Asymptotic limit for the stokes and navier-stokes problems in a planar domain with a vanishing hole

Alexandre Munnier

ASYMPTOTIC LIMIT FOR THE STOKES AND NAVIER–STOKES PROBLEMS IN A PLANAR DOMAIN WITH A VANISHING HOLE

ALEXANDRE MUNNIER

Abstract. We show that the eigenvalues of the Stokes operator in a domain with a small hole converge to the eigenvalues of the Stokes operator in the whole domain, when the diameter of the hole tends to 0. The convergence of the eigenspaces and the convergence of the Stokes semigroup are also established. Concerning the Navier–Stokes equations, we prove that the vorticity of the solution in the perforated domain converges as the hole shrinks to a point r to the vorticity of the solution in the punctured domain (i.e. the whole domain with the point r removed). The main ingredients of the analysis are a suitable decomposition of the vorticity space, the formalism elaborated in [7] and some basics of potential theory.

1. Introduction

Let $\mathcal{F}$ be an open, bounded and smooth domain in $\mathbb{R}^2$. The Lebesgue space $\mathbf{L}^2(\mathcal{F}) = L^2(\mathcal{F}, \mathbb{R}^2)$ and the Sobolev space $\mathbf{H}^1_0(\mathcal{F}) = H^1_0(\mathcal{F}, \mathbb{R}^2)$ are equipped with their usual scalar products and the Hilbert spaces:

$$\mathbf{J}_0(\mathcal{F}) = \big\{u \in \mathbf{L}^2(\mathcal{F}) \,:\, \nabla\cdot u = 0 \text{ in } \mathcal{F},\, u\cdot n = 0 \text{ on } \partial\mathcal{F}\big\} \qquad \text{and} \qquad \mathbf{J}_1(\mathcal{F}) = \mathbf{J}_0(\mathcal{F}) \cap \mathbf{H}^1_0(\mathcal{F}), \tag{1.1}$$

are provided respectively with the scalar products:

$$(u,v)_{\mathbf{J}_0(\mathcal{F})} = (u,v)_{\mathbf{L}^2(\mathcal{F})} \qquad \text{for all } u,v \in \mathbf{J}_0(\mathcal{F}), \tag{1.2a}$$

$$(u,v)_{\mathbf{J}_1(\mathcal{F})} = (\nabla^\perp\cdot u, \nabla^\perp\cdot v)_{L^2(\mathcal{F})} \qquad \text{for all } u,v \in \mathbf{J}_1(\mathcal{F}). \tag{1.2b}$$

In (1.1), n stands for the unit outer normal vector to $\partial\mathcal{F}$. In (1.2b) and subsequently in the paper, for every $x = (x_1, x_2) \in \mathbb{R}^2$, the notation $x^\perp$ is used to represent the vector $(-x_2, x_1)$. Identifying $\mathbf{J}_0(\mathcal{F})$ with its dual space $\mathbf{J}_0'(\mathcal{F})$ and denoting by $\mathbf{J}_{-1}(\mathcal{F})$ the dual space of $\mathbf{J}_1(\mathcal{F})$ with respect to the pivot $\mathbf{J}_0(\mathcal{F})$, we obtain a Gelfand triple of Hilbert spaces:

$$\mathbf{J}_1(\mathcal{F}) \subset \mathbf{J}_0(\mathcal{F}) \subset \mathbf{J}_{-1}(\mathcal{F}),$$

both inclusions being continuous and dense. The Stokes operator $\mathsf{J}_\mathcal{F}$ is the unbounded operator on $\mathbf{J}_0(\mathcal{F})$ of domain

$$D(\mathsf{J}_\mathcal{F}) = \big\{u \in \mathbf{J}_1(\mathcal{F}) \,:\, (u,\cdot)_{\mathbf{J}_1(\mathcal{F})} \in \mathbf{J}_0'(\mathcal{F})\big\},$$

and defined for every $u \in D(\mathsf{J}_\mathcal{F})$ by means of the Riesz representation Theorem by:

$$(\mathsf{J}_\mathcal{F} u, \cdot)_{\mathbf{J}_0(\mathcal{F})} = (u,\cdot)_{\mathbf{J}_1(\mathcal{F})}.$$

The spectrum of $\mathsf{J}_\mathcal{F}$ consists in a sequence of monotonically ordered positive eigenvalues $(\lambda_k^\mathcal{F})_{k\geqslant 1}$ that tends to $+\infty$. The eigenvalues are counted with their multiplicity. They meet the Courant–Fischer min-max principle:

$$\lambda_k^\mathcal{F} = \min_{V\in W_k^J} \max_{\substack{\theta\in V\\ \theta\neq 0}} \frac{\|\theta\|^2_{\mathbf{J}_1(\mathcal{F})}}{\|\theta\|^2_{\mathbf{J}_0(\mathcal{F})}} \qquad \text{for all } k\in\mathbb{N},\, k\geqslant 1, \tag{1.3}$$

where W_k^J stands for the set of all the subspaces of dimension k in $\mathbf{J}_1(\mathcal{F})$. For every positive integer k, we denote by $I_k^\mathcal{F}$ the set of all the indices j such that $\lambda_j^\mathcal{F} = \lambda_k^\mathcal{F}$ and by $\Lambda_k^\mathcal{F}$ the eigenspace associated with the eigenvalue $\lambda_k^\mathcal{F}$. This implies in particular that $\dim \Lambda_k^\mathcal{F} = \#I_k^\mathcal{F}$ and that $\Lambda_j^\mathcal{F} = \Lambda_k^\mathcal{F}$ if $j \in I_k^\mathcal{F}$. To every eigenvalue (still counted with algebraic multiplicity) we associate an eigenfunction $u_k^\mathcal{F}$ chosen in such a way that the family $\big\{u_k^\mathcal{F},\, k\geqslant 1\big\}$ is a Riesz Basis orthonormal in $\mathbf{J}_0(\mathcal{F})$ and orthogonal in $\mathbf{J}_1(\mathcal{F})$.

Our first purpose is to study the behavior of the spectrum of $\mathsf{J}_\mathcal{F}$ when the domain $\mathcal{F}$ has a small hole whose diameter tends to 0. More precisely, the geometry considered is as follows: $\mathcal{G}$ is an open, bounded and simply connected domain with a smooth boundary denoted by Γ. For every ε (a real number ranging from 0 to some positive real number ε_0) we define a smooth and simply connected domain $\mathcal{O}_\varepsilon$

Date: November 21, 2020.

(subsequently referred to as the "hole" in the domain or the "obstacle" in the fluid). The boundary of $\mathcal{O}_\varepsilon$ is denoted by Σ_ε and we assume that there exists a point $r \in \mathcal{G}$ such that:

$$\overline{\mathcal{D}(r,\varepsilon_0)} \subset \mathcal{G} \quad \text{and} \quad \overline{\mathcal{O}_\varepsilon} \subset \mathcal{D}(r,\varepsilon) \qquad \text{for all } \varepsilon \in (0,\varepsilon_0),$$

where $\mathcal{D}(r,\varepsilon)$ stands for the disk of center r and radius ε. As ε tends to 0, we shall write that $\mathcal{O}_\varepsilon$ "shrinks" (or vanishes) to a point. The perforated domain (occupied by the fluid) is $\mathcal{F}_\varepsilon = \mathcal{G} \setminus \overline{\mathcal{O}_\varepsilon}$ and thereby its boundary is the disjoint union $\Gamma \cup \Sigma_\varepsilon$. Notice that the notion of "vanishing" obstacle as defined here is more general than the one considered in [4] or [6] for instance, where the obstacle is the homothetic image of a reference configuration. Our definition is roughly the same as in [3] (where the more intricated case of moving obstacles is addressed).

The statement of the first main result yet requires making precise the notion of subspace convergence: For every $\varepsilon \in (0,\varepsilon_0)$, let W_ε be a closed subspace in a Hilbert space H and let the orthogonal projection on W_ε be denoted by Π_{W_ε}. Let W be another closed subspace of H and let the orthogonal projection on W be denoted by Π_W. We shall write that $W_\varepsilon \longrightarrow W$ as $\varepsilon \longrightarrow 0$ when:

$$\sup_{\substack{\theta\in H\\ \theta\neq 0}} \frac{\|\Pi_W\theta - \Pi_{W_\varepsilon}\theta\|_H}{\|\theta\|_H} \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0.$$

Extending the functions by 0 inside $\mathcal{O}_\varepsilon$, we can assume that for every $\varepsilon \in (0,\varepsilon_0)$, $\mathbf{J}_0(\mathcal{F}_\varepsilon)$ is a closed subspace of $\mathbf{J}_0(\mathcal{G})$ and that $\mathbf{J}_1(\mathcal{F}_\varepsilon)$ is a closed subspace of $\mathbf{J}_1(\mathcal{G})$. In the same manner, the eigenspaces $\Lambda_k^{\mathcal{F}_\varepsilon}$ can be considered as closed subspaces of $\mathbf{J}_1(\mathcal{G})$.

Theorem 1.1 (Convergence of eigenvalues and eigenspaces)**.** *Assume that $\mathcal{O}_\varepsilon$ shrinks to a point as ε goes to 0. Then, for every positive integer k:*

$$\text{(1.4a)} \qquad \lambda_k^{\mathcal{F}_\varepsilon} \longrightarrow \lambda_k^{\mathcal{G}} \qquad \text{as } \varepsilon \longrightarrow 0,$$

$$\text{(1.4b)} \qquad \bigoplus_{j\in I_k^{\mathcal{G}}} \Lambda_j^{\mathcal{F}_\varepsilon} \longrightarrow \Lambda_k^{\mathcal{G}} \quad \text{in} \quad \mathbf{J}_0(\mathcal{G}) \quad \text{as } \varepsilon \longrightarrow 0.$$

We emphasize that in (1.4b) the sum ranges over all the indices j such that $\lambda_j^{\mathcal{G}} = \lambda_k^{\mathcal{G}}$ (because some eigenvalues can be different when $\varepsilon > 0$ and may eventually meet when $\varepsilon = 0$).

The asymptotic behavior of solutions of the (stationary) Stokes equations in a domain with a small hole has been widely investigated; see for instance [1] and references therein. However, the asymptotic limit of the eigenvalues and eigenspaces of the Stokes operator has not been dealt with so far.

The proof of Theorem 1.1 rests on a restatement of the Stokes operator in term of so–called non–primitive variables (stream function and vorticity). This task was carried out in the paper [7] and briefly summarized later on. Then, the conclusion of the theorem derives from a suitable decomposition of the vorticity space (established in Section 3) for a perforated domain.

The convergence results (1.4) allow quite easily deriving convergence results for the Stokes semigroup. Denote by $\{\mathsf{T}_{\mathcal{G}}(t),\, t \geqslant 0\}$ the Stokes semigroup whose infinitesimal generator is $\mathsf{J}_{\mathcal{G}}$ (the Stokes operator for the domain $\mathcal{G}$). For every $\theta \in \mathbf{J}_0(\mathcal{G})$, we have the classical expression:

$$\text{(1.5)} \qquad \mathsf{T}_{\mathcal{G}}(t)\theta = \sum_{j\geqslant 1} \big(\theta, u_j^{\mathcal{G}}\big)_{\mathbf{J}_0(\mathcal{G})} e^{-\lambda_j^{\mathcal{G}} t} u_j^{\mathcal{G}}, \qquad t \geqslant 0.$$

In the same manner and for every $\varepsilon \in (0,\varepsilon_0)$, we can define for the domain $\mathcal{F}_\varepsilon$ the semigroup $\{\mathsf{T}_{\mathcal{F}_\varepsilon}(t),\, t \geqslant 0\}$ whose infinitesimal generator is the Stokes operator $\mathsf{J}_{\mathcal{F}_\varepsilon}$. Thus:

$$\text{(1.6)} \qquad \mathsf{T}_{\mathcal{F}_\varepsilon}(t)\theta = \sum_{j\geqslant 1} \big(\theta, u_j^{\mathcal{F}_\varepsilon}\big)_{\mathbf{J}_0(\mathcal{G})} e^{-\lambda_j^{\mathcal{F}_\varepsilon} t} u_j^{\mathcal{F}_\varepsilon}, \qquad t \geqslant 0,$$

and this expression makes sense for every $\theta \in \mathbf{J}_0(\mathcal{G})$.

Corollary 1.2. *The following limit holds for every $T > 0$ and every $\theta \in \mathbf{J}_0(\mathcal{G})$:*

$$\text{(1.7a)} \qquad \sup_{t\in[0,T]} e^{\lambda_1^{\mathcal{G}} t} \big\|\mathsf{T}_{\mathcal{G}}(t)\theta - \mathsf{T}_{\mathcal{F}_\varepsilon}(t)\theta\big\|_{\mathbf{J}_0(\mathcal{G})} \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

For every compact set $K \subset \mathbf{J}_0(\mathcal{G})$ and every $T > 0$:

$$\text{(1.7b)} \qquad \sup_{\substack{t\in[0,T]\\ \theta\in K,\, \theta\neq 0}} e^{\lambda_1^{\mathcal{G}} t} \frac{\big\|\mathsf{T}_{\mathcal{G}}(t)\theta - \mathsf{T}_{\mathcal{F}_\varepsilon}(t)\theta\big\|_{\mathbf{J}_0(\mathcal{G})}}{\|\theta\|_{\mathbf{J}_0(\mathcal{G})}} \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

The exponential decay property:

$$\|\mathsf{T}_{\mathcal{G}}(t)\theta\|_{\mathbf{J}_0(\mathcal{G})} \leqslant \|\theta\|_{\mathbf{J}_0(\mathcal{G})} e^{-\lambda_1^{\mathcal{G}} t} \qquad \text{for all } \theta \in \mathbf{J}_0(\mathcal{G}),$$

explains the role played by the term $e^{\lambda_1^{\mathcal{G}} t}$ in the estimate (1.7).

We turn now our attention to the Navier–Stokes equations. For every $\varepsilon \in (0, \varepsilon_0)$, let u^0_ε be given in $\mathbf{J}_0(\mathcal{F}_\varepsilon)$ and let u_ε be the unique function in

$$L^2(\mathbb{R}_+; \mathbf{J}_1(\mathcal{F}_\varepsilon)) \cap C(\mathbb{R}_+; \mathbf{J}_0(\mathcal{F}_\varepsilon)) \cap H^1(\mathbb{R}_+; \mathbf{J}_{-1}(\mathcal{F}_\varepsilon)),$$

that solves the following Cauchy problem for every $\theta \in \mathbf{J}_1(\mathcal{F}_\varepsilon)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_\varepsilon, \theta)_{\mathbf{J}_0(\mathcal{F}_\varepsilon)} + \nu(u_\varepsilon, \theta)_{\mathbf{J}_1(\mathcal{F}_\varepsilon)} - ((u_\varepsilon \cdot \nabla)\theta, u_\varepsilon)_{\mathbf{L}^2(\mathcal{F}_\varepsilon)} = 0 \quad \text{on } \mathbb{R}_+ \tag{1.8a}$$

$$u_\varepsilon(0) = u^0_\varepsilon \quad \text{in } \mathcal{F}_\varepsilon. \tag{1.8b}$$

Theorem 1.3. *Assume that there exists $u^0 \in \mathbf{J}_0(\mathcal{G})$ such that $u^0_\varepsilon \rightharpoonup u^0$ weak in $\mathbf{J}_0(\mathcal{G})$ (here and subsequently, u^0_ε and u_ε are extended by 0 inside $\mathcal{O}_\varepsilon$). Then, as ε goes to 0:*

$$u_\varepsilon \rightharpoonup u \text{ weak-}\star \text{ in } L^\infty(\mathbb{R}_+; \mathbf{J}_0(\mathcal{G})), \tag{1.9a}$$

$$u_\varepsilon \longrightarrow u \quad \text{strong in } L^2_{\ell oc}(\mathbb{R}_+; \mathbf{J}_0(\mathcal{G})), \tag{1.9b}$$

$$u_\varepsilon \rightharpoonup u \quad \text{weak in } L^2(\mathbb{R}_+; \mathbf{J}_1(\mathcal{G})), \tag{1.9c}$$

where the function u belongs to:

$$L^2(\mathbb{R}_+; \mathbf{J}_1(\mathcal{G})) \cap C(\mathbb{R}_+; \mathbf{J}_0(\mathcal{G})) \cap H^1(\mathbb{R}_+; \mathbf{J}_{-1}(\mathcal{G})),$$

and solves the Cauchy problem for every $\theta \in \mathbf{J}_1(\mathcal{G})$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(u, \theta)_{\mathbf{J}_0(\mathcal{G})} + \nu(u, \theta)_{\mathbf{J}_1(\mathcal{G})} - ((u \cdot \nabla)\theta, u)_{\mathbf{L}^2(\mathcal{G})} = 0 \quad \text{on } \mathbb{R}_+ \tag{1.10a}$$

$$u(0) = u^0 \quad \text{in } \mathcal{G}. \tag{1.10b}$$

Let now χ be in $\mathscr{D}(\mathcal{G} \setminus \{r\})$ and denote by ω^0_ε the vorticity of the velocity field u^0_ε. Assume that for every $\varepsilon \in (0, \varepsilon_0)$, ω^0_ε is in $V_0(\mathcal{F}_\varepsilon)$ and that the quantity $\|\chi\omega^0_\varepsilon\|_{L^2(\mathcal{G})}$ is uniformly bounded. Then, as ε goes to 0:

$$\chi\omega_\varepsilon \rightharpoonup \chi\omega \qquad \text{weak-}\star \text{ in } L^\infty(\mathbb{R}_+; L^2(\mathcal{G})), \tag{1.11a}$$

$$\nabla(\chi\omega_\varepsilon) \rightharpoonup \nabla(\chi\omega) \quad \text{weak in } L^2(\mathbb{R}_+; \mathbf{L}^2(\mathcal{G})), \tag{1.11b}$$

where $\omega = \nabla^\perp \cdot u$ and $\omega_\varepsilon = \nabla^\perp \cdot u_\varepsilon$.

