceq_1}_i \sum_{v \in V^1_-(i,w) \setminus \{i\}} (b_v - \ell_v) \geq 0$$

holds. Assume now that Property (b) does not hold. Consequently, there are $u_1, u_2 \in V^1_-(o, w)$ with $u_1 \prec_1 u_2$ such that $\ell_{u_1} > 0$ and $\ell_{u_2} < b_{u_2}$ hold. Thus, $y^{\preceq_1}_{u_1} = 1$ and

$$\sum_{v \in V^1_-(u_1,w) \setminus \{u_1\}} (b_v - \ell_v) > 0$$

holds, which implies $f_2(\ell, y^{\preceq_1}) > 0$.

If ℓ satisfies Property (b), then we set $y_u^{\preceq_l} = 0$ for all $u \in V_-^l(o, t_1^l) \setminus \{t_1^l\}$. Otherwise, we set $y_u^{\preceq_l} = 1$. Furthermore, for $i \in V_-^l(t_1^l, w)$,

$$\sum_{v \in V_-^l(i,w) \setminus \{i\}} (b_v - \ell_v) = 0$$

holds due to Property (b). Consequently, $f_2(\ell, y^{\preceq_l}) = 0$ holds. $\square$

Lemma 5.6. Let z be feasible for (17) and

$$f_3(\ell, x^{\succeq_r}) := \sum_{i \in V_+^r(o,w)} x_i^{\succeq_r} \sum_{v \in V_+^r(o,i) \setminus \{i\}} (b_v - \ell_v). \tag{20}$$

Then, there exists $x^{\succeq_r}$ such that $f_3(\ell, x^{\succeq_r}) = 0$ holds if and only if ℓ satisfies Property (c).

Lemma 5.7. Let z be feasible for (17) and

$$f_4(\ell, y^{\preceq_r}, y^{\succeq_r}) = \sum_{\substack{i,j \in V_-^r(o,w) \setminus \{w\}: \\ i \preceq_r j}} y_i^{\preceq_r} y_j^{\succeq_r} \sum_{v \in V_-^r(i,j) \setminus \{i,j\}} (b_v - \ell_v). \tag{21}$$

Then, there exists $(y^{\preceq_r}, y^{\succeq_r})$ such that $f_4(\ell, y^{\preceq_r}, y^{\succeq_r}) = 0$ holds if and only if ℓ satisfies Property (d).

In the following, we consider $f_1, \ldots, f_4$ as specified in Lemmas 5.4–5.7. As a next step, we show that changing the nomination ℓ on the boundary nodes of Properties (a)–(d) does not affect the values of $f_1, \ldots, f_4$, since the corresponding products of binary variables are zero.

Lemma 5.8. Let z be an optimal solution of (17) and let $u_1, u_3 \in V_+^l(o, w)$ with $u_1 \preceq_l u_3$ be nodes such that $\ell_{u_1} > 0$, $\ell_{u_3} > 0$, $\ell_u = 0$ for all $u \in (V_+^l(o, u_1) \cup V_+^l(u_3, w)) \setminus \{o, u_1, u_3\}$. Suppose further that z' is feasible for (17) with

$$\ell'_{u_1} > 0, \quad \ell'_{u_3} > 0, \quad \ell'_u = \ell_u, \quad u \in V_+^l(o, w) \setminus \{o, u_1, u_3\}.$$

Then, $f_1(\ell', x^{\preceq_l}, x^{\succeq_l}) = f_1(\ell, x^{\preceq_l}, x^{\succeq_l})$ holds.

Proof. Optimality of z and the choice of u_1 and u_3 imply $x_u^{\preceq_l} = 0$ for all $u \in V_+^l(o, u_1) \setminus \{o, u_1\}$ and $x_u^{\succeq_l} = 0$ for all $u \in V_+^l(u_3, w) \setminus \{u_3\}$. Hence, for $i, j \in V_+^l(o, w) \setminus \{o\}$ with $i \preceq_l j$ we have

$$x_i^{\preceq_l} x_j^{\succeq_l} \sum_{v \in V_+^l(i,j) \setminus \{i,j\}} (b_v - \ell_v) = 0,$$

whenever u_1 or u_3 is in $V_+^l(i, j) \setminus \{i, j\}$, because then $x_i^{\preceq_l} x_j^{\succeq_l} = 0$. Consequently, a change of ℓ_{u_1} or ℓ_{u_3} does not change $f_1(\ell, x^{\preceq_l}, x^{\succeq_l})$. $\square$

Lemma 5.9. Let z be an optimal solution of (17) and let $v_1 \in V_-^l(o, w)$ be a node such that $\ell_{v_1} > 0$ and $\ell_v = 0$ for all $v \in V_-^l(o, v_1) \setminus \{v_1\}$. Suppose further that z' is feasible for (17) with

$$\ell'_{v_1} > 0, \quad \ell'_v = \ell_v, \quad u \in V_-^l(o, w) \setminus \{v_1, w\}.$$

Then, $f_2(\ell', y^{\preceq_l}) = f_2(\ell, y^{\preceq_l})$ holds.

Proof. Optimality of z and the choice of v_1 imply $y_v^{\preceq_1} = 0$ for all $v \in V_-^l(o, v_1) \setminus \{v_1\}$. Hence,

$$y_i^{\preceq_1} \sum_{v \in V_-^l(i, w) \setminus \{i\}} (b_v - \ell_v) = 0$$

holds whenever $v_1 \in V_-^l(i, w) \setminus \{i\}$. Thus, a change of ℓ_{v_1} does not change $f_2(\ell, y^{\preceq_1})$. $\qquad\square$

Lemma 5.10. Let z be an optimal solution of (17) and let $u_1 \in V_+^r(o, w)$ be a node such that $\ell_{u_1} > 0$ and $\ell_u = 0$ for all $u \in V_+^r(u_1, w) \setminus \{u_1\}$. Suppose further that z' is feasible for (17) with

$$\ell'_{u_1} > 0, \quad \ell'_u = \ell_u, \quad u \in V_+^r(o, w) \setminus \{o, u_1\}.$$

Then, $f_3(\ell, x^{\succeq_r}) = f_3(\tilde{\ell}, x^{\succeq_r})$ holds.

Lemma 5.11. Let z be an optimal solution of (17) and let $v_1, v_3 \in V_-^r(o, w)$ with $v_1 \preceq_r v_3$ be nodes such that $\ell_{v_1} > 0$, $\ell_{v_3} > 0$, $\ell_u = 0$ for all $u \in (V_-^r(o, v_1) \cup V_-^r(v_3, w)) \setminus \{v_1, v_3, w\}$. Suppose further that z' is feasible for (17) with

$$\ell'_{v_1} > 0, \quad \ell'_{v_3} > 0, \quad \ell'_u = \ell_u, \quad u \in V_-^r(o, w) \setminus \{v_1, v_3, w\}.$$

Then, $f_4(\ell', y^{\preceq_r}, y^{\succeq_r}) = f_4(\ell, y^{\preceq_r}, y^{\succeq_r})$ holds.

The two last proofs can again be found in Appendix A. We next show that there is an optimal solution of (14) that satisfies Properties (a)–(d). More precisely, we prove that the optimal value of (17) is zero by individually treating $f_1, \ldots, f_4$. The final result then easily follows from Lemmas 5.4–5.7.

Lemma 5.12. If z is an optimal solution of (17), then $f_1(\ell, x^{\preceq_1}, x^{\succeq_1}) = 0$ holds.

