Smallest Universal Covers for Families of Triangles

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Abstract

A universal cover for a family $\mathcal{T}$ of triangles is a convex shape that contains a congruent copy of each triangle $T \in \mathcal{T}$. We conjecture that for any family $\mathcal{T}$ of triangles (of bounded area) there is a *triangle* that forms a universal cover for $\mathcal{T}$ of smallest possible area. We prove this conjecture for all families of two triangles, and for the family of triangles that fit in the unit circle.

1 Introduction

A *universal cover* for a given family $\mathcal{T}$ of objects is a convex set Z that contains a congruent copy of every element $T \in \mathcal{T}$. A *smallest* universal cover is a universal cover of the smallest area (there can be multiple smallest universal covers).

Perhaps the oldest question on universal covers was asked by Lebesgue in 1914: what is the smallest area of a convex set Z that can be used to cover a congruent copy of any set of diameter one in the plane? Lebesgue's problem was first studied by Pál [8], who found that the area of a smallest universal cover is at least 0.8257 and at most 0.8454. Both bounds were improved by several authors, the current best upper bound is around 0.844 [3], the best lower bound is around 0.832 [5], so the problem is still open.

Moser asked for the smallest universal cover for the family of curves of length one. The problem is interesting both for open and closed curves, and both versions are still open. The survey by Wetzel [9] and the book by Brass et al. [4] list these and other results related to universal covers.

Among the few problems on universal covers that are solved are questions where $\mathcal{T}$ is a family of *triangles*. It is known that the smallest universal cover for the family of all triangles of perimeter one, as well as for the family of all triangles of diameter one, is itself a triangle [6, 7]. For the family of all triangles of diameter one, the smallest universal cover similar to a prescribed triangle is also known [10].

We conjecture that this is not a coincidence, and that there is always a triangle that forms a smallest universal cover. More formally, we define a family $\mathcal{T}$ of triangles to be *bounded* if there is a constant D such that no element of $\mathcal{T}$ has diameter larger than D, and state the following conjecture:

▶ Conjecture 1. For any bounded family $\mathcal{T}$ of triangles there is a triangle Z that is a smallest universal cover for $\mathcal{T}$.

If true, this would mirror the situation for translation covers of line segments: the smallest convex translation cover for *any* family of line segments can be chosen to be a triangle [2].

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Our results. Our first result (Theorem 4) describes the triangle T^* that is the unique smallest universal cover for the family $\mathcal{T}_\circ$ of all triangles that fit into the unit circle. This complements the previous results by Kovalev [7] and Füredi and Wetzel [6], as the radius of the circumcircle is, next to diameter and perimeter, a natural "size" for triangles.

It turns out that T^* can be defined by two specific triangles in $\mathcal{T}_\circ$. In other words, T^* is already the smallest universal cover for a *two-element subfamily* of $\mathcal{T}_\circ$. One can notice from the constructions in [6, 7] that the same is true for the family of all triangles of diameter one, and for the family of all triangles of perimeter one. We show that Conjecture 1 holds for any family of triangles with this property, by proving that the smallest universal cover for a family of any *two triangles* can be chosen to be a triangle (Theorem 7).

Hence, if, for an arbitrary triangle family, a smallest universal cover can be determined by a subfamily consisting of two triangles, then Conjecture 1 is implied by Theorem 7. However, we show that not all families of triangles have this property. We give an example of a family $\mathcal{T}_3$ of three triangles such that each proper subfamily has a universal cover smaller than a smallest universal cover of $\mathcal{T}_3$ (Theorem 8).

2 Preliminaries

We will say that a convex shape X *fits into* a convex shape Y if there is a shape $X' \subseteq Y$ congruent to X (that is, X' is the image of X under translation, rotation, and reflection). We say that X *maximally fits into* Y if X fits into Y, but there is no shape X' that is similar to X and larger than X and fits into Y. The following lemma has been well known; see, for instance, Agarwal et al. [1] for a proof.

▶ **Lemma 2.** *If a triangle X maximally fits into a convex polygon Y, then there are at least four incidences between vertices of X and edges of Y.*

Since the triangle X has only three vertices, one of them must be involved in two incidences, that is, it must coincide with a vertex of Y. Of particular interest to us is the special case where Y is a triangle as well. In this case the lemma implies that a vertex of X coincides with a vertex of Y and that an edge of X lies on an edge of Y. There are three cases, depicted in Figure 1. An immediate consequence is the following:

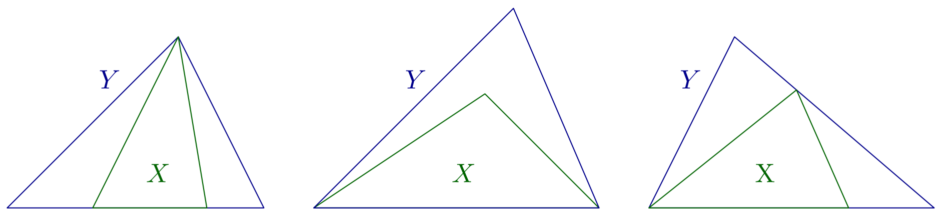

Figure 1 The three cases where X maximally fits into Y.