Although stated in a different and more intricate contexte (exterior domain or moving obstacles), the convergence results (1.9) meet those obtained in [4] and [3] and to this extent cannot be considered as new. However, we shall provide a completely different and more simple proof based on the stream–vorticity formulation of the Navier–Stokes equations introduced in [7] and involving a different compactness argument. In contrast, the convergence (1.11) of the vorticity is new.

As already mentioned in [7], the analysis of the solutions to the Navier–Stokes equations in a planar domain is tightly related to the analysis of the harmonic functions and more precisely on some L^2–mass concentration properties near the boundaries of the domain. This provides an efficient and original strategy to deal with the problem.

The rest of the paper is organized as follows: The next Section is a short summary of results from [7]. Section 3 is dedicated to technical lemmas addressing mainly L^2–mass concentration properties of harmonic functions in a perforated domain. A theorem describing the structure of the vorticity space is also provided. The proofs of Theorem 1.1 and Corollary 1.2 are carried out in Section 4 and the last section contains the proof of Theorem 1.3.

2. The Navier–Stokes equations in non–primitive variables

Let us put aside for a while the perforated domain $\mathcal{F}_\varepsilon$ and consider back as in the Introduction the more general domain simply denoted by $\mathcal{F}$. In addition of being smooth and bounded, the domain $\mathcal{F}$ is also assumed to be N–connected (N a nonnegative integer). The boundary of $\mathcal{F}$ can be split into a disjoint union of smooth Jordan curves:

$$\partial\mathcal{F} = \Big(\bigcup_{k=1}^N \Sigma_k\Big) \cup \Gamma. \tag{2.1}$$

The curves Σ_k for $k \in \{1,\dots,N\}$ are the inner boundaries of $\mathcal{F}$ while Γ is the outer boundary. The Hilbert spaces:

$$S_0(\mathcal{F}) = \{\psi \in H^1(\mathcal{F}) \,:\, \psi|_\Gamma = 0 \quad \text{and} \quad \psi|_{\Sigma_j} = c_j, \quad c_j \in \mathbb{R}, \quad j = 1,\dots,N\}, \tag{2.2a}$$

$$S_1(\mathcal{F}) = \left\{\psi \in S_0(\mathcal{F}) \cap H^2(\mathcal{F}) \,:\, \frac{\partial \psi}{\partial n}\Big|_{\partial\mathcal{F}} = 0\right\}, \tag{2.2b}$$

are provided with the scalar products:

$$(\psi_1,\psi_2)_{S_0(\mathcal{F})} = (\nabla\psi_1,\nabla\psi_2)_{\mathbf{L}^2(\mathcal{F})} \qquad \text{for all } \psi_1,\psi_2 \in S_0(\mathcal{F}), \tag{2.2c}$$

$$(\psi_1,\psi_2)_{S_1(\mathcal{F})} = (\Delta\psi_1,\Delta\psi_2)_{L^2(\mathcal{F})} \qquad \text{for all } \psi_1,\psi_2 \in S_1(\mathcal{F}). \tag{2.2d}$$

The space $S_1(\mathcal{F})$ is continuously and densely embedded in $S_0(\mathcal{F})$. Using the latter as pivot space and denoting by $S_{-1}(\mathcal{F})$ the dual of $S_1(\mathcal{F})$, we obtain a Gelfand triple of Hilbert spaces:

$$S_1(\mathcal{F}) \subset S_0(\mathcal{F}) \subset S_{-1}(\mathcal{F}).$$

This provides the suitable functional framework to deal with the Navier–Stokes equations in stream function formulation. Thus:

Theorem 2.1 (Well posedness of the weak NS equations in stream function formulation)**.** *For any $\psi^0 \in S_0(\mathcal{F})$, there exists a unique function:*

$$\psi \in H^1(\mathbb{R}_+; S_{-1}(\mathcal{F})) \cap C(\mathbb{R}_+; S_0(\mathcal{F})) \cap L^2(\mathbb{R}_+; S_1(\mathcal{F})),$$

satisfying for every $\theta \in S_1(\mathcal{F})$ the Cauchy problem:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi,\theta)_{S_0(\mathcal{F})} + \nu(\psi,\theta)_{S_1(\mathcal{F})} - (D^2\theta\nabla^\perp\psi,\nabla\psi)_{\mathbf{L}^2(\mathcal{F})} = 0 \qquad \text{on } \mathbb{R}_+, \tag{2.3a}$$

$$\psi(0) = \psi^0 \quad \text{in } \mathcal{F}, \tag{2.3b}$$

where $D^2\theta$ is the Hessian tensor field of θ in $\mathcal{G}$.

The proof of this result (as the proofs of all the results stated in this section) can be found in [7].

We shall now established the expression of the Navier–Stokes equations in vorticity formulation. Let $\mathfrak{H}(\mathcal{F})$ stand for the closed space of the harmonic functions h in $L^2(\mathcal{F})$ verifying, for every smooth Jordan curve $\mathscr{C}$ included in $\mathcal{F}$:

$$\int_{\mathscr{C}} \frac{\partial h}{\partial n}\,\mathrm{d}s = 0, \tag{2.4}$$

where n is the unit normal vector to the curve $\mathscr{C}$ (when $\mathcal{F}$ is simply connected, this condition is automatically satisfied by any harmonic function in $L^2(\mathcal{F})$). In the flux condition (2.4), the normal derivative of h on $\mathscr{C}$ is well defined as an element of $H^{-1/2}(\mathscr{C})$ (so the integral should be understood as a duality bracket). Next, we introduce $V_0(\mathcal{F}) = \mathfrak{H}(\mathcal{F})^\perp$ so that:

$$L^2(\mathcal{F}) = V_0(\mathcal{F}) \overset{\perp}{\oplus} \mathfrak{H}(\mathcal{F}). \tag{2.5}$$

We recall that (see [2], [7] for a proof):

Proposition 2.2. *The following operators are isometries:*

$$\nabla^\perp : S_0(\mathcal{F}) \longrightarrow \mathbf{J}_0(\mathcal{F}), \qquad \psi \longmapsto \nabla^\perp\psi, \tag{2.6a}$$

$$\nabla^\perp : S_1(\mathcal{F}) \longrightarrow \mathbf{J}_1(\mathcal{F}), \qquad \psi \longmapsto \nabla^\perp\psi, \tag{2.6b}$$

$$\Delta : S_1(\mathcal{F}) \longrightarrow V_0(\mathcal{F}), \qquad \psi \longmapsto \Delta\psi. \tag{2.6c}$$

The inverse of the last operator is usually referred to as the Biot-Savart operator.

For a velocity field u in the space $\mathbf{J}_1(\mathcal{F})$, the function ψ such that $\nabla^\perp\psi = u$ is the associated stream function and $\omega = \Delta\psi = \nabla^\perp \cdot u$ is the vorticity field. From Proposition 2.2, it can easily be deduced that the formulation (2.3) of the Navier–Stokes equations is equivalent to the classical one, (1.8).

The operator (2.6c) being an isomorphism, the space $V_0(\mathcal{F})$ will be called in the sequel the vorticity space. This space is provided with the classical L^2–scalar product that is denoted by $(\cdot,\cdot)_{V_0(\mathcal{F})}$.

Definition 2.3 (The projectors $\mathsf{P}_\mathcal{F}$ and $\mathsf{Q}_\mathcal{F}$)**.** *The orthogonal projector onto $V_0(\mathcal{F})$ in $L^2(\mathcal{F})$ is denoted by $\mathsf{P}_\mathcal{F}$ and the orthogonal projection from $H^1(\mathcal{F})$ onto $S_0(\mathcal{F})$ for the semi-norm $(\nabla\cdot,\nabla\cdot)_{\mathbf{L}^2(\mathcal{F})}$ is denoted by $\mathsf{Q}_\mathcal{F}$.*

The projectors $\mathsf{P}_\mathcal{F}$ and $\mathsf{Q}_\mathcal{F}$ will play an important role in the proof of Theorem 1.3. They enjoy the following properties:

Proposition 2.4. *For every positive integer k, the operators $\mathsf{P}_{\mathcal{F}}$ and $\mathsf{Q}_{\mathcal{F}}$ map continuously $H^k(\mathcal{F})$ into $H^k(\mathcal{F})$. The mapping $\mathsf{P}_{\mathcal{F}} : S_0(\mathcal{F}) \longmapsto V_1(\mathcal{F})$ is invertible and its inverse is $\mathsf{Q}_{\mathcal{F}} : V_1(\mathcal{F}) \longmapsto S_0(\mathcal{F})$.*

The function space $V_1(\mathcal{F}) = V_0(\mathcal{F}) \cap H^1(\mathcal{F})$ provided with the scalar product:

$$(\omega_1, \omega_2)_{V_1(\mathcal{F})} = (\nabla \mathsf{Q}_{\mathcal{F}}\omega_1, \nabla \mathsf{Q}_{\mathcal{F}}\omega_2)_{\mathbf{L}^2(\mathcal{F})} \qquad \text{for all } \omega_1, \omega_2 \in V_1(\mathcal{F}),$$

is densely and continuously included in $V_0(\mathcal{F})$. The dual of $V_1(\mathcal{F})$ using $V_0(\mathcal{F})$ as pivot space is denoted by $V_{-1}(\mathcal{F})$ so that:

$$V_1(\mathcal{F}) \subset V_0(\mathcal{F}) \subset V_{-1}(\mathcal{F}),$$

is a Gelfand triple of Hilbert spaces. We recall:

Theorem 2.5 (well posedness of the strong NS equations in vorticity formulation [7])**.** *For every vorticity field ω^0 in $V_0(\mathcal{F})$, there exists a unique function ω in the space:*

$$L^2(\mathbb{R}_+; V_1(\mathcal{F})) \cap C(\mathbb{R}_+; V_0(\mathcal{F})) \cap H^1(\mathbb{R}_+; V_{-1}(\mathcal{F})),$$

satisfying the following Cauchy problem for every $\theta \in V_1(\mathcal{F})$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\omega, \theta)_{V_0(\mathcal{F})} + \nu(\omega, \theta)_{V_1(\mathcal{F})} - (\omega \nabla^\perp \psi, \nabla \mathsf{Q}_{\mathcal{F}}\theta)_{\mathbf{L}^2(\mathcal{F})} = 0 \qquad \text{on } \mathbb{R}_+, \tag{2.7a}$$

$$\omega(0) = \omega^0 \quad \text{in } \mathcal{F}, \tag{2.7b}$$

where ψ is the stream function deduced from ω by means of the Biot-Savart operator.

Remark 2.6. *We emphasize that in the equation* (2.7a)*, the dissipative term is $\|\omega\|^2_{V_1(\mathcal{F})} = \|\nabla \mathsf{Q}_{\mathcal{F}}\omega\|^2_{\mathbf{L}^2(\mathcal{F})}$ and not $\|\nabla \omega\|^2_{\mathbf{L}^2(\mathcal{F})}$ as it could be envisioned.*

An other consequence of Proposition 2.2 is the restatement of the Courant–Fischer min-max principle (1.3) for the eigenvalues of the Stokes operator, in term of stream functions. Thus, for every positive integer k, we have:

$$\lambda_k^{\mathcal{F}} = \min_{V \in W_k^{\mathcal{F}}} \max_{\substack{\theta \in V \\ \theta \neq 0}} \frac{\|\theta\|^2_{S_1(\mathcal{F})}}{\|\theta\|^2_{S_0(\mathcal{F})}}, \tag{2.8}$$

where $W_k^{\mathcal{F}}$ stands the set of all the subspaces of $S_1(\mathcal{F})$ of dimension k. We can defined as well a family $\{\psi_k^{\mathcal{F}},\, k \geqslant 1\}$ in $S_1(\mathcal{F})$ that is an orthonormal Riesz basis in $S_0(\mathcal{F})$ and orthogonal in $S_1(\mathcal{F})$ and such that:

$$(\psi_k^{\mathcal{F}}, \theta)_{S_1(\mathcal{F})} = \lambda_k^{\mathcal{F}} (\psi_k^{\mathcal{F}}, \theta)_{S_0(\mathcal{F})} \qquad \text{for all } \theta \in S_1(\mathcal{F}).$$

3. Decomposition of the vorticity space

In this section we consider a fixed perforated domain $\mathcal{F} = \mathcal{G} \setminus \overline{\mathcal{O}}$ with $\overline{\mathcal{O}} \subset \mathcal{G}$, the sets $\mathcal{G}$ and $\mathcal{O}$ being open, bounded and simply connected, with smooth boundaries denoted respectively by Γ and Σ (see the left hand side picture of Fig. 1). We are not yet interested in letting the obstacle $\mathcal{O}$ shrink into a point but we shall assume that, loosely speaking, $\mathcal{O}$ is small enough or far enough from the boundary Γ for allowing an annulus encircling $\mathcal{O}$ to be included in $\mathcal{F}$. The precise statement of the geometric hypothesizes on the domains is rather technical:

Hypothesis 3.1. *There exist four concentric disks $\mathcal{D}_e$, $\mathcal{D}_i$, $\mathcal{D}_+$ and $\mathcal{D}_-$ (see Fig. 2) such that:*

$$\overline{\mathcal{G}} \subset \mathcal{D}_e, \quad \overline{\mathcal{D}_i} \subset \mathcal{G}, \quad \overline{\mathcal{D}_+} \subset \mathcal{D}_i, \quad \overline{\mathcal{D}_-} \subset \mathcal{D}_+ \quad \text{and} \quad \overline{\mathcal{O}} \subset \mathcal{D}_-.$$

The radii of the disks $\mathcal{D}_e, \mathcal{D}_i, \mathcal{D}_+$ and $\mathcal{D}_-$ are denoted respectively by R_e, R_i, R_+ and R_- (R_e and R_i are fixed while R_+ and R_- are meant to tend to 0*). They are such that:*

$$R_- = R_e e^{-\delta^2}, \quad R_+ = R_e e^{-\delta} \qquad \text{for some } \delta > \delta_0 \text{ with } \delta_0 = 2 + \ln(R_e/R_i). \tag{3.1}$$

In the case where the obstacle $\mathcal{O}_\varepsilon$ shrinks into a point, the hypothesis above is obviously satisfied for ε small enough providing that:

$$\delta \leqslant \sqrt{\ln(R_e/\varepsilon)}. \tag{3.2}$$

We define the annuli $\mathcal{C}_e = \mathcal{D}_e \setminus \overline{\mathcal{D}_+}$ and $\mathcal{C}_+ = \mathcal{D}_+ \setminus \overline{\mathcal{D}_-}$. When the obstacle shrinks, the analysis requires distinguishing between the behavior of some quantities near the boundary Σ and far from this boundary. To this purpose, we introduce also the domains $\mathcal{F}_\Sigma = \mathcal{D}_+ \setminus \overline{\mathcal{O}}$ (a vanishing neighborhood of the boundary Σ in $\mathcal{F}$ where some boundary layer phenomena will take place) and $\mathcal{F}_\Gamma = \mathcal{F} \setminus \overline{\mathcal{D}_+}$ its supplement in $\mathcal{F}$ (see the right hand side picture of Fig. 1). The size of the "boundary layer" $\mathcal{F}_\Sigma$ is given by the identities (3.1).

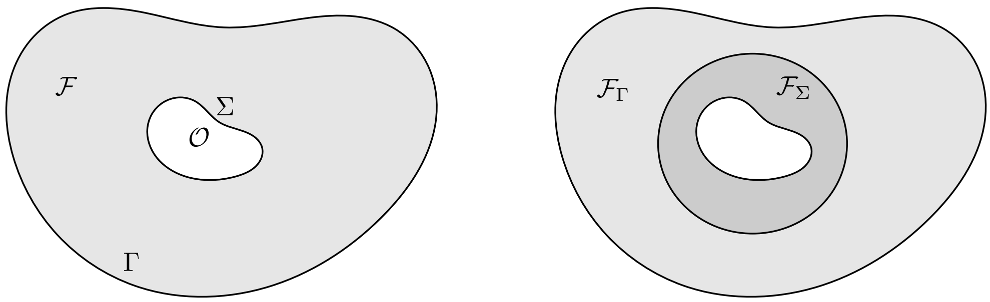

FIGURE 1. On the left: The domains and the boundaries; $\mathcal{G} = \mathcal{F} \cup \overline{\mathcal{O}}$. On the right: the partition of $\mathcal{F}$ into $\mathcal{F}_\Sigma$ (a neighborhood of the boundary Σ) and $\mathcal{F}_\Gamma$.

