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Zeinab Bakhtiari[a], Hans van Ditmarsch[a], Umberto Rivieccio[b]

[a] *CNRS, LORIA, Université de Lorraine, France*
[b] *DIMAp, Universidade Federal do Rio Grande do Norte, Brazil*

Abstract

Baltag, Moss, and Solecki proposed an expansion of classical modal logic, called *logic of epistemic actions and knowledge* (**EAK**), in which one can reason about knowledge and change of knowledge. Kurz and Palmigiano showed how duality theory provides a flexible framework for modelling such epistemic changes, allowing one to develop dynamic epistemic logics on a weaker propositional basis than classical logic (for example an intuitionistic basis). In this paper we show how the techniques of Kurz and Palmigiano can be further extended to define and axiomatize a *bilattice logic of epistemic actions and knowledge* (**BEAK**). Our propositional basis is a modal expansion of the well-known four-valued logic of Belnap and Dunn, which is a system designed for handling inconsistent as well as potentially conflicting information. These features, we believe, make our framework particularly promising from a computer science perspective.

Keywords: Dynamic epistemic logic, bilattices, modal logic, algebraic models, duality

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> *You are lost on Place Stanislas in the historical center of Nancy and you need to catch a train. So you accost a friendly and French looking person and there you go, pointing to the right: "Is this the way to the railway station?" "Oui." (Yes.) Merci, etc., you each go your way, but, a few moments later, while still remaining in some doubt, you ask another person, and then pointing in the opposite direction: "Is this the way to the railway station?" "Oui." What will you do?*

(First lesson: when asking directions, never *suggestively point in one direction.) You will probably resolve the inconsistency by yet further communication (or consultation of a map, say) before you continue on your way. And sure enough, the next person you ask does not even answer the question and shrugs her shoulders before walking on. Inconsistent or absent responses in dynamic interaction are just as common as inconsistency in static information. Propositions that can be true, false, both (true and false), or neither are modelled with bilattices. In this work we investigate the dynamic modal logic of bilattices, where not only propositions but also actions have four-valued features.*

1. Introduction

In the past decades, reasoning about *knowledge* and *information change* has gained a prominent place in various areas of artificial intelligence and computer science such as distributed systems [26], protocol verification [27], and game theory [3]. In these areas agents have to deal with *incomplete* and *inconsistent* information. For example, in distributed systems, agents receive information from multiple sources that may be inconsistent. Moreover, in real-world situations, agents do not have complete information about all aspects of the world and their reasoning power is bounded by thresholds such as time and limited memory [11]. Under such circumstances, applying a classical approach to model information change may not be appropriate because it suffers from the *logical omniscience* problem [46]; that is, the agents know all the consequences of what they know. As a result, they cannot hold contradictory knowledge without "knowing" every sentence of the language, because a contradiction classically entails any formula. Several approaches have been proposed to formalize inconsistent and incomplete information in the literature, see e.g. [6, 7, 31, 30, 11, 12]. To set the stage for future discussion, we are going to review the most closely related works. In [6] Belnap proposed a *four-valued logic* whose semantics in-

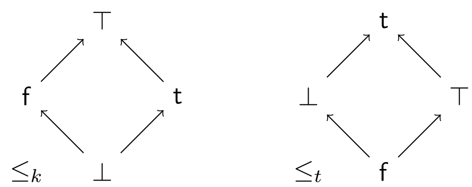


Figure 1: The four-element Belnap lattice in its two orders, the bilattice FOUR

volves, besides the classical truth values t and f, two intermediate values: $\top$ (both true and false) for handling inconsistent information and $\bot$ (neither true nor false) for incomplete information. In this logic, each atomic formula can be assigned one of the four values chosen from the set $\mathbf{4} = \{\mathsf{t}, \mathsf{f}, \bot, \top\}$. Belnap observed that his four values can be arranged in a lattice in two ways: ordering them either by information degree (the knowledge order $\leq_k$) or by the truth degree (the truth order $\leq_t$). The set $\mathbf{4}$ together with $\leq_k$ and $\leq_t$ forms two complete lattices, which are shown in Figure 1. Given two truth values x and y, $x \leq_t y$ can be read as "y is at least as true as x", while $x \leq_k y$ means that "y contains at least as much information as x".

Belnap's four-valued logic inspired Levesque to address the logical omniscience problem. In [31] he proposed a logic of explicit and implicit belief. Explicit beliefs are actively entertained by the agent, whereas implicit beliefs include the logical consequences of her explicit beliefs. This logic has a modality for explicit belief and a modality for implicit belief. The interpretation of these operators is based on situation semantics. Unlike in possible worlds, in a situation a sentence can be true, false, both true and false (*incoherent situation*), or neither true nor false (*incomplete situation*). From our perspective, [31] establishes a significant link between many-valued logics and epistemic logics. An objection raised against Levesque's model is that it is restricted to a single agent environment and therefore does not account for nested beliefs [37]. Fagin and Halpern address multi-agent belief in their logic of knowledge (or belief) and awareness [12]. The semantics of this awareness logic is based on possible worlds and does not allow the agents to have contradictory knowledge, but the awareness function at each possible world provides an effect that is similar to an incomplete situation. In [42, 43], Sim compares the approaches of [31] and [12] in detail and shows that the situations of [31] and the Kripke models with (un)awareness of [12] can be associated with a model based on a *bilattice* structure.

Bilattices are algebraic structures introduced by Ginsberg [22] to unify logical formalisms for default reasoning and non-monotonic reasoning. A bilattice is a set B equipped with two partial orders, the knowledge order ($\leq_k$) and the truth order ($\leq_t$), such that $(B, \leq_k)$ and $(B, \leq_t)$ are both complete lattices. The partial orders $\leq_k$ and $\leq_t$ have similar interpretations as in Belnap's logic. Belnap's four-element lattice is the smallest non-trivial bilattice. It is called FOUR. See Figure 1.

Bilattices have found applications in different research areas such as logic programming [14], semantics of natural language questions [35] and philosophical logic [13, 17]. In the 1990s Arieli and Avron [1, 2] carried bilattices to a new stage introducing bilattice-based logical systems that are suitable for non-monotonic and paraconsistent reasoning. Later on, Jung and Rivieccio [28] introduced a modal expansion of the logic of [1] that can be used to reason about knowledge, belief, time, and obligation. The formulas of this logic are interpreted in Kripke frames wherein both the accessibility relation and the valuation function are four-valued. Four-valued accessibility relations go back to Fitting [15, 16], who suggested a family of many-valued modal logics and generalized Kripke models involving many-valued accessibility relations. He argued in [16] that many-valued accessibility relations are natural to formalize that some worlds alternative to the real world are more relevant than others.

A similar formalism to that of [28] was proposed in [36]. They studied a Belnapian version of the modal logic **K**. The semantics of this logic is based on Kripke models where valuations are four-valued (as in [28]), however, the accessibility relation is classical. Because of this, the modal operators of [36] differ from those of [28], although the propositional base of both logics is the same. The formalism of [28] is the more general, because one can define the modal operators of [36] in the language [28], but not the other way round [28, Prop. 2].

In this work we develop a bilattice-based modal logic with dynamic operators that enable us to reason about information change in the presence of incomplete or inconsistent information. We build our logic by combining the *logic of epistemic action and knowledge* (**EAK**) of [5] with the bilattice-valued modal logic of [28]. The logic **EAK** extends classical modal logic with an operator for reasoning about the effects of epistemic actions, as represented by action models. Epistemic actions are events by which agents receive new information about the world, whilst leaving the facts of the world itself unchanged. An action model is a relational structure similar to a Kripke model, where the accessibility relation between two actions (points in the action model domain) represent an agent's uncertainty as to which action actually occurred. The structure of action models should of course fit that of Kripke models, with four-valued accessibility relations. How to give intuitive interpretations to such four-valued action models is non-trivial, and we will give this ample attention.

Formally, epistemic changes are modeled via the so-called *product update* construction on the Kripke models that provides a relational semantics for **EAK**. Through the product update, a Kripke model encoding the current epistemic setup of a group of agents is replaced by an updated model.

An adequate formal treatment of **EAK** and dynamic epistemic logics, from a syntactic as well as a semantic point of view, faces non-trivial technical problems which become even more serious when moving to a non-classical setting [18]. Such problems can be addressed in an algebraic framework. An elegant and versatile approach to the algebraic treatment of dynamic epistemic logic has been developed in a recent series of papers [29, 32, 38, 39, 9], in which the authors define non-classical counterparts of dynamic logics.

The contribution of the present paper is that we extend the methods of [29, 38, 39] by introducing a suitable notion of product update on relational and algebraic models of bilattice modal logic, thus obtaining a semantics and a complete axiomatization for a bilattice-based version of **EAK** (called **BEAK**). We restrict ourselves to the single-agent setting, but the multi-agent generalization is straightforward. We provide motivating examples for our logic.

The paper is organized as follows. Section 2 recalls the necessary definitions and results on modal bilattice logic. It provides the static modal fragment on which we build our bilattice-based dynamic epistemic logic. Section 3 expounds the technical details of the update mechanism on the algebraic structures (modal bilattices), and introduces an algebraic semantics and a relational semantics for our logic, that are then shown to be equivalent via duality. In Section 4 we introduce a Hilbert-style calculus for **BEAK**, and we show its soundness and completeness. Completeness is shown by a reduction to the static fragment. Section 5 gives a detailed case study illustrating the usage of epistemic dynamics in a bilattice setting. Readers wishing to sharpen their intuitions on knowledge (change) and bilattices, or wanting to ascertain the relevance of our framework for such settings, are suggested to read this section earlier.

2. Bilattice modal logic

In this section we introduce bilattice modal logic and recall facts and definitions that will be needed to develop our bilattice-based logic of epistemic actions and knowledge. We refer the reader to [28, 40] for further details, as well as for background discussion and motivation on bilattices (see also Section 5).

Bilattice modal logic is defined by four-valued Kripke models (W, R, V), in which both valuations and the accessibility relation $R\colon W \times W \to \mathsf{FOUR}$ take values into the four-element Belnap bilattice FOUR (Figure 1). The sentential language is that of classical modal logic (augmented with constants representing the elements of FOUR), but propositional connectives and modal operators are interpreted using the algebraic operations of FOUR. This logic can be easily extended to define bilattice-based logics where the modal operators are intended to model epistemic attitudes of agents, for example a four-valued analogue of modal logic **S5**.

The non-modal base of bilattice modal logic is the four-valued logic introduced by Arieli and Avron [1], which can be defined using the four-element Belnap bilattice.

We view FOUR as an algebra having operations $(\wedge, \vee, \otimes, \oplus, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top)$ of type $(2, 2, 2, 2, 2, 1, 0, 0, 0, 0)$. We note that both reducts $(\mathsf{FOUR}, \wedge, \vee, \mathsf{f}, \mathsf{t})$ and $(\mathsf{FOUR}, \otimes, \oplus, \bot, \top)$ are bounded distributive lattices, where the lattice orders are denoted, respectively, by $\leq_t$ (*truth order*) and $\leq_k$ (*knowledge order*). We have, moreover, a binary *weak implication* operation $\supset$ defined by $x \supset y := y$ if $x \in \{\mathsf{t}, \top\}$ and $x \supset y := \mathsf{t}$ otherwise. *Negation* is a unary operation $\neg$ having $\bot$ and $\top$ as fixed points and such that $\neg \mathsf{f} = \mathsf{t}$ and $\neg \mathsf{t} = \mathsf{f}$. We call it *bilattice negation*.

The operations $\otimes$ and $\oplus$ need not be included in the primitive signature because they can be retrieved as terms in the language $(\wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top)$. Thus, we will consider them as abbreviations of the terms shown below, together with the following defined operations:

$$
\begin{aligned}
x \otimes y &:= (x \wedge \bot) \vee (y \wedge \bot) \vee (x \wedge y) \\
x \oplus y &:= (x \wedge \top) \vee (y \wedge \top) \vee (x \wedge y) \\
{\sim} x &:= x \supset \mathsf{f} \\
x \to y &:= (x \supset y) \wedge (\neg y \supset \neg x) \\
x * y &:= \neg (y \to \neg x) \\
x \Leftrightarrow y &:= (x \supset y) \wedge (y \supset x) \\
x \leftrightarrow y &:= (x \to y) \wedge (y \to x).
\end{aligned}
$$

The operation $\sim$ provides an alternative negation (that one might call *two-valued negation*, to distinguish it from the bilattice negation $\neg$; note that ${\sim} x$ only takes values t and f), while $\to$ is an alternative implication called *strong implication*, which is adjoint to the operation $*$, called *strong conjunction* or *fusion*.

The logic of bilattices (**LB**) of [1] can then be introduced as the propositional

$(\supset \mathbf{1})$	$\varphi \supset (\psi \supset \varphi)$	$(\supset \mathsf{f})$	$\mathsf{f} \supset \varphi$
$(\supset \mathbf{2})$	$(\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))$	$(\supset \top)$	$\varphi \supset \top$
$(\supset \mathbf{3})$	$((\varphi \supset \psi) \supset \varphi) \supset \varphi$	$(\supset \bot)$	$\bot \supset \varphi$
$(\neg\neg)$	$\varphi \Leftrightarrow \neg\neg\varphi$	$(\neg\wedge)$	$\neg(\varphi \wedge \psi) \Leftrightarrow (\neg\varphi \vee \neg\psi)$
$(\wedge\supset)$	$(\varphi \wedge \psi) \supset \varphi$ $\quad (\varphi \wedge \psi) \supset \psi$	$(\neg\vee)$	$\neg(\varphi \vee \psi) \Leftrightarrow (\neg\varphi \wedge \neg\psi)$
$(\supset\wedge)$	$\varphi \supset (\psi \supset (\varphi \wedge \psi))$	$(\neg\supset)$	$\neg(\varphi \supset \psi) \Leftrightarrow (\varphi \wedge \neg\psi)$
$(\supset \mathsf{t})$	$\varphi \supset \mathsf{t}$		
$(\supset\vee)$	$\varphi \supset (\varphi \vee \psi)$ $\quad \psi \supset (\varphi \vee \psi)$	**(MP)**	from φ and $\varphi \supset \psi$ infer ψ
$(\vee\supset)$	$(\varphi \supset \chi) \supset ((\psi \supset \chi) \supset ((\varphi \vee \psi) \supset \chi))$		

Table 1: The proof system **LB** of bilattice logic [1]

logic defined by the matrix $(\mathsf{FOUR}, \{\mathsf{t}, \top\})$ as follows. Starting from a countable set of propositional variables p, one constructs the formula algebra $\mathbf{Fm} = (Fm, \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top)$ in the usual way. Given formulas $\Gamma, \{\varphi\} \subseteq Fm$, we define $\Gamma \vDash \varphi$ iff, for all homomorphisms $V\colon \mathbf{Fm} \to \mathsf{FOUR}$, if $V(\gamma) \in \{\mathsf{t}, \top\}$ for all $\gamma \in \Gamma$, then also $V(\varphi) \in \{\mathsf{t}, \top\}$. Arieli and Avron [1] provided an axiomatization for **LB**, which is given in Table 1. The axioms of **LB** are the axioms of classical logic in the language $(\wedge, \vee, \supset, \mathsf{f}, \mathsf{t})$, plus the axioms that characterize the interaction of negation with other operators and constants.