▶ **Corollary 3.** *If a triangle X fits into a triangle Y, then we can place X in Y such that an edge of X lies on an edge of Y.*

We will let $|PQ|$ denote the length of the segment PQ, while $|X|$ denotes the area of a convex shape X; we also use $|ABC|$ to denote the area of the triangle $\triangle ABC$ and similarly for convex polygons with more than three corners.

3 Triangles contained in the unit circle

Let $T_0 = \triangle ABC$ be the equilateral triangle of side length $\sqrt{3}$. This is the largest equilateral triangle that fits into the unit circle. Then, for $60° \leqslant \theta < 90°$, we let $T_1(\theta) = \triangle DEF$ be the isosceles triangle whose circumradius is one and whose base angles are θ. We place $T_1(\theta)$ such that its long edge DE is aligned with the edge AB of T_0 and its corner F lies on the edge AC; see Figure 2. We define $T_2(\theta)$ as $\triangle ADC$.

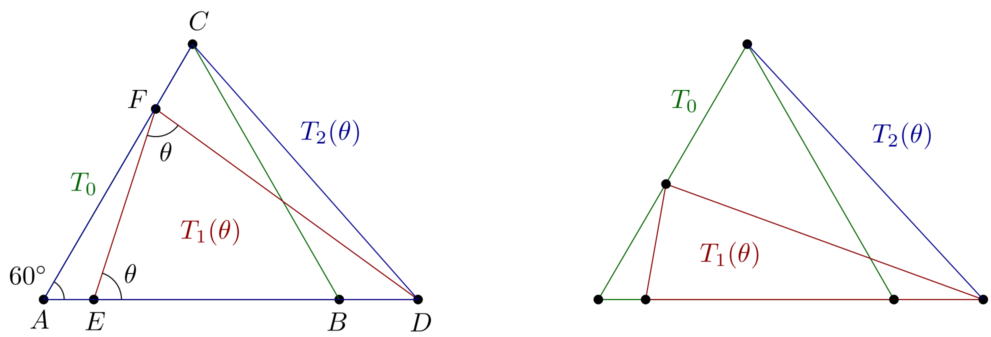

Figure 2 Construction of $T_2(\theta) = \triangle ADC$, for two different values of the angle θ.

We now define $T^* = T_2(80°)$. Our first main result will be the following:

▶ **Theorem 4.** *The triangle T^* is the unique smallest universal cover for the family of all triangles that fit in the unit-radius circle.*

One may wonder what makes 80° special. The reason is that it is for $\theta = 80°$ that $T_1(\theta)$ maximally fits into $T_2(\theta)$ in two distinct ways. To see this, consider the height HD in T^*, and reflect both A and F about the line HD, obtaining points $\tilde{C}$ and G: Calculation shows that $\angle FDC \approx 27.52° > 20°$, resulting in Figure 3. Since $\angle A\tilde{C}D = \angle CAD = 60°$, we obtain an equilateral triangle $\triangle AD\tilde{C}$. We also have $|DG| = |DF|$ and $\angle GDH = \angle FDH = 10°$, so $\triangle FDG$ is congruent to $\triangle EDF = T_1(80°)$.

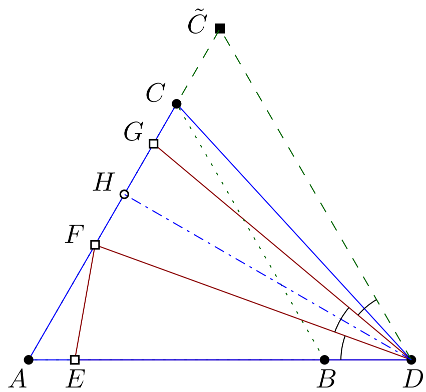

Figure 3 $T^* = \triangle ADC$.

For proving the optimality of T^*, the following lemma is useful, which is an adaptation of a result by Füredi and Wetzel [6, Theorem 5].