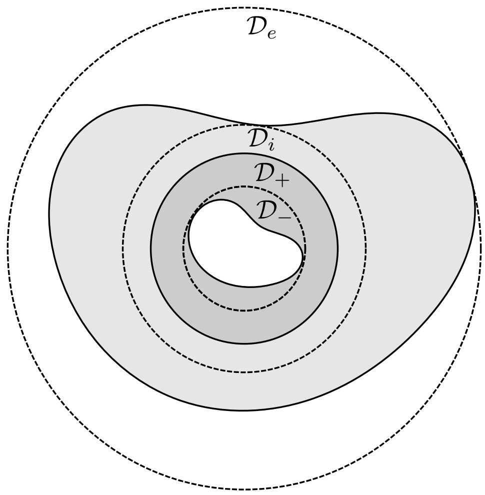

FIGURE 2. The four concentric disks introduced in Hypothesis 3.1. We recall also the definition of the annuli $\mathcal{C}_e = \mathcal{D}_e \setminus \overline{\mathcal{D}_+}$ and $\mathcal{C}_+ = \mathcal{D}_+ \setminus \overline{\mathcal{D}_-}$.

Considering the decomposition (2.5), we may imagine that the structure of $V_0(\mathcal{G})$ is tightly related to the structure of $\mathfrak{H}(\mathcal{G})$. The analysis of this latter space is carried out in a series of Lemmas. In the sequel, we shall denote by $f^\star$ a function in $L^2(\mathcal{G})$ obtained by extending by 0 a function f of $L^2(\mathcal{F})$ or $L^2(\mathcal{O})$.

Lemma 3.2. *Both following assertions hold true:*

(1) *The space $\mathfrak{H}(\mathcal{O})$ is the closure in $L^2(\mathcal{O})$ of the space $\{h|_\mathcal{O} : h \in \mathfrak{H}(\mathcal{G})\}$.*
(2) *The space $\mathfrak{H}_\mathcal{G}(\mathcal{F}) = \{h|_\mathcal{F} : h \in \mathfrak{H}(\mathcal{G})\}$ is closed in $L^2(\mathcal{G})$. We denote by $\mathfrak{H}_0(\mathcal{F})$ its orthogonal complement in $\mathfrak{H}(\mathcal{F})$ so that:*

$$\mathfrak{H}(\mathcal{F}) = \mathfrak{H}_\mathcal{G}(\mathcal{F}) \overset{\perp}{\oplus} \mathfrak{H}_0(\mathcal{F}). \tag{3.3}$$

The space $\mathfrak{H}_0(\mathcal{F})$ is never reduced to $\{0\}$ and thereby

$$\mathfrak{H}_0^\star(\mathcal{F}) = \{h^\star : h \in \mathfrak{H}_0(\mathcal{F})\}, \tag{3.4}$$

is a nonempty closed subspace of $V_0(\mathcal{G})$.

Proof. Concerning the first assertion, assume that there exists a function $h_\mathcal{O} \in \mathfrak{H}(\mathcal{O})$ such that:

$$\int_\mathcal{O} h_\mathcal{O} h_\mathcal{G} \, \mathrm{d}x = 0 \qquad \text{for all } h_\mathcal{G} \in \mathfrak{H}(\mathcal{G}).$$

This means that $h_\mathcal{O}^\star$ (i.e. $h_\mathcal{O}$ extended by 0 in $\mathcal{F}$) belongs to $V_0(\mathcal{G})$ and therefore (using the isometry (2.6c) of Proposition 2.6), there exists a function $\psi \in H^2_0(\mathcal{G})$ such that $\Delta\psi = h_\mathcal{O}^\star$. But in $\mathcal{F}$ the function ψ is harmonic and on the boundary Γ it satisfies $\psi|_\Gamma = 0$ and $\partial\psi/\partial n|_\Gamma = 0$. The unique continuation principle for harmonic functions asserts that $\psi = 0$ in $\mathcal{F}$. Since ψ belongs to $H^2_0(\mathcal{G})$, we deduce that $\psi|_\Sigma = \partial\psi/\partial n|_\Sigma = 0$ and therefore that $h_\mathcal{O}$ is in $V_0(\mathcal{O})$ (using again the isometry (2.6c) of Proposition 2.6). By definition $V_0(\mathcal{O}) = \mathfrak{H}(\mathcal{O})^\perp$ and hence $h_\mathcal{O} = 0$.

Addressing the second assertion of the lemma, observe that the convergence of a sequence of functions in $\mathfrak{H}_{\mathcal{G}}(\mathcal{F})$ entails in particular the convergence of the traces of these functions in $H^{-1/2}(\Gamma)$ and therefore of the functions in $\mathfrak{H}(\mathcal{G})$. This proves that $\mathfrak{H}_{\mathcal{G}}(\mathcal{F})$ is closed.

Let $h_{\mathcal{G}}$ be a nonconstant function in $\mathfrak{H}(\mathcal{G})$ and define $g_0 = h_{\mathcal{G}}|_\Sigma$ and $g_1 = \partial h_{\mathcal{G}}/\partial n|_\Sigma$ (notice that g_0 cannot be constant because this would entails that $h_{\mathcal{G}}$ is constant in $\mathcal{O}$ and then also in $\mathcal{G}$). Let ψ be the biharmonic function in $\mathcal{F}$ such that $\psi|_\Gamma = 0$, $\partial\psi/\partial n|_\Gamma = 0$ and $\partial\psi/\partial n|_\Sigma = g_1$, $\psi|_\Sigma = g_0 + c$ with c a constant such that $\int_\Sigma \partial(\Delta\psi)/\partial n \,\mathrm{d}s = 0$ (this condition makes sense since $\partial(\Delta\psi)/\partial n \in H^{-1/2}(\Sigma)$). Then, for every $h'_{\mathcal{G}} \in \mathfrak{H}(\mathcal{G})$:

$$\int_{\mathcal{F}} \Delta\psi\, h'_{\mathcal{G}}\,\mathrm{d}x = \int_\Sigma \Big(g_1\, h'_{\mathcal{G}} - g_0\,\frac{\partial h'_{\mathcal{G}}}{\partial n}\Big)\,\mathrm{d}s = \int_{\mathcal{O}} \Big(\Delta h'_{\mathcal{G}}\, h_{\mathcal{G}} - h'_{\mathcal{G}}\Delta h_{\mathcal{G}}\Big)\,\mathrm{d}x = 0,$$

and therefore $\Delta\psi$ belongs to $\mathfrak{H}_0(\mathcal{F})$. Moreover the function ψ cannot be harmonic in $\mathcal{F}$ because its boundary conditions on Γ would imply that $\psi = 0$ in $\mathcal{F}$ (according to the unique continuation principle) and then that g_0 is constant. The proof of the lemma is now complete. □

We shall establish now some L^2–mass concentration properties for harmonic functions. We remind that R_i and R_e are fixed (and depend only on $\mathcal{G}$) while R_+ and R_- are meant to tend to 0.

Lemma 3.3. *Under Hypothesis 3.1, the following estimate holds:*

$$\sup_{h\in\mathfrak{H}(\mathcal{G})} \frac{\|h\|_{L^2(\mathcal{O})}}{\|h\|_{L^2(\mathcal{F})}} \leqslant \sqrt{\frac{R_-^2}{R_i^2 - R_-^2}}.$$

Proof. In $\mathcal{D}_i$, the function h can be expanded in polar coordinates as:

$$h(r,\theta) = a_0 + \sum_{n\geqslant 1} r^n\big[a_n\cos(n\theta) + b_n\sin(n\theta)\big],$$

where $(a_n)_{n\geqslant 0}$ and $(b_n)_{n\geqslant 1}$ are two sequences of real numbers. Straightforward computations lead to:

$$\|h\|^2_{L^2(\mathcal{D}_-)} = \left(\frac{R_-}{R_i}\right)^2 \left[\pi a_0^2 R_i^2 + \pi\sum_{n\geqslant 1}\left(\frac{a_n^2+b_n^2}{2n+2}\right) R_i^{2n+2}\left(\frac{R_-}{R_i}\right)^{2n}\right] \leqslant \left(\frac{R_-}{R_i}\right)^2 \|h\|^2_{L^2(\mathcal{D}_i)}.$$

We denote by $\mathcal{C}_i$ the annulus $\mathcal{D}_i\setminus\overline{\mathcal{D}_-}$ and, because of the inclusions $\overline{\mathcal{O}}\subset\mathcal{D}_-$ and $\mathcal{C}_i\subset\mathcal{F}$, we have for every $h\in\mathfrak{H}(\mathcal{G})$:

$$\frac{\|h\|^2_{L^2(\mathcal{O})}}{\|h\|^2_{L^2(\mathcal{F})}} \leqslant \frac{\|h\|^2_{L^2(\mathcal{D}_-)}}{\|h\|^2_{L^2(\mathcal{C}_i)}} = \frac{\|h\|^2_{L^2(\mathcal{D}_-)}}{1-\|h\|^2_{L^2(\mathcal{D}_-)}}.$$

The conclusion follows. □

We introduce the exterior domain $\mathcal{K} = \mathbb{R}^2\setminus\overline{\mathcal{O}}$ and the space:

$$\mathfrak{H}_{\mathcal{K}}(\mathcal{F}) = \Big\{h|_{\mathcal{F}}\,:\, h\in L^2_{\ell oc}(\mathcal{K}),\ h \text{ harmonic in } \mathcal{K} \text{ and } \lim_{|x|\to+\infty} h(x) = 0\Big\}.$$

The asymptotic behavior of the functions in $\mathfrak{H}_{\mathcal{K}}(\mathcal{F})$ entails in particular that the flux condition (2.4) is satisfied for every $h\in\mathfrak{H}_{\mathcal{K}}(\mathcal{F})$ and every smooth Jordan curve $\mathscr{C}$ included in $\mathcal{K}$.

The space $\mathfrak{H}(\mathcal{F})\cap H^1(\mathcal{F})$ is denoted by $\mathfrak{H}^1(\mathcal{F})$. Similarly, we define

$$\mathfrak{H}^1_{\mathcal{G}}(\mathcal{F}) = \big\{h|_{\mathcal{F}}\,:\, h\in\mathfrak{H}^1(\mathcal{G})\big\} = \mathfrak{H}_{\mathcal{G}}(\mathcal{F})\cap H^1(\mathcal{F}),$$
$$\mathfrak{H}^1_{\mathcal{K}}(\mathcal{F}) = \big\{h|_{\mathcal{F}}\,:\, h\in\mathfrak{H}^1(\mathcal{K})\big\} = \mathfrak{H}_{\mathcal{K}}(\mathcal{F})\cap H^1(\mathcal{F}),$$

where $\mathfrak{H}^1(\mathcal{K}) = \mathfrak{H}(\mathcal{K})\cap H^1_{\ell oc}(\mathcal{K})$. Notice in particular that every function in $\mathfrak{H}^1(\mathcal{K})$ has finite Dirichlet energy i.e. $\|\nabla h\|_{\mathbf{L}^2(\mathcal{K})} < +\infty$. It is worth observing that the spaces $\mathfrak{H}^1(\mathcal{G})$ and $\mathfrak{H}^1(\mathcal{K})$ can also be defined by means of single layer potentials. We refer for instance to the book [9] for details on the single layer potential. Basics on this topic are also available in a section of [10]. Thus, we denote by $\mathsf{S}_\Gamma : H^{-1/2}(\Gamma)\longrightarrow\mathfrak{H}^1(\mathcal{G})$ the single layer potential on Γ defined for every $q\in L^2(\Gamma)$ (and extended by density in $H^{-1/2}(\Gamma)$) by:

$$\mathsf{S}_\Gamma q(x) = \frac{1}{2\pi}\int_\Gamma \ln|x-s|q(s)\,\mathrm{d}s \qquad \text{for all } x\in\mathcal{G}.$$

The operator S_Γ is an isomorphism. Define now $\widehat{H}^{-1/2}(\Sigma) = \big\{q \in H^{-1/2}(\Sigma) \,:\, \int_\Sigma q\,\mathrm{d}s = 0\big\}$ and the operator $\mathsf{S}_\Sigma : \widehat{H}^{-1/2}(\Sigma) \longrightarrow \mathfrak{H}^1(\mathcal{K})$, for every $q \in L^2(\Sigma) \cap \widehat{H}^{-1/2}(\Sigma)$, by:

$$\mathsf{S}_\Sigma q(x) = \frac{1}{2\pi}\int_\Sigma \ln|x-s| q(s)\,\mathrm{d}s \qquad \text{for all } x \in \mathcal{K}.$$

Then, the operator S_Σ is also an isomorphism.

We gave earlier a first decomposition (see (3.3)) of the space $\mathfrak{H}(\mathcal{F})$. We shall provide below an other one in terms of the spaces $\mathfrak{H}_\mathcal{G}(\mathcal{F})$ and $\mathfrak{H}_\mathcal{K}(\mathcal{F})$.

Lemma 3.4. *The space $\mathfrak{H}(\mathcal{F})$ admits the following (non-orthogonal) decomposition:*

$$\mathfrak{H}(\mathcal{F}) = \mathfrak{H}_\mathcal{G}(\mathcal{F}) \oplus \mathfrak{H}_\mathcal{K}(\mathcal{F}). \tag{3.5}$$

Moreover, under Hypothesis 3.1, the following equivalent estimates hold for the functions in $\mathfrak{H}_\mathcal{K}(\mathcal{F})$:

$$\inf_{h\in\mathfrak{H}_\mathcal{K}(\mathcal{F})} \frac{\|h\|_{L^2(\mathcal{F}_\Sigma)}}{\|h\|_{L^2(\mathcal{F})}} \geqslant \sqrt{1-\frac{1}{\delta}} \quad \text{and} \quad \sup_{h\in\mathfrak{H}_\mathcal{K}(\mathcal{F})} \frac{\|h\|_{L^2(\mathcal{F}_\Gamma)}}{\|h\|_{L^2(\mathcal{F})}} \leqslant \sqrt{\frac{1}{\delta}}. \tag{3.6a}$$

Concerning the functions in $\mathfrak{H}_\mathcal{G}(\mathcal{F})$, they satisfy the estimates:

$$\sup_{h\in\mathfrak{H}_\mathcal{G}(\mathcal{F})} \frac{\|h\|_{L^2(\mathcal{F}_\Sigma)}}{\|h\|_{L^2(\mathcal{F})}} \leqslant \sqrt{\frac{R_+^2}{R_i^2 - R_-^2}} \quad \text{and} \quad \inf_{h\in\mathfrak{H}_\mathcal{G}(\mathcal{F})} \frac{\|h\|_{L^2(\mathcal{F}_\Gamma)}}{\|h\|_{L^2(\mathcal{F})}} \geqslant \sqrt{\frac{R_i^2 - (R_-^2 + R_+^2)}{R_i^2 - R_-^2}}. \tag{3.6b}$$

According to (3.5), every function $h \in \mathfrak{H}(\mathcal{F})$ can be decomposed into a sum $h_\mathcal{G} + h_\mathcal{K}$ with $h_\mathcal{G} \in \mathfrak{H}_\mathcal{G}(\mathcal{F})$ and $h_\mathcal{K} \in \mathfrak{H}_\mathcal{K}(\mathcal{F})$. Suppose now that $\mathcal{O}$ shrinks into a point (i.e. δ tends to $+\infty$ or equivalently R_+ tends to 0). Then, (3.6b) means that the function $h_\mathcal{G}$ concentrates (as far as the L^2–norm is concerned) far from the boundary Σ, namely in $\mathcal{F}_\Gamma$ while, according to the estimate (3.6a), the function $h_\mathcal{K}$ concentrates in $\mathcal{F}_\Sigma$, that is to say along the boundary Σ. So, although the decomposition (3.5) is not orthogonal, it becomes in some sens "more and more" orthogonal as $\mathcal{O}$ shrinks.

Proof. Every function in $\mathfrak{H}^1(\mathcal{F})$ is the sum of a single layer potential with a density supported by Γ (a function of $\mathfrak{H}^1_\mathcal{G}(\mathcal{F})$) and a single layer potential with a density supported by Σ (a function of $\mathfrak{H}^1_\mathcal{K}(\mathcal{F})$). The identity (3.5) follows by density.

Let now h be in $\mathfrak{H}_\mathcal{K}(\mathcal{F})$, $h \neq 0$. We begin with the obvious estimate:

$$\frac{\|h\|_{L^2(\mathcal{F}_\Sigma)}}{\|h\|_{L^2(\mathcal{F}_\Gamma)}} \geqslant \frac{\|h\|_{L^2(\mathcal{C}_+)}}{\|h\|_{L^2(\mathcal{C}_e)}}. \tag{3.7}$$

In the exterior domain $\mathbb{R}^2 \setminus \overline{\mathcal{D}_-}$, the harmonic function h can be expanded in polar coordinates as:

$$h(r,\theta) = \sum_{k\geqslant 1} r^{-k}\big(a_k\cos(k\theta) + b_k\sin(k\theta)\big),$$

where $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ are two sequences of real numbers. It follows that, in the annulus $\mathcal{C}_+$:

$$\|h\|^2_{L^2(\mathcal{C}_+)} = \pi(a_1^2+b_1^2)\ln(R_+/R_-) + \sum_{k\geqslant 2}\pi\left(\frac{a_k^2+b_k^2}{2k-2}\right)\left(\frac{1}{R_+}\right)^{2k-2}\left[\left(\frac{R_+}{R_-}\right)^{2k-2} - 1\right], \tag{3.8a}$$

while in the annulus $\mathcal{C}_e$:

$$\|h\|^2_{L^2(\mathcal{C}_e)} = \pi(a_1^2+b_1^2)\ln(R_e/R_+) + \sum_{k\geqslant 2}\pi\left(\frac{a_k^2+b_k^2}{2k-2}\right)\left(\frac{1}{R_+}\right)^{2k-2}\left[1-\left(\frac{R_+}{R_e}\right)^{2k-2}\right]. \tag{3.8b}$$

We deduce from both identities (3.8) that:

$$\|h\|^2_{L^2(\mathcal{C}_e)} \leqslant \max\left\{\frac{\ln(R_+/R_e)}{\ln(R_-/R_+)}, (R_+/R_e)^2\right\}\|h\|^2_{L^2(\mathcal{C}_+)},$$

and from (3.1) that:

$$(R_+/R_e)^2 \leqslant \frac{\ln(R_+/R_e)}{\ln(R_-/R_+)} \leqslant \frac{1}{\delta - 1}.$$

All together, with (3.7) and since $\|h\|^2_{L^2(\mathcal{F})} = \|h\|^2_{L^2(\mathcal{F}_\Sigma)} + \|h\|^2_{L^2(\mathcal{F}_\Gamma)}$, we prove (3.6a).