Proof. Let z be an optimal solution of (17). By contradiction, we assume that $f_1(\ell, x^{\preceq_1}, x^{\succeq_1}) > 0$ holds. Lemma 5.4 implies that ℓ does not satisfy Property (a). Consequently, there are entries $u_1, u_2, u_3 \in V_+^l(o, w) \setminus \{o\}$ with $u_1 \prec_l u_2 \prec_l u_3$ such that $\ell_{u_1} > 0, \ell_{u_2} < b_{u_2}$, and $\ell_{u_3} > 0$. If $q_a > 0$ for $a \in \delta^{\mathrm{out}}(o) \cap P^l(o, w)$, we replace $u_1 = o$. Otherwise, we choose $u_1 \neq o$ such that $\ell_u = 0$ holds for all $u \in V_+^l(o, u_1) \setminus \{o, u_1\}$ and we choose u_3 such that $\ell_u = 0$ holds for all $u \in V_+^l(u_3, w) \setminus \{u_3\}$. We now consider a flow decomposition as in Lemma 3.1. Due to $q \geq 0$, an exit $v_3 \in V_-^l(u_3, w)$ with $q(P^l(u_3, v_3)) > 0$ exists. Moreover, by the choice of u_1, there is an exit $v_1 \in V_-^l(u_1, w)$ with $\ell_v = 0$ for all $v \in V_-^l(o, v_1) \setminus \{v_1\}$ and $q(P^l(u_1, v_1)) > 0$. We need to distinguish two cases.

Case 1: $v_1 \prec_l u_2$ holds. We now decrease $q(P^l(u_3, v_3))$ by $\varepsilon > 0$ and increase $q(P^l(u_2, v_3))$ by the same amount ε. This increases the potential difference $\Pi_{P^l(o, w)}(q)$ due to $u_2 \prec_l u_3$. Thus, we decrease $q(P^l(u_1, v_1))$ by $\tilde{\varepsilon} > 0$. Due to Lemma 3.4, we can choose ε and $\tilde{\varepsilon}$ such that $\Pi_{P^l(o, w)}(q)$ stays the same as before the modification and $\ell_{u_1} > 0, \ell_{u_2} \leq b_{u_2}, \ell_{u_3} > 0, \ell_{v_1} > 0$ holds. In particular, the binary variables of z stay the same. Due to this and Lemmas 5.9–5.11, the values of f_2, f_3, and f_4 stay the same. Moreover, the modified solution satisfies Constraints (17b). Furthermore, by this modification we decrease q_a for $a \in P^l(u_1, v_1)$, increase q_a for $a \in P^l(u_2, u_3)$, and the remaining arc flows stay the same. Hence, since $u_1 \prec_l v_1 \prec_l u_2 \prec_l u_3$ and by Lemma 3.4 (d), we possibly increase the potential difference between w_1 and w_2 and Constraint (17d) is still satisfied. Consequently, z is still feasible for (17). Due to this modification, we decrease $\ell_{u_1} > 0$ and $\ell_{u_3} > 0$ and increase ℓ_{u_2}. By Lemma 5.8, considering only the decrease of ℓ_{u_1} and ℓ_{u_3} does not change the objective function value. In contrast, the increase of ℓ_{u_2} decreases

f_1 because

$$x_{u_1}^{\preceq_1} x_{u_3}^{\succeq_1} \sum_{v \in V_+^1(u_1,u_3)\setminus\{u_1,u_3\}} (b_v - \ell_v)$$

decreases. Thus, the modification decreases the objective function value, which contradicts the optimality of the original solution.

Case 2: $u_2 \prec_1 v_1$ holds. We now decrease $q(P^1(u_1,v_1))$ by $\varepsilon > 0$ and increase $q(P^1(u_2,v_1))$ by the same amount ε. This decreases the potential difference $\Pi_{P^1(o,w)}(q)$ due to $u_1 \prec_1 u_2$. Thus, we now decrease $q(P^1(u_3,v_3))$ by $\tilde{\varepsilon} > 0$ and increase $q(P^1(u_2,v_3))$ by the same amount $\tilde{\varepsilon}$, which increases the potential difference $\Pi_{P^1(o,w)}(q)$ due to $u_2 \prec_1 u_3$. Due to Lemma 3.4, we can choose ε and $\tilde{\varepsilon}$ such that $\Pi_{P^1(o,w)}(q)$ stays the same and $\ell_{u_1} > 0, \ell_{u_2} \leq b_{u_2}, \ell_{u_3} > 0$ holds. In analogy to Case 1, the function values of f_2, f_3, and f_4 stay the same and the modified solution satisfies Constraints (17b). Further, the modification only decreases q_a for $a \in P^1(u_1,u_2)$ and increases flow q_a for $a \in P^1(u_2,u_3)$. The remaining arc flows stay the same. Hence, since $u_1 \prec_1 u_2 \prec_1 u_3$ and by Lemma 3.4 (d), we possibly increase the potential difference between w_1 and w_2 and Constraint (17d) is still satisfied. Consequently, z is feasible for (17) after modification. In analogy to Case 1, the modification decreases f_1, which contradicts the optimality of the original solution. $\qquad\square$

Lemma 5.13. If z is an optimal solution of (17), then $f_2(\ell, y^{\preceq_1}) = 0$ holds.

Proof. Let z be an optimal solution of (17). By contradiction, we assume that $f_2(\ell, y^{\preceq_1}) > 0$ holds. Lemma 5.5 implies that ℓ does not satisfy Property (b). Consequently, there are exits $v_1, v_2 \in V_-^1(o,w)$ with $v_1 \prec_1 v_2$, $\ell_{v_1} > 0$ and $\ell_{v_2} < b_{v_2}$. We now choose v_1 such that $\ell_u = 0$ holds for all $u \in V_-^1(o,v_1) \setminus \{v_1\}$ and v_2 such that $\ell_u = b_u$ holds for all $u \in V_-^1(v_1,v_2) \setminus \{v_1, v_2\}$. Next, let an entry $u_1 \in V_+^1(o,w)$ be given so that $\ell_{u_1} > 0$, $\ell_u = 0$ for all $u \in V_+^1(o,u_1) \setminus \{o,u_1\}$, and in a flow decomposition as by Lemma 3.1, $q(P^1(u_1,v_1)) > 0$ holds. Due to Lemma 3.4 and $v_1 \prec_1 v_2$, we can decrease $q(P^1(u_1,v_1))$ and increase $q(P^1(u_1,v_2))$ such that $\Pi_{P^1(o,w)}(q)$ remains the same and $0 < \ell_{u_1} \leq b_{u_1}, \ell_{v_1} > 0, 0 < \ell_{v_2} \leq b_{v_2}$ hold. Thus, the binary variables of z stay the same. Furthermore, by Lemmas 5.8, 5.10, and 5.11 the values of f_1, f_3, and f_4 stay the same. The modified solution satisfies Constraints (17b) and we only decrease q_a for $a \in P^1(u_1,v_1)$ and increase q_a for $a \in P^1(v_1,v_2)$. The remaining arc flows are unchanged. Then, since $u_1 \prec_1 v_1 \prec_1 v_2$ and by Lemma 3.4 (d), Constraint (17d) is still satisfied. Consequently, z is still feasible for (17). Due to this modification, we decrease $\ell_{v_1} > 0$ and increase ℓ_{v_2}. By Lemma 5.9, considering only the decrease of ℓ_{v_1} does not change the objective function value. In contrast, the increase of ℓ_{v_2} decreases f_2 because

$$y_{v_1}^{\preceq_1} \sum_{v \in V_-^1(v_1,w)\setminus\{v_1\}} (b_v - \ell_v)$$

decreases. Thus, the modification decreases the objective function value, which contradicts the optimality of the original solution. $\qquad\square$

Lemma 5.14. If z is an optimal solution of (17), then $f_3(\ell, x^{\succeq_r}) = 0$ holds.

Lemma 5.15. If z is an optimal solution of (17), then $f_4(\ell, y^{\preceq_r}, y^{\succeq_r}) = 0$ holds.