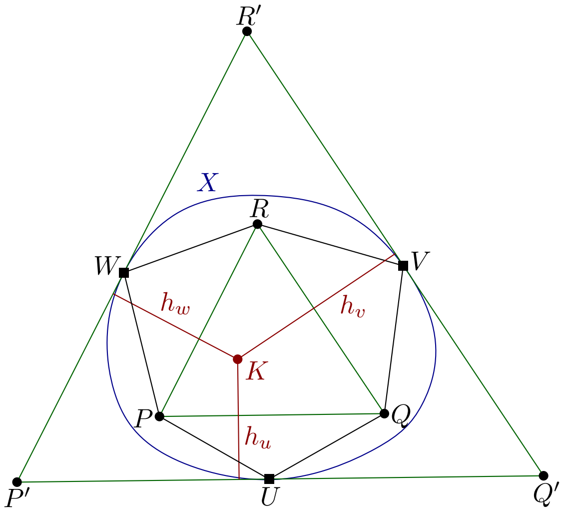

Figure 4 Proof of Lemma 5.

▶ **Lemma 5.** *Let $\mathcal{T}$ be a family of triangles, and let Z be a universal cover for $\mathcal{T}$. Let $S \in \mathcal{T}$, and let S' be the smallest universal cover for $\mathcal{T}$ that is similar to S. If*

$$\frac{|S'|}{|S|} = \left(\frac{|Z|}{|S|}\right)^2,$$

then Z is a smallest universal cover for $\mathcal{T}$.

Proof. Let $\triangle PQR = S$, and let X be a universal cover for $\mathcal{T}$. We can assume $S \subseteq X$. We draw tangents to X that are parallel to the edges of S, obtaining a triangle $\triangle P'Q'R'$ that encloses X and that is similar to S; see Figure 4. By the assumption, this implies that $|P'Q'R'| \geqslant |S'|$, and therefore

$$\frac{|P'Q'|}{|PQ|} \geqslant \frac{|Z|}{|S|}.$$

Let U, V, and W be points of X on the three edges of $\triangle P'Q'R'$, let K be any point inside S, and let h_u, h_v, and h_w be the distances from K to the lines $P'Q'$, $Q'R'$, and $R'P'$, respectively. We then have

$$\begin{aligned}
|X| \geqslant |PUQVRW| &= \frac{1}{2}\big(|PQ|h_u + |QR|h_v + |RP|h_w\big) \\
&= \frac{|PQ|}{|P'Q'|} \cdot \frac{1}{2}\big(|P'Q'|h_u + |Q'R'|h_v + |R'P'|h_w\big) \\
&= \frac{|PQ|}{|P'Q'|}|P'Q'R'| = \frac{|PQ|}{|P'Q'|}\left(\frac{|P'Q'|}{|PQ|}\right)^2|PQR| \\
&= \frac{|P'Q'|}{|PQ|}|S| \geqslant \frac{|Z|}{|S|}|S| = |Z|.
\end{aligned}$$ ◀

The following is a special case of Lemma 5.

▶ **Corollary 6.** *Let S and T be two triangles where T does not fit into S, and let Z be a universal cover for $\{S, T\}$. Let S' be the smallest triangle similar to S such that T fits in S'. If*

$$\frac{|S'|}{|S|} = \left(\frac{|Z|}{|S|}\right)^2,$$

then Z is a smallest universal cover for the family $\{S, T\}$.

Now we are ready to sketch the proof of Theorem 4. It is not too hard to show that T^* is indeed a universal cover for triangles in the unit circle. We then notice that the triangle $AD\tilde{C}$ is the smallest equilateral triangle into which $T_1(80°)$ fits. Thus, for optimality, we can apply Corollary 6 with $S = T_0 = \triangle ABC$, $T = T_1(80°) = \triangle DEF$, and $Z = T^* = \triangle ADC$ (recall Figure 3). In fact, there are many smallest universal covers (of the same area) for the family $\{T_0, T_1(80°)\}$. However, we can show that any smallest universal cover that accommodates $T_1(\theta)$ for every θ should coincide with T^*; the proof is omitted. The uniqueness then follows immediately.

4 Two triangles

In the following theorem, we describe how to find a triangle that is a smallest universal cover for a given family of two triangles.