Finally, for every h in $\mathfrak{H}(\mathcal{G})$, proceeding as in the proof of Lemma 3.3, we easily show that:

$$\frac{\|h\|_{L^2(\mathcal{F}_\Sigma)}}{\|h\|_{L^2(\mathcal{F})}} \leqslant \frac{\|h\|_{L^2(\mathcal{D}_+)}}{\|h\|_{L^2(\mathcal{C}_i)}} \leqslant \sqrt{\frac{R_+^2}{R_i^2 - R_-^2}},$$

and we conclude the proof. □

Slight modifications in the proof of the lemma 3.4 lead to the statement of its counterpart in terms of the H^1–norm.

Lemma 3.5. *The following estimate holds true:*

$$\sup_{h\in\mathfrak{H}^1(\mathcal{K})}\frac{\|\nabla h\|_{\mathbf{L}^2(\mathbb{R}^2\setminus\overline{\mathcal{G}})}}{\|\nabla h\|_{\mathbf{L}^2(\mathcal{F})}}\leqslant\sqrt{\frac{R_-^2}{R_i^2-R_-^2}}.\tag{3.9}$$

The space $\mathfrak{H}^1(\mathcal{F})$ admits the following non-orthogonal decomposition:

$$\mathfrak{H}^1(\mathcal{F})=\mathfrak{H}^1_{\mathcal{G}}(\mathcal{F})\oplus\mathfrak{H}^1_{\mathcal{K}}(\mathcal{F}).\tag{3.10}$$

Moreover, under Hypothesis 3.1, the following (equivalent) estimates hold for the functions in $\mathfrak{H}^1_{\mathcal{K}}(\mathcal{F})$:

$$\inf_{h\in\mathfrak{H}^1_{\mathcal{K}}(\mathcal{F})}\frac{\|\nabla h\|_{\mathbf{L}^2(\mathcal{F}_\Sigma)}}{\|\nabla h\|_{\mathbf{L}^2(\mathcal{F})}}\geqslant\sqrt{1-\left(\frac{R_-}{R_+}\right)^2}\qquad\sup_{h\in\mathfrak{H}^1_{\mathcal{K}}(\mathcal{F})}\frac{\|\nabla h\|_{\mathbf{L}^2(\mathcal{F}_\Gamma)}}{\|\nabla h\|_{\mathbf{L}^2(\mathcal{F})}}\leqslant\left(\frac{R_-}{R_+}\right).\tag{3.11a}$$

Regarding the functions in $\mathfrak{H}^1_{\mathcal{G}}(\mathcal{F})$, we have:

$$\sup_{h\in\mathfrak{H}^1_{\mathcal{G}}(\mathcal{F})}\frac{\|h\|_{L^2(\mathcal{F}_\Sigma)}}{\|h\|_{L^2(\mathcal{F})}}\leqslant\sqrt{\frac{R_+^2}{R_i^2-R_+^2}}.\tag{3.11b}$$

According to the space decompositions (3.3) and (3.5), every function h_0 of $\mathfrak{H}_0(\mathcal{F})$ can be decomposed as:

$$h_0=h_{\mathcal{K}}-\Pi_{\mathcal{G}}h_{\mathcal{K}},\tag{3.12}$$

where $\Pi_{\mathcal{G}}$ is the orthogonal projection on $\mathfrak{H}_{\mathcal{G}}(\mathcal{F})$ in $\mathfrak{H}(\mathcal{F})$. Notice that in (3.12):

$$\|h_{\mathcal{K}}\|^2_{L^2(\mathcal{F})}=\|h_0\|^2_{L^2(\mathcal{F})}+\|\Pi_{\mathcal{G}}h_{\mathcal{K}}\|^2_{L^2(\mathcal{F})}.$$

As $\mathcal{O}$ shrinks, h_0 tends to $h_{\mathcal{K}}$ in (3.12), or equivalently $\Pi_{\mathcal{G}}h_{\mathcal{K}}$ tends to 0. More precisely, we claim:

Lemma 3.6. *For every $h_{\mathcal{K}}\in\mathfrak{H}_{\mathcal{K}}(\mathcal{F})$, under Hypothesis 3.1, the following estimate holds:*

$$\|\Pi_{\mathcal{G}}h_{\mathcal{K}}\|_{L^2(\mathcal{F})}\leqslant 2\sqrt{\frac{\delta_0}{\delta}}\|h_{\mathcal{K}}\|_{L^2(\mathcal{F})}.\tag{3.13}$$

Proof. The L^2–norm of $\Pi_{\mathcal{G}}h_{\mathcal{K}}$ can be expressed as:

$$\|\Pi_{\mathcal{G}}h_{\mathcal{K}}\|_{L^2(\mathcal{F})}=\sup_{\substack{\theta\in\mathfrak{H}_{\mathcal{G}}(\mathcal{F})\\ \theta\neq 0}}\frac{1}{\|\theta\|_{L^2(\mathcal{F})}}(h_{\mathcal{K}},\theta)_{L^2(\mathcal{F})},$$

and we have:

$$\frac{1}{\|\theta\|_{L^2(\mathcal{F})}}|(h_{\mathcal{K}},\theta)_{L^2(\mathcal{F})}|\leqslant\|h_{\mathcal{K}}\|_{L^2(\mathcal{F}_\Sigma)}\frac{\|\theta\|_{L^2(\mathcal{F}_\Sigma)}}{\|\theta\|_{L^2(\mathcal{F})}}+\frac{\|h_{\mathcal{K}}\|_{L^2(\mathcal{F}_\Gamma)}}{\|h_{\mathcal{K}}\|_{L^2(\mathcal{F})}}\frac{\|\theta\|_{L^2(\mathcal{F}_\Gamma)}}{\|\theta\|_{L^2(\mathcal{F})}}\|h_{\mathcal{K}}\|_{L^2(\mathcal{F})}.$$

Using the estimate (3.6b) for the first term in the right hand side and the estimate (3.6a) for the second term, we obtain that:

$$\|\Pi_{\mathcal{G}}h_{\mathcal{K}}\|_{L^2(\mathcal{F})}\leqslant\left(\sqrt{\frac{R_+^2}{R_i^2-R_-^2}}+\sqrt{\frac{1}{\delta}}\right)\|h_{\mathcal{K}}\|_{L^2(\mathcal{F})},$$

and (3.13) follows, taking into account (3.1). □

We can now address the structure of the space $V_0(\mathcal{G})$.

Lemma 3.7. *For every $h_{\mathcal{O}}$ in $\mathfrak{H}(\mathcal{O})$, there exists a unique function $\mathsf{T}h_{\mathcal{O}}$ in $\mathfrak{H}_{\mathcal{G}}(\mathcal{F})$ such that the function ω defined in $\mathcal{G}$ by:*

$$\omega|_{\mathcal{O}}=h_{\mathcal{O}}\qquad\text{and}\qquad\omega|_{\mathcal{F}}=\mathsf{T}h_{\mathcal{O}},$$

belongs to $V_0(\mathcal{G})$. Under Hypothesis 3.1, the mapping $\mathsf{T}:h_{\mathcal{O}}\in\mathfrak{H}(\mathcal{O})\longmapsto\mathsf{T}h_{\mathcal{O}}\in\mathfrak{H}_{\mathcal{G}}(\mathcal{F})$ is bounded and:

$$\|\mathsf{T}h_{\mathcal{O}}\|_{L^2(\mathcal{F})}\leqslant\sqrt{\frac{R_-^2}{R_i^2-R_-^2}}\|h_{\mathcal{O}}\|_{L^2(\mathcal{O})}\qquad\text{for all } h_{\mathcal{O}}\in\mathfrak{H}(\mathcal{O}).\tag{3.14}$$

It follows in particular that

$$W_0(\mathcal{G})=\Big\{\omega\in L^2(\mathcal{G})\,:\,\omega|_{\mathcal{O}}=h_{\mathcal{O}},\quad\omega|_{\mathcal{F}}=\mathsf{T}h_{\mathcal{O}},\quad h_{\mathcal{O}}\in\mathfrak{H}(\mathcal{O})\Big\},\tag{3.15}$$

is a closed subspace of $V_0(\mathcal{G})$.

Thus, the space $W_0(\mathcal{G})$ contains piecewise harmonic functions in $\mathcal{G}$ that are orthogonal in $L^2(\mathcal{G})$ to the harmonic functions in $\mathcal{G}$.

Proof. Let $h_{\mathcal{O}}$ be given in $\mathfrak{H}(\mathcal{O})$ and define ψ in $H^2_0(\mathcal{G})$ by setting $\Delta\psi = h_{\mathcal{O}}$ in $\mathcal{O}$ with $\psi = c$ on Σ (a constant that will be fixed later) and $\Delta^2\psi = 0$ in $\mathcal{F}$ with $\psi|_\Gamma = 0$, $\partial\psi/\partial n|_\Gamma = 0$. Choose the constant c such that $\int_\Sigma \partial(\Delta\psi|_{\mathcal{F}})/\partial n\,\mathrm{d}s = 0$ (the normal derivative of $\Delta\psi|_{\mathcal{F}}$ on Σ belongs to $H^{-1/2}(\Sigma)$). Then define $\mathsf{T}h_{\mathcal{O}}$ as the orthogonal projection in $\mathfrak{H}(\mathcal{F})$ of $\Delta\psi|_{\mathcal{F}}$ on the subspace $\mathfrak{H}_{\mathcal{G}}(\mathcal{F})$. This proves existence. Uniqueness is deduced from the first point of Lemma 3.2.

Notice that the function $\mathsf{T}h_{\mathcal{O}}$ can equivalently be defined either by:

$$\mathsf{T}h_{\mathcal{O}} = \operatorname{argmin}\Big\{\|\theta\|_{L^2(\mathcal{F})}\ :\ \theta\in\mathfrak{H}(\mathcal{F}),\ \int_{\mathcal{F}}\theta h\,\mathrm{d}x + \int_{\mathcal{O}} h_{\mathcal{O}} h\,\mathrm{d}x = 0\quad\forall\, h\in\mathfrak{H}(\mathcal{G})\Big\},$$

or by $\mathsf{T}h_{\mathcal{O}} = \Delta\psi_{\mathcal{G}}|_{\mathcal{F}}$ where:

$$\psi_{\mathcal{G}} = \operatorname{argmin}\big\{\|\Delta\psi\|_{L^2(\mathcal{G})}\ :\ \psi\in H^2_0(\mathcal{G}),\ \Delta\psi|_{\mathcal{O}} = h_{\mathcal{O}}\big\}.$$

To prove the estimate (3.14), let $h_{\mathcal{O}}$ be given in $\mathfrak{H}(\mathcal{O})$. By definition of the space $\mathfrak{H}_{\mathcal{G}}(\mathcal{F})$, the function $\mathsf{T}h_{\mathcal{O}}$ (defined in $\mathcal{F}$) can be extended in the whole domain $\mathcal{G}$ in such a way that it belongs to $\mathfrak{H}(\mathcal{G})$. Keeping the same notation for this extended function, it follows that:

$$\|\mathsf{T}h_{\mathcal{O}}\|^2_{L^2(\mathcal{F})} + \int_{\mathcal{O}} h_{\mathcal{O}}\mathsf{T}h_{\mathcal{O}}\,\mathrm{d}x = 0.$$

Invoking now Lemma 3.3, we obtain that:

$$\|\mathsf{T}h_{\mathcal{O}}\|^2_{L^2(\mathcal{F})} \leqslant \|h_{\mathcal{O}}\|_{L^2(\mathcal{O})}\|\mathsf{T}h_{\mathcal{O}}\|_{L^2(\mathcal{O})} \leqslant \|h_{\mathcal{O}}\|_{L^2(\mathcal{O})}\sqrt{\frac{R_-^2}{R_i^2-R_-^2}}\|\mathsf{T}h_{\mathcal{O}}\|_{L^2(\mathcal{F})},$$

and the proof is completed. □

By extending the functions by 0 in $\mathcal{O}$ or $\mathcal{F}$, we define $V_0^\star(\mathcal{F})$ and $V_0^\star(\mathcal{O})$, two closed subspaces of $V_0(\mathcal{G})$. These spaces, together with $\mathfrak{H}_0^\star(\mathcal{F})$ (defined in (3.4)) and $W_0(\mathcal{F})$ (defined in (3.15)) enter the decomposition of $V_0(\mathcal{G})$.

Theorem 3.8. *The vorticity space $V_0(\mathcal{G})$ admits the following orthogonal decomposition:*

$$V_0(\mathcal{G}) = V_0^\star(\mathcal{F})\overset{\perp}{\oplus} V_0^\star(\mathcal{O})\overset{\perp}{\oplus}\mathfrak{H}_0^\star(\mathcal{F})\overset{\perp}{\oplus} W_0(\mathcal{G}). \tag{3.16}$$

Under Hypothesis 3.1 and for every $\omega\in V_0(\mathcal{G})$:

$$\|\omega-\omega_{\mathcal{F}}\|_{L^2(\mathcal{G})}\longrightarrow 0\qquad \text{as } \delta \text{ tends to } +\infty, \tag{3.17}$$

where $\omega_{\mathcal{F}}$ is the orthogonal projection of ω on $V_0^\star(\mathcal{F})$. Moreover, for every $\delta > 4\delta_0$ (what means that $\mathcal{O}$ is small enough), we have the following L^2 estimate outside the boundary layer $\mathcal{F}_\Sigma$:

$$\|\omega-\omega_{\mathcal{F}}\|_{L^2(\mathcal{F}_\Gamma)} \leqslant \sqrt{\frac{2\delta_0}{\delta-4\delta_0}}\|\omega\|_{L^2(\mathcal{F})}. \tag{3.18}$$

In (3.18), the domain $\mathcal{F}_\Gamma$ depends also on δ since $\mathcal{F}_\Gamma = \mathcal{G}\setminus\overline{\mathcal{D}_+}$ and the radius $R_+ = R_e e^{-\delta}$ of the disk $\mathcal{D}_+$ tends to 0 when δ goes to $+\infty$. Recall that when the obstacle shrinks (i.e. when $\mathcal{O}=\mathcal{O}_\varepsilon$), we can choose $\delta = \sqrt{\ln(R_e/\varepsilon)}$ for Hypothesis 3.1 to be satisfied.

Proof. Let ω be in $V_0(\mathcal{G})$ and define $\omega_{\mathcal{O}} = \omega|_{\mathcal{O}}$. The function $\omega_{\mathcal{O}}$ can be decomposed as $\omega^0_{\mathcal{O}} + \omega^{\mathfrak{H}}_{\mathcal{O}}$ with $\omega^0_{\mathcal{O}}\in V_0(\mathcal{O})$ and $\omega^{\mathfrak{H}}_{\mathcal{O}}\in\mathfrak{H}(\mathcal{O})$ (because by definition $V_0(\mathcal{O})$ is the orthogonal complement of $\mathfrak{H}(\mathcal{O})$ in $L^2(\mathcal{O})$). Then extend $\omega_{\mathcal{O}}$ (keeping the same notation) by setting $\omega_{\mathcal{O}} = \mathsf{T}\omega^{\mathfrak{H}}_{\mathcal{O}}$ in $\mathcal{F}$ and notice that $\omega_{\mathcal{O}}$, this extended function, belongs to $V_0(\mathcal{G})$. Introduce now $\omega_{\mathcal{F}} = \omega-\omega_{\mathcal{O}}$ in $\mathcal{G}$. This function is in $V_0(\mathcal{G})$ (like ω and $\omega_{\mathcal{O}}$) and equal to zero in $\mathcal{O}$. Therefore, its restriction to $\mathcal{F}$ belongs to $(\mathfrak{H}_{\mathcal{G}}(\mathcal{F}))^\perp = V_0(\mathcal{F})\oplus\mathfrak{H}_0(\mathcal{F})$ and the proof of the identity (3.16) is completed.