Again, the proofs for the results concerning f_3 and f_4 can be found in Appendix A. Finally, we obtain the main structural property for nodes w_1 and w_2 on different sides of G by combining the previous lemmas.
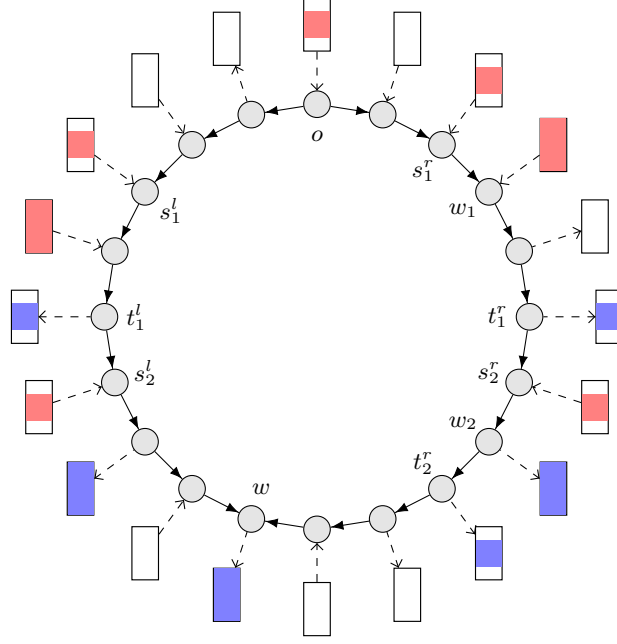
FIGURE 4. Configuration of s and t nodes with $o \preceq_{\mathrm{r}} w_1 \prec_{\mathrm{r}} w_2 \preceq_{\mathrm{r}} w$. Boxes qualitatively illustrate the amount of the booking that is nominated.

Theorem 5.16. Let $(o, w) \in V_+ \times V_-$ be fixed, $w_1 \in P^{\mathrm{l}}(o, w)$, and $w_2 \in P^{\mathrm{r}}(o, w)$. Then, an optimal solution (ℓ, q, π) of (14) exists that satisfies Properties (a)–(d).

Proof. The zero nomination is feasible for Problem (14). Furthermore, the feasible region of the latter problem is compact and thus, an optimal solution is attained. Consequently, Problem (17) has an optimal solution, which is attained. Due to Lemmas 5.12–5.15 and Lemmas 5.4–5.7, an optimal solution (ℓ, q, π, x, y) of Problem (17) exists that satisfies Properties (a)–(d). Additionally, the solution (ℓ, q, π) is also optimal for Problem (14). □

5.2. **Nodes on the Same Side of** G. Assume $w_1, w_2 \in P^{\mathrm{l}}(o, w)$ or $w_1, w_2 \in P^{\mathrm{r}}(o, w)$ holds. We can w.l.o.g. assume that $w_1, w_2 \in P^{\mathrm{r}}(o, w)$ holds. If $w_2 \prec_{\mathrm{r}} w_1$ holds, then from $q \geq 0$ in Problem (14) it follows that $\Pi_{P^{\mathrm{r}}(w_1, w_2)}(q) \leq 0$ is valid. Thus, the zero nomination is an optimal solution for Problem (14). Consequently, we now assume that $w_1 \prec_{\mathrm{r}} w_2$ holds.

We want to show that an optimal solution (ℓ, q, π) of Problem (14) exists such that Properties (a), (b), (d), and (a) w.r.t. $P^{\mathrm{r}}(o, w)$, i.e., two entries $s_1^r, s_2^r \in V_+^{\mathrm{r}}(o, w)$ with $s_1^r \preceq_{\mathrm{r}} s_2^r$ exists such that

$$\ell_v = 0, \quad v \in \left(V_+^{\mathrm{r}}(o, s_1^r) \cup V_+^{\mathrm{r}}(s_2^r, w) \right) \setminus \{o, s_1^r, s_2^r\},$$
$$\ell_v = b_v, \quad v \in V_+^{\mathrm{r}}(s_1^r, s_2^r) \setminus \{s_1^r, s_2^r\},$$

is satisfied. Figure 4 illustrates a possible node configuration. To this end, we

introduce an optimization problem similar to (16), which is given by

$$\min_{\ell,q,\pi,x,y} \quad f_1(\ell, x^{\preceq_1}, x^{\succeq_1}) + f_2(\ell, y^{\preceq_1}) + f_3(\ell, x^{\preceq_r}, x^{\succeq_r}) + f_4(\ell, y^{\preceq_r}, y^{\succeq_r}) \tag{22a}$$

$$\text{s.t.} \quad \text{(1a), (3b), (14b), (14c), (16b)–(16h), (17d),}$$

$$Mx_v^{\preceq_r} \geq \sum_{u \in V_+^r(o,v)\setminus\{o\}} \ell_u, \quad v \in V_+^r(o,v)\setminus\{o\}, \tag{22b}$$

$$x_v^{\preceq_r} \in \{0,1\}, \quad v \in V. \tag{22c}$$

Note that an analogous variant of Lemma 5.3 is also valid for Problem (22).

We specify the parts of the objective function of (22) as follows: the functions f_1, f_2, and f_4 are defined as in Lemmas 5.4, 5.5, and 5.7. The function f_3 is defined in analogy to Lemma 5.4 w.r.t. P^r. We note that f_i for $i = 1, \ldots, 4$ also inherit the corresponding properties of Lemmas 5.4–5.11. We now prove that the optimal objective value of (22) is zero.

Lemma 5.17. If z is an optimal solution of (22), then $f_1(\ell, x^{\preceq_1}, x^{\succeq_1}) = 0$ holds.

Proof. The claim follows in analogy to Lemma 5.12. In doing so, we note that the modifications in the proof of Lemma 5.12 only affect nodes of $P^l(o,w)$. Consequently, we do not change the potential difference between w_1 and w_2 due to $w_1, w_2 \in P^r(o,w)$. □

Lemma 5.18. If z is an optimal solution of (22), then $f_2(\ell, y^{\preceq_1}) = 0$ holds.

Proof. The claim follows in analogy to Lemma 5.13. □

To show analogous results for f_3 and f_4, we make use of an auxiliary lemma.

Lemma 5.19. An optimal solution z of (22) exists such that $\ell_v = 0$ for all $v \in V_-^r(o, w_1)$ and $\ell_u = 0$ for all $u \in V_+^r(w_2, w)$ is satisfied.

Proof. We choose an optimal solution z of (22) such that

$$\sum_{v \in V_-^r(o,w_1)} \ell_v + \sum_{u \in V_+^r(w_2,w)} \ell_u$$

is minimal. Note that every addend is nonnegative. By contradiction, we assume that

$$\sum_{v \in V_-^r(o,w_1)} \ell_v + \sum_{u \in V_+^r(w_2,w)} \ell_u > 0$$

holds.