In [28] it was proposed to expand this logic semantically with modal operators, by considering *four-valued Kripke models*, i.e., structures (W, R, V) such that R and V are both four-valued. That is, one defines $R\colon W \times W \to \mathsf{FOUR}$ and $V\colon \mathbf{Fm} \times W \to \mathsf{FOUR}$. We call (W, R) a *four-valued Kripke frame*. Valuations are required to be homomorphisms in their first argument, so they preserve all non-modal connectives (including the four constants). The modal operator $\Diamond$ is defined as follows: for every $w \in W$ and every $\varphi \in Fm$,

$$V(\Diamond\varphi, w) := \bigvee\{R(w, w') * V(\varphi, w') \mid w' \in W\}$$

where $\bigvee$ denotes the infinitary version of $\vee$ in FOUR and $*$ is the strong conjunction introduced above. The dual operator $\Box$ is defined as $V(\Box\varphi, w) :=$

$\bigwedge\{R(w,w') \to V(\varphi,w') \mid w' \in W\}$, where $\bigwedge$ denotes the infinitary version of $\wedge$ in FOUR, and $\to$ is the strong implication.[1] It holds that $V(\Box\varphi,w) = V(\neg\Diamond\neg\varphi,w)$ for all $w \in W$ and all valuations V, that is, the two modal operators are inter-definable as in the classical case. In the present paper we take $\Diamond$ as primitive, and use $\Box$ as a shorthand for $\neg\Diamond\neg$.

A modal consequence relation can now be introduced in the usual way. We say that a point $w \in W$ of a four-valued model $M = (W,R,V)$ satisfies a formula $\varphi \in Fm$ iff $V(\varphi,w) \in \{\mathsf{t},\top\}$, and we write $M,w \vDash \varphi$. For a set of formulas $\Gamma \subseteq Fm$, we write $M,w \vDash \Gamma$ to mean that $M,w \vDash \gamma$ for each $\gamma \in \Gamma$. The (local) consequence $\Gamma \vDash \varphi$ holds if, for every model $M = (W,R,V)$ and every $w \in W$, it is the case that $M,w \vDash \Gamma$ implies $M,w \vDash \varphi$. For $\emptyset \models \varphi$ we write $\models \varphi$ (for 'φ is valid').

This consequence relation inherits from the non-modal fragment the deduction theorem in the following form: $\Gamma \vDash \varphi$ if and only if there is a finite $\Gamma' \subseteq \Gamma$ such that $\vDash \bigwedge\Gamma' \supset \varphi$, where $\bigwedge\Gamma' := \bigwedge\{\gamma \in \Gamma'\}$. This will remain true for the dynamic expansion **BEAK**. It implies that in our axiomatization task we can without loss of generality restrict our attention to valid formulas. This consequence relation is axiomatized in [28], in the logic that we call here **LB**$\boldsymbol{\Box}$. The axiomatization of **LB**$\boldsymbol{\Box}$ is displayed in Table 2.

A derivation is a sequences of formulas such that every formula is an instantiation of an axiom or the result of applying a rule to formulas prior in the sequence. If φ occurs in a derivation we write $\vdash \varphi$, for 'φ is a theorem'. By $\Gamma \vdash \varphi$ we mean that there is a finite subset Γ' of Γ such that $\vdash \bigwedge\Gamma' \supset \varphi$.

The *necessitation rule* "from φ, infer $\Box\varphi$" does not hold [28, Section III.A], and the *normality axiom* $\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$ also does not hold [8]. We give a brief account of completeness results for **LB**$\boldsymbol{\Box}$, and of the proof of algebraic

[1] Alternative definitions of $\Box$ and $\Diamond$ are discussed in [28, p. 440]. If one were to replace $\to$ by $\supset$ in the definition of $\Box$, i.e., $V(\Box\varphi,w) := \bigwedge\{R(w,w') \supset V(\varphi,w') \mid w' \in W\}$, then this is equivalent to $V(\Box\varphi,w) := \bigwedge\{V(\varphi,w') \mid w' \in W, R(w,w') \in \{\mathsf{t},\top\}\}$. This alternative is employed in [36]. As shown in [28], it is less expressive than the version with $\to$.

($\Box$**t**)	$\Box \mathsf{t} \leftrightarrow \mathsf{t}$
($\Box\wedge$)	$\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$
($\Box\bot$)	$\Box(\bot \to \varphi) \leftrightarrow (\bot \to \Box\varphi)$
($\Box-$**monotonicity**)	from $\varphi \to \psi$ infer $\Box\varphi \to \Box\psi$

Table 2: The proof system for **LB$\Box$** consists of all axioms and rules of **LB** (Table 1) plus these three axioms and rule [28].

completeness, as we will build on them later on. We begin with completeness with respect to Kripke models (relational completeness).

Theorem 1 (Relational completeness [28, Theorem 19])
For all $\Gamma, \{\varphi\} \subseteq Fm$, $\Gamma \vdash \varphi$ *iff* $\Gamma \models \varphi$. $\dashv$

In order to state the algebraic completeness theorem we need to introduce a class of algebras providing an alternative semantics for bilattice modal logic. A *modal bilattice* is an algebra $\mathbf{B} = (B, \wedge, \vee, \supset, \sim, \Diamond, \mathsf{f}, \mathsf{t}, \bot, \top)$ (where we may label the operations and constants with the name of the algebra, as in $\Diamond_{\mathbf{B}}$, $\mathsf{f}_{\mathbf{B}}$, etc., to distinguish them from those in other algebras) such that the $\Diamond$-free reduct of $\mathbf{B}$ is an *implicative bilattice* (i.e., satisfies exactly all identities that are valid in FOUR) and the following identities are satisfied: (i) $\Diamond\mathsf{f} = \mathsf{f}$, (ii) $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$, (iii) $\Box(x \supset \bot) = \Diamond x \supset \bot$. It is easy to show that identities (i)-(iii) correspond, respectively, to axioms (i)-(iii) of our calculus, and that the presentation of modal bilattices given here is equivalent to the one in [28].

We say that a subset $F \subseteq B$ of a modal bilattice $\mathbf{B}$ is a *bifilter* if F is a lattice filter of the truth lattice such that $\top \in F$ (in which case it follows that F is also a filter of the knowledge lattice). Given a pair $(\mathbf{B}, F)$ and formulas $\Gamma, \{\varphi\} \subseteq Fm$, we write $\Gamma \models_{(B,F)} \varphi$ to mean that, for every modal bilattice homomorphism $V\colon \mathbf{Fm} \to \mathbf{B}$, if $V(\gamma) \in F$ for all $\gamma \in \Gamma$, then also $V(\varphi) \in F$. A valid formula φ is one satisfying $V(\varphi) \geq_t \top_{\mathbf{B}}$ for every $\mathbf{B}$ and V. We can then state algebraic completeness.

Theorem 2 (Algebraic completeness [28, Theorem 10]) *For all* $\Gamma, \{\varphi\} \subseteq Fm$, $\Gamma \vdash \varphi$ *iff* $\Gamma \vDash_{(\mathbf{B},F)} \varphi$ *for any modal bilattice* $\mathbf{B}$ *and any bifilter* $F \subseteq B$.$\dashv$

Just as with classical modal logic, the relational and the algebraic semantics for bilattice modal logic are related via a Stone-type duality [28, Theorem 18]. In the case of bilattices, another key ingredient that greatly simplifies the picture is the so-called *twist structure* representation, which works as follows. Let $\mathbf{A} = (A, \wedge, \vee, \sim, \Diamond^+, \Diamond^-, 0, 1)$ be a *bimodal Boolean algebra* [28, Definition 11], i.e., a structure such that $(A, \Diamond^+)$ and $(A, \Diamond^-)$ are both modal Boolean algebras [10], and no relation between $\Diamond^+$ and $\Diamond^-$ is assumed. The dual operators $\Box^+$ and $\Box^-$ are defined in the usual way by setting $\Box^+ x := {\sim}\Diamond^+{\sim}x$ and $\Box^- x := {\sim}\Diamond^-{\sim}x$. The *twist structure over* $\mathbf{A}$ is defined as the algebra $\mathbf{A}^{\bowtie} = (A \times A, \wedge, \vee, \supset, \neg, \Diamond, \mathsf{f}, \mathsf{t}, \bot, \top)$ with operations given, for all $(a_1, a_2), (b_1, b_2) \in A \times A$, by:

$$
\begin{aligned}
(a_1, a_2) \wedge (b_1, b_2) &:= (a_1 \wedge b_1,\ a_2 \vee b_2) \\
(a_1, a_2) \vee (b_1, b_2) &:= (a_1 \vee b_1,\ a_2 \wedge b_2) \\
(a_1, a_2) \supset (b_1, b_2) &:= ({\sim}a_1 \vee b_1, a_1 \wedge b_2) \\
\neg(a_1, a_2) &:= (a_2,\ a_1) \\
\Diamond(a_1, a_2) &:= (\Diamond^+ a_1,\ \Box^+ a_2 \wedge {\sim}\Diamond^- a_1) \\
\mathsf{f} &:= (0, 1) \\
\mathsf{t} &:= (1, 0) \\
\bot &:= (0, 0) \\
\top &:= (1, 1)
\end{aligned}
$$

Any twist structure is a modal bilattice. Conversely, *any modal bilattice is isomorphic to a twist structure* [28, Theorem 12]. This means that instead of working directly with modal bilattices, we will work with twist structures.

The twist structure construction allows us to relate four-valued Kripke frames and modal bilattices via Jónsson-Tarski duality for classical modal logic (see, e.g., [23]). Given a modal bilattice $\mathbf{B}$ viewed as a twist structure $\mathbf{A}^{\bowtie}$, we can consider the structure $(\mathbf{A}_\bullet, R^+, R^-)$, where $(\mathbf{A}_\bullet, R^+)$ and $(\mathbf{A}_\bullet, R^-)$ are the classical Kripke frames associated to the modal Boolean algebras $(\mathbf{A}, \Diamond^+)$ and $(\mathbf{A}, \Diamond^-)$ according to the Jónsson-Tarski duality. The relations R^+ and R^- can obviously be combined into one four-valued relation R by letting $R(w, w') = \mathsf{t}$ iff $(w, w') \in R^+ \cap R^-$, $R(w, w') = \top$ iff $(w, w') \in R^+ \backslash R^-$, $R(w, w') = \bot$ iff $(w, w') \in R^- \backslash R^+$, and $R(w, w') = \mathsf{f}$ iff $(w, w') \notin R^+ \cup R^-$. In this way we obtain a four-valued Kripke frame $(\mathbf{A}_\bullet, R)$. Conversely, every four-valued Kripke frame $\mathcal{F} = (W, R)$ can be viewed as a pair of frames (W, R^+, R^-) by defining $(w, w') \in R^+$ iff $R(w, w') \in \{\mathsf{t}, \top\}$ and $(w, w') \in R^-$ iff $R(w, w') \in \{\mathsf{t}, \bot\}$. Thus, according to Jónsson-Tarski duality, we obtain classical frames $\mathcal{F}_+ = (W, R^+)$ and $\mathcal{F}_- = (W, R^-)$, to which one associates modal Boolean algebras $(\mathcal{F}_+)^+ = (\mathcal{P}(W), \cap, \cup, \sim, \Diamond^+)$ and $(\mathcal{F}_-)^+ = (\mathcal{P}(W), \cap, \cup, \sim, \Diamond^-)$ according to Jónsson-Tarski duality, in which $\sim$ is the Boolean complementation and the operations $\Diamond^+$, $\Diamond^-$ are defined, for each $U \subseteq W$, by $\Diamond^+ U := (R^+)^{-1}[U]$ and $\Diamond^- U := (R^-)^{-1}[U]$. Since $(\mathcal{F}_+)^+$ and $(\mathcal{F}_-)^+$ share the same carrier set, we actually have a bimodal Boolean algebra $\mathcal{F}^+ = (\mathcal{P}(W), \cap, \cup, \sim, \Diamond^+, \Diamond^-)$, from which a modal bilattice $(\mathcal{F}^+)^{\bowtie}$ can be obtained via the twist structure construction. We then define the complex algebra of a Kripke frame $\mathcal{F} = (W, R^+, R^-)$ as the twist structure $\mathcal{F}^\bullet := (\mathcal{P}(W), \cap, \cup, \sim, \Diamond^+, \Diamond^-)^{\bowtie}$. The correspondence between four-valued Kripke frames and modal bilattices extends to Kripke models and algebraic models, which implies that the relational and the algebraic semantics for bilattice modal logic are indeed equivalent [28].

3. Epistemic updates on bilattices

3.1. Language of **BEAK** *and action models*

In this section we introduce the bilattice logic of epistemic action and knowledge (**BEAK**) through an algebraic semantics and through a relational semantics. The algebraic semantics of **BEAK** draws inspiration from the algebraic analysis of epistemic updates and pseudo-quotients on modal bilattices. We use duality to define a relational semantics based on four-valued Kripke models. We then show that the relational semantics is equivalent to the algebraic semantics.

Definition 3 (Language of BEAK and action models) *Let* AtProp *be a countable set of propositional variables. The set* $\mathcal{L}$ *of formulas* φ *of* **BEAK** *is defined over the signature* $(\wedge, \vee, \supset, \neg, \Diamond, \langle\alpha\rangle, \mathsf{f}, \mathsf{t}, \bot, \top)$*, with new inductive construct* $\langle\alpha\rangle\varphi$*, where* α *is a four-valued* action model over $\mathcal{L}$*, defined as a tuple* $\alpha = (K, k, R_\alpha, Pre_\alpha)$ *such that* K *is a finite non-empty set,* $k \in K$*,* $R_\alpha \colon K \times K \to \mathsf{FOUR}$ *and* $Pre_\alpha \colon K \to \mathcal{L}$*.* $\dashv$

The point of action model α will always be k. If we wish to emphasize the point we may write $\alpha(k)$ or α_k instead of α. We overload the use of Pre_α: it is not merely the function Pre_α, but also stands for the formula $Pre_\alpha(k)$ (where the context easily disambiguates use). When shifting points to $l \in K$ with $l \neq k$ we are explicit and always write $\alpha(l)$ (or α_l) and $Pre_\alpha(l)$. Derived connectives $\sim, \Box, \oplus, \otimes, \to, *, \leftrightarrow$ are defined as before. Moreover, we let $[\alpha]\varphi := \neg\langle\alpha\rangle\neg\varphi$.