▶ **Theorem 7.** *Let S and T be triangles. Then there is a triangle Z that is a smallest universal cover for the family $\{S, T\}$.*

Proof. If S fits in T or if T fits in S, the statement is true, so we assume that this is not the case. Let S' be the smallest triangle similar to S such that T fits in S'. This implies that T maximally fits into S', so by Lemma 2 there are three cases. We denote S by $\triangle ABC$, S' by $\triangle A'B'C'$, and T by $\triangle PQR$.
Case 1. P and Q lie on the edge $A'B'$, and $R = C'$; see Figure 5. We first observe

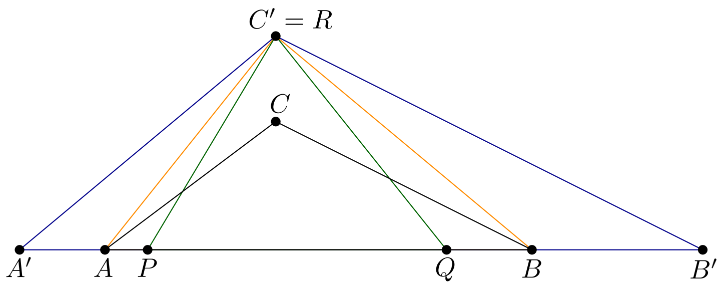

Figure 5 Proof of Theorem 7 - Case 1.

that $|AB| > |PQ|$: otherwise, we can place the segment AB inside the segment PQ, which causes C to fall inside T, and S to fit into T, a contradiction. We can therefore place AB inside $A'B'$ so that it covers PQ and C lies inside T. Then the triangle $Z = \triangle ABR$ is a universal cover for S and T. Then

$$\frac{|Z|}{|S|} = \frac{|ABR|}{|ABC|} = \frac{|A'C'|}{|AC|} \qquad \text{and} \qquad \frac{|S'|}{|S|} = \left(\frac{|A'C'|}{|AC|}\right)^2,$$

so by Corollary 6 Z is a smallest universal cover for $\{S, T\}$.
Case 2. PQ coincides with $A'B'$; see Figure 6. Let h_R be the height of R in T, let h_C be the height of C in S. We have $h_C > h_R$, since otherwise S fits into T by a placement in which $C = R$ and AB is parallel to PQ. We can therefore place $S = \triangle ABC$ such that A and B are on the segment $A'B'$ and C is on the segment RC'. Then $Z = \triangle PQC$ is a smallest universal cover for $\{S, T\}$ by Corollary 6 since

$$\frac{|Z|}{|S|} = \frac{|PQC|}{|ABC|} = \frac{|PQ|}{|AB|} = \frac{|A'B'|}{|AB|} \qquad \text{and} \qquad \frac{|S'|}{|S|} = \left(\frac{|A'B'|}{|AB|}\right)^2.$$

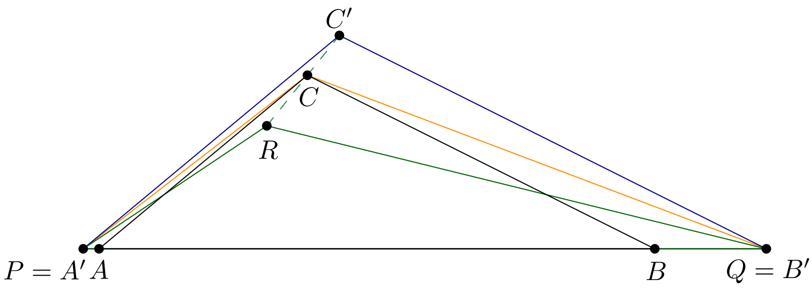

Figure 6 Proof of Theorem 7 - Case 2.

Case 3. P coincides with A', Q lies on the edge $A'B'$, and R lies on the edge $B'C'$. Let again h_R be the height of R in T, let h_C be the height of C in S.

If $h_C \geqslant h_R$, then we can place $S = \triangle ABC$ such that $B = B'$, A lies on $A'B'$, and C lies on the segment RC'; see Figure 7. Then $Z = \triangle PBC$ is a smallest universal cover for $\{S, T\}$

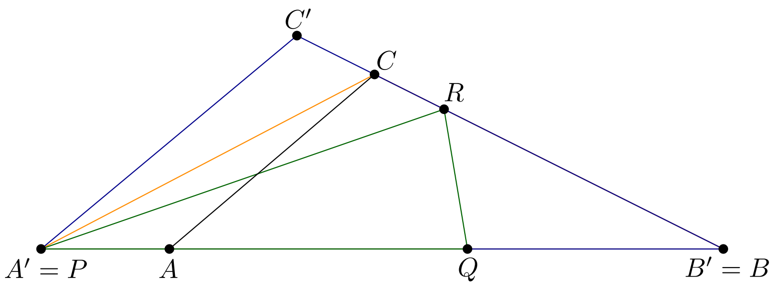

Figure 7 Proof of Theorem 7 - Case 3 when $h_C \geqslant h_R$.