Case 1: There exists $v \in V_-^r(o, w_1)$ with $\ell_v > 0$. We now choose v such that $\ell_{v'} = 0$ for all $v' \in V_-^r(o, v) \setminus \{v\}$ is satisfied. Consequently, an entry $u \in V_+^r(o, v)$ exists such that $\ell_{u'} = 0$ holds for all $u' \in V_+^r(o, u) \setminus \{o\}$ and in a flow decomposition, such as in Lemma 3.1, $q(P^r(u, v)) > 0$ is satisfied. We can now decrease the latter by $\varepsilon > 0$ such that $\ell_u > 0$ and $\ell_v > 0$ holds. This decreases the potential drop $\Pi_{P^r(o,w)}(q)$. Due to Lemmas 5.17 and 5.18, we can assume that $q(P^l(s_1^l, t_1^l)) > 0$ holds. By using Lemma 3.4, we can now decrease the latter by $\tilde{\varepsilon}$ and choose ε such that $\Pi_{P^l(o,w)}(q) = \Pi_{P^r(o,w)}(q)$ holds and $\ell_u, \ell_v, \ell_{s_1^l}, \ell_{t_1^l}$ are positive. Moreover, Lemmas 5.8–5.11 imply that the solution obtained after the modifications is still feasible and optimal for (22). In doing so, we note that the modifications do not change any flow of $P^r(w_1, w_2)$ and thus, the potential difference between w_1 and w_2

stays the same. This is a contradiction to the choice of z because

$$\sum_{v \in V_-^{\mathrm{r}}(o,w_1)} \ell_v + \sum_{u \in V_+^{\mathrm{r}}(w_2,w)} \ell_u$$

is decreased in the modified solution.

Case 2: There is $u \in V_+^{\mathrm{r}}(w_2,w)$ with $\ell_u > 0$. We now choose u such that $\ell_{u'} = 0$ for all $u' \in V_+^{\mathrm{r}}(u,w)$. Due to $q \geq 0$, an exit $v \in V_-^{\mathrm{r}}(u,w)$ exists such that $\ell_{v'} = 0$ holds for all $v' \in V_-^{\mathrm{r}}(v,w) \setminus \{v,w\}$ and $q(P^{\mathrm{r}}(u,v)) > 0$. In analogy to Case 1, the claim follows by decreasing the flow $q(P^{\mathrm{r}}(u,v))$ by $\varepsilon > 0$ and $q(P^{\mathrm{l}}(s_1^l,t_1^l))$ by $\tilde{\varepsilon} > 0$. $\qquad\square$

Lemma 5.20. If z is an optimal solution of (22), then $f_3(\ell,x^{\preceq_{\mathrm{r}}},x^{\succeq_{\mathrm{r}}}) = 0$ holds.

Proof. Let z be an optimal solution of (22) that satisfies Lemma 5.19. By contradiction, we assume that $f_3(\ell,x^{\preceq_{\mathrm{r}}},x^{\succeq_{\mathrm{r}}}) > 0$ holds. Lemma 5.4 implies that ℓ does not satisfy Property (a) w.r.t. P^{r}. Consequently, there are entries $u_1,u_2,u_3 \in V_+^{\mathrm{r}}(o,w)\setminus\{o\}$ with $u_1 \prec_{\mathrm{r}} u_2 \prec_{\mathrm{r}} u_3$ such that $\ell_{u_1} > 0, \ell_{u_2} < b_{u_2}$, and $\ell_{u_3} > 0$ hold. If $q_a > 0$ for $a \in \delta^{\mathrm{out}}(o) \cap P^{\mathrm{r}}(o,w)$, we replace $u_1 = o$. Otherwise, we choose $u_1 \neq o$ such that $\ell_u = 0$ holds for all $u \in V_+^{\mathrm{r}}(o,u_1) \setminus \{o,u_1\}$ and we choose u_3 such that $\ell_u = 0$ holds for all $u \in V_+^{\mathrm{r}}(u_3,w)\setminus\{u_3\}$. We now consider a flow decomposition as in Lemma 3.1. Due to $q \geq 0$, an exit $v_3 \in V_-^{\mathrm{r}}(u_3,w)$ with $q(P^{\mathrm{r}}(u_3,v_3)) > 0$ exists. By the choice of u_1, there is an exit $v_1 \in V_-^{\mathrm{r}}(u_1,w)$ with $\ell_v = 0$ for all $v \in V_-^{\mathrm{r}}(o,v_1) \setminus \{v_1\}$ and $q(P^{\mathrm{r}}(u_1,v_1)) > 0$. Consequently, $v_1 \preceq_{\mathrm{r}} v_3$ holds. We now distinguish two cases.

Case 1: $u_2 \preceq_{\mathrm{r}} w_1$. Due to Lemma 5.19, $w_1 \preceq_{\mathrm{r}} v_1$ holds. Consequently, we can decrease $q(P^{\mathrm{r}}(u_1,v_1)) > 0$ by $\varepsilon > 0$ and we increase $q(P^{\mathrm{r}}(u_2,v_1))$ by the same amount such that $\ell_{u_1} > 0$ and $\ell_{u_2} \leq b_{u_2}$ holds. Since $u_1 \prec_{\mathrm{r}} u_2$ holds, this modification decreases the potential difference $\Pi_{P^{\mathrm{r}}(o,w)}(q)$ but the flow on arcs of $P^{\mathrm{r}}(w_1,w_2)$ stays the same due to $u_2 \preceq_{\mathrm{r}} w_1$. Consequently, $\Pi_{P^{\mathrm{r}}(w_1,w_2)}(q)$ is unchanged. From the proof of Lemma 5.12, it follows that this modification decreases f_3. In analogy to Case 1 of Lemma 5.19, we can now decrease $\Pi_{P^{\mathrm{l}}(o,w)}(q)$ by modifying $\ell_{s_1^l}$ and $\ell_{t_1^l}$ such that $\Pi_{P^{\mathrm{r}}(o,w)}(q) = \Pi_{P^{\mathrm{l}}(o,w)}(q)$ holds without changing the values of f_i for $i = 1,\dots,4$. This is a contradiction to the optimality of z because we have decreased f_3 in the first part of the modification.

Case 2: $w_1 \prec_{\mathrm{r}} u_2$. Due to $u_2 \prec_{\mathrm{r}} u_3$ and Lemma 3.4, we can decrease $q(P^{\mathrm{r}}(u_3,v_3))$ by $\varepsilon > 0$ and increase $q(P^{\mathrm{r}}(u_2,v_3))$ by $0 < \tilde{\varepsilon} \leq \varepsilon$ such that $\Pi_{P^{\mathrm{r}}(o,w)}(q) = \Pi_{P^{\mathrm{l}}(o,w)}(q)$, $\ell_{u_3} > 0$, $\ell_{v_3} > 0$, and $\ell_{u_2} \leq b_{u_2}$ holds. Consequently, the binary variables of z stay the same. By using Lemmas 5.8, 5.9, and 5.11, the values f_1, f_2, and f_4 stay the same as well. The modified solution satisfies Constraints (17b). Further, the modification only increases q_a for $a \in P^{\mathrm{r}}(u_2,u_3)$ and decreases the flow q_a for $a \in P^{\mathrm{r}}(u_3,v_3)$. The remaining arc flows stay unchanged. Due to $w_1 \prec_{\mathrm{r}} u_2 \prec_{\mathrm{r}} u_3 \prec_{\mathrm{r}} w_2$ and Lemma 3.4 (d), we possibly increase the potential difference between w_1 and w_2 and thus, Constraint (17d) is still satisfied. Case 1 of Lemma 5.12 implies that the previous modification decreases f_3, which is a contradiction to the optimality of z. $\qquad\square$

Lemma 5.21. If z is an optimal solution of (22), then $f_4(\ell,y^{\preceq_{\mathrm{r}}},y^{\succeq_{\mathrm{r}}}) = 0$ holds.