We refer the reader to [5] for further explanations and motivation on action models. Two considerations justify a *four-valued* relation R_α. Firstly, in the **EAK** setting action models are Kripke frames, and in bilattice modal logic Kripke frames are four-valued [28, 40]. Secondly, our choice is more general from a mathematical point of view, because by restricting the range of values of R_α to t and f we obtain two-valued action models as a special case.

The dynamic formulas have the same meaning as in **EAK**. The meaning of $\langle\alpha\rangle\varphi$ is: there is an execution of action model α after which φ is the case.

As mentioned, defining non-classical counterparts of dynamic logics is more easily accomplished with algebraic tools, which provide us with a flexible semantics that can be accommodated with incomplete and contradictory information.

3.2. Algebraic semantics for **BEAK**

Epistemic updates are modeled on the algebraic counterpart of the logic through the *pseudo-quotient* construction. The definition of pseudo-quotient adopted is that of [38, 39]. In the **EAK** setting, we also need to introduce a suitable notion of *intermediate structure.* First, we need to define an action model over a modal bilattice.

Definition 4 (Action model over a modal bilattice) *An* action model *over a modal bilattice* $\mathbf{B}$ *is a tuple* $a = (K, k, R_a, Pre_a)$ *such that* K *is a finite non-empty set,* $k \in K$*,* $R_a \colon K \times K \to \mathsf{FOUR}$ *is a four-valued accessibility relation and* $Pre_a \colon K \to \mathbf{B}$*.* $\dashv$

The notation for action models over a logical language (interpreted on relational models) and the notation for action models over a modal bilattice (i.e., defined algebraically) is of course very similar. The reason for that similar notation is that we will later show a correspondence between the relational semantics and the algebraic semantics. We should further point out that action models over a logical language are presented as a somewhat hybrid syntactic/semantic (language/models) object, whereas action models over bilattices are pure semantics. (An alternative presentation wherein action models *frames* are semantic, and action models are syntax would also have been possible.)

Note that, as for action models α, by convention the point of an action model a over a modal bilattice is k, unless explicitly indicated otherwise. Given a modal bilattice $\mathbf{B}$ and an action model $a = (K, k, R_a, Pre_a)$ over $\mathbf{B}$, we can consider the direct power $\mathbf{B}^K$ having as carrier set the collection of maps B^K. Obviously $\mathbf{B}^K$ is an algebra in the same equational class, but as [29] points out, in the cases in which $\mathbf{B}$ is the complex algebra of some Kripke frame $\mathcal{F} = (W, R^+, R^-)$, it is not a suitable candidate for an intermediate structure, because it only depends on $\mathbf{B}$ and K, ignoring the rest of the information carried by the action model. In order to avoid this, we let all non-modal operations on B^K be defined as in a direct power, and we lift the modal operator.

Definition 5 (Intermediate structure) *For every Kripke frame* $\mathcal{F} = (W, R^+, R^-)$*, and every action model* $a = (K, k, R_a^+, R_a^-, Pre_a)$ *over the complex algebra of* $\mathcal{F}$*, the* intermediate structure *is defined as* $\coprod_a \mathcal{F} = (\coprod_K W, R^+ \times R_a^+, R^- \times R_a^-)$ *where* $\coprod_K W$ *is the* $|K|$*-fold coproduct of* W *(which is isomorphic, as a set, to* *the Cartesian product $W \times K$), and*

$$(R^+ \times R_a^+)((w,i),(u,j)) \quad \textit{iff} \quad R^+(w,u) \ \textit{and} \ R_a^+(i,j);$$

$$(R^- \times R_a^-)((w,i)(u,j)) \quad \textit{iff} \quad R^-(w,u) \ \textit{and} \ R_a^-(i,j).$$

Also, the updated frame structure $\mathcal{F}_a$ is defined as the subframe of $\coprod_a \mathcal{F}$ the domain of which is the subset

$$W_a = \{(w,j) \in \coprod_K W \mid w \in Pre_a(j)\}.$$

$\dashv$

Note that the precondition part of the action model is not used in the definition of the intermediate structure, but only plays a role in the subsequently defined *updated* frame $\mathcal{F}_a$.

The following dual characterization of intermediate structures on modal bilattices is based on the ideas used in [29].

Definition 6 (Intermediate algebra) *For every modal bilattice* $\mathbf{B}$ *and every action model* $a = (K, k, R_a, Pre_a)$ *over* $\mathbf{B}$*, the* intermediate algebra $\prod_a \mathbf{B}$ *is the algebra in which all non-modal operations are defined as in the direct power* $\mathbf{B}^K$*, and the modal operator is given, for each* $f \in B^K$ *and* $j \in K$*, by*

$$\Diamond_{\prod_a \mathbf{B}} f(j) = \bigvee \{\Diamond_{\mathbf{B}} f(i) \mid i \in K \ \textit{and} \ R_a(j,i) \in \{\mathtt{t}, \top\}\}$$

where $\bigvee$ *denotes the join in* $\mathbf{B}$ *(because the set* K *is finite, it always exists).* $\dashv$

Proposition 7 *Let* $a = (K, k, R_a, Pre_a)$ *be an action model over a modal bilattice* $\mathbf{B}$*. Then the intermediate bilattice* $\prod_a \mathbf{B}$ *is a modal bilattice in which the dual operator is given, for each* $f \in B^K$ *and* $j \in K$*, by*

$$\Box_{\prod_a \mathbf{B}} f(j) = \bigwedge \{\Box_{\mathbf{B}} f(i) \mid i \in K \ \textit{and} \ R_a(j,i) \in \{\mathtt{t}, \top\}\}.$$

Taking advantage of the twist-structure representation one can equivalently restate Definition 5 as follows. We view a modal bilattice $(\mathbf{B}, \Diamond)$ as a twist structure over some bimodal Boolean algebra $(\mathbf{A}, \Diamond^+, \Diamond^-)$. Similarly, we view the four-valued relation R_a as a pair of two-valued relations R_a^+, R_a^- defined, for all $i, j \in K$, as $R_a^+(i,j)$ iff $R_a(i,j) \in \{\mathtt{t}, \top\}$ and $R_a^-(i,j)$ iff $R_a(i,j) \in \{\mathtt{t}, \bot\}$. Then the intermediate bilattice is given by the modal twist structure over $(\mathbf{A}^K, \Diamond^+_{\prod_a \mathbf{A}}, \Diamond^-_{\prod_a \mathbf{A}})$ where, for each $g \in A^K$ and $j \in K$,

$$\Diamond^+_{\prod_a \mathbf{A}} g(j) = \bigvee \{\Diamond^+ g(i) \mid R_a^+(j,i)\}, \qquad \Diamond^-_{\prod_a \mathbf{A}} g(j) = \bigvee \{\Diamond^- g(i) \mid R_a^-(j,i)\}.$$

The following result shows that this notion of complex algebra is consistent with our algebraic definition of intermediate structure (Definition 6).

Theorem 8 *Let* $\mathbf{B}$ *be the complex algebra of some frame* $\mathcal{F} = (W, R^+, R^-)$ *and let* $a = (K, k, R_a^+, R_a^-, Pre_a)$ *be an action model over* $\mathbf{B}$*. Then the algebra* $(\prod_a \mathbf{B}, \Diamond_{\prod_a \mathbf{B}})$ *is isomorphic to the complex algebra of* $\coprod_a \mathcal{F}$*.* $\dashv$

Proof. By definition, $\mathbf{B} = (\mathcal{P}(W), \cap, \cup, \sim, \Diamond^+, \Diamond^-)^{\bowtie}$. To simplify notation, let $\mathbf{A} = (\mathcal{P}(W), \cap, \cup, \sim, \Diamond^+, \Diamond^-)$, and let $a^+ = (K, k, R_a^+, Pre_a^+)$ and $a^- = (K, k, R_a^-, Pre_a^-)$ be two standard action models over $\mathbf{A}$, where Pre_a^+ and Pre_a^- are first and second components of Pre_a according to the twist-structure representation of $\mathbf{B}$. Then $\prod_a \mathbf{B} = (A^K, \cap, \cup, \sim, \Diamond_{\prod_{a^+} A}, \Diamond_{\prod_{a^-} A})^{\bowtie}$, where $\prod_{a^+} \mathbf{A}$ and $\prod_{a^-} \mathbf{A}$ are the intermediate algebras of $\mathbf{A}$ for the action models a^+ and a^- [29, Section 3]. By [29, Proposition 3.1], it follows that $(A^K, \cap, \cup, \sim, \Diamond_{\prod_{a^+} \mathbf{A}})$ and $(A^K, \cap, \cup, \sim, \Diamond_{\prod_{a^-} \mathbf{A}})$ are isomorphic to the complex algebras of $\coprod_{a^+} \mathcal{F}$ and $\coprod_{a^-} \mathcal{F}$, respectively. Then

$$(\mathcal{P}(W \times K), \cap, \cup, \sim, \Diamond^+, \Diamond^-) \cong (A^K, \cap, \cup, \sim, \Diamond_{\prod_{a^+} \mathbf{A}}, \Diamond_{\prod_{a^-} \mathbf{A}})$$

which means that $(\coprod_a \mathcal{F})^\bullet \cong (\prod_a \mathbf{B}, \Diamond_{\prod_a \mathbf{B}})$.

At this point we can apply the definition of pseudo-quotient from [38, 39] to obtain a suitable notion of quotient of an intermediate structure. Given an action model $a = (K, k, R_a, Pre_a)$ over a modal bilattice $\mathbf{B}$, we define the relation $\equiv_a$ on $\prod_a \mathbf{B}$ as follows:

$$\text{For all } f, g \in B^K : \qquad f \equiv_a g \quad \text{iff} \quad f \ \wedge \sim\sim Pre_a = g \ \wedge \sim\sim Pre_a.$$

Because $Pre_a \in \prod_a \mathbf{B}$ and $\wedge, \sim$ are algebraic operations of $\prod_a \mathbf{B}$, it follows from [38, Fact 2.2] that $\equiv_a$ is a congruence of the non-modal reduct of $\prod_a \mathbf{B}$. We will denote the equivalence class of $f \in B^K$ by $[f]_a$ (or simply by $[f]$ when there is no risk of confusion) and the quotient set $\mathbf{B}^K/\equiv_a$ by $\mathbf{B}_a$. Note that, as mentioned in [38] in the similar context of public announcements, the double negations appearing in the definition of the congruence $\equiv_a$ ensure that elements that interpret equivalent propositions in $\prod_a \mathbf{B}$ are indeed identified.

Proposition 9 ([38, Fact 2.3]) *Let $a = (K, k, R_a, Pre_a)$ be an action model over a modal bilattice $\mathbf{B}$. Then:*

(i) $[f \wedge {\sim\sim} Pre_a] = [f]$ *for every* $f \in \prod_a \mathbf{B}$. *Hence, for every* $f \in \prod_a \mathbf{B}$, *there exists a unique* $g \in \prod_a \mathbf{B}$ *such that* $g \in [f]$ *and* $g \leq_t Pre_a$.

(ii) *For all* $f, g \in \prod_a \mathbf{B}$, *we have* $[f] \leq_t [g]$ *iff* $f \wedge {\sim\sim} Pre_a \leq_t g \wedge {\sim\sim} Pre_a$.

Item (i) above implies that each $\equiv_a$-equivalence class has a canonical representative, which is the unique element in the given class which is below the element ${\sim\sim} Pre_a$ in the truth order. Hence we can define an (injective) map $i' : \mathbf{B}_a \to \prod_a \mathbf{B}$ by $[f] \mapsto f \wedge {\sim\sim} Pre_a$ for all $[f] \in \mathbf{B}_a$. Denoting by $\pi\colon \prod_a \mathbf{B} \to \mathbf{B}_a$ the canonical projection map, we clearly have that the composition $\pi \circ i'$ is the identity on $\mathbf{B}_a$.

As we will learn in the next section 3.2, the map $i' : \mathbf{B}_a \to \prod_a \mathbf{B}$ plays a key role in the definition of interpretation of **BEAK** formulas on algebraic models. In the next theorem we dually characterize i' in terms of the inclusion map $i\colon \mathcal{F}_a \to \coprod_a \mathcal{F}$.

Assume a modal bilattice $\mathbf{B}$ is the complex algebra of some frame $\mathcal{F}$ (i.e., $\mathbf{B} = \mathcal{F}^\bullet$) and $a = (K, k, R_a, Pre_a)$ is an action model over $\mathbf{B}$. Then we know from [39, Section 5.2] that there is a modal bilattice isomorphism $\nu\colon (\mathcal{F}_a)^\bullet \to \mathbf{B}_a$. Letting $\mu := \nu^{-1}$, we have the following characterization.

Theorem 10 *Let $\mathbf{B} = \mathcal{F}^\bullet$ for some frame $\mathcal{F}$, and let $a = (K, k, R_a, Pre_a)$ be an action model over $\mathbf{B}$. Then $i'(b) = i[\mu(b)]$ for every $b \in \mathbf{B}_a$. It follows that $i[c] = i'(\nu(c))$ for every $c \in (\mathcal{F}_a)^\bullet$.* $\dashv$

Proof. Apply [29, Proposition 3.6] to the twist structure over $\mathbf{B}_a$.

Modal operator(s) on the pseudo-quotient can now be introduced as another application of the definitions in [38, 39]. For every action model $a = (K, k, R_a, Pre_a)$ over a modal bilattice $\mathbf{B}$ and every $f \in \prod_a \mathbf{B}$, we let

$$\Diamond_a[f] := [\Diamond_{\prod_a \mathbf{B}}(f \wedge \sim\sim Pre_a)].$$

The dual operator is given by $\Box_a[f] := \neg\Diamond_a\neg[f]$.