by Corollary 6 since

$$\frac{|Z|}{|S|} = \frac{|PBC|}{|ABC|} = \frac{|PB|}{|AB|} = \frac{|A'B'|}{|AB|} \quad \text{and} \quad \frac{|S'|}{|S|} = \left(\frac{|A'B'|}{|AB|}\right)^2.$$

It remains to consider the case where $h_C < h_R$. Then we place $S = \triangle ABC$ such that C is on the edge PR while A and B are on $A'B'$; see Figure 8. We let $Z = \triangle PBR$. We

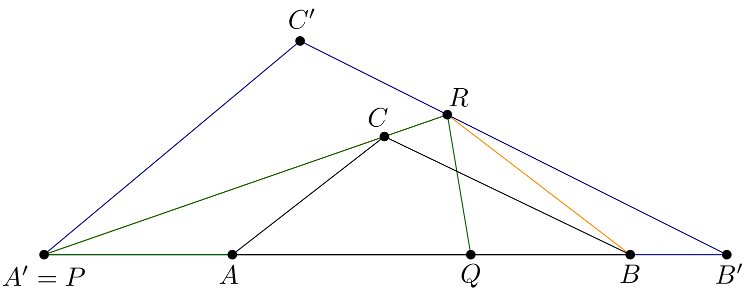

Figure 8 Proof of Theorem 7 - Case 3 when $h_C < h_R$.

observe that $|CBR| = |CBB'|$, since the two triangles have the same base and the same height, as $B'C'$ is parallel to BC. Therefore

$$\frac{|Z|}{|S|} = \frac{|PBR|}{|ABC|} = \frac{|PB'C|}{|ABC|} = \frac{|PB'|}{|AB|} = \frac{|A'B'|}{|AB|}.$$

On the other hand,

$$\frac{|S'|}{|S|} = \left(\frac{|A'B'|}{|AB|}\right)^2,$$

so Corollary 6 again implies that Z is a smallest universal cover for $\{S, T\}$. ◀

5 Two triangles are not enough

▶ **Theorem 8.** *There exists a three-element family $\mathcal{T}_3 = \{\triangle ABC, \triangle DEF, \triangle GHI\}$ whose smallest universal cover is larger than a smallest universal cover for any two of the triangles.*

Proof. (Sketch) We start by constructing three triangles as follows:

- $\triangle ABC$ is an equilateral triangle of side length 2 and thus of height $\sqrt{3}$;
- $\triangle DEF$ is an isosceles triangle where $|DF| = |EF|$, $|DE| = 6$, and the height of F is $\sqrt{3}/(1+\varepsilon)$;
- $\triangle GHI$ is an isosceles triangle with $|GI| = |HI|$, the height of G and H is ε, and the projection KI of HI on GI has length $6 - \varepsilon$.

By applying Theorem 7, we can conclude that, for each proper subfamily of $\mathcal{T}_3$, a smallest universal cover is of area at most $3\sqrt{3}$. Now we assume for a contradiction that a universal cover for $\mathcal{T}_3$ of area $3\sqrt{3}$ exists, and proceed as in the proof of Lemma 5. ◀

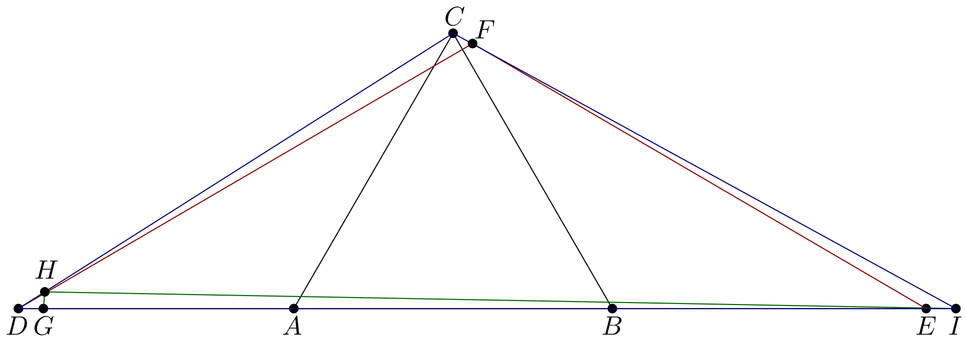

Figure 9 A family of three triangles whose every proper subfamily has a smaller universal cover.

We conjecture that if we arrange the three triangles as in Figure 9 (so that H lies on CD and F lies on CI), then $\triangle CDI$ is the unique smallest universal cover for $\mathcal{T}_3$.

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