Proof. Let z be an optimal solution of (22) that satisfies Lemma 5.19. By contradiction, we assume that $f_4(\ell,y^{\preceq_{\mathrm{r}}},x^{\succeq_{\mathrm{r}}}) > 0$ holds. Lemma 5.7 implies that ℓ does not satisfy Property (d). Consequently, there are exits $v_1,v_2,v_3 \in V_-^{\mathrm{r}}(o,w) \setminus \{w\}$ with $v_1 \prec_{\mathrm{r}} v_2 \prec_{\mathrm{r}} v_3, \ell_{v_1} > 0, \ell_{v_2} < b_{v_2}$, and $\ell_{v_3} > 0$. Furthermore, we choose v_1 such that $\ell_v = 0$ holds for all $v \in V_-^{\mathrm{r}}(o,v_1) \setminus \{v_1\}$. If $q_a > 0$ for $a \in \delta^{\mathrm{in}}(w) \cap P^{\mathrm{r}}(o,w)$, we replace $v_3 = w$. Otherwise, we choose $v_3 \neq w$ such that $\ell_v = 0$ holds for all

$v \in V_-^l(v_3, w) \setminus \{v_3, w\}$. We now consider a flow decomposition as in Lemma 3.1. Due to $q \geq 0$, there is an entry $u_3 \in V_+^r(o, v_3)$ with $\ell_u = 0$ for all $u \in V_+^r(u_3, w) \setminus \{u_3\}$ and $q(P^r(u_3, v_3)) > 0$. Furthermore, an entry $u_1 \in V_+^r(o, w)$ with $\ell_u = 0$ for all $u \in V_+^r(o, u_1) \setminus \{o, u_1\}$ exists that satisfies $q(P^r(u_1, v_1)) > 0$. Due to Lemma 5.19, $w_1 \prec_r v_1 \prec_r v_2 \prec_r v_3$ holds. We now distinguish two cases.

Case 1: $v_2 \preceq_r w_2$. Consequently, $v_1 \prec_r v_2 \preceq_r w_2$ holds. We can now decrease $q(P^r(u_1, v_1))$ by $\varepsilon > 0$ and increase $q(P^r(u_1, v_2))$ by $0 < \tilde{\varepsilon} \leq \varepsilon$ such that $\Pi_{P^r(o,w)}$ stays the same and $\ell_{u_1} > 0, \ell_{v_1} > 0$, and $\ell_{v_2} \leq b_{v_2}$ holds. In particular, the binary variables of z stay the same after the modification. Due to this and Lemmas 5.8 and 5.10, the values of f_1, f_2, and f_3 stay unchanged. The modified solution satisfies Constraints (17b). Further, this modification only decreases q_a for $a \in P^r(u_1, v_1)$ and increases arc flows q_a for $a \in P^r(v_1, v_2)$. The remaining arc flows stay the same. Hence, since $w_1 \prec_r v_1 \prec_r v_2 \preceq_r w_2$ and by Lemma 3.4 (d), we possibly increase the potential difference between w_1 and w_2 and Constraint (17d) is still satisfied. Consequently, z is still a feasible for (22). In analogy to Case 1 of Lemma 5.15, it follows that the modification decreases f_4, which is a contradiction to the optimality of z.

Case 2: $w_2 \prec_r v_2$. Consequently, $u_3 \prec_r w_2 \prec_r v_2 \prec_r v_3$ holds. We can now apply Case 2 of Lemma 5.15. In doing so, we keep in mind that $w_1 \prec_r v_1$ and $w_2 \prec_r v_2 \prec_r v_3$ hold which ensures that z still satisfies (17d) after the applied modifications. $\square$

Finally, we obtain a result for the present case that is analogous to Theorem 5.16.

Theorem 5.22. Let $(o, w) \in V_+ \times V_-$ be fixed and $w_1, w_2 \in P^r(o, w)$. Then, an optimal solution (ℓ, q, π) of (14) exists that satisfies Properties (a), (b), (d) and (a) w.r.t. P^r.

Proof. The zero nomination is feasible for Problem (14) and it is optimal if $w_2 \preceq_r w_1$ holds. Furthermore, the feasible region of the latter problem is compact and thus, an optimal solution is attained. Consequently, Problem (22) has an optimal solution, which is attained. Due to Lemmas 5.17–5.21 and Lemmas 5.4–5.7, for $w_1 \prec_r w_2$ an optimal solution of Problem (22) exists that satisfies Properties (a), (b), (d) and, (a) w.r.t. P^r. Additionally, the solution is also optimal for Problem (14). $\square$

6. A Polynomial-Time Algorithm

Exploiting the special structure of nominations that maximize the potential difference between a pair of nodes, we now show that the feasibility of a booking can be checked in polynomial time on a cycle. First, we obtain an estimate on the number of arithmetic operations necessary to detect the existence of an infeasible nomination, or otherwise certify its non-existence. In a second step, we then translate this result to the Turing model of computation, resulting in a polynomial-time algorithm for deciding the feasibility of a booking. For doing so, we make the following non-restrictive assumption on the rationality of the problem data.

Assumption 6.1. We consider a booking $b \in \mathbb{Q}^V$ and assume that $\Lambda_a \in \mathbb{Q}$ for all $a \in A$ and $\pi_u^-, \pi_u^+ \in \mathbb{Q}$ for all $u \in V$. Additionally, we assume that the encoding lengths are bounded from above by τ.

As a consequence of Corollary 5.2, a booking b is feasible if and only if, for every $(w_1, w_2) \in V^2$ and $(o, w) \in V_+ \times V_-$,

$$\pi_{w_1} - \pi_{w_2} > \pi_{w_1}^+ - \pi_{w_2}^-, \quad \text{(1a), (3b), (14b), (14c)} \tag{23}$$

admits no solution. We now make several observations. First, recall that A' is obtained from A by orienting all arcs from o to w. Then, given (14c), the right-hand

sides of (14b) simplify to $\Lambda_a q_a^2$ for all $a \in A'$. Second, we eliminate the potentials π by aggregating the resulting constraints along $P^{\mathrm{l}}(o, w)$ and $P^{\mathrm{r}}(o, w)$. We only treat the situation corresponding to Section 5.2 in which $o \preceq_{\mathrm{r}} w_1 \prec_{\mathrm{r}} w_2 \preceq_{\mathrm{r}} w$ is valid, since it has the highest number of s and t nodes necessary to set up the structural properties and thus represents the worst case in terms of complexity. The situation corresponding to Section 5.1 with w_1 and w_2 on different paths w.r.t. o and w can however be treated in a similar way. We obtain

$$\sum_{a \in P^{\mathrm{r}}(w_1, w_2)} \Lambda_a q_a^2 > \pi_{w_1}^+ - \pi_{w_2}^-,$$

$$\sum_{a \in P^{\mathrm{l}}(o, w)} \Lambda_a q_a^2 - \sum_{a \in P^{\mathrm{r}}(o, w)} \Lambda_a q_a^2 = 0.$$

It is well-known that if the nomination is balanced, the rank of the flow conservation constraints (1a) is $|V| - 1$, resulting in a single degree of freedom in the case of a cycle. Thus, we introduce $\ell_w = \ell_w^{\mathrm{l}} + \ell_w^{\mathrm{r}}$ to take into account the supply to the flow-meeting point w along P^{l} and P^{r} separately. Then, for $a = (u, v) \in A'$, (1a) leads to

$$q_a = \begin{cases} -\displaystyle\sum_{v' \in P^{\mathrm{l}}(v, w) \setminus \{w\}} \sigma_{v'} \ell_{v'} + \ell_w^{\mathrm{l}}, & \text{if } a \in P^{\mathrm{l}}(o, w), \\ -\displaystyle\sum_{v' \in P^{\mathrm{r}}(v, w) \setminus \{w\}} \sigma_{v'} \ell_{v'} + \ell_w^{\mathrm{r}}, & \text{if } a \in P^{\mathrm{r}}(o, w). \end{cases}$$