Proposition 11 ([38, Fact 2.4]) *For every action model $a = (K, k, R_a, Pre_a)$ over a modal bilattice $\mathbf{B}$, the algebra $(\mathbf{B}_a, \Diamond_a)$ is a modal bilattice.* $\dashv$

Definition 12 (Algebraic model) *An* algebraic model of **BEAK** *is a tuple $M = (\mathbf{B}, V)$ such that $\mathbf{B}$ is bilattice and $V : \mathsf{AtProp} \to \mathbf{B}$. For every algebraic model M and every action model α over $\mathcal{L}$, we let $\prod_\alpha M = (\prod_\alpha \mathbf{B}, \prod_\alpha V)$, where $\prod_\alpha \mathbf{B} = \prod_a \mathbf{B}$ and $a = (K, k, R_a, Pre_a)$ is the action model over $\mathbf{B}$ induced by α via V with $Pre_a = V \circ Pre_\alpha$. Moreover, we let $(\prod_\alpha V)(p) := \prod_a V(p)$ for every $p \in \mathsf{AtProp}$. Likewise, we define $M_a := (\mathbf{B}_a, \pi \circ \prod_\alpha V)$.* $\dashv$

Definition 13 (Algebraic semantics for BEAK) *Given an algebraic model $M = (\mathbf{B}, V)$, the extension map $[\![.]\!] : \mathcal{L} \to \mathbf{B}$ is defined as follows:*

$$\begin{aligned}
[\![p]\!]_M &:= V(p) && \\
[\![c]\!]_M &:= c_{\mathbf{B}} && \text{for } c \in \{\mathsf{f}, \mathsf{t}, \bot, \top\} \\
[\![\circ\varphi]\!]_M &:= \circ_{\mathbf{B}}[\![\varphi]\!]_M && \text{for } \circ \in \{\neg, \Diamond\} \\
[\![\varphi \bullet \psi]\!]_M &:= [\![\varphi]\!]_M \bullet_{\mathbf{B}} [\![\psi]\!]_M && \text{for } \bullet \in \{\wedge, \vee, \supset\} \\
[\![\langle\alpha\rangle\varphi]\!]_M &:= \sim\sim[\![Pre_\alpha]\!]_M \wedge_{\mathbf{B}} \pi_k \circ i'([\![\varphi]\!]_{M_\alpha}) && \\
[\![[\alpha]\varphi]\!]_M &:= [\![Pre_\alpha]\!]_M \supset_{\mathbf{B}} \pi_k \circ i'([\![\varphi]\!]_{M_\alpha}) &&
\end{aligned}$$

where $i' : \mathbf{B}_a \to \prod_a \mathbf{B}$ is defined as above and $\pi_k : \prod_\alpha \mathbf{B} \to \mathbf{B}$ is the projection onto the k-th coordinate. $\dashv$

We define $\Gamma \models_{\mathbf{BEAK}} \varphi$ iff, for every algebraic model $M = (\mathbf{B}, V)$ and every bifilter $F \subseteq B$, we have that $[\![\gamma]\!]_M \in F$ for all $\gamma \in \Gamma$ implies $[\![\varphi]\!]_M \in F$.

3.3. Relational semantics for **BEAK**

We will now use this algebraic semantics and duality theory to introduce a relational semantics for **BEAK**. We mimic the algebraic update construction using duality. Again, the line of argument is as in [38, 39].

Let $M = (\mathcal{F}, V)$ be a Kripke model with underlying frame $\mathcal{F} = (W, R^+, R^-)$, and let $\alpha = (K, k, R^+_\alpha, R^-_\alpha, Pre_\alpha)$ be a four-valued action model over $\mathcal{L}$. In exactly the same way as Definition 5, we can define the Kripke frame $\coprod_\alpha \mathcal{F} := (\coprod_K W,\, R^+ \times R^+_\alpha, R^- \times R^-_\alpha)$.

For any formula $\varphi \in \mathcal{L}$, we denote $V^+(\varphi) := \{w \in W \mid V(\varphi, w) \in \{\mathsf{t}, \top\}\}$ and $V^-(\varphi) := \{w \in W \mid V(\varphi, w) \in \{\mathsf{f}, \top\}\}$. The intermediate model $M \times \alpha$ is then defined as the coproduct structure

$$\coprod_\alpha M = (\coprod_\alpha \mathcal{F}, V^*)$$

where $V^*\colon \mathsf{AtProp} \times \coprod_K W \to \mathsf{FOUR}$ is given, for every $p \in \mathsf{AtProp}$ and every $(w, k) \in \coprod_K W$, by

$$V^*(p, (w, k)) = V(p, w).$$

Finally, we define the updated model $M_\alpha := (W_\times, R^+_\times, R^-_\times, V_\times)$ as follows:

$$\begin{aligned} W_\times &:= \{(w, j) \in \coprod_K W \mid w \in V^+(Pre_\alpha(j))\} \\ R^+_\times &:= (R^+ \times R^+_\alpha) \cap (W_\times \times W_\times) \\ R^-_\times &:= (R^- \times R^-_\alpha) \cap (W_\times \times W_\times) \\ V_\times(p, (w, j)) &:= V^*(p, (w, j)) \restriction_{\mathsf{AtProp} \times W_\times} \end{aligned}$$

The intermediate model $M \times \alpha$ above is of course very similar to the intermediate structure of Definition 5, and the (frame underlying the) updated model M_α is similar to the the update frame structure $\mathcal{F}_a$ of Definition 5. In particular, note that the relations $R^+ \times R^+_\alpha$ in the former and $R^+ \times R^+_a$ in the latter correspond, and also $R^- \times R^-_\alpha$ in the former and $R^- \times R^-_a$ in the latter. Then, in the former, $R^+_\times$ as the restriction of $R^+ \times R^+_a$ to the domain $W_\times$ consisting of the (world, action) pairs satisfying the precondition of that action in that world), corresponds, in the latter, to the (unnamed) restriction of $R^+ \times R_a^+$ to the domain W_a.

We can extend the valuation $V_\times$ supplied by M_α to arbitrary formulas in the usual way. In particular, the notion of satisfaction for dynamic **BEAK** modalities can now be defined relationally.

Definition 14 (Relational semantics for BEAK)

$$M, w \models \langle\alpha\rangle\varphi \quad \textit{iff} \quad M, w \models Pre_\alpha \textit{ and } M_\alpha, (w, k) \models \varphi \qquad \dashv$$

Since $M, w \models \varphi$ iff $M, w \models {\sim\sim}\varphi$, the above definition is in keeping with the algebraic semantics of Definition 13.

3.4. Equivalence of the algebraic and the relational semantics for **BEAK**

To prove the equivalence of the algebraic and the relational semantics for **BEAK** we use the method of [29], also applied in [39].

We first take a closer look at the modal bilattices that arise as complex algebras of Kripke frames, which we call *perfect modal bilattices* [39, Section 5.2].

A modal Boolean algebra $(\mathbf{A}, \Diamond)$ is called *perfect* if (i) $\mathbf{A}$ is complete, (ii) atomic, i.e. $\mathbf{A}$ is completely join-generated by its set of *atoms* $\mathsf{At}(\mathbf{A}) := \{x \in A \mid x \neq 0 \text{ and, for all } y \in A,\, y < x \text{ implies } y = 0\}$, and $\Diamond$ preserves infinitary joins. In a similar way, modal bilattices arising from four-valued Kripke frames correspond to twist structures over complete and atomic bimodal Boolean algebras, have the form $(\mathbf{A}, \Diamond^+, \Diamond^-)^{\bowtie}$ where $\mathbf{A}$ is a bimodal Boolean algebra that is complete and atomic.

We define a *perfect bimodal Boolean algebra* as a bimodal Boolean algebra $(\mathbf{A}, \Diamond^+, \Diamond^-)^{\bowtie}$ such that $(\mathbf{A}, \Diamond^+)$ and $(\mathbf{A}, \Diamond^-)$ are both perfect modal Boolean algebras. As [38] has pointed out, it follows from the duality for classical logic that the complex algebra of a Kripke frame (W, R), that $(\mathcal{P}(W), \cap, \cup, \sim, \Diamond^+, \Diamond^-)$ is a perfect bimodal Boolean algebra.

A bilattice $\mathbf{B}$ is called *perfect* iff $\mathbf{B} = \mathbf{A}^{\bowtie}$, where $\mathbf{A}$ is a perfect bimodal Boolean algebra.

Now we show by duality that there is a 1-1 correspondence between twist structures over perfect bimodal Boolean algebras and four-valued Kripke frames.

Let $\mathbf{B} = \mathbf{A}^{\bowtie}$, where $\mathbf{A} = (A, \wedge, \vee, \sim, \Diamond^+, \Diamond^-, 0, 1)$ is a bimodal Boolean algebra. We define a Kripke frame $\mathbf{B}_\bullet = (\mathsf{At}(\mathbf{A}), R^+, R^-)$ where the relations R^+ and R^- are given, for all $x, y \in \mathsf{At}(\mathbf{A})$, by

$$R^+(x, y) \quad \text{iff} \quad x \leq \Diamond^+ y, \qquad\qquad R^-(x, y) \quad \text{iff} \quad x \leq \Diamond^- y.$$

The following result summarizes the duality between perfect modal bilattices and four-valued Kripke frames [39, Proposition 5.5].

Proposition 15 *For every four-valued Kripke frame $\mathcal{F}$ and every perfect modal bilattice $\mathbf{B}$, we have that $\mathcal{F} \cong (\mathcal{F}^\bullet)_\bullet$ and that $\mathbf{B} \cong (\mathbf{B}_\bullet)^\bullet$.*

Our next aim is to show that, for every perfect modal bilattice $\mathbf{B} = \mathbf{A}^{\bowtie}$ and every action model $a = (K, k, R_a^+, R_a^-, Pre_a)$ over $\mathbf{B}$, we have $(\mathbf{B}_a)_\bullet \cong (\mathbf{B}_\bullet)_{\bar{a}}$, where $\bar{a} := (K, k, R_a^+, R_a^-, \overline{Pre_a})$ is the action model over the complex algebra of $\mathbf{B}_\bullet$ and $\overline{Pre_a}\colon K \to (\mathbf{B}_\bullet)^\bullet$ is defined as

$$\overline{Pre_a}\colon j \mapsto (\{y \in \mathsf{At}(\mathbf{A}) \mid y \leq Pre_a^+(j)\}, \{y \in \mathsf{At}(\mathbf{A}) \mid y \leq Pre_a^-(j)\})$$

where Pre_a^+ and Pre_a^- are the components of Pre_a according to the twist structure presentation of $\prod_a \mathbf{B}$. We then have the following.

Proposition 16 (Cf. [29, Fact 4.8]) *For every perfect modal bilattice $\mathbf{B}$ and every action model $a = (K, k, R_a^+, R_a^-, Pre_a)$ over $\mathbf{B}$, we have : (i) $(\prod_a \mathbf{B})_\bullet \cong \prod_a(\mathbf{B}_\bullet)$, and (ii) $(\mathbf{B}_a)_\bullet \cong (\mathbf{B}_\bullet)_{\bar{a}}$.* $\dashv$

Proof. Let $\mathbf{A} = (A, \wedge, \vee, \sim, \Diamond^+, \Diamond^-)$. We can assume without loss of generality that $\mathbf{B} = \mathbf{A}^{\bowtie}$. Let $a = (K, k, R_a^+, R_a^-, Pre_a)$ be an action model over $\mathbf{B}$. It can be easily seen that $\prod_a \mathbf{B} \cong (\prod_a \mathbf{A}, \Diamond^+_{\prod_a \mathbf{A}}, \Diamond^-_{\prod_a \mathbf{A}})^{\bowtie}$, in which $(\prod_a \mathbf{A}, \Diamond^+_{\prod_a \mathbf{A}}, \Diamond^-_{\prod_a \mathbf{A}})$ is a bimodal boolean algebra [28, Definition 11]. Then $(\prod_a \mathbf{A}, \Diamond^+_{\prod_a \mathbf{A}})$ and $(\prod_a \mathbf{A}, \Diamond^-_{\prod_a \mathbf{A}})$ are modal boolean algebras. Fact 4.8 in [29] can be applied to the present setting, because a bimodal boolean algebra is made of two boolean algebras that share the same non-modal boolean structure, and the two diamonds are not related in any non-trivial way. Applied to the present setting it then states that $(\mathsf{At}(\prod_a \mathbf{A}), R^+) \cong (\prod_a \mathbf{A}, \Diamond^+_{\prod_a \mathbf{A}})$, and likewise that $(\mathsf{At}(\prod_a \mathbf{A}), R^-) \cong (\prod_a \mathbf{A}, \Diamond^-_{\prod_a \mathbf{A}})$. These isomorphisms show that indeed $(\prod_a \mathbf{B})_\bullet \cong \coprod_a (\mathbf{B})_\bullet$.

For (ii) it is enough to apply [29, Fact 4.8] to the twist structure corresponding to $\mathbf{B}_a$.

As shown above, the identification between the two relational structures implies that the mechanism of epistemic updates remains essentially unchanged when moving from the classical to a bilattice setting.

We are now going to rewrite the definition of satisfaction for formulas of type $\langle\alpha\rangle\varphi$. To this end we introduce the notation

$$M \xrightarrow{\iota_k} \coprod_\alpha M \xhookleftarrow{i} M_\alpha$$

where the map $i : M_\alpha \to \coprod_\alpha M$ is the submodel embedding and $\iota_k : M \to \coprod_\alpha M$ is the embedding of M into its k-colored copy. This is the copy corresponding to the distinguished point k of α. The satisfaction condition for $\langle\alpha\rangle$-formulas (Definition (14)) can be equivalently written as follows, where $M_\alpha = (W_\times, R^+_\times, R^-_\times, V_\times)$: $w \in V^+(\langle\alpha\rangle\varphi)$ iff $\exists x \in W_\times$ such that $i(x) = \iota_k(w) \in (V^*)^+(Pre_\alpha)$ and $x \in V^+_\times(\varphi)$. Since the map $i : M_\alpha \hookrightarrow \coprod_\alpha M$ is injective, we have $x \in V^+_\times(\varphi)$ iff $\iota_k(w) = i(x) \in i(V^+_\times(\varphi))$, iff $w \in \iota_k^{-1}(i(V^+_\times(\varphi)))$. Hence we have $w \in V^+(\langle\alpha\rangle\varphi)$ iff $w \in V^+(Pre_\alpha) \cap \iota_k^{-1}[i(V^+_\times(\varphi))]$, i.e., $V^+(\langle\alpha\rangle\varphi) = V^+(Pre_\alpha) \cap \iota_k^{-1}(i(V^+_\times(\varphi)))$.

As observed earlier, $V^+(\varphi) = V^+(\sim\sim\varphi)$ for any $\varphi \in Fm$ and any valuation V. Satisfaction of a formula in bilattice modal logic only depends, for each valuation V, on its positive part $V^+(\varphi)$. This implies that the result of [39, Prop. 5.1] indeed extends to any **BEAK** formula:

Theorem 17 *For every Kripke model $(\mathcal{F}, V)$, s in the domain of $\mathcal{F}$, and formula φ of* **BEAK***: (i) $(\mathcal{F}, V), s \models \varphi$ iff $s \in V^+(\varphi)$, and (ii) $(\mathcal{F}, V) \vDash \varphi$ iff $(\mathcal{F}^\bullet, V^\bullet) \vDash \varphi$.* $\dashv$

With this statement of the equivalence of the relational semantics and the algebraic semantics of **BEAK** we close the section.