Consider again a function ω in $V_0(\mathcal{G})$. According to (3.16), ω can be decomposed into the orthogonal sum:

$$\omega = \omega_{\mathcal{F}} + \omega_{\mathcal{O}} + \omega_{\mathfrak{H}} + \omega_W, \tag{3.19}$$

with $\omega_{\mathcal{F}}\in V_0^\star(\mathcal{F})$, $\omega_{\mathcal{O}}\in V_0^\star(\mathcal{O})$, $\omega_{\mathfrak{H}}\in\mathfrak{H}_0^\star(\mathcal{F})$ and $\omega_W\in W_0(\mathcal{G})$. In this sum, every term depends on δ. Notice first that $\omega|_{\mathcal{O}} = \omega_{\mathcal{O}}|_{\mathcal{O}} + \omega_W|_{\mathcal{O}}$ and this sum is orthogonal in $L^2(\mathcal{O})$. The dominated convergence theorem yields the convergence toward 0 of $\omega_{\mathcal{O}}|_{\mathcal{O}}$ and $\omega_W|_{\mathcal{O}}$ in $L^2(\mathcal{O})$. The latter convergence combined with Lemma 3.7 yields the convergence of ω_W toward 0 in $L^2(\mathcal{G})$. Let us turn our attention now to the term $\omega_{\mathfrak{H}}$ in (3.19). By definition of the orthogonal projection, we have:

$$\|\omega_{\mathfrak{H}}|_{\mathcal{F}}\|_{L^2(\mathcal{F})} = \sup_{\substack{h_0\in\mathfrak{H}_0(\mathcal{F})\\ h_0\neq 0}} \frac{1}{\|h_0\|_{L^2(\mathcal{F})}}(\omega, h_0)_{L^2(\mathcal{F})}. \tag{3.20}$$

For every $h_0 \in \mathfrak{H}_0(\mathcal{F})$ decomposed as in (3.12):

$$\frac{1}{\|h_0\|_{L^2(\mathcal{F})}}\int_{\mathcal{F}} \omega h_0 \,\mathrm{d}x = \frac{\|h_{\mathcal{K}}\|_{L^2(\mathcal{F})}}{\|h_0\|_{L^2(\mathcal{F})}}\left[\int_{\mathcal{F}} \omega \frac{h_{\mathcal{K}}}{\|h_{\mathcal{K}}\|_{L^2(\mathcal{F})}}\,\mathrm{d}x - \int_{\mathcal{F}} \omega \frac{\Pi_{\mathcal{G}} h_{\mathcal{K}}}{\|h_{\mathcal{K}}\|_{L^2(\mathcal{F})}}\,\mathrm{d}x\right]. \tag{3.21}$$

However, in the decomposition (3.12), by definition of the space $\mathfrak{H}_{\mathcal{G}}(\mathcal{F})$, the function $\Pi_{\mathcal{G}} h_{\mathcal{K}}$ can be supposed to be in $\mathfrak{H}(\mathcal{G})$ (we keep the same notation). From this observation, we deduce that in (3.21):

$$-\int_{\mathcal{F}} \omega \frac{\Pi_{\mathcal{G}} h_{\mathcal{K}}}{\|h_{\mathcal{K}}\|_{L^2(\mathcal{F})}}\,\mathrm{d}x = \int_{\mathcal{O}} \omega \frac{\Pi_{\mathcal{G}} h_{\mathcal{K}}}{\|h_{\mathcal{K}}\|_{L^2(\mathcal{F})}}\,\mathrm{d}x. \tag{3.22}$$

According to (3.13), we have when $\delta > 4\delta_0$:

$$\frac{\|h_{\mathcal{K}}\|_{L^2(\mathcal{F})}}{\|h_0\|_{L^2(\mathcal{F})}} \leqslant \sqrt{\frac{\delta}{\delta - 4\delta_0}}. \tag{3.23a}$$

On the other hand, according to (3.6a):

$$\left|\int_{\mathcal{F}} \omega \frac{h_{\mathcal{K}}}{\|h_{\mathcal{K}}\|_{L^2(\mathcal{F})}}\,\mathrm{d}x\right| \leqslant \|\omega\|_{L^2(\mathcal{F}_\Gamma)}\sqrt{\frac{1}{\delta}} + \|\omega\|_{L^2(\mathcal{F}_\Sigma)}, \tag{3.23b}$$

and the second term tends to 0 according to the dominated convergence theorem. Finally, considering the last term in (3.21):

$$\left|\int_{\mathcal{O}} \omega \frac{\Pi_{\mathcal{G}} h_{\mathcal{K}}}{\|h_{\mathcal{K}}\|_{L^2(\mathcal{F})}}\,\mathrm{d}x\right| \leqslant \|\omega\|_{L^2(\mathcal{O})}\frac{\|\Pi_{\mathcal{G}} h_{\mathcal{K}}\|_{L^2(\mathcal{O})}}{\|\Pi_{\mathcal{G}} h_{\mathcal{K}}\|_{L^2(\mathcal{F})}}\frac{\|\Pi_{\mathcal{G}} h_{\mathcal{K}}\|_{L^2(\mathcal{F})}}{\|h_{\mathcal{K}}\|_{L^2(\mathcal{F})}}, \tag{3.23c}$$

and according to Lemma 3.3 and Lemma 3.6:

$$\frac{\|\Pi_{\mathcal{G}} h_{\mathcal{K}}\|_{L^2(\mathcal{O})}}{\|\Pi_{\mathcal{G}} h_{\mathcal{K}}\|_{L^2(\mathcal{F})}} \leqslant \sqrt{\frac{R_-^2}{R_i^2 - R_-^2}} \qquad \text{and} \qquad \frac{\|\Pi_{\mathcal{G}} h_{\mathcal{K}}\|_{L^2(\mathcal{F})}}{\|h_{\mathcal{K}}\|_{L^2(\mathcal{F})}} \leqslant 2\sqrt{\frac{\delta_0}{\delta}}. \tag{3.23d}$$

Using the estimates (3.23) in the equality (3.21), we conclude the proof of the first convergence result (3.17). Let us address now the estimate (3.18). Considering back the decomposition (3.19) , we have:

$$(\omega - \omega_{\mathcal{F}})|_{\mathcal{F}_\Gamma} = \omega_{\mathfrak{H}}|_{\mathcal{F}_\Gamma} + \omega_W|_{\mathcal{F}_\Gamma}. \tag{3.24}$$

According to the decomposition (3.12), there exists $\omega_{\mathcal{K}} \in h_{\mathcal{K}}$ such that:

$$\omega_{\mathfrak{H}} = \omega_{\mathcal{K}} - \Pi_{\mathcal{G}}\omega_{\mathcal{K}}, \tag{3.25}$$

whence we deduce that:

$$\|\omega_{\mathfrak{H}}\|_{L^2(\mathcal{F}_\Gamma)} \leqslant \|\omega_{\mathcal{K}}\|_{L^2(\mathcal{F}_\Gamma)} + \|\Pi_{\mathcal{G}}\omega_{\mathcal{K}}\|_{L^2(\mathcal{F})}.$$

Using (3.6a) for the first term in the right hand side and (3.13) for the second, we get :

$$\|\omega_{\mathfrak{H}}\|_{L^2(\mathcal{F}_\Gamma)} \leqslant 3\sqrt{\frac{\delta_0}{\delta}}\|\omega_{\mathcal{K}}\|_{L^2(\mathcal{F})}. \tag{3.26a}$$

On the other hand, proceeding as for (3.23a), we obtain that:

$$\|\omega_{\mathcal{K}}\|_{L^2(\mathcal{F})} \leqslant \sqrt{\frac{\delta}{\delta - 4\delta_0}}\|\omega_{\mathfrak{H}}\|_{L^2(\mathcal{F})} \leqslant \sqrt{\frac{\delta}{\delta - 4\delta_0}}\|\omega\|_{L^2(\mathcal{F})}. \tag{3.26b}$$

Combining both estimates (3.26) yields:

$$\|\omega_{\mathfrak{H}}\|_{L^2(\mathcal{F}_\Gamma)} \leqslant 3\sqrt{\frac{\delta_0}{\delta - 4\delta_0}}\|\omega\|_{L^2(\mathcal{F})}. \tag{3.27a}$$

Going back to (3.24) and recalling the definition (3.15) of $W_0(\mathcal{G})$, it comes:

$$\|\omega_W\|_{L^2(\mathcal{F})} = \|\mathsf{T}(\omega_W|_{\mathcal{O}})\|_{L^2(\mathcal{F})}.$$

We can then apply Lemma 3.7 to obtain:

$$\|\omega_W\|_{L^2(\mathcal{F})} \leqslant \sqrt{\frac{R_-^2}{R_i^2 - R_-^2}}\|\omega_W|_{\mathcal{O}}\|_{L^2(\mathcal{O})} \leqslant \sqrt{\frac{R_-^2}{R_i^2 - R_-^2}}\|\omega\|_{L^2(\mathcal{F})}. \tag{3.27b}$$

Using both estimates (3.27) in (3.24), we obtain (3.18) and complete the proof. ☐

The last lemma of this section is the cornerstone of the proof of Theorem 1.3. It concerns also the behavior of harmonic functions. Thus, let ψ be in $S_0(\mathcal{G})$ (this space is defined in (2.2)). The function $\psi|_{\mathcal{F}}$ can be decomposed as

$$\psi|_{\mathcal{F}} = \mathsf{Q}_{\mathcal{F}}\psi + h_{\mathcal{F}}, \tag{3.28}$$

where $h_{\mathcal{F}}$ belongs to $\mathfrak{H}^1(\mathcal{F})$ and the projector $\mathsf{Q}_{\mathcal{F}}$ is introduced in Definition 2.3. We shall now prove that when the domain $\mathcal{O}$ shrinks (or more precisely when R_- tends to 0), the H^1–norm in $\mathcal{F}_\Gamma$ of the harmonic function $h_{\mathcal{F}}$ tends to 0.

Lemma 3.9. *Under Hypothesis 3.1, there exists a constant* $\mathbf{c}_{[\mathcal{G}]}$ *such that, for every* $\psi \in S_0(\mathcal{G})$*:*

$$\|\nabla(\psi - \mathsf{Q}_{\mathcal{F}}\psi)\|_{\mathbf{L}^2(\mathcal{F}_\Gamma)} \leqslant \mathbf{c}_{[\mathcal{G}]}\sqrt{\frac{R_-^2}{R_i^2 - R_-^2}}\|\psi\|_{S_0(\mathcal{G})}.$$

Proof. According to the identity (3.10) in Lemma 3.5, the harmonic function $h_{\mathcal{F}}$ in (3.28) can be decomposed as:

$$h_{\mathcal{F}} = h_{\mathcal{G}} + h_{\mathcal{K}}. \tag{3.29}$$

Forming the scalar product of (3.29) with $h_{\mathcal{K}}$ in $H^1(\mathcal{F})$, we deduce that:

$$\|\nabla h_{\mathcal{K}}\|^2_{\mathbf{L}^2(\mathcal{F})} \leqslant \|\nabla h_{\mathcal{F}}\|_{\mathbf{L}^2(\mathcal{F})}\|\nabla h_{\mathcal{K}}\|_{\mathbf{L}^2(\mathcal{F})} + \|\nabla h_{\mathcal{G}}\|_{\mathbf{L}^2(\mathcal{F}_\Gamma)}\|\nabla h_{\mathcal{K}}\|_{\mathbf{L}^2(\mathcal{F}_\Gamma)} + \|\nabla h_{\mathcal{G}}\|_{\mathbf{L}^2(\mathcal{F}_\Sigma)}\|\nabla h_{\mathcal{K}}\|_{\mathbf{L}^2(\mathcal{F}_\Sigma)}.$$

We use now the estimates (3.11) to obtain:

$$\|\nabla h_{\mathcal{K}}\|_{\mathbf{L}^2(\mathcal{F})} \leqslant \left(\left(\frac{R_-}{R_+}\right) + \sqrt{\frac{R_+^2}{R_i^2 - R_+^2}}\right)\|\nabla h_{\mathcal{G}}\|_{\mathbf{L}^2(\mathcal{F})} + \|\nabla h_{\mathcal{F}}\|_{\mathbf{L}^2(\mathcal{F})}.$$

The very same estimate holds true inverting the roles played by $h_{\mathcal{G}}$ and $h_{\mathcal{K}}$, whence:

$$\|\nabla h_{\mathcal{K}}\|_{\mathbf{L}^2(\mathcal{F})} \leqslant \left(1 - \left(\frac{R_-}{R_+}\right) - \sqrt{\frac{R_+^2}{R_i^2 - R_+^2}}\right)^{-1}\|\nabla h_{\mathcal{F}}\|_{\mathbf{L}^2(\mathcal{F})} \leqslant \mathbf{c}\|\nabla h_{\mathcal{F}}\|_{\mathbf{L}^2(\mathcal{F})}, \tag{3.30}$$

the second inequality resulting from (3.1). The decomposition (3.28) being orthogonal, $\|\nabla h_{\mathcal{F}}\|_{\mathbf{L}^2(\mathcal{F})} \leqslant \|\psi\|_{S_0(\mathcal{G})}$ and therefore there exists a constant $\mathbf{c}$ such that:

$$\|\nabla h_{\mathcal{G}}\|_{\mathbf{L}^2(\mathcal{F})} \leqslant \mathbf{c}\|\psi\|_{S_0(\mathcal{G})} \quad \text{and} \quad \|\nabla h_{\mathcal{K}}\|_{\mathbf{L}^2(\mathcal{F})} \leqslant \mathbf{c}\|\psi\|_{S_0(\mathcal{G})}. \tag{3.31}$$

The combination of the estimate (3.9) with (3.31) yields:

$$\|\nabla h_{\mathcal{K}}\|_{\mathbf{L}^2(\mathbb{R}^2\setminus\overline{\mathcal{G}})} \leqslant \sqrt{\frac{R_-^2}{R_i^2 - R_-^2}}\|\nabla h_{\mathcal{K}}\|_{\mathbf{L}^2(\mathcal{F})} \leqslant \mathbf{c}\sqrt{\frac{R_-^2}{R_i^2 - R_-^2}}\|\psi\|_{S_0(\mathcal{G})}. \tag{3.32}$$

Applying T_Γ (the trace operator on Γ valued in $H^{1/2}(\Gamma)$) to the identity (3.28), taking into account (3.29), we obtain:

$$0 = \mathrm{T}_\Gamma\, h_{\mathcal{F}} = \mathrm{T}_\Gamma\, h_{\mathcal{G}} + \mathrm{T}_\Gamma\, h_{\mathcal{K}}. \tag{3.33}$$

Let us recall now some elementary results of potential theory (we refer again to the book [9] or to the dedicated section in [10]). The flux condition $\int_\Gamma \partial h_{\mathcal{K}}/\partial n\, \mathrm{d}s = 0$ entails that the trace of $h_{\mathcal{K}}$ on Γ belongs to the following subspace of $H^{1/2}(\Gamma)$:

$$\widehat{H}^{1/2}(\Gamma) = \left\{\gamma \in H^{1/2}(\Gamma)\,:\, \int_\Gamma \gamma\, \mathfrak{e}_\Gamma\, \mathrm{d}s = 0\right\},$$

where $\mathfrak{e}_\Gamma$ stands for the equilibrium density of Γ. For any $\gamma \in \widehat{H}^{1/2}(\Gamma)$, there exists a unique function h_γ harmonic in $\mathbb{R}^2 \setminus \mathcal{G}$ such that $\mathrm{T}_\Gamma\, h_\gamma = \gamma$ and $\|\nabla h_\gamma\|_{\mathbf{L}^2(\mathbb{R}^2\setminus\mathcal{G})} < +\infty$. In $\widehat{H}^{1/2}(\Gamma)$, the norm:

$$\|\gamma\|_{\widehat{H}^{1/2}(\Gamma)} = \|\nabla h_\gamma\|_{\mathbf{L}^2(\mathbb{R}^2\setminus\mathcal{G})},$$

is equivalent to the usual norm of $H^{1/2}(\Gamma)$.