As a consequence of the previous discussion, we need to check that the system of polynomials

$$\sum_{a = (u, v) \in P^{\mathrm{r}}(w_1, w_2)} \Lambda_a \left(-\sum_{v' \in P^{\mathrm{r}}(v, w) \setminus \{w\}} \sigma_{v'} \ell_{v'} + \ell_w^{\mathrm{r}} \right)^2 > \pi_{w_1}^+ - \pi_{w_2}^-, \tag{24a}$$

$$\sum_{a = (u, v) \in P^{\mathrm{l}}(o, w)} \Lambda_a \left(-\sum_{v' \in P^{\mathrm{l}}(v, w) \setminus \{w\}} \sigma_{v'} \ell_{v'} + \ell_w^{\mathrm{l}} \right)^2$$

$$- \sum_{a = (u, v) \in P^{\mathrm{r}}(o, w)} \Lambda_a \left(-\sum_{v' \in P^{\mathrm{r}}(v, w) \setminus \{w\}} \sigma_{v'} \ell_{v'} + \ell_w^{\mathrm{r}} \right)^2 = 0, \tag{24b}$$

$$-\sum_{v' \in P^{\mathrm{l}}(v, w) \setminus \{w\}} \sigma_{v'} \ell_{v'} + \ell_w^{\mathrm{l}} \geq 0, \quad (u, v) \in P^{\mathrm{l}}, \tag{24c}$$

$$-\sum_{v' \in P^{\mathrm{r}}(v, w) \setminus \{w\}} \sigma_{v'} \ell_{v'} + \ell_w^{\mathrm{r}} \geq 0, \quad (u, v) \in P^{\mathrm{r}}, \tag{24d}$$

$$\ell_w^{\mathrm{l}} + \ell_w^{\mathrm{r}} = \ell_w, \quad \ell \in N(b), \tag{24e}$$

admits no solution.

We now reduce the dimension of (24) to obtain a system of polynomials with a constant number of constraints and variables independent of the problem size. Hence, we make use of the structure analyzed in Section 5.2 for potential-difference maximizing nominations.

In what follows, we consider a configuration of Properties (a), (b), (d) and (a) w.r.t. P^{r}, determined by $s_1^l, s_2^l \in V_+^{\mathrm{l}}(o, w)$, $t_1^l \in V_-^{\mathrm{l}}(o, w)$, $s_1^r, s_2^r \in V_+^{\mathrm{r}}(o, w)$ as well as $t_1^r, t_2^r \in V_-^{\mathrm{r}}(o, w)$, and the corresponding partially fixed $\ell \in N(b)$.

Lemma 6.2. There exists a system of polynomials equivalent to (24e) that has at most 9 variables and 16 constraints, independent of the size of the cycle.

Proof. First, observe that we can substitute the nomination entries for o and w using

$$\ell_w = \ell_w^l + \ell_w^r, \quad \ell_o = -\sum_{u \in V \setminus \{o\}} \sigma_u \ell_u.$$

Fixing nomination entries either to their booking bound or to zero, as by Properties (a)–(d), it is easy to observe that $\ell_{s_1^l}, \ell_{s_2^l}, \ell_{t_1^l}, \ell_{s_1^r}, \ell_{s_2^r}, \ell_{t_1^r}, \ell_{t_2^r}, \ell_w^l, \ell_w^r$, are the only remaining 9 variables. Note that in some situations these variables may coincide. In particular, there are at most 14 constraints corresponding to the booking bounds, namely $0 \leq \ell_u \leq b_u$ for all $u \in \left\{ s_1^l, s_2^l, t_1^l, s_1^r, s_2^r, t_1^r, t_2^r \right\}$.

The number of additional constraints due to o and w depend on the configuration under consideration. If $o \notin \left\{ s_1^l, s_1^r \right\}$, then the additional constraint $\ell_o = b_o$ is necessary. If $w \notin \left\{ t_1^l, t_2^r \right\}$, then $\ell_w = b_w$ is required. □

A combinatorial analysis of (24c) and (24d) also leads to the following constant number of constraints.

Lemma 6.3. There exists a system of polynomials equivalent to (24c) and (24d) with at most 24 constraints, independent of the size of the cycle. This system can be determined in $O(|A|)$ time.

Proof. Let us first consider (24d). There are four s and t nodes on $P^r(o, w)$, namely $s_1^r, s_2^r, t_1^r, t_2^r$. Thus, assuming that constants have been moved to the right-hand sides in (24d), there can be at most 2^4 left-hand sides with different constant right-hand sides. For every left-hand side, it is sufficient to impose a single constraint admitting the maximum constant on the right-hand side. This is easily achieved by iterating over all arcs of $P^r(o, w)$. Similarly, (24c) can be reduced to a system with 2^3 constraints. □

The following result now is a direct consequence of the two previous results.

Theorem 6.4. System (24) can be reduced in $O(|A|)$ time to a system of polynomials with at most 9 variables and 42 constraints.

Next, we apply a general decision algorithm for the existence of solutions for systems of polynomial equations and inequalities, given by Algorithm 14.16 in [5], to estimate the number of arithmetic operations necessary to decide the existence of a solution for (24). Note that this algorithm can in particular handle strict inequalities as required to determine a violation of the potential difference bounds; see, e.g., Notation 11.31 in [5]. We then obtain the following result.

Theorem 6.5. Suppose Assumption 6.1 holds. Then, the existence of a solution of (24) can be decided in $O((\log|V| + \tau)|V_+|^4|V_-|^3)$ time.

Proof. Algorithm 14.16 in [5] has a complexity in the arithmetic computation model of $s^k d^{O(k)}$, where s is the number of constraints, k is the number of variables, and d is the highest degree of the polynomials. For a given configuration of Properties (a), (b), (d), and (a) w.r.t. P^r, the number of variables and constraints in (24) can be reduced to a constant by Theorem 6.4 and $d = 2$. Consequently, the existence of a solution for this reduced system can be checked in $O(1)$ arithmetic operations.

Under Assumption 6.1, the encoding length of the rational coefficients of (24) are bounded by $O(\log|V| + \tau)$. This can easily be deduced by analyzing the constant term in, e.g., (24a). Given the constant number of variables and constraints in the reduced version of (24), the encoding length of integer coefficients after scaling is still bounded by $O(\log|V| + \tau)$. In this case, the encoding length of coefficients

appearing in intermediate computations and the output of Algorithm 14.16 in [5] are also bounded by $O(\log|V| + \tau)$. From a discussion in Chapter 1 of [17], the existence of a solution to the reduced version of System (24) can then be checked in $O(\log|V| + \tau)O(1) = O(\log|V| + \tau)$ time on a Turing machine.

By Lemmas 5.17–5.21, a solution of (24) exists if and only if there is a configuration of Properties (a), (b), (d), and (a) w.r.t. P^r, such that a solution of the reduced version of (24) exists. Consequently, the result follows by iterating over all combinations of $s_1^l, s_2^l, t_1^l, s_1^r, s_2^r, t_1^r, t_2^r$. $\square$

Furthermore, iterating this procedure over all $(o, w) \in V_+ \times V_-$, we obtain the final result for validating a booking on a cycle, which ensures that checking the feasibility of a booking on a cycle can be done in polynomial time.

Corollary 6.6. Under Assumption 6.1, the feasibility of booking $b \in \mathbb{Q}_{\geq 0}^V$ can be checked in $O((\log|V| + \tau)|V_+|^5|V_-|^4)$ time on a cycle.

We close this section with a short remark on how our results can be applied to other types of utility networks, e.g., to water distribution or power networks.

Remark 6.7. The structural properties derived in Sections 2–5 can be applied to potential-based networks if the following assumptions hold: The potentials satisfy (1) where for any arc $a \in A$, the right-hand side of (1b) is a function $\phi_a : \mathbb{R} \to \mathbb{R}$ that may depend on the arc flow q_a and that is continuous, strictly increasing, and odd, i.e., $\phi_a(-q_a) = -\phi_a(q_a)$. Consequently, our structural results hold for many different networks such as water, hydrogen, or lossless DC (direct current) power flow networks, if the physics model is chosen appropriately; see [16]. In particular, we can reduce the optimization problem (3), where we replace the right-hand side of (1b) by ϕ_a, to a fixed inequality system for all these potential-based networks as shown in Section 6. However, the presented complexity result is only valid in the case in which the potential function $\phi_a(q_a)$ is a polynomial in the variables $|q_a|$ and q_a that is strictly increasing and odd.