($\langle\boldsymbol{\alpha}\rangle$**-constants**)	$\langle\alpha\rangle\mathsf{f} \leftrightarrow \mathsf{f} \qquad \langle\alpha\rangle\mathsf{t} \leftrightarrow {\sim\sim} Pre_\alpha$
	$\langle\alpha\rangle\top \leftrightarrow (Pre_\alpha \wedge \top) \qquad \langle\alpha\rangle\bot \leftrightarrow \neg(Pre_\alpha \supset \bot)$
($\langle\boldsymbol{\alpha}\rangle$**-atoms**)	$\langle\alpha\rangle p \leftrightarrow ({\sim\sim} Pre_\alpha \wedge p)$
($\langle\boldsymbol{\alpha}\rangle\boldsymbol{\wedge}$)	$\langle\alpha\rangle(\varphi \wedge \psi) \leftrightarrow (\langle\alpha\rangle\varphi \wedge \langle\alpha\rangle\psi)$
($\langle\boldsymbol{\alpha}\rangle\boldsymbol{\vee}$)	$\langle\alpha\rangle(\varphi \vee \psi) \leftrightarrow (\langle\alpha\rangle\varphi \vee \langle\alpha\rangle\psi)$
($\langle\boldsymbol{\alpha}\rangle\boldsymbol{\supset}$)	$\langle\alpha\rangle(\varphi \supset \psi) \leftrightarrow ({\sim\sim} Pre_\alpha \wedge (\langle\alpha\rangle\varphi \supset \langle\alpha\rangle\psi))$
($\langle\boldsymbol{\alpha}\rangle\boldsymbol{\neg}$)	$\langle\alpha\rangle\neg\varphi \leftrightarrow ({\sim\sim} Pre_\alpha \wedge \neg\langle\alpha\rangle\varphi))$
($\langle\boldsymbol{\alpha}\rangle\boldsymbol{\Diamond}$)	$\langle\alpha\rangle\Diamond\varphi \leftrightarrow ({\sim\sim} Pre_\alpha \wedge \bigvee\{\Diamond\langle\alpha_j\rangle\varphi \mid \mathsf{R}_\alpha(k,j) \in \{\mathsf{t}, \top\}\})$
(**RE**)	from $\varphi \leftrightarrow \psi$ infer $\chi[\varphi/p] \leftrightarrow \chi[\psi/p]$

Table 3: The axiomatization $\mathbb{BEAK}$ for the logic **BEAK** consists of all rules and axioms of the axiomatization for **LB**$\Box$ (see Tables 1 and 2) and the above axioms and rule.

4. Axiomatization

In this section we give a Hilbert-style proof system for **BEAK** on the class of four-valued frames, and we show that it is sound and complete. The proof system $\mathbb{BEAK}$ for the logic **BEAK** consists of all the rules and axioms given in the Tables 1, 2, and 3. Table 1 contains the propositional part, Table 2 contains the (static) modal part, and Table 3 contains the dynamic (modal) part. In the rule **RE**, called 'replacement of equivalents', $\chi[\varphi/p]$ means uniform substitution of all occurrences of p in χ by φ (this can be easily defined inductively).

The axioms in Table 3 are the expected reduction rules for any logical structure following a dynamic modality for action model execution. Clearly, as in **BEAK** we have constants, we have axioms for the reduction of each of those constants. But there is nothing surprising about them. The other axioms may look more familiar to the reader informed about dynamic epistemic logics, except for the occasional need of the ${\sim\sim}$ binding of preconditions Pre_α: this is to ensure the restriction of the possible values of ${\sim\sim} Pre_\alpha$, namely to t and f only.

In the axiom ($\langle\boldsymbol{\alpha}\rangle\boldsymbol{\Diamond}$), note that $\alpha = \alpha_k$ with designated action k. This axiom is the typical reduction for modality $\Diamond$ after update $\langle\cdot\rangle$ in dynamic epistemic logics [45, 34]: after an update α (i.e., α_k) the agent considers it possible that φ, if and only if α is executable and for some conceivable α_j (i.e., for some α_j with $\mathsf{R}_\alpha(k,j)$) the agent considers it possible that after α_j, φ. The non-typical part of the reduction is that $\mathsf{R}_\alpha(k,j) \in \{\mathsf{t}, \top\}$. This is to be expected in a four-valued relational setting, and of course similar to the restriction of announcements φ to values $\mathsf{t}, \top$ in bilattice public announcement logic [39].

We recall mentioning in the introduction that the multi-agent generalization of our work is straightforward. This is a good moment to see why the multi-agent generalization of the axiomatization is straightforward. If we were to replace the *single* axiom ($\boldsymbol{\langle\alpha\rangle\Diamond}$) by, for each agent n, axioms

$$(\boldsymbol{\langle\alpha\rangle\Diamond^n}) \quad \langle\alpha\rangle\Diamond^n\varphi \leftrightarrow ({\sim}{\sim}Pre_\alpha \wedge \bigvee\{\Diamond^n\langle\alpha_j\rangle\varphi \mid \mathsf{R}^n_\alpha(k,j) \in \{\mathsf{t}, \top\}\})$$

where R^n_α denotes the relation in α *for agent* n, and if we were to similarly replace the $\mathbf{LB\Box}$ axioms of Table 2 by the set of their multi-agent equivalents, then we already have the axiomatization of the multi-agent setting. There are no interaction axioms for different agents.

The derivation rule 'replacement of equivalents' (**RE)** was erroneously missing in previous axiomatizations of non-classical dynamic epistemic logics [32, 29, 38, 39]. In the absence of (**RE**), the reduction strategy of $\mathbb{BEAK}$ to its static fragment, as sketched in the proof of Theorem 24, later, would not succeed.[2]

The axiom $\langle\alpha\rangle p \leftrightarrow ({\sim}{\sim}Pre_\alpha \wedge p)$, called ($\boldsymbol{\langle\alpha\rangle}$**-atoms**), guarantees that the value of atoms is preserved after update. Such an axiom is often formulated both for positive and for negative atoms (i.e., for literals). The axiom for negative atoms is indeed a theorem of our axiomatization. We show its derivation as an example.

[2] The rule **RE** is needed because we use an inside-out reduction strategy. For the alternative outside-in reduction strategy, **RE** is not needed, but then one needs a reduction axiom of shape "$\langle\alpha\rangle\langle\beta\rangle\varphi \leftrightarrow \dots$" as well as a rule "from $\varphi \rightarrow \psi$ infer $\langle\alpha\rangle\varphi \rightarrow \langle\alpha\rangle\psi$" ($\alpha$-monotonicity). For classical dynamic epistemic logics, for the special case of public announcement logics, such variants are discussed in detail in [47].

Example 18 *$\langle\alpha\rangle\neg p \leftrightarrow {\sim\sim}Pre_\alpha \wedge \neg p$ is a theorem of $\mathbb{BEAK}$.*

$\langle\alpha\rangle\neg p \leftrightarrow {\sim\sim}Pre_\alpha \wedge \neg\langle\alpha\rangle p$	$(\boldsymbol{\langle\alpha\rangle\neg})$
$\langle\alpha\rangle\neg p \leftrightarrow \neg({\sim\sim}Pre_\alpha) \vee \neg p$	**LB**, ($\boldsymbol{\langle\alpha\rangle}$***-atoms***)
$\neg({\sim\sim}Pre_\alpha) \leftrightarrow {\sim}Pre_\alpha$	$(\boldsymbol{\neg\supset}), p \wedge \mathsf{t} \leftrightarrow p$
$\langle\alpha\rangle\neg p \leftrightarrow ({\sim\sim}Pre_\alpha \wedge ({\sim}Pre_\alpha \vee \neg p))$	**LB**
$\langle\alpha\rangle\neg p \leftrightarrow ({\sim\sim}Pre_\alpha \wedge {\sim}Pre_\alpha) \vee ({\sim\sim}Pre_\alpha \wedge \neg p)$	**LB**
$({\sim\sim}Pre_\alpha \wedge {\sim}Pre_\alpha) \leftrightarrow \mathsf{f}$	**LB**
$\langle\alpha\rangle\neg p \leftrightarrow (\mathsf{f} \vee ({\sim\sim}Pre_\alpha \wedge \neg p))$	**LB**
$\langle\alpha\rangle\neg p \leftrightarrow ({\sim\sim}Pre_\alpha \wedge \neg p)$	$p \leftrightarrow p \vee \mathsf{f}$ $\dashv$

We now proceed by showing soundness and completeness. The following lemmas are needed to establish that $\mathbb{BEAK}$ is sound with respect to the algebraic semantics. Most proofs are straightforward adaptations of the lemmas from [39].

Lemma 19 ([39], Lemma 6.1) *Let $M = (\mathbf{B}, V)$ be an algebraic model and φ a formula such that $[\![\varphi]\!]_{M_\alpha} = \pi([\![\varphi]\!]_{\prod_\alpha M})$ for any four-valued action α over $\mathcal{L}$. Then:*

(i) $[\![\langle\alpha\rangle\varphi]\!]_M = {\sim\sim}[\![Pre_\alpha]\!]_M \wedge [\![\varphi]\!]_M$

(ii) $[\![[\alpha]\varphi]\!]_M = [\![Pre_\alpha]\!]_M \supset [\![\varphi]\!]_M$

Lemma 20 ([39], Fact 6.2) *Let $\mathbf{B}$ be a modal bilattice and a be a four-valued action model over $\mathbf{B}$, and let $i' : \mathbf{B}_a \to \prod_a \mathbf{B}$ be given by $[g] \mapsto g \wedge {\sim\sim}Pre_a$. Then for every $[b], [c] \in \mathbf{B}_a$:*

(i) $i'([b] \wedge [c]) = i'([b]) \wedge i'([c])$;

(ii) $i'([b] \vee [c]) = i'([b]) \vee i'([c])$;

(iii) $i'([b] \supset [c]) = {\sim\sim}Pre_a \wedge (i'([b]) \supset i'([c]))$;

(iv) $i'(\neg[b]) = {\sim\sim}Pre_a \wedge \neg i'([b])$;

(v) $i'(\Box_a[b]) = {\sim\sim}Pre_a \wedge \Box_{\prod_a \mathbf{B}}(Pre_a \supset i'([b]))$;

(vi) $i'(\Diamond_a[b]) = {\sim\sim}Pre_a \wedge \Diamond_{\prod_a \mathbf{B}}(i'([b]) \wedge {\sim\sim}Pre_a)$.

Lemma 21 ([39, Lemma 6.3]) *Let $M = (\mathbf{B}, V)$ be an algebraic model with underlying modal bilattice $\mathbf{B} = (B, \wedge, \vee, \supset, \neg, \Diamond, \Box, \mathsf{f}, \mathsf{t}, \bot, \top)$. For every four-valued action model α over $\mathcal{L}$ and all formulas φ and ψ:*

(i) $[\![\langle\alpha\rangle(\varphi \vee \psi)]\!]_M = [\![\langle\alpha\rangle\varphi]\!]_M \vee [\![\langle\alpha\rangle\psi]\!]_M$;

(ii) $[\![\langle\alpha\rangle(\varphi \wedge \psi)]\!]_M = [\![\langle\alpha\rangle\varphi]\!]_M \wedge [\![\langle\alpha\rangle\psi]\!]_M$;

(iii) $[\![\langle\alpha\rangle(\varphi \supset \psi)]\!]_M = \sim\sim[\![Pre_\alpha]\!]_M \wedge \neg[\![\langle\alpha\rangle\varphi]\!]_M$;

(iv) $[\![\langle\alpha\rangle\neg\varphi]\!]_M = \sim\sim[\![Pre_\alpha]\!]_M \wedge \neg[\![\langle\alpha\rangle\varphi]\!]_M$;

(v) $[\![\langle\alpha\rangle\Diamond\varphi]\!]_M = \sim\sim[\![Pre_\alpha]\!]_M \wedge \bigvee\{\Diamond_{\mathbf{B}}([\![\langle\alpha_j\rangle\varphi]\!]_M) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\}$;

(vi) $[\![\langle\alpha\rangle\Box\varphi]\!]_M = \sim\sim[\![Pre_\alpha]\!]_M \wedge \bigwedge\{\Box_{\mathbf{B}}([\![[\alpha_j]\varphi]\!]_M) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\}$.

Proof. We only show non-trivial items (v) and (vi). Concerning (v), first observe that:

$$\begin{aligned}
&\pi_k \circ i'([\![\Diamond\varphi]\!]_{M_\alpha}) = \pi_k(\sim\sim Pre_\alpha \wedge \Diamond_{\prod_\alpha \mathbf{B}}(\sim\sim Pre_\alpha \wedge i'([\![\varphi]\!]_{M_\alpha})))\\
&= \sim\sim Pre_a \wedge \bigvee \left\{\Diamond_{\mathbf{B}}(\sim\sim Pre_a(j) \wedge i'([\![\varphi]\!]_{M_\alpha}))(j) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\right\}\\
&= \sim\sim[\![Pre_\alpha]\!]_M \wedge \bigvee \left\{\Diamond_{\mathbf{B}}(\sim\sim Pre_a(j)) \wedge i'([\![\varphi]\!]_{M_\alpha}))(j) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\right\}\\
&= \sim\sim[\![Pre_\alpha]\!]_M \wedge \bigvee \left\{\Diamond_{\mathbf{B}}(\sim\sim[\![Pre_\alpha(j)]\!]_M \wedge \pi_j \circ i'([\![\varphi]\!]_{M_\alpha}))(j) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\right\}\\
&= \sim\sim[\![Pre_\alpha]\!]_M \wedge \bigvee \left\{\Diamond_{\mathbf{B}}(\sim\sim[\![Pre_\alpha(j)]\!]_M \wedge \pi_j \circ i'([\![\varphi]\!]_{M_{\alpha_j}}))(j) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\right\}\\
&= \sim\sim[\![Pre_\alpha]\!]_M \wedge \bigvee \left\{\Diamond_{\mathbf{B}}([\![\langle\alpha_j\rangle\varphi]\!]_M) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\right\}.
\end{aligned}$$

To justify the equality between lines 4 and 5 above, note that M_α is independent from the point of α, i.e., $(M_\alpha =)\ M_{\alpha_k} = M_{\alpha_j}$. Then:

$$\begin{aligned}
&[\![\langle\alpha\rangle\Diamond\varphi]\!]_M = \sim\sim[\![Pre_\alpha]\!]_M \wedge \pi_k \circ i'([\![\Diamond\varphi]\!]_{M_\alpha})\\
&= \sim\sim[\![Pre_\alpha]\!]_M \wedge \sim\sim[\![Pre_\alpha]\!]_M \wedge \bigvee \left\{\Diamond_{\mathbf{B}}([\![\langle\alpha_j\rangle\varphi]\!]_M) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\right\}\\
&= \sim\sim[\![Pre_\alpha]\!]_M \wedge \bigvee \left\{\Diamond_{\mathbf{B}}([\![\langle\alpha_j\rangle\varphi]\!]_M) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\right\}.
\end{aligned}$$