Combining (3.32) and (3.33), we deduce first that:

$$\|\,\mathrm{T}_\Gamma\, h_{\mathcal{G}}\|_{H^{1/2}(\Gamma)} \leqslant \mathbf{c}\sqrt{\frac{R_-^2}{R_i^2 - R_-^2}}\|\psi\|_{S_0(\mathcal{G})},$$

and next (considering the function $h_{\mathcal{G}}$ as defined in the whole domain $\mathcal{G}$) that:

$$\|\nabla h_{\mathcal{G}}\|_{\mathbf{L}^2(\mathcal{G})} \leqslant \mathbf{c}_{[\mathcal{G}]}\| \operatorname{T}_\Gamma h_{\mathcal{G}}\|_{H^{1/2}(\Gamma)} \leqslant \mathbf{c}_{[\mathcal{G}]}\sqrt{\frac{R_-^2}{R_i^2-R_-^2}}\|\psi\|_{S_0(\mathcal{G})}. \tag{3.34a}$$

The second estimate in (3.11a) together with (3.31) leads to:

$$\|\nabla h_{\mathcal{K}}\|_{\mathbf{L}^2(\mathcal{F}_\Gamma)} \leqslant \mathbf{c}\left(\frac{R_-}{R_+}\right)\|\psi\|_{S_0(\mathcal{G})}. \tag{3.34b}$$

Using both estimates (3.34) in the identity (3.29), we conclude the proof of the lemma. □

4. Proof of Theorem 1.1 and Corollary 1.2

Proof of of Theorem 1.1. For every positive integer k, define $W_k^{\mathcal{G}}$ the set of the subspaces of $S_1(\mathcal{G})$ of dimension k. Similarly $W_k^{\mathcal{F}_\varepsilon}$ stands for the set of the subspaces of $S_1(\mathcal{F}_\varepsilon)$ of dimension k (the spaces S_0 and S_1 are defined in (2.2)). Then, the Courant–Fischer min-max principle (2.8) for the eigenvalues of the Stokes operator read as follows for the domains $\mathcal{G}$ and $\mathcal{F}_\varepsilon$ respectively:

$$\lambda_k^{\mathcal{G}} = \min_{V\in W_k^{\mathcal{G}}}\max_{\substack{\theta\in V\\ \theta\neq 0}}\frac{\|\theta\|^2_{S_1(\mathcal{G})}}{\|\theta\|^2_{S_0(\mathcal{G})}} \qquad \text{and} \qquad \lambda_k^{\mathcal{F}_\varepsilon} = \min_{V\in W_k^{\mathcal{F}_\varepsilon}}\max_{\substack{\theta\in V\\ \theta\neq 0}}\frac{\|\theta\|^2_{S_1(\mathcal{F}_\varepsilon)}}{\|\theta\|^2_{S_0(\mathcal{F}_\varepsilon)}}.$$

Notice that every function of $S_0(\mathcal{F}_\varepsilon)$ (or $S_1(\mathcal{F}_\varepsilon)$) can be seen as a function in $S_0(\mathcal{G})$ (or $S_1(\mathcal{G})$) with the same norm once extended by the suitable constant inside $\mathcal{O}_\varepsilon$. We can then consider that $S_0(\mathcal{F}_\varepsilon)\subset S_0(\mathcal{G})$ and $S_1(\mathcal{F}_\varepsilon)\subset S_1(\mathcal{G})$. From the inclusion $W_k^{\mathcal{F}_\varepsilon}\subset W_k^{\mathcal{G}}$ we deduce straightforwardly that $\lambda_k^{\mathcal{G}}\leqslant\lambda_k^{\mathcal{F}_\varepsilon}$.

Denote by $\{\psi_1^{\mathcal{G}},\dots,\psi_k^{\mathcal{G}}\}$ an orthonormal family in $S_0(\mathcal{G})$ (and orthogonal in $S_1(\mathcal{G})$) made of the k first eigenfunctions of the Stokes operator and let $\mathcal{W}_k^{\mathcal{G}}$ be the subspace spanned by the stream functions $\psi_j^{\mathcal{G}}$. We denote by $\Pi_{\mathcal{F}_\varepsilon}$ the orthogonal projection from $S_1(\mathcal{G})$ onto $S_1(\mathcal{F}_\varepsilon)$ and $\Pi^\perp_{\mathcal{F}_\varepsilon}=\operatorname{Id}-\Pi_{\mathcal{F}_\varepsilon}$. From the convergence result (3.17) of Theorem 3.8 and (2.6c) of Proposition 2.2, we deduce that:

$$\eta_k(\varepsilon)=\max_{\substack{\theta\in\mathcal{W}_k^{\mathcal{G}}\\ \theta\neq 0}}\frac{\|\Pi^\perp_{\mathcal{F}_\varepsilon}\theta\|_{S_1(\mathcal{G})}}{\|\theta\|_{S_1(\mathcal{G})}}\longrightarrow 0\quad\text{as}\quad\varepsilon\longrightarrow 0.$$

Considering now the norm of $S_0(\mathcal{G})$ we have also:

$$\max_{\substack{\theta\in\mathcal{W}_k^{\mathcal{G}}\\ \theta\neq 0}}\frac{\|\Pi^\perp_{\mathcal{F}_\varepsilon}\theta\|_{S_0(\mathcal{G})}}{\|\theta\|_{S_0(\mathcal{G})}}\leqslant\eta_k(\varepsilon)\sqrt{\frac{\lambda_k^{\mathcal{G}}}{\lambda_1^{\mathcal{G}}}}.$$

Then, by direct computation, we show that for $\eta_k(\varepsilon)$ small enough:

$$\max_{\substack{\theta\in\mathcal{W}_k^{\mathcal{G}}\\ \theta\neq 0}}\left|\frac{\|\theta\|^2_{S_1(\mathcal{G})}}{\|\theta\|^2_{S_0(\mathcal{G})}}-\frac{\|\Pi_{\mathcal{F}_\varepsilon}\theta\|^2_{S_1(\mathcal{G})}}{\|\Pi_{\mathcal{F}_\varepsilon}\theta\|^2_{S_0(\mathcal{G})}}\right|\leqslant\mathbf{c}_{[\lambda_1^{\mathcal{G}},\lambda_k^{\mathcal{G}}]}\eta_k(\varepsilon),$$

where $\mathbf{c}_{[\lambda_1^{\mathcal{G}},\lambda_k^{\mathcal{G}}]}$ is a positive constant depending on $\lambda_1^{\mathcal{G}}$ and $\lambda_k^{\mathcal{G}}$ only. For $\eta_k(\varepsilon)$ small enough, the family $\{\Pi_{\mathcal{F}_\varepsilon}\psi_1^{\mathcal{G}},\dots,\Pi_{\mathcal{F}_\varepsilon}\psi_k^{\mathcal{G}}\}$ is free and spanned a subspace of $S_1(\mathcal{F}_\varepsilon)$ of dimension k. If follows that:

$$\lambda_k^{\mathcal{F}_\varepsilon}\leqslant\max_{\substack{\theta\in\mathcal{W}_k^{\mathcal{G}}\\ \theta\neq 0}}\frac{\|\Pi_{\mathcal{F}_\varepsilon}\theta\|^2_{S_1(\mathcal{G})}}{\|\Pi_{\mathcal{F}_\varepsilon}\theta\|^2_{S_0(\mathcal{G})}}\leqslant\max_{\substack{\theta\in\mathcal{W}_k^{\mathcal{G}}\\ \theta\neq 0}}\frac{\|\theta\|^2_{S_1(\mathcal{G})}}{\|\theta\|^2_{S_0(\mathcal{G})}}+\max_{\substack{\theta\in\mathcal{W}_k^{\mathcal{G}}\\ \theta\neq 0}}\left|\frac{\|\theta\|^2_{S_1(\mathcal{G})}}{\|\theta\|^2_{S_0(\mathcal{G})}}-\frac{\|\Pi_{\mathcal{F}_\varepsilon}\theta\|^2_{S_1(\mathcal{G})}}{\|\Pi_{\mathcal{F}_\varepsilon}\theta\|^2_{S_0(\mathcal{G})}}\right|\leqslant\lambda_k^{\mathcal{G}}+\mathbf{c}_{[\lambda_1^{\mathcal{G}},\lambda_k^{\mathcal{G}}]}\eta_k(\varepsilon),$$

and the proof of (1.4a) is completed.

Let us address the result (1.4b) about the convergence of the eigenspaces. We consider again a Riesz orthonormal basis $\{\psi_j^{\mathcal{G}},\,j\geqslant 1\}$ of $S_0(\mathcal{G})$ made of eigenfunctions of the Stokes operator in $\mathcal{G}$ (in stream function formulation). Similarly, for every ε, we introduce $\{\psi_j^{\mathcal{F}_\varepsilon},\,j\geqslant 1\}$ a Riesz orthonormal basis in $S_0(\mathcal{F}_\varepsilon)$ made of eigenfunctions of the Stokes operator in $\mathcal{F}_\varepsilon$. Let a positive integer k be given and let $m\notin I_k^{\mathcal{G}}$ (recall that $I_k^{\mathcal{G}}$ is the set of all the indices j such that $\lambda_j^{\mathcal{G}}=\lambda_k^{\mathcal{G}}$). Then:

$$\big(\psi_k^{\mathcal{F}_\varepsilon},\psi_m^{\mathcal{G}}\big)_{S_0(\mathcal{G})}=\frac{1}{\lambda_m^{\mathcal{G}}}\big(\psi_k^{\mathcal{F}_\varepsilon},\Pi_{\mathcal{F}_\varepsilon}\psi_m^{\mathcal{G}}\big)_{S_1(\mathcal{G})}=\frac{\lambda_k^{\mathcal{F}_\varepsilon}}{\lambda_m^{\mathcal{G}}}\big(\psi_k^{\mathcal{F}_\varepsilon},\Pi_{\mathcal{F}_\varepsilon}\psi_m^{\mathcal{G}}\big)_{S_0(\mathcal{G})}.$$

It follows that:

$$\left(\psi_k^{\mathcal{F}_\varepsilon},\left(1-\frac{\lambda_k^{\mathcal{F}_\varepsilon}}{\lambda_m^{\mathcal{G}}}\right)\psi_m^{\mathcal{G}}+\frac{\lambda_k^{\mathcal{F}_\varepsilon}}{\lambda_m^{\mathcal{G}}}\Pi^\perp_{\mathcal{F}_\varepsilon}\psi_m^{\mathcal{G}}\right)_{S_0(\mathcal{G})}=0. \tag{4.1}$$

According to (1.4a), for ε small enough, the eigenvalue $\lambda_k^{\mathcal{F}_\varepsilon}$ is closed to $\lambda_k^{\mathcal{G}}$ and therefore $\lambda_k^{\mathcal{F}_\varepsilon} \neq \lambda_m^{\mathcal{G}}$. We deduce first that:

$$\big(\psi_k^{\mathcal{F}_\varepsilon}, \psi_m^{\mathcal{G}}\big)_{S_0(\mathcal{G})} = \frac{\lambda_k^{\mathcal{F}_\varepsilon}}{\lambda_k^{\mathcal{F}_\varepsilon} - \lambda_m^{\mathcal{G}}} \big(\psi_k^{\mathcal{F}_\varepsilon}, \Pi_{\mathcal{F}_\varepsilon}^{\perp} \psi_m^{\mathcal{G}}\big)_{S_0(\mathcal{G})}, \tag{4.2}$$

and then, summing over all the indices $m \notin I_k^{\mathcal{G}}$:

$$\sum_{m \notin I_k^{\mathcal{G}}} \big(\psi_k^{\mathcal{F}_\varepsilon}, \psi_m^{\mathcal{G}}\big)^2_{S_0(\mathcal{G})} = \big(\lambda_k^{\mathcal{F}_\varepsilon}\big)^2 \sum_{m \notin I_k^{\mathcal{G}}} \frac{\big(\psi_k^{\mathcal{F}_\varepsilon}, \Pi_{\mathcal{F}_\varepsilon}^{\perp} \psi_m^{\mathcal{G}}\big)^2_{S_0(\mathcal{G})}}{\big(\lambda_k^{\mathcal{F}_\varepsilon} - \lambda_m^{\mathcal{G}}\big)^2}.$$

It is known (see for instance [5]) that $\lambda_m^{\mathcal{G}} = \mathscr{O}(m)$ as $m \to +\infty$. On the other hand, for every m:

$$\big|\big(\psi_k^{\mathcal{F}_\varepsilon}, \Pi_{\mathcal{F}_\varepsilon}^{\perp} \psi_m^{\mathcal{G}}\big)^2_{S_0(\mathcal{G})}\big| \leqslant \frac{1}{\lambda_1^{\mathcal{G}}} \|\Pi_{\mathcal{F}_\varepsilon}^{\perp} \psi_m^{\mathcal{G}}\|^2_{S_1(\mathcal{G})},$$

and this quantity tends to 0 along with ε according to the convergence result (3.17) of Theorem 3.8. The dominated convergence Theorem ensures next that:

$$\sum_{m \notin I_k^{\mathcal{G}}} \big(\psi_k^{\mathcal{F}_\varepsilon}, \psi_m^{\mathcal{G}}\big)^2_{S_0(\mathcal{G})} \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0. \tag{4.3}$$

Define now $\underline{\Lambda}_k^{\mathcal{F}_\varepsilon} = \bigoplus_{j \in I_k^{\mathcal{G}}} \Lambda_j^{\mathcal{F}_\varepsilon}$ (recall that $\Lambda_j^{\mathcal{F}_\varepsilon}$ is the eigenspace associated to the eigenvalue $\lambda_j^{\mathcal{F}_\varepsilon}$). Then, for every $\theta \in S_0(\mathcal{G})$:

$$\Pi^{\perp}_{\Lambda_k^{\mathcal{G}}} \Pi_{\underline{\Lambda}_k^{\mathcal{F}_\varepsilon}} \theta = \sum_{m \notin I_k^{\mathcal{G}}} \left(\sum_{j \in I_k^{\mathcal{G}}} (\theta, \psi_j^{\mathcal{F}_\varepsilon})_{S_0(\mathcal{G})} (\psi_j^{\mathcal{F}_\varepsilon}, \psi_m^{\mathcal{G}})_{S_0(\mathcal{G})} \right) \psi_m^{\mathcal{G}},$$

whence we deduce that:

$$\|\Pi^{\perp}_{\Lambda_k^{\mathcal{G}}} \Pi_{\underline{\Lambda}_k^{\mathcal{F}_\varepsilon}} \theta\|^2_{S_0(\mathcal{G})} \leqslant \|\theta\|^2_{S_0(\mathcal{G})} \sum_{j \in I_k^{\mathcal{G}}} \left(\sum_{m \notin I_k^{\mathcal{G}}} (\psi_j^{\mathcal{F}_\varepsilon}, \psi_m^{\mathcal{G}})^2_{S_0(\mathcal{G})} \right), \tag{4.4}$$

and the double sum in the right hand side tends to 0 as ε goes to 0 according to (4.3).

Let consider back the identity (4.1), switching the indices k and m:

$$\left(\psi_m^{\mathcal{F}_\varepsilon}, \left(1 - \frac{\lambda_m^{\mathcal{F}_\varepsilon}}{\lambda_k^{\mathcal{G}}}\right) \psi_k^{\mathcal{G}} + \frac{\lambda_m^{\mathcal{F}_\varepsilon}}{\lambda_k^{\mathcal{G}}} \Pi_{\mathcal{F}_\varepsilon}^{\perp} \psi_k^{\mathcal{G}}\right)_{S_0(\mathcal{G})} = 0. \tag{4.5}$$

Denote by k^- the lowest index in $I_k^{\mathcal{G}}$ and by k^+ the largest index. Recall that the indice m is assumed not belonging to $I_k^{\mathcal{G}}$. It means that either $m \leqslant k^- - 1$ and we can assume that for ε small enough $\lambda_m^{\mathcal{F}_\varepsilon}$ is closed to $\lambda_m^{\mathcal{G}}$ or $m \geqslant k^+ + 1$ and for every ε, $\lambda_m^{\mathcal{F}_\varepsilon} \geqslant \lambda_{k^+ + 1}^{\mathcal{G}} > \lambda_k^{\mathcal{G}}$. In either case, for every ε small enough, $\lambda_m^{\mathcal{F}_\varepsilon} \neq \lambda_k^{\mathcal{G}}$. We deduce that:

$$\big(\psi_m^{\mathcal{F}_\varepsilon}, \psi_k^{\mathcal{G}}\big)_{S_0(\mathcal{G})} = \frac{\lambda_m^{\mathcal{F}_\varepsilon}}{\lambda_m^{\mathcal{F}_\varepsilon} - \lambda_k^{\mathcal{G}}} \big(\psi_m^{\mathcal{F}_\varepsilon}, \Pi_{\mathcal{F}_\varepsilon}^{\perp} \psi_k^{\mathcal{G}}\big)_{S_0(\mathcal{G})},$$

and then, summing over all the indices $m \notin I_k^{\mathcal{G}}$:

$$\sum_{m \notin I_k^{\mathcal{G}}} (\psi_k^{\mathcal{G}}, \psi_m^{\mathcal{F}_\varepsilon})^2_{S_0(\mathcal{G})} = \sum_{m \notin I_k^{\mathcal{G}}} \left(\frac{\lambda_m^{\mathcal{F}_\varepsilon}}{\lambda_m^{\mathcal{F}_\varepsilon} - \lambda_k^{\mathcal{G}}} \right)^2 (\Pi_{\mathcal{F}_\varepsilon}^{\perp} \psi_k^{\mathcal{G}}, \psi_m^{\mathcal{F}_\varepsilon})^2_{S_0(\mathcal{G})}.$$

In the right hand side, the first term in the sum is uniformly bounded (with respect to ε and m) and for the second, Parseval's identity yields:

$$\sum_{m \geqslant 1} (\Pi_{\mathcal{F}_\varepsilon}^{\perp} \psi_k^{\mathcal{G}}, \psi_m^{\mathcal{F}_\varepsilon})^2_{S_0(\mathcal{G})} = \|\Pi_{\mathcal{F}_\varepsilon}^{\perp} \psi_k^{\mathcal{G}}\|^2_{S_0(\mathcal{G})} \leqslant \frac{1}{\lambda_1^{\mathcal{G}}} \|\Pi_{\mathcal{F}_\varepsilon}^{\perp} \psi_k^{\mathcal{G}}\|^2_{S_1(\mathcal{G})}.$$

Altogether, we have proved that

$$\sum_{m \notin I_k^{\mathcal{G}}} (\psi_k^{\mathcal{G}}, \psi_m^{\mathcal{F}_\varepsilon})^2_{S_0(\mathcal{G})} \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