However, the overall question of deciding the feasibility of a booking discussed in this paper is rather specific and tailored to the European gas market system since, e.g., the market design for electricity is different to the one for gas in Europe.

7. Conclusion

In this work, we prove that deciding the feasibility of a booking in the European entry-exit gas market model is in P for the special case of cycle networks. To the best of our knowledge, this is the first in-depth complexity analysis in this context that considers a nonlinear flow model and a network topology that is not a tree. Our approach requires the combination of both the cyclic structure of the network and properties of the underlying nonlinear potential-based flow model with a general decision algorithm from real algebraic geometry. We show that the size of a polynomial equality and inequality system for deciding the feasibility of a booking is constant and, in particular, does not depend on the size of the cycle. Thus, a general algorithm for solving this system can serve as a constant-time oracle used in an enumeration of polynomial complexity.

Although our theoretical result moves the frontier of knowledge about the hardness of deciding the feasibility of bookings in the European entry-exit gas market, it still remains an open question to exactly determine the frontier between easy and hard cases if a nonlinear and potential-based flow model is considered. Although we believe that the problem is hard on general networks, no hardness results are known so far. Since both trees and single cycle networks are now well understood, a

possibility is to consider more general classes of networks. Thus, a reasonable next step could be networks consisting of a single cycle with trees on it or, even more generally, cactus graphs. In our opinion, it is promising to combine the techniques used on trees and cycles in order to solve this larger graph class.

Finally, although the present paper is a very specific one, we hope that the structural insights gained can be later put together with other insights to obtain more general techniques for (adjustable) robust and nonlinear flow problems.

Acknowledgments

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Appendix A. Omitted Proofs

Proof of Lemma 5.6. Let z be feasible for (17). For $i \in V_+^{\mathrm{r}}(o, w)$,

$$x_i^{\succeq_{\mathrm{r}}} \sum_{v \in V_+^1(o,i) \setminus \{i\}} (b_v - \ell_v) \geq 0$$

holds. Assume now that Property (c) does not hold. Consequently, there are $u_1, u_2 \in V_+^{\mathrm{r}}(o, w)$ with $u_1 \prec_{\mathrm{r}} u_2$ such that $\ell_{u_1} < b_{u_1}$ and $\ell_{u_2} > 0$ hold. Consequently,

$x_{u_2}^{\succeq_r} = 1$ and

$$\sum_{v \in V_+^1(o,u_2) \setminus \{u_2\}} (b_v - \ell_v) > 0$$

holds. Thus, $f_3(\ell, x^{\succeq_r}) > 0$.

If ℓ satisfies Property (c), then we set $x_u^{\succeq_r} = 0$ for all $u \in V_+^r(s_1^r, w) \setminus \{s_1^r\}$, otherwise we set $x_u^{\succeq_r} = 1$. Furthermore, for $i \in V_+^r(o, s_1^r)$,

$$\sum_{v \in V_+^r(o,i) \setminus \{i\}} (b_v - \ell_v) = 0$$

holds due to Property (c). Consequently, $f_3(\ell, x^{\succeq_r}) = 0$.   □

Proof of Lemma 5.7. Let z be feasible for (17). For $i, j \in V_-^r(o, w)$ where $i \preceq_r j$

$$y_i^{\prec_r} y_j^{\succeq_r} \sum_{v \in V_-^r(i,j) \setminus \{i,j\}} (b_v - \ell_v) \geq 0$$

holds. Assume now that Property (d) does not hold. Consequently, there are $u_1, u_2, u_3 \in V_-^r(o, w) \setminus \{w\}$ with $u_1 \prec_r u_2 \prec_r u_3$ such that $\ell_{u_1} > 0, \ell_{u_2} < b_{u_2}$, and $\ell_{u_3} > 0$ hold. Thus, $y_{u_1}^{\prec_r} = y_{u_3}^{\succeq_r} = 1$ and

$$\sum_{v \in V_-^r(u_1,u_3)) \setminus \{u_1,u_3\}} (b_v - \ell_v) > 0$$

holds. Thus, $f_4(\ell, y^{\prec_r}, y^{\succeq_r}) > 0$.

If ℓ satisfies Property (d), then we set $y_v^{\prec_r} = 0$ for all $v \in V_-^r(o, t_1^r) \setminus \{t_1^r\}$, otherwise we set $y_v^{\prec_r} = 1$. Additionally, we set $y_v^{\succeq_r} = 1$ for all $v \in V_-^r(o, t_2^r)$ and otherwise we set $y_v^{\succeq_r} = 0$. Consequently, for $i \in V_-^r(o, t_1^r) \setminus \{t_1^r\}$ or $j \in V_-^r(t_2^r, w) \setminus \{t_2^r\}$, the equality $y_i^{\prec_1} y_j^{\succeq_r} = 0$ holds and for $i, j \in V_-^r(t_1^r, t_2^r)$,

$$\sum_{v \in V_-^r(i,j) \setminus \{i,j\}} (b_v - \ell_v) = 0$$

holds due to Property (d). Consequently, $f_4(\ell, y^{\prec_r}, y^{\succeq_r}) = 0$.   □

Proof of Lemma 5.10. Optimality of z and the choice of u_1 imply $x_u^{\succeq_r} = 0$ for all $u \in V_+^r(u_1, w) \setminus \{u_1\}$. Hence,

$$x_i^{\succeq_r} \sum_{v \in V_+^r(o,i) \setminus \{i\}} (b_v - \ell_v) = 0,$$

whenever $u_1 \in V_+^r(o, i) \setminus \{i\}$. Thus, a change of ℓ_{u_1} does not change $f_3(\ell, x^{\succeq_r})$.   □

Proof of Lemma 5.11. Optimality of z and the choice of v_1 and v_3 imply $y_u^{\prec_r} = 0$ for all $u \in V_-^r(o, v_1) \setminus \{o, v_1\}$ and $y_u^{\succeq_r} = 0$ for all $u \in V_-^r(v_3, w) \setminus \{v_3, w\}$. Hence,

$$y_i^{\prec_r} y_j^{\succeq_r} \sum_{v \in V_-^r(i,j) \setminus \{i,j\}} (b_v - \ell_v) = 0,$$

whenever v_1 or v_3 are in $V_-^r(i, j) \setminus \{i, j\}$. Consequently, a change of ℓ_{v_1} or ℓ_{v_3} does not change $f_4(\ell, y^{\prec_r}, y^{\succeq_r})$.   □