To show item (vi), we preliminarily observe that

$$\begin{aligned}
&\pi_k \circ i'([\![\Box\varphi]\!]_{M_\alpha}) = \pi_k(\sim\sim Pre_a \wedge \Box_{\prod_\alpha \mathbf{B}}(Pre_a \supset i'([\![\varphi]\!]_{M_\alpha})))\\
&= \sim\sim Pre_a \wedge \bigwedge \left\{\Box_{\mathbf{B}}(Pre_a(j) \supset i'([\![\varphi]\!]_{M_\alpha}))(j) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\right\}
\end{aligned}$$

$$= \sim\sim[\![Pre_\alpha]\!]_M \wedge \bigwedge \{\Box_{\mathbf{B}}(Pre_a(j) \supset i'([\![\varphi]\!]_{M_\alpha}))(j) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\}$$

$$= \sim\sim[\![Pre_\alpha]\!]_M \wedge \bigwedge\{\Box_{\mathbf{B}}([\![Pre_\alpha(j)]\!]_M \supset \pi_j \circ i'([\![\varphi]\!]_{M_\alpha}))(j) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\}$$

$$= \sim\sim[\![Pre_\alpha]\!]_M \wedge \bigwedge\{\Box_{\mathbf{B}}([\![Pre_\alpha(j)]\!]_g M \supset \pi_j \circ i'([\![\varphi]\!]_{M_{\alpha_j}}))(j) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\}$$

$$= \sim\sim[\![Pre_\alpha]\!]_M \wedge \bigwedge\{\Box_{\mathbf{B}}([\![[\alpha_j]\varphi]\!]_M) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\}.$$

Hence:

$$[\![\langle\alpha\rangle\Box\varphi]\!]_M = \sim\sim[\![Pre_\alpha]\!]_M \wedge \pi_k \circ i'([\![\Box\varphi]\!]_{M_\alpha})$$

$$= \sim\sim[\![Pre_\alpha]\!]_M \wedge (\sim\sim[\![Pre_\alpha]\!]_M \wedge \bigwedge \{\Box_{\mathbf{B}}([\![[\alpha_j]\varphi]\!]_M) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\})$$

$$= \sim\sim[\![Pre_\alpha]\!]_M \wedge \bigwedge \{\Box_{\mathbf{B}}([\![[\alpha_j]\varphi]\!]_M) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\}.$$

The next lemma is also helpful for the intuition linking the relational and algebraic setting, but is not strictly necessary in the completeness proof, wherein we use that $\langle\alpha\rangle\varphi$ is a primitive language construct and $[\alpha]\varphi$ a derived one. Item (v) of this lemma justifies our usage of $[\alpha]\varphi$ as an abbreviation for $\neg\langle\alpha\rangle\neg\varphi$.

Lemma 22 ([39, Fact 6.4]) *Let $M = (\mathbf{B}, V)$ be an algebraic model with underlying modal bilattice $\mathbf{B} = (\mathbf{B}, \wedge, \vee, \supset, \neg, \Diamond, \Box, \mathsf{f}, \mathsf{t}, \bot, \top)$. For every action model α over $\mathcal{L}$ and all formulas φ and ψ in $\mathcal{L}$:*

(i) $[\![[\alpha](\varphi \wedge \psi)]\!]_M = [\![[\alpha]\varphi]\!]_M \wedge [\![[\alpha]\psi]\!]_M$

(ii) $[\![[\alpha](\varphi \vee \psi)]\!]_M = [\![[\alpha]Pre_\alpha]\!]_M \supset ([\![\langle\alpha\rangle\varphi]\!]_M \vee [\![\langle\alpha\rangle\psi]\!]_M)$

(iii) $[\![[\alpha](\varphi \supset \psi)]\!]_M = [\![\langle\alpha\rangle\varphi]\!]_M \supset [\![\langle\alpha\rangle\psi]\!]_M$

(iv) $[\![[\alpha]\neg\varphi]\!]_M = \neg[\![\langle\alpha\rangle\varphi]\!]_M$

(v) $[\![[\alpha]\varphi]\!]_M = [\![\neg\langle\alpha\rangle\neg\varphi]\!]_M$

(vi) $[\![[\alpha]\Diamond\varphi]\!]_M = [\![Pre_\alpha]\!]_M \supset \bigvee \{\Diamond_{\mathbf{B}}([\![[\alpha_j]\varphi]\!]_M) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\}$

(vii) $[\![[\alpha]\Box\varphi]\!]_M = [\![Pre_\alpha]\!]_M \supset \bigwedge \{\Box_{\mathbf{B}}([\![[\alpha_j]\varphi]\!]_M) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\}$

Proof. The items of interest are (vi) and (vii). Item (vi):

$$[\![[\alpha]\Diamond\varphi]\!]_M = [\![Pre_\alpha]\!]_M \supset \pi_k \circ i'([\![\Diamond\varphi]\!]_{M_\alpha})$$

$$= [\![Pre_\alpha]\!]_M \supset (\sim\sim[\![Pre_\alpha]\!]_M \wedge \bigvee \{\Diamond_{\mathbf{B}}([\![\langle\alpha_j\rangle\varphi]\!]_M) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\})$$

$$= [\![Pre_\alpha]\!]_M \supset \bigvee \left\{\Diamond_{\mathbf{B}}([\![\langle\alpha_j\rangle\varphi]\!]_M) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\right\} \qquad (*)$$

Equivalence $(*)$ holds since in every modal bilattice we have that $a \supset {\sim}{\sim} a = \mathsf{t}$ and that $(a \supset b) \wedge (a \supset c) = a \supset (b \wedge c)$. Concerning item (vii):

$$[\![[\alpha]\Diamond\varphi]\!]_M = [\![Pre_\alpha]\!]_M \supset \pi_k \circ i'([\![\Box\varphi]\!]_{M_\alpha}) \qquad (**)$$
$$[\![Pre_\alpha]\!]_M \supset \bigwedge \left\{\Box_{\mathbf{B}}([\![[\alpha_j]\varphi]\!]_M) \mid R_\alpha(k,j) \in \{\mathsf{t}, \top\}\right\}$$

For equivalence $(**)$ we refer to Lemma 21.vii.

Lemma 23 *The rule* **RE** *is sound: if* $\models \varphi \leftrightarrow \psi$ *then* $\models \chi[\varphi/p] \leftrightarrow \chi[\psi/p]$. $\dashv$

Proof. Let $\varphi, \psi \in \mathcal{L}$ be such that $\models \varphi \leftrightarrow \psi$. We will prove that for all $\chi \in \mathcal{L}$, and any Kripke model $M = (W, R, V)$ and state $w \in W$:

$$V(\chi[\varphi/p], w) = V(\chi[\psi/p], w)$$

The proof is by induction on χ.
— The case where χ is a logical constant or an atomic proposition is immediate.
— If $\chi = \gamma \bullet \delta$, where $\bullet \in \{\wedge, \vee, \supset\}$, or $\chi = \neg\gamma$, use that V is a homomorphism in its first argument with respect to bilattice operators.
— If $\chi = \Diamond\gamma$, then

$$\begin{aligned} V(\Diamond\gamma[\varphi/p], w) &= \bigvee\{R(w,w') * V(\gamma[\varphi/p], w') \mid w' \in W\} \\ &= \bigvee\{R(w,w') * V(\gamma[\psi/p], w') \mid w' \in W\} \qquad \text{(Inductive hyp.)} \\ &= V(\Diamond\gamma[\psi/p], w). \end{aligned}$$

— Finally, let $\chi = \langle\alpha\rangle\gamma$. We show that $V^+(\langle\alpha\rangle\gamma[\varphi/p]) = V^+(\langle\alpha\rangle\gamma[\psi/p])$ and that $V^-(\langle\alpha\rangle\gamma[\varphi/p]) = V^-(\langle\alpha\rangle\gamma[\psi/p])$. Let $M_\alpha = (W_\times, R_\times^+, R_\times^-, V_\times)$. By inductive hypothesis, for every $(w,k) \in W_\times$, $V_\times(\gamma[\varphi/p], (w,k)) = V_\times(\gamma[\psi/p], (w,k))$. Therefore, $V_\times^+(\gamma[\varphi/p]) = V_\times^+(\gamma[\psi/p])$ and $V_\times^-(\gamma[\varphi/p]) = V_\times^-(\gamma[\psi/p])$. Also, by inductive hypothesis, $Pre_\alpha[\varphi/p] = Pre_\alpha[\psi/p]$. Hence,

$$\begin{aligned} V^+(\langle\alpha\rangle\gamma[\varphi/p]) &= V^+(Pre_\alpha[\varphi/p]) \cap \iota_k^{-1}(i(V_\times^+(\gamma[\varphi/p]))) \\ &= V^+(Pre_\alpha[\psi/p]) \cap \iota_k^{-1}(i(V_\times^+(\gamma[\psi/p]))) \end{aligned}$$

$$= V^+(\langle\alpha\rangle\gamma[\psi/p]).$$

A similar argument shows that $V^-(\langle\alpha\rangle\gamma[\varphi/p]) = V^-(\langle\alpha\rangle\gamma[\psi/p])$. Therefore, $V(\langle\alpha\rangle\gamma[\varphi/p]) = V(\langle\alpha\rangle\gamma[\psi/p])$.

We now get to the announced completeness result.

Theorem 24 *The proof system* $\mathbb{BEAK}$ *is sound and complete with respect to algebraic and relational models.* $\dashv$

Proof. The soundness of the preservation of logical constants and propositional variables follows from Lemma 20. The soundness of the remaining axioms is proved in Lemma 21. The soundness of **RE** is proved in Lemma 23.

The proof of completeness is analogous to that of classical and intuitionistic **EAK**, and follows from the reducibility of $\mathbb{BEAK}$ to bilattice modal logic.

Let φ be valid. Let us assume that we only use primitive connectives of $\mathcal{L}$ (so, for example, $\langle\alpha\rangle$ but not $[\alpha]$). Consider some innermost occurrence $\langle\alpha\rangle\psi$ of a dynamic modality in φ, where ψ is in the static language. The axioms of $\mathbb{BEAK}$ make it possible to transform $\langle\alpha\rangle\psi$ into an equivalent formula without a dynamic modality:

We 'push' the dynamic modality down the generation tree of the formula, through the static connectives, until it binds a proposition letter or a constant symbol. There, the dynamic modality disappears, thanks to an application of the appropriate axiom preserving proposition letters or constants, and, crucially, applying the **RE** rule (we *replace* a subformula in a larger expression by an equivalent formula without the dynamic modality).

This process is repeated for all the dynamic modalities of φ, so as to obtain a formula φ' which is provably equivalent to φ. Since φ is valid by assumption, and since provable equivalence preserves validity, by soundness we can conclude that φ' is valid. By Theorem 2, we can conclude that φ' is a theorem in bilattice modal logic and thus in $\mathbb{BEAK}$. Therefore, as φ and φ' are provably equivalent, φ is also a theorem. This concludes the proof.

5. Case study: Knowledge of inconsistency and incompleteness

A good image for a recipient of possibly inconsistent information is the database. You are Hendrik Edeling, a breeder of tulips. Consider the database D1-Acuminata containing information on the colour of a particular tulip that is a candidate for selective breeding. It may contain the information that the tulip is red, or that it is not red, or it may lack this information, or it may, inconsistently so, contain the information that it is both red and not red. In other words, the proposition p *for 'the tulip is red' can have one of the four values* $\mathtt{t}, \mathtt{f}, \top, \bot$*. Let us now consider the perspective of Edeling wishing to consult the database. And let us assume that Edeling is uncertain which of the four states* $\mathtt{t}, \mathtt{f}, \top, \bot$ *the database is in, with respect to the proposition* p*. That makes four possible worlds that he is unable to distinguish. If he now queries the database and get 'yes' as an answer to the query '*p*?', he can rule out two of these four possibilities and keep the worlds wherein* p *has the value* t *and the value* $\top$*. So this is a way to process a* public announcement of the proposition p. *Now a further query to narrow down the options would be querying the database on the value of* $\neg p$*, or, more properly said, querying it on the falsity of* p*. A confirmation that* p *is false reduces Hendrik Edeling's uncertainty because the only remaining world satisfying it, is the one where the value of* p *it* $\top$*. In another sense, Hendrik has become more uncertain again, because he has confirmation that the database is inconsistent. We could also have communicated directly (in one formula) to Edeling that the database is inconsistent. Or that it is consistent, or that it is incomplete (value* $\bot$*). How? Please read on.*

Given initial uncertainty about p*, Edeling may also have to interact with his colleague Saartje Burgerhart, another renowned tulip expert. Maybe even a competitor! Consider the action of Burgerhart being informed that the database is lacking information on* p *(the datebase is incomplete), while Edeling remains uncertain whether she gets this information.*

The information that the agents receive may also be modal. Suppose that Hendrik is being told that $p \wedge \neg\Box p$*: "The tulip is red but you don't know this!"*

Unlike in two-valued modal logic, this formula may remain true after its announcement. It need not be an unsuccessful update. How come? Again, please read on.

Bilattice modal logic. We model the D1-Acuminata database containing information on that tulip as a world. The proposition that the tulip is red is p. There are four possible worlds. We use mnemonic names for the worlds: $\mathsf{p}_\bot$ is the world where $V(p) = \bot$, p_t is the world where $V(p) = \mathsf{t}$, p_f is the world where $V(p) = \mathsf{f}$, and $\mathsf{p}_\top$ is the world where $V(p) = \top$. Uncertainty about the four worlds is represented by the following model M. The box enclosing the worlds means that they are indistinguishable (the accessibility relation R is the universal relation on this domain) for Hendrik Edeling.

M: $\boxed{\mathsf{p}_\bot \quad \mathsf{p}_\mathsf{f} \quad \mathsf{p}_\mathsf{t} \quad \mathsf{p}_\top}$

We can now evaluate, for example, that $M, \mathsf{p}_\mathsf{t} \models p$, or that $M, \mathsf{p}_\top \models p$ (we recall that $M, w \models \varphi$ means that $V(\varphi, w) \in \{\mathsf{t}, \top\}$). We do not have that $M, \mathsf{p}_\mathsf{t} \models \Box p$, as both p and $\neg p$ are considered possible. Hendrik is uncertain about p. A public announcement $p!$ restricts the model to the p_t and $\mathsf{p}_\top$ state.