Noticing now that:

$$\|\Pi_{\Lambda_k^{\mathcal{G}}}\Pi^{\perp}_{\underline{\Lambda}_k^{\mathcal{F}_\varepsilon}}\theta\|^2_{S_0(\mathcal{G})}\leqslant\|\theta\|^2_{S_0(\mathcal{G})}\sum_{k\in I_k}\left(\sum_{m\notin I_k}(\psi_k^{\mathcal{G}},\psi_m^{\mathcal{F}_\varepsilon})^2_{S_0(\mathcal{G})}\right),$$

we deduce with (4.4) that, for every $\theta\in S_0(\mathcal{G})$, $\theta\neq 0$:

$$\frac{\|\Pi_{\underline{\Lambda}_k^{\mathcal{F}_\varepsilon}}\theta-\Pi_{\Lambda_k^{\mathcal{G}}}\theta\|^2_{S_0(\mathcal{G})}}{\|\theta\|^2_{S_0(\mathcal{G})}}=\frac{\|\Pi^{\perp}_{\Lambda_k^{\mathcal{G}}}\Pi_{\underline{\Lambda}_k^{\mathcal{F}_\varepsilon}}\theta\|^2_{S_0(\mathcal{G})}+\|\Pi_{\Lambda_k^{\mathcal{G}}}\Pi^{\perp}_{\underline{\Lambda}_k^{\mathcal{F}_\varepsilon}}\theta\|^2_{S_0(\mathcal{G})}}{\|\theta\|^2_{S_0(\mathcal{G})}}\longrightarrow 0\quad\text{as }\varepsilon\longrightarrow 0,$$

and the proof of (1.4b) is completed. □

Proof of Corollary 1.2. The semigroup of the Stokes operator in the domain $\mathcal{G}$ reads:

$$\mathsf{T}_{\mathcal{G}}(t)\theta=\sum_{j\geqslant 1}\big(\theta,\psi_j^{\mathcal{G}}\big)_{S_0(\mathcal{G})}e^{-\lambda_j^{\mathcal{G}}t}\psi_j^{\mathcal{G}},\qquad\text{for all }t\geqslant 0\text{ and }\theta\in S_0(\mathcal{G}).\tag{4.6}$$

Notice that although we use the stream function formulation, we keep the same notation as in (1.6). In (4.6), we reuse the Riesz orthonormal basis $\{\psi_j^{\mathcal{G}},\,j\geqslant 1\}$ of $S_0(\mathcal{G})$ made of eigenfunctions of the Stokes operator in $\mathcal{G}$ that was introduced in the proof of Theorem 1.1 above. With similar notation for the Stokes semigroup in the domain $\mathcal{F}_\varepsilon$, we have, for every $\theta\in S_0(\mathcal{F}_\varepsilon)$ and every $t\geqslant 0$:

$$e^{\lambda_1^{\mathcal{G}}t}\big(\mathsf{T}_{\mathcal{G}}(t)\theta-\mathsf{T}_{\mathcal{F}_\varepsilon}(t)\theta\big)=\sum_{j\geqslant 1}e^{(\lambda_1^{\mathcal{G}}-\lambda_j^{\mathcal{G}})t}\Big(\big(\theta,\psi_j^{\mathcal{G}}\big)_{S_0(\mathcal{G})}\psi_j^{\mathcal{G}}-\big(\theta,\psi_j^{\mathcal{F}_\varepsilon}\big)_{S_0(\mathcal{G})}e^{(\lambda_j^{\mathcal{G}}-\lambda_j^{\mathcal{F}_\varepsilon})t}\psi_j^{\mathcal{F}_\varepsilon}\Big).$$

For every positive integer k, define the spaces $\overline{\Lambda}_k^{\mathcal{G}}=\bigoplus_{j=1}^k\Lambda_j^{\mathcal{G}}$ and $\overline{\Lambda}_k^{\mathcal{F}_\varepsilon}=\bigoplus_{j=1}^k\underline{\Lambda}_j^{\mathcal{F}_\varepsilon}$ where we recall that $\underline{\Lambda}_k^{\mathcal{F}_\varepsilon}=\bigoplus_{j\in I_k^{\mathcal{G}}}\Lambda_j^{\mathcal{F}_\varepsilon}$.

Let now θ be fixed in $S_0(\mathcal{G})$ and any $\zeta>0$ be given. Let N be an integer large enough such that:

$$\|\Pi^{\perp}_{\overline{\Lambda}_N^{\mathcal{G}}}\theta\|_{S_0(\mathcal{G})}\leqslant\zeta.$$

According to Theorem 1.1, for ε small enough:

$$\|\Pi^{\perp}_{\overline{\Lambda}_N^{\mathcal{F}_\varepsilon}}\theta-\Pi^{\perp}_{\overline{\Lambda}_N^{\mathcal{G}}}\theta\|_{S_0(\mathcal{G})}=\|\Pi_{\overline{\Lambda}_N^{\mathcal{F}_\varepsilon}}\theta-\Pi_{\overline{\Lambda}_N^{\mathcal{G}}}\theta\|_{S_0(\mathcal{G})}\leqslant\zeta.$$

Denote by N^+ the largest index in $I_N^{\mathcal{G}}$ and notice now that:

$$\begin{aligned}\|e^{\lambda_1^{\mathcal{G}}t}\big(\mathsf{T}_{\mathcal{G}}(t)\theta-\mathsf{T}_{\mathcal{F}_\varepsilon}(t)\theta\big)\|_{S_0(\mathcal{G})}\leqslant\sum_{j=1}^{N^+}\Big\|\big(\theta,\psi_j^{\mathcal{G}}\big)_{S_0(\mathcal{G})}\psi_j^{\mathcal{G}}-\big(\theta,\psi_j^{\mathcal{F}_\varepsilon}\big)_{S_0(\mathcal{G})}e^{(\lambda_j^{\mathcal{G}}-\lambda_j^{\mathcal{F}_\varepsilon})t}\psi_j^{\mathcal{F}_\varepsilon}\Big\|_{S_0(\mathcal{G})}\\+\|\Pi^{\perp}_{\overline{\Lambda}_N^{\mathcal{G}}}\theta\|_{S_0(\mathcal{G})}+\|\Pi^{\perp}_{\overline{\Lambda}_N^{\mathcal{F}_\varepsilon}}\theta\|_{S_0(\mathcal{G})}.\end{aligned}$$

Invoking again Theorem 1.1, the sum in the right hand side can be made smaller than ζ assuming that ε is small enough. It follows that for N large enough and ε small enough:

$$\|e^{\lambda_1^{\mathcal{G}}t}\big(\mathsf{T}_{\mathcal{G}}(t)\theta-\mathsf{T}_{\mathcal{F}_\varepsilon}(t)\theta\big)\|_{S_0(\mathcal{G})}\leqslant 4\zeta,$$

which concludes the proof of (1.7a).

To prove (1.7b), it suffices to notice that:

$$\sup_{\substack{\theta\in K\\ \theta\neq 0}}\frac{\|\Pi^{\perp}_{\overline{\Lambda}_N^{\mathcal{G}}}\theta\|_{S_0(\mathcal{G})}}{\|\theta\|_{S_0(\mathcal{G})}}\longrightarrow 0\quad\text{as}\quad N\longrightarrow+\infty,$$

and also (according to Theorem 1.1, (1.4b)):

$$\sup_{\substack{\theta\in S_0(\mathcal{G})\\ \theta\neq 0}}\frac{\|\Pi^{\perp}_{\overline{\Lambda}_N^{\mathcal{F}_\varepsilon}}\theta-\Pi^{\perp}_{\overline{\Lambda}_N^{\mathcal{G}}}\theta\|_{S_0(\mathcal{G})}}{\|\theta\|_{S_0(\mathcal{G})}}\longrightarrow 0\quad\text{as}\quad N\longrightarrow+\infty,$$

and the conclusion follows. □

5. Proof of Theorem 1.3

For every $\varepsilon \in (0,\varepsilon_0)$, the solution to the Navier–Stokes equations in $\mathcal{F}_\varepsilon$ with initial data $\psi^0_\varepsilon \in S_0(\mathcal{F}_\varepsilon)$ (as defined in Theorem 2.1) is denoted by ψ_ε. The function ψ_ε belongs to the space:

$$H^1(\mathbb{R}_+; S_{-1}(\mathcal{F}_\varepsilon)) \cap C(\mathbb{R}_+; S_0(\mathcal{F}_\varepsilon)) \cap L^2(\mathbb{R}_+; S_1(\mathcal{F}_\varepsilon)).$$

It satisfies $\psi_\varepsilon(0) = \psi^0_\varepsilon$ and for every $\theta \in S_1(\mathcal{F}_\varepsilon)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi_\varepsilon,\theta)_{S_0(\mathcal{G})} + \nu(\psi_\varepsilon,\theta)_{S_1(\mathcal{G})} = (D^2\theta\nabla^\perp\psi_\varepsilon, \nabla\psi_\varepsilon)_{\mathbf{L}^2(\mathcal{G})} \quad \text{on } \mathbb{R}_+. \tag{5.1}$$

One easily verifies that for any $\varepsilon \in (0,\varepsilon_0)$ and any $t \geqslant 0$:

$$\|\psi_\varepsilon(t)\|_{S_0(\mathcal{G})} \leqslant \|\psi^0_\varepsilon\|_{S_0(\mathcal{G})} e^{-\nu\lambda_1^{\mathcal{G}} t} \quad \text{and} \quad \int_0^{+\infty} \|\psi_\varepsilon(s)\|^2_{S_1(\mathcal{G})}\,\mathrm{d}s \leqslant \frac{1}{2\nu}\|\psi^0_\varepsilon\|^2_{S_0(\mathcal{G})}. \tag{5.2}$$

Assume now that $\psi^0_\varepsilon \rightharpoonup \psi^0$ weak in $S_0(\mathcal{G})$ as ε goes to 0. Then, there exists a function $\psi \in L^\infty(\mathbb{R}_+; S_0(\mathcal{G})) \cap L^2(\mathbb{R}_+; S_1(\mathcal{G}))$ such that up to a subsequence,

$$\psi_\varepsilon \rightharpoonup \psi \text{ weak–}\star \text{ in } L^\infty(\mathbb{R}_+; S_0(\mathcal{G})) \quad \text{and} \quad \psi_\varepsilon \rightharpoonup \psi \text{ weak in } L^2(\mathbb{R}_+; S_1(\mathcal{G})).$$

We shall now establish some estimates for the time derivative of ψ_ε and apply a classical result of compactness (see for instance [8, Theorem 5.1] in the book of Lions). From this point on, our proof differs from the one in [4], where the authors deduce compactness from Arzelà–Ascoli Theorem.

Thus, let Ω be a fixed smooth subdomain included in every $\mathcal{F}_\varepsilon$ (for ε small enough) such that $\mathcal{G}\setminus\overline{\Omega}$ has a finite number of connected components. As usual, every function $\psi \in S_1(\Omega)$ can be seen as a function of $S_1(\mathcal{G})$ once extended by suitable constants in $\mathcal{G}\setminus\overline{\Omega}$. For every $\theta \in S_1(\Omega)$, we deduce from (5.1) that:

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}(\psi_\varepsilon,\theta)_{S_0(\mathcal{G})}\right| \leqslant \nu\|\psi_\varepsilon\|_{S_1(\mathcal{G})}\|\theta\|_{S_1(\Omega)} + \|D^2\theta\|_{L^2(\mathcal{G})}\|\nabla\psi_\varepsilon\|^2_{\mathbf{L}^4(\mathcal{G})} \quad \text{a.e. on } \mathbb{R}_+. \tag{5.3a}$$

On the one hand:

$$\|D^2\theta\|_{L^2(\mathcal{G})} \leqslant \mathbf{c}_{[\mathcal{G}]}\|\Delta\theta\|_{L^2(\Omega)} = \mathbf{c}_{[\mathcal{G}]}\|\theta\|_{S_1(\Omega)}. \tag{5.3b}$$

On the other hand, using Ladyzhenskaya's inequality (see [11, Lemma 3.3, page 291]), we get:

$$\|\nabla\psi_\varepsilon(t)\|^2_{\mathbf{L}^4(\mathcal{G})} \leqslant \mathbf{c}\|\psi_\varepsilon(t)\|_{S_1(\mathcal{G})}\|\psi_\varepsilon(t)\|_{S_0(\mathcal{G})} \quad \text{for a.e. } t \in \mathbb{R}_+. \tag{5.3c}$$

Combining (5.2) and (5.3), we infer that $\partial_t(\mathsf{Q}_\Omega\psi_\varepsilon)$ is uniformly bounded in $L^2(\mathbb{R}_+; S_{-1}(\Omega))$. By definition of Q_Ω as a projector (see Definition 2.3), we have:

$$\|\mathsf{Q}_\Omega\psi_\varepsilon(t)\|_{S_0(\Omega)} \leqslant \|\nabla\psi_\varepsilon(t)\|_{\mathbf{L}^2(\Omega)} \leqslant \|\psi_\varepsilon(t)\|_{S_0(\mathcal{G})} \quad \text{for a.e. } t \in \mathbb{R}_+. \tag{5.4a}$$

On the other hand, according to Proposition 2.4, $\mathsf{Q}_\Omega\psi_\varepsilon(t)$ is in $S_0(\Omega)\cap H^2(\Omega)$ for a.e. t and:

$$\|\Delta\mathsf{Q}_\Omega\psi_\varepsilon(t)\|_{L^2(\Omega)} = \|\Delta\psi_\varepsilon(t)\|_{L^2(\Omega)} \leqslant \|\psi_\varepsilon(t)\|_{S_1(\mathcal{G})} \quad \text{for a.e. } t \in \mathbb{R}_+. \tag{5.4b}$$

From (5.4) and (5.2) we deduce that the functions $\mathsf{Q}_\Omega\psi_\varepsilon$ (for every $\varepsilon \in (0,\varepsilon_0)$) are uniformly bounded in $L^2(\mathbb{R}_+; S_0(\Omega)\cap H^2(\Omega))$ where the Hilbert space $S_0(\Omega)\cap H^2(\Omega)$ is provided with the norm:

$$\left[\|\theta\|^2_{S_0(\Omega)} + \|\Delta\theta\|^2_{L^2(\Omega)}\right]^{1/2}.$$

We can now apply the result of compactness [8, Theorem 5.1] with the triple of compactly embedded Hilbert spaces:

$$S_0(\Omega)\cap H^2(\Omega) \subset S_0(\Omega) \subset S_{-1}(\Omega).$$

We deduce that the family of functions $(\mathsf{Q}_\Omega\psi_\varepsilon)_{\varepsilon>0}$ is precompact in $L^2_{\ell oc}(\mathbb{R}_+; S_0(\Omega))$ and thereby that, up to a subsequence:

$$\mathsf{Q}_\Omega\psi_\varepsilon \longrightarrow \mathsf{Q}_\Omega\psi \quad \text{strong in } L^2_{\ell oc}(\mathbb{R}_+; S_0(\mathcal{G})). \tag{5.5}$$

We consider now a geometric configuration as described in the beginning of Section 3 with in particular two concentric disks $\mathcal{D}_+$ and $\mathcal{D}_-$ of radii R_+ and R_- given in (3.1) for some δ large enough such that, for every ε small enough $\overline{\mathcal{O}}_\varepsilon \subset \mathcal{D}_-$ (what means that the condition (3.2) between δ and ε is satisfied). We choose $\Omega = \mathcal{G}\setminus\overline{\mathcal{D}_-}$ and we denote $\Omega_\Gamma = \Omega\setminus\overline{\mathcal{D}_+}$. Let us recall also the definition of the annulus $\mathcal{C}_+ = \mathcal{D}_+\setminus\overline{\mathcal{D}_-}$. According to Lemma 3.9 (in which Ω plays the role of $\mathcal{F}$, Ω_Γ of $\mathcal{F}_\Gamma$ and $\mathcal{C}_+$ of $\mathcal{F}_\Sigma$), we have:

$$\|\nabla(\psi_\varepsilon - \mathsf{Q}_\Omega\psi_\varepsilon)(t)\|_{\mathbf{L}^2(\Omega_\Gamma)} \leqslant \mathbf{c}_{[\mathcal{G}]}\sqrt{\frac{R_-^2}{R_i^2 - R_-^2}}\|\psi_\varepsilon(t)\|_{S_0(\mathcal{G})} \quad \text{for a.e. } t \in \mathbb{R}_+. \tag{5.6}$$