Proof of Lemma 5.14. Let z be an optimal solution of (17). By contradiction, we assume that $f_3(\ell, x^{\succeq_r}) > 0$ holds. Lemma 5.6 implies that ℓ does not satisfy Property (c). Consequently, there are entries $u_1, u_2 \in V_+^r(o, w)$ with $u_1 \prec_r u_2$, $\ell_{u_1} < b_{u_1}$, and $\ell_{u_2} > 0$. We now choose u_1 such that $\ell_u = b_u$ holds for all $u \in V_+^r(o, u_1) \setminus \{u_1\}$ and u_2 such that $\ell_u = 0$ holds for all $u \in V_+^r(u_2, w) \setminus \{u_2\}$. Due to the latter, there is an exit $v_2 \in V_-^r(u_2, w)$ with $\ell_{v_2} > 0$ and $\ell_v = 0$ for all $v \in V_-^r(v_2, w) \setminus \{v_2, w\}$. Furthermore, we can assume w.l.o.g. that in a

flow decomposition, see Lemma 3.1, $q(P^{\mathrm{r}}(u_2, v_2)) > 0$ holds. Due to Lemma 3.4 and $u_1 \prec_{\mathrm{r}} u_2$, we can decrease $q(P^{\mathrm{r}}(u_2, v_2))$ and increase $q(P^{\mathrm{r}}(u_1, v_2))$ such that $\Pi_{P^{\mathrm{r}}(o,w)}(q)$ stays the same as before the modification and $0 < \ell_{u_1} \leq b_{u_1}, \ell_{u_2} > 0, \ell_{v_2} > 0$ hold. Thus, the binary variables of z stay the same. Furthermore, by Lemmas 5.8, 5.9, and 5.11 the values of f_1, f_2, and f_4 stay the same. The modified solution satisfies Constraints (17b). The modification only decreases q_a for $a \in P^{\mathrm{r}}(u_2, v_2)$, increases q_a for $a \in P^{\mathrm{r}}(u_1, u_2)$, and the remaining arc flows stay the same. Hence, since $u_1 \prec_{\mathrm{r}} u_2 \prec_{\mathrm{r}} v_2$ and by Lemma 3.4 (d), Constraint (17d) is still satisfied. Consequently, z is still feasible for (17). Due to this modification, we increase $\ell_{u_1} > 0$ and decrease ℓ_{u_2}. By Lemma 5.10, considering only the decrease of ℓ_{u_2} does not change the objective value. In contrast, the increase of ℓ_{u_1} decreases f_3 because

$$x_{u_2}^{\succeq_{\mathrm{r}}} \sum_{v \in V_+^{\mathrm{r}}(o, u_2) \setminus \{u_2\}} (b_v - \ell_v)$$

decreases. Thus, the modification decreases the objective value, which is a contradiction to the optimality of the original solution. $\qquad\square$

Proof of Lemma 5.15. Let z be an optimal solution of (17). By contradiction, we assume that $f_4(\ell, y^{\preceq_{\mathrm{r}}}, y^{\succeq_{\mathrm{r}}}) > 0$ holds. Lemma 5.7 implies that ℓ does not satisfy Property (d). Consequently, there are exits $v_1, v_2, v_3 \in V_-^{\mathrm{r}}(o, w) \setminus \{w\}$ with $v_1 \prec_{\mathrm{r}} v_2 \prec_{\mathrm{r}} v_3$, $\ell_{v_1} > 0, \ell_{v_2} < b_{v_2}$, and $\ell_{v_3} > 0$. Furthermore, we choose v_1 such that $\ell_v = 0$ holds for all $v \in V_-^{\mathrm{r}}(o, v_1) \setminus \{v_1\}$. If $q_a > 0$ for $a \in \delta^{\mathrm{in}}(w) \cap P^{\mathrm{r}}(o, w)$, we replace $v_3 = w$. Otherwise, we choose $v_3 \neq w$ such that $\ell_v = 0$ holds for all $v \in V_-^{\mathrm{l}}(v_3, w) \setminus \{v_3, w\}$. We now consider a flow decomposition such as in Lemma 3.1. Due to $q \geq 0$, there is an entry $u_3 \in V_+^{\mathrm{r}}(o, v_3)$ with $\ell_u = 0$ for all $u \in V_+^{\mathrm{r}}(u_3, w) \setminus \{u_3\}$ and $q(P^{\mathrm{r}}(u_3, v_3)) > 0$. Furthermore, an entry $u_1 \in V_+^{\mathrm{r}}(o, w)$ with $\ell_u = 0$ for all $u \in V_+^{\mathrm{r}}(o, u_1) \setminus \{o, u_1\}$ exists which satisfies $q(P^{\mathrm{r}}(u_1, v_1)) > 0$. We now distinguish two cases.

Case 1: $v_2 \prec_{\mathrm{r}} u_3$ holds. We now decrease $q(P^{\mathrm{r}}(u_1, v_1))$ by $\varepsilon > 0$ and increase $q(P^{\mathrm{l}}(u_1, v_2))$ by the same amount ε. This increases the potential difference $\Pi_{P(o,w)}(q)$. Thus, we decrease $q(P^{\mathrm{l}}(u_3, v_3))$ by $\tilde{\varepsilon} > 0$. Due to Lemma 3.4, we can choose ε and $\tilde{\varepsilon}$ such that $\Pi_{P(o,w)}(q)$ stays the same and $\ell_{v_1} > 0, 0 < \ell_{v_2} \leq b_{v_2}, \ell_{u_3} > 0, \ell_{v_3} > 0$ hold. Thus, the binary variables of z stay the same. Furthermore, by Lemmas 5.8–5.10, the values of f_1, f_2, and f_3 stay the same. The modified solution satisfies Constraints (17b). Further, the modification only decreases q_a for $a \in P^{\mathrm{r}}(u_3, v_3)$, increases q_a for $a \in P^{\mathrm{r}}(v_1, v_2)$, and the remaining arc flows stay the same. Hence, since $v_1 \prec_{\mathrm{r}} v_2 \prec_{\mathrm{r}} u_3 \prec_{\mathrm{r}} v_3$ and by Lemma 3.4 (d), Constraint (17d) is still satisfied. Consequently, z is still feasible for (17). Due to this modification, we decrease $\ell_{v_1} > 0$ and $\ell_{v_3} > 0$ and increase ℓ_{v_2}. By Lemma 5.11, considering only the decrease of ℓ_{v_1} and ℓ_{v_3} does not change the objective value. In contrast, the increase of ℓ_{v_2} decreases f_4 because

$$y_{v_1}^{\prec_{\mathrm{r}}} y_{v_3}^{\succeq_{\mathrm{r}}} \sum_{v \in V_-^{\mathrm{r}}(v_1, v_3) \setminus \{v_1, v_3\}} (b_v - \ell_v)$$

decreases. Thus, the modification decreases the objective value, which contradicts the optimality of the original solution.

Case 2: $u_3 \prec_{\mathrm{r}} v_2$ holds. We now decrease $q(P^{\mathrm{r}}(u_3, v_3))$ by $\varepsilon > 0$ and increase $q(P^{\mathrm{r}}(u_3, v_2))$ by the same amount ε. This decreases the potential difference $\Pi_{P^{\mathrm{r}}(o,w)}(q)$. Thus, we decrease $q(P^{\mathrm{r}}(u_1, v_1))$ by $\tilde{\varepsilon} > 0$ and increase $q(P^{\mathrm{r}}(u_1, v_2))$ by the same amount $\tilde{\varepsilon}$ which increases the potential difference $\Pi_{P^{\mathrm{r}}(o,w)}(q)$. Due to Lemma 3.4, we can choose ε and $\tilde{\varepsilon}$ such that $\Pi_{P^{\mathrm{r}}(o,w)}(q)$ stays the same and $\ell_{v_1} > 0, 0 < \ell_{v_2} \leq b_{v_2}, \ell_{v_3} > 0$ hold. In particular, the binary variables of z stay the same. Furthermore, by Lemmas 5.8–5.10, the values of f_1, f_2, and f_3 stay the

same. The modified solution satisfies Constraints (17b). Further, the modification only decreases q_a for $a \in P^{\mathrm{r}}(v_2, v_3)$, increases q_a for $a \in P^{\mathrm{r}}(v_1, v_2)$, and the remaining arc flows stay the same. Hence, since $v_1 \prec_{\mathrm{r}} v_2 \prec_{\mathrm{r}} v_3$ and by Lemma 3.4 (d), Constraint (17d) is still satisfied.

Consequently, z is still feasible for (17). In analogy to Case 1, the modification decreases f_4, which contradicts the optimality of the original solution. $\qquad\square$

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