M: $\boxed{\mathsf{p}_\bot \quad \mathsf{p}_\mathsf{f} \quad \mathsf{p}_\mathsf{t} \quad \mathsf{p}_\top} \quad \stackrel{p!}{\Rightarrow} \quad M_p$: $\boxed{\mathsf{p}_\mathsf{t} \quad \mathsf{p}_\top}$

A public announcement is a singleton action model with reflexive access. Instead of writing that α is a public announcement of φ we write $\varphi!$; and for the corresponding model update we write M_φ instead of M_α. We can justify the restriction to t and $\top$ by considering this semantics of announcement to be the response to a query $p?$. In both cases the answer will be 'yes'. In two-valued public announcement logic, we are used to having $[p!]\Box p$ as a validity for propositional variables. This is no longer the case in our setting. In particular, $M, s \not\models [p!]\Box p$. We recall the semantics of $\Box$, namely

$$V(\Box\varphi, w) := \bigwedge\{R(w, w') \to V(\varphi, w') \mid w' \in W\},$$

where $\bigwedge$ denotes the infinitary version of $\wedge$ in FOUR and $\to$ is the strong implication. So far, our models have two-valued accessibility relations, i.e.,

$(w, w') \in R$ or $(w, w') \notin R$ for all pairs in M. This means that $\Box p$ takes the value $\bigwedge\{\mathsf{t} \to V(p, w) : w \in \{\mathsf{p}_\bot, \mathsf{p}_\mathsf{t}, \mathsf{p}_\mathsf{f}, \mathsf{p}_\top\}\}$. As $V(p, \mathsf{p}_\top) = \top$, and $\mathsf{t} \to \top = \mathsf{f}$ (the other three values are t), $\Box p$ is therefore *false* (in any state of M) and not true. The intuition behind this is that in bilattice modal logic $\Box\varphi$ *is false if* φ *is false in an accessible world.* It is therefore not necessarily the case that $\Box\varphi$ *is true if* φ *is true in all accessible worlds.* In fact, if in one or more of those accessible worlds φ has the value $\top$ (as in our example model M), then φ is also false in an accessible world, and thus we are done for. From $M_p, s \not\models \Box p$ then follows, using that $(M, s \models [\alpha]\varphi$ iff $(M, s \models Pre_\alpha$ implies $M_\alpha, s \models \varphi))$, that $M, s \not\models [p!]\Box p$.

Now consider the announcement of $p \wedge \neg\Box p$. This formula is known as the Moore sentence [33, 44]. In two-valued public announcement logic, as the result of truthfully announcing it, it becomes false; $[(p \wedge \neg\Box p)!]\neg(p \wedge \neg\Box p)$ is valid in public announcement logic. It is not valid in **BEAK**. Similarly to above, we have:

$$M\colon \boxed{\mathsf{p}_\bot \quad \mathsf{p}_\mathsf{f} \quad \mathsf{p}_\mathsf{t} \quad \mathsf{p}_\top} \quad \overset{(p\wedge\neg\Box p)!}{\Rightarrow} \quad M_{p\wedge\neg\Box p}\colon \boxed{\mathsf{p}_\mathsf{t} \quad \mathsf{p}_\top}$$

Thus, because in $M_{p\wedge\neg\Box p}$ we have that $R(\mathsf{p}_\mathsf{t}, \mathsf{p}_\top) = \mathsf{t}$ and that $V(p, \mathsf{p}_\top) = \top$, it follows that $M_{p\wedge\neg\Box p}, \mathsf{p}_\mathsf{t} \not\models \Box p$. In fact, we now have that $M_{p\wedge\neg\Box p}, \mathsf{p}_\mathsf{t} \models p \wedge \neg\Box p$ and thus the (from a modal logical perspective) surprising result that:

$[(p \wedge \neg\Box p)!](p \wedge \neg\Box p)$ *is satisfiable in* **BEAK**.

Having seen some simple examples of announcements and of formulas, and modal formulas, let us present some simple announcements on the status quo of a database, with regard to p.

- the database is consistent: announcement of $\sim(p \wedge \neg p)$
- the database is inconsistent: announcement of $p \wedge \neg p$
- the database is complete: announcement of $p \vee \neg p$
- the database is incomplete: announcement of $\sim(p \vee \neg p)$

The four-valued truth tables of these formulas are illustrative.

$\sim$	$(p$	$\wedge$	$\neg$	$p)$
t	$\bot$	$\bot$	$\bot$	$\bot$
t	f	f	t	f
t	t	f	f	t
f	$\top$	$\top$	$\top$	$\top$
$\Uparrow$				

p	$\wedge$	$\neg$	p
$\bot$	$\bot$	$\bot$	$\bot$
f	f	t	f
t	f	f	t
$\top$	$\top$	$\top$	$\top$
	$\Uparrow$		

p	$\vee$	$\neg$	p
$\bot$	$\bot$	$\bot$	$\bot$
f	t	t	f
t	t	f	t
$\top$	$\top$	$\top$	$\top$
	$\Uparrow$		

$\sim$	$(p$	$\vee$	$\neg$	$p)$
t	$\bot$	$\bot$	$\bot$	$\bot$
f	f	t	t	f
f	t	t	f	t
f	$\top$	$\top$	$\top$	$\top$
$\Uparrow$				

Again, we do not necessarily have that after these announcements, the formulas of the announcement are known: $[\sim(p \vee \neg p)!]\Box\sim(p \vee \neg p)$ and $[\sim(p \wedge \neg p)!]\Box\sim(p \vee \neg p)$ are *valid*, but $[(p \vee \neg p)!]\Box(p \vee \neg p)$ and $[(p \wedge \neg p)!]\Box(p \wedge \neg p)$ are *invalid*. (Although $[\sim\sim(p \vee \neg p)!]\Box\sim\sim(p \vee \neg p)$ and $[\sim\sim(p \wedge \neg p)!]\Box\sim\sim(p \vee \neg p)$ are valid.)

It is illustrative to see announcements as answers of queries to the database. When Hendrik queries the database with p? then the answer he gets will be 'yes' if the state of the database is t or $\top$, when he queries the database with $\neg p$? then the answer he gets will be 'yes' if the state of the database is f or $\top$. This is like Fitting's Rosencrantz and Guildenstern (R and G) setting in [15]. In question-answer analysis in two-valued logic [25], a question induces a *partition* on the domain, and a yes/no question, such as a question φ? on the truth of φ, a *dichotomy*. Fitting's Rosencrantz and Guildenstern other answer is 'no'. That is, for either of them, a classical dichotomy. However, it is tempting to see a *question in four-valued logic* differently, namely as inducing (a set of subsets that is) a *partial cover* of the domain. It is a *cover*, as two subsets may have non-empty intersection (namely when they contain worlds where φ has the value $\top$). It is *partial*, as some worlds may not be in any subset, namely when φ has the value $\bot$. If the world has no information on φ (value $\bot$), then 'there is no answer' or, differently said, the answer is: "I don't know." This becomes like the introductory example where you were trying to find your way to the railway station in Nancy. That example also serves to illustrate another, we think, interesting feature of four-valued question-answer analysis: if the value of φ is $\top$, then the answer to the question $?\varphi$ is 'yes' (so not 'yes and no'); whereas the answer to the question $\neg\varphi?$ is also 'yes'. Knowledge of inconsistency is a higher order feature for a database: whereas consulting memory directly is more straightforward: if you already have the answer 'yes', why trying to rule out the answer 'no'? In other words, questions become *leading questions*. We do not know if this analysis of questions in four-valued logics is common in inquisitive semantics [25].

Roles in dynamic epistemics *To understand dynamic epistemics, also on bilattices, it is important to distinguish different roles: (i) the agent/object/process identified with a propositional variable (the holder of the information), (ii) the agent being uncertain about the proposition, and (iii) the provider of reliable new information (on the proposition), the dynamic part. In our tulip example we have distinguished (i) (the database) from (ii) (Hendrik Edeling), but not (i) from (iii) (the database is queried and provides the answers). In the railway station example (i) and (iii) are separate: accidental pedestrians perform the role of (iii). It is common to view the source of new information, the 'announcing agent', as an anonymous oracle or trusted authority (Hendrik Edeling's system manager, so to speak). In the tulip example we can even think of the different roles as different components of 'the database' as hardware: (i) is RAM, (ii) is the CPU, and (iii) is the interface. In multi-agent examples (where each agent* a *has her own* $\Box^i$ *in the logical language) it is also easier to separate roles.*

Truth values or possible worlds? *If we see Hendrik Edeling as the database, we can consider the value of* p *his uncertainty. Initially the value of* p *is* $\bot$. *It then* changes *into* t *once Hendrik gathers positive information on* p, *and may fur-*

ther change into $\top$ *if he additionally receives negative information on* p*. These are so-called* factual (ontic) changes*. But if we see Hendrik as different from the database, then his uncertainty is between four worlds of a Kripke model, where a world represents the fixed value of* p *in the database. Receiving information now means restricting this model in order to finally find out the true value of* p*. This is* informational (epistemic) change*. The former is quite different from the latter. Factual change can also be modelled in dynamic epistemics, but is outside our scope.*

Multi-agent knowledge and actions. As mentioned, our framework generalizes to a multi-agent modal setting, wherein instead of the modality $\Box$ we have modalities $\Box^i$, for each agent i. *Knowledge* modalities come with an accessibility relation that is an equivalence relation (and that, so far, is two-valued; four-valued accessibility relations will be considered next). Other scenarios are conceivable, for example for belief, intentions or obligations, or time (with temporal modalities).

Hendrik Edeling has a colleague Sara Burgerhart who is another expert on Acuminata tulips and who may also have access to the same database. We model some scenarios and give typical formulas. Elementary checks on their adequacy are left to the reader. The equivalence classes of the accessibility relation for Hendrik are depicted as solid boxes and for Sara they are depicted as dashed boxes. Modality $\Box^h$ represents Hendrik's knowledge and $\Box^s$ represents Sarah's knowledge.

- Sara knows that the tulip is red.

  $\boxed{\mathsf{p_t}}$

  $\mathsf{p_t} \models \Box^s p$

- Sara knows whether the tulip is red. Hendrik is uncertain whether she knows that. (And we should now add; "and they are both aware of this scenario." We will refrain from doing so from now on.) Sara says to Hendrik: "I know that the tulip is red."

$\boxed{\mathsf{p_t} \quad \mathsf{p_f} \quad \mathsf{p_\perp} \;\; \mathsf{p_\top}} \xRightarrow{\Box^s p!} \boxed{\mathsf{p_t}}$

$\mathsf{p_t} \models \neg\Box^h(\Box^s p \vee \Box^s \neg p) \wedge [\Box^s p!]\Box^h p$

- Sara knows whether the tulip is red. Hendrik is uncertain whether she knows that. Sara says to Hendrik: "I know whether the tulip is red."

  $\boxed{\mathsf{p_t} \quad \mathsf{p_f} \quad \mathsf{p_\perp} \;\; \mathsf{p_\top}} \xRightarrow{(\Box^s p \vee \Box^s \neg p)!} \boxed{\mathsf{p_t} \quad \mathsf{p_f}}$

  ($\mathsf{p_f}$ or) $\mathsf{p_t} \models \neg\Box^h(\Box^s p \vee \Box^s \neg p) \wedge [(\Box^s p \vee \Box^s \neg p)!](\neg\Box^h p \wedge \Box^h(\Box^s p \vee \Box^s \neg p))$

- Sara knows that the database is consistent, but she doesn't know that it is incomplete.

  $\mathsf{p_\perp} \quad \mathsf{p_f} \quad \mathsf{p_t}$

  $\mathsf{p_\perp} \models \Box^s{\sim}(p \wedge \neg p) \wedge \neg\Box^s{\sim}(p \vee \neg p)$

- Sara knows whether the database is consistent.

  $\mathsf{p_\perp} \quad \mathsf{p_f} \quad \mathsf{p_t} \qquad \mathsf{p_\top}$

  $\mathsf{p_\perp} \models \Box^s{\sim}(p \wedge \neg p) \vee \Box^s{\sim}{\sim}(p \wedge \neg p)$ is valid on this model.

- Sara knows whether the database is consistent. Hendrik does not. Sara says to Hendrik: "The system manager just informed me that the database is consistent."

  $\boxed{\mathsf{p_\perp} \quad \mathsf{p_f} \quad \mathsf{p_t} \qquad \mathsf{p_\top}} \xRightarrow{\Box^i{\sim}(p \wedge \neg p)!} \boxed{\mathsf{p_\perp} \quad \mathsf{p_f} \quad \mathsf{p_t}}$

- Sara and Hendrik are both uncertain about the status of the database. The system manager says: "I will now inform Sara whether the database is consistent." He proceeds to do so, but by whispering into her ear, so that Hendrik cannot hear what he says to Sara.

  $\boxed{\mathsf{p_\perp} \quad \mathsf{p_f} \quad \mathsf{p_t} \quad \mathsf{p_\top}} \xRightarrow{\alpha} \boxed{\mathsf{p_\perp} \quad \mathsf{p_f} \quad \mathsf{p_t} \qquad \mathsf{p_\top}}$

  Here, α represent the whisper action. This is non-deterministic choice between action models α_k and α_l (and where for $\alpha_k \cup \alpha_l$ we write α), where $\alpha_k = (K, k, R_\alpha, Pre_\alpha)$ such that $K = \{k, l\}$, $Pre_\alpha(k) = {\sim}(p \wedge \neg p)$; $Pre_\alpha(l) = {\sim}{\sim}(p \wedge \neg p)$; $R^s_\alpha(k,k) = \mathtt{t}$, $R^s_\alpha(l,l) = \mathtt{t}$, $R^s_\alpha(k,l) = \mathtt{f}$, and

$R^s_\alpha(l,k) = \mathsf{f}$; $R^h_\alpha(i,j) = \mathsf{t}$ for all $i,j \in \{k,l\}$. Action model α_l is the same as α_k but with the other designated point.

In all the above, we only considered a single propositional variable, p. However, we can also consider situations wherein Hendrik Edeling is the expert on p, and controls that database, whereas Sara Burgerhart (possibly) has information on the tulip's petals. Are they round and wide, or are they narrow and sleek? Let that be a proposition q. (In fact Acuminata tulips have sleek petals — they approach more the Turkish ideal tulip than the Dutch ideal tulip.) We could think of her as controlling another database. And both databases could contain thousands of items of possibly inconsistent information. The scenarios merely represent the most elementary setting to reason about database consistency and completeness by interacting agents.

Four-valued accessibility relations. Our framework does not only permit four-valued propositions but also four-valued relations. Using Fitting's [15] fitting words:

> *Now, two kinds of judgements are possible. 1) A is true in situation w; and 2) w is a situation that should be considered.*

Where A is any proposition, for which we tend to write φ, and where we call w a world. Fitting considered many-valued logics in general, whereas we are in bilattice logic, with judgements on truth and falsity. In other words, if $R(w,u) = \mathsf{t}$ then u is in, and if $R(w,u) = \mathsf{f}$ then u is out.