The same estimate holds true replacing ψ_ε by ψ. We shall now let ε tends to 0 and δ tends to $+\infty$ (or equivalently R_+ tends to 0) and prove that ψ_ε converges strongly to ψ in $L^2_{\ell oc}(\mathbb{R}_+;S_0(\mathcal{G}))$, i.e. remove the projector Q_Ω in (5.5). On the one hand, a.e. on $\mathbb{R}_+$:

$$\|\psi-\psi_\varepsilon\|^2_{S_0(\mathcal{G})}=\|\nabla(\psi_\varepsilon-\psi)\|^2_{\mathbf{L}^2(\mathcal{D}_+)}+\|\nabla(\psi_\varepsilon-\psi)\|^2_{\mathbf{L}^2(\Omega_\Gamma)},\tag{5.7}$$

and the first term in the right hand side tends to 0 in $L^1(\mathbb{R}_+)$ uniformly with respect to ε as δ tends to $+\infty$ according to the dominated convergence Theorem and the uniform estimate (5.2). Concerning the second term, we write that:

$$\begin{aligned}\|\nabla(\psi_\varepsilon(t)-\psi(t))\|_{\mathbf{L}^2(\Omega_\Gamma)}\leqslant\|\nabla(\psi_\varepsilon(t)-\mathsf{Q}_\Omega\psi_\varepsilon(t))\|_{\mathbf{L}^2(\Omega_\Gamma)}+\|\nabla(\mathsf{Q}_\Omega\psi_\varepsilon(t)-\mathsf{Q}_\Omega\psi(t))\|_{\mathbf{L}^2(\Omega_\Gamma)}\\+\|\nabla(\mathsf{Q}_\Omega\psi(t)-\psi(t))\|_{\mathbf{L}^2(\Omega_\Gamma)},\end{aligned}\tag{5.8}$$

and therefore, with (5.6), for every pair (δ,ε) satisfying (3.2), we have a.e. on $\mathbb{R}_+$:

$$\|\nabla(\psi_\varepsilon-\psi)\|_{\mathbf{L}^2(\Omega_\Gamma)}\leqslant\mathbf{c}_{[\mathcal{G}]}\sqrt{\frac{R_-^2}{R_i^2-R_-^2}}\big(\|\psi_\varepsilon\|_{S_0(\mathcal{G})}+\|\psi\|_{S_0(\mathcal{G})}\big)+\|\nabla(\mathsf{Q}_\Omega\psi_\varepsilon-\mathsf{Q}_\Omega\psi)\|_{\mathbf{L}^2(\Omega_\Gamma)}.$$

We can now choose δ large enough in such a way that the first term in the right hand side is small in $L^2(\mathbb{R}_+)$ for every ε satisfying (3.2). Then (δ being fixed) for every $T>0$ we can make the last term in the right hand side also small in $L^2(0,T)$ by choosing ε small enough, according to (5.5). All together with (5.7), we can now conclude that (up to a subsequence in ε):

$$\psi_\varepsilon\longrightarrow\psi\quad\text{strong in }L^2_{\ell oc}(\mathbb{R}_+;S_0(\mathcal{G})).$$

It is classical (see for instance [8, pages 76–77]) to combine this convergence result with the estimate (5.3c) and obtain that, for every $\theta\in S_1(\mathcal{G})$:

$$(D^2\theta,\nabla^\perp\psi_\varepsilon,\nabla\psi_\varepsilon)_{\mathbf{L}^2(\mathcal{G})}\longrightarrow(D^2\theta,\nabla^\perp\psi,\nabla\psi)_{\mathbf{L}^2(\mathcal{G})}\quad\text{weak in }L^2_{\ell oc}(\mathbb{R}_+)\quad\text{as}\quad\varepsilon\longrightarrow 0.$$

For every $\varepsilon\in(0,\varepsilon_0)$ and a.e. $t\in\mathbb{R}_+$, denote by $\xi_\varepsilon(t)$ the linear form $\partial_t\psi_\varepsilon(t)$ in $S_{-1}(\mathcal{F}_\varepsilon)$. The same computations as for (5.3) lead for every $\theta\in S_1(\mathcal{F}_\varepsilon)$ to:

$$\big|\langle\xi_\varepsilon,\theta\rangle_{S_{-1}(\mathcal{F}_\varepsilon),S_1(\mathcal{F}_\varepsilon)}\big|\leqslant\nu\|\psi_\varepsilon\|_{S_1(\mathcal{G})}\|\theta\|_{S_1(\mathcal{F}_\varepsilon)}+\|D^2\theta\|_{L^2(\mathcal{F}_\varepsilon)}\|\nabla\psi_\varepsilon\|^2_{\mathbf{L}^4(\mathcal{G})}\quad\text{a.e. on }\mathbb{R}_+.\tag{5.9}$$

Since $S_1(\mathcal{F}_\varepsilon)$ is a closed subspace of $S_1(\mathcal{G})$, we can extend by 0 every linear form $\xi_\varepsilon(t)$ on $S_1(\mathcal{G})$ (consider an orthogonal supplement of $S_1(\mathcal{F}_\varepsilon)$ in $S_1(\mathcal{G})$, which is actually made explicit in Theorem 3.8). We still denote by $\xi_\varepsilon(t)$ this extended form. From (5.9) and the uniform estimates (5.2) and (5.3c), we deduce that ξ_ε is uniformly bounded with respect to ε in $L^2(\mathbb{R}_+;S_{-1}(\mathcal{G}))$ and therefore that there exists $\xi\in L^2(\mathbb{R}_+;S_{-1}(\mathcal{G}))$ such that (up to a subsequence):

$$\xi_\varepsilon\rightharpoonup\xi\quad\text{weak in }L^2(\mathbb{R}_+;S_{-1}(\mathcal{G})).\tag{5.10}$$

Consider again an open set $\Omega=\mathcal{G}\setminus\overline{\mathcal{D}_-}$ and let θ be in $\mathscr{D}(\mathbb{R};S_1(\Omega))$. Then, for every ε small enough $\Omega\subset\mathcal{F}_\varepsilon$ and:

$$\int_{\mathbb{R}_+}(\psi_\varepsilon(s),\partial_t\theta(s))_{S_0(\mathcal{G})}\,\mathrm{d}s=(\psi^0_\varepsilon,\theta(0))_{S_0(\mathcal{G})}-\int_{\mathbb{R}_+}\big\langle\xi_\varepsilon(s),\theta(s)\big\rangle_{S_{-1}(\mathcal{G}),S_1(\mathcal{G})}\,\mathrm{d}s.$$

Letting ε go to 0, we obtain that:

$$\int_{\mathbb{R}_+}(\psi(s),\partial_t\theta(s))_{S_0(\mathcal{G})}\,\mathrm{d}s=(\psi^0,\theta(0))_{S_0(\mathcal{G})}-\int_{\mathbb{R}_+}\big\langle\xi(s),\theta(s)\big\rangle_{S_{-1}(\mathcal{G}),S_1(\mathcal{G})}\,\mathrm{d}s.$$

This identity being true for every $\theta\in\mathscr{D}(\mathbb{R}_+;S_1(\Omega))$ and for every Ω, we conclude with Theorem 3.8 (Ω plays the role of $\mathcal{F}$) that it is also true for every $\theta\in\mathscr{D}(\mathbb{R}_+;S_1(\mathcal{G}))$ and therefore that $\xi=\partial_t\psi$ and $\psi(0)=\psi^0$.

Consider back an open set $\Omega=\mathcal{G}\setminus\overline{\mathcal{D}_-}$ and let θ be in $S_1(\Omega)$. Then, we can pass to the limit in every term of the equation (5.1) and obtain that ψ belongs to the space:

$$L^2(\mathbb{R}_+;S_1(\mathcal{G}))\cap C(\mathbb{R}_+;S_0(\mathcal{G}))\cap H^1(\mathbb{R}_+;S_{-1}(\mathcal{G})),$$

and solves the Cauchy problem:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\psi,\theta)_{S_0(\mathcal{G})}+\nu(\psi,\theta)_{S_1(\mathcal{G})}&=(D^2\theta\nabla^\perp\psi,\nabla\psi)_{\mathbf{L}^2(\mathcal{G})}&&\text{on }\mathbb{R}_+,\\ \psi(0)&=\psi^0&&\text{in }\mathcal{G}.\end{aligned}$$

This is true for every Ω and therefore, invoking again Theorem 3.8, this is true for every $\theta\in S_1(\mathcal{G})$.

We shall prove now the convergence results (1.11). For every ε, the vorticity field ω_ε belongs to the space:

$$H^1(\mathbb{R}_+;V_{-1}(\mathcal{F}_\varepsilon))\cap C(\mathbb{R}_+;V_0(\mathcal{F}_\varepsilon)\cap L^2(\mathbb{R}_+;V_1(\mathcal{F}_\varepsilon)),$$

and solves for every $\theta\in V_1(\mathcal{F}_\varepsilon)$, the Cauchy problem:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\omega_\varepsilon,\theta)_{V_0(\mathcal{F}_\varepsilon)}+\nu(\omega_\varepsilon,\theta)_{V_1(\mathcal{F}_\varepsilon)}+(\omega_\varepsilon\nabla^\perp\psi_\varepsilon,\nabla\mathsf{Q}_{\mathcal{F}_\varepsilon}\theta)_{\mathbf{L}^2(\mathcal{F}_\varepsilon)}=0\quad\text{on }\mathbb{R}_+,\tag{5.11a}$$

$$\omega_\varepsilon(0)=\omega_\varepsilon^0\quad\text{in }\mathcal{F}_\varepsilon.\tag{5.11b}$$

Let now χ be in $\mathscr{D}(\mathcal{G}\setminus\{r\})$. Then, for every ε small enough, the domain $\mathcal{O}_\varepsilon$ and the support $\mathcal{U}$ of the function χ are disjoint. We choose $\theta=\theta_\varepsilon=\mathsf{P}_{\mathcal{F}_\varepsilon}\chi^2\omega_\varepsilon$ in (5.11a) and this function is indeed in $V_1(\mathcal{F}_\varepsilon)$ because $\mathsf{Q}_{\mathcal{F}_\varepsilon}\mathsf{P}_{\mathcal{F}_\varepsilon}\chi^2\omega_\varepsilon=\chi^2\omega_\varepsilon$ belongs to $S_0(\mathcal{F}_\varepsilon)$. On the one hand, we have:

$$(\omega_\varepsilon,\theta_\varepsilon)_{V_0(\mathcal{F}_\varepsilon)}=(\omega_\varepsilon,\chi^2\omega_\varepsilon)_{V_0(\mathcal{F}_\varepsilon)}=\|\chi\omega_\varepsilon\|^2_{L^2(\mathcal{G})}.\tag{5.12a}$$

On the other hand:

$$\begin{aligned}(\nabla\mathsf{Q}_{\mathcal{F}_\varepsilon}\mathsf{P}_{\mathcal{F}_\varepsilon}\chi^2\omega_\varepsilon,\nabla\mathsf{Q}_{\mathcal{F}_\varepsilon}\omega_\varepsilon)_{\mathbf{L}^2(\mathcal{F}_\varepsilon)}&=(\nabla\chi^2\omega_\varepsilon,\nabla\omega_\varepsilon)_{\mathbf{L}^2(\mathcal{G})}\\&=\|\nabla(\chi\omega_\varepsilon)\|^2_{L^2(\mathcal{G})}-\|\omega_\varepsilon\nabla\chi\|^2_{L^2(\mathcal{G})}.\end{aligned}\tag{5.12b}$$

Using (5.12) in (5.11a), we obtain:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\chi\omega_\varepsilon\|^2_{L^2(\mathcal{G})}+\nu\|\nabla(\chi\omega_\varepsilon)\|^2_{L^2(\mathcal{G})}=\\ \nu\|\omega_\varepsilon\nabla\chi\|^2_{L^2(\mathcal{G})}-(\omega_\varepsilon\nabla^\perp\psi_\varepsilon,\chi\omega_\varepsilon\nabla\chi)_{\mathbf{L}^2(\mathcal{G})}-(\chi\omega_\varepsilon\nabla^\perp\psi_\varepsilon,\nabla(\chi\omega_\varepsilon))_{\mathbf{L}^2(\mathcal{G})}.\end{aligned}\tag{5.13}$$

Applying Hölder's inequality to the second term in the right hand side term leads to:

$$|(\omega_\varepsilon\nabla^\perp\psi_\varepsilon,\chi\omega_\varepsilon\nabla\chi)_{\mathbf{L}^2(\mathcal{G})}|\leqslant\|\nabla\chi\|_{L^\infty(\mathcal{G})}\|\omega_\varepsilon\|_{L^2(\mathcal{G})}\|\nabla\psi_\varepsilon\|_{\mathbf{L}^4(\mathcal{G})}\|\chi\omega_\varepsilon\|_{L^4(\mathcal{G})},\tag{5.14a}$$

Then, Ladyzhenskaya's inequality yields:

$$\|\nabla\psi_\varepsilon\|_{\mathbf{L}^4(\mathcal{G})}\leqslant\mathbf{c}\|\psi_\varepsilon\|^{1/2}_{S_0(\mathcal{G})}\|\omega_\varepsilon\|^{1/2}_{L^2(\mathcal{G})},\tag{5.14b}$$

$$\|\chi\omega_\varepsilon\|_{L^4(\mathcal{G})}\leqslant\mathbf{c}\|\chi\omega_\varepsilon\|^{1/2}_{L^2(\mathcal{G})}\|\nabla(\chi\omega_\varepsilon)\|^{1/2}_{\mathbf{L}^2(\mathcal{G})}.\tag{5.14c}$$

Putting together the estimates (5.14) and applying Young's inequality, we obtain:

$$\begin{aligned}|(\omega_\varepsilon\nabla^\perp\psi_\varepsilon,\chi\omega_\varepsilon\nabla\chi)_{\mathbf{L}^2(\mathcal{G})}|\leqslant\frac{\nu}{2}\|\nabla(\chi\omega_\varepsilon)\|^2_{\mathbf{L}^2(\mathcal{G})}\\+\mathbf{c}_{[\nu]}\|\nabla\chi\|^2_{L^\infty(\mathcal{G})}\|\omega_\varepsilon\|^2_{L^2(\mathcal{G})}+\mathbf{c}_{[\nu]}\|\psi_\varepsilon\|^2_{S_0(\mathcal{G})}\|\omega_\varepsilon\|^2_{L^2(\mathcal{G})}\|\chi\omega_\varepsilon\|^2_{L^2(\mathcal{G})}\end{aligned}\tag{5.15a}$$

The last term in the right hand side of (5.13) can be dealt with the same way:

$$|(\chi\omega_\varepsilon\nabla^\perp\psi_\varepsilon,\nabla(\chi\omega_\varepsilon))_{\mathbf{L}^2(\mathcal{G})}|\leqslant\frac{\nu}{2}\|\nabla(\chi\omega_\varepsilon)\|^2_{\mathbf{L}^2(\mathcal{G})}+\mathbf{c}_{[\nu]}\|\psi_\varepsilon\|^2_{S_0(\mathcal{G})}\|\omega_\varepsilon\|^2_{L^2(\mathcal{G})}\|\chi\omega_\varepsilon\|^2_{L^2(\mathcal{G})}.\tag{5.15b}$$

Going back to (5.13) and using both estimates (5.15), we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\chi\omega_\varepsilon\|^2_{L^2(\mathcal{G})}+\Big(\nu\lambda^d_{\mathcal{U}}-\mathbf{c}_{[\nu]}\|\psi_\varepsilon\|^2_{S_0(\mathcal{G})}\|\omega_\varepsilon\|^2_{L^2(\mathcal{G})}\Big)\|\chi\omega_\varepsilon\|^2_{L^2(\mathcal{G})}\leqslant\mathbf{c}_{[\nu]}\|\nabla\chi\|^2_{L^\infty(\mathcal{G})}\|\omega_\varepsilon\|^2_{L^2(\mathcal{G})},$$

where $\lambda^d_{\mathcal{U}}$ is the smallest eigenvalue of the Dirichlet Laplacian in the domain $\mathcal{U}$ (the support of the function χ). We can now apply Grönwall's inequality and thanks to the uniform estimates (5.2), we prove first that $\|\chi\omega_\varepsilon\|_{L^2(\mathcal{G})}$ is uniformly bounded in $L^\infty(\mathbb{R}_+)$ and next, with (5.13), that $\|\nabla(\chi\omega_\varepsilon)\|^2_{L^2(\mathcal{G})}$ is uniformly bounded in $L^2(\mathbb{R}_+)$. The convergence results (1.11) follow and the proof is completed.

References

[1] F. Caubet, C. Conca, and M. Godoy. On the detection of several obstacles in 2D Stokes flow: topological sensitivity and combination with shape derivatives. *Inverse Probl. Imaging*, 10(2):327–367, 2016.

[2] J.-L. Guermond and L. Quartapelle. Equivalence of **u**-p and ζ-ψ formulations of the time-dependent Navier-Stokes equations. *Internat. J. Numer. Methods Fluids*, 18(5):471–487, 1994.

[3] J. He and D. Iftimie. A small solid body with large density in a planar fluid is negligible. *Journal of Dynamics and Differential Equations*, 31(3):1671–1688, Sep 2019.

[4] D. Iftimie, M. C. Lopes Filho, and H. J. Nussenzveig Lopes. Two-dimensional incompressible viscous flow around a small obstacle. *Math. Ann.*, 336(2):449–489, 2006.

[5] A. A. Ilyin. On the spectrum of the Stokes operator. *Funktsional. Anal. i Prilozhen.*, 43(4):14–25, 2009.

[6] C. Lacave and T. Takahashi. Small moving rigid body into a viscous incompressible fluid. *Archive for Rational Mechanics and Analysis*, 223(3):1307–1335, Mar 2017.

[7] J. Lequeurre and A. Munnier. Vorticity and stream function formulations for the 2d navier–stokes equations in a bounded domain. *Journal of Mathematical Fluid Mechanics*, 22(2):15, 2020.

[8] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod; Gauthier-Villars, Paris, 1969.

[9] W. McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.

[10] A. Munnier and K. Ramdani. Conformal mapping for cavity inverse problem: an explicit reconstruction formula. *Applicable Analysis*, 96(1):108–129, 2017.

[11] R. Temam. *Navier-Stokes equations. Theory and numerical analysis*. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. Studies in Mathematics and its Applications, Vol. 2.