Let a Kripke model $M = (W, R, V)$ be given with a two-valued relation R ($(w,w') \in R$ or $(w,w') \notin R$). Let W' be a set of worlds disjoint from W. Consider M' with domain $W \cup W'$ and define a relation R' such $R'(w,w') = R'(w',w) = R'(w',w') = \mathsf{f}$ for any world $w' \in W'$ and any $w \in W$. Then for all φ, $M, w \models \varphi$ iff $M', w \models \varphi$. This follows easily, as f is the bottom of the truth order $\leq_t$. For non-modal formulas it is obvious that $M, w \models \varphi$ iff $M', w \models \varphi$; for modal formulas we can observe that $V(\Diamond\varphi, w) = \bigvee\{R'(w,w') * V(\varphi,w') \mid w' \in W\} = \bigvee\{R'(w,w') * V(\varphi,w') \mid w' \in W \cup W'\}$, because when $w' \in W'$ we have that $R'(w, w') * V(\varphi, w') = \mathsf{f}$. Thus, this conjunct does not affect the value of the join. Similarly, $R'(w, w') \to V(\varphi, w') = \mathsf{t}$ does not affect the value of the meet defining $V(\Box\varphi, w)$.

Not surprisingly, with values $\bot$ or $\top$ for pairs in the accessibility relation it becomes harder to appeal to our modelling intuitions. For example, what does it mean that 'Hendrik Edeling considers world w possible' has value $\top$? Does he then consider it possible and impossible at the same time? Our previous visualization with boxes is no longer suitable, and from now on we depict all pairs in the accessibility relation explicitly as arrows, labelled with the value of that pair in R (so, for example, below we have that $R(\mathsf{p_f}, \mathsf{p_t}) = \top$).

$$\begin{tikzcd} \mathsf{p_f} \arrow[loop above, "\mathsf{t}"] \arrow[r, bend left, "\top"] & \mathsf{p_t} \arrow[loop above, "\top"] \arrow[l, bend left, "\mathsf{t}"] \end{tikzcd}$$

We could interpret this by saying that Hendrik's beliefs are more inclined towards p being false than towards p being true, as $\top$ is lower in the $\leq_{\mathsf{t}}$ hierarchy than t (worlds considered t are more *plausible* than worlds considered $\top$). Still, $\top$ access is good enough to get to know p. Compare the following three (distinct) models:

$$M : \begin{tikzcd} \mathsf{p_t} \arrow[loop above, "\mathsf{t}"] \end{tikzcd} \qquad M' : \begin{tikzcd} \mathsf{p_t} \arrow[loop above, "\top"] \end{tikzcd} \qquad M'' : \begin{tikzcd} \mathsf{p_\top} \arrow[loop above, "\mathsf{t}"] \end{tikzcd}$$

In M and M', $\Box p$ is true, whereas in M'', $\Box p$ is false. (In M', $V(\Box p, \mathsf{p_t}) = R(\mathsf{p_t}, \mathsf{p_t}) \to V(p, \mathsf{p_t}) = \top \to \mathsf{t} = \mathsf{t}$; whereas in M'', $V(\Box p, \mathsf{p_\top}) = R(\mathsf{p_\top}, \mathsf{p_\top}) \to V(p, \mathsf{p_\top}) = \mathsf{t} \to \top = \mathsf{f}$.) The latter is easily explained: $\Box p$ is false if there is an accessible world where p is false. And value $\top$ means that p is (also) false. To understand that $\Box p$ is true in M', it is sufficient to observe that the $\mathsf{p_t}$ world *is considered*. It is in. That it is simultaneously out does not hurt. So Hendrik still knows that tulips are red.

What properties are satisfied by Kripke models with four-valued relations that are used to interpret knowledge modalities? Are they still equivalence relations? Take transitivity: if $(w, w') \in R$ and $(w', w'') \in R$ then $(w, w'') \in R$;

but if $(w, w') \notin R$ and $(w', w'') \notin R$ then we need not have that $(w, w'') \notin R$ (for example, suppose $w'' = w$). Transitivity plays a role in the four-valued logic **BS4** of [36] (employing two-valued relations), and transitivity of four-valued relations is summarily discussed in [16] in the context of combining knowledge of different experts. The answer to our questions is in the logic, not in the structures: for transitivity we need the properties enforcing the validity of $\Box\varphi \to \Box\Box\varphi$. We can achieve this with simple means. First, an example.

Hendrik Edeling knows the colour of the tulips in the Acuminata database. They are red, or white, or blue. His model of uncertainty is

$$\boxed{r \quad r'} \quad \boxed{w \quad w'} \quad \boxed{b \quad b'}$$

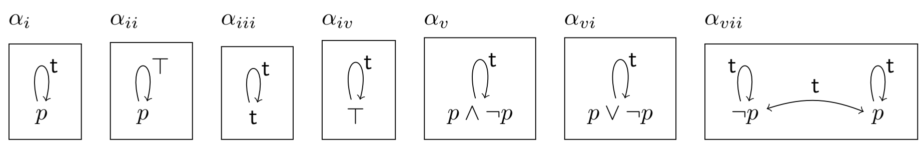
There are three equivalence classes, and all pairs are either in or out (for $(x, y) \in R$ read $R(x, y) = \mathsf{t}$ and for $(x, y) \notin R$ read $R(x, y) = \mathsf{f}$). We have that: $R(r, w) = \mathsf{f}$ and $R(w, r') = \mathsf{f}$ but $R(r, r') = \mathsf{t}$; $R(r, w) = \mathsf{f}$ and $R(w, b) = \mathsf{f}$ and $R(r, b) = \mathsf{f}$; $R(r, w) = \mathsf{f}$ and $R(w, w') = \mathsf{t}$ and $R(r, w') = \mathsf{f}$. However, $R(x, y) = \mathsf{t}$ and $R(y, z) = \mathsf{t}$ imply $R(x, z) = \mathsf{t}$. That is only what matters: t or $\top$ related worlds should relate the same to all other worlds.

The structural requirements to enforce the validity of the properties of knowledge are as follows.

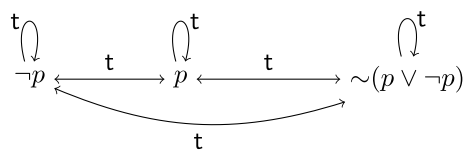
- If $R(w, x) = \mathsf{t}$ and $R(x, y) = \mathsf{t}$, then $R(w, y) = \mathsf{t}$.
- If $R(w, x) = \top$ and $R(x, y) = \top$, then $R(w, y) = \top$.
- If $R(w, x) = \mathsf{t}$ and $R(x, y) = \top$, then $R(w, y) = \top$.
- If $R(w, x) = \top$ and $R(x, y) = \mathsf{t}$, then $R(w, y) = \mathsf{t}$.
- $R(w, w) \in \{\mathsf{t}, \top\}$.
- If $R(w, x) \in \{\mathsf{t}, \top\}$ and $R(w, y) = i$, then $R(x, y) = i$ (where $i = \bot, \mathsf{t}, \mathsf{f}, \top$).

These cannot be properly called 'frame properties', as the manipulation of the pairs in the relation depends on their *values* in a given model. If these properties are fulfilled, then the schemata $\Box\varphi \to \varphi$, $\Box\varphi \to \Box\Box\varphi$, and $\Diamond\varphi \to \Box\Diamond\varphi$ are all valid (this is easy to see). Similarly, we get $\Box^i\varphi \to \Box^i\Box^i\varphi$, etc., for multi-agent bilattice epistemic logic.

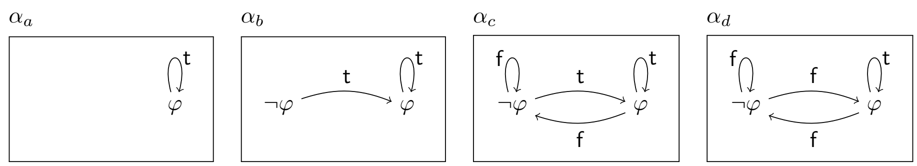
Four-valued action models. In our logical framework not only the accessibility relations of Kripke models are four-valued but also the accessibility relations of action models. Let us see some variations on the announcement of p. We have replaced the *names* of action models by their *preconditions*. (The boxes only serve to separate models and have no meaning.)



Action α_i is the public announcement of p (and also its correspondent in bilattice logic [39]). The difference between α_i and α_{ii} is that, when executed on a two-valued Kripke model, all links between worlds get value $\top$ instead of t; and in both cases the domain is restricted to the p-worlds (i.e., the $V(p) \in \{\mathsf{t}, \top\}$ worlds). For example, in a Kripke model M with two indistinguishable, two-valued, p and $\neg p$ worlds, both $[\alpha_i]\Box p$ and $[\alpha_{ii}]\Box p$ are true. The difference between i and ii only appears when evaluating knowledge of inconsistencies: $[\alpha_i]\Box(p \wedge \neg p)$ is false whereas $[\alpha_{ii}]\Box(p \wedge \neg p)$ is true, as $\mathsf{t} \to \top = \mathsf{f}$ whereas $\top \to \top = \top$. Action models α_{iii} and α_{iv} result (when executed on a given model) in isomorphic models: they have the same update effect (namely, none at all); the worlds preserved by precondition t are the same as those preserved by precondition $\top$ (a public announcement of φ restricts the domain to worlds where $\varphi \in \{\mathsf{t}, \top\}$; trivially, $\mathsf{t} \in \{\mathsf{t}, \top\}$ and $\top \in \{\mathsf{t}, \top\}$). These actions are 'clock ticks': executed on any model, the result will be isomorphic to it. Actions α_v and α_{vi} we have already discussed: these are the announcements that p is inconsistent, respectively, that there is complete information about p. Action α_{vii} has the same update effect as α_{vi}. Actions α_{vi} and α_{vii} are different from α_{iii} and α_{iv}: the latter two preserve $\perp$ worlds at their execution, the former two not. Given that, an interesting eighth version, with the same update effect as α_{iii} and α_{iv}, is:



Now consider the following four alternative depictions as action models of a public announcement of φ. The rightmost of the two points (in case there are two) is the designated point.



Again, α_a $(= \alpha_i)$ is the standard. Action model α_b is known as the Gerbrandy-style *conscious update* [21]. Instead of eliminating *worlds* that do not satisfy the announcement formula, it eliminates *arrows* (pairs in the accessibility relation) that do not point to worlds satisfying the announcement. An obvious 'four-valued completion' of this action model is α_c. A less obvious four-valued completion of α_a is α_d. Clearly the update effect of α_a and α_d is the same, and also the update effect of α_b and α_c. Actions α_a and α_b have also the same update effect (where it is important that the φ-world is the point of α_b; the correspondence only holds when the announcement is true). This does not change for bilattice modal logic (it is about accessibility). Thus, all four describe essentially the same action!

Similarly to above we could add a third point to α_d with preconditions $\sim(\neg\varphi \vee \varphi)$, however in this case equally f-*accessible* from and to the other points, and while keeping the φ point as the designated one. Let this be α_e. Again, α_e has the same update effect as all the others. But executing α_e does not restrict the domain of the model. On any model, we get the same result (logically indistinguishable results) by arrow elimination when executing α_e as by world elimination when executing α_a. This can be applied to any action model: given any Kripke model M with domain W and action model α with domain K, once having computed the $|K|$-fold coproduct of W (cartesian product $W \times K$), we need not restrict the *domain* as when computing M_α, but it suffices to restrict the *accessibility relation*, i.e., we need to make enough $R(v, w)$ swap their value from t to f.

> **What is an announcement in four-valued logic?** *As well known, public* announcement *logic is a misnomer, it is rather a logic of public, information changing,* events*. Various communicative phenomena including (informative) announcements count as (information changing) events: (a) if you say something that is heard by all (that is, an oral observation of a public announcement); (b) a visual observation (by all) of a property of surrounding objects, for example, when you see a red tulip blossoming in the fields; (c) written information observed by all, such as a teacher writing* $1 + 1 = 2$ *on a blackboard, or an envelope containing information on* p*, opened in public. (Some events called 'public announcements' are not information changing events at all, but factual changing events, as in "I hereby declare Donald Trump to be the president of the USA." We exclude those from consideration.) Not all of these make sense in a setting where inconsistency or incompleteness plays a role. Direct observations are hardly ever inconsistent. A tulip is red. Or it is not red. Now it may be red or orange, or something indefinable in between. But then we would say that the proposition 'the tulip is red' is in between true and false; we would not say that it is simultaneously true and false. A visual illusion might count as a contradictory observation (*$\top$*): is the image below that of a young or of an old woman?*

And what would it mean that a direct observation is absent ($\perp$)? Whereas the contents of a letter can easily be contradictory or absent. You open it. It contains a leaf, with p written on it. Or the leaf contains $\neg p$. Or there was no leaf enclosed. Or two, one with p and the other with $\neg p$.

6. Conclusions and future research

We proposed a four-valued bilattice-based modal logic including dynamic modalities for the consequences of actions. Our logic is suitable for reasoning about inconsistent and incomplete information, and about change of information in such settings. We have presented an axiomatization of the logic and shown completeness using algebraic logic and duality theory. We hope that our logic may be useful in computer science applications.

The present paper is part of an ongoing enterprise that aims, on the applied logic side, at extending dynamic epistemic logics beyond classical reasoning and, on the theoretical side, at achieving a better understanding of the very mechanism of epistemic updates.[3] From the latter point of view, an intriguing direction for future research is the investigation of the most general conditions for the algebraic/duality theoretic machinery to be applicable to epistemic updates. The papers [32, 29, 38, 39, 9] have shown that a uniform methodology, with few ad hoc adjustments, can be extended from the classical setting to those of intuitionistic, bilattice and finite-valued Łukasiewicz modal logics. Other logics are likely to be easily dealt with, for example positive (i.e. negation-free) modal logic and semilattice-based modal systems. The question then arises what could be minimal requirements of algebraic/relational semantics that would allow for a uniform definition of epistemic updates, perhaps one that does not heavily rely (as is so far the case) on the particular algebraic language involved. For example, since the pseudo-quotient construction involves the definition of a (partial)

[3] Added in proofs: a valuable recent addition is the four-valued public announcement logic of [41].

congruence by certain algebraic terms, we may wonder what kind of terms we should postulate in an abstract setting. Algebraic logic may turn out to be helpful here, and in particular the results from the general theory of the algebraization of logics that establish a link between logical filters (theories of a logic) and congruences of the associated algebraic semantics.

The Hilbert-style axiomatization $\mathbb{BEAK}$, although complete, may not be very suitable for proof search. We do not know its complexity, but similar calculi for dynamic epistemic logics tend to be NP or (in the multi-agent case) PSPACE. Also, maybe more importantly, it is not very constructive, as common for such calculi. Recent advances in proof calculi for dynamic epistemic logics, typically from an algebraic perspective [18, 20, 19, 24, 4], may be applicable to the bilattice dynamic epistemic logic presented in this work